Polyvector-valued Gauge Field Theories and Quantum Mechanics in Noncommutative Clifford Spaces

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Abstract

The basic ideas and results behind polyvector-valued gauge field theories and Quantum Mechanics in Noncommutative Clifford spaces are presented. The construction of Noncommutative Clifford-space gravity as polyvector-valued gauge theories of twisted diffeomorphisms in Clifford-spaces would require quantum Hopf algebraic deformations of Clifford algebras.

Clifford algebras are deeply related and essential tools in many aspects in Physics. The Extended Relativity theory in Clifford-spaces (C-spaces) is a natural extension of the ordinary Relativity theory [3] whose generalized polyvector-valued coordinates are Clifford-valued quantities which incorporate lines, areas, volumes, hyper-volumes.... degrees of freedom associated with the collective particle, string, membrane, p-brane,... dynamics of p-loops (closed p-branes) in D-dimensional target spacetime backgrounds.

It was recently shown [1] how an unification of Conformal Gravity and a $U(4) \times U(4)$ Yang-Mills theory in four dimensions could be attained from a Clifford Gauge Field Theory in C-spaces (Clifford spaces) based on the (complex) Clifford $Cl(4, C)$ algebra underlying a complexified four dimensional spacetime (8 real dimensions). Clifford-space tensorial-gauge fields generalizations of Yang-Mills theories allows to predict the existence of new particles (bosons, fermions) and tensor-gauge fields of higher-spins in the 10 TeV regime [2]. Tensorial Generalized Yang-Mills in C-spaces (Clifford spaces) based on poly-vector valued (anti-symmetric tensor fields) gauge fields $\mathcal{A}_M(\mathbf{X})$ and field strengths $\mathcal{F}_{MN}(\mathbf{X})$ have been studied in [2], [3] where $\mathbf{X} = X_M \Gamma^M$ is a C-space polyvector valued coordinate

$$\mathbf{X} \;=\; \sigma\, \mathbf{1} \;+\; x_\mu\, \gamma^\mu \;+\; x_{\mu_1\mu_2}\, \gamma^{\mu_1} \wedge \gamma^{\mu_2} \;+\; x_{\mu_1\mu_2\mu_3}\, \gamma^{\mu_1} \wedge \gamma^{\mu_2} \wedge \gamma^{\mu_3} \;+\; ...... \;+$$

$$x_{\mu_1\mu_2\mu_3......\mu_d}\ \gamma^{\mu_1}\wedge\gamma^{\mu_2}\wedge\ \gamma^{\mu_3}.......\ \wedge\gamma^{\mu_d} \quad (1)$$

In order to match dimensions in each term of (1) a length scale parameter must be suitably introduced. In [3] we introduced the Planck scale as the expansion parameter in (1). The scalar component σ of the C-space poly-vector valued coordinate $\mathbf{X}$ was interpreted by [4] as a Stuckelberg time-like parameter that solves the problem of time in Cosmology in a very elegant fashion.

A Clifford gauge field theory in the C-space associated with the ordinary $4D$ spacetime requires $\mathcal{A}_M(\mathbf{X}) = \mathcal{A}_M^A(\mathbf{X})\ \Gamma_A$ that is a poly-vector valued gauge field where M represents the poly-vector index associated with the C-space, and whose gauge group $\mathcal{G}$ is itself based on the Clifford algebra $Cl(3,1)$ of the tangent space spanned by 16 generators Γ_A. The expansion of the poly-vector Clifford-algebra-valued gauge field $\mathcal{A}_M^A$, for $fixed$ values of A, is of the form

$$\mathcal{A}_M^A\,\Gamma^M\ =\ \Phi^A + \mathcal{A}_\mu^A\,\gamma^\mu + \mathcal{A}_{\mu_1\mu_2}^A\,\gamma^{\mu_1}\wedge\gamma^{\mu_2} + \mathcal{A}_{\mu_1\mu_2\mu_3}^A\,\gamma^{\mu_1}\wedge\gamma^{\mu_2}\wedge\gamma^{\mu_3} + ... \quad (2)$$

The index A spans the 16-dim Clifford algebra $Cl(3,1)$ of the tangent space such as

$$\Phi^A\ \Gamma_A\ =\ \Phi\ +\ \Phi^a\ \Gamma_a\ +\ \Phi^{ab}\ \Gamma_{ab}\ +\ \Phi^{abc}\ \Gamma_{abc}\ +\ \Phi^{abcd}\ \Gamma_{abcd}. \quad (3a)$$

$$\mathcal{A}_\mu^A\ \Gamma_A\ =\ \mathcal{A}_\mu\ +\ \mathcal{A}_\mu^a\ \Gamma_a\ +\ \mathcal{A}_\mu^{ab}\ \Gamma_{ab}\ +\ \mathcal{A}_\mu^{abc}\ \Gamma_{abc}\ +\ \mathcal{A}_\mu^{abcd}\ \Gamma_{abcd}. \quad (3b)$$

$$\mathcal{A}_{\mu\nu}^A\ \Gamma_A\ =\ \mathcal{A}_{\mu\nu}\ +\ \mathcal{A}_{\mu\nu}^a\ \Gamma_a\ +\ \mathcal{A}_{\mu\nu}^{ab}\ \Gamma_{ab}\ +\ \mathcal{A}_{\mu\nu}^{abc}\ \Gamma_{abc}\ +\ \mathcal{A}_{\mu\nu}^{abcd}\ \Gamma_{abcd}. \quad (3c)$$

etc......

In order to match dimensions in each term of (2) another length scale parameter must be suitably introduced. For example, since $\mathcal{A}_{\mu\nu\rho}^A$ has dimensions of $(length)^{-3}$ and $\mathcal{A}_\mu^A$ has dimensions of $(length)^{-1}$ one needs to introduce another length parameter in order to match dimensions. This length parameter does not need to coincide with the Planck scale. The Clifford-algebra-valued gauge field $\mathcal{A}_\mu^A(x^\mu)\Gamma_A$ in ordinary spacetime is naturally embedded into a far richer object $\mathcal{A}_M^A(\mathbf{X})\Gamma_A$ in C-spaces. The advantage of recurring to C-spaces associated with the $4D$ spacetime manifold is that one can have a (complex) Conformal Gravity, Maxwell and $U(4)\times U(4)$ Yang-Mills unification in a very geometric fashion as provided by [1]

Field theories in Noncommutative spacetimes have been the subject of intense investigation in recent years, see [8] and references therein. Star Product deformations of Clifford Gauge Field Theories based on ordinary Noncommutative spacetimes are straightforward generalizations of the work by [5]. The wedge star product of two Clifford-valued one-forms is defined as

$$\mathbf{A}\ \wedge_*\ \mathbf{A}\ =\ \Big(\ (\mathcal{A}_\mu^A\ *\ \mathcal{A}_\nu^B)\ \Gamma_A\ \Gamma_B\ \Big)\ dx^\mu\wedge dx^\nu\ =$$

$$\frac{1}{2} \ \left(\ (\mathcal{A}^A_\mu *_s \mathcal{A}^B_\nu) \ [\Gamma_A, \ \Gamma_B] \ + \ (\mathcal{A}^A_\mu *_a \mathcal{A}^B_\nu) \ \{\Gamma_A, \ \Gamma_B\} \ \right) \ dx^\mu \ \wedge \ dx^\nu. \quad (4)$$

In the case when the coordinates don't commute $[x^\mu, x^\nu] = \theta^{\mu\nu}$ (constants), the cosine (symmetric) star product is defined by [5]

$$f *_s g \ \equiv \ \frac{1}{2} (f * g + g * f) \ = \ f \ g + \ \left(\frac{i}{2}\right)^2 \ \theta^{\mu\nu} \ \theta^{\kappa\lambda} \ (\partial_\mu \ \partial_\kappa f) \ (\partial_\nu \ \partial_\lambda g) \ + \ O(\theta^4). \quad (5)$$

and the sine (anti-symmetric Moyal bracket) star product is

$$f *_a g \ \equiv \ \frac{1}{2} (f * g - g * f) \ = \ \left(\frac{i}{2}\right) \ \theta^{\mu\nu} \ (\partial_\mu f) \ (\partial_\nu g) \ + \left(\frac{i}{2}\right)^3 \ \theta^{\mu\nu} \ \theta^{\kappa\lambda} \ \theta^{\alpha\beta} \ (\partial_\mu \ \partial_\kappa \ \partial_\alpha f) \ (\partial_\nu \ \partial_\lambda \ \partial_\beta g) \ + \ O(\theta^5). \quad (6)$$

Notice that both commutators *and* anticommutators of the gammas appear in the star deformed products in (4). The star product deformations of the gauge field strengths in the case of the $U(2,2)$ gauge group were given by [5] and the expressions for the star product deformed action are very cumbersome .

In this letter we proceed with the construction of Polyvector-valued Gauge Field Theories in *noncommutative* Clifford Spaces (C-spaces) which are polyvector-valued *extensions* and *generalizations* of the ordinary *noncommutative* spacetimes. We begin firstly by writing the commutators $[\Gamma_A, \Gamma_B]$. For $pq = odd$ one has [7]

$$[\ \gamma_{b_1 b_2 .....b_p}, \ \gamma^{a_1 a_2 ......a_q} \ ] \ = \ 2\gamma^{a_1 a_2 ......a_q}_{b_1 b_2 .....b_p} \ - \frac{2p!q!}{2!(p-2)!(q-2)!} \ \delta^{[a_1 a_2}_{[b_1 b_2} \ \gamma^{a_3 ....a_q]}_{b_3 .....b_p]} \ + \ \frac{2p!q!}{4!(p-4)!(q-4)!} \ \delta^{[a_1 ....a_4}_{[b_1 ....b_4} \ \gamma^{a_5 ....a_q]}_{b_5 .....b_p]} \ - \ ...... \quad (7)$$

for $pq = even$ one has

$$[\ \gamma_{b_1 b_2 .....b_p}, \ \gamma^{a_1 a_2 ......a_q} \ ] \ = \ - \ \frac{(-1)^{p-1} 2p!q!}{1!(p-1)!(q-1)!} \ \delta^{[a_1}_{[b_1} \ \gamma^{a_2 a_3 ....a_q]}_{b_2 b_3 .....b_p]} \ - \frac{(-1)^{p-1} 2p!q!}{3!(p-3)!(q-3)!} \ \delta^{[a_1 ....a_3}_{[b_1 ....b_3} \ \gamma^{a_4 ....a_q]}_{b_4 .....b_p]} \ + \ ...... \quad (8)$$

The anti-commutators for $pq = even$ are

$$\{ \ \gamma_{b_1 b_2 .....b_p}, \gamma^{a_1 a_2 ......a_q} \ \} \ = \ 2\gamma^{a_1 a_2 ......a_q}_{b_1 b_2 .....b_p} \ - \frac{2p!q!}{2!(p-2)!(q-2)!} \ \delta^{[a_1 a_2}_{[b_1 b_2} \ \gamma^{a_3 ....a_q]}_{b_3 .....b_p]} \ + \ \frac{2p!q!}{4!(p-4)!(q-4)!} \ \delta^{[a_1 ....a_4}_{[b_1 ....b_4} \ \gamma^{a_5 ....a_q]}_{b_5 .....b_p]} \ - \ ...... \quad (9)$$

and the anti-commutators for $pq = odd$ are

$$\{ \gamma_{b_1 b_2 .....b_p}, \ \gamma^{a_1 a_2 ......a_q} \} \ = \ - \frac{(-1)^{p-1} 2p!q!}{1!(p-1)!(q-1)!} \ \delta^{[a_1}_{[b_1} \ \gamma^{a_2 a_3 ....a_q]}_{b_2 b_3 .....b_p]} \ - \frac{(-1)^{p-1} 2p!q!}{3!(p-3)!(q-3)!} \ \delta^{[a_1 ....a_3}_{[b_1 ....b_3} \ \gamma^{a_4 ....a_q]}_{b_4 .....b_p]} \ + \ ...... \tag{10}$$

For instance,

$$\mathcal{J}^a_b \ = \ [\gamma_b, \gamma^a] \ = \ 2\gamma^a_b; \quad \mathcal{J}^{a_1 a_2}_{b_1 b_2}, \ = \ [\gamma_{b_1 b_2}, \gamma^{a_1 a_2}] \ = \ - 8 \ \delta^{[a_1}_{[b_1} \ \gamma^{a_2]}_{b_2]}. \tag{11}$$

$$\mathcal{J}^{a_1 a_2 a_3}_{b_1 b_2 b_3} \ = \ [\gamma_{b_1 b_2 b_3}, \gamma^{a_1 a_2 a_3}] \ = \ 2 \ \gamma^{a_1 a_2 a_3}_{b_1 b_2 b_3} \ - 36 \ \delta^{[a_1 a_2}_{[b_1 b_2} \ \gamma^{a_3]}_{b_3]}. \tag{12}$$

$$\mathcal{J}^{a_1 a_2 a_3 a_4}_{b_1 b_2 b_3 b_4} \ = \ [\gamma_{b_1 b_2 b_3 b_4}, \gamma^{a_1 a_2 a_3 a_4}] \ = \ -32 \, \delta^{[a_1}_{[b_1} \ \gamma^{a_2 a_3 a_4]}_{b_2 b_3 b_4]} + 192 \, \delta^{[a_1 a_2 a_3}_{[b_1 b_2 b_3} \ \gamma^{a_4]}_{b_4]}. \tag{13}$$

etc...

The second step is to write down the *noncommutative* algebra associated with the noncommuting poly-vector-valued coordinates in $D = 4$ and which can be obtained from the Clifford algebra (7-10) by performing the following replacements (and relabeling indices)

$$\gamma^\mu \leftrightarrow X^\mu, \quad \gamma^{\mu_1 \mu_2} \leftrightarrow X^{\mu_1 \mu_2}, \quad .... \ \gamma^{\mu_1 \mu_2 .....\mu_n} \leftrightarrow X^{\mu_1 \mu_2 ....\mu_n}. \tag{14}$$

When the spacetime metric components $g_{\mu\nu}$ are *constant*, from the replacements (14) and the Clifford algebra (7-10) (after one relabels indices), one can then construct the following *noncommutative* algebra among the poly-vector-valued coordinates in $D = 4$, and *obeying* the Jacobi identities, given by the relations

$$[\ X^{\mu_1}, \ X^{\mu_2} \ ]_* \ = \ X^{\mu_1} \ * \ X^{\mu_2} \ - \ X^{\mu_2} \ * \ X^{\mu_1} \ = \ 2 \ X^{\mu_1 \mu_2}. \tag{15}$$

$$[\ X^{\mu_1 \mu_2}, \ X^\nu \ ]_* \ = \ 4 \ (\ g^{\mu_2 \nu} \ X^{\mu_1} \ - \ g^{\mu_1 \nu} \ X^{\mu_2} \ ). \tag{16}$$

$$[\ X^{\mu_1 \mu_2 \mu_3}, \ X^\nu \ ]_* \ = \ 2 \, X^{\mu_1 \mu_2 \mu_3 \nu}, \quad [\ X^{\mu_1 \mu_2 \mu_3 \mu_4}, \ X^\nu \ ]_* \ = \ -8 \, g^{\mu_1 \nu} \ X^{\mu_2 \mu_3 \mu_4} \ \pm ...... \tag{17}$$

$$[\ X^{\mu_1 \mu_2}, \ X^{\nu_1 \nu_2} \ ]_* \ = \ - 8 \ g^{\mu_1 \nu_1} \ X^{\mu_2 \nu_2} \ + \ 8 \ g^{\mu_1 \nu_2} \ X^{\mu_2 \nu_1} \ + \ 8 \ g^{\mu_2 \nu_1} \ X^{\mu_1 \nu_2} \ - \ 8 \ g^{\mu_2 \nu_2} \ X^{\mu_1 \nu_1}. \tag{18}$$

$$[\ X^{\mu_1 \mu_2 \mu_3}, \ X^{\nu_1 \nu_2} \ ]_* \ = \ 12 \ g^{\mu_1 \nu_1} \ X^{\mu_2 \mu_3 \nu_2} \ \pm \ .... \tag{19}$$

$$[\ X^{\mu_1 \mu_2 \mu_3}, \ X^{\nu_1 \nu_2 \nu_3} \ ]_* \ = \ - 36 \ G^{\mu_1 \mu_2 \ \nu_1 \nu_2} \ X^{\mu_3 \nu_3} \ \pm \ ...... \tag{20}$$

$$[\,X^{\mu_1\mu_2\mu_3\mu_4},\ X^{\nu_1\nu_2}\,]_* \ = \ -16\ g^{\mu_1\nu_1}\ X^{\mu_2\mu_3\mu_4\nu_2}\ \pm ...... \tag{21}$$

$$[\,X^{\mu_1\mu_2\mu_3\mu_4},\ X^{\nu_1\nu_2}\,]_* \ = \ -16\ g^{\mu_1\nu_1}\ X^{\mu_2\mu_3\mu_4\nu_2}\ +\ 16\ g^{\mu_1\nu_2}\ X^{\mu_2\mu_3\mu_4\nu_1}\ -\ .... \tag{22}$$

$$[\,X^{\mu_1\mu_2\mu_3\mu_4},\ X^{\nu_1\nu_2\nu_3}\,]_* \ = \ 48\,G^{\mu_1\mu_2\mu_3\ \nu_1\nu_2\nu_3}\ X^{\mu_4} - 48\,G^{\mu_1\mu_2\mu_4\ \nu_1\nu_2\nu_3}\ X^{\mu_3} + ... \tag{23}$$

$$[\,X^{\mu_1\mu_2\mu_3\mu_4},\ X^{\nu_1\nu_2\nu_3\nu_4}\,]_* \ = \ 192\ G^{\mu_1\mu_2\mu_3\ \nu_1\nu_2\nu_3}\ X^{\mu_4\nu_4}\ -\ .... \tag{24}$$

etc...... where

$$G^{\mu_1\mu_2......\mu_n\ \nu_1\nu_2......\nu_n}\ =\ g^{\mu_1\nu_1}\ g^{\mu_2\nu_2}\ ....\ g^{\mu_n\nu_n}\ +\ signed\ permutations \tag{25}$$

The metric components $G^{\mu_1\mu_2......\mu_n\ \nu_1\nu_2......\nu_n}$ in C-space can also be written as a determinant of the $n\times n$ matrix $\mathbf{G}$ whose entries are $g^{\mu_I\nu_J}$

$$det\ \mathbf{G}_{n\times n}\ =\ \frac{1}{n!}\ \epsilon_{i_1i_2.....i_n}\ \epsilon_{j_1j_2....j_n}\ g^{\mu_{i_1}\nu_{j_1}}\ g^{\mu_{i_2}\nu_{j_2}}\ ....\ g^{\mu_{i_n}\nu_{j_n}}. \tag{26}$$

$i_1, i_2, ..., i_n \subset I = 1,2,....., D$ and $j_1, j_2, ..., j_n \subset J = 1,2,....., D$. One must also include in the C-space metric G^{MN} the (Clifford) scalar-scalar component G^{00} (that could be related to the dilaton field) and the pseudo-scalar/pseudo-scalar component $G^{\mu_1\mu_2.....\mu_D\ \nu_1\nu_2......\nu_D}$ (that could be related to the axion field).

One must emphasize that when the spacetime metric components $g_{\mu\nu}$ are *no* longer *constant*, the noncommutative algebra among the poly-vector-valued coordinates in $D=4$, does *not* longer *obey* the Jacobi identities. For this reason we restrict our construction to a flat spacetime background $g_{\mu\nu}=\eta_{\mu\nu}$.

The noncommutative conditions on the polyvector coordinates in condensed notation can be written as

$$[\,X^M,\ X^N\,]_* \ = \ X^M{*}X^N - X^N{*}X^M \ = \ \Omega^{MN}(X)\ =\ f^{MN}{}_L\,X^L\ =\ f^{MNL}\,X_L \tag{27}$$

the structure constants f^{MNL} are antisymmetric under the exchange of polyvector valued indices. An immediate consequence of the noncommutativity of coordinates is

$$[\,\hat{X}^{\mu_1},\ \hat{X}^{\mu_2}\,]\ =\ 2\ \hat{X}^{\mu_1\mu_2}\ \Rightarrow\ \Delta X^\mu\ \Delta X^\nu\ \geq\ \frac{1}{2}\ |\ <\ \hat{X}^{\mu\nu}\ >\ |=\ X^{\mu\nu} \tag{28}$$

Hence, the bivector area coordinates $X^{\mu\nu}$ in C-space can be seen as a measure of the noncommutative nature of the "quantized" spacetime coordinates $\hat{X}^\mu$.

The third step is to define the noncommutative star product of functions of X as

$$(A_1 \ * \ A_2)(Z) \ = \ exp \left(\ \frac{1}{2} \ \Omega^{MN} \ \partial_{X^M} \ \partial_{Y^N} \ \right) \ A_1(X) \ A_2(Y)|_{X=Y=Z} \ =$$

$$\sum_{n=0}^{\infty} \frac{(\frac{1}{2})^n}{n!} \ \Omega^{M_1N_1} \ \Omega^{M_2N_2} \ .... \ \Omega^{M_nN_n} \ (\partial^n_{M_1M_2......M_n} \ A_1) \ (\partial^n_{N_1N_2......N_n} \ A_2) + ... \tag{29}$$

where the ellipsis in (29) are the terms involving derivatives acting on Ω^{MN} and

$$\partial^n_{M_1M_2......M_n} \ A_1(Z) \ \equiv \ \partial_{M_1} \ \partial_{M_2} \ ...... \ \partial_{M_n} \ A_1(Z). \tag{30a}$$

$$\partial^n_{N_1N_2......N_n} \ A_2(Z) \ \equiv \ \partial_{N_1} \ \partial_{N_2} \ ...... \ \partial_{N_n} \ A_2(Z). \tag{30b}$$

Derivatives on Ω^{mn} appear in the ordinary Moyal star product when Ω^{mn} depends on the phase space coordinates. For instance, the Moyal star product when the symplectic structure $\Omega^{mn}(\vec{q},\vec{p})$ is *not* constant is given by

$$A \ * \ B = \ A \ exp \left(\ \frac{i\hbar}{2} \ \Omega^{mn} \ \overleftarrow{\partial}_m \ \overrightarrow{\partial}_n \ \right) \ B \ =$$

$$A \ B + i\hbar \ \Omega^{mn} \ (\partial_m A \ \partial_n B) + \ \frac{(i\hbar)^2}{2} \ \Omega^{m_1n_1} \ \Omega^{m_2n_2} \ (\partial^2_{m_1m_2} \ A) \ (\partial^2_{n_1n_2} \ B) \ +$$

$$\frac{(i\hbar)^2}{3} [\ \Omega^{m_1n_1} \ (\partial_{n_1} \ \Omega^{m_2n_2}) \ (\partial_{m_1}\partial_{n_2} A \ \partial_{m_2} B \ - \ \partial_{m_2} A \ \partial_{m_1}\partial_{n_2} B \ ) \ ] + O(\hbar^3). \tag{31}$$

Due to the derivative terms $\partial_{n_1}\Omega^{m_2n_2}$ the star product is associative up to second order only [6] $(f * g) * h = f * (g * h) + O(\hbar^3)$. The derivatives terms acting on $\Omega^{MN}(X)$ in (29) are

$$\frac{\partial X_L}{\partial X^M} \ = \ \frac{\partial \ (G_{LN} \ X^N)}{\partial X^M}. \tag{32a}$$

if, and only if, G_{LN} is X-independent, like in a flat C-space, from eq-(32) one gets

$$\frac{\partial X_L}{\partial X^M} \ = \ G_{LN} \ \frac{\partial X^N}{\partial X^M} \ = \ G_{LN} \ \delta^N_M \ = \ G_{LM} \tag{32b}$$

Due to the antisymmetry of the structure constants f^{MNL} the non-zero values for f^{MNL} require $M \neq N \neq L$, such that $G_{LM} = 0$ for a diagonal G_{LM}. Therefore, for flat (X-independent) diagonal metrics in C-space, the derivatives terms acting on $\Omega^{MN}(X)$ in eq- (29) are zero and the star product is associative and noncommutative in this special case. For more general metrics in C-space the star product will no longer be associative as it occurs in eq-(31) .

The C-space differential form associated with the polyvector-valued Clifford gauge field is

$$\mathbf{A} \ = \ \mathcal{A}_M \ dX^M \ = \ \Phi \ d\sigma \ + \ \mathcal{A}_\mu \ dx^\mu \ + \ \mathcal{A}_{\mu\nu} \ dx^{\mu\nu} \ + \ .... \ +$$

$$\mathcal{A}_{\mu_1\mu_2.....\mu_d}\ dx^{\mu_1\mu_2.......\mu_d}.. \qquad (33a)$$

where $\Phi = \Phi^A\ \Gamma_A,\ \mathcal{A}_\mu = \mathcal{A}^A_\mu\ \Gamma_A,\ \mathcal{A}_{\mu\nu} = \mathcal{A}^A_{\mu\nu}\ \Gamma_A, ...$ The C-space differential form associated with the polyvector-valued field-strength is

$$\mathbf{F}\ =\ F_{MN}\ dX^M\ \wedge\ dX^N\ =\ F_{0\ \mu}\ d\sigma\ \wedge\ dx^\mu\ +\ F_{0\ \mu_1\mu_2}\ d\sigma\ \wedge\ dx^{\mu_1\mu_2}\ +\ ....$$

$$F_{0\ \nu_1\nu_2.....\nu_d}\ d\sigma \wedge dx^{\nu_1\nu_2......\nu_d} + F_{\mu\nu}\ dx^\mu \wedge dx^\nu + F_{\mu_1\mu_2\ \nu_1\nu_2}\ dx^{\mu_1\mu_2} \wedge dx^{\nu_1\nu_2} + ...$$

$$+\ F_{\mu_1\mu_2.....\mu_{d-1}\ \nu_1\nu_2.....\nu_{d-1}}\ dx^{\mu_1\mu_2......\mu_{d-1}}\ \wedge\ dx^{\nu_1\nu_2.......\nu_{d-1}}. \qquad (33b)$$

The field strength is antisymmetric under the exchange of poly-vector indices $F_{MN} = -F_{NM}$. For this reason one has $F_{00} = 0$ and $F_{12....d\ 12.....d} = 0$. Finally, given the noncommutative conditions on the poly-vector coordinates (27), the components of the Clifford-algebra valued field strength $F^C_{MN}\Gamma_C$ in $Noncommutative$ C-spaces are

$$F_{[MN]}\ =\ F^C_{[MN]}\ \Gamma_C\ =\ (\ \partial_M\ \mathcal{A}^C_N\ -\ \partial_N\ \mathcal{A}^C_M\ )\ \Gamma_C\ +$$

$$\frac{1}{2}\ (\ \mathcal{A}^A_M * \mathcal{A}^B_N\ -\ \mathcal{A}^B_N * \mathcal{A}^A_M\ )\ \{\ \Gamma_A,\ \Gamma_B\ \} + \frac{1}{2}\ (\ \mathcal{A}^A_M * \mathcal{A}^B_N\ +\ \mathcal{A}^B_N * \mathcal{A}^A_M\ )\ [\ \Gamma_A,\ \Gamma_B\ ]. \qquad (34)$$

The commutators $[\ \Gamma_A,\ \Gamma_B\ ]$ and anti-commutators $\{\ \Gamma_A,\ \Gamma_B\ \}$ in eqs-(34), where A, B are polyvector-valued indices, can be read from the relations in eqs-(7-10) . Notice that both the standard commutators *and* anticommutators of the gammas appear in the terms containing the star deformed products of (34) and which define the Clifford-algebra valued field strength in noncommutative C-spaces; i.e. if the products of fields were to commute one would have had only the Lie algebra commutator $\mathcal{A}^A_M \mathcal{A}^J_B\ [\Gamma_A, \Gamma_B]$ pieces without the anti-commutator $\{\Gamma_A, \Gamma_B\}$ contributions in the r.h.s of eq-(34).

We should remark that one is *not* deforming the Clifford algebra involving $[\ \Gamma_A,\ \Gamma_B\ ]$ and $\{\ \Gamma_A,\ \Gamma_B\ \}$ in eq-(34) but it is the "point" product algebra $\mathcal{A}^A_M * \mathcal{A}^B_N$ of the fields which is being deformed. (Quantum) q-Clifford algebras have been studied by [9]. The symmetrized star product is

$$\mathcal{A}^A_M\ \ *_s\ \ \mathcal{A}^B_N\ \equiv\ \frac{1}{2}\left(\mathcal{A}^A_M * \mathcal{A}^B_N + \mathcal{A}^B_N * \mathcal{A}^A_M\right)\ =\ \mathcal{A}^A_M\ \mathcal{A}^B_N\ +$$

$$X^{\mu\nu}\ X^{\kappa\lambda}\ (\partial_\mu\ \partial_\kappa\ \mathcal{A}^A_M)\ (\partial_\nu\ \partial_\lambda\ \mathcal{A}^B_N)\ +\ ...... \qquad (35)$$

the antisymmetrized (Moyal bracket) star product is

$$\mathcal{A}^A_M *_a \mathcal{A}^B_N\ \equiv\ \frac{1}{2}\left(\mathcal{A}^A_M * \mathcal{A}^B_N - \mathcal{A}^B_N * \mathcal{A}^A_M\right)\ =\ X^{\mu\nu}\ (\partial_\mu\ \mathcal{A}^A_M)\ (\partial_\nu\ \mathcal{A}^B_N)\ +$$

$$X^{\mu\nu}\ X^{\kappa\lambda}\ X^{\alpha\beta}\ (\partial_\mu\ \partial_\kappa\ \partial_\alpha\ \mathcal{A}^A_M)\ (\partial_\nu\ \partial_\lambda\ \partial_\beta\ \mathcal{A}^B_N)\ +\ .... \qquad (36)$$

It is important to emphasize, as it is customary in Moyal star products, that the poly-vector coordinates appearing in the r.h.s of eqs-(35-36) are treated as c-numbers (as if they were commuting) while it is the product of functions appearing in the l.h.s of (35-36) which are *noncommutative*.

Star products in noncommutative C-space lead to *many more terms* in eqs-(35-36) than in ordinary noncommutative spaces. For example, there are derivatives terms involving polyvectors which do *not* appear in ordinary noncommutative spaces, like

$$-4\ g^{\mu_1\nu_1}\ X^{\mu_2\nu_2}\ \frac{\partial \mathcal{A}^A_M}{\partial X^{\mu_1\mu_2}}\ \frac{\partial \mathcal{A}^B_N}{\partial X^{\nu_1\nu_2}}\ \pm\ .... \tag{37a}$$

$$2\ (\ g^{\mu_2\nu}\ X^{\mu_1}\ -\ g^{\mu_1\nu}\ X^{\mu_2}\ )\ \frac{\partial \mathcal{A}^A_M}{\partial X^{\mu_1\mu_2}}\ \frac{\partial \mathcal{A}^B_N}{\partial X^{\nu}}. \tag{37b}$$

$$X^{\mu_1\mu_2\mu_3\nu}\ \frac{\partial \mathcal{A}^A_M}{\partial X^{\mu_1\mu_2\mu_3}}\ \frac{\partial \mathcal{A}^J_B}{\partial X^{\nu}}. \tag{37c}$$

$$96\ G^{\mu_1\mu_2\mu_3\ \nu_1\nu_2\nu_3}\ X^{\mu_4\nu_4}\ \frac{\partial \mathcal{A}^A_M}{\partial X^{\mu_1\mu_2\mu_3\mu_4}}\ \frac{\partial \mathcal{A}^B_N}{\partial X^{\nu_1\nu_2\nu_3\nu_4}},\quad etc\ ...... \tag{37d}$$

There is a *subalgebra* of the noncommutative polyvector-valued coordinates algebra (27) involving only X^μ and the bivector coordinates $X^{\mu\nu}$ when the spacetime metric components $g_{\mu\nu}$ are *constant*. However, because $[X^{\mu_1\mu_2}, X^\nu] \neq 0$ one must not confuse the algebra in this case with the ordinary Θ-noncommutative algebra $[X^{\mu_1}, X^{\mu_2}] = \Theta^{\mu_1\mu_2}$ where the components of $\Theta^{\mu_1\mu_2}$ are comprised of *constants* such that $[\Theta^{\mu_1\mu_2}, X^\nu] = 0$.

The analog of a Yang-Mills action in C-spaces is

$$S\ =\ \int\ [DX]\ <\ F^A_{MN}\ \Gamma_A\ *\ F^B_{PQ}\ \Gamma_B\ >\ G^{MP}\ G^{NQ}. \tag{38}$$

where $<\ \Gamma_A\ \Gamma_B\ >$ denotes the Clifford-scalar part of the Clifford geometric product of two generators. As mentioned in the introduction suitable powers of a length scale must be included in the expansion of the terms inside the integrand in order to have consistent dimensions (the action is dimensionless). The action (38) becomes

$$\int\ [DX]\ (\ F_{MN}\ *\ F^{MN}\ +\ F^a_{MN}\ *\ F^{MN}_a\ +$$

$$F^{a_1a_2}_{MN}\ *\ F^{MN}_{a_1a_2}\ +\ ....\ +\ F^{a_1a_2.....a_d}_{MN}\ *\ F^{MN}_{a_1a_2.....a_d}\ ). \tag{39}$$

the measure in C-space is given by

$$DX\ =\ d\sigma\ \prod dx^\mu\ \prod\ dx^{\mu_1\mu_2}\ \prod dx^{\mu_1\mu_2\mu_3}\ .....\ dx^{\mu_1\mu_2.......\mu_d}. \tag{40a}$$

The Clifford-valued gauge field $\mathcal{A}_M$ transforms under star gauge transformations according to $\mathcal{A}'_M\ =\ U_*^{-1} * \mathcal{A}_M * U_* + U_*^{-1} * \partial_M U_*$. The field strength F transforms covariantly $F'_{MN} = U_*^{-1} * F_{MN} * U_*$ such that the action (39) is star gauge invariant. $U_* = exp_*(\xi(X)) = exp_*(\xi^A(X)\Gamma_A)$ is defined via a star power series expansion $U_* = \sum_n \frac{1}{n!}(\xi(X))^n_*$ where $(\xi(X))^n_* = \xi(X) * \xi(X) * ..... * \xi(X)$.

The integral $\int F * F = \int F \ F$ + total derivatives. If the fields vanish fast enough at infinity and/or there are no boundaries, the contribution of the total derivative terms are zero. In this way one proves the star gauge invariance of the action (39) under infinitesimal $\delta F = [F, \xi]$ transformations due to the relations

$$\delta_\xi S = \int < F * [F, \ \xi] > = \int < F \ [F, \ \xi] > + \ total \ derivatives =$$

$$\int \frac{1}{2} < [F^2, \ \xi] > = \frac{1}{2} \int < F^2 \ \xi - \xi \ F^2 > + \ total \ derivatives = 0. \tag{40b}$$

after using the Liebnitz rule and the cyclic property of the scalar part of the geometric product

$$< F^2 \ \xi - \xi \ F^2 > =$$

$$F^A \ F^B \ \xi^C < \Gamma_A \ \Gamma_B \ \Gamma_C > - \xi^C \ F^A \ F^B < \Gamma_C \ \Gamma_A \ \Gamma_B > = 0. \tag{40c}$$

In ordinary commutative C-spaces one can perform the mode expansion in integer powers of the poly-vector coordinates

$$\mathcal{A}_M(X) = \mathcal{A}_M(\sigma, \mathbf{x}^\mu, x^{\mu_1\mu_2}, ..., x^{\mu_1\mu_2......\mu_d}) =$$

$$\sum_{n_I} \mathcal{A}_{M,n_I}(\mathbf{x}^\mu) \ \sigma^{n_o} \ (x^{12})^{n_{12}} \ ...... \ (x^{123})^{n_{123}} \ .... \ (x^{12......d})^{n_{123......d}}. \tag{41}$$

The sum over the spacetime dependent fields $\mathcal{A}_{M,n_I}(\mathbf{x}^\mu)$ is taken over the infinite number of integer-valued modes associated with the collection set of integers

$$n_I = n_o, n_{12}, \ ......, n_{123}, \ ...., n_{1234}, \ ...., n_{12....d}. \tag{42}$$

In Noncommutative C-spaces we may replace the ordinary products of the poly-vector valued coordinates in eq-(41) for their star products.

To finalize we provide a description of QM in Noncommutative C-spaces based on Yang's Noncommutative phase space algebra [10]. There is a *subalgebra* of the C-space operator-valued coordinates which is *isomorphic* to the Noncommutative Yang's $4D$ spacetime algebra [10]. This can be seen after establishing the following correspondence between the C-space vector/bivector (area-coordinates) algebra, associated to the $6D$ angular momentum (Lorentz) algebra, and the Yang's spacetime algebra via the $SO(6)$ generators Σ^{ij} in $6D$ $(i, j = 1, 2, 3......, 6)$ as follows [11]

$$i \ \hbar \ \Sigma^{\mu\nu} \leftrightarrow i \ \frac{\hbar}{\lambda^2} \ \hat{X}^{\mu\nu}, \quad i \ \Sigma^{56} \leftrightarrow i \ \frac{R}{\lambda} \ \mathcal{N}. \tag{43a}$$

$$i \ \lambda \ \Sigma^{\mu 5} \leftrightarrow i \ \hat{X}^\mu, \quad i \ \Sigma^{\mu 6} \leftrightarrow i \ \frac{R}{\hbar} \ \hat{P}^\mu \tag{43b}$$

where the indices $\mu, \nu = 1, 2, 3, 4$. The scales λ and R are a lower and upper scale respectively, like the Planck and Hubble scale. The $SO(6)$ algebra $[\Sigma^{ij}, \ \Sigma^{kl}] =$

$-g^{ik}\Sigma^{jl} + ...$ can be recast in terms of a *noncommutative* phase space algebra as

$$[\hat{P}^\mu,\ \mathcal{N}] \ = \ -\, i\ \eta^{66}\ \frac{\hbar}{R^2}\hat{X}^\mu, \quad [\hat{X}^\mu,\ \mathcal{N}] \ = \ i\ \eta^{55}\ \frac{\lambda^2}{\hbar}\ \hat{P}^\mu. \tag{44}$$

$$[\hat{X}^\mu,\ \hat{X}^\nu] \ = \ -i\ \eta^{55}\ \hat{X}^{\mu\nu}, \ \ [\hat{P}^\mu,\ \hat{P}^\nu] \ = \ -i\ \eta^{66}\ \frac{\hbar^2}{R^2\ \lambda^2}\ \hat{X}^{\mu\nu}, \ \ \hat{X}^{\mu\nu} \ = \ \lambda^2\ \Sigma^{\mu\nu}. \tag{45}$$

$$[\hat{X}^\mu,\ \hat{P}^\mu] \ = \ i\ \hbar\ \eta^{\mu\nu}\ \frac{\lambda}{R}\ \Sigma^{56} \ = \ i\ \hbar\ \eta^{\mu\nu}\ \mathcal{N}, \quad [\hat{X}^{\mu\nu},\ \mathcal{N}] \ = \ 0. \tag{46}$$

The last relation is the *modified* Heisenberg algebra in $4D$ since $\mathcal{N}$ does *not* commute with X^μ nor P^μ. The remaining *nonvanishing* commutation relations are

$$[\Sigma^{\mu\nu},\ \hat{X}^\rho] \ = \ -\, i\ \eta^{\mu\rho}\ \hat{X}^\nu \ + i\ \eta^{\nu\rho}\hat{X}^\mu \tag{47a}$$

$$[\Sigma^{\mu\nu},\ \hat{P}^\rho] \ = \ -\, i\ \eta^{\mu\rho}\ \hat{P}^\nu \ + i\ \eta^{\nu\rho}\hat{P}^\mu. \tag{47b}$$

$$[\Sigma^{\mu\nu},\ \Sigma^{\rho\tau}] \ = \ -\, i\ \eta^{\mu\rho}\ \Sigma^{\nu\tau} \ + i\ \eta^{\nu\rho}\ \Sigma^{\mu\tau} \ - \ .... \tag{47c}$$

the last relation is the same as that in eq-(18) after reabsorbing factors of 2 in the definition of $\Sigma^{\mu\nu}$. Eqs-(44-47) are the defining relations of the Yang's Noncommutative $4D$ spacetime algebra involving the $8D$ phase-space variables X^μ, P^μ and the angular momentum (Lorentz) generators $\Sigma^{\mu\nu}$ in $4D$. The above commutators obey the Jacobi identities. An immediate consequence of Yang's noncommutative algebra is that now one has a modified products of uncertainties

$$\Delta X^\mu\ \Delta P^\nu \ \geq \ \frac{\hbar}{2}\ \eta^{\mu\nu}\ ||<\ \Sigma^{56}\ >||; \quad \Delta X^\mu\ \Delta X^\nu \ \geq \ \frac{\lambda^2}{2}\ ||<\ \Sigma^{\mu\nu}\ >||$$

$$\Delta P^\mu\ \Delta P^\nu \ \geq \ \frac{1}{2}(\frac{\hbar}{R})^2\ ||<\ \Sigma^{\mu\nu}\ >||. \tag{48}$$

The Noncommutative phase space Yang's algebra in $4D$ can be generalized to the Noncommutative Clifford phase space algebra associated to the $4D$ spacetime after following the same prescription as in eqs-(43) by invoking higher dimensions ($12D$ in this case instead of $6D$) as follows

$$X^\mu \ \leftrightarrow \ \lambda\ \Gamma^\mu \wedge \Gamma^5, \quad P^\mu \ \leftrightarrow \ \frac{\hbar}{R}\ \Gamma^\mu \wedge \Gamma^6. \tag{49}$$

$$X^{\mu_1\mu_2} \ \leftrightarrow \ \Upsilon^{[\mu_1\mu_2]\ [57]} \ \neq \ \lambda^2\ \Gamma^{\mu_1} \wedge \Gamma^{\mu_2} \wedge \Gamma^5 \wedge \Gamma^7$$

$$P^{\mu_1\mu_2} \ \leftrightarrow \ \Upsilon^{[\mu_1\mu_2]\ [68]} \ \neq \ (\frac{\hbar}{R})^2\ \Gamma^{\mu_1} \wedge \Gamma^{\mu_2} \wedge \Gamma^6 \wedge \Gamma^8. \tag{50}$$

$$X^{\mu_1\mu_2\mu_3} \ \leftrightarrow \ \Upsilon^{[\mu_1\mu_2\mu_3]\ [579]} \ \neq \ \lambda^3\ \Gamma^{\mu_1} \wedge \Gamma^{\mu_2} \wedge \Gamma^{\mu_3} \wedge \Gamma^5 \wedge \Gamma^7 \wedge \Gamma^9$$

$$P^{\mu_1\mu_2\mu_3} \leftrightarrow \Upsilon^{[\mu_1\mu_2\mu_3]\ [6810]} \neq (\frac{\hbar}{R})^3\ \Gamma^{\mu_1} \wedge \Gamma^{\mu_2} \wedge \Gamma^{\mu_3} \wedge \Gamma^6 \wedge \Gamma^8 \wedge \Gamma^{10}. \quad (51)$$

$$X^{\mu_1\mu_2\mu_3\mu_4} \leftrightarrow \Upsilon^{[\mu_1\mu_2\mu_3\mu_4]\ [57911]} \neq \lambda^4\ \Gamma^{\mu_1} \wedge \Gamma^{\mu_2} \wedge \Gamma^{\mu_3} \wedge \Gamma^{\mu_4} \wedge \Gamma^5 \wedge \Gamma^7 \wedge \Gamma^9 \wedge \Gamma^{11}$$

$$P^{\mu_1\mu_2\mu_3\mu_4} \leftrightarrow \Upsilon^{[\mu_1\mu_2\mu_3\mu_4]\ [681012]} \neq (\frac{\hbar}{R})^4\ \Gamma^{\mu_1} \wedge \Gamma^{\mu_2} \wedge \Gamma^{\mu_3} \wedge \Gamma^{\mu_4} \wedge \Gamma^6 \wedge \Gamma^8 \wedge \Gamma^{10} \wedge \Gamma^{12}. \quad (52)$$

The indices $\mu_1, \mu_2, \mu_3, \mu_4$ range from $1, 2, 3, 4$. The extra indices span 8 additional directions (dimensions) leaving a total dimension of $4 + 8 = 12$. The *noncommutative* Clifford phase space algebra commutators are defined in terms of the algebra

$$[\Upsilon^{MN},\ \Upsilon^{PQ}] = -i\, G^{MP}\ \Upsilon^{NQ}\ + i\, G^{MQ}\ \Upsilon^{NP}\ + i\, G^{NP}\ \Upsilon^{MQ}\ - i\, G^{NQ}\ \Upsilon^{MP} \quad (53)$$

The generators obey $\Upsilon^{MN} = -\Upsilon^{NM}$, and $G^{MN} = G^{NM}$ under an exchange of multi-indices $M \leftrightarrow N$.

The algebra (53) has the same structure as a *generalized spin algebra* and satisfies the Jacobi identities. We must stress that

$$[\Upsilon^{MN},\ \Upsilon^{PQ}] \neq [\ [\Gamma^M, \Gamma^N],\ [\Gamma^P, \Gamma^Q]\ ]. \quad (54)$$

except in the special case when M, N, P, Q are all *bivector* indices : hence we must *emphasize* that the generalized spin algebra (53) *is not isomorphic* to the noncommutative algebra of eqs-(15-24) ! For example, from the commutator

$$[\Upsilon^{[\mu_1\mu_2\mu_3]\ [579]},\ \Upsilon^{[\nu_1\nu_2\nu_3]\ [6810]}] = -i\ G^{[\mu_1\mu_2\mu_3]\ [\nu_1\nu_2\nu_3]}\ \Upsilon^{[579]\ [6810]}. \quad (55a)$$

one can infer the Weyl-Heisenberg algebra commutator

$$[X^{\mu_1\mu_2\mu_3},\ P^{\nu_1\nu_2\nu_3}] = -i\ \hbar^3\ G^{[\mu_1\mu_2\mu_3]\ [\nu_1\nu_2\nu_3]}\ \Upsilon^{[579]\ [6810]}. \quad (55b)$$

From the commutator

$$[\Upsilon^{[\mu_1\mu_2\mu_3]\ [579]},\ \Upsilon^{[\nu_1\nu_2\nu_3]\ [579]}] = -i\ G^{[579]\ [579]}\ \Upsilon^{[\mu_1\mu_2\mu_3]\ [\nu_1\nu_2\nu_3]}. \quad (56a)$$

one can infer the commutator among the tri-vector coordinates

$$[X^{\mu_1\mu_2\mu_3},\ X^{\nu_1\nu_2\nu_3}] = -i\ \lambda^6\ G^{[579]\ [579]}\ \Upsilon^{[\mu_1\mu_2\mu_3]\ [\nu_1\nu_2\nu_3]}. \quad (56b)$$

where $\Upsilon^{[\mu_1\mu_2\mu_3]\ [\nu_1\nu_2\nu_3]}$ is a generalized angular momentum (spin) generator. From the commutator

$$[\Upsilon^{[\mu_1\mu_2\mu_3]\ [579]},\ \Upsilon^{[579]\ [6810]}] = i\ G^{[579]\ [579]}\ \Upsilon^{[\mu_1\mu_2\mu_3]\ [6810]}. \quad (57a)$$

one can infer the commutator

$$[X^{\mu_1\mu_2\mu_3},\ \Upsilon^{[579]\ [6810]}]\ =\ i\ \lambda^6\ \frac{1}{\hbar^3}\ G^{[579]\ [579]}\ P^{\mu_1\mu_2\mu_3}. \qquad (57b)$$

which *exchanges* the $X^{\mu_1\mu_2\mu_3}$ for $P^{\mu_1\mu_2\mu_3}$, etc ..... Therefore, eqs-(55,56,57) are the suitable tri-vector analog of eqs-(44,45,46). Clearly, the above non-vanishing commutators *differ* from those in eqs-(15-24) and will modify the QM wave equations when one introduces potential terms like $V(X) = g(X * X * .... * X)$. QM in ordinary (commutative) C-spaces can be found in [11].

Having provided the basic ideas and results behind polyvector gauge field theories in Noncommutative Clifford spaces, the construction of Noncommutative Clifford-space gravity as polyvector valued gauge theories of twisted diffeomorphisms in C-spaces will be the subject of future investigations. It would require quantum Hopf algebraic deformations of Clifford algebras [9]. Such theory is far richer than gravity in Noncommutative spacetimes [12].

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