Getting path integrals physically and technically right

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Abstract

Feynman's Lagrangian path integral was an outgrowth of Dirac's vague surmise that Lagrangians have a role in quantum mechanics. Lagrangians implicitly incorporate Hamilton's first equation of motion, so their use contravenes the uncertainty principle, but they are relevant to semiclassical approximations and relatedly to the ubiquitous case that the Hamiltonian is quadratic in the canonical momenta, which accounts for the Lagrangian path integral's "success". Feynman also invented the Hamiltonian path integral, which is fully compatible with the uncertainty principle. This paper refines that path integral to automatically enforce standard endpoint stipulations on the paths over which it integrates, which makes proof of its key decomposition property straightforward. Orthogonal path expansion using "Taylor-normalized" Legendre polynomials in time enables that path integral to be evaluated unambiguously through first order in its elapsed time. This, together with its decomposition property, shows that the path integral satisfies the Schrödinger equation with a unique quantization of its classical Hamiltonian. A widespread misconception regarding that uniqueness is traced to the erroneous belief that widely separated path endpoint stipulations are not fulfilled for arbitrarily short nonzero elapsed times. The paper also obtains the quantum amplitude for any stipulated configuration or momentum path, which turns out to be an unrestricted functional integral over, respectively, all momentum or all configuration paths. The first of these results is directly compared with Feynman's mistaken Lagrangian-action hypothesis for such a configuration path amplitude, with special heed to the case that the Hamiltonian is quadratic in the canonical momenta.

Introduction

The incorporation of the correspondence principle into quantum mechanics has proceeded along two profound and elegant parallel tracks, namely Dirac's canonical commutation rules and Feynman's path integrals. It is, however, unfortunately the case that from their inceptions the *prescribed implementations* of both of these have had some physically unrefined aspects—albeit these conceivable stumbling blocks turn out to be of little or no *practical* consequence in light of the fact that the Hamiltonians which have been of interest are almost invariably quadratic forms in the canonical momenta and as well usually consist of sums of terms which themselves depend either on *only* the canonical coordinates or on *only* the canonical momenta, which makes their unique quantization unmistakably obvious. In this paper we nonetheless show that the *physically called-for* refinements of the prescribed implementations of both the canonical commutation rules and the path integrals result in the unique quantization of *all* classical Hamiltonians rather than *only* those which have heretofore been of practical interest. This endows quantum mechanics with a degree of coherence and consistency which is entirely comparable to that of classical mechanics, and also renders fully transparent its precise relationship to the latter.

Whereas the called-for refinement of Dirac's canonical commutation rule prescription is the straightforward strengthening of its classical correspondence to the maximum that is still self-consistent, the physical issue which besets Feynman's prescribed *Lagrangian* path integral is more drastic. Because Lagrangians *implicitly* incorporate Hamilton's first equation of motion, they likewise *implicitly* contravene the uncertainty principle, which makes their utilization in rigorous quantum theory impermissible—albeit they *do* play a role in semiclassical approximations and, relatedly, in the practically ubiquitous special circumstance that the Hamiltonian is a *quadratic form in the canonical momenta.* In general, however, the *Lagrangian* path integral must be regarded as invalid, and should be replaced by the *Hamiltonian phase-space path integral*, also invented by Feynman, which is fully compatible with the uncertainty principle. This paper upgrades the technical efficacy of that path integral's adherence to the standard configuration or momentum endpoint restrictions on the phase-space paths over which it integrates, which consequently makes demonstration of its key decomposition property entirely straightforward. A widespread misconception that the Hamiltonian phase-space path integral does *not* yield a unique result is traced to *misapprehension* of the fact that these endpoint restrictions on the permitted paths may be *arbitrarily* specified (*and* are fulfilled) *regardless* of *how short* the nonzero time interval allotted to those paths may be.

Through orthogonal path expansion using specially "Taylor-normalized" scaled and translated Legendre polynomials in time, the Hamiltonian phase-space path integral is calculated through first order in its time interval, which yields the unique quantization of its classical Hamiltonian. That result turns out to be in complete accord with the unambiguous quantization of that classical Hamiltonian which emerges from the strengthened, but still self-consistent, variant of Dirac's canonical commutation rule prescription that is mentioned above. From its expression through first order in its time interval, together with its decomposition property, it is readily shown that the Hamiltonian phase-space path integral satisfies the Schrödinger equation.

This paper also obtains the formal quantum amplitude for a *specified* configuration-space path or a *specified* momentum-space path as an *unrestricted* functional integral over, respectively, *all* momentum-space paths or *all* configuration-space paths. The first of these two results is then instructively *directly* compared and contrasted with Feynman's mistaken *Lagrangian-action hypothesis* for such a specified configuration-space path amplitude, with special attention given to the case that the Hamiltonian is a quadratic form in the canonical momenta.

The Lagrangian path integral

In the preface to *Quantum Mechanics and Path Integrals* by R. P. Feynman and A. R. Hibbs [1], which treats *only* the *Lagrangian* path integral, the reader encounters the revelation that, "Over the succeeding years, ... Dr. Feynman's approach to teaching the subject of quantum mechanics evolved somewhat away from the initial path integral approach. At the present time, it appears that the operator technique is both deeper and more powerful for the solution of more general quantum-mechanical problems." Unfortunately, no recognizable elaboration of this cautionary note regarding the *Lagrangian* path integral is to be found in the book's main text. But in what might be construed as a muffled echo of this theme, we *do* learn in the second paragraph of page 33 of the book that to define the "normalizing factor" $1/A$ which is required to *convert* the Dirac-inspired very short-time Lagrangian-action phase factor [2] into the *actual* very short-time quantum mechanical propagator in configuration representation "seems to be a very difficult problem and we do not know how to do it in general terms" [1]. This makes it clear that the authors, *contrary* to a widely held impression, did *not succeed* in making *Lagrangian* path integration into a systematic alternate approach to quantum mechanics—which one could suppose may have been reason enough for Feynman to have *turned away* from teaching it.

On page 33 Feynman and Hibbs interpret this "normalizing factor" $1/A$ as *also* being the "path measure normalization factor", which, when paired with each of multiple integrations over configuration space (at successive, narrowly spaced points in time), converts the whole lot of those integrations into an actual integration over *all paths* in the limit that the spacing of the successive time points is taken to zero. For the *particular* class of one-degree-of-freedom Lagrangians which have the form, $L(\dot{q}, q, t) = \frac{1}{2}m\dot{q}^2 - V(q, t)$—to which corresponds the class of quantized Hamiltonians that have the form, $\widehat{H}(t) = \widehat{p}^2/(2m) + V(\widehat{q}, t)$—Feynman and Hibbs point out on page 33 that the factor $1/A$ comes out to equal $\sqrt{m/(2\pi i\hbar\delta t)}$, as that particular quantity properly converts the δt-time-interval Lagrangian-action phase factor into the *actual* δt-time-interval *quantum mechanical propagator in configuration representation.* Feynman and Hibbs fail, however, to scrutinize the issue of whether this object can pass muster as *also* being the "path measure normalization factor" which they have, on page 33, *explicitly* claimed it must be. One notes immediately that this particular $1/A$ depends on the particle mass m, whereas the *set* of *all* paths could not possibly depend on anything other than the time interval on which they are defined and the constraints on their endpoints. The "measure normalization factor" for such paths could also feature constants of mathematics and of nature, but that *set* of *all* paths clearly *does not change in the slightest* if a *different* value is selected for the particle's *mass*! The particle mass is a *parameter* of the Lagrangian, which is supposed to be at the heart of the path *integrand*—the *measure* aspect of any integral is always supposed to be *independent* of the choice of *integrand*! Furthermore, "measure normalization factors" are, by their nature, supposed to be *positive* numbers, whereas this particular $1/A$ is complex-valued! It can only be concluded that the "Lagrangian path integral" simply *cannot* make sense as a "path integral" at all! It is a great pity that Feynman failed to recognize these *surface* anomalies of the "Lagrangian path integral" immediately, as digging deeper only unearths ever worse ones.

Feynman does not seem to have reflected at all on the fact that mechanical systems that are described by *configuration* Lagrangians $L(\dot{q}, q, t)$ can in most instances *also* be described by *momentum* Lagrangians $L(\dot{p}, p, t)$. Indeed, if $L(\dot{q}, q, t) = \frac{1}{2}m\dot{q}^2 - V(q, t)$, then it turns out that $L(\dot{p}, p, t) = -\dot{p}F^{-1}(\dot{p}; t) - V(F^{-1}(\dot{p}; t)) - p^2/(2m)$, where $F(q; t) \stackrel{\text{def}}{=} -\partial V(q, t)/\partial q$. Unpleasant though this $L(\dot{p}, p, t)$ appears for *general* $V(q, t)$, it greatly simplifies when $V(q, t)$ is a *quadratic form* in q, e.g., for the harmonic oscillator $V(q, t) = \frac{1}{2}kq^2$, $L(\dot{p}, p, t) = \dot{p}^2/(2k) - p^2/(2m)$. Indeed it will pretty much be for *only* those $V(q, t)$ which *are* quadratic forms in q that the very short-time quantum mechanical propagator in *momentum representation*, which is simply a *Fourier transformation* of the one in configuration representation, will bear much resemblance to the desired very short-time *momentum* Lagrangian-action phase factor that arises from the quite ugly $L(\dot{p}, p, t)$ given above—the good correspondence in the quadratic form cases is an instance of the fact that the Fourier transformation of an exponentiated quadratic form generally comes out to itself be an exponentiated quadratic form times a simple factor (albeit that factor is by *no means* assured to make sense in the role of "path measure normalization factor", as we have seen above). When $V(q, t)$ is *not* a quadratic form in q, it will usually be quite impossible to transparently relate the Fourier transformation of the very short-time quantum propagator in configuration representation to the very short-time Lagrangian-action phase factor which arises from the fraught $L(\dot{p}, p, t)$ given above. The burden of reconciling the two will then have been loaded *entirely* onto the shoulders of the $1/A$ factor, whose role as a *fudge factor* will thus have been starkly exposed (its forlorn cause as a "path measure normalization factor" will certainly *not* have been furthered).

The *inability* of the *Lagrangian* approach to cope in all but very fortuitous circumstances with the *Fourier transformations* that take the quantum mechanics configuration representation to its momentum representation and conversely, suggest a *fundamental incompatability* of Lagrangians with the canonical *commutation rule*, $\widehat{q}\widehat{p} - \widehat{p}\widehat{q} = i\hbar\mathrm{I}$, as that *underlies* the Fourier relation between those representations. It *also*, of course, is the *heart* of the *uncertainty principle.* Now Feynman took pains to try to move well away from classical dynamics by attempting (albeit not so successfully!) to *integrate* quantum amplitudes over *all* paths, so it does not seem likely that conflicts with the above quantum canonical commutation rule could be rooted in that aspect of his approach. We have, however, just seen that, aside from Lagrangians of quadratic form, the relationships between $L(\dot{q}, q, t)$ and $L(\dot{p}, p, t)$ exhibit *no* indication of compatibility with that commutation rule. This seems to hint that there may be something *intrinsic to Lagrangians* that is generally incompatible with the quantum momentum-configuration commutation rule. So might $L(\dot{\mathbf{q}}, \mathbf{q}, t)$ *itself* have a property that *clashes* with the *uncertainty principle*? It turns out that one need *not* look very far to locate *that* culprit: Dirac (and later Feynman) *simply failed* to bear in mind the *basic fact* that to *any* configuration path $\mathbf{q}(t)$, $L(\dot{\mathbf{q}}, \mathbf{q}, t)$ *automatically associates a uniquely determined momentum path* $\mathbf{p}(t) = \nabla_{\dot{\mathbf{q}}(t)} L(\dot{\mathbf{q}}(t), \mathbf{q}(t), t)$, a relation that is *patently incompatible* with the uncertainty principle!

Dirac's vague 1933 surmise about the role of the *Lagrangian* in quantum mechanics [2] has clearly done a long-lived disservice to physics, but Feynman and also all those who sought to educate themselves in Feynman's Lagrangian path integral results were as well scientifically obliged to ponder and pursue any apparently dubious peculiarities which emanate from them. H. Bethe blurted out that there are no paths in quantum mechanics upon hearing Feynman's ideas for the first time at a Cornell University seminar. While this initial visceral reaction cannot be defended as stated, it seems clear that discomfort concerning the uncertainty principle was percolating in Bethe's mind. It is a very great pity that Bethe did not *persist* in pondering that discomfort, seeking to pin down and confront its source.

The Hamiltonian actions and the phase-space path integral concept

Feynman not only originated the Lagrangian path integral idea, he was also the first to publish the idea of the *Hamiltonian phase-space path integral*—which he deeply buried in Appendix B of his major 1951 paper [3]. Apparently he attached little importance to it, and it conceivably slipped from his mind by 1965, as there is *no mention* of it in the book by Feynman and Hibbs. Perhaps Feynman had a reflexive aversion to all Hamiltonian approaches because of the fact that the Hamiltonian *density* in field theories is *not* Lorentz-invariant, whereas the Lagrangian density is—that would have been a pity: the full *action density in Hamiltonian form* is *also* a Lorentz invariant; indeed the Lagrangian density is merely a *restricted version* of this. For quantum theory the Hamiltonian is far superior, as it does *not* harbor the uncertainty principle trap that is *implicit* in the Lagrangian. To be sure, *either one* of the *two* classical Hamiltonian equations of motion *does* implicitly contradict the uncertainty principle (indeed, the Lagrangian is a version of the Hamiltonian action integrand that has been *restricted* according to *one* of the classical Hamiltonian equations of motion). But if we firmly drop *both* classical Hamiltonian equations of motion, $\mathbf{q}(t)$ and $\mathbf{p}(t)$ become *independent* argument functions of the Hamiltonian action functional, and thus do *not* challenge the uncertainty principle.

The path integral concept in this context then becomes one of summing quantum amplitudes over *all phase-space paths.* This states what must be done a bit too expansively, however, as we know that in order to obtain a physically *useful* summed amplitude, we must *restrict* the $\mathbf{q}(t)$ paths to ones which all have the *same* value $\mathbf{q}_i$ at the initial time t_i and *also* all have the *same* value $\mathbf{q}_f$ at the final time t_f. An *alternate* useful restriction is, of course, to require the $\mathbf{p}(t)$ paths to all have the *same* value $\mathbf{p}_i$ at the initial time t_i and *also* to all have the *same* value $\mathbf{p}_f$ at the final time t_f. As is well known, when the configuration paths $\mathbf{q}(t)$ are endpoint-restricted as just described, the two classical Hamiltonian equations of motion result from setting to zero the first-order variation with respect to $[\mathbf{q}(t), \mathbf{p}(t)]$ of the Hamiltonian action functional,

$$S_H([\mathbf{q}(t), \mathbf{p}(t)]; t_f, t_i) \stackrel{\text{def}}{=} \int_{t_i}^{t_f} dt\ (\dot{\mathbf{q}}(t) \cdot \mathbf{p}(t) - H(\mathbf{q}(t), \mathbf{p}(t), t))\,, \tag{1a}$$

whereas when it is the *momentum* paths $\mathbf{p}(t)$ that are endpoint-restricted as described above, the *same* two classical Hamiltonian equations of motion result from setting to zero the first-order variation with respect to $[\mathbf{q}(t), \mathbf{p}(t)]$ of the very slightly *different* Hamiltonian action functional,

$$S'_H([\mathbf{q}(t), \mathbf{p}(t)]; t_f, t_i) \stackrel{\text{def}}{=} \int_{t_i}^{t_f} dt\ (-\mathbf{q}(t) \cdot \dot{\mathbf{p}}(t) - H(\mathbf{q}(t), \mathbf{p}(t), t))\,. \tag{1b}$$

We are, to be sure, interested in *summing* the quantum amplitudes for *all* the appropriately endpoint-restricted phase-space paths rather than in finding *which* of those paths is the *classical* one by the variational approach. Nevertheless, in order to honor the *correspondence principle*, we must make it a path summand *requirement* that the *dominant* path as $\hbar \to 0$, i.e., the path of *stationary phase*, *matches* the classical path. For that reason, we must be careful to *also* match the very slightly *different* actions, S_H or S'_H, respectively, to their appropriate *corresponding* configuration or momentum endpoint restrictions, respectively, *even* in the summands of our *path sums* over *quantum amplitudes*—which, in standard fashion, are taken to be proportional to the exponential of $(i/\hbar)$ times the *action* of the path in question.

We also note that that the *values* which the two endpoint-restriction vectors $\mathbf{q}_i$ and $\mathbf{q}_f$ (or, alternately, $\mathbf{p}_i$ and $\mathbf{p}_f$) are *permitted* to assume are *completely arbitrary* and *mutually independent.* We shall, in fact, in quantum mechanical practice frequently be *integrating* over the *full range* of either or both of $\mathbf{q}_i$ and $\mathbf{q}_f$ (or, alternately, of either or both of $\mathbf{p}_i$ and $\mathbf{p}_f$), so this utter freedom of choice is, in fact, a *necessity*—in the language of quantum mechanics the *range* of both $\mathbf{q}_i$ and $\mathbf{q}_f$ (or, alternately, of both $\mathbf{p}_i$ and $\mathbf{p}_f$) must, for *each*, describe a *complete set* of quantum states. The statements just made are neither modified nor qualified *in the slightest* when the *positive* quantity $|t_f - t_i|$ is made *increasingly small.* In other words, $|\mathbf{q}_f - \mathbf{q}_i|$ (or, alternately, $|\mathbf{p}_f - \mathbf{p}_i|$) remains *unbounded* no matter *how small* the positive value of $|t_f - t_i|$ may be. There will *always exist* an infinite number of paths which *adhere* to the endpoint restrictions no matter *how large* $|\mathbf{q}_f - \mathbf{q}_i|$ is or *how small* a positive value $|t_f - t_i|$ assumes. Indeed, given *any velocity* $\mathbf{v}(t)$ that is defined for $t \in [t_i, t_f]$ and *which satisfies* $\int_{t_i}^{t_f} dt\, \mathbf{v}(t) = \mathbf{q}_f - \mathbf{q}_i$, the path,

$$\mathbf{q}(t) = \mathbf{q}_i + \int_{t_i}^{t} dt'\, \mathbf{v}(t'),$$

obviously qualifies. *One* such velocity $\mathbf{v}(t)$ is, of course, the *constant* one, $(\mathbf{q}_f - \mathbf{q}_i)/(t_f - t_i)$, and to it may be added an *arbitrary* number of terms of the form, $\mathbf{v}^{(n)}(t_i)((t-t_i)^n/n! - (t_f-t_i)^n/(n+1)!)$, $n = 1, 2, \ldots$. These *utterly elementary* observations have, in fact, *completely eluded the grasp* of an astonishing number of "experts" in the field of path integrals. Time and again it is implicitly or explicitly *insisted* that,

$$\lim_{|t_f-t_i|\to 0} |\mathbf{q}_f - \mathbf{q}_i| = 0,$$

which is then taken to justify the resort to *completely unsound approximations*, in some instances even a *vast class* of these [5, 6]. This last approach can produce variegated results that are not merely wrong, but even mutually incompatible!

The endpoint-restriction *configuration* vectors $\mathbf{q}_i$ and $\mathbf{q}_f$ are, of course, as well part and parcel of the *Lagrangian* path integral, and on their page 38, Feynman and Hibbs make a variation of the blunder just described. Their Equation (2-33) on that page shows a very clear instance of $\mathbf{q}_i$ and $\mathbf{q}_f$ being *independently* integrated, *each* over its *full range.* That *notwithstanding*, just below their *very next* Equation (2-34), they effectively claim that for sufficiently small $|t_f - t_i|$, the error expression $|\mathbf{q}(t) - \frac{1}{2}(\mathbf{q}_f + \mathbf{q}_i)|$ is *first-order* in $|t_f - t_i|$ for all t in the interval $[t_i, t_f]$. Of course $\mathbf{q}(t)$ obeys the usual two fundamental endpoint restrictions $\mathbf{q}(t_i) = \mathbf{q}_i$ and $\mathbf{q}(t_f) = \mathbf{q}_f$. These constraints immediately imply that the above error expression is equal to $\frac{1}{2}|\mathbf{q}_f - \mathbf{q}_i|$ *both* at $t = t_i$ *and* at $t = t_f$. But their *independent* integrations over the *full ranges* of $\mathbf{q}_i$ and $\mathbf{q}_f$ in their *adjacent* Equation (2-33) make it extremely obvious that $\frac{1}{2}|\mathbf{q}_f - \mathbf{q}_i|$ *has no upper bound*! Moreover, this conclusion is clearly *utterly independent* of *how small* a positive value $|t_f - t_i|$ may have!

Having no upper bound is *a very long way indeed* from being first-order in $|t_f - t_i|$ as $|t_f - t_i| \to 0$! This massive blunder by the ostensible ultimate experts in the field drives home the lesson that all scientists bear the obligation to ponder and pursue apparently dubious peculiarities *irrespective* of their pedigree. Science has nothing to gain from the perpetuation of unrecognized mistakes *whatever* their source. The Lagrangian path integral is, of course, deficient because that approach violates the uncertainty principle, i.e., it is *physically* wrong. So adding a gross *mathematical* mistake *on top* of that doesn't really much matter. The critical issue with this particular category of mathematical blunder is that it has *also* infiltrated the Hamiltonian phase-space path integral, which has *no* known deficiencies of physical principle, and the *manner* of the blunder's intrusion has *completely obfuscated* the unique, straightforward result which the Hamiltonian path integral in fact yields.

The key consequences of the Hamiltonian phase-space path integral were first *correctly* worked out in a groundbreaking paper by Kerner and Sutcliffe [4]. That paper was quickly taken to task by L. Cohen [5] *because it failed to take into account the full consequences of the "fact" that* $\lim_{|t_f-t_i|\to 0} |\mathbf{q}_f - \mathbf{q}_i| = 0$! Cohen's "fact" is, of course, as we have gone to great pains above to demonstrate, a *baneful fiction*! A consequence of the toxic assumption that $\lim_{|t_f-t_i|\to 0} |\mathbf{q}_f - \mathbf{q}_i| = 0$ is, according to Cohen and his followers Tirapegui et al. [6], that for all sufficiently small positive values of $|t_f - t_i|$, the term $H(\mathbf{q}(t), \mathbf{p}(t), t)$ in the integrand of the Hamiltonian action in Eq. (1a) may, for *all* t in the interval $[t_i, t_f]$, always be replaced by *any constant-in-time* "discretization" entity of the form $h(\mathbf{q}_f, \mathbf{q}_i, \bar{\mathbf{p}}, \bar{t})$, where $\bar{\mathbf{p}}$ can be regarded as a type of average value of $\mathbf{p}(t)$ for t in the interval $[t_i, t_f]$, $\bar{t}$ is some *fixed* element of that interval, and h is *any* smooth function that satisfies $h(\mathbf{q}, \mathbf{q}, \mathbf{p}, t) = H(\mathbf{q}, \mathbf{p}, t)$. Thus, $H(\frac{1}{2}(\mathbf{q}_f + \mathbf{q}_i), \bar{\mathbf{p}}, \bar{t})$—which is effectively the same as the *bad approximation* to $\mathbf{q}(t)$ by $\frac{1}{2}(\mathbf{q}_f + \mathbf{q}_i)$ advocated by Feynman and Hibbs—is one such "discretization". The quasi-*optimized* "discretization" $\frac{1}{2}(H(\mathbf{q}_f, \bar{\mathbf{p}}, \bar{t}) + H(\mathbf{q}_i, \bar{\mathbf{p}}, \bar{t}))$ is nonetheless *also* a *bad approximation*, as can be verified by examining its differences from $H(\mathbf{q}(t), \mathbf{p}(t), t)$ at the two endpoints $t = t_i$ and $t = t_f$ when $\mathbf{q}_f$ is assumed to be *arbitrarily different* from $\mathbf{q}_i$. The remaining members of this *vast* class of "discretizations" are *bad approximations* as well, as similar arguments about how badly they can miss at one or the other or both of those two time endpoints shows. One upshot of the *misguided imposition* of this vast "discretization" class of *unsound approximations* on the Hamiltonian phase-space path integral is to foster the *false impression* that the Hamiltonian path integral does *not* yield a *unique* result—indeed that it even paradoxically simultaneously yields quite a few *mutually incompatible* results! The *correct* treatment of this path integral *does* in fact yield a unique result; it is merely the fact that *different* members of this *vast class* of *unsound* "discretization" *approximations* can *differ substantially from each other* that lies behind the pedestrian phenomenon that two *different* unsound "discretization" approximations can produce two *sufficiently different* wrong results such that they are in fact *mutually incompatible.* Tirapegui et al. [6] actually set to work *categorizing* this vast class of *unsound* "discretization" approximations and their frequently mutually incompatible results—*all* of which are, in fact, nothing more than the *counterproductive fruit* of Cohen's *completely erroneous assertion* that $\lim_{|t_f-t_i|\to 0} |\mathbf{q}_f - \mathbf{q}_i| = 0$!

Formulating efficacious Hamiltonian path integrals

With the burden of Cohen's counterproductive mathematical lapse—which has been permitted to block understanding for far too many decades—lifted, we turn our attention to trying for formulate the Hamiltonian phase-space path integral in a way that is as sound, efficacious, and understandable technically as it is physically. This implies, in particular, that we *begin* with the concept of summing quantum amplitudes over all phase-space paths, *not* that we *stumble* on it in conseqence of first having written down a great many repeated integrations over configuration (or momentum) space. This *direct* approach to phase-space path summing means that our thinking will, *from the beginning*, be oriented toward the concept of *functional integration.* The *technical* challenge for such an approach will then clearly revolve around the sheer awkwardness of reconciling the set-of-measure-zero *endpoint restrictions* with the requirement that the *functional integration* must nonetheless perform its task in a mathematically sensible and understandable way. This is simply too difficult to achieve in one go, so we *begin* by writing the phase-space path integral in the standard merely *schematic* form, where those problematic endpoint restrictions are *only* expressed in words (almost as a wish or prayer!),

$$K_H(\mathbf{q}_f, t_f; \mathbf{q}_i, t_i) = \int \mathcal{D}^{(t\in[t_i,t_f])}_{[\mathbf{q}(t),\mathbf{p}(t)]} \exp(iS_H([\mathbf{q}(t),\mathbf{p}(t)]; t_f, t_i)/\hbar), \tag{2}$$

where it is *understood* that the $\mathbf{q}(t)$ paths that enter into the *functional integral* on the right hand side of Eq. (2) are *restricted* by the endpoint conditions $\mathbf{q}(t_i) = \mathbf{q}_i$ and $\mathbf{q}(t_f) = \mathbf{q}_f$. It now behooves us, of course, to discover mathematical machinery which gives full, proper *effect* to that *understanding*! We recall that a time-honored way to introduce restrictions on the variables of *ordinary* integrations is to insert into the *integrands* Dirac delta functions whose *arguments* reflect the *equations* describing those restrictions. Now the singularity character of the Dirac delta function is finely tuned to the *measure* of *ordinary* integration—certainly *not* to that of our *functional* integral, so we shall tentatively, and with trepidation, *experiment* with that recipe,

$$K_H(\mathbf{q}_f, t_f; \mathbf{q}_i, t_i) = \int \mathcal{D}^{(t\in[t_i,t_f])}_{[\mathbf{q}(t),\mathbf{p}(t)]} \, \delta^{(n)}(\mathbf{q}_f - \mathbf{q}(t_f)) \exp(iS_H([\mathbf{q}(t),\mathbf{p}(t)]; t_f, t_i)/\hbar) \, \delta^{(n)}(\mathbf{q}(t_i) - \mathbf{q}_i). \tag{3}$$

If we get the path integral $K_H(\mathbf{q}_f, t_f; \mathbf{q}_i, t_i)$ right, it will represent the quantum amplitude for the transition from the configuration eigenstate $|\mathbf{q}_i\rangle$ that was prepared at the initial time t_i to, at the subsequent time t_f, the the configuration eigenstate $|\mathbf{q}_f\rangle$—under the influence of the quantum dynamics described by the Hamiltonian H. Now when $t_f \to t_i$, the state $|\mathbf{q}_i\rangle$ will not have had *time* to evolve dynamically *at all*; so in this degenerate case, the transition amplitude will just be the *overlap amplitude* of $|\mathbf{q}_f\rangle$ with $|\mathbf{q}_i\rangle$, namely $\langle\mathbf{q}_f|\mathbf{q}_i\rangle$, and this, in turn, is simply given by the Dirac "continuum orthnormalization" of these two states, and therefore equals $\delta^{(n)}(\mathbf{q}_f - \mathbf{q}_i)$. So we shall *require* our path integral to have this correct "zero elapsed time" limit, i.e.,

$$\lim_{t_f\to t_i} K_H(\mathbf{q}_f, t_f; \mathbf{q}_i, t_i) = \delta^{(n)}(\mathbf{q}_f - \mathbf{q}_i). \tag{4}$$

Let us now see how Eq. (3) fares with this *requirement.* When we take the limit $t_f \to t_i$ on the right hand side of Eq. (3), we note from Eq. (1a) that $S_H([\mathbf{q}(t),\mathbf{p}(t)]; t_f, t_i) \to 0$. Therefore, we obtain,

$$\lim_{t_f\to t_i} K_H(\mathbf{q}_f, t_f; \mathbf{q}_i, t_i) = \int \mathcal{D}_{[\mathbf{q}(t_i),\mathbf{p}(t_i)]} \, \delta^{(n)}(\mathbf{q}_f - \mathbf{q}(t_i)) \, \delta^{(n)}(\mathbf{q}(t_i) - \mathbf{q}_i). \tag{5a}$$

It is entirely plausible to interpret $\int \mathcal{D}_{[\mathbf{q}(t_i),\mathbf{p}(t_i)]}$ as *ordinary* integration over phase space, albeit with an unknown measure normalization factor N^{-1}. Therefore we obtain,

$$\lim_{t_f\to t_i} K_H(\mathbf{q}_f, t_f; \mathbf{q}_i, t_i) = N^{-1} \int d^n\mathbf{p} \int d^n\mathbf{q} \, \delta^{(n)}(\mathbf{q}_f - \mathbf{q}) \, \delta^{(n)}(\mathbf{q} - \mathbf{q}_i) = \infty \times \delta^{(n)}(\mathbf{q}_f - \mathbf{q}_i)/N, \tag{5b}$$

which we could *force* to take the form of the result of Eq. (4) by setting N to the appropriate value of infinity!? The *Ansatz* given by Eq. (3) has *another* uncomfortable property, which is perhaps not unrelated to this normalization quandary, namely that the functional *integrand* now has sensitivity to variations of one of its argument functions on sets of measure zero (this applies to its argument function $\mathbf{q}(t)$ on the sets $\{t_i\}$, $\{t_f\}$, and $\{t_i, t_f\}$). Traditional functionals, such as the action of Eq. (1a), are what we shall call "distributed" functionals, by which we mean that they are *insensitive* to variations of their argument functions on sets of measure zero. It seems plausible that "distributed" functionals might be more readily *functionally integrated* than those which are *not* "distributed".

The above discussion would suggest that we might be much better off if, instead of *insisting* on functionally integrating *directly* over $\mathbf{q}(t)$—which is subject to those vexing endpoint restrictions that apply to a time set of measure zero—we could *instead* functionally integrate over a *different* argument function that is *closely related* to $\mathbf{q}(t)$, but for which those endpoint restrictions that apply to $\mathbf{q}(t)$ *translate* into "distributed" restrictions in terms of the *other* function, i.e., restrictions that do *not fix* the *other* function's values on *any* set of measure zero.

Although the discussion in the above paragraph may seem no more than vague, wishful musing, if we look at the integrand of the of the action functional in Eq. (1a), we see that it *also* depends on $\mathbf{q}(t)$ *through* $\dot{\mathbf{q}}(t)$. Could $\dot{\mathbf{q}}(t)$ be our wished-for function? Let us define, $\mathbf{v}(t) \stackrel{\text{def}}{=} \dot{\mathbf{q}}(t)$. Then we can enforce the endpoint restriction $\mathbf{q}(t_i) = \mathbf{q}_i$ by simply writing,

$$\mathbf{q}(t) = \mathbf{q}_i + \int_{t_i}^{t} dt'\, \mathbf{v}(t'), \tag{6a}$$

which places *no* restriction whatever on $\mathbf{v}(t)$! However, if we now *as well* require that $\mathbf{q}(t_f) = \mathbf{q}_f$, then Eq. (6a) implies that,

$$\int_{t_i}^{t_f} dt\, \mathbf{v}(t) = \mathbf{q}_f - \mathbf{q}_i, \tag{6b}$$

which is *indeed* a "distributed" restriction on $\mathbf{v}(t)$! So we have replaced the *two* endpoint restrictions on $\mathbf{q}(t)$ by the *single* restriction of Eq. (6b) on $\mathbf{v}(t)$, and that restriction is mercifully a "distributed" one. The "price" to be paid for this is to functionally integrate over $[\mathbf{v}(t), \mathbf{p}(t)]$ *instead of* over $[\mathbf{q}(t), \mathbf{p}(t)]$, and to replace all occurrences of $\mathbf{q}(t)$ according to Eq. (6a). In fact, if we *firmly enforce* Eq. (6b) (e.g., with a Dirac delta function inserted into the functional integrand) we can *just as well* replace all occurrences of $\mathbf{q}(t)$ according to,

$$\mathbf{q}(t) = \mathbf{q}_f + \int_{t_f}^{t} dt'\, \mathbf{v}(t'). \tag{6c}$$

If we go back to the tentatively proposed functional integral of Eq. (3) and proceed to replace all occurrences of $\mathbf{q}(t)$ in its integrand according to Eq. (6a), we find that the delta function factor $\delta^{(n)}(\mathbf{q}(t_i) - \mathbf{q}_i)$ has become redundant (it turns to $\delta^{(n)}(\mathbf{0})$, an infinity which we proceed to drop—conceivably it could be precisely the unwelcome infinity that appears in Eq. (5b)). With these changes in its integrand, plus the change to functionally integrating over $[\mathbf{v}(t), \mathbf{p}(t)]$ *instead of* over $[\mathbf{q}(t), \mathbf{p}(t)]$, Eq. (3) gives birth to,

$$K_H(\mathbf{q}_f, t_f; \mathbf{q}_i, t_i) =$$
$$\int \mathcal{D}^{(t\in[t_i,t_f])}_{[\mathbf{v}(t),\mathbf{p}(t)]}\, \delta^{(n)}\!\Big(\mathbf{q}_f - \mathbf{q}_i - \int_{t_i}^{t_f} dt\, \mathbf{v}(t)\Big) \exp\Big(iS_H\Big(\Big[\mathbf{q}_i + \int_{t_i}^{t} dt'\, \mathbf{v}(t'), \mathbf{p}(t)\Big]; t_f, t_i\Big)/\hbar\Big), \tag{7a}$$

where, explicitly,

$$S_H\Big(\Big[\mathbf{q}_i + \int_{t_i}^{t} dt'\, \mathbf{v}(t'), \mathbf{p}(t)\Big]; t_f, t_i\Big) = \int_{t_i}^{t_f} dt\, \Big(\mathbf{v}(t)\cdot\mathbf{p}(t) - H\Big(\mathbf{q}_i + \int_{t_i}^{t} dt'\, \mathbf{v}(t'), \mathbf{p}(t), t\Big)\Big). \tag{7b}$$

We immediately see that the integrand of the functional integral in Eq. (7a) is a *distributed functional*, which realizes a key goal. The *only* exceptional feature of the functional integral of Eq. (7a) is the occurrence of the Dirac delta function factor in its integrand, but even *this* factor is *itself* a distributed functional. We now show that this delta function factor plays the major role in producing the *required* "zero elapsed time limit" given in Eq. (4) for our path integral of Eq. (7a).

It is clear from Eq. (7b) that the action functional in the integrand of the functional integral given in Eq. (7a) vanishes in the limit that $t_f \to t_i$. Therefore we immeditately obtain from Eq. (7a) that,

$$\lim_{t_f\to t_i} K_H(\mathbf{q}_f, t_f; \mathbf{q}_i, t_i) = \delta^{(n)}(\mathbf{q}_f - \mathbf{q}_i) \lim_{t_f\to t_i} \int \mathcal{D}^{(t\in[t_i,t_f])}_{[\mathbf{v}(t),\mathbf{p}(t)]} \tag{8}$$

We shall show below that that for $t_f \neq t_i$, $\int \mathcal{D}^{(t\in[t_i,t_f])}_{[\mathbf{v}(t),\mathbf{p}(t)]}$ with integrand unity is *independent* of t_i and t_f. Assuming it is also finite, since we are free to choose an over-all *normalization factor* for the functional integration *measure* associated with $\mathcal{D}^{(t\in[t_i,t_f])}_{[\mathbf{v}(t),\mathbf{p}(t)]}$, we make this measure normalization factor choice be such that $\int \mathcal{D}^{(t\in[t_i,t_f])}_{[\mathbf{v}(t),\mathbf{p}(t)]}$ with integrand unity assumes the *value* unity. Then Eq. (8) above immediately implies Eq. (4). That $\int \mathcal{D}^{(t\in[t_i,t_f])}_{[\mathbf{v}(t),\mathbf{p}(t)]}$ may be taken as finite, and that it is independent of t_i and t_f for $t_f \neq t_i$, obviously will not be clear until a more detailed understanding of the nature of functional path integration is attained, a matter to which we now turn.

Normalized multiple integration over the orthogonal components of paths

To integrate over the space of N-dimensional vectors $\mathbf{X}$, we simply perform a (possibly) normalized multiple integration over any of its complete sets of N mutually orthogonal components,

$$\int d^N\mathbf{X} = M_N \int dX_1 \int dX_2 \ldots \int dX_N.$$

Here $\mathbf{X} = \Sigma_{k=1}^{N} \mathbf{b}_k X_k$, where the complete set of N *basis vectors* $\mathbf{b}_k$ are *mutually orthogonal*, i.e., they satisfy $\mathbf{b}_k \cdot \mathbf{b}_{k'} = 0$ if $k \neq k'$, and therefore $X_k = \mathbf{b}_k \cdot \mathbf{X}/\mathbf{b}_k \cdot \mathbf{b}_k$. Now our paths $(\mathbf{v}(t), \mathbf{p}(t))$, being *functions* defined on the interval $t \in [t_i, t_f]$, *also* have complete sets of mutually orthogonal components, namely their components with respect to complete sets of mutually orthogonal discrete real-valued *basis functions* $B_k(t)$, $k = 0, 1, 2, \ldots$, on the interval $[t_i, t_f]$ that satisfy,

$$\int_{t_i}^{t_f} dt\, B_k(t) B_{k'}(t) = 0 \text{ if } k \neq k'.$$

We can expand any of our paths in terms such basis functions,

$$(\mathbf{v}(t), \mathbf{p}(t)) = \Sigma_{k=0}^{\infty} B_k(t)(\mathbf{v}_k, \mathbf{p}_k),$$

where that path's orthogonal functional components $(\mathbf{v}_k, \mathbf{p}_k)$, $k = 0, 1, 2, \ldots$, with respect to this basis set are given by,

$$(\mathbf{v}_k, \mathbf{p}_k) = \int_{t_i}^{t_f} dt\, B_k(t)(\mathbf{v}(t), \mathbf{p}(t)) / \int_{t_i}^{t_f} dt\ (B_k(t))^2.$$

Given such orthogonal functional components of our paths, we are now in a position to *as well* write functional path integration in terms of multiple integration over them *and* a generalized normalization methodology,

$$\int \mathcal{D}^{(t \in [t_i, t_f])}_{[\mathbf{v}(t), \mathbf{p}(t)]} =$$

$$\lim_{K \to \infty} \lim_{(V_0, P_0, V_1, P_1, \ldots, V_K, P_K \to \infty)} M_K(V_0, P_0, V_1, P_1, \ldots, V_K, P_K) \int_{\{|\mathbf{v}_0| \leq V_0\}} d^n\mathbf{v}_0 \int_{\{|\mathbf{p}_0| \leq P_0\}} d^n\mathbf{p}_0 \times$$

$$\int_{\{|\mathbf{v}_1| \leq V_1\}} d^n\mathbf{v}_1 \int_{\{|\mathbf{p}_1| \leq P_1\}} d^n\mathbf{p}_1 \cdots \int_{\{|\mathbf{v}_K| \leq V_K\}} d^n\mathbf{v}_K \int_{\{|\mathbf{p}_K| \leq P_K\}} d^n\mathbf{p}_K,$$

where we have built enough flexibility into its normalization to be sure that $\int \mathcal{D}^{(t \in [t_i, t_f])}_{[\mathbf{v}(t), \mathbf{p}(t)]}$ with integrand unity can be be made finite. It is as well clear that this object has no dependence on t_i or t_f for $t_f \neq t_i$.

A commonly invoked *slight variation* of the above *complete* discrete basis set approach involves a *sequence* of *incomplete* discrete *approximation* basis sets to the intuitively appealing complete *continuum* basis set of delta functions in time, $B_{t_c}(t) \stackrel{\text{def}}{=} \delta(t - t_c)$, where $t_c \in [t_i, t_f]$. Given a partition of the time interval $[t_i, t_f]$ into $K + 1$ disjoint time subintervals, where $K = 0, 1, 2, \ldots$, we can *approximate* $B_{t_c}(t)$ by $B^K_{t_c}(t)$, which, for t in any of the $K + 1$ disjoint time subintervals of $[t_i, t_f]$ equals the *inverse* of the *duration* of that time subinterval when t_c is *also* in that subinterval, but equals zero otherwise. Obviously there are *only* $K + 1$ *distinct* such approximating functions $B^K_{t_c}(t)$, so we may define $B^K_k(t) \stackrel{\text{def}}{=} B^K_{t_c}(t)$, where t_c is *any* time element of time subinterval number k, $k = 0, 1, \ldots, K$. It is clear that $B^K_k(t)$ is *orthogonal* to $B^K_{k'}(t)$ for $k \neq k'$. One develops in this way a *sequence* in K of *incomplete* orthogonal basis sets that each have *only* $K + 1$ members. When $K \to \infty$, the intuitively appealing *continuum* basis set of delta functions $B_{t_c}(t) = \delta(t - t_c)$, which is, of course, *complete*, will be recovered *provided* that care is taken to ensure that the *durations* of *all* of the *individual time subintervals* of partition number K tend toward *zero* in that limit.

The momentum path integral

Before we go on to derive the other important properties of the configuration path integral $K_H(\mathbf{q}_f, t_f; \mathbf{q}_i, t_i)$, which is defined by Eq. (7a) as a functional integral with a *distributed* integrand, we wish to take note of the fact that in *exactly* the same way as $K_H(\mathbf{q}_f, t_f; \mathbf{q}_i, t_i)$ was developed from the action functional of Eq. (1a) together with the *configuration* path endpoint restrictions $\mathbf{q}(t_i) = \mathbf{q}_i$ and $\mathbf{q}(t_f) = \mathbf{q}_f$ that are *classically* appropriate to that action, so too can we develop an entirely analogous definition of the *momentum* path integral $K'_H(\mathbf{p}_f, t_f; \mathbf{p}_i, t_i)$ from the action functional of Eq. (1b) together with *momentum* path endpoint restrictions $\mathbf{p}(t_i) = \mathbf{p}_i$ and $\mathbf{p}(t_f) = \mathbf{p}_f$, which are *classically* appropriate to that *latter* action. By following precisely analogous steps for the development of a tractable, efficacious definition of the momentum path integral $K'_H(\mathbf{p}_f, t_f; \mathbf{p}_i, t_i)$ as those we have followed in arriving at the definition of $K_H(\mathbf{q}_f, t_f; \mathbf{q}_i, t_i)$ given by Eq. (7a), we obtain the following analogous definition of the momentum path integral $K'_H(\mathbf{p}_f, t_f; \mathbf{p}_i, t_i)$ as a functional integral with a *distributed* integrand,

$$K'_H(\mathbf{p}_f, t_f; \mathbf{p}_i, t_i) =$$

$$\int \mathcal{D}^{(t\in[t_i,t_f])}_{[\mathbf{q}(t),\mathbf{f}(t)]}\, \delta^{(n)}\!\Big(\mathbf{p}_f - \mathbf{p}_i - \textstyle\int_{t_i}^{t_f} dt\, \mathbf{f}(t)\Big) \exp\Big(iS'_H\Big(\Big[\mathbf{q}(t), \mathbf{p}_i + \textstyle\int_{t_i}^{t} dt'\, \mathbf{f}(t')\Big]; t_f, t_i\Big)/\hbar\Big), \tag{9a}$$

where, explicitly,

$$S'_H\Big(\Big[\mathbf{q}(t), \mathbf{p}_i + \textstyle\int_{t_i}^{t} dt'\, \mathbf{f}(t')\Big]; t_f, t_i\Big) = \textstyle\int_{t_i}^{t_f} dt\, \Big(-\mathbf{q}(t)\cdot\mathbf{f}(t) - H\Big(\mathbf{q}(t), \mathbf{p}_i + \textstyle\int_{t_i}^{t} dt'\, \mathbf{f}(t'), t\Big)\Big). \tag{9b}$$

For *every one* of the properties of $K_H(\mathbf{q}_f, t_f; \mathbf{q}_i, t_i)$ that we shall derive below, there corresponds closely analogous property of $K'_H(\mathbf{p}_f, t_f; \mathbf{p}_i, t_i)$, which can be derived by tightly analogous steps. We shall therefore *never* go through these derivations for $K'_H(\mathbf{p}_f, t_f; \mathbf{p}_i, t_i)$ in explicit fashion (the reader is invited to do this), but shall write down some of the results. As the reader might possibly already have anticipated, it transpires at the end of quite a long calculational road that $K'_H(\mathbf{p}_f, t_f; \mathbf{p}_i, t_i)$ is the standard quantum mechanics unitary Fourier transformation of $K_H(\mathbf{q}_f, t_f; \mathbf{q}_i, t_i)$ from configuration to momentum representation—underlying *both* of these path integrals is the *same* abstract *operator* of quantum mechanics, namely that of time evolution. But we digress—there is a *list* of intermediate results that needs to be demonstrated for these path integrals before their place in quantum mechanics can be made clear.

Space-time reversal and decomposition properties of the path integral

Our first intermediate result for the path integral of Eq. (7a) is that,

$$K_H(\mathbf{q}_i, t_i; \mathbf{q}_f, t_f) = (K_H(\mathbf{q}_f, t_f; \mathbf{q}_i, t_i))^*, \tag{10a}$$

which is readily shown from Eqs. (7) if one bears in mind the relation that is implied when Eqs. (6a) and (6c) are taken together (this relation is readily *proved* from Eq. (6b), which is *enforced* in Eq. (7a) by the delta function factor). For more straightforward translation into the language of quantum mechanics operators, Eq. (10a) is normally *rephrased* to become a relation between time reversal and *Hermitian* conjugation,

$$K_H(\mathbf{q}_f, t_i; \mathbf{q}_i, t_f) = (K_H(\mathbf{q}_i, t_f; \mathbf{q}_f, t_i))^*. \tag{10b}$$

Our next intermediate result is the *decomposition property* of the path integral $K_H(\mathbf{q}_f, t_f; \mathbf{q}_i, t_i)$, i.e., that for any $t_c \in [t_i, t_f]$,

$$K_H(\mathbf{q}_f, t_f; \mathbf{q}_i, t_i) = \textstyle\int d^n\mathbf{q}\, K_H(\mathbf{q}_f, t_f; \mathbf{q}, t_c) K_H(\mathbf{q}, t_c; \mathbf{q}_i, t_i). \tag{11}$$

Note that in the two special cases $t_c = t_i$ and $t_c = t_f$, Eq. (11) follows immediately from Eq. (4), so we are free to assume that $t_c \neq t_i$ and $t_c \neq t_f$ in the remainder of the demonstration of Eq. (11) which is set out below. That demonstration is notationally unwieldy but otherwise rather straightforward, being mainly a consequence of the properties of the delta-function constraint that appears in the functional integrand for $K_H(\mathbf{q}_f, t_f; \mathbf{q}_i, t_i)$ on the right hand side of Eq. (7a). We develop the demonstration by focusing on that *subset* of the paths $(\mathbf{v}(t), \mathbf{p}(t))$ entering into the right hand side of Eq. (7a) for which a specified configuration value $\mathbf{q}$ is *attained* at a specified intermediate time t_c, i.e., we for now *restrict* our attention to those $(\mathbf{v}(t), \mathbf{p}(t))$ that enter into the right hand side of Eq. (7a) which, *in addition*, satisfy,

$$\mathbf{q}(t_c) \stackrel{\text{def}}{=} \mathbf{q}_i + \textstyle\int_{t_i}^{t_c} dt\, \mathbf{v}(t) = \mathbf{q}, \tag{12}$$

where $t_c \in [t_i, t_f]$, $t_c \neq t_i$ and $t_c \neq t_f$. For any such path $(\mathbf{v}(t), \mathbf{p}(t))$, it is a completely straightforward exercise in the elementary decomposition of integrals to demonstrate from Eq. (7b) that,

$$S_H\left(\left[\mathbf{q}_i + \int_{t_i}^{t} dt'\,\mathbf{v}(t'), \mathbf{p}(t)\right]; t_f, t_i\right) =$$

$$S_H\left(\left[\mathbf{q}_i + \int_{t_i}^{t} dt'\,\mathbf{v}(t'), \mathbf{p}(t)\right]; t_c, t_i\right) + S_H\left(\left[\mathbf{q} + \int_{t_c}^{t} dt'\,\mathbf{v}(t'), \mathbf{p}(t)\right]; t_f, t_c\right). \tag{13}$$

We can now *enforce* the restriction of Eq. (12) on the paths that enter into the right hand side of Eq. (7a) by inserting the delta function factor $\delta^{(n)}\Big(\mathbf{q} - \mathbf{q}_i - \int_{t_i}^{t_c} dt\,\mathbf{v}(t)\Big)$ into the path integrand on the right hand side of Eq. (7a). However, since,

$$\int d^n\mathbf{q}\,\delta^{(n)}\Big(\mathbf{q} - \mathbf{q}_i - \int_{t_i}^{t_c} dt\,\mathbf{v}(t)\Big) = 1,$$

we readily *recover* $K_H(\mathbf{q}_f, t_f; \mathbf{q}_i, t_i)$ from its thus *path-restricted* version by simply *integrating* the latter over the entire range of $\mathbf{q}$. In this way Eq. (7a) is reexpressed as,

$$K_H(\mathbf{q}_f, t_f; \mathbf{q}_i, t_i) =$$

$$\int d^n\mathbf{q} \int \mathcal{D}^{(t\in[t_i,t_f])}_{[\mathbf{v}(t),\mathbf{p}(t)]}\,\delta^{(n)}\Big(\mathbf{q} - \mathbf{q}_i - \int_{t_i}^{t_c} dt\,\mathbf{v}(t)\Big)\ \delta^{(n)}\Big(\mathbf{q}_f - \mathbf{q}_i - \int_{t_i}^{t_f} dt\,\mathbf{v}(t)\Big)\, e^{iS_H([\mathbf{q}_i + \int_{t_i}^{t} dt'\,\mathbf{v}(t'),\,\mathbf{p}(t)]; t_f, t_i)/\hbar}. \tag{14a}$$

Now by inserting the action decomposition of Eq. (13) into Eq. (14a), and by making explicit the effect of the restriction imposed by the first delta function of Eq. (14a) on its second delta function, we obtain,

$$K_H(\mathbf{q}_f, t_f; \mathbf{q}_i, t_i) =$$

$$\int d^n\mathbf{q} \int \mathcal{D}^{(t\in[t_i,t_f])}_{[\mathbf{v}(t),\mathbf{p}(t)]}\,\delta^{(n)}\Big(\mathbf{q} - \mathbf{q}_i - \int_{t_i}^{t_c} dt\,\mathbf{v}(t)\Big)\, e^{iS_H([\mathbf{q}_i + \int_{t_i}^{t} dt'\,\mathbf{v}(t'),\,\mathbf{p}(t)]; t_c, t_i)/\hbar} \times$$

$$\delta^{(n)}\Big(\mathbf{q}_f - \mathbf{q} - \int_{t_c}^{t_f} dt\,\mathbf{v}(t)\Big)\, e^{iS_H([\mathbf{q} + \int_{t_c}^{t} dt'\,\mathbf{v}(t'),\,\mathbf{p}(t)]; t_f, t_c)/\hbar}, \tag{14b}$$

which obviously can be reexpressed as,

$$K_H(\mathbf{q}_f, t_f; \mathbf{q}_i, t_i) =$$

$$\int d^n\mathbf{q} \int \mathcal{D}^{(t\in[t_i,t_c])}_{[\mathbf{v}(t),\mathbf{p}(t)]}\,\delta^{(n)}\Big(\mathbf{q} - \mathbf{q}_i - \int_{t_i}^{t_c} dt\,\mathbf{v}(t)\Big)\, e^{iS_H([\mathbf{q}_i + \int_{t_i}^{t} dt'\,\mathbf{v}(t'),\,\mathbf{p}(t)]; t_c, t_i)/\hbar} \times$$

$$\int \mathcal{D}^{(t\in[t_c,t_f])}_{[\mathbf{v}(t),\mathbf{p}(t)]}\,\delta^{(n)}\Big(\mathbf{q}_f - \mathbf{q} - \int_{t_c}^{t_f} dt\,\mathbf{v}(t)\Big)\, e^{iS_H([\mathbf{q} + \int_{t_c}^{t} dt'\,\mathbf{v}(t'),\,\mathbf{p}(t)]; t_f, t_c)/\hbar}, \tag{14c}$$

which, in turn, we readily recognize as,

$$K_H(\mathbf{q}_f, t_f; \mathbf{q}_i, t_i) = \int d^n\mathbf{q}\,K_H(\mathbf{q}, t_c; \mathbf{q}_i, t_i) K_H(\mathbf{q}_f, t_f; \mathbf{q}, t_c). \tag{14d}$$

Eq. (14d), aside from the transposition of the two factors in the integrand, is of course the same as Eq. (11), which we have now demonstrated to hold for all $t_c \in [t_i, t_f]$. It will ultimately turn out that this path integral decomposition property in fact holds *with no restriction whatsoever* on t_c. This, however, cannot be established without our *next* intermediate result, which is the *actual evaluation* of the path integral $K_H(\mathbf{q}_f, t_f; \mathbf{q}_i, t_i)$ through *first order* in its elapsed time $(\delta t)_{fi} \stackrel{\text{def}}{=} (t_f - t_i)$.

Path integral evaluation through first order in its elapsed time

From Eq. (4) it is seen that we, of course, already know the value of $K_H(\mathbf{q}_f, t_f; \mathbf{q}_i, t_i)$ through *zeroth order* in the elapsed time $(\delta t)_{fi}$. In order to work it out through first order in $(\delta t)_{fi}$, we will need to expand its integrand functional $I_H([\mathbf{v}(t), \mathbf{p}(t)]; \mathbf{q}_f, t_f; \mathbf{q}_i, t_i)$ out in orders of $(\delta t)_{fi}$. To be able to carry that out systematically, it is essential that the orthogonal basis functions $B_k(t)$ with which we expand all the paths $(\mathbf{v}(t), \mathbf{p}(t))$ have the property that for $t \in [t_i, t_f]$, $B_k(t)$ be of order $O(((\delta t)_{fi})^k)$. One such basis set is obtained by taking $B_0(t) = 1$

and,

$$B_k(t) = (t-(t_f+t_i)/2)^k/k! + \Sigma_{j=1}^k c_k^{(j)}(t-(t_f+t_i)/2)^{k-j}((t_f-t_i)/2)^j \text{ for } k = 1, 2, \ldots,$$

where the k dimensionless $c_k^{(1)}, \ldots, c_k^{(k)}$ are *recursively determined* by the k *orthogonality* requirements that,

$$\int_{t_i}^{t_f} dt\, B_k(t)B'_k(t) = 0 \text{ for } k' = 0, 1, \ldots, k-1.$$

With *dimensionless* $c_k^{(j)}$, $j = 1, 2, \ldots, k$, it is clear that $B_k(t)$ is of order $O(((\delta t)_{fi})^k)$ for $t \in [t_i, t_f]$, as we have required, and the above scheme for $B_k(t)$ *does indeed* produce dimensionless $c_k^{(j)}$ because,

$$\int_{t_i}^{t_f} dt\,(t-(t_f+t_i)/2)^N = ((t_f-t_i)/2)^{N+1}(1+(-1)^N)/(N+1).$$

We note that when $t_f \to t_i$, i.e., the "degenerate interval limit", $B_k(t) \to (t-t_i)^k/k!$, which are the well-known polynomials for the orders of the *Taylor expansion* of functions about the *single point* t_i. Since $B_0(t) = 1$, we also note that,

$$(\mathbf{v}_0, \mathbf{p}_0) = \int_{t_i}^{t_f} dt\,(\mathbf{v}(t), \mathbf{p}(t))/(t_f-t_i) = (\bar{\mathbf{v}}, \bar{\mathbf{p}}),$$

which is the path's *mean value* over the interval $[t_i, t_f]$, and which we henceforth conveniently *abbreviate* as simply $(\mathbf{v}, \mathbf{p})$. It is also convenient to note that the first three $B_k(t)$ are,

$$B_0(t) = 1,\;\; B_1(t) = (t-(t_f+t_i)/2),\;\; B_2(t) = (t-(t_f+t_i)/2)^2/2 - ((t_f-t_i)/2)^2/6,$$

which illustrates the key fact that $B_k(t)$ is of order $O(((\delta t)_{fi})^k)$ for $t \in [t_i, t_f]$. In view of their properties, we can call the $B_k(t)$ scaled, translated Legendre polynomials with Taylor-like normalizations.

Path integrand expansion through first order in its elapsed time

We turn now to the *integrand* functional $I_H([\mathbf{v}(t), \mathbf{p}(t)]; \mathbf{q}_f, t_f; \mathbf{q}_i, t_i)$ of the path integral $K_H(\mathbf{q}_f, t_f; \mathbf{q}_i, t_i)$, which, according to Eqs. (7a) and (7b), is given by,

$$I_H([\mathbf{v}(t), \mathbf{p}(t)]; \mathbf{q}_f, t_f; \mathbf{q}_i, t_i) = \delta^{(n)}\Big(\mathbf{q}_f - \mathbf{q}_i - \int_{t_i}^{t_f} dt\,\mathbf{v}(t)\Big)\, e^{i\int_{t_i}^{t_f} dt\,(\mathbf{v}(t)\cdot\mathbf{p}(t) - H(\mathbf{q}_i + \int_{t_i}^{t} dt'\,\mathbf{v}(t'), \mathbf{p}(t), t))/\hbar},$$

and which we wish to evaluate through first order in $(\delta t)_{fi} \stackrel{\text{def}}{=} (t_f - t_i)$. Since,

$$\int_{t_i}^{t_f} dt\,\mathbf{v}(t) = (t_f - t_i)\bar{\mathbf{v}} = (\delta t)_{fi}\mathbf{v},$$

we have that,

$$\delta^{(n)}\Big(\mathbf{q}_f - \mathbf{q}_i - \int_{t_i}^{t_f} dt\,\mathbf{v}(t)\Big) = \delta^{(n)}(\mathbf{q}_f - \mathbf{q}_i - (\delta t)_{fi}\mathbf{v}),$$

which implies that any occurrence of $\mathbf{v}$ in $I_H([\mathbf{v}(t), \mathbf{p}(t)]; \mathbf{q}_f, t_f; \mathbf{q}_i, t_i)$ is to be replaced by $(\mathbf{q}_f - \mathbf{q}_i)/(\delta t)_{fi}$. As one example, given our orthogonal basis expansion,

$$(\mathbf{v}(t), \mathbf{p}(t)) = (\mathbf{v}, \mathbf{p}) + \Sigma_{k=1}^{\infty} B_k(t)(\mathbf{v}_k, \mathbf{p}_k),$$

for $t \in [t_i, t_f]$, we have that,

$$\mathbf{v}(t) = \mathbf{v} + (t-(t_f+t_i)/2)\mathbf{v}_1 + O((\delta t)_{fi}^2) = \mathbf{v} + (t - t_i - (\delta t)_{fi}/2)\mathbf{v}_1 + O((\delta t)_{fi}^2),$$

but we actually must write,

$$\mathbf{v}(t) = (\mathbf{q}_f - \mathbf{q}_i)/(\delta t)_{fi} + (t - t_i - (\delta t)_{fi}/2)\mathbf{v}_1 + O((\delta t)_{fi}^2) = (\mathbf{q}_f - \mathbf{q}_i)/(\delta t)_{fi} + O((\delta t)_{fi}),$$

for occurrences of $\mathbf{v}(t)$ in the functional integrand $I_H([\mathbf{v}(t), \mathbf{p}(t)]; \mathbf{q}_f, t_f; \mathbf{q}_i, t_i)$. However, for such occurrences of $\mathbf{p}(t)$, we may simply write,

$$\mathbf{p}(t) = \mathbf{p} + (t - t_i - (\delta t)_{fi}/2)\mathbf{p}_1 + O((\delta t)_{fi}^2).$$

In particular, we have that,

$$\mathbf{v}(t)\cdot\mathbf{p}(t) = (\mathbf{q}_f - \mathbf{q}_i)\cdot\mathbf{p}/(\delta t)_{fi} + (t - t_i - (\delta t)_{fi}/2)(\mathbf{q}_f - \mathbf{q}_i)\cdot\mathbf{p}_1/(\delta t)_{fi} + O((\delta t)_{fi}).$$

Now,

$$\int_{t_i}^{t_f} dt\, \mathbf{v}(t)\cdot\mathbf{p}(t) = \int_{t_i}^{t_i+(\delta t)_{fi}} dt\, \mathbf{v}(t)\cdot\mathbf{p}(t),$$

and,

$$\int_{t_i}^{t_i+(\delta t)_{fi}} dt\, (t - t_i - (\delta t)_{fi}/2) = 0.$$

Therefore,

$$\int_{t_i}^{t_f} dt\, \mathbf{v}(t)\cdot\mathbf{p}(t) = (\mathbf{q}_f - \mathbf{q}_i)\cdot\mathbf{p} + O((\delta t)_{fi}^2).$$

We also have that,

$$\mathbf{q}_i + \int_{t_i}^{t} dt'\, \mathbf{v}(t') = \mathbf{q}_i + (\mathbf{q}_f - \mathbf{q}_i)(t - t_i)/(\delta t)_{fi} + O((\delta t)_{fi}^2).$$

Therefore we obtain that,

$$\int_{t_i}^{t_f} dt\, (-H(\mathbf{q}_i + \int_{t_i}^{t} dt'\, \mathbf{v}(t'), \mathbf{p}(t), t)) =$$

$$\Big(-\int_{t_i}^{t_i+(\delta t)_{fi}} dt\, H(\mathbf{q}_i + (\mathbf{q}_f - \mathbf{q}_i)(t - t_i)/(\delta t)_{fi}, \mathbf{p}, t_i)\Big) + O((\delta t)_{fi}^2) =$$

$$\Big(-(\delta t)_{fi}\int_0^1 d\lambda\, H(\mathbf{q}_i + \lambda(\mathbf{q}_f - \mathbf{q}_i), \mathbf{p}, t_i)\Big) + O((\delta t)_{fi}^2),$$

where we have changed the variable of the integration from t to $\lambda \stackrel{\text{def}}{=} (t - t_i)/(\delta t)_{fi}$.

Assembling the above results yields,

$$I_H([\mathbf{v}(t), \mathbf{p}(t)]; \mathbf{q}_f, t_i + (\delta t)_{fi}; \mathbf{q}_i, t_i) =$$

$$\delta^{(n)}(\mathbf{q}_f - \mathbf{q}_i - (\delta t)_{fi}\mathbf{v})\; e^{i(\mathbf{q}_f-\mathbf{q}_i)\cdot\mathbf{p}/\hbar - i((\delta t)_{fi}/\hbar)\int_0^1 d\lambda\, H(\mathbf{q}_i+\lambda(\mathbf{q}_f-\mathbf{q}_i),\, \mathbf{p},\, t_i)}(1 + O((\delta t)_{fi}^2)). \tag{15a}$$

From this it is apparent that $I_H([\mathbf{v}(t), \mathbf{p}(t)]; \mathbf{q}_f, t_i + (\delta t)_{fi}; \mathbf{q}_i, t_i)$ through first order in $(\delta t)_{fi}$ is *independent* of $\mathbf{v}_k$ and $\mathbf{p}_k$ for all $k = 1, 2, \ldots$. In view of our previously given normalized multiple integration expression for $\int \mathcal{D}^{(t\in[t_i,t_f])}_{[\mathbf{v}(t),\mathbf{p}(t)]}$, we therefore can write,

$$K_H(\mathbf{q}_f, t_i + (\delta t)_{fi}; \mathbf{q}_i, t_i) =$$

$$M \int d^n\mathbf{v} \int d^n\mathbf{p}\;\; \delta^{(n)}(\mathbf{q}_f - \mathbf{q}_i - (\delta t)_{fi}\mathbf{v})\; e^{i(\mathbf{q}_f-\mathbf{q}_i)\cdot\mathbf{p}/\hbar - i((\delta t)_{fi}/\hbar)\int_0^1 d\lambda\, H(\mathbf{q}_i+\lambda(\mathbf{q}_f-\mathbf{q}_i),\, \mathbf{p},\, t_i)}(1 + O((\delta t)_{fi}^2)), \tag{15b}$$

where,

$$M = \lim_{K\to\infty} \lim_{(V_1,P_1,\ldots,V_K,P_K\to\infty)} M_K(V_1, P_1, \ldots, V_K, P_K)\times$$

$$\int_{\{|\mathbf{v}_1|\le V_1\}} d^n\mathbf{v}_1 \int_{\{|\mathbf{p}_1|\le P_1\}} d^n\mathbf{p}_1 \cdots \int_{\{|\mathbf{v}_K|\le V_K\}} d^n\mathbf{v}_K \int_{\{|\mathbf{p}_K|\le P_K\}} d^n\mathbf{p}_K.$$

The delta function factor $\delta^{(n)}(\mathbf{q}_f - \mathbf{q}_i - (\delta t)_{fi}\mathbf{v})$ in Eq. (15b) allows its $\int d^n\mathbf{v}$ integration to be immediately carried out, yielding an overall factor of $|(\delta t)_{fi}|^{-n}$. We therefore let $M = N|(\delta t)_{fi}|^n$, and thereby obtain,

$$K_H(\mathbf{q}_f, t_i + (\delta t)_{fi}; \mathbf{q}_i, t_i) = N\int d^n\mathbf{p}\;\; e^{i(\mathbf{q}_f-\mathbf{q}_i)\cdot\mathbf{p}/\hbar - i((\delta t)_{fi}/\hbar)\int_0^1 d\lambda\, H(\mathbf{q}_i+\lambda(\mathbf{q}_f-\mathbf{q}_i),\, \mathbf{p},\, t_i)}(1 + O((\delta t)_{fi}^2)), \tag{15c}$$

from which we readily calculate that,

$$\lim_{(\delta t)_{fi}\to 0} K_H(\mathbf{q}_f, t_i + (\delta t)_{fi}; \mathbf{q}_i, t_i) = N(2\pi\hbar)^n\delta^{(n)}(\mathbf{q}_f - \mathbf{q}_i).$$

The requirement of Eq. (4) therefore determines that $N = (2\pi\hbar)^{-n}$. With that we obtain from Eq. (15c) the unique result for the path integral $K_H(\mathbf{q}_f, t_i + (\delta t)_{fi}; \mathbf{q}_i, t_i)$ through first order in $(\delta t)_{fi}$,

$$K_H(\mathbf{q}_f, t_i + (\delta t)_{fi}; \mathbf{q}_i, t_i) = \delta^{(n)}(\mathbf{q}_f - \mathbf{q}_i) - i((\delta t)_{fi}/\hbar)Q_H(t_i; \mathbf{q}_f; \mathbf{q}_i) + O((\delta t)_{fi}^2), \tag{16a}$$

where,

$$Q_H(t_i;\mathbf{q}_f;\mathbf{q}_i) \stackrel{\text{def}}{=} \int_0^1 d\lambda\,(2\pi\hbar)^{-n}\int d^n\mathbf{p}\,H(\mathbf{q}_i+\lambda(\mathbf{q}_f-\mathbf{q}_i),\mathbf{p},t_i)e^{i(\mathbf{q}_f-\mathbf{q}_i)\cdot\mathbf{p}/\hbar}. \tag{16b}$$

It is easily demonstrated that $Q_H(t_i;\mathbf{q}_f;\mathbf{q}_i)$ is *Hermitian*, i.e. that,

$$Q_H(t_i;\mathbf{q}_f;\mathbf{q}_i) = (Q_H(t_i;\mathbf{q}_i;\mathbf{q}_f))^*. \tag{16c}$$

The quantized Hamiltonian operator

At this point we wish to mention the results for the momentum path integral $K'_H(\mathbf{p}_f,t_f;\mathbf{p}_i,t_i)$ which parallel those that we have demonstrated for the configuration path integral $K_H(\mathbf{q}_f,t_f;\mathbf{q}_i,t_i)$. In the zero elapsed time limit, the former of course tends toward $\delta^{(n)}(\mathbf{p}_f-\mathbf{p}_i)$. The former also has its time reversal equal to its Hermitian conjugate, and as well manifests the decomposition property. Finally, to first order in the elapsed time, it satisfies relations that are highly analogous to those of Eqs. (16), namely,

$$K'_H(\mathbf{p}_f,t_i+(\delta t)_{fi};\mathbf{p}_i,t_i) = \delta^{(n)}(\mathbf{p}_f-\mathbf{p}_i) - i((\delta t)_{fi}/\hbar)Q'_H(t_i;\mathbf{p}_f;\mathbf{p}_i) + O((\delta t)^2_{fi}), \tag{17a}$$

where,

$$Q'_H(t_i;\mathbf{p}_f;\mathbf{p}_i) \stackrel{\text{def}}{=} \int_0^1 d\lambda\,(2\pi\hbar)^{-n}\int d^n\mathbf{q}\,H(\mathbf{q},\mathbf{p}_i+\lambda(\mathbf{p}_f-\mathbf{p}_i),t_i)e^{-i(\mathbf{p}_f-\mathbf{p}_i)\cdot\mathbf{q}/\hbar}. \tag{17b}$$

It is easily demonstrated that $Q'_H(t_i;\mathbf{p}_f;\mathbf{p}_i)$ is *Hermitian*, i.e. that,

$$Q'_H(t_i;\mathbf{p}_f;\mathbf{p}_i) = (Q'_H(t_i;\mathbf{p}_i;\mathbf{p}_f))^*. \tag{17c}$$

A key relationship between $Q'_H(t_i;\mathbf{p}_f;\mathbf{p}_i)$ and $Q_H(t_i;\mathbf{q}_f;\mathbf{q}_i)$ is that,

$$\int d^n\mathbf{p}_f d^n\mathbf{p}_i\,\langle\mathbf{q}_f|\mathbf{p}_f\rangle Q'_H(t_i;\mathbf{p}_f;\mathbf{p}_i)\langle\mathbf{p}_i|\mathbf{q}_i\rangle = Q_H(t_i;\mathbf{q}_f;\mathbf{q}_i), \tag{18}$$

where we have used the standard quantum mechanics notation for the overlap amplitude between a configuration state and a momentum state, i.e., $\langle\mathbf{q}|\mathbf{p}\rangle = e^{i\mathbf{p}\cdot\mathbf{q}/\hbar}/(2\pi\hbar)^{n/2}$ and $\langle\mathbf{p}|\mathbf{q}\rangle = (\langle\mathbf{q}|\mathbf{p}\rangle)^*$. To carry out the verification of Eq. (18), it is useful to make the $d\lambda$-integration that arises from $Q'_H(t_i;\mathbf{p}_f;\mathbf{p}_i)$ via Eq. (17b) the *outermost* integration, and then change integration variables from the $(\mathbf{p}_f,\mathbf{p}_i)$ pair to the $\mathbf{p}=\mathbf{p}_i+\lambda(\mathbf{p}_f-\mathbf{p}_i)$ and $\mathbf{p}_-=(\mathbf{p}_f-\mathbf{p}_i)$ pair. This variable transformation has unit Jacobian, and the $d^n\mathbf{p}_-$-integration will give rise to a delta function which, in turn, permits the $d^n\mathbf{q}$-integration that arises from $Q'_H(t_i;\mathbf{p}_f;\mathbf{p}_i)$ via Eq. (17b) to be carried out. The upshot is to leave only the $d^n\mathbf{p}$-integration and the $d\lambda$-integration, both of which indeed occur in $Q_H(t_i;\mathbf{q}_f;\mathbf{q}_i)$, which is itself, of course, the result being sought. With this outline of the procedure, we leave the remaining straightforward details of verifying Eq. (18) to the reader.

Eq. (18) demonstrates that $Q_H(t_i;\mathbf{q}_f;\mathbf{q}_i)$ and $Q'_H(t_i;\mathbf{p}_f;\mathbf{p}_i)$ are, respectively, the configuration and momentum representations of the *very same quantum mechanical operator*, which we shall now denote as $\widehat{H}(t_i)$. Therefore,

$$Q_H(t_i;\mathbf{q}_f;\mathbf{q}_i) = \langle\mathbf{q}_f|\widehat{H}(t_i)|\mathbf{q}_i\rangle, \tag{19a}$$

and,

$$Q'_H(t_i;\mathbf{p}_f;\mathbf{p}_i) = \langle\mathbf{p}_f|\widehat{H}(t_i)|\mathbf{p}_i\rangle. \tag{19b}$$

The *unique* operator $\widehat{H}(t_i)$ was first obtained from from the Hamiltonian phase-space path integral by Kerner and Sutcliffe [4], but it was first mooted by Born and Jordan [8] in their pre-Dirac version of quantum mechanics. Born and Jordan's theory featured commutation rules which were more elaborate than those of Dirac, but those rules were nevertheless still *not* sufficiently strong to *uniquely* pin down the operator $\widehat{H}(t_i)$ of Eqs. (19). Therefore Born and Jordan's discovery of $\widehat{H}(t_i)$ must be regarded as fascinatingly fortuitous rather than wholly systematic. Dirac, with his Poisson bracket insight into quantum commutators, had a very good chance to pin $\widehat{H}(t_i)$ down uniquely, but truly ironically he ended up choosing commutation rules that were even much *weaker* [7] than those of his predecessors Born and Jordan! Kerner [9] was apparently the first to work out the slightly strengthened canonical commutation rule that Dirac *ought*, by rights, to have lit upon, but very unfortunately Kerner failed to publish that work. We shall briefly develop the highly satisfactory canonical commutation rule that Dirac *missed* at the end of this paper.

The path integral Schrödinger equation in operator notation

First, however, we must finish the development of the path integral. At this point it beomes very convenient to reexpress all the work done so far in operator notation. Thus the the configuration path integral *defines* the quantum mechanics operator $U_H(t_f;t_i)$ via its *configuration representation matrix elements*,

$$\langle \mathbf{q}_f|U_H(t_f;t_i)|\mathbf{q}_i\rangle \stackrel{\text{def}}{=} K_H(\mathbf{q}_f,t_f;\mathbf{q}_i,t_i), \tag{20a}$$

and the momentum path integral *defines* the quantum mechanics operator $U'_H(t_f;t_i)$ via its *momentum representation matrix elements*,

$$\langle \mathbf{p}_f|U'_H(t_f;t_i)|\mathbf{p}_i\rangle \stackrel{\text{def}}{=} K'_H(\mathbf{p}_f,t_f;\mathbf{p}_i,t_i). \tag{20b}$$

Eqs. (16) through (20) show that $U_H(t_i+(\delta t)_{fi};t_i)$ and $U'_H(t_i+(\delta t)_{fi};t_i)$ agree with each other through first order in $(\delta t)_{fi}$,

$$U_H(t_i+(\delta t)_{fi};t_i) = \mathrm{I} - i((\delta t)_{fi}/\hbar)\widehat{H}(t_i) + O((\delta t)^2_{fi}), \tag{21a}$$

and,

$$U'_H(t_i+(\delta t)_{fi};t_i) = \mathrm{I} - i((\delta t)_{fi}/\hbar)\widehat{H}(t_i) + O((\delta t)^2_{fi}). \tag{21b}$$

Eqs. (16c), (17c) and (19) show that $\widehat{H}(t_i)$ is Hermitian,

$$\widehat{H}(t_i) = \widehat{H}^\dagger(t_i). \tag{22}$$

Eqs. (20a) and (10b) show that the time reversal of $U_H(t_f;t_i)$ equals its Hermitian conjugate,

$$U_H(t_i;t_f) = U^\dagger_H(t_f;t_i), \tag{23a}$$

and we analogously know that the same is true of $U'_H(t_f;t_i)$,

$$U'_H(t_i;t_f) = U'^\dagger_H(t_f;t_i). \tag{23b}$$

Eqs. (20a) and (11) show that for $t_c \in [t_i,t_f]$, the decomposition property of $U_H(t_f;t_i)$ holds,

$$U_H(t_f;t_i) = U_H(t_f;t_c)U_H(t_c;t_i), \tag{24a}$$

and we analogously know that under this same condition the decomposition property of $U'_H(t_f;t_i)$ holds,

$$U'_H(t_f;t_i) = U'_H(t_f;t_c)U'_H(t_c;t_i). \tag{24b}$$

With the above information we shall (somewhat tediously) be able to verify that $U_H(t;t_0)$ satisfies a linear first-order differential equation in time that involves $\widehat{H}(t)$. Since the above information for $U'_H(t;t_0)$ is *completely identical* to that for $U_H(t;t_0)$, $U'_H(t;t_0)$ will satisfy the *same* differential equation. With the fact that $U_H(t_0;t_0) = \mathrm{I} = U'_H(t_0;t_0)$, this differential equation can be rewritten as an integral equation, which, in turn, can be iterated to develop the *same* formal series solution for *both* $U_H(t;t_0)$ and $U'_H(t;t_0)$ in terms of $\widehat{H}(t)$. This will demonstrate that $U'_H(t;t_0) = U_H(t;t_0)$ (the above information already tells us that this is true through first order in $(t-t_0)$). We now turn to the somewhat long-winded matter of calculating the time derivative of $U_H(t;t_0)$.

To calculate that time derivative, we must calculate the difference $U_H(t+\delta t;t_0) - U_H(t;t_0)$ to first order in δt. We shall carry this out in all cases by applying Eq. (21a) *after* the way to such an application has been *cleared* by *first* applying the decomposition property given by Eq. (24a)—this, however, is *restricted* by the requirement that $t_c \in [t_i, t_f]$. The most straightforward of the two cases we need to consider will be the one that $(\delta t)(t-t_0) \geq 0$. This permits the decomposition property of $U_H(t+\delta t;t_0)$ to be used with a minimum of mental gymnastics, followed by straighforward application of Eq. (21a), i.e.,

$$U_H(t+\delta t;t_0) - U_H(t;t_0) = U_H(t+\delta t;t)U_H(t;t_0) - U_H(t;t_0) = -i(\delta t/\hbar)\widehat{H}(t)U_H(t;t_0).$$

The less straightforward case is the one that $(\delta t)(t-t_0) < 0$. In that case we are *obliged* to use the decomposition property of $U_H(t;t_0)$ *rather* than that of $U_H(t+\delta t;t_0)$. Doing so, we obtain,

$$U_H(t+\delta t;t_0) - U_H(t;t_0) = U_H(t+\delta t;t_0) - U_H(t;t+\delta t)U_H(t+\delta t;t_0) =$$

$$U_H(t+\delta t;t_0) - U_H(t+\delta t-\delta t;t+\delta t)U_H(t+\delta t;t_0) = -i(\delta t/\hbar)\widehat{H}(t+\delta t)U_H(t+\delta t;t_0),$$

where the last equality of course results from the application of Eq. (21a) to the first factor of the *second* term of the very awkward expression on its left hand side. The extra terms involving δt in the *arguments* of the operators on the right hand side of that last equality will obviously *not affect* the limiting result as $\delta t \to 0$. Therefore we have established that,

$$dU_H(t;t_0)/dt = -(i/\hbar)\widehat{H}(t)U_H(t;t_0), \tag{25a}$$

which we recognize as the Schrödinger equation for the operator $U_H(t;t_0)$. Bearing in mind that $U_H(t_0;t_0) = \mathrm{I}$, we can reexpress Eq. (25a) as an inhomogeneous linear integral equation,

$$U_H(t;t_0) = \mathrm{I} - (i/\hbar)\int_{t_0}^{t} dt_1\, \widehat{H}(t_1)U_H(t_1;t_0), \tag{25b}$$

which we can readily *iterate*, thereby developing the at least *formal* series expansion solution of $U_H(t;t_0)$ in terms of $\widehat{H}(t)$,

$$U_H(t;t_0) = \mathrm{I} + (-i/\hbar)\int_{t_0}^{t} dt_1\, \widehat{H}(t_1) + \sum_{n=2}^{\infty} (-i/\hbar)^n \int_{t_0}^{t} dt_1\, \widehat{H}(t_1)\int_{t_0}^{t_1} dt_2\, \widehat{H}(t_2)\cdots\int_{t_0}^{t_{n-1}} dt_n\, \widehat{H}(t_n). \tag{25c}$$

It is clear that $U'_H(t;t_0)$ will have the *identical* formal series expansion solution in terms of $\widehat{H}(t)$ as the right hand side of Eq. (25c), because $U'_H(t;t_0)$ obeys *exactly* the same *relations*, as shown by Eqs. (21) through (24), as those obeyed by $U_H(t;t_0)$, and it was on the *basis* of those *relations* that the formal series expansion solution of Eq. (25c) was *developed*. Therefore, notwithstanding the detailed *convergence* properties of the *particular* formal series expansion solution that is given by Eq. (25c), we can nevertheless conclude that $U_H(t;t_0)$ and $U'_H(t;t_0)$ *must* be *identical* operator-valued functionals of the operator-valued argument function $\widehat{H}(t)$, and thus that $U'_H(t;t_0) = U_H(t;t_0)$. This now unified *operator version of the path integral*, $U_H(t;t_0)$, is obviously the *time evolution operator* of quantum mechanics.

The fact that $U_H(t;t_0)$ satisfies the Schrödinger equation finally permits one to demonstrate that its decomposition property is satisfied *without restriction*, with the *unitarity* of $U_H(t;t_0)$ then following as a simple corollary. The Schrödinger equation specifically permits one to demonstrate that the derivative with respect to the variable t of $U_H(t_f;t)U_H(t;t_i)$ always vanishes, so that it must for *all* t have the *same* value that it has when $t = t_f$ or $t = t_i$, which is readily seen to be $U_H(t_f;t_i)$, yielding its unrestricted decomposition property. Furthermore, if we take into consideration the fact that the time reversal of $U_H(t;t_0)$ equals its Hermitian conjugate, as given by Eq. (23a), then $U^\dagger_H(t;t_0)U_H(t;t_0) = U_H(t_0;t)U_H(t;t_0)$, which, because of the unrestricted decomposition property, equals $U_H(t_0;t_0)$, which in turn equals I, thus demonstrating the *unitarity* of $U_H(t;t_0)$. Now we turn to the demonstration that the derivative with respect to t of $U_H(t_f;t)U_H(t;t_i)$ always vanishes, which we carry out in a series of steps that involve the Schrödinger equation, the fact that the time reversal of $U_H(t;t_0)$ equals its Hermitian conjugate, and the fact that the Hamiltonian operator $\widehat{H}(t)$

is Hermitian. First we note that $U_H(t_f;t)U_H(t;t_i) = U_H^\dagger(t;t_f)U_H(t;t_i)$. Then,

$$d(U_H^\dagger(t;t_f)U_H(t;t_i))/dt = ((-i/\hbar)\widehat{H}(t)U_H(t;t_f))^\dagger U_H(t;t_i) + (-i/\hbar)U_H^\dagger(t;t_f)\widehat{H}(t)U_H(t;t_i) =$$

$$(i/\hbar)U_H(t_f;t)\widehat{H}(t)U_H(t;t_i) + (-i/\hbar)U_H(t_f;t)\widehat{H}(t)U_H(t;t_i) = 0.$$

Quantum amplitudes for individual configuration or momentum paths

Looking at the interpretations that we have given to the path integrals of Eq. (2) and Eq. (7a), it is entirely reasonable to interpret the *unrestricted* functional integral over *only momentum paths* $\mathbf{p}(t)$,

$$A_H([\mathbf{q}(t)];t_f,t_i) \stackrel{\text{def}}{=} \int\mathcal{D}^{(t\in[t_i,t_f])}_{[\mathbf{p}(t)]} \exp(iS_H([\mathbf{q}(t),\mathbf{p}(t)];t_f,t_i)/\hbar),$$

as the quantum amplitude that the dynamical system traverses the arbitrarily specified *configuration path* $\mathbf{q}(t)$ for $t \in [t_i,t_f]$. If we now also consider the interpretation we have given to Eq. (9a), we see that the amplitude that the dynamical system traverses the arbitrarily specified *momentum path* $\mathbf{p}(t)$ for $t \in [t_i,t_f]$ ought to similarly be given by the *unrestricted* functional integral over *only configuration paths* $\mathbf{q}(t)$,

$$A'_H([\mathbf{p}(t)];t_f,t_i) \stackrel{\text{def}}{=} \int\mathcal{D}^{(t\in[t_i,t_f])}_{[\mathbf{q}(t)]} \exp(iS'_H([\mathbf{q}(t),\mathbf{p}(t)];t_f,t_i)/\hbar).$$

Now we note from Eq. (1a) that the *unrestricted* variation of the classical action $S_H([\mathbf{q}(t),\mathbf{p}(t)];t_f,t_i)$ with respect to the *momentum path* $\mathbf{p}(t)$ yields the *first* classical Hamiltonian equation, and from Eq. (1b) that the *unrestricted* variation of the classical action $S'_H([\mathbf{q}(t),\mathbf{p}(t)];t_f,t_i)$ with respect to the *configuration path* $\mathbf{q}(t)$ yields the *second* classical Hamiltonian equation. We therefore see that our above *unrestricted* functional integrals for $A_H([\mathbf{q}(t)];t_f,t_i)$ and $A'_H([\mathbf{p}(t)];t_f,t_i)$ are the *precise embodiments* of the principle that the *quantization of classical dynamics* is achieved by substituting *superposition* of the exponential of $(i/\hbar)$ times the classical action for *variation* of that action. (Additionally, of course, that classical action must *not* be one that *implicitly violates* the uncertainty principle!) This *validates* the interpretation of $A_H([\mathbf{q}(t)];t_f,t_i)$ as the orthodox *quantum amplitude* that the dynamical system traverses the specified *configuration path* $\mathbf{q}(t)$ for $t \in [t_i,t_f]$ and of $A'_H([\mathbf{p}(t)];t_f,t_i)$ as the orthodox *quantum amplitude* that the dynamical system traverses the specified *momentum path* $\mathbf{p}(t)$ for $t \in [t_i,t_f]$. The *dominant* stationary phase $\mathbf{p}(t)$ *momentum* path that *contributes* to $A_H([\mathbf{q}(t)];t_f,t_i)$ is readily seen to be the one that comes from *algebraically* solving the *first* classical Hamiltonian equation, i.e.,

$$\dot{\mathbf{q}}(t) = \nabla_{\mathbf{p}(t)}H(\mathbf{q}(t),\mathbf{p}(t),t),$$

whereas the *dominant* stationary phase $\mathbf{q}(t)$ *configuration* path that *contributes* to $A'_H([\mathbf{p}(t)];t_f,t_i)$ is seen to be the one that comes from *algebraically* solving the *second* classical Hamiltonian equation, i.e.,

$$\dot{\mathbf{p}}(t) = -\nabla_{\mathbf{q}(t)}H(\mathbf{q}(t),\mathbf{p}(t),t).$$

If we now wish to obtain the amplitude $K_H(\mathbf{q}_f,t_f;\mathbf{q}_i,t_i)$ described below Eq. (2), we clearly must superpose the *configuration* $A_H([\mathbf{q}(t)];t_f,t_i)$ over *all* the $\mathbf{q}(t)$ that satisfy the *restrictions* $\mathbf{q}(t_i) = \mathbf{q}_i$ and $\mathbf{q}(t_f) = \mathbf{q}_f$. The mathematically efficacious approach to superposing the $A_H([\mathbf{q}(t)];t_f,t_i)$, over only those $\mathbf{q}(t)$ which conform to these *endpoint restrictions* has previously been discussed at great length, and clearly will result in Eq. (7a). The discussion just given concerning obtaining $K_H(\mathbf{q}_f,t_f;\mathbf{q}_i,t_i)$ from $A_H([\mathbf{q}(t)];t_f,t_i)$ can simply be paraphrased for the process of obtaining $K'_H(\mathbf{p}_f,t_f;\mathbf{p}_i,t_i)$ from $A'_H([\mathbf{p}(t)];t_f,t_i)$, and equally clearly will result in Eq. (9a).

We also can now trace the *nub* of the problem with the Feynman-Dirac *Lagrangian*-action hypothesis for $A_H([\mathbf{q}(t)];t_f,t_i)$: when they exponentiate $(i/\hbar)$ times the *Lagrangian* action for that configuration path $\mathbf{q}(t)$, they generate *precisely* the phase factor which is *only the integrand* corresponding to *one particular* momentum path of the above-given *functional integral* for $A_H([\mathbf{q}(t)];t_f,t_i)$ over *all* momentum paths—that *particular* momentum path $\mathbf{p}(t)$ is given by,

$$\mathbf{p}(t) = \nabla_{\dot{\mathbf{q}}(t)}L(\dot{\mathbf{q}}(t),\mathbf{q}(t),t).$$

Now from classical dynamics one easily verifies that the *particular* momentum path which Feynman and Dirac *inadvertently* chose is in fact the *strongest contributor* to the *actually required sum* over momentum paths, i.e., it is the one that algebraically satisfies the first classical Hamiltonian equation—this explains why the

Lagrangian path integral can be coerced into "working" under certain favorable conditions, and *also* why, even under the *most favorable* of those conditions (i.e., Hamiltonians which are quadratic forms in $\mathbf{p}(t)$, whose Gaussian-phase functional integrals over the $\mathbf{p}(t)$ *automatically* produce the *dominant* phase factor), they *still* require an *additional* factor, courtesy of the fact that *integration* over even *Gaussian* phases yields *not only* the dominant phase factor, but a *non-phase* factor as well (in any *subsequent* integration over *configuration paths* this factor is *in fact not*, as Feynman's *wrong* Lagrangian approach drove him to mistakenly conclude, some *totally ad hoc* measure "normalizing factor", but a *completely natural part of the integrand*). The Lagrangian path integral is thus seen to be a shakily defined *relative* of *systematic* semiclassical asmptotic approximations to the Hamiltonian phase-space path integral.

The strengthened, self-consistent canonical commutation rule

The unique quantization given by Eq. (16b) or Eq. (17b) could *very well* have been discovered by Dirac when he was formulating his canonical commutation rule in 1925 [7], or at any time thereafter that he should have chosen to revisit that work. We now briefly explore just what it was that Dirac *failed to light on* during an entire lifetime (see reference [10] for greater detail). We note that the canonical commutation rules which Dirac ended up postulating in 1925 (after some struggling) can be gathered into the single formula,

$$[c_1\mathrm{I} + \mathbf{k}_1 \cdot \widehat{\mathbf{q}} + \mathbf{l}_1 \cdot \widehat{\mathbf{p}}, c_2\mathrm{I} + \mathbf{k}_2 \cdot \widehat{\mathbf{q}} + \mathbf{l}_2 \cdot \widehat{\mathbf{p}}] = i\hbar(\mathbf{k}_1 \cdot \mathbf{l}_2 - \mathbf{l}_1 \cdot \mathbf{k}_2)\mathrm{I}, \tag{26a}$$

where c_1 and c_2 are constant scalars, and $\mathbf{k}_1$, $\mathbf{l}_1$, $\mathbf{k}_2$, $\mathbf{l}_2$ are constant vectors. The above equation can be reexpressed in the much more suggestive form,

$$[\overbrace{c_1 + \mathbf{k}_1 \cdot \mathbf{q} + \mathbf{l}_1 \cdot \mathbf{p}}, \overbrace{c_2 + \mathbf{k}_2 \cdot \mathbf{q} + \mathbf{l}_2 \cdot \mathbf{p}}] = i\hbar\, \overbrace{\{c_1 + \mathbf{k}_1 \cdot \mathbf{q} + \mathbf{l}_1 \cdot \mathbf{p}, c_2 + \mathbf{k}_2 \cdot \mathbf{q} + \mathbf{l}_2 \cdot \mathbf{p}\}}, \tag{26b}$$

where the overbrace denotes the *quantization* of the classical dynamical variable beneath it, and the vertical curly brackets of course denote the *classical* Poisson bracket. (We use overbraces to denote quantization *only* where the orthodox "hat" accent $\widehat{\ }$, which is the standard way to denote quantization, *fails* to be sufficiently *wide*.) Eq. (26b) is compellingly elegant in light of Dirac's amazing groundbreaking demonstration that the quantum mechanical analog of the classical Poisson bracket *must* be $(-i/\hbar)$ times the commutator bracket [7]. Indeed it rather strongly *suggests* the possibility of *extending* Dirac's Eq. (26b) to simply read,

$$[\overbrace{F_1(\mathbf{q}, \mathbf{p})}, \overbrace{F_2(\mathbf{q}, \mathbf{p})}] = i\hbar\, \overbrace{\{F_1(\mathbf{q}, \mathbf{p}), F_2(\mathbf{q}, \mathbf{p})\}}. \tag{27}$$

We note that Dirac's Eq. (26b) is simply the *restriction* of Eq. (27) to $F_i(\mathbf{q}, \mathbf{p})$, $i = 1, 2$, that are both *inhomogeneous linear functions* of phase space. Another, equivalent way to express this restriction is to say that *all* second-order partial derivatives of the $F_i(\mathbf{q}, \mathbf{p})$, $i = 1, 2$, *must vanish*. Dirac was very tempted by Eq. (27), but upon playing with it he found to his consternation that it *overdetermined* the quantization of classical dynamical variables, and thus would be *self-inconsistent* as a postulate [7]. Dismayed, he retreated to the *restriction* on the $F_i(\mathbf{q}, \mathbf{p})$ that results in Eq. (26a), which, however, *cannot* determine the *order* of noncommuting factors *at all*! Far better that abject *underdetermination* of the quantization of classical dynamical variables than the outright *self-inconsistency* of their *overdetermination* was undoubtedly the thought that ran through Dirac's mind.

But could there be a "middle way" that skirts the overdetermination *without* having to settle for *not* determining the order of noncommuting factors *at all*? Very unfortunately, Dirac apparently never revisited this issue after 1925. If one plays with polynomial forms of the $F_i(\mathbf{q}, \mathbf{p})$, one realizes that the *overdetermination* does *not* occur if *no* monomials that are dependent on *both* $\mathbf{q}$ *and* $\mathbf{p}$ are present. This tells us that Dirac's condition that *all* second-order partial derivatives of the $F_i(\mathbf{q}, \mathbf{p})$ must vanish is *excessive*; that to prevent the self-inconsistent overdetermination of quantization it is *quite enough* to require that *only* the *mixed* $\mathbf{q}, \mathbf{p}$ second-order partial derivatives of the $F_i(\mathbf{q}, \mathbf{p})$ must vanish, i.e., that,

$$\nabla_{\mathbf{p}} \nabla_{\mathbf{q}} F_i(\mathbf{q}, \mathbf{p}) = 0,\ i = 1, 2, \tag{28a}$$

which has the general solution, $F_i(\mathbf{q}, \mathbf{p}) = f_i(\mathbf{q}) + g_i(\mathbf{p})$, $i = 1, 2$. Therefore, if we merely *replace* Dirac's Eq. (26b) by,

$$[\overbrace{f_1(\mathbf{q}) + g_1(\mathbf{p})}, \overbrace{f_2(\mathbf{q}) + g_2(\mathbf{p})}] = i\hbar\, \overbrace{\{f_1(\mathbf{q}) + g_1(\mathbf{p}), f_2(\mathbf{q}) + g_2(\mathbf{p})\}}\,, \tag{28b}$$

we will *still* have a canonical commutation rule that does *not* provoke the self-inconsistent *overdetermination* of classical dynamical variables. But does it make any dent in the gross *nondetermination* of the ordering of noncommuting factors that characterizes Dirac's Eq. (26b)? The question of whether a proposed approach fully determines the quantization of *all* classical dynamical variables can be boiled down to the issue of whether it fully determines the quantization of the class of exponentials $\exp(i(\mathbf{k}\cdot\mathbf{q} + \mathbf{l}\cdot\mathbf{p}))$, because if it *does*, the *linearity* of quantization, combined with Fourier expansion, then determines the quantization of *all* dynamical variables. It is apparent that the only truly *new* consequence of Eq. (28b) versus Dirac's Eq. (26b) is that,

$$[f(\widehat{\mathbf{q}}), g(\widehat{\mathbf{p}})] = i\hbar\, \overbrace{\nabla_{\mathbf{q}} f(\mathbf{q}) \cdot \nabla_{\mathbf{p}} g(\mathbf{p})}\,. \tag{28c}$$

Putting now $f(\mathbf{q}) = e^{i\mathbf{k}\cdot\mathbf{q}}$ and $g(\mathbf{p}) = e^{i\mathbf{l}\cdot\mathbf{p}}$, we see that Eq. (28c) yields,

$$\overbrace{e^{i(\mathbf{k}\cdot\mathbf{q}+\mathbf{l}\cdot\mathbf{p})}} = (i/(\hbar\mathbf{k}\cdot\mathbf{l}))[e^{i\mathbf{k}\cdot\widehat{\mathbf{q}}}, e^{i\mathbf{l}\cdot\widehat{\mathbf{p}}}], \tag{29}$$

which clearly answers the question concerning *full* determination of quantization in the affirmative! It now remains to be worked out how the unique, self-consistent quantization that results from marginally *extending* Dirac's *excessively restricted* canonical commutation rule of Eq. (26b) to the *slightly less restricted* canonical quantization rule of Eq. (28b) in fact *compares* with the unique quantization rule of Eq. (16b), which is a *key consequence* of the Hamiltonian phase-space path integral. To carry out the comparison, it is very helpful to use the identity,

$$[e^{i\mathbf{k}\cdot\widehat{\mathbf{q}}}, e^{i\mathbf{l}\cdot\widehat{\mathbf{p}}}] = \textstyle\int_0^1 d\lambda\, d(e^{i\lambda\mathbf{k}\cdot\widehat{\mathbf{q}}} e^{i\mathbf{l}\cdot\widehat{\mathbf{p}}} e^{i(1-\lambda)\mathbf{k}\cdot\widehat{\mathbf{q}}})/d\lambda, \tag{30a}$$

which is simply a consequence of the fundamental theorem of the calculus. Now if we carry out the differentiation under the integral sign, there results,

$$\begin{aligned} &[e^{i\mathbf{k}\cdot\widehat{\mathbf{q}}}, e^{i\mathbf{l}\cdot\widehat{\mathbf{p}}}] = \\ &\textstyle\int_0^1 d\lambda\, e^{i\lambda\mathbf{k}\cdot\widehat{\mathbf{q}}}[i\mathbf{k}\cdot\widehat{\mathbf{q}}, e^{i\mathbf{l}\cdot\widehat{\mathbf{p}}}] e^{i(1-\lambda)\mathbf{k}\cdot\widehat{\mathbf{q}}} = -i\hbar\mathbf{k}\cdot\mathbf{l} \int_0^1 d\lambda\, e^{i\lambda\mathbf{k}\cdot\widehat{\mathbf{q}}} e^{i\mathbf{l}\cdot\widehat{\mathbf{p}}} e^{i(1-\lambda)\mathbf{k}\cdot\widehat{\mathbf{q}}}, \end{aligned} \tag{30b}$$

Combining this identity with the quantization result of Eq. (29) yields,

$$\overbrace{e^{i(\mathbf{k}\cdot\mathbf{q}+\mathbf{l}\cdot\mathbf{p})}} = \textstyle\int_0^1 d\lambda\, e^{i\lambda\mathbf{k}\cdot\widehat{\mathbf{q}}} e^{i\mathbf{l}\cdot\widehat{\mathbf{p}}} e^{i(1-\lambda)\mathbf{k}\cdot\widehat{\mathbf{q}}}. \tag{31}$$

We note here that the *form* of Eq. (31) is that of a rule for the *ordering* of noncommuting factors—and that rule has a characteristically Born-Jordan [8] *appearance*, i.e., *all* of the orderings of the class that it embraces appear with *equal weight*. H. Weyl, a mathematician who liked to dabble in the new quantum mechanics, thought it highly plausible that Nature would *select* the *most symmetric* of that class of orderings [11], i.e., the one for which $\lambda = \frac{1}{2}$, but Eq. (31) has it that Nature does *not select* amongst orderings *at all*, that it *instead* achieves an *alternate* kind of symmetry through utter *nondiscrimination* amongst orderings (an echo, perhaps, of the need to sum over *all* paths). Now in order to compare the quantization given by Eq. (31) to the result of the integration which is called for by Eq. (16b), we must first obtain the *configuration representation* of the former, which is facilitated by the well-known result that,

$$\langle\mathbf{q}_f|e^{i\mathbf{l}\cdot\widehat{\mathbf{p}}}|\mathbf{q}_i\rangle = \delta^{(n)}(\mathbf{q}_f + \hbar\mathbf{l} - \mathbf{q}_i).$$

Using this, we obtain from Eq. (31) that,

$$\langle\mathbf{q}_f|\,\overbrace{e^{i(\mathbf{k}\cdot\mathbf{q}+\mathbf{l}\cdot\mathbf{p})}}\,|\mathbf{q}_i\rangle = \textstyle\int_0^1 d\lambda\, e^{i\mathbf{k}\cdot(\mathbf{q}_i + \lambda(\mathbf{q}_f - \mathbf{q}_i))}\ \delta^{(n)}(\mathbf{q}_f + \hbar\mathbf{l} - \mathbf{q}_i), \tag{32}$$

which result, it is readily verified, is *also* produced by the path integral quantization formula of Eq. (16b) when $e^{i(\mathbf{k}\cdot\mathbf{q}+\mathbf{l}\cdot\mathbf{p})}$ is substituted for the classical Hamiltonian.

We do not really need to go further than this to have demonstrated that the quantization produced by the path integral is the *same* as that produced by the mildly extended canonical commutation rule of Eq. (28b). The reader may find it interesting, however, to follow out the full consequences of combining the *linearity* of quantization with the *Fourier expansion* of an *arbitrary* classical dynamical variable $F(\mathbf{q},\mathbf{p})$, which together formally imply that,

$$\langle \mathbf{q}_f|\,\overbrace{F(\mathbf{q},\mathbf{p})}\,|\mathbf{q}_i\rangle = \\ (2\pi)^{-2n}\int d^n\mathbf{q}' d^n\mathbf{p}'\, F(\mathbf{q}',\mathbf{p}')\int d^n\mathbf{k} d^n\mathbf{l}\, e^{-i(\mathbf{k}\cdot\mathbf{q}'+\mathbf{l}\cdot\mathbf{p}')}\langle\mathbf{q}_f|\,\overbrace{e^{i(\mathbf{k}\cdot\mathbf{q}+\mathbf{l}\cdot\mathbf{p})}}\,|\mathbf{q}_i\rangle. \tag{33a}$$

The next step is, of course, to substitute the unambiguous result for the quantization of the exponential $e^{i(\mathbf{k}\cdot\mathbf{q}+\mathbf{l}\cdot\mathbf{p})}$, which was obtained in Eq. (32) from the mildly extended canonical commutation rule of Eq. (28b), for the last factor of the integrand on the right hand side of Eq. (33a). We leave it to the reader to then plow through all the integrations that can be carried out in closed form to obtain,

$$\langle \mathbf{q}_f|\,\overbrace{F(\mathbf{q},\mathbf{p})}\,|\mathbf{q}_i\rangle = \int_0^1 d\lambda\,(2\pi\hbar)^{-n}\int d^n\mathbf{p}\, F(\mathbf{q}_i+\lambda(\mathbf{q}_f-\mathbf{q}_i),\mathbf{p})e^{i(\mathbf{q}_f-\mathbf{q}_i)\cdot\mathbf{p}/\hbar}, \tag{33b}$$

which is precisely the *same* quantization result as is obtained from the Hamiltonian phase-space path integral, namely that given by Eq. (16b), when $F(\mathbf{q},\mathbf{p})$ is subtituted for the classical Hamiltonian. Dirac's 1925 postulation of Eqs. (26) as *the* canonical commutation rule is thus seen to be a *purely historical aberration*. One can only suppose that if Dirac had *kept working* over the years on trying to obtain a more satisfactory canonical commutation rule than the abjectly deficient Eqs. (26), he would *surely* have eventually lit upon their slight extension to Eq. (28b), which *removes* their vexing ordering ambiguity *without* imperiling their self-consistency. The Hamiltonian phase-space path integral's utterly straightforward unique quantization *ought* to have been the needed wake-up call to the physics community on this issue, but by then the result of Dirac's inadequate work had become so *ingrained* that it was mentioned by Cohen [5] in his last paragraph as another reason to call into question the correct path integral results of Kerner and Sutcliffe [4]. Cohen's mention of the "usual" ambiguity of quantization may have been one of Kerner's motivations to revisit Dirac's canonical commutation rule. He soon came up with the mild extension to Eq. (28b) and showed it to produce the very same Born-Jordan [8] quantization as does the Hamiltonian phase-space path integral [9]. Stunningly, however, Kerner *never published* those results! Neither did he *ever reply* in print nor at any scholarly forum to the meritless $\lim_{|t_f-t_i|\to 0}|\mathbf{q}_f-\mathbf{q}_i| = 0$ objection that Cohen raised regarding his groundbreaking paper with Sutcliffe on the consequences of the Hamiltonian phase-space path integral. Pressed on why, he said that he "did not want to pick a fight with Leon Cohen" [9]. Kerner's apparently shy, retiring nature came within a hair of *denying* physics the gifts that his mind had produced. To read page after page of solemn classification by Tirapegui et al. [6] of wrong "discretization" results that flow from Cohen's lapse is to utterly despair of Kerner's choice of silence.

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