

A NOTE ON THE MELLIN CONVOLUTION OF FUNCTIONS AND ITS RELATION TO RIESZ CRITERION AND RIEMANN HYPOTHESIS

Jose Javier Garcia Moreta Graduate student of Physics at the UPV/EHU (University of Basque country) In Solid State Physics Address: Address: Practicantes Adan y Grijalba 2 5 G P.O 644 48920 Portugalete Vizcaya (Spain) Phone: (00) 34 685 77 16 53 E-mail: josegarc2002@yahoo.es

• ABSTRACT: In this paper we study how the Mellin convolution of functions f and g (f * g) and how is related to the Riesz criterion for the Riemann Hypothesis, the idea is to stablish a Fredholm integral equation of First kind for the Riesz function and the Hardy function.

1. MELLIN CONVOLUTION OF FUNCTIONS:

Given two functions f and g , we can define the Mellin Convolution (f * g) by the two equivalent integral forms

$$\left(f^*g\right) = \int_0^\infty \frac{dt}{t} f\left(\frac{x}{t}\right) g(t^{-1}) = \int_0^\infty \frac{dt}{t} f\left(xt\right) g(t) \qquad (1.1)$$

This operator is linear $(f^*(\lambda g + \mu h)) = \lambda (f^*g) + \mu (f^*h)$, and associative $(f^*(g^*h)) = (f^*g)^*h$, another main property is the 'Convolution Theorem', if we define the Mellin transform of f and g, $\hat{F}(s) = \int_{0}^{\infty} dt f(t) t^{s-1}$ and $\hat{G}(s) = \int_{0}^{\infty} dt g(t) t^{s-1}$ this theorem tells that the Mellin transform of the Convolution theorem is the product of the Mellin transforms of f and g $M[(f^*g) = \hat{F}(s)\hat{G}(-s)$

<u>Proof:</u> If we take the integral $\int_{0}^{\infty} x^{s-1} dx$ to both sides , and introduce the change of variable z = xt inside of the double integral involving x and t.

$$\int_{0}^{\infty} (f * g)(x) x^{s-1} dx = \int_{0}^{\infty} dt \int_{0}^{\infty} dx f(xt) g(t) \frac{dt}{t} x^{s-1} = \hat{F}(s) \int_{0}^{\infty} dt g(t) t^{-s-1}$$
(1.2)

The last factor is just $\hat{F}(s)\hat{G}(-s)$, so our Convolution Theorem is proved, the main and direct application of this theorem is the solution of the Convolution integral equation $\int_{0}^{\infty} dx K(xt) y(t) \frac{dt}{t} = h(x)$ for some given K(xt) and h(x), then the solution for y(t) can be described by the Mellin-Cahen integral

$$y(t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{\hat{H}(-s)}{\hat{K}(-s)} \frac{ds}{t^s} \qquad \int_{0}^{\infty} dt K(t) t^{s-1} = \hat{K}(s) \qquad \int_{0}^{\infty} dt h(t) t^{s-1} = \hat{H}(s)$$
(1.3)

Here 'c' is a real constant chosen in a manner so all the poles of $\frac{\dot{H}}{\dot{K}}(-s)$ lie on the left of the line in the complex plane defined by Re (c).

2. RIESZ AND HARDY CRITERION AND RIEMANN HYPOTHESIS:

Given an infinite sum $\sum_{n=0}^{\infty} a_n$ we can define its Borel transform and its 'sum' B(S)

$$B(a_n, x) = \sum_{n=0}^{\infty} \frac{a_n}{\Psi(n+1)} x^n \quad \Psi(n+1) = \int_0^{\infty} dt g(t) t^n \quad B(S) = \int_0^{\infty} dt g(t) B(a_n, x) \quad (2.1)$$

(2.1) can give the sum of the series (providing the integral is convergent and well defined) no matter if the series is convergent or divergent, for example it gives the correct value of the Grandi's series $1-1+1-1+1-1+\ldots = -1/2$, and the correct expansion for the exponential integral $\int_{x}^{\infty} \frac{e^{-t}}{t} dt$, by evaluation of the divergent series (except when x=0) $\sum_{n=0}^{\infty} (-x)^n n!$ via the Laplace transform $\int_{0}^{\infty} \frac{e^{-t}}{1+xt} dt$, this results with $\Psi(n+1) = n!$ were known to Borel, the general result (2.1) appeared in the paper by Leopold Nachbin as a generalization of Borel series to the case when $B(a_n, x) \neq O(e^{ax})$ (see reference [5]). Appart from the linearity of the Borel transform and its sum B(S), we have another interesting property S = B(S), the usual 'sum' in the sense of addition gives the same result as the Borel transform B(S)

Example: $\frac{1}{1+x} = \sum_{n=0}^{\infty} (-x)^n$ its Borel transform is $B(a_n, xt) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} (xt)^n = e^{-xt}$ and its sum $B(S, x) = \int_0^{\infty} dt e^{-t(1+x)} = \frac{1}{1+x}$ S = B (S) are equal just because the power series expansion of $(1+x)^{-1}$ is convergent on the interval (-1,1)

We will exploit this property in order to study and give an integral equation for the following Hardy and Riesz criterion for the Riemann Hypothesis

• Riesz criterion and Riemann Hypothesis:

Hardy [3] and Riesz provided two different criteria for the truth of Riemann Hypothesis based on the growth of two different power series $\sum_{n=0}^{\infty} a_n x^n$ involving the Riemann zeta:

$$Riesz(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^n}{(n-1)! \zeta(2n)} = O(x^{1/4+\varepsilon}) \qquad H(x) = \sum_{n=1}^{\infty} \frac{(-1)^n x^n}{n! \zeta(2n+1)} = O(x^{-1/4+\varepsilon}) \quad (2.2)$$

Here ε is any positive real number, and $O(x^{1/4+\varepsilon})$ means that there is a positive constant 'C' so $f(x) \le Cx^{1/4+\varepsilon}$, that is if Riemann Hypothesis is true then $Riesz(x) \le Cx^{1/4+\varepsilon}$ and $H(x) \le Dx^{-1/4+\varepsilon}$ for both positive 'C' and 'D' (see [7]).

Using the Mellin convolution of two functions (f * g) and the properties of convolution theorem, we can give an integral equation representation for the Riesz and Hardy functions via a Fredholm integral of first kind with Kernel $\left[\sqrt{\frac{x}{t}}\right]$ in the form: $1 - e^{-x} = \int_{0}^{\infty} \frac{dt}{t} \left[\sqrt{\frac{x}{t}}\right] Riesz(t)$ $\sqrt{x}(e^{-x} - 1) = \int_{0}^{\infty} \frac{dt}{\sqrt{t}} \left[\sqrt{\frac{x}{t}}\right] \left(\frac{H}{2} + t\dot{H}\right)$ x >0 (2.3)

• With
$$\dot{H} = \frac{dH}{dt}$$
, from the representation $\zeta(2s) = s \int_{0}^{1} dt \left[\frac{1}{\sqrt{t}} \right] t^{s-1}$ we have that

R $iesz(t) = O(t^{1/2+\varepsilon})$ due to the 'Prime Number theorem' .In order to give a proof for the Riesz function integral equation inside (2.3) we should recall the properties of the Mellin integral transforms for Riesz function , $M\left\{Riesz(t)\right\} = \frac{\Gamma(s+1)}{\zeta(-2s)}$, the inverse of the Floor function [x] and the Riemann Zeta are related by the integral $\zeta(2s) = s \int_{0}^{1} dt \left[\frac{1}{\sqrt{t}}\right] t^{s-1}$ valid for Re(s) > 1, using the properties and definition of

Gamma function for positive 's' $\int_{0}^{\infty} dt e^{-t} t^{s-1} = \Gamma(s)$ and the 'Mellin Convolution

theorem' we find $-\Gamma(s) = \frac{\Gamma(s+1)}{\zeta(-2s)} \cdot \frac{\zeta(-2s)}{-s} = -\Gamma(s)$, so we can give an inmediate

justification for the first integral equation inside (2.3) for the Riesz function. A proof based on the fact that for a infinite (and convergent) power series its Borel generalized transform and its 'normal' or usual definition of sum are equal so S = B(s), is the following :

- First we use the known expansion $1 e^{-x} = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n!}$ (2.4)
- We use the Mellin integral representation valid for $n \ge 1$ $\zeta(2n) = n \int_{0}^{1} dt \left[\frac{1}{\sqrt{t}} \right] t^{n-1}$
- The Riesz(t) function has the power series $Riesz(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^n}{(n-1)! \zeta(2n)}$, now if we use that S= B(S) since the power series defining the Riesz function is

convergent we get the result

$$1 - e^{-x} = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n!} = \int_0^1 \frac{dt}{t} \left[\frac{1}{\sqrt{t}} \right] \left(\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\zeta(2n)} \frac{(xt)^n}{(n-1)!} \right) \quad (2.5)$$

The final expression inside (2.5) is just the Riesz(xt) so with a change of variable y=xt and the fact that for x < 1, then [x] = 0, we can see that the series expansion of Riesz(t) solves the integral equation (2.3) for 'x' positive , in order to give a proof for Hardy's series we must use the properties

$$x\frac{dH}{dx} = \sum_{n=1}^{\infty} \frac{(-1)^n n x^n}{n! \zeta(2n+1)} \qquad \qquad \frac{\zeta(2n+1)}{n+1/2} = \int_0^1 dt \left[\frac{1}{\sqrt{t}}\right] t^{n-1/2} \qquad n \ge 1$$
(2.6)

In order to prove (2.3) we have used simply the fact S = B(S), in a previous paper [2] we investigated a similar procedure to solve integral equation with Kernel

 $g(s) = \int_{0} dt K(st) f(t)$, and applied this to solve and obtain an expression for the Prime

counting function $\pi(x) = \sum_{p \le x} 1$ via the series expansion $\sum_{n=1}^{\infty} \frac{a_n}{\zeta(n+1)\Gamma(n+1)} (\log x)^n$, since the Prime number theorem imposes a bound to the Prime number counting fucntion $\lim_{x \to \infty} \frac{\pi(x)\log x}{x} = 1$ then for big 'n' $a_n \approx \frac{1}{n}$ (formally), these $\{a_n\}$ come from the expansion of $\log \zeta(s)$ into a power series $\sum_{n=0}^{\infty} a_n s^{-n}$ valid for Re (s) > 1 with $a_n = \int \frac{dz}{dt} \log \zeta(z)$ for a certain contour 'C' inside complex plane, the proof again is

 $a_n = \int_C \frac{dz}{z^{n+1}} \log \zeta(z)$ for a certain contour 'C' inside complex plane, the proof again is based on the fact S= B(S)

$$\log \zeta(s) = \sum_{n=0}^{\infty} a_n s^{-n} = \int_0^{\infty} \frac{dt}{e^t - 1} \left(\sum_{n=0}^{\infty} a_n \frac{(ts^{-1})^n}{\zeta(n+1)n!} \right) = s \int_0^{\infty} \frac{dy}{e^{sy} - 1} \left(\sum_{n=0}^{\infty} a_n \frac{y^n}{\zeta(n+1)n!} \right)$$
(2.7)

To prove (2.7) we have used the Generalized Borel transform involving the Zeta function $\zeta(n+1)\Gamma(n+1) = \int_{0}^{\infty} \frac{dt}{e^{t}-1}t^{n}$, and an appropriate change of variable, the last expression inside (2.7) is $\pi(e^{t})$ due to the Fredholm integral equation of First kind satisfied by the Prime Number counting function $\frac{\log \zeta(s)}{s} = \int_{0}^{\infty} \frac{dt}{e^{st}-1} \frac{\pi(e^{t})}{t}$ as it can be seen in , Apostol [1] (for other issues involving Number theory treated on this paper , this is perhaps the best introductory reference).

An alternative definition of equations (2.3) exists if we take the representation of Riemann Zeta as the Mellin transform involving the fractional part of x

$$\int_{0}^{\infty} dt \, frac\left(\frac{1}{t}\right) t^{s-1} dt = -\frac{\zeta(s)}{s} \qquad frac\left(t\right) = \frac{1}{2} - \frac{1}{\pi} \sum_{k=1}^{\infty} \frac{\sin(2\pi kx)}{k} \quad (2.8)$$

In (2.8) the 'fractional part function' is understood in terms of Fourier Analysis to be proportional to the sawtooth function, its derivative is the distribution $-2\sum_{k=1}^{\infty} \cos(2\pi kx)$, which is zero everywhere except for $x = 0, 1, 2, 3, 4, \dots$ using again the property of the Borel resummation algorithm S = B(S) the integral equations for Riesz and Hardy functions become

$$e^{-x} - 1 = \int_{0}^{\infty} \frac{dt}{t} \operatorname{frac}\left(\sqrt{\frac{x}{t}}\right) \operatorname{Riesz}(t) \quad \sqrt{x}(1 - e^{-x}) = \int_{0}^{\infty} \frac{dt}{\sqrt{t}} \operatorname{frac}\left(\sqrt{\frac{x}{t}}\right) \left(\frac{H}{2} + t\dot{H}\right) \quad (2.9)$$

The main advantage of using (2.8) instead of (2.6) is that (2.6) is easier for practical calculations involving the Fourier expansion in order to check if the RH is true so $Riesz(z) = O(z^{1/4+\varepsilon})$, another problem comes as $x \to 0$ for $\left[\frac{1}{\sqrt{x}}\right]$ this problem can be avoided using the Fourier representation for the fractional part of 'x', the fact that we can take the Mellin transform of $\left[\frac{1}{\sqrt{x}}\right]$ or $frac\left(\frac{1}{\sqrt{t}}\right)$ in order to define the Borel sum B(S) comes from the 'regularized' integral $\int_{0}^{\infty} t^{s-1} dt = 0$ valid for every s and the definition of the fractional part as t - [t].

According to Wolf [7] the solution to the Riesz function in(2.9) as $x \to \infty$ should be Ries(x) = $Cx^{1/4} \sin\left(\phi - \frac{\gamma_1}{2}\log x\right)$ with $\gamma_1 \approx 14.134725...$ (imaginary part of the first non-trivial zero for Riemann Zeta), $C = 7.775 \cdot 10^{-5}$ and $\phi = -0.54916$.. some computer calculations are still to be made in order to check if this is correct. If we insert this formula inside (2.9) and use the Mellin representation of the Riemann zeta function

$$\zeta(s) = s \int_{0}^{\infty} dt frac\left(\frac{1}{t}\right) t^{s-1} \text{ , since } \zeta\left(\frac{1}{2} \pm i\gamma\right) = 0 \text{ and using Euler's formula for the sine}$$

as
$$x \to \infty$$
, $e^{-x} \to 0$ $1 \approx C \frac{\zeta\left(\frac{1}{2} - i\gamma\right)}{i + 2\gamma} x^{1/4 - i\gamma/2} e^{i\phi} - C \frac{\zeta\left(\frac{1}{2} + i\gamma\right)}{i - 2\gamma} x^{1/4 + i\gamma/2} e^{-i\phi} \approx 0$ (2.10)

So we get 0 = 1 ;; this apparent contradiction would come from the fact that Wolf's formula is only valid for big 'x' [8] and we have taken the regularized value for the divergent Mellin transform $\int_{0}^{\infty} t^{s-1}dt = 0 = \int_{0}^{\infty} t^{s-1}H(t)dt$. A final question is , could we reproduce from (2.9) , and using the Mellin transform technique, the equalities ??

$$Riesz(x) = x \sum_{n=1}^{\infty} \frac{\mu(n)}{n^2} \exp\left(-\frac{x}{n^2}\right) \text{ and } xe^{-x} = \sum_{n=1}^{\infty} Riesz\left(\frac{x}{n^2}\right) \quad (2.11)$$

Both formulae in (2.11) were known to Riesz , and can be proved in the following way , using the Mellin transform inside (2.9) we get $\Gamma(s) = \frac{\zeta(-2s)}{s} R(s)$, rearranging terms $s\Gamma(s) = \Gamma(s+1) = \zeta(-2s)R(s)$ or simply $\frac{\Gamma(s+1)}{\zeta(-2s)} = R(s) = \int_{0}^{\infty} dt Riesz(t)t^{s-1}$, now using the following representation (Perron formula [1] for Dirichlet series)

$$F(s) = s \int_{0}^{\infty} \frac{dt}{x^{s+1}} f(t) \quad F(s)\zeta(s) = s \int_{0}^{\infty} \frac{dt}{x^{s+1}} \sum_{n=1}^{\infty} f\left(\frac{x}{n}\right) \quad \frac{F(s)}{\zeta(s)} = s \int_{0}^{\infty} \frac{dt}{x^{s+1}} \sum_{n=1}^{\infty} \mu(n) f\left(\frac{x}{n}\right) \quad (2.12)$$

And $\frac{1}{\zeta(s)} = s \int_{0}^{\infty} \frac{dt}{x^{s+1}} M(t)$, the proof of (2.11) is inmediate, however (2.9) have the

advantage of avoid a summation involving the Möebius function defined as follow $\mu(n) \begin{cases} 0 & \text{if n has repeated prime factor} \\ (-1)^k & \text{if n is product of k different primes} \end{cases} \text{ and } \mu(1) = 1$

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