DARK PARTICLES

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ABSTRACT

This paper relates Freidmann's equation of general relativity to Heisenberg's uncertainty equation and Langevin's equation of quantum and statistical mechanics. The connection is made by showing their solutions are equal when a hypothesized "dark" particle carries a reduced Planck mass. The solutions are found by defining a general relativistic length scale L and a quantum mechanical length scale ℓ and deriving the energy density ρ under three regimes (equation of state, $w = \frac{1}{3}, -\frac{1}{3}, and - 1$). The density is solved via the introduction of a spring force that places the particle in the ground state of the harmonic oscillator with energy equal to the temperature of the particle. Black body radiation is found trapped by the strong gravitational fields of the particle and has the right relic density and behavior to explain the open questions of dark energy and dark matter. While more questions remain, further investigation seems justified under Occam's razor.

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I. INTRODUCTION

By solving Freidmann's equation next to Heisenberg's uncertainty equation and showing the solutions are equal in three different regimes, a robust theory unites the two and comes together in a holistic solution known as the Langevin equation.

We intend for the approach to be simple and thus preferred over other more complicated explanations of observed physical phenomena like the acceleration of the Universe and its missing matter.

A. Definitions

The Freidmann equation relates a length scale L to the energy density, ρ [1].

$$\left(\frac{(\dot{L})}{L}\right)^2 = \frac{8\pi G}{3}\rho(L)$$

Similarly, the quantum mechanical length scale ℓ defined below as $\ell(t) = \sqrt{2}\Delta x_{Quantum}(t)$ is related to $(\dot{\ell})(t)$ through the Heisenberg uncertainty relation [2] when,

$$\Delta p_x(t) = m\Delta v_x(t) = m\Delta \dot{x}_{Quantum}(t)$$

From here we can write

$$\Delta x_{Quantum}(t) \cdot \Delta \dot{x}_{Quantum}(t) = \frac{\hbar}{2m}$$

$$\ell(t) \cdot (\dot{\ell})(t) = \frac{\hbar}{m}$$

We show that $L = \ell$ (when the mass is the reduced Planck mass) by first deriving in the appendix the density of the "dark particle." Exploring the mathematics and physics of this hypothetical particle begins with the ordinary quantum mechanical harmonic oscillator in the ground state where the one dimension ground state quantum mechanical energy is equal to the temperature.

$$E_{x0} = \frac{\hbar\omega_0}{2} = k_B T$$

While the frequency of the harmonic oscillator is equal to $\omega_0/2\pi$, we find a de-coherence time period equal to one over twice the temperature (which also obeys the Heisenberg uncertainty relation with the ground state energy) .

$$\tau = \frac{\hbar}{2E_{x0}} = \frac{\hbar}{2k_B T}$$

To define the length scale associated with τ (which will determine among other things the volume in which this energy resides), an Ansatz is used to assume dark particles come as a pair with equal and independent probability distributions.

Now define $L(t)$ as the distance a pair of particles, traveling in opposite directions at the speed of light, can traverse in time τ. Calling $L(t)$ the general relativistic length scale of the dark particle pair is consistent with our understanding of how the length "scales" as it is proportional to the inverse of the temperature [1].

$$L = 2c\tau = \frac{\hbar c}{k_B T}$$

Using this length scale, the density is found by integrating the energy over its probability distribution (which is equal to the three dimensional ground state energy, E_0) and dividing by the volume, $V = L^3$.

$$\rho = \frac{\int_0^\infty \hbar\omega\, p(\omega)d\omega}{V} = \frac{E_0}{V} = \frac{3E_{x0}}{V} = \frac{3(k_B T)^4}{\hbar^3 c^5}$$

Define a second length scale $\ell(t)$, the quantum mechanical length scale, as the standard deviation between the dark particle pair. In this case the distance between the pair of particles with equal and independent probability distributions has a variance $\ell^2(t)$ that is equal to 2 times the variance of one particle.

$$\ell^2(t) \ = 2(\Delta x)^2{}_{Quantum}(t)$$

Taking the time derivative we have

$$(\dot{\ell})(t) \ = \sqrt{2}(\dot{\Delta x})_{Quantum}(t)$$

We are now in a position to solve (under three regimes) for $L(t)$ using the Freidmann equation of general relativity and $\ell(t)$ using the Heisenberg uncertainty relation of quantum mechanics. One regime occurs when the temperature is completely variable, a second that represents the curvature of space, and a third when the temperature is constant. The magnitude of the mass of the dark particle is shown explicitly in the first regime to be the reduced Planck mass when $L(t)$ and $\ell(t)$ (of the same form) are set equal to each other.

Throughout the paper the definition of these quantities above, $\tau, L, \ell, and\ \rho$ do not change. They will have different solutions based on what is variable and what is constant, but τ for example will always be one over twice the temperature in Planck units.

II. THREE REGIMES

A. Variable Temperature Regime $w = 1/3$

When the temperature is variable and inversely proportional to the length scale we can re-write the density as,

$$\rho(L) = \frac{3\hbar}{c(L(t))^4}$$

Simple calculus solves the Freidmann equation when the density is dominated by this equation of state, w $= 1/3$.

$$\left(\frac{(\dot{L})}{L}\right)^2 = \frac{8\pi G}{3}\rho(L) = \frac{8\pi G\hbar}{c(L(t))^4}$$

Solving for $L(t)$

$$L(t) = \left(\frac{32\pi G\hbar}{c}\right)^{1/4}\sqrt{t}$$

The above is the general relativistic solution for the scale factor of one particle pair.

Solving for $\ell(t)$ begins with the Heisenberg uncertainty relation [2].

$$\Delta(\dot{x})_{Linear}(t) = \frac{\hbar}{2m\Delta x_{Linear}(t)}$$

Solving the differential equation gives the known result of quantum diffusion derived in appendix A and studied in [3,4].

$$\int 2\Delta x_{Linear}(t)d\big(\Delta x_{Linear}(t)\big) = \int \frac{\hbar}{m}dt$$

$$\Delta x_{linear}(t) = \left(\frac{\hbar}{m}t\right)^{\frac{1}{2}}$$

Leading to the quantum mechanical length scale,

$$\ell(t) = \left(\frac{2\hbar}{m}t\right)^{\frac{1}{2}}$$

Equating the general relativistic length scale, $L(t)$, to the quantum mechanical length scale, $\ell(t)$, solves for the square of the mass of the particle.

$$m^2 = \frac{\hbar c}{8\pi G}$$

The obvious solution is the positive reduced Planck mass. This solution is interesting as a fundamental particle with this mass will be dominated by gravitational effects if its charge is less than, $\sqrt{\hbar c\varepsilon_0/2} \cong 2.34q_{electron}$, if it has any charge at all; and because it is precisely the Planck mass divided by $\sqrt{8\pi}$ is simplifies many of the expressions in general relativity [5].

B. Curvature Regime $w = -1/3$

While the quantum diffusion described above in the first regime has a variance that is linear in time, another type of quantum diffusion has a variance that is quadratic in time [2] as shown in appendix B. In this section we will show that the same solution is available from the Freidmann equation. We can write

$$\Delta x^2_{Quadratic}(t) = \frac{k_B T_0}{m} t^2$$

Using a variation of the derivation of Friedmann's equation as given by Liddle [1], we equate the average gravitational potential energy of a sphere with radius r to the three dimensional potential energy of the harmonic oscillator with $k_B T_0/2$ Joules per degree of freedom hypothesized in appendix A

$$\overline{PE_{gravity}} = \overline{\frac{-GMm}{r}} = \overline{PE_{Hooks\ law}} = \frac{3k_B T_0}{2}$$

When $M = 4\pi r^3 \rho_{curve}/3$

$$\overline{\frac{-GMm}{r}} = \frac{-4\pi G\rho_{curve} m\overline{r^2}}{3}$$

We can re-write $\overline{r^2}$ in terms of our length scale where $L^2 = 2\overline{x^2}$

$$\overline{r^2} = \overline{x^2} + \overline{y^2} + \overline{z^2} = 3\overline{x^2} = \frac{3L^2}{2}$$

If the temperature is kept constant at T_0 we can re-write this in the form of the Freidmann equation.

$$\left(\frac{(\dot{L})}{L}\right)^2 = \frac{8\pi G}{3}\rho_{curve} = -\frac{2k_B T_0}{m}\left(\frac{1}{L}\right)^2$$

It is not yet clear how to handle the negative sign (however later when ρ_{curve} is added to the other densities it all balances out and the solution is real).

$$L = \sqrt{\frac{-2k_B T_0}{m}}\, t$$

Looking at the absolute value we have as we should,

$$|L^2(t)| = |\ell^2(t)| = 2\left|\Delta x^2_{Quadratic}(t)\right|$$

This regime is interesting as it provides the explanation for the spring force hypothesized in the appendix.

$$F = -\frac{d}{dr}PE(r) = -\frac{d}{dr}\frac{-4\pi G\rho_{curve} m r^2}{3}$$
$$= \frac{8\pi G\rho_{curve}}{3} mr = -\frac{2k_B T_0}{m}\left(\frac{1}{L}\right)^2 mr$$

Plugging in for the magnitude of $|L^2(t)|$ evaluated at $t = \tau = \hbar/2k_B T$ we derive $-F_{particle}$ from appendix A and thus validate our 2nd Ansatz.

$$F = -\frac{m}{\tau^2} r$$

The thermal energy curves the space around the particle which provides the resistive spring force.

C. Constant Temperature Regime $w = -1$

In this regime the temperature is kept constant at T_0. The general relativistic length scale in this regime is solved by directly replacing it with the quantum mechanical length scale into the density.

$$\rho\left(L = \sqrt{2}\Delta x_{Constant}\right) = \frac{3\hbar}{c4(\Delta x)^4_{Constant}}$$

Shown in appendix B, when $\tau_0 = \hbar/2k_B T_0$ we have,

$$(\Delta x)^2_{Constant} = (\Delta x)^2 + \tfrac{(\Delta p_x)^2}{m^2}\tau_0^2 = \frac{\hbar^2}{2mk_B T_0}$$

With this replacement and substituting in the reduced Planck mass, the Friedmann equation becomes

$$\left(\frac{(\dot{L})}{L}\right)^2 = \frac{8\pi G}{3}\rho = \frac{(k_B T_0)^2}{\hbar^2}$$

Solving for $L(t)$ yields,

$$L(t) = \frac{\hbar}{\sqrt{mk_B T_0}} e^{\frac{k_B T_0}{\hbar}t}$$

IV. PARTICLE LIFECYCLE

A. Langevin Equation

Each one of these regimes is interesting in its own right, however considering them holistically, solves the lifecycle of the particle and is consistent with the Langevin equation.

$$\left(\frac{(\dot{L})}{L}\right)^2 = \frac{\hbar^2}{m^2(L(t))^4} - \frac{2k_BT_0}{m}\left(\frac{1}{L}\right)^2 + \frac{(k_BT_0)^2}{\hbar^2}$$

The three terms have an equation of state, $w = \frac{1}{3}, -\frac{1}{3}, -1$ respectivly. This equation is easily solved.

$$\frac{(\dot{L})}{L} = \left(\frac{\hbar}{mL^2} - \frac{k_BT_0}{\hbar}\right)$$

With a little calculus the solution is

$$L^2 = \frac{\hbar^2}{mk_BT_0}\left(1 - e^{-2k_BT_0t/\hbar}\right)$$

With $k_BT_0 = \hbar/(2\tau_0)$ and the diffusion constant from appendix B, $D = \hbar/(2m)$, this is re-written,

$$L^2 = 4D\tau_0\left(1 - e^{-t/\tau_0}\right)$$

Tying this solution back to the quantum mechanical length scale, $L^2 = \ell^2 = 2\Delta x^2_{Quantum}(t)$ gives

$$\Delta x^2_{Langevin}(t) = 2D\tau_0\left(1 - e^{-t/\tau_0}\right)$$

One will recognize this as the solution to Langevin's equation when the noisy driving force is correlated with the particle displacement (derived in appendix B) [6].

B. Balanced (Zero) Density

One amazing feature of looking at the density from all three regimes together is that it approaches zero as time passes. When time first starts out, the density is huge given the inverse dependence on the fourth power of the length scale.

$$\rho_{Holistic}(L) =$$

$$\frac{3}{8\pi G}\left(\frac{\hbar^2}{m^2(L(t))^4} - \frac{2k_BT_0}{m}\left(\frac{1}{L}\right)^2 + \frac{(k_BT_0)^2}{\hbar^2}\right)$$

However at $L's$ asymptotic value the three separate densities perfectly cancel each other.

$$\lim_{L\to\hbar/\sqrt{mk_BT_0}} \rho_{Holistic}(L) = 0$$

One might ask however if this energy density is a candidate for the missing energy of the Universe: i.e. dark matter under the the temperature is variable where $w = 1/3$ [1]; and dark energy when the temperature is constant where $w = -1$ [7]. The answer is not directly. As we just argued the density approaches zero as time passes. However indirectly we find that the strong local gravitation fields produce a radiation density of bosons that separately adds to the density. This radiation density contributes to distant gravitational effects and could be of the right magnitude to explain the relic density of dark energy.

III. RADIATION FIELD

Due to the high gravitational fields of the dark particle, one would expect the particle to be coupled to a radiation field [8]; and this is what is found (as detailed in appendix C).

We implicitly assumed in the beginning of the paper that the dark particles have a non-zero mass. However if we look at a density of radiation where the photons have zero mass we see that only the first term in the holistic energy density remains.

$$\rho = \frac{3\hbar}{c(L(t))^4} - \frac{6mk_BT_0}{c\hbar}\left(\frac{1}{L}\right)^2 + 3m^2\frac{(k_BT_0)^2}{c\hbar^3}$$

$$\rho_{zero-mass} \propto \frac{3\hbar}{c(L(t))^4} = \frac{3(k_BT)^4}{\hbar^3c^5}$$

Furthermore when the probability distribution on the energy of the harmonic oscillator hypothesized in appendix A is associated with bosons instead of fermions (where multiple particles can occupy each mode), the density needs to be summed over these multiple states. Doing so and further accounting for time dilation and red-shift effects radiating in the high gravitational fields of the particle, the resulting radiation density trapped in the gravitational fields is exactly the black body distribution (as detailed in appendix C). When T_{DP} is the temperature of the radiation trapped by the dark particles we have:

$$\rho_{radiation}(\omega)d\omega = \frac{\hbar\omega^3}{\pi^2c^3}\frac{1}{\left(e^{\hbar\omega/kT_{DP}} - 1\right)}d\omega$$

$$\rho_{radiation} = \int_0^\infty u(\omega)d\omega = \frac{\pi^2(k_B T_{DP})^4}{15\hbar^3 c^3}$$

In normal black body radiation, a macroscopic cavity provides the confinement of the radiation [9]. Here it is the gravitational effects from the mass of the particle that confines the radiation.

IV. DISCUSSION

A. Dark Matter and Dark Energy

The density and length scales discussed above are associated with individual dark particle pairs. However with their theoretical derivation behind us we can get to a discussion of their application to the open questions of dark energy and dark matter by extrapolating to the density and length scale of the Universe. A more thorough analysis is scheduled in due time, yet the gist of the argument is simple enough to relay here.

We will assume the reader is familiar with dark energy [7] and the observations it intends to explain. However to quickly set the stage, cosmological observations early last century by Hubble indicated the Universe is expanding. More recent observations of Type Ia supernova suggest the Universe is not just expanding but is also accelerating [1].

The most popular explanation for these findings is an elusive energy density with an equation of state, $w < -1/3$ [7] coined "Dark Energy" making up ~73% [10] of the energy of the Universe. Despite many attempts to explain dark energy's origin [11], those attempts have fallen short [7,12]. Yet there are still many theories under review [13]; with the most accepted being the Lambda Cold Dark Matter model, $\Lambda\mathrm{CDM}$ where the Lambda, the Cosmological Constant provides a negative pressure $w = -1$ [7].

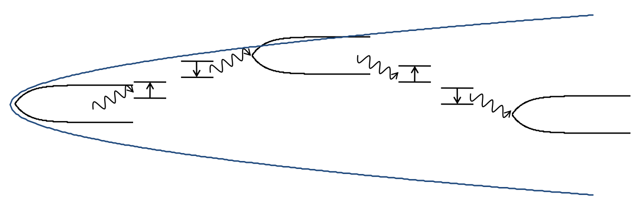
Another set of observations (that seem to be independent of the expansion and acceleration observations) show a discrepancy between the amounts of luminous matter we can visually account for and the amounts of mass we can infer from gravitational effects, such as speed of galaxy rotation [1]. "Dark Matter," the explanation for this discrepancy, is a positive pressure energy density that clumps near other baryonic matter, is devoid of interactions with photons [1], and makes up ~22% of the energy of the Universe [10].

We hypothesize that under different conditions experienced in the Universe, dark particles can explain both dark energy and dark matter.

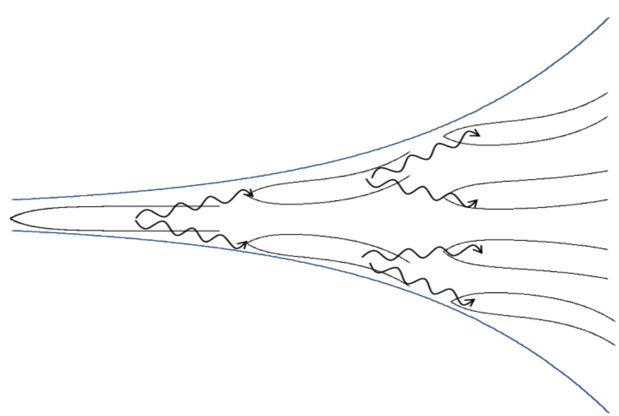
A. Constant and Variable Temperature

Now hypothesize that a local group of dark particle's are able to exchange heat with the local surroundings when hydrogen atoms or other sinks are nearby to capture the radiation from its gravitational binding. In this case the temperature of the dark particles is variable and thus inversely proportional to the length scale of the local Universe leading to an aggregate energy density local to the hydrogen atoms with an equation of state, $w = 1/3$.

Dark particles randomly walk as it exchanges heat with available local hydrogen atoms, thus allowing the temperature of its trapped radiation to equilibrate with the external radiation field. Figure 1 below is an artist rendition of what this process might look like.



However when the dark particles are isolated away from any sinks, no radiation can escape and the dark particles radiation has no way to release heat or change their temperature. In this case the particles' radiation density is constant and the equation of state is $w = -1$. The total energy scales linearly with the volume (which is exponentially increasing) as work is done on the system as it expands. The radiated photons in turn exponentially generate new dark particles to fill the space. Figure 2 below is an artist rendition of what this regime might look like.



B. Application to our Universe

When the Universe was in the Dark Ages, where a preponderance of hydrogen atoms was available [14] the dark particles could equilibrate with the comic microwave background. After the Dark Ages when re-ionization happed, the dark particles became frozen at that temperature with no way to release or absorb any hear. Today the aggregate effect of dark particles' radiation that are devoid of local neutral hydrogen atoms provide a negative pressure to the Universe with a constant density equal to $\rho_{\text{Dark Energy}}$. Using the Six-Parameter ΛCDM Fit from the 7-year WAMP data [10] we have $T_{today} = 2.725K$ and $z_{Re-ionization} = 10.5$,

$$\begin{aligned}\rho_{\text{Dark Energy}} &= \frac{\pi^2}{15}\frac{(k_B T_{DP})^4}{\hbar^3 c^5} \\ &= \frac{\pi^2}{15}\frac{\left(k_B(1+z_{Re-ionization})T_{today}\right)^4}{\hbar^3 c^5} \\ &= 8.09 \cdot 10^{-27}\ \frac{kg}{m^3} = constant\end{aligned}$$

We can compare this to the to the accepted relic density by using the ΛCDM model with $\Omega_\Lambda = 0.734$ and $\mathrm{H}_0 = 71.0\frac{km}{s}/Mpc$ [10].

$$\rho_{\Lambda CDM} = \Omega_\Lambda \rho_{\text{critical}} = 6.95 \cdot 10^{-27}\ \frac{kg}{m^3}$$

The explanation given here (based on dark particles) for dark energy is only off by 16% from ΛCDM; a huge improvement from the 10^{120} discrepancy found in other derivations of the vacuum energy density [11].

The argument continues for dark matter; *in local regions near other baryonic matter* these dark particles find the sinks needed to exchange heat and thus allow the temperature of the radiation and the density to vary. In this environment the dark particles' radiation has a positive pressure energy density that presents an attractive gravitational force explaining the observed gravitational effects of dark matter and why it clumps near other baryonic matter.

In the case of galaxy rotation, dark particles provides an explanation why the velocity of rotation does not scale like one over the square root of the radius which would be consistent with a large mass at the center of the galaxy [1]. But rather the mass appear to be spread throughout the galaxy. In the dark particle explanation, the gas of the galaxy (which is spread throughout the galaxy) couples the dark particles to the local temperature giving a local positive pressure to the galaxy. The combination of the temperature of the gas as a function of radius and the density of coupling sites as a function of radius, could explain why the velocity of rotation goes is flat.

Two opposite examples are in agreement with the argument here. First VIRGOI21 is an example where a preponderance of both hydrogen atoms and dark matter is found [15]. Oppositely globular clusters are regions where starlight is very prevalent. In this case we would expect any atoms, or other loose baryonic matter to be fully ionized and thus not able to provide the coupling mechanism to allow the local dark particles' radiation to cool. The local region would have no positive pressure dark particles explaining why no dark matter is measured within globular clusters [16].

IV. CONCLUSION

While the mathematics of quantum diffusion can get complicated [3] and that of general relativity even more complicated [5], the mathematics presented here is intended to not be more complicated than calculus. Rather the intent is to present the framework and the right values for the physical parameters necessary to join Freidmann's equation to Heisenberg's Uncertainty equation.

The only Ansatzes used are 1) dark particles coming as pairs and 2) a spring force that resists motion. Given that this simple development requires no major change to quantum mechanics or general relativity and has the potential to explain dark matter or dark energy, more investigation is justified under Occam's razor [17].

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APPENDIX A – ENERGY DENSITY

A. The Process

This analysis hypothesizes that a particle comes into existence with its pair and the two will independently diffuse in space in each of the three position dimensions. To start with, the magnitude squared of the wavefunction will be the Gaussian distribution

$$x \sim p(x)dx = \frac{1}{\sqrt{2\pi\Delta x^2}} \cdot e^{-\frac{x^2}{2\Delta x^2}} dx$$

We can now calculate the distribution on the momentum p_x, p_y, and p_z. The distribution on p_x will also be a Gaussian with a variance that satisfies the Heisenberg uncertainty principle [2].

$$(\Delta p_x)^2 = \frac{\hbar^2}{4(\Delta x)^2}$$

$$p_x \sim p(p_x)dp_x = \frac{1}{\sqrt{2\pi\Delta {p_x}^2}} \cdot e^{-\frac{{p_x}^2}{2\Delta {p_x}^2}} dp_x$$

Thus the probability distribution on x and p_x (with y & z and p_y & p_z being respectively equal), will be the Gaussian distribution with mean zero and variance $(\Delta x)^2$ and $(\Delta p_x)^2$ respectively.

The total energy of the particle will have both position and momentum components. The additon of the kinetic energy should be familiar $\hbar\omega_{px} = \frac{1}{2m}(p_x)^2$. However there is also a potential energy term $\hbar\omega_x = \frac{m}{2}\left(\frac{x}{\tau}\right)^2$ resulting from the inertial mass. Assuming the particle is at rest relative to a reference frame, the particle will originate at rest and begin to move a total displacement, x in time τ. From Netwon's second Law [18] the force on the particle will be

$$F_{particle} = ma = m\frac{\Delta v}{\Delta t} = m\frac{\frac{\Delta x}{\Delta t}}{\Delta t} = \frac{mx}{(\tau)^2}$$

With the particle being at rest on average we can look at Newton's third Law [18] or stating Newton's first Law [18] backward, i.e. given a particle at rest, the sum of the forces acting on the particle is zero, we have

$$F_{spring} = -F_{particle} = \frac{-mx}{(\tau)^2}$$

Here we see Hook's Law which has an associated potential energy $\hbar\omega_x = \frac{m}{2}\left(\frac{x}{\tau}\right)^2$ leading to the harmonci osscilator with 1-D groud state energy, E_{x0} [19].

$$E_{x0} = \frac{\hbar\omega_0}{2} = \frac{\hbar}{2}\sqrt{\frac{m/\tau^2}{m}} = k_B T$$

Considering the y and z dimensions as well, we are led to the three dimensional harmonic oscillator.

$$Energy = \hbar\omega = \hbar\omega_x + \hbar\omega_{px} + \hbar\omega_y + \hbar\omega_{py} + \hbar\omega_z + \hbar\omega_{pz}$$

$$Energy = \hbar\omega = \frac{m}{2}\left(\frac{x}{\tau}\right)^2 + \frac{1}{2m}(p_x)^2 + \frac{m}{2}\left(\frac{y}{\tau}\right)^2 + \frac{1}{2m}(p_y)^2 + \frac{m}{2}\left(\frac{z}{\tau}\right)^2 + \frac{1}{2m}(p_z)^2$$

If we assume the process is ergodic and that the particle is at a equilibrium temperature T the equipartition theorem can be evoked where, the average energy is equal to $k_B T/2$ for each quadratic dimension [19].

$$\overline{\frac{m}{2}\left(\frac{x}{\tau}\right)^2} = \overline{\frac{1}{2m}(p_x)^2} = \frac{k_B T}{2}$$

Since the average of x and p_x is equal to zero, the average of x^2 and ${p_x}^2$ is equal to the variance $(\Delta x)^2$ and $(\Delta p_x)^2$

$$(\Delta p_x)^2 = m \cdot k_B T$$

$$(\Delta x)^2 = \frac{\hbar^2}{4m \cdot k_B T}$$

$$\tau = \frac{\hbar}{2k_B T}$$

$$\overline{\hbar\omega} = \frac{6 \cdot k_B T}{2}$$

After τ seconds, the particle will be found in phase space in each of the three dual dimensions: (x, p_x), (y, p_y), and (z, p_z).

Given the relationship $\omega_x = ax^2\ (a > 0)$ and with $x \sim p(x)$) as above, the distribution on ω_x is [20],

$$\omega_x \sim p(\omega_x)d\omega_x = \frac{1}{\sqrt{2\pi\Delta x^2 a\omega_x}} e^{\frac{-\omega_x}{2a\Delta x^2}} d\omega_x$$

Thus from the probability distributions on x, y, and z given above the distribution on $\omega_x$is derived

$$\omega_x \sim p(\omega_x) d\omega_x = \sqrt{\frac{\hbar}{\pi k_B T \omega_x}} e^{\frac{-\hbar\omega_x}{k_B T}} d\omega_x$$

Fortunately the distribution on ω_{px}, given p_x, p_y, and p_z above, is the same

$$\omega_{px} \sim p(\omega_{px}) d\omega_{px} = \sqrt{\frac{\hbar}{\pi k_B T \omega_{px}}} e^{\frac{-\hbar\omega_{px}}{k_B T}} d\omega_{px}$$

Since ω is simply the sum of the independent variables ω_x, ω_y, ω_z and ω_{px}, ω_{py}, ω_{pz} we can solve for the distribution on ω as the convolution of the other distributions [21].

$$\omega \sim p(\omega) = p(\omega_x) * p(\omega_{px}) * p(\omega_y) * p(\omega_{py}) * p(\omega_z) * p(\omega_{pz})$$

$$\omega \sim p(\omega) d\omega = \frac{\hbar^3\omega^2}{2(k_B T)^3} e^{\frac{-\hbar\omega}{k_B T}} d\omega$$

To find the average energy density, integrate over frequency and divide by the volume of space that the particle and it's pair could possibly fall within. The volumes should be the space a photon can traverse in time τ in either direction, i.e. $(2c\tau)^3$

$$\rho = \frac{\int_0^\infty \frac{\hbar\omega}{c^2} p(\omega) d\omega}{V} = \frac{\int_0^\infty \frac{\hbar\omega}{c^2} \frac{\hbar^3\omega^2}{2(k_B T)^3} e^{\frac{-\hbar\omega}{k_B T}} d\omega}{c^2 (2c\tau)^3} = \frac{3(k_B T)^4}{\hbar^3 c^5}$$

APPENDIX B – QUANTUM DIFFUSION

A. Linear Variance

This diffusion is also called imaginary diffusion as it can be derived by taking the quantum mechanical kinetic energy Hamiltonian and making a Minkowski transformation to imaginary time.

$$H = \frac{p^2}{2m}$$

$$i\hbar \frac{\partial}{\partial t} = -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2}$$

$$it \rightarrow t$$

$$\frac{\partial}{\partial t} = \frac{\hbar}{2m} \frac{\partial^2}{\partial x^2}$$

$$D = \frac{\hbar}{2m}$$

D can also be derived from Einstein's kinetic theory [6].

$$D = \mu k_B T = \frac{v}{F} k_B T = \frac{v}{m(v/\tau)} k_B T = \frac{\hbar}{2m}$$

Plugging into the variance we have

$$(\Delta x)^2_{Linear} = \frac{\hbar}{m} \cdot t$$

This quantum diffusion has also been studied by Belavkin [3] and Nelson [4].

B. Quadratic Variance

A second type of quantum diffusion occurs due to multiple frequencies evolving at different rates. While this derivation will follow the method outlined by Bohm [22], the precise result is found in Shankar [2].

$$\Psi(p_x) = \frac{1}{\sqrt{\sqrt{2\pi\Delta p_x{}^2}}} \cdot e^{-\frac{p_x{}^2}{4\Delta p_x{}^2}}$$

$$\psi(x) = \int_{-\infty}^{\infty} e^{i(p_x x - E(p_x)t)/\hbar} \Psi(p_x)\, dp_x$$

Where

$$E(p_x) = \frac{p_x{}^2}{2m}$$

Solving the integral and taking the magnitude squared results in a probability distribution with a variance that is grows in quadratic time.

$$\psi^*(x)\psi(x) = \frac{1}{\sqrt{2\pi\left(\Delta x^2 + \frac{k_B T_0}{m} t^2\right)}} \cdot e^{-\frac{x^2}{2\left(\Delta x^2 + \frac{k_B T_0}{m} t^2\right)}} dx$$

Considering only the quadratic term $\Delta x^2_{Quadratic}$ we have

$$\Delta x^2_{Quadratic}(t) = \frac{k_B T_0}{m} t^2$$

C. Constant Variance

When the temperature is constant at T_0, and $\tau_0 = \hbar/2k_B T_0$ we have,

$$(\Delta x)^2_{Constant} = \frac{\hbar}{m} \cdot \tau_0 = (\Delta x)^2 + \frac{(\Delta p_x)^2}{m^2} \tau_0{}^2 = \frac{\hbar^2}{2m k_B T_0}$$

D. Langevan Equation

The Langevan equation begins with the force equation with both a resistive force that opposes and is proportional to the velocity and a noise driving force [6].

$$m(\ddot{x}) = -\frac{m}{\tau}(\dot{x}) + F_{Noise}$$

Einstein's relation can be used to show that $\tau = \hbar/2k_BT$ as we defined above [6]. Multiplying by x and using the product rule of differentiation yields,

$$mx(\ddot{x}) = m\frac{d}{dt}\big(x(\dot{x})\big) - m\big((\dot{x})\big)^2 = -\frac{m}{\tau}x(\dot{x}) + xF_{Noise}$$

In the classical case F_{Noise} is uncorrelated with x, however here the force is the same as we used in deriving the harmonic oscillator in appendix A,

$$F_{Noise} = F_{spring} = \frac{-mx}{(\tau)^2}$$

$$m\frac{d}{dt}\big(x(\dot{x})\big) = -\frac{m}{\tau}x(\dot{x}) + m(\dot{x})^2 - \frac{mx^2}{(\tau)^2}$$

Taking the time average and using the equipartition theorem [19] on the kinetic and potential energy leaves,

$$m\frac{d}{dt}\left(\overline{x(\dot{x})}\right) = -\frac{m}{\tau}\overline{x(\dot{x})} + k_BT - k_BT$$

$$\overline{x(\dot{x})} = Ce^{-t/\tau}$$

Again a break from the classical derivation when $\overline{x(\dot{x})}(t=0)$. Classically this initial condition is zero, however we want the Langevin equation to be quantum mechanically correct and obey the Heisenberg uncertainty relation.

$$\overline{x(\dot{x})}(t=0) = \frac{\hbar}{2m}$$

$$\overline{x(\dot{x})} = \frac{\hbar}{2m}e^{-t/\tau}$$

From here the rest of the derivation is the same as the classical Langevin derivation

$$\frac{d}{dt}\left(\overline{x^2}\right) = 2\overline{x(\dot{x})} = \frac{\hbar}{m}e^{-t/\tau}$$

$$\overline{x^2} = \Delta x^2_{Langevin} = \int_0^t \frac{\hbar}{m}e^{-t'/\tau}dt = \frac{\hbar}{m}\tau\left(1 - e^{-t/\tau}\right) = 2D\tau\left(1 - e^{-t/\tau}\right)$$

APPENDIX C – BLACK BODY RADIATION

The energy density in appendix A was related to fermions. In the fermions' case, there can be only one particle per quantum state. However a radiation density can have more than one particle occupy each quantum state. The energy will be radiated by a photon of energy $\hbar\omega$; since there can actually be M photons per mode (as a photon is a boson) the total energy is $M\hbar\omega$. Again enlisting the help of the equipartition theorem, we solve for M [19].

$$\begin{aligned} Energy = M \cdot \hbar\omega &= \frac{m}{2}\left(\frac{x}{\tau}\right)^2 + \frac{1}{2m}(p_x)^2 + \frac{m}{2}\left(\frac{y}{\tau}\right)^2 \\ &+ \frac{1}{2m}\left(p_y\right)^2 + \frac{m}{2}\left(\frac{z}{\tau}\right)^2 + \frac{1}{2m}(p_z)^2 \end{aligned}$$

$$M \cdot \overline{\hbar\omega} = \frac{6 \cdot k_BT}{2}$$

$$M \cdot \overline{\frac{m}{2}\left(\frac{x}{\tau}\right)^2} = M \cdot \overline{\frac{1}{2m}(p_x)^2} = \frac{k_BT}{2}$$

$$(\Delta p_x)^2 = \frac{m \cdot k_BT}{M}$$

$$(\Delta x)^2 = \frac{M\hbar^2}{4m \cdot k_BT}$$

$$\tau = \frac{M\hbar}{2k_BT}$$

Plugging these into our equation for the probability the energy is ω, $p_{radiation}(\omega)d\omega$

$$p_{radiation}(\omega)d\omega = \frac{M^3\hbar^3\omega^2}{2(k_BT)^3}e^{\frac{-M\hbar\omega}{k_BT}}d\omega$$

Next to find the density we need Poynting's theorem. The energy $\hbar\omega$ built up by the particle's diffusion is transferred to the photon field every τ seconds. As the radiation moves away from the particle, general relativistic effects red-shifts the energy and dilates the cycle duration. Using Poynting's theorem as a form of conservation of energy [23], the delta in radiated power between when the photon is emitted in the high gravitational fields near the particle and the radiation power a distance $c\tau$ away from the particle is found. This delta can be thought of as the energy density trapped by the gravitational fields.

When there is no current we have,

$$P = \frac{d}{dt}E = -S \cdot A$$

In figure 3 below, the inner circle has a radius equal the particle's spatial step as defined by Heisenberg uncertainty principle.

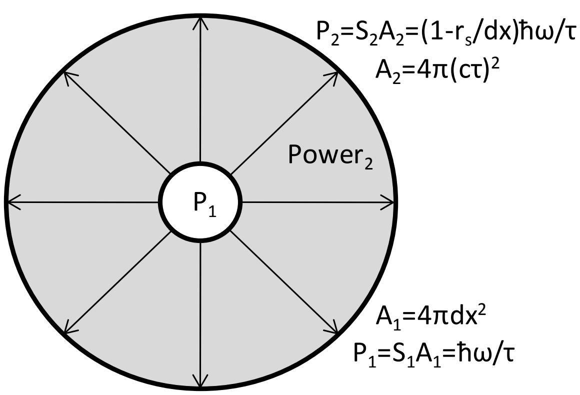
$$dx = cdt = \frac{\hbar}{2mc}$$

When the particle radiates the power flowing through this hypothetical inner circle is equal to the energy of the photon divided by the cycle time of the diffusion process, $\hbar\omega/\tau$. The outer circle has a radius equal to the extent of a photon traveling for τ seconds which if $mc^2 \gg k_B T$ is beyond the range of the particle's general relativistic effects.

The power flux at this point in space is reduced by the factor $(1 - r_s/dx)/(1 - r_s/c\tau)$, where r_s is the Schwarzchild radius. This reduction in power flux is due to the energy being gravitationally red-shifted by the amount $\sqrt{1 - r_s/dx}/\sqrt{1 - r_s/c\tau}$, and the time being dilated by the factor, $\sqrt{1 - r_s/c\tau}/\sqrt{1 - r_s/dx}$ [24]. When $mc^2 \gg k_B T$ we have,

$$S_1 \cdot A_1 = \frac{\hbar\omega}{\tau}$$

$$S_2 \cdot A_2 = \frac{\hbar\omega}{\tau}(1 - r_s/dx)$$



The power between the inner radius and the outer radius (the grey area in figure 3 above) is

$$P_2 - P_1 = S_1 \cdot A_1 - S_2 \cdot A_2 = \frac{\hbar\omega}{\tau}\frac{r_s}{dx} = \frac{\hbar\omega}{2\pi\tau}$$

$P_2 - P_1$ divided by the area of the outer radius, represents the power flux. Thus $(P_2 - P_1)/A_2$ divided by the speed of light and multiplied by the probability distribution on the radiated energy is the average energy density of the radiation field gravitationally trapped by the mass of the particle.

$$\begin{aligned}\rho_{radiation-M}(\omega)d\omega &= \frac{\hbar\omega}{2\pi\tau}\frac{p_{radiation}(\omega)d\omega}{c \cdot 4\pi(c\tau)^2} \\ &= \frac{\hbar\omega^3}{2\pi^2 c^3}e^{\frac{-M\hbar\omega}{k_B T}}d\omega\end{aligned}$$

The last step is to sum over all states M from 1 to ∞, and both degrees of polarization [8] since all are possible.

$$\begin{aligned}\rho_{radiation}(\omega)d\omega &= \sum_{pol=1,-1}\sum_{M=1}^{\infty}\rho_{radiation-M}(\omega)d\omega \\ &= \frac{\hbar\omega^3 d\omega}{\pi^2 c^3}\sum_{M=1}^{\infty}e^{\frac{-M\hbar\omega}{k_B T}} \\ &= \frac{\hbar\omega^3}{\pi^2 c^3}\frac{1}{\left(e^{\hbar\omega/kT} - 1\right)}d\omega\end{aligned}$$

One will recognize the energy density of Black Body radiation, and of course integrating over ω gives the total average energy density [8]

$$\rho_{radiation} = \int_0^\infty \rho_{radiation}(\omega)d\omega = \frac{\pi^2 (k_B T)^4}{15\hbar^3 c^3}$$

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