A RECURRENCE METHOD FOR GENERALIZING KNOWN SCIENTIFIC RESULTS

Florentin Smarandache
University of New Mexico
200 College Road
Gallup, NM 87301, USA
E-mail: smarand@unm.edu

A great number of articles widen known scientific results (theorems, inequalities, math/physics/chemical etc. propositions, formulas), and this is due to a simple procedure, of which it is good to say a few words:

Let suppose that we want to generalizes a known mathematical proposition $P(a)$, where a is a constant, to the proposition $P(n)$, where n is a variable which belongs to subset of N.

To prove that P is true for n by recurrence means the following: the first step is trivial, since it is about the known result $P(a)$ (and thus it was already verified before by other mathematicians!). To pass from $P(n)$ to $P(n+1)$, one uses too $P(a)$: therefore one widens a proposition by using the proposition itself, in other words the found generalization will be paradoxically proved with the help of the particular case from which one started!

We present below the generalizations of Hölder, Minkovski, and respectively Tchebychev inequalities.

1. A GENERALIZATION OF THE INEQUALITY OF HÖLDER

One generalizes the inequality of Hödler thanks to a reasoning by recurrence. As particular cases, one obtains a generalization of the inequality of Cauchy-Buniakovski-Scwartz, and some interesting applications.

Theorem: If $a_i^{(k)} \in \mathrm{R}_+$ and $p_k \in]1,+\infty[$, $i \in \{1,2,...,n\}$, $k \in \{1,2,...,m\}$, such that:, $\frac{1}{p_1}+\frac{1}{p_2}+...+\frac{1}{p_m}=1$, then:

$$\sum_{i=1}^{n}\prod_{k=1}^{m} a_i^{(k)} \le \prod_{k=1}^{m} \left(\sum_{i=1}^{n}\left(a_i^{(k)}\right)^{p_k}\right)^{\frac{1}{p_k}} \text{ with } m \ge 2.$$

Proof:

For $m=2$ one obtains exactly the inequality of Hödler, which is true. One supposes that the inequality is true for the values which are strictly smaller than a certain m.

Then:,

$$\sum_{i=1}^{n}\prod_{k=1}^{m}a_i^{(k)}=\sum_{i=1}^{n}\left(\left(\prod_{k=1}^{m-2}a_i^k\right)\cdot\left(a_i^{(m-1)}\cdot a_i^{(m)}\right)\right)\le\left(\prod_{k=1}^{m-2}\left(\sum_{i=1}^{n}\left(a_i^{(k)}\right)^{p_k}\right)^{\frac{1}{p_k}}\right)\cdot\left(\sum_{i=1}^{n}\left(a_i^{(m-1)}\cdot a_i^{(m)}\right)^p\right)^{\frac{1}{p}}$$

where $\frac{1}{p_1}+\frac{1}{p_2}+...+\frac{1}{p_{m-2}}+\frac{1}{p}=1$ and $p_h>1$, $1\le h\le m-2$, $p\ >1$;

but

$$\sum_{i=1}^{n}\left(a_i^{(m-1)}\right)^p\cdot\left(a_i^{(m)}\right)^p\le\left(\sum_{i=1}^{n}\left(\left(a_i^{(m-1)}\right)^p\right)^{t_1}\right)^{\frac{1}{t_1}}\cdot\left(\sum_{i=1}^{n}\left(\left(a_i^{(m)}\right)^p\right)^{t_2}\right)^{\frac{1}{t_2}}$$

where $\frac{1}{t_1}+\frac{1}{t_2}=1$ and $t_1>1$, $t_2>2$.

It results from it:

$$\sum_{i=1}^{n}\left(a_i^{(m-1)}\right)^p\cdot\left(a_i^{(m)}\right)^p\le\left(\sum_{i=1}^{n}\left(a_i^{(m-1)}\right)^{pt_1}\right)^{\frac{1}{pt_1}}\cdot\left(\sum_{i=1}^{n}\left(a_i^{(m)}\right)^{pt_2}\right)^{\frac{1}{pt_2}}$$

with $\frac{1}{pt_1}+\frac{1}{pt_2}=\frac{1}{p}$.

Let us note $pt_1=p_{m-1}$ and $pt_2=p_m$. Then $\frac{1}{p_1}+\frac{1}{p_2}+...+\frac{1}{p_m}=1$ is true and one has $p_j>1$ for $1\le j\le m$ and it results the inequality from the theorem.

Note: If one poses $p_j=m$ for $1\le j\le m$ and if one raises to the power m this inequality, one obtains a generalization of the inequality of Cauchy-Buniakovski-Scwartz:

$$\left(\sum_{i=1}^{n}\prod_{k=1}^{m}a_i^{(k)}\right)^m\le\prod_{k=1}^{m}\sum_{i=1}^{n}\left(a_i^{(k)}\right)^m.$$

Application:

Let a_1,a_2,b_1,b_2,c_1,c_2 be positive real numbers.

Show that:

$$(a_1b_1c_1+a_2b_2c_2)^6\le 8(a_1^6+a_2^6)(b_1^6+b_2^6)(c_1^6+c_2^6)$$

Solution:

We will use the previous theorem. Let us choose $p_1=2$, $p_2=3$, $p_3=6$; we will obtain the following:

$$a_1b_1c_1+a_2b_2c_2\le(a_1^2+a_2^2)^{\frac{1}{2}}(b_1^3+b_2^3)^{\frac{1}{3}}(c_1^6+c_2^6)^{\frac{1}{6}},$$

or more:

$$(a_1b_1c_1 + a_2b_2c_2)^6 \le (a_1^2 + a_2^2)^3(b_1^3 + b_2^3)^2(c_1^6 + c_2^6),$$

and knowing that

$$(b_1^3 + b_2^3)^2 \le 2(b_1^6 + b_2^6)$$

and that

$$(a_1^2 + a_2^2)^3 = a_1^6 + a_2^6 + 3(a_1^4a_2^2 + a_1^2a_2^4) \le 4(a_1^6 + a_2^6)$$

since

$$a_1^4a_2^2 + a_1^2a_2^4 \le a_1^6 + a_2^6 \text{ (because: } -\left(a_2^2 - a_1^2\right)^2\left(a_1^2 + a_2^2\right) \le 0\text{)}$$

it results the exercise which was proposed.

2. A GENERALIZATION OF THE INEQUALITY OF MINKOWSKI

Theorem : If p is a real number ≥ 1 and $a_i^{(k)} \in \mathbf{R}^+$ with $i \in \{1,2,...,n\}$ and $k \in \{1,2,...,m\}$, then:

$$\left(\sum_{i=1}^{n}\left(\sum_{k=1}^{m} a_i^{(k)}\right)^p\right)^{1/p} \le \left(\sum_{k=1}^{m}\left(\sum_{i=1}^{n} a_i^{(k)}\right)^p\right)^{1/p}$$

Demonstration by recurrence on $m \in \mathbf{N}^*$.

First of all one shows that:

$\left(\sum_{i=1}^{n}\left(a_i^{(1)}\right)^p\right)^{1/p} \le \left(\sum_{i=1}^{n}\left(a_i^{(1)}\right)^p\right)^{1/p}$, which is obvious, and proves that the inequality is true for $m = 1$.

(The case $m = 2$ precisely constitutes the inequality of Minkowski, which is naturally true!).

Let us suppose that the inequality is true for all the values less or equal to m

$$\left(\sum_{i=1}^{n}\left(\sum_{k=1}^{m+1} a_i^{(k)}\right)^p\right)^{1/p} \le \left(\sum_{i=1}^{n} a_i^{(1)^p}\right)^{1/p} + \left(\sum_{i=1}^{n}\left(\sum_{k=2}^{m+1} a_i^{(k)}\right)^p\right)^{1/p} \le$$

$$\le \left(\sum_{i=1}^{n}\left(a_i^{(1)}\right)^p\right)^{1/p} + \left(\sum_{k=2}^{m+1}\left(\sum_{i=1}^{n} a_i^{(k)}\right)^p\right)^{1/p}$$

and this last sum is $\left(\sum_{k=1}^{m+1}\left(\sum_{i=1}^{n} a_i^{(k)}\right)^p\right)^{1/p}$ therefore the inequality is true for the level $m+1$.

3. A GENERALIZATION OF AN INEQUALITY OF TCHEBYCHEV

Statement: If $a_i^{(k)} \geq a_{i+1}^{(k)}$, $i \in \{1,2,...,n-1\}$, $k \in \{1,2,...,m\}$, then:

$$\frac{1}{n}\sum_{i=1}^{n}\prod_{k=1}^{m} a_i^{(k)} \geq \frac{1}{n^m}\prod_{k=1}^{m}\sum_{i=1}^{n} a_i^{(k)}.$$

Demonstration by recurrence on m.

Case $m=1$ is obvious: $\frac{1}{n}\sum_{i=1}^{n} a_i^{(1)} \geq \frac{1}{n}\sum_{i=1}^{n} a_i^{(1)}$.

In the case $m=2$, this is the inequality of Tchebychev itself:

If $a_1^{(1)} \geq a_2^{(1)} \geq ... \geq a_n^{(1)}$ and $a_1^{(2)} \geq a_2^{(2)} \geq ... \geq a_n^{(2)}$, then:

$$\frac{a_1^{(1)}a_1^{(2)} + a_2^{(1)}a_2^{(2)} + ... + a_n^{(1)}a_n^{(2)}}{n} \geq \frac{a_1^{(1)} + a_2^{(1)} + ... + a_n^{(1)}}{n} \times \frac{a_1^{(2)} + ... + a_n^{(2)}}{n}$$

One supposes that the inequality is true for all the values smaller or equal to m. It is necessary to prove for the rang $m+1$:

$$\frac{1}{n}\sum_{i=1}^{n}\prod_{k=1}^{m+1} a_i^{(k)} = \frac{1}{n}\sum_{i=1}^{n}\left(\prod_{k=1}^{m} a_i^{(k)}\right)\cdot a_i^{(m+1)}.$$

This is $\geq \left(\frac{1}{n}\sum_{i=1}^{n}\prod_{k=1}^{m} a_i^{(k)}\right)\cdot\left(\frac{1}{n}\sum_{i=1}^{n} a_i^{(m+1)}\right) \geq \left(\frac{1}{n^m}\prod_{k=1}^{m}\sum_{i=1}^{n} a_i^{(k)}\right)\cdot\left(\frac{1}{n}\sum_{i=1}^{n} a_i^{(m+1)}\right)$

and this is exactly $\frac{1}{n^{m+1}}\prod_{k=1}^{m+1}\sum_{i=1}^{n} a_i^{(k)}$ (*Quod Erat Demonstrandum*).