A PROOF OF RIEMANN HYPOTHESIS USING THE GROWTH OF MERTENS FUNCTION $M(x)$

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Abstract. A study of growth of $M(x)$ as $x \to \infty$ is one of the most useful approach to the Riemann hypophotesis(RH). It is very known that the RH is equivalent to which $M(x) = O(x^{1/2+\varepsilon})$ for $\varepsilon > 0$. Also Littlewood proved that "the RH is equivalent to the statement that $\lim_{x\to\infty} M(x)x^{-1/2-\varepsilon} = 0$, for every $\varepsilon > 0$".[1] To use growth of $M(x)$ approaches zero as $x \to \infty$, I simply prove that the Riemann hypothesis is valid. Now Riemann hypothesis is not hypothesis any longer.

1. Introduction

The Riemann zeta-function $\zeta(s)$ is the function of complex numbers s $(s \neq 1)$. There are infinitely many zeros at the negative even integers such that at $(s = -2,\ s = -4,\ s = -6,\ \cdots)$ These are called the trivial zeros. The Riemann hypothesis(RH) is related the non-trivial zeros, and states that:

$$\text{"All non-trivial zeros of Riemann zeta-function } \zeta(s) \text{ have real part } \frac{1}{2}.\text{"}$$

The RH has been implied strong bounds on the growth of many arithmetic functions. Among them, our most interesting function is Mertens function.

1.1. **Mertens function :** $M(n)$ is defined as follows :

$$M(n) = \sum_{k=1}^{n} \mu(k)$$

where $\mu(k)$ is the Möbius function. [1, 2]

The inverse of the Riemann zeta function is expressed that the Dirichlet series generates the Möbius function by Euler product.

$$\frac{1}{\zeta(s)} = \prod_{p_k}^{\infty}(1 - \frac{1}{p_k^s}) = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s} \tag{1.1}$$

where $\Re(s) > 1$, p_k is the k-th prime number

Mertens function, $M(x)$ is closely linked with the positions of zeroes of the Riemann zeta-function, $\zeta(s)$. When we define $M(0) = 0$, their relation is expressed as follows : [3]

$$\frac{1}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s} = \sum_{n=1}^{\infty} \frac{M(n) - M(n-1)}{n^s}$$

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$$= \sum_{n=1}^{\infty} \frac{M(n)}{n^s} - \sum_{n=1}^{\infty} \frac{M(n-1)}{(n)^s} = \sum_{n=1}^{\infty} \frac{M(n)}{n^s} - \sum_{n=1}^{\infty} \frac{M(n)}{(n+1)^s}$$

$$= \sum_{n=1}^{\infty} M(n)(\frac{1}{n^s} - \frac{1}{(n+1)^s}) = \sum_{n=1}^{\infty} M(n) \int_n^{n+1} \frac{s}{x^{s+1}} dx$$

$$= s \sum_{n=1}^{\infty} \int_n^{n+1} \frac{M(x)}{x^{s+1}} = s \int_1^{\infty} \frac{M(x)}{x^{s+1}} dx$$

since $M(x)$ is constant on each interval $[n, n+1)$

$$\frac{1}{\zeta(s)} = s \int_1^{\infty} M(x) x^{-s-1} dx \tag{1.2}$$

The equation (1.2) shows that a relation of the Mertens function and zeros of the Riemann zeta-function very well.[3]

If $|M(x)| < C|x^{1/2}|$ for $C > 0$, then

$$|\frac{M(x)}{x^{s+1}}| < |\frac{C\sqrt{x}}{x^{s+1}}| = \frac{C}{\sqrt{x}} |\frac{1}{x^s}| = \frac{C}{\sqrt{x}} \frac{1}{x^{\Re(s)}} = \frac{C}{x^{\Re(s)+1/2}}$$

This means that $\Re(s) > 1/2$ because, the right integral in equation (1.2) would converge provided which $\Re(s) + 1/2 > 1$. According to this result, it can define a function analytic in $\Re(s) > 1/2$ and extend an analytic continuation of $1/\zeta(s)$ from $\Re(s) > 1$ to $\Re(s) > 1/2$. It means that $\zeta(s)$ have no zeros for $\Re(s) > 1/2$ and also for $\Re(s) < 1/2$ by symmetry. Thus, all non-trivial zeros must have real part one-half.[3] $|M(x)| < C|x^{1/2}|$ called Mertens conjecture is a condition stronger than RH. Actually, the RH is equivalent to a condition that $M(x) = O(x^{1/2+\varepsilon})$ for all $\varepsilon > 0$.[2, 4] Also according to a chapter 12 in the reference[1], a necessary and sufficient condition for the RH is

$$\lim_{x \to \infty} \frac{M(x)}{x^{1/2+\varepsilon}} = 0, \text{ for every } \varepsilon > 0 \text{ , proven by Littlewood.} \tag{1.3}$$

I just will prove that equation (1.3) is valid using the growth of $M(x)$, for a proof of the RH.

2. The Growth of Mertens Function

While I was studying about the growth of $M(x)$ as $x \to \infty$, I found a fact that the equation (1.1) is very similar to $\sum_{n=1}^{\infty} \mu(n)$.

$$\sum_{n=1}^{\infty} \frac{\mu(n)}{n^s} = \prod_{p_k}^{\infty} (1 - \frac{1}{p_k^s})$$

VS

$$\sum_{n=1}^{\infty} \mu(n) = \ ???$$

If we can remove $\frac{1}{n^s}$ in the equation (1.1), can we know about $\sum_{n=1}^{\infty} \mu(n)$? The solution was found very easily. Look at the equation (2.1).

$$\prod_{k=1}^{\infty} (1 - \frac{p_k}{p_k}) = 0 \text{ , where } p_k \text{ is the } k\text{-th prime number.} \tag{2.1}$$

Actually, it is seem that means nothing at all. However, I want to call that it is one of the Golden Keys for opening locked RH. Because, it shows that the growth of $M(x)$ approaches zero as $x \to \infty$.

Theorem 2.1. *A Golden Key of the Riemann Hypothesis*

$$\prod_{k=1}^{\infty}(1-\frac{p_k}{p_k}) = \sum_{n=1}^{\infty}\mu(n) = \lim_{n\to\infty} M(n) = 0$$

Proof.

$$\prod_{k=1}^{\infty}(1-\frac{p_k}{p_k}) = 0$$

$$= (1-\frac{2}{2})(1-\frac{3}{3})(1-\frac{5}{5})(1-\frac{7}{7})(1-\frac{11}{11})(1-\frac{13}{13})(1-\frac{17}{17})(1-\frac{19}{19})\dots$$

$$= 1-\frac{2}{2}-\frac{3}{3}-\frac{5}{5}+\frac{6}{6}-\frac{7}{7}+\frac{10}{10}-\frac{11}{11}-\frac{13}{13}+\frac{14}{14}+\frac{15}{15}-\frac{17}{17}-\frac{19}{19}+\frac{21}{21}+\dots$$

$$= 1+\frac{-2}{2}+\frac{-3}{3}+\frac{0}{4}+\frac{-5}{5}+\frac{6}{6}+\frac{-7}{7}+\frac{0}{8}+\frac{0}{9}+\frac{10}{10}+\frac{-11}{11}+\frac{0}{12}+\frac{-13}{13}+\frac{14}{14}+\frac{15}{15}+\frac{0}{16}+\dots$$

$$= \frac{1\mu(1)}{1}+\frac{2\mu(2)}{2}+\frac{3\mu(3)}{3}+\frac{4\mu(4)}{4}+\frac{5\mu(5)}{5}+\frac{6\mu(6)}{6}+\frac{7\mu(7)}{7}+\frac{8\mu(8)}{8}+\frac{9\mu(9)}{9}+\frac{10\mu(10)}{10}+\dots$$

$$= \sum_{n=1}^{\infty}\frac{n\mu(n)}{n} = \sum_{n=1}^{\infty}\mu(n) = \lim_{n\to\infty} M(n) = 0$$

□

How do you think about the convergence of the growth of $M(x)$? Maybe most people have believed that the growth of $M(x)$ must be diverged as $x \to \infty$. However, the theorem(2.1) shows that the growth of $M(x)$ approaches zero as $x \to \infty$.

3. The Probability of Möbius Function

The theorem(2.1) shows some results about probability of Möbius function as following :

Corollary 3.1.

$$Pr(\mu(n) = +1) = Pr(\mu(n) = -1)$$

where n is the natural number.

Proof.

$$\text{Because, } \lim_{n\to\infty} M(n) = 0$$

Therefore, the numbers of -1 and $+1$ of $\mu(n)$ are equal.

□

Corollary 3.2.

$$Pr(\mu(n) = +1) = \frac{3}{\pi^2}\ ,\ Pr(\mu(n) = -1) = \frac{3}{\pi^2}\ \textit{and}\ Pr(\mu(n) = 0) = 1-\frac{6}{\pi^2}$$

where n is the natural number.

Proof.

Using the inclusion-exclusion principle,

a probability of the total square-free numbers is defined as follows :

$$Pr(\mu(n) \neq 0) = (1 - \frac{1}{2^2})(1 - \frac{1}{3^2})(1 - \frac{1}{5^2})(1 - \frac{1}{7^2})(1 - \frac{1}{11^2}) \cdots$$

$$= \prod_{k=1}^{\infty}(1 - \frac{1}{p_k^2}) = \frac{1}{\zeta(2)} = \frac{6}{\pi^2}$$

Because, $Pr(\mu(n) = -1) = Pr(\mu(n) = +1)$

and

$$Pr(\mu(n) = -1) + Pr(\mu(n) = +1) + Pr(\mu(n) = 0) = 1$$

Therefore, $Pr(\mu(n) = +1) = \frac{3}{\pi^2}$, $Pr(\mu(n) = -1) = \frac{3}{\pi^2}$, $Pr(\mu(n) = 0) = 1 - \frac{6}{\pi^2}$

□

Denjoy's proposal an another probabilistic condition that is equilvalent to RH with probability one.[1] It has some suppositions which square-free numbers are random sequences and independent events with symmetrical distribution. In other words if a square-free number is taken at random and has an equal probability of containing an odd or an even number of distinct prime divisors, $M(x) = O(x^{1/2+\varepsilon})$ and the RH is true with probability one. From corollary (3.1) and (3.2), we can verify a fact that $Pr(\mu(n) = +1)$ and $Pr(\mu(n) = -1)$ are equal. These are providing the plausible evidences for the Riemann Hypothesis.

4. A Proof of Riemann's Hypothesis

Theorem 4.1.

All non-trivial zeros of $\zeta(s)$ have real part one-half.

Proof.

Using theorem (2.1), $\lim_{x \to \infty} M(x) = 0$

$$\Downarrow$$

$$\lim_{x \to \infty} \frac{M(x)}{x^{1/2+\varepsilon}} = 0, \text{ for every } \varepsilon > 0$$

This condition is equivalent to the Riemann hypothesis.[1]

Therefore, the Riemann hypothesis is true.

□

5. Conclusion

I very simply prove the RH using the growth of $M(x)$ approaches zero as $x \to \infty$. From now on, Riemann hypothesis is not his hypothesis any longer. It is reborn an obvious theorem.

The $M(x)$ closely linked with the positions of zeroes of $\zeta(s)$ have some questions still. Their relation has been very known that the RH is equivalent to $M(x) = O(x^{1/2+\varepsilon})$.[2, 4] I think that this relation is very similar to $|\pi(x) - Li(x)| = O(\sqrt{x}\ logx)$ called Koch's result. RH is proven using the growth of $M(x)$ approaches zero as $x \to \infty$. This condition is fairly stronger than $O(x^{1/2+\varepsilon})$. If Koch's result and $M(x)$ are closely related, I conjecture that $\lim_{x\to\infty} |\pi(x) - Li(x)| = 0$ alike the growth of $M(x)$.

Conjecture 1.

$$|Li(x) - \pi(x)| < C\sqrt{x}\ logx\ ,\ where\ C \geq 0$$

$$\lim_{x\to\infty} C = 0$$

Today, the precise version of Koch's result is that $|Li(x) - \pi(x)| < \pi/8\sqrt{x}\ logx$ where $x > 2657$ proven by Schoenfeld.

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