Funcoids and Reloids*

a generalization of proximities and uniformities

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Abstract

It is a part of my Algebraic General Topology research.

In this article I introduce the concepts of *funcoids* which generalize proximity spaces and *reloids* which generalize uniform spaces. The concept of funcoid is generalized concept of proximity, the concept of reloid is cleared from superfluous details (generalized) concept of uniformity. Also funcoids and reloids are generalizations of binary relations whose domains and ranges are filters (instead of sets).

Also funcoids and reloids can be considered as a generalization of (oriented) graphs, this provides us with a common generalization of analysis and discrete mathematics.

The concept of continuity is defined by an algebraic formula (instead of old messy epsilon-delta notation) for arbitrary morphisms (including funcoids and reloids) of a partially ordered category. In one formula are generalized continuity, proximity continuity, and uniform continuity.

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1 Common

1.1 Draft status

This article is a draft.

This text refers to a preprint edition of [14]. Theorem number clashes may appear due editing both of these manuscripts.

1.2 Earlier works

Some mathematicians were researching generalizations of proximities and uniformities before me but they have failed to reach the right degree of generalization which is presented in this work allowing to represent properties of spaces with algebraic (or categorical) formulas.

Proximity structures were introduced by Smirnov in [4].

Some references to predecessors:

- In [5], [6], [11], [2], [17] are studied generalized uniformities and proximities.
- Proximities and uniformities are also studied in [9], [10], [16], [18], [19].
- [7] and [8] contains recent progress in quasi-uniform spaces. [8] has a very long list of related literature.

Some works ([15]) about proximity spaces consider relationships of proximities and compact topological spaces. In this work is not done the attempt to define or research their generalization, compactness of funcoids or reloids. It seems potentially productive to attempt to borrow the definitions and procedures from the above mentioned works. I hope to do this study in a separate article.

[3] studies mappings between proximity structures. (In this work no attempt to research mappings between funcoids is done.) [12] researches relationships of quasi-uniform spaces and topological spaces. [1] studies how proximity structures can be treated as uniform structures and compactification regarding proximity and uniform spaces.

1.3 Used concepts, notation and statements

The set of functions from a set A to a set B is denoted as B^A.

I will often skip parentheses and write $f x$ instead of $f(x)$ to denote the result of a function f acting on the argument x.

I will denote $\langle f\rangle X = \{ f\alpha \mid \alpha \in X\}$ for a set X.

For simplicity I will assume that all sets in consideration are subsets of universal set $\mho$.

1.3.1 Filters

In this work the word *filter* will refer to a filter on a set $\mho$ (in contrast to [14] where are considered filters on arbitrary posets). Note that I do not require filters to be proper.

I will call the set of filters ordered reverse to set-theoretic inclusion of filters *the set of filter objects* $\mathfrak{F}$ and its element *filter objects* (f.o. for short). I will denote $\operatorname{up}\mathcal{F}$ the filter corresponding to a filter object $\mathcal{F}$. So we have $\mathcal{A} \subseteq \mathcal{B} \Leftrightarrow \operatorname{up}\mathcal{A} \supseteq \operatorname{up}\mathcal{B}$ for every filter objects $\mathcal{A}$ and $\mathcal{B}$. We also will equate filter objects corresponding to principal filters with corresponding sets. (Thus we have $\mathscr{P}\mho \subseteq \mathfrak{F}$.) See [14] for formal definition of filter objects in the framework of ZF. Filters (and filter objects) are studied in the work [14].

Prior reading of [14] is needed to fully understand this work.

Filter objects corresponding to ultrafilters are atoms of the lattice $\mathfrak{F}$ and will be called *atomic filter objects*.

Also we will need to introduce the concept of *generalized filter base*.

Definition 1. *Generalized filter base* is a set $S \in \mathscr{P}\mathfrak{F} \setminus \{\emptyset\}$ such that

$$\forall \mathcal{A}, \mathcal{B} \in S \exists \mathcal{C} \in S : \mathcal{C} \subseteq \mathcal{A} \cap^{\mathfrak{F}} \mathcal{B}.$$

Proposition 2. Let S is a generalized filter base. If $\mathcal{A}_1, \ldots, \mathcal{A}_n \in S$ ($n \in \mathbb{N}$), then

$$\exists \mathcal{C} \in S : \mathcal{C} \subseteq \mathcal{A}_1 \cap^{\mathfrak{F}} \ldots \cap^{\mathfrak{F}} \mathcal{A}_n.$$

Proof. Can be easily proved by induction. □

Theorem 3. If S is a generalized filter base, then $\operatorname{up}\bigcap^{\mathfrak{F}} S = \bigcup \langle \operatorname{up}\rangle S$.

Proof. Obviously $\operatorname{up}\bigcap^{\mathfrak{F}} S \supseteq \bigcup \langle \operatorname{up}\rangle S$. Reversely, let $K \in \operatorname{up}\bigcap^{\mathfrak{F}} S$; then $K = A_1 \cap \ldots \cap A_n$ where $A_i \in \operatorname{up}\mathcal{A}_i$ where $\mathcal{A}_i \in S$, $i = 1, \ldots, n$, $n \in \mathbb{N}$; so exists $\mathcal{C} \in S$ such that $\mathcal{C} \subseteq \mathcal{A}_1 \cap \ldots \cap \mathcal{A}_n \subseteq A_1 \cap \ldots \cap A_n = K$, $K \in \operatorname{up}\mathcal{C}$, $K \in \bigcup \langle \operatorname{up}\rangle S$. □

Corollary 4. If S is a generalized filter base, then $\bigcap^{\mathfrak{F}} S = \emptyset \Leftrightarrow \emptyset \in S$.

Proof. $\bigcap^{\mathfrak{F}} S = \emptyset \Leftrightarrow \emptyset \in \operatorname{up}\bigcap^{\mathfrak{F}} S \Leftrightarrow \emptyset \in \bigcup \langle \operatorname{up}\rangle S \Leftrightarrow \exists \mathcal{X} \in S : \emptyset \in \operatorname{up}\mathcal{X} \Leftrightarrow \emptyset \in S$. □

2 Partially ordered dagger categories

2.1 Partially ordered categories

Definition 5. I will call a *partially ordered (pre)category* a (pre)category together with partial order $\subseteq$ on each of its Hom-sets with the additional requirement that

$$f_1 \subseteq f_2 \wedge g_1 \subseteq g_2 \Rightarrow g_1 \circ f_1 \subseteq g_2 \circ f_2$$

for every morphisms f_1, g_1, f_2, g_2 such that $\mathrm{Src}\, f_1 = \mathrm{Src}\, f_2 \wedge \mathrm{Dst}\, f_1 = \mathrm{Dst}\, f_2 = \mathrm{Src}\, g_1 = \mathrm{Src}\, g_2 \wedge \mathrm{Dst}\, g_1 = \mathrm{Dst}\, g_2$.

2.2 Dagger categories

Definition 6. I will call a *dagger precategory* a precategory together with an involutive contravariant identity-on-objects prefunctor $x \mapsto x^\dagger$.

In other words, a *dagger precategory* is a precategory equipped with a function $x \mapsto x^\dagger$ on its set of morphisms which reverses the source and the destination and is subject to the following identities for every morphisms f and g:

1. $f^{\dagger\dagger} = f$;
2. $(g \circ f)^\dagger = f^\dagger \circ g^\dagger$.

Definition 7. I will call a *dagger category* a category together with an involutive contravariant identity-on-objects functor $x \mapsto x^\dagger$.

In other words, a *dagger category* is a category equipped with a function $x \mapsto x^\dagger$ on its set of morphisms which reverses the source and the destination and is subject to the following identities for every morphisms f and g and object A:

1. $f^{\dagger\dagger} = f$;
2. $(g \circ f)^\dagger = f^\dagger \circ g^\dagger$;
3. $(1_A)^\dagger = 1_A$.

Theorem 8. If a category is a dagger precategory then it is a dagger category.

Proof. We need to prove only that $(1_A)^\dagger = 1_A$. Really

$$(1_A)^\dagger = (1_A)^\dagger \circ 1_A = (1_A)^\dagger \circ (1_A)^{\dagger\dagger} = ((1_A)^\dagger \circ 1_A)^\dagger = (1_A)^{\dagger\dagger} = 1_A. \qquad \square$$

For a partially ordered dagger (pre)category I will additionally require (for every morphisms f and g)

$$f^\dagger \subseteq g^\dagger \Leftrightarrow f \subseteq g.$$

An example of dagger category is the category **Rel** whose objects are sets and whose morphisms are binary relations between these sets with usual composition of binary relations and with $f^\dagger = f^{-1}$.

Definition 9. A morphism f of a dagger category is called *unitary* when it is an isomorphism and $f^\dagger = f^{-1}$.

Definition 10. *Symmetric* (endo)morphism of a dagger precategory is such a morphism f that $f = f^\dagger$.

Definition 11. *Transitive* (endo)morphism of a precategory is such a morphism f that $f = f \circ f$.

Theorem 12. The following conditions are equivalent for a morphism f of a dagger precategory:

1. f is symmetric and transitive.
2. $f = f^\dagger \circ f$.

Proof.

(1) ⇒ (2). If f is symmetric and transitive then $f^\dagger \circ f = f \circ f = f$.

(2) ⇒ (1). $f^\dagger = (f^\dagger \circ f)^\dagger = f^\dagger \circ f^{\dagger\dagger} = f^\dagger \circ f = f$, so f is symmetric. $f = f^\dagger \circ f = f \circ f$, so f is transitive. □

2.2.1 Some special classes of morphisms

Definition 13. For a partially ordered dagger category I will call *monovalued* morphism such a morphism f that $f \circ f^\dagger \sqsubseteq 1_{\mathrm{Dst}\, f}$.

Definition 14. For a partially ordered dagger category I will call *entirely defined* morphism such a morphism f that $f^\dagger \circ f \sqsupseteq 1_{\mathrm{Src}\, f}$.

Definition 15. For a partially ordered dagger category I will call *injective* morphism such a morphism f that $f^\dagger \circ f \sqsubseteq 1_{\mathrm{Src}\, f}$.

Definition 16. For a partially ordered dagger category I will call *surjective* morphism such a morphism f that $f \circ f^\dagger \sqsupseteq 1_{\mathrm{Dst}\, f}$.

Remark 17. Easy to show that this is a generalization of monovalued, entirely defined, injective, and surjective binary relations as morphisms of the category **Rel**.

Obvious 18. "Injective morphism" is a dual of "monovalued morphism" and "surjective morphism" is a dual of "entirely defined morphism".

Definition 19. For a given partially ordered dagger category C the *category of monovalued (entirely defined, injective, surjective) morphisms* of C is the category with the same set of objects as of C and the set of morphisms being the set of monovalued (entirely defined, injective, surjective) morphisms of C with the composition of morphisms the same as in C.

We need to prove that these are really categories, that is that composition of monovalued (entirely defined) morphisms is monovalued (entirely defined) and that identity morphisms are monovalued and entirely defined.

Proof. We will prove only for monovalued morphisms and entirely defined morphisms, as injective and surjective morphisms are their duals.

Monovalued. Let f and g are monovalued morphisms, $\mathrm{Dst}\, f = \mathrm{Src}\, g$. $(g \circ f) \circ (g \circ f)^\dagger = g \circ f \circ f^\dagger \circ g^\dagger \sqsubseteq g \circ 1_{\mathrm{Dst}\, f} \circ g^\dagger = g \circ 1_{\mathrm{Src}\, g} \circ g^\dagger = g \circ g^\dagger \sqsubseteq 1_{\mathrm{Dst}\, g} = 1_{\mathrm{Dst}(g \circ f)}$. So $g \circ f$ is monovalued.

That identity morphisms are monovalued follows from the following: $1_A \circ (1_A)^\dagger = 1_A \circ 1_A = 1_A = 1_{\mathrm{Dst}\, 1_A} \sqsubseteq 1_{\mathrm{Dst}\, 1_A}$.

Entirely defined. Let f and g are entirely defined morphisms, $\mathrm{Dst}\, f = \mathrm{Src}\, g$. $(g \circ f)^\dagger \circ (g \circ f) = f^\dagger \circ g^\dagger \circ g \circ f \sqsupseteq f^\dagger \circ 1_{\mathrm{Src}\, g} \circ f = f^\dagger \circ 1_{\mathrm{Dst}\, f} \circ f = f^\dagger \circ f \sqsupseteq 1_{\mathrm{Src}\, f} = 1_{\mathrm{Src}(g \circ f)}$. So $g \circ f$ is entirely defined.

That identity morphisms are entirely defined follows from the following: $(1_A)^\dagger \circ 1_A = 1_A \circ 1_A = 1_A = 1_{\mathrm{Src}\, 1_A} \sqsupseteq 1_{\mathrm{Src}\, 1_A}$. □

Definition 20. I will call a *bijective* morphism a morphism which is entirely defined, monovalued, injective, and surjective.

Obvious 21. Bijective morphisms form a full subcategory.

Proposition 22. If a morphism is bijective then it is an isomorphism.

Proof. Let f is bijective. Then $f \circ f^{\dagger} \subseteq 1_{\mathrm{Dst}\, f}$, $f^{\dagger} \circ f \supseteq 1_{\mathrm{Src}\, f}$, $f^{\dagger} \circ f \subseteq 1_{\mathrm{Src}\, f}$, $f \circ f^{\dagger} \supseteq 1_{\mathrm{Dst}\, f}$. Thus $f \circ f^{\dagger} = 1_{\mathrm{Dst}\, f}$ and $f^{\dagger} \circ f = 1_{\mathrm{Src}\, f}$ that is $f^{\dagger}$ is an inverse of f. □

3 Funcoids

3.1 Informal introduction into funcoids

Funcoids are a generalization of proximity spaces and a generalization of pretopological spaces. Also funcoids are a generalization of binary relations.

That funcoids are a common generalization of "spaces" (proximity spaces, (pre)topological spaces) and binary relations (including monovalued functions) makes them smart for describing properties of functions in regard of spaces. For example the statement "f is a continuous function from a space μ to a space ν" can be described in terms of funcoids as the formula $f \circ \mu \subseteq \nu \circ f$ (see below for details).

Most naturally funcoids appear as a generalization of proximity spaces.

Let δ be a proximity that is certain binary relation so that $A \,\delta\, B$ is defined for every sets A and B. We will extend it from sets to filter objects by the formula:

$$\mathcal{A} \,\delta'\, \mathcal{B} \Leftrightarrow \forall A \in \operatorname{up} \mathcal{A}, B \in \operatorname{up} \mathcal{B}: A \,\delta\, B.$$

Then (as will be proved below) exist two functions $\alpha, \beta \in \mathfrak{F}^{\mathfrak{F}}$ such that

$$\mathcal{A} \,\delta'\, \mathcal{B} \Leftrightarrow \mathcal{B} \cap^{\mathfrak{F}} \alpha \mathcal{A} \neq \emptyset \Leftrightarrow \mathcal{A} \cap^{\mathfrak{F}} \beta \mathcal{B} \neq \emptyset.$$

The pair $(\alpha; \beta)$ is called *funcoid* when $\mathcal{B} \cap^{\mathfrak{F}} \alpha \mathcal{A} \neq \emptyset \Leftrightarrow \mathcal{A} \cap^{\mathfrak{F}} \beta \mathcal{B} \neq \emptyset$. So funcoids are a generalization of proximity spaces.

Funcoids consist of two components the first α and the second β. The first component of a funcoid f is denoted as $\langle f \rangle$ and the second component is denoted as $\langle f^{-1} \rangle$. (The similarity of this notation with the notation for the image of a set under a function is not a coincidence, we will see that in the case of discrete funcoids (see below) these coincide.)

One of the most important properties of a funcoid is that it is uniquely determined by just one of its components. That is a funcoid f is uniquely determined by the function $\langle f \rangle$. Moreover a funcoid f is uniquely determined by $\langle f \rangle|_{\mathscr{P}\mho}$ that is by values of function $\langle f \rangle$ on sets.

Next we will consider some examples of funcoids determined by specified values of the first component on sets.

Funcoids as a generalization of pretopological spaces: Let α be a pretopological space that is a map $\alpha \in \mathfrak{F}^{\mho}$. Then we define $\alpha' X \overset{\text{def}}{=} \bigcup^{\mathfrak{F}} \{\alpha x \mid x \in X\}$ for every set X. We will prove that there exists a unique funcoid f such that $\alpha' = \langle f \rangle|_{\mathscr{P}\mho}$. So funcoids are a generalization of pretopological spaces. Funcoids are also a generalization of preclosure operators: For every preclosure operator p exists unique funcoid such that $\langle f \rangle|_{\mathscr{P}\mho} = p$; in this case $\langle f \rangle|_{\mathscr{P}\mho} \in \mathscr{P}\mho^{\mathscr{P}\mho}$.

For every binary relation p exists unique funcoid f such that $\forall X \in \mathscr{P}\mho: \langle f \rangle X = \langle p \rangle X$ (where $\langle p \rangle$ is defined in the introduction), recall that a funcoid is uniquely determined by the values of its first component on sets. I will call such funcoids *discrete*. So funcoids are a generalization of binary relations.

Composition of binary relations (i.e. of discrete funcoids) complies with the formulas:

$$\langle g \circ f \rangle = \langle g \rangle \circ \langle f \rangle \quad \text{and} \quad \langle (g \circ f)^{-1} \rangle = \langle f^{-1} \rangle \circ \langle g^{-1} \rangle.$$

By the same formulas we can define composition of every two funcoids.

Also funcoids can be reversed (like reversal of X and Y in a binary relation) by the formula $(\alpha; \beta)^{-1} = (\beta; \alpha)$. In particular case if μ is a proximity we have $\mu^{-1} = \mu$ because proximities are symmetric.

Funcoids behave similarly to (multivalued) functions but acting on filter objects instead of acting on sets. Below will be defined domain and image of a funcoid (the domain and the image of a funcoid are filter objects).

3.2 Basic definitions

Definition 23. Let's call *a funcoid* a pair $(\alpha; \beta)$ where $\alpha, \beta \in \mathfrak{F}^{\mathfrak{F}}$ such that

$$\forall \mathcal{X}, \mathcal{Y} \in \mathfrak{F}: (\mathcal{Y} \cap^{\mathfrak{F}} \alpha\mathcal{X} \neq \emptyset \Leftrightarrow \mathcal{X} \cap^{\mathfrak{F}} \beta\mathcal{Y} \neq \emptyset).$$

Definition 24. $\langle(\alpha; \beta)\rangle \overset{\text{def}}{=} \alpha$ for a funcoid $(\alpha; \beta)$.

Definition 25. $(\alpha; \beta)^{-1} = (\beta; \alpha)$ for a funcoid $(\alpha; \beta)$.

Proposition 26. If f is a funcoid then f^{-1} is also a funcoid.

Proof. Follows from symmetry in the definition of funcoid. □

Obvious 27. $(f^{-1})^{-1} = f$ for a funcoid f.

Definition 28. The relation $[f] \in \mathscr{P}\mathfrak{F}^2$ is defined by the formula (for every filter objects $\mathcal{X}$, $\mathcal{Y}$ and funcoid f)

$$\mathcal{X}[f]\mathcal{Y} \overset{\text{def}}{=} \mathcal{Y} \cap^{\mathfrak{F}} \langle f\rangle\mathcal{X} \neq \emptyset.$$

Obvious 29. $\mathcal{X}[f]\mathcal{Y} \Leftrightarrow \mathcal{Y} \cap^{\mathfrak{F}} \langle f\rangle\mathcal{X} \neq \emptyset \Leftrightarrow \mathcal{X} \cap^{\mathfrak{F}} \langle f^{-1}\rangle\mathcal{Y}$ for every filter objects $\mathcal{X}$, $\mathcal{Y}$ and funcoid f.

Obvious 30. $[f^{-1}] = [f]^{-1}$ for a funcoid f.

Theorem 31.

1. For given value of $\langle f\rangle$ exists no more than one funcoid f.
2. For given value of $[f]$ exists no more than one funcoid f.

Proof. Let f and g are funcoids.

Obviously $\langle f\rangle = \langle g\rangle \Rightarrow [f] = [g]$ and $\langle f^{-1}\rangle = \langle g^{-1}\rangle \Rightarrow [f] = [g]$. So enough to prove that $[f] = [g] \Rightarrow \langle f\rangle = \langle g\rangle$.

Provided that $[f] = [g]$ we have $\mathcal{Y} \cap^{\mathfrak{F}} \langle f\rangle\mathcal{X} \neq \emptyset \Leftrightarrow \mathcal{X}[f]\mathcal{Y} \Leftrightarrow \mathcal{X}[g]\mathcal{Y} \Leftrightarrow \mathcal{Y} \cap^{\mathfrak{F}} \langle g\rangle\mathcal{X} \neq \emptyset$ and consequently $\langle f\rangle\mathcal{X} = \langle g\rangle\mathcal{X}$ for every f.o. $\mathcal{X}$ and $\mathcal{Y}$ because the set of filter objects is separable [14], thus $\langle f\rangle = \langle g\rangle$. □

Proposition 32. $\langle f\rangle(\mathcal{I} \cup^{\mathfrak{F}} \mathcal{J}) = \langle f\rangle\mathcal{I} \cup^{\mathfrak{F}} \langle f\rangle\mathcal{J}$ for every funcoid f and $\mathcal{I}, \mathcal{J} \in \mathfrak{F}$.

Proof.

$$\begin{aligned}
\star\langle f\rangle(\mathcal{I} \cup^{\mathfrak{F}} \mathcal{J}) &= \\
\{\mathcal{Y} \in \mathfrak{F} \mid \mathcal{Y} \cap^{\mathfrak{F}} \langle f\rangle(\mathcal{I} \cup^{\mathfrak{F}} \mathcal{J}) \neq \emptyset\} &= \\
\{\mathcal{Y} \in \mathfrak{F} \mid (\mathcal{I} \cup^{\mathfrak{F}} \mathcal{J}) \cap^{\mathfrak{F}} \langle f^{-1}\rangle\mathcal{Y} \neq \emptyset\} &= \text{(by corollary 10 in [14])} \\
\{\mathcal{Y} \in \mathfrak{F} \mid (\mathcal{I} \cap^{\mathfrak{F}} \langle f^{-1}\rangle\mathcal{Y}) \cup^{\mathfrak{F}} (\mathcal{J} \cap^{\mathfrak{F}} \langle f^{-1}\rangle\mathcal{Y}) \neq \emptyset\} &= \\
\{\mathcal{Y} \in \mathfrak{F} \mid \mathcal{I} \cap^{\mathfrak{F}} \langle f^{-1}\rangle\mathcal{Y} \neq \emptyset \vee \mathcal{J} \cap^{\mathfrak{F}} \langle f^{-1}\rangle\mathcal{Y} \neq \emptyset\} &= \\
\{\mathcal{Y} \in \mathfrak{F} \mid \mathcal{Y} \cap^{\mathfrak{F}} \langle f\rangle\mathcal{I} \neq \emptyset \vee \mathcal{Y} \cap^{\mathfrak{F}} \langle f\rangle\mathcal{J} \neq \emptyset\} &= \\
\{\mathcal{Y} \in \mathfrak{F} \mid (\mathcal{Y} \cap^{\mathfrak{F}} \langle f\rangle\mathcal{I}) \cup^{\mathfrak{F}} (\mathcal{Y} \cap^{\mathfrak{F}} \langle f\rangle\mathcal{J}) \neq \emptyset\} &= \text{(by corollary 10 in [14])} \\
\{\mathcal{Y} \in \mathfrak{F} \mid \mathcal{Y} \cap^{\mathfrak{F}} (\langle f\rangle\mathcal{I} \cup^{\mathfrak{F}} \langle f\rangle\mathcal{J}) \neq \emptyset\} &= \\
\star(\langle f\rangle\mathcal{I} \cup^{\mathfrak{F}} \langle f\rangle\mathcal{J}).
\end{aligned}$$

Thus $\langle f\rangle(\mathcal{I} \cup^{\mathfrak{F}} \mathcal{J}) = \langle f\rangle\mathcal{I} \cup^{\mathfrak{F}} \langle f\rangle\mathcal{J}$ because $\mathfrak{F}$ is separable. □

3.2.1 Composition of funcoids

Definition 33. *Composition* of funcoids is defined by the formula

$$(\alpha_2; \beta_2) \circ (\alpha_1; \beta_1) = (\alpha_2 \circ \alpha_1; \beta_1 \circ \beta_2).$$

Proposition 34. If f, g are funcoids then $g\circ f$ is funcoid.

Proof. Let $f=(\alpha_1;\beta_1)$, $g=(\alpha_2;\beta_2)$. For every $\mathcal{X},\mathcal{Y}\in\mathfrak{F}$ we have

$\mathcal{Y}\cap^{\mathfrak{F}}(\alpha_2\circ\alpha_1)\mathcal{X}\neq\emptyset\Leftrightarrow\mathcal{Y}\cap^{\mathfrak{F}}\alpha_2\alpha_1\mathcal{X}\neq\emptyset\Leftrightarrow\alpha_1\mathcal{X}\cap^{\mathfrak{F}}\beta_2\mathcal{Y}\neq\emptyset\Leftrightarrow\mathcal{X}\cap^{\mathfrak{F}}\beta_1\beta_2\mathcal{Y}\neq\emptyset\Leftrightarrow\mathcal{X}\cap^{\mathfrak{F}}(\beta_1\circ\beta_2)\mathcal{Y}\neq\emptyset$.

So $(\alpha_2\circ\alpha_1;\beta_1\circ\beta_2)$ is a funcoid. □

Obvious 35. $\langle g\circ f\rangle=\langle g\rangle\circ\langle f\rangle$ for every funcoids f and g.

Proposition 36. $(h\circ g)\circ f=h\circ(g\circ f)$ for every funcoids f, g, h.

Proof.
$\langle(h\circ g)\circ f\rangle=\langle h\circ g\rangle\circ\langle f\rangle=(\langle h\rangle\circ\langle g\rangle)\circ\langle f\rangle=\langle h\rangle\circ(\langle g\rangle\circ\langle f\rangle)=\langle h\rangle\circ\langle g\circ f\rangle=\langle h\circ(g\circ f)\rangle$. □

Theorem 37. $(g\circ f)^{-1}=f^{-1}\circ g^{-1}$ for every funcoids f and g.

Proof. $\left\langle(g\circ f)^{-1}\right\rangle=\left\langle f^{-1}\right\rangle\circ\left\langle g^{-1}\right\rangle=\left\langle f^{-1}\circ g^{-1}\right\rangle$. □

3.3 Funcoid as continuation

Theorem 38. For every funcoid f and filter objects $\mathcal{X}$ and $\mathcal{Y}$

1. $\langle f\rangle\mathcal{X}=\bigcap^{\mathfrak{F}}\langle\langle f\rangle\rangle\mathrm{up}\,\mathcal{X}$;
2. $\mathcal{X}[f]\mathcal{Y}\Leftrightarrow\forall X\in\mathrm{up}\,\mathcal{X},Y\in\mathrm{up}\,\mathcal{Y}: X[f]Y$.

Proof. 2. $\mathcal{X}[f]\mathcal{Y}\Leftrightarrow\mathcal{Y}\cap^{\mathfrak{F}}\langle f\rangle\mathcal{X}\neq\emptyset\Leftrightarrow\forall Y\in\mathrm{up}\,\mathcal{Y}: Y\cap^{\mathfrak{F}}\langle f\rangle\mathcal{X}\neq\emptyset\Leftrightarrow\forall Y\in\mathrm{up}\,\mathcal{Y}:\mathcal{X}[f]Y$.
Analogously $\mathcal{X}[f]\mathcal{Y}\Leftrightarrow\forall X\in\mathrm{up}\,\mathcal{X}: X[f]\mathcal{Y}$. Combining these two equivalences we get

$$\mathcal{X}[f]\mathcal{Y}\Leftrightarrow\forall X\in\mathrm{up}\,\mathcal{X},Y\in\mathrm{up}\,\mathcal{Y}: X[f]Y.$$

1. $\mathcal{Y}\cap^{\mathfrak{F}}\langle f\rangle\mathcal{X}\neq\emptyset\Leftrightarrow\mathcal{X}[f]\mathcal{Y}\Leftrightarrow\forall X\in\mathrm{up}\,\mathcal{X}: X[f]\mathcal{Y}\Leftrightarrow\forall X\in\mathrm{up}\,\mathcal{X}:\mathcal{Y}\cap^{\mathfrak{F}}\langle f\rangle X\neq\emptyset$.
Let's denote $W=\left\{\mathcal{Y}\cap^{\mathfrak{F}}\langle f\rangle X\mid X\in\mathrm{up}\,\mathcal{X}\right\}$. We will prove that W is a generalized filter base. To prove this enough to show that $V=\{\langle f\rangle X\mid X\in\mathrm{up}\,\mathcal{X}\}$ is a generalized filter base.
Let $\mathcal{P},\mathcal{Q}\in V$. Then $\mathcal{P}=\langle f\rangle A$, $\mathcal{Q}=\langle f\rangle B$ where $A,B\in\mathrm{up}\,\mathcal{X}$; $A\cap B\in\mathrm{up}\,\mathcal{X}$ and $\mathcal{R}\subseteq\mathcal{P}\cap^{\mathfrak{F}}\mathcal{Q}$ for $\mathcal{R}=\langle f\rangle(A\cap B)\in V$. So V is a generalized filter base and thus W is a generalized filter base.
$\emptyset\notin W\Leftrightarrow\bigcap^{\mathfrak{F}}W\neq\emptyset$ by the corollary 4 of the theorem 3. That is

$$\forall X\in\mathrm{up}\,\mathcal{X}:\mathcal{Y}\cap^{\mathfrak{F}}\langle f\rangle X\neq\emptyset\Leftrightarrow\mathcal{Y}\cap^{\mathfrak{F}}\bigcap{}^{\mathfrak{F}}\langle\langle f\rangle\rangle\mathrm{up}\,\mathcal{X}\neq\emptyset.$$

Comparing with the above, $\mathcal{Y}\cap^{\mathfrak{F}}\langle f\rangle\mathcal{X}\neq\emptyset\Leftrightarrow\mathcal{Y}\cap^{\mathfrak{F}}\bigcap^{\mathfrak{F}}\langle\langle f\rangle\rangle\mathrm{up}\,\mathcal{X}\neq\emptyset$. So $\langle f\rangle\mathcal{X}=\bigcap^{\mathfrak{F}}\langle\langle f\rangle\rangle\mathrm{up}\,\mathcal{X}$ because the lattice of filter objects is separable. □

Theorem 39.

1. A function $\alpha\in\mathfrak{F}^{\mathscr{P}\mho}$ conforming to the formulas (for every $I,J\in\mathscr{P}\mho$)

$$\alpha\emptyset=\emptyset,\quad\alpha(I\cup J)=\alpha I\cup^{\mathfrak{F}}\alpha J$$

can be continued to the function $\langle f\rangle$ for a unique funcoid f;

$$\langle f\rangle\mathcal{X}=\bigcap{}^{\mathfrak{F}}\langle\alpha\rangle\mathrm{up}\,\mathcal{X}\tag{1}$$

for every filter object $\mathcal{X}$.

2. A relation $\delta\in\mathscr{P}(\mathscr{P}\mho)^2$ conforming to the formulas (for every $I,J,K\in\mathscr{P}\mho$)

$$\begin{array}{ll}\neg(\emptyset\,\delta\,I), & I\cup J\,\delta\,K\Leftrightarrow I\,\delta\,K\vee J\,\delta\,K,\\ \neg(I\,\delta\,\emptyset), & K\,\delta\,I\cup J\Leftrightarrow K\,\delta\,I\vee K\,\delta\,J\end{array}\tag{2}$$

can be continued to the relation $[f]$ for a unique funcoid f;

$$\mathcal{X}[f]\mathcal{Y}\Leftrightarrow\forall X\in\mathrm{up}\,\mathcal{X},Y\in\mathrm{up}\,\mathcal{Y}: X\,\delta\,Y\tag{3}$$

for every filter objects $\mathcal{X}$, $\mathcal{Y}$.

Proof. Existence of no more than one such funcoids and formulas (1) and (3) follow from the previous theorem.

2. Let define $\alpha \in \mathfrak{F}^{\mathscr{P}\mho}$ by the formula $\partial(\alpha X) = \{Y \in \mathscr{P}\mho \mid X \,\delta\, Y\}$ for every $X \in \mathscr{P}\mho$. (It is obvious that $\{Y \in \mathscr{P}\mho \mid X \,\delta\, Y\}$ is a free star.) Analogously can be defined $\beta \in \mathfrak{F}^{\mathscr{P}\mho}$ by the formula $\partial(\beta X) = \{X \in \mathscr{P}\mho \mid X \,\delta\, Y\}$. Let's continue α and β to $\alpha' \in \mathfrak{F}^{\mathfrak{F}}$ and $\beta' \in \mathfrak{F}^{\mathfrak{F}}$ by the formulas

$$\alpha'\mathcal{X} = \bigcap{}^{\mathfrak{F}} \langle\alpha\rangle \mathrm{up}\,\mathcal{X} \quad \text{and} \quad \beta'\mathcal{X} = \bigcap{}^{\mathfrak{F}} \langle\beta\rangle \mathrm{up}\,\mathcal{X}$$

and δ to $\delta' \in \mathscr{P}\mathfrak{F}^2$ by the formula

$$\mathcal{X} \,\delta'\, \mathcal{Y} \Leftrightarrow \forall X \in \mathrm{up}\,\mathcal{X}, Y \in \mathrm{up}\,\mathcal{Y}\colon X \,\delta\, Y.$$

$\mathcal{Y} \cap^{\mathfrak{F}} \alpha'\mathcal{X} \neq \emptyset \Leftrightarrow \mathcal{Y} \cap^{\mathfrak{F}} \bigcap^{\mathfrak{F}} \langle\alpha\rangle \mathrm{up}\,\mathcal{X} \neq \emptyset \Leftrightarrow \bigcap^{\mathfrak{F}} \langle \mathcal{Y} \cap^{\mathfrak{F}} \rangle \langle\alpha\rangle \mathrm{up}\,\mathcal{X} \neq \emptyset$. Let's prove that

$$W = \langle \mathcal{Y} \cap^{\mathfrak{F}} \rangle \langle\alpha\rangle \mathrm{up}\,\mathcal{X}$$

is a generalized filter base: To prove it is enough to show that $\langle\alpha\rangle\mathrm{up}\ \mathcal{X}$ is a generalized filter base. If $\mathcal{A}, \mathcal{B} \in \langle\alpha\rangle \mathrm{up}\,\mathcal{X}$ then exist $X_1, X_2 \in \mathrm{up}\,\mathcal{X}$ such that $\mathcal{A} = \alpha X_1$ and $\mathcal{A} = \alpha X_2$.

Then $\alpha(X_1 \cap X_2) \in \langle\alpha\rangle \mathrm{up}\,\mathcal{X}$. So $\langle\alpha\rangle \mathrm{up}\,\mathcal{X}$ is a generalized filter base and thus W is a generalized filter base.

Accordingly the corollary 4 of the theorem 3, $\bigcap^{\mathfrak{F}} \langle \mathcal{Y} \cap^{\mathfrak{F}} \rangle \langle\alpha\rangle \mathrm{up}\,\mathcal{X} \neq \emptyset$ is equivalent to

$$\forall X \in \mathrm{up}\,\mathcal{X}\colon \mathcal{Y} \cap^{\mathfrak{F}} \alpha X \neq \emptyset,$$

what is equivalent to $\forall X \in \mathrm{up}\,\mathcal{X}, Y \in \mathrm{up}\,\mathcal{Y}\colon Y \cap^{\mathfrak{F}} \alpha X \neq \emptyset \Leftrightarrow \forall X \in \mathrm{up}\,\mathcal{X}, Y \in \mathrm{up}\,\mathcal{Y}\colon Y \in \partial(\alpha X) \Leftrightarrow \forall X \in \mathrm{up}\,\mathcal{X}, Y \in \mathrm{up}\,\mathcal{Y}\colon X \,\delta\, Y$. Combining the equivalencies we get $\mathcal{Y} \cap^{\mathfrak{F}} \alpha'\mathcal{X} \neq \emptyset \Leftrightarrow X \,\delta'\, Y$. Analogously $\mathcal{X} \cap^{\mathfrak{F}} \beta'\mathcal{Y} \neq \emptyset \Leftrightarrow X \,\delta'\, Y$. So $\mathcal{Y} \cap^{\mathfrak{F}} \alpha'\mathcal{X} \neq \emptyset \Leftrightarrow \mathcal{X} \cap^{\mathfrak{F}} \beta'\mathcal{Y} \neq \emptyset$, that is $(\alpha'; \beta')$ is a funcoid. From the formula $\mathcal{Y} \cap^{\mathfrak{F}} \alpha'\mathcal{X} \neq \emptyset \Leftrightarrow X \,\delta'\, Y$ follows that $[(\alpha'; \beta')]$ is a continuation of δ.

1. Let define the relation $\delta \in \mathscr{P}(\mathscr{P}\mho)^2$ by the formula $X \,\delta\, Y \Leftrightarrow Y \cap^{\mathfrak{F}} \alpha X \neq \emptyset$.

That $\neg(\emptyset \,\delta\, I)$ and $\neg(I \,\delta\, \emptyset)$ is obvious. We have $I \cup J \,\delta\, K \Leftrightarrow (I \cup J) \cap^{\mathfrak{F}} \alpha K \neq \emptyset \Leftrightarrow (I \cup^{\mathfrak{F}} J) \cap^{\mathfrak{F}} \alpha K \neq \emptyset \Leftrightarrow (I \cap^{\mathfrak{F}} \alpha K) \cup^{\mathfrak{F}} (J \cup^{\mathfrak{F}} \alpha K) \neq \emptyset \Leftrightarrow I \cap^{\mathfrak{F}} \alpha K \neq \emptyset \vee J \cup^{\mathfrak{F}} \alpha K \neq \emptyset \Leftrightarrow I \,\delta\, K \vee J \,\delta\, K$ and $K \,\delta\, I \cup J \Leftrightarrow K \cap^{\mathfrak{F}} \alpha(I \cup J) \neq \emptyset \Leftrightarrow K \cap^{\mathfrak{F}} (\alpha I \cup^{\mathfrak{F}} \alpha J) \neq \emptyset \Leftrightarrow (K \cap^{\mathfrak{F}} \alpha I) \cup^{\mathfrak{F}} (K \cap^{\mathfrak{F}} \alpha J) \neq \emptyset \Leftrightarrow K \cap^{\mathfrak{F}} \alpha I \neq \emptyset \vee K \cap^{\mathfrak{F}} \alpha J \neq \emptyset \Leftrightarrow K \,\delta\, I \vee K \,\delta\, J$.

That is the formulas (2) are true.

Accordingly the above δ can be continued to the relation $[f]$ for some funcoid f.

$\forall X, Y \in \mathscr{P}\mho\colon (Y \cap^{\mathfrak{F}} \langle f\rangle X \neq \emptyset \Leftrightarrow X[f]Y \Leftrightarrow Y \cap^{\mathfrak{F}} \alpha X \neq \emptyset)$, consequently $\forall X \in \mathscr{P}\mho\colon \alpha X = \langle f\rangle X$. So $\langle f\rangle$ is a continuation of α. □

Note that by the last theorem to every proximity δ corresponds a unique funcoid. So funcoids are a generalization of (quasi-)proximity structures.

Reverse funcoids can be considered as a generalization of conjugate quasi-proximity.

Definition 40. Any (multivalued) function f will be considered as a funcoid, where by definition $\langle f\rangle\mathcal{X} = \bigcap^{\mathfrak{F}} \langle\langle f\rangle\rangle \mathrm{up}\,\mathcal{X}$ for every $\mathcal{X} \in \mathfrak{F}$.

Using the last theorem it is easy to show that this definition is monovalued and does not contradict to former stuff.

Definition 41. Funcoids corresponding to a binary relation are called *discrete funcoids*.

We may equate discrete funcoids with corresponding binary relations by the method of appendix B in [14]. This is useful for describing relationships of funcoids and binary relations, such as for the formulas of continuous functions and continuous funcoids (see below). For simplicity I will not dive here into formal definition of equating discrete funcoids with binary relations (by the method shown in appendix B in [14]) but we simply will (informally) assume that discrete funcoids can be equated with binary relations.

I will denote FCD the set of funcoids or the category of funcoids (see below) dependently on context.

3.4 Lattice of funcoids

Definition 42. $f \subseteq g \stackrel{\text{def}}{=} [f] \subseteq [g]$ for $f, g \in \mathsf{FCD}$.

Thus FCD is a poset. (Taken into account that $[f] \neq [g]$ if $f \neq g$.)

Definition 43. I will call the *filtrator of funcoids* (see [14] for the definition of filtrators) the filtrator $(\mathsf{FCD}; \mathscr{P}\mho^2)$.

Conjecture 44. The filtrator of funcoids is:

1. with separable core;
2. with co-separable core.

Theorem 45. The set of funcoids is a complete lattice. For every $R \in \mathscr{P}\mathsf{FCD}$ and $X, Y \in \mathscr{P}\mho$

1. $X\left[\bigcup^{\mathsf{FCD}} R\right] Y \Leftrightarrow \exists f \in R: X[f]Y$;
2. $\left\langle \bigcup^{\mathsf{FCD}} R \right\rangle X = \bigcup^{\mathfrak{F}} \{\langle f \rangle X \mid f \in R\}$.

Proof. Accordingly [13] to prove that it is a complete lattice enough to prove existence of all joins.

2. $\alpha X \stackrel{\text{def}}{=} \bigcup^{\mathfrak{F}} \{\langle f \rangle X \mid f \in R\}$. We have $\alpha \emptyset = \emptyset$;

$$\begin{aligned}
\alpha(I \cup J) &= \bigcup\nolimits^{\mathfrak{F}} \{\langle f \rangle (I \cup J) \mid f \in R\} \\
&= \bigcup\nolimits^{\mathfrak{F}} \{\langle f \rangle (I \cup^{\mathfrak{F}} J) \mid f \in R\} \\
&= \bigcup\nolimits^{\mathfrak{F}} \{\langle f \rangle I \cup^{\mathfrak{F}} \langle f \rangle J \mid f \in R\} \\
&= \bigcup\nolimits^{\mathfrak{F}} \{\langle f \rangle I \mid f \in R\} \cup^{\mathfrak{F}} \bigcup\nolimits^{\mathfrak{F}} \{\langle f \rangle J \mid f \in R\} \\
&= \alpha I \cup^{\mathfrak{F}} \alpha J.
\end{aligned}$$

So α can be continued to $\langle h \rangle$ for a funcoid h. Obviously

$$\forall f \in R: h \supseteq f. \tag{4}$$

And h is the least funcoid for which holds the condition (4). So $h = \bigcup^{\mathsf{FCD}} R$.

1. $X\left[\bigcup^{\mathsf{FCD}} R\right] Y \Leftrightarrow Y \cap^{\mathfrak{F}} \left\langle \bigcup^{\mathsf{FCD}} R \right\rangle X \neq \emptyset \Leftrightarrow Y \cap^{\mathfrak{F}} \bigcup^{\mathfrak{F}} \{\langle f \rangle X \mid f \in R\} \neq \emptyset \Leftrightarrow \exists f \in R: Y \cap^{\mathfrak{F}} \langle f \rangle X \neq \emptyset \Leftrightarrow \exists f \in R: X[f]Y$ (used the theorem 40 in [14]). □

In the next theorem, compared to the previous one, the class of infinite unions is replaced with lesser class of finite unions and simultaneously class of sets is changed to more wide class of filter objects.

Theorem 46. For every funcoids f and g and a filter object $\mathcal{X}$

1. $\left\langle f \cup^{\mathsf{FCD}} g \right\rangle \mathcal{X} = \langle f \rangle \mathcal{X} \cup^{\mathfrak{F}} \langle g \rangle \mathcal{X}$;
2. $\left[f \cup^{\mathsf{FCD}} g\right] = [f] \cup [g]$.

Proof.

1. Let $\alpha \mathcal{X} \stackrel{\text{def}}{=} \langle f \rangle \mathcal{X} \cup^{\mathfrak{F}} \langle g \rangle \mathcal{X}$; $\beta \mathcal{Y} \stackrel{\text{def}}{=} \langle f^{-1} \rangle \mathcal{Y} \cup^{\mathfrak{F}} \langle g^{-1} \rangle \mathcal{Y}$ for every $\mathcal{X}, \mathcal{Y} \in \mathfrak{F}$. Then

$$\begin{aligned}
\mathcal{Y} \cap^{\mathfrak{F}} \alpha \mathcal{X} \neq \emptyset &\Leftrightarrow \mathcal{Y} \cap^{\mathfrak{F}} \langle f \rangle \mathcal{X} \neq \emptyset \vee \mathcal{Y} \cap^{\mathfrak{F}} \langle g \rangle \mathcal{X} \neq \emptyset \\
&\Leftrightarrow \mathcal{X} \cap^{\mathfrak{F}} \langle f^{-1} \rangle \mathcal{Y} \neq \emptyset \vee \mathcal{X} \cap^{\mathfrak{F}} \langle g^{-1} \rangle \mathcal{Y} \neq \emptyset \\
&\Leftrightarrow \mathcal{X} \cap^{\mathfrak{F}} \beta \mathcal{Y} \neq \emptyset.
\end{aligned}$$

So $h = (\alpha; \beta)$ is a funcoid. Obviously $h \supseteq f$ and $h \supseteq g$. If $p \supseteq f$ and $p \supseteq g$ for some funcoid p then $\langle p\rangle\mathcal{X} \supseteq \langle f\rangle\mathcal{X} \cup^{\mathfrak{F}} \langle g\rangle\mathcal{X} = \langle h\rangle\mathcal{X}$ that is $p \supseteq h$. So $f \cup^{\mathsf{FCD}} g = h$.

2. $\mathcal{X}\left[f \cup^{\mathsf{FCD}} g\right]\mathcal{Y} \Leftrightarrow \mathcal{Y} \cap^{\mathfrak{F}} \left\langle f \cup^{\mathsf{FCD}} g\right\rangle\mathcal{X} \neq \emptyset \Leftrightarrow \mathcal{Y} \cap^{\mathfrak{F}} (\langle f\rangle\mathcal{X} \cup^{\mathfrak{F}} \langle g\rangle\mathcal{X}) \neq \emptyset \Leftrightarrow \mathcal{Y} \cap^{\mathfrak{F}} \langle f\rangle\mathcal{X} \neq \emptyset \vee \mathcal{Y} \cap^{\mathfrak{F}} \langle g\rangle\mathcal{X} \neq \emptyset \Leftrightarrow \mathcal{X}[f]\mathcal{Y} \vee \mathcal{X}[g]\mathcal{Y}$ for every $\mathcal{X}, \mathcal{Y} \in \mathfrak{F}$. □

3.5 More on composition of funcoids

Proposition 47. $[g \circ f] = [g] \circ \langle f\rangle = \left\langle g^{-1}\right\rangle^{-1} \circ [f]$ for $f, g \in \mathsf{FCD}$.

Proof. $\mathcal{X}[g \circ f]\mathcal{Y} \Leftrightarrow \mathcal{Y} \cap^{\mathfrak{F}} \langle g \circ f\rangle\mathcal{X} \neq \emptyset \Leftrightarrow \mathcal{Y} \cap^{\mathfrak{F}} \langle g\rangle\langle f\rangle\mathcal{X} \neq \emptyset \Leftrightarrow \langle f\rangle\mathcal{X}[g]\mathcal{Y} \Leftrightarrow \mathcal{X}([g] \circ \langle f\rangle)\mathcal{Y}$ for every $\mathcal{X}, \mathcal{Y} \in \mathfrak{F}$. $[g \circ f] = \left[(f^{-1} \circ g^{-1})^{-1}\right] = \left[f^{-1} \circ g^{-1}\right]^{-1} = (\left[f^{-1}\right] \circ \left\langle g^{-1}\right\rangle)^{-1} = \left\langle g^{-1}\right\rangle^{-1} \circ [f]$. □

The following theorem is a variant for funcoids of the statement (which defines compositions of relations) that $x(g \circ f)z \Leftrightarrow \exists y(x f y \wedge y g z)$ for every x and z and every binary relations f and g.

Theorem 48. For every $\mathcal{X}, \mathcal{Z} \in \mathfrak{F}$ and $f, g \in \mathsf{FCD}$

$$\mathcal{X}[g \circ f]\mathcal{Z} \Leftrightarrow \exists y \in \operatorname{atoms}^{\mathfrak{F}}\mho: (\mathcal{X}[f]y \wedge y[g]\mathcal{Z}).$$

Proof.

$$\begin{aligned}
\exists y \in \operatorname{atoms}^{\mathfrak{F}}\mho: (\mathcal{X}[f]y \wedge y[g]\mathcal{Z}) &\Leftrightarrow \exists y \in \operatorname{atoms}^{\mathfrak{F}}\mho: (\mathcal{Z} \cap^{\mathfrak{F}} \langle g\rangle y \neq \emptyset \wedge y \cap^{\mathfrak{F}} \langle f\rangle\mathcal{X} \neq \emptyset)\\
&\Leftrightarrow \exists y \in \operatorname{atoms}^{\mathfrak{F}}\mho: (\mathcal{Z} \cap^{\mathfrak{F}} \langle g\rangle y \neq \emptyset \wedge y \subseteq \langle f\rangle\mathcal{X})\\
&\Rightarrow \mathcal{Z} \cap^{\mathfrak{F}} \langle g\rangle\langle f\rangle\mathcal{X} \neq \emptyset\\
&\Leftrightarrow \mathcal{X}[g \circ f]\mathcal{Z}.
\end{aligned}$$

Reversely, if $\mathcal{X}[g \circ f]\mathcal{Z}$ then $\langle f\rangle\mathcal{X}[g]\mathcal{Z}$, consequently exists $y \in \operatorname{atoms}^{\mathfrak{F}}\langle f\rangle\mathcal{X}$ such that $y[g]\mathcal{Z}$; we have $\mathcal{X}[f]y$. □

Theorem 49. If f, g, h are funcoids then

1. $f \circ (g \cup^{\mathsf{FCD}} h) = f \circ g \cup^{\mathsf{FCD}} f \circ h$;
2. $(g \cup^{\mathsf{FCD}} h) \circ f = g \circ f \cup^{\mathsf{FCD}} h \circ f$.

Proof. I will prove only the first equality because the other is analogous.

For every $\mathcal{X}, \mathcal{Z} \in \mathfrak{F}$

$$\begin{aligned}
\mathcal{X}\left[f \circ (g \cup^{\mathsf{FCD}} h)\right]\mathcal{Z} &\Leftrightarrow \exists y \in \operatorname{atoms}^{\mathfrak{F}}\mho: (\mathcal{X}\left[g \cup^{\mathsf{FCD}} h\right]y \wedge y[f]\mathcal{Z})\\
&\Leftrightarrow \exists y \in \operatorname{atoms}^{\mathfrak{F}}\mho: ((\mathcal{X}[g]y \vee \mathcal{X}[h]y) \wedge y[f]\mathcal{Z})\\
&\Leftrightarrow \exists y \in \operatorname{atoms}^{\mathfrak{F}}\mho: (\mathcal{X}[g]y \wedge y[f]\mathcal{Z} \vee \mathcal{X}[h]y \wedge y[f]\mathcal{Z})\\
&\Leftrightarrow \exists y \in \operatorname{atoms}^{\mathfrak{F}}\mho: (\mathcal{X}[g]y \wedge y[f]\mathcal{Z}) \vee \exists y \in \operatorname{atoms}^{\mathfrak{F}}\mho: (\mathcal{X}[h]y \wedge y[f]\mathcal{Z})\\
&\Leftrightarrow \mathcal{X}[f \circ g]\mathcal{Z} \vee \mathcal{X}[f \circ h]\mathcal{Z}\\
&\Leftrightarrow \mathcal{X}\left[f \circ g \cup^{\mathsf{FCD}} f \circ h\right]\mathcal{Z}.
\end{aligned}$$

□

3.6 Domain and range of a funcoid

Definition 50. Let $\mathcal{A} \in \mathfrak{F}$. The *identity funcoid* $I^{\mathsf{FCD}}_{\mathcal{A}} = (\mathcal{A} \cap^{\mathfrak{F}}; \mathcal{A} \cap^{\mathfrak{F}})$.

Proposition 51. The identity funcoid is a funcoid.

Proof. We need to prove that $(\mathcal{A} \cap^{\mathfrak{F}} \mathcal{X}) \cap^{\mathfrak{F}} \mathcal{Y} \neq \emptyset \Leftrightarrow (\mathcal{A} \cap^{\mathfrak{F}} \mathcal{Y}) \cap^{\mathfrak{F}} \mathcal{X} \neq \emptyset$ what is obvious. □

Obvious 52. $(I_{\mathcal{A}}^{\mathsf{FCD}})^{-1} = I_{\mathcal{A}}^{\mathsf{FCD}}$.

Obvious 53. $\mathcal{X}[I_{\mathcal{A}}^{\mathsf{FCD}}]\mathcal{Y} \Leftrightarrow \mathcal{A} \cap^{\mathfrak{F}} \mathcal{X} \cap^{\mathfrak{F}} \mathcal{Y} \neq \emptyset$ for any $\mathcal{X}, \mathcal{Y} \in \mathfrak{F}$.

Definition 54. I will define *restricting* of a funcoid f to a filter object $\mathcal{A}$ by the formula $f|_{\mathcal{A}}^{\mathsf{FCD}} \stackrel{\text{def}}{=} f \circ I_{\mathcal{A}}^{\mathsf{FCD}}$.

Definition 55. *Image* of a funcoid f will be defined by the formula $\operatorname{im} f = \langle f \rangle \mho$.

Domain of a funcoid f is defined by the formula $\operatorname{dom} f = \operatorname{im} f^{-1}$.

Proposition 56. $\langle f \rangle \mathcal{X} = \langle f \rangle (\mathcal{X} \cap^{\mathfrak{F}} \operatorname{dom} f)$ for every $f \in \mathsf{FCD}$, $\mathcal{X} \in \mathfrak{F}$.

Proof. For every filter object $\mathcal{Y}$ we have $\mathcal{Y} \cap^{\mathfrak{F}} \langle f \rangle (\mathcal{X} \cap^{\mathfrak{F}} \operatorname{dom} f) \neq \emptyset \Leftrightarrow \mathcal{X} \cap^{\mathfrak{F}} \operatorname{dom} f \cap^{\mathfrak{F}} \langle f^{-1} \rangle \mathcal{Y} \neq \emptyset \Leftrightarrow \mathcal{X} \cap^{\mathfrak{F}} \operatorname{im} f^{-1} \cap^{\mathfrak{F}} \langle f^{-1} \rangle \mathcal{Y} \neq \emptyset \Leftrightarrow \mathcal{X} \cap^{\mathfrak{F}} \langle f^{-1} \rangle \mathcal{Y} \neq \emptyset \Leftrightarrow \mathcal{Y} \cap^{\mathfrak{F}} \langle f \rangle \mathcal{X} \neq \emptyset$. Thus $\langle f \rangle \mathcal{X} = \langle f \rangle (\mathcal{X} \cap^{\mathfrak{F}} \operatorname{dom} f)$ because the lattice of filter objects is separable. □

Proposition 57. $\mathcal{X} \cap^{\mathfrak{F}} \operatorname{dom} f \neq \emptyset \Leftrightarrow \langle f \rangle \mathcal{X} \neq \emptyset$ for every $f \in \mathsf{FCD}$, $\mathcal{X} \in \mathfrak{F}$.

Proof. $\mathcal{X} \cap^{\mathfrak{F}} \operatorname{dom} f \neq \emptyset \Leftrightarrow \mathcal{X} \cap^{\mathfrak{F}} \langle f^{-1} \rangle \mho \neq \emptyset \Leftrightarrow \mho \cap^{\mathfrak{F}} \langle f \rangle \mathcal{X} \neq \emptyset \Leftrightarrow \langle f \rangle \mathcal{X} \neq \emptyset$. □

Corollary 58. $\operatorname{dom} f = \bigcup^{\mathfrak{F}} \{ a \in \operatorname{atoms}^{\mathfrak{F}} \mho \mid \langle f \rangle a \neq \emptyset \}$.

Proof. This follows from the fact that $\mathfrak{F}$ is an atomistic lattice. □

3.7 Category of funcoids

I will define the category FCD of funcoids:

- The set of objects is $\mathfrak{F}$.
- The set of morphisms from a filter object $\mathcal{A}$ to a filter object $\mathcal{B}$ is the set of triples $(f; \mathcal{A}; \mathcal{B})$ where f is a funcoid such that $\operatorname{dom} f \subseteq \mathcal{A}$, $\operatorname{im} f \subseteq \mathcal{B}$.
- Composition of morphisms is defined in the natural way.
- Identity morphism of a filter object $\mathcal{A}$ is $(I_{\mathcal{A}}^{\mathsf{FCD}}; \mathcal{A}; \mathcal{A})$.

To prove that it is really a category is trivial.

3.8 Specifying funcoids by functions or relations on atomic filter objects

Theorem 59. For every funcoid f and filter objects $\mathcal{X}$ and $\mathcal{Y}$

1. $\langle f \rangle \mathcal{X} = \bigcup^{\mathfrak{F}} \langle \langle f \rangle \rangle \operatorname{atoms}^{\mathfrak{F}} \mathcal{X}$;
2. $\mathcal{X}[f]\mathcal{Y} \Leftrightarrow \exists x \in \operatorname{atoms}^{\mathfrak{F}} \mathcal{X}, y \in \operatorname{atoms}^{\mathfrak{F}} \mathcal{Y} : x[f]y$.

Proof. 1.

$$\begin{aligned}
\mathcal{Y} \cap^{\mathfrak{F}} \langle f \rangle \mathcal{X} \neq \emptyset &\Leftrightarrow \mathcal{X} \cap^{\mathfrak{F}} \langle f^{-1} \rangle \mathcal{Y} \neq \emptyset \\
&\Leftrightarrow \exists x \in \operatorname{atoms}^{\mathfrak{F}} \mathcal{X} : x \cap^{\mathfrak{F}} \langle f^{-1} \rangle \mathcal{Y} \neq \emptyset \\
&\Leftrightarrow \exists x \in \operatorname{atoms}^{\mathfrak{F}} \mathcal{X} : \mathcal{Y} \cap^{\mathfrak{F}} \langle f \rangle x \neq \emptyset.
\end{aligned}$$

$\partial \langle f \rangle \mathcal{X} = \bigcup \langle \partial \rangle \langle \langle f \rangle \rangle \operatorname{atoms}^{\mathfrak{F}} \mathcal{X} = \partial \bigcup^{\mathfrak{F}} \langle \langle f \rangle \rangle \operatorname{atoms}^{\mathfrak{F}} \mathcal{X}$.

2. If $\mathcal{X}[f]\mathcal{Y}$, then $\mathcal{Y} \cap^{\mathfrak{F}} \langle f \rangle \mathcal{X} \neq \emptyset$, consequently exists $y \in \operatorname{atoms}^{\mathfrak{F}} \mathcal{Y}$ such that $y \cap^{\mathfrak{F}} \langle f \rangle \mathcal{X} \neq \emptyset$, $\mathcal{X}[f]y$. Repeating this second time we get that there exist $x \in \operatorname{atoms}^{\mathfrak{F}} \mathcal{X}$ such that $x[f]y$. From this follows

$$\exists x \in \operatorname{atoms}^{\mathfrak{F}} \mathcal{X}, y \in \operatorname{atoms}^{\mathfrak{F}} \mathcal{Y} : x[f]y.$$

The reverse is obvious. □

Theorem 60.

1. A function $\alpha \in \mathfrak{F}^{\operatorname{atoms}^{\mathfrak{F}} \mho}$ such that (for every $a \in \operatorname{atoms}^{\mathfrak{F}} \mho$)

$$\alpha a \subseteq \bigcap{}^{\mathfrak{F}} \left\langle \bigcup{}^{\mathfrak{F}} \circ \langle \alpha \rangle \circ \operatorname{atoms}^{\mathfrak{F}} \right\rangle \operatorname{up} a \quad (5)$$

can be continued to the function $\langle f \rangle$ for a unique funcoid f;

$$\langle f \rangle \mathcal{X} = \bigcup{}^{\mathfrak{F}} \langle \alpha \rangle \operatorname{atoms}^{\mathfrak{F}} \mathcal{X} \quad (6)$$

for every filter object $\mathcal{X}$.

2. A relation $\delta \in \mathscr{P}(\operatorname{atoms}^{\mathfrak{F}} \mho)^2$ such that (for every $a, b \in \operatorname{atoms}^{\mathfrak{F}} \mho$)

$$\forall X \in \operatorname{up} a, Y \in \operatorname{up} b \, \exists x \in \operatorname{atoms}^{\mathfrak{F}} X, y \in \operatorname{atoms}^{\mathfrak{F}} Y : x \, \delta \, y \Rightarrow a \, \delta \, b \quad (7)$$

can be continued to the relation $[f]$ for a unique funcoid f;

$$\mathcal{X} [f] \mathcal{Y} \Leftrightarrow \exists x \in \operatorname{atoms}^{\mathfrak{F}} \mathcal{X}, y \in \operatorname{atoms}^{\mathfrak{F}} \mathcal{Y} : x \, \delta \, y \quad (8)$$

for every filter objects $\mathcal{X}$, $\mathcal{Y}$.

Proof. Existence of no more than one such funcoids and formulas (6) and (8) follow from the previous theorem.

1. Consider the function $\alpha' \in \mathfrak{F}^{\mho}$ defined by the formula (for every $X \in \mathscr{P}\mho$)

$$\alpha' X = \bigcup{}^{\mathfrak{F}} \langle \alpha \rangle \operatorname{atoms}^{\mathfrak{F}} X.$$

Obviously $\alpha' \emptyset = \emptyset$. For every $I, J \in \mathscr{P}\mho$

$$\begin{aligned}
\alpha'(I \cup J) &= \bigcup{}^{\mathfrak{F}} \langle \alpha' \rangle \operatorname{atoms}^{\mathfrak{F}} (I \cup J) \\
&= \bigcup{}^{\mathfrak{F}} \langle \alpha' \rangle (\operatorname{atoms}^{\mathfrak{F}} I \cup \operatorname{atoms}^{\mathfrak{F}} J) \\
&= \bigcup{}^{\mathfrak{F}} \left(\langle \alpha' \rangle \operatorname{atoms}^{\mathfrak{F}} I \cup \langle \alpha' \rangle \operatorname{atoms}^{\mathfrak{F}} J \right) \\
&= \bigcup{}^{\mathfrak{F}} \langle \alpha' \rangle \operatorname{atoms}^{\mathfrak{F}} I \cup^{\mathfrak{F}} \bigcup{}^{\mathfrak{F}} \langle \alpha' \rangle \operatorname{atoms}^{\mathfrak{F}} J. \\
&= \alpha' I \cup^{\mathfrak{F}} \alpha' J.
\end{aligned}$$

Let continue α' till a funcoid f (by the theorem 25): $\langle f \rangle \mathcal{X} = \bigcap{}^{\mathfrak{F}} \langle \alpha' \rangle \operatorname{up} \mathcal{X}$.

Let's prove the reverse of (5):

$$\begin{aligned}
\bigcap{}^{\mathfrak{F}} \left\langle \bigcup{}^{\mathfrak{F}} \circ \langle \alpha \rangle \circ \operatorname{atoms}^{\mathfrak{F}} \right\rangle \operatorname{up} a &= \bigcap{}^{\mathfrak{F}} \left\langle \bigcup{}^{\mathfrak{F}} \circ \langle \alpha \rangle \right\rangle \left\langle \operatorname{atoms}^{\mathfrak{F}} \right\rangle \operatorname{up} a \\
&\subseteq \bigcap{}^{\mathfrak{F}} \left\langle \bigcup{}^{\mathfrak{F}} \circ \langle \alpha \rangle \right\rangle \{\{a\}\} \\
&= \bigcap{}^{\mathfrak{F}} \left\{ \left(\bigcup{}^{\mathfrak{F}} \circ \langle \alpha \rangle \right) \{a\} \right\} \\
&= \bigcap{}^{\mathfrak{F}} \left\{ \bigcup{}^{\mathfrak{F}} \langle \alpha \rangle \{a\} \right\} \\
&= \bigcap{}^{\mathfrak{F}} \left\{ \bigcup{}^{\mathfrak{F}} \{\alpha a\} \right\} = \bigcap{}^{\mathfrak{F}} \{\alpha a\} = \alpha a.
\end{aligned}$$

Finally,

$$\alpha a = \bigcap{}^{\mathfrak{F}} \left\langle \bigcup{}^{\mathfrak{F}} \circ \langle \alpha \rangle \circ \operatorname{atoms}^{\mathfrak{F}} \right\rangle \operatorname{up} a = \bigcap{}^{\mathfrak{F}} \langle \alpha' \rangle \operatorname{up} a = \langle f \rangle a,$$

so $\langle f \rangle$ is a continuation of α.

2. Consider the relation $\delta' \in \mathscr{P}(\mathscr{P}\mho)^2$ defined by the formula (for every $X, Y \in \mathscr{P}\mho$)

$$X \, \delta' \, Y \Leftrightarrow \exists x \in \operatorname{atoms}^{\mathfrak{F}} X, y \in \operatorname{atoms}^{\mathfrak{F}} Y : x \, \delta \, y.$$

Obviously $\neg(X \, \delta' \, \emptyset)$ and $\neg(\emptyset \, \delta' \, Y)$.

$$\begin{aligned}
(I \cup J) \, \delta' \, Y &\Leftrightarrow \exists x \in \operatorname{atoms}^{\mathfrak{F}} (I \cup J), y \in \operatorname{atoms}^{\mathfrak{F}} Y : x \, \delta \, y \\
&\Leftrightarrow \exists x \in \operatorname{atoms}^{\mathfrak{F}} I \cup \operatorname{atoms}^{\mathfrak{F}} J, y \in \operatorname{atoms}^{\mathfrak{F}} Y : x \, \delta \, y \\
&\Leftrightarrow \exists x \in \operatorname{atoms}^{\mathfrak{F}} I, y \in \operatorname{atoms}^{\mathfrak{F}} Y : x \, \delta \, y \vee \exists x \in \operatorname{atoms}^{\mathfrak{F}} J, y \in \operatorname{atoms}^{\mathfrak{F}} Y : x \, \delta \, y \\
&\Leftrightarrow I \, \delta' \, Y \vee J \, \delta' \, Y;
\end{aligned}$$

analogously $X\,\delta'\,(I\cup J)\Leftrightarrow X\,\delta'\,I\vee X\,\delta'\,J$. Let's continue δ' till a funcoid f (by the theorem 25):

$$\mathcal{X}[f]\mathcal{Y}\Leftrightarrow\forall X\in\operatorname{up}\mathcal{X},Y\in\operatorname{up}\mathcal{Y}\colon X\,\delta'\,Y$$

The reverse of (7) implication is trivial, so

$$\forall X\in\operatorname{up}a,Y\in\operatorname{up}b\,\exists x\in\operatorname{atoms}^{\mathfrak{F}}X,y\in\operatorname{atoms}^{\mathfrak{F}}Y\colon x\,\delta\,y\Leftrightarrow a\,\delta\,b.$$

$\forall X\in\operatorname{up}a,Y\in\operatorname{up}b\,\exists x\in\operatorname{atoms}^{\mathfrak{F}}X,y\in\operatorname{atoms}^{\mathfrak{F}}Y\colon x\,\delta\,y\Leftrightarrow\forall X\in\operatorname{up}a,Y\in\operatorname{up}b\colon X\,\delta'\,Y\Leftrightarrow a[f]b$.

So $a\,\delta\,b\Leftrightarrow a[f]b$, that is $[f]$ is a continuation of δ. □

One of uses of the previous theorem is the proof of the following theorem:

Theorem 61. If R is a set of funcoids, $x,y\in\operatorname{atoms}^{\mathfrak{F}}\mho$, then

1. $\langle\bigcap^{\mathsf{FCD}}R\rangle x=\bigcap^{\mathfrak{F}}\{\langle f\rangle x\mid f\in R\}$;
2. $x\left[\bigcap^{\mathsf{FCD}}R\right]y\Leftrightarrow\forall f\in R\colon x[f]y$.

Proof. 2. Let denote $x\,\delta\,y\Leftrightarrow\forall f\in R\colon x[f]y$.

$$\begin{gathered}\forall X\in\operatorname{up}a,Y\in\operatorname{up}b\,\exists x\in\operatorname{atoms}^{\mathfrak{F}}X,y\in\operatorname{atoms}^{\mathfrak{F}}Y\colon x\,\delta\,y\Leftrightarrow\\ \forall f\in R,X\in\operatorname{up}a,Y\in\operatorname{up}b\,\exists x\in\operatorname{atoms}^{\mathfrak{F}}X,y\in\operatorname{atoms}^{\mathfrak{F}}Y\colon x[f]y\Rightarrow\\ \forall f\in R,X\in\operatorname{up}a,Y\in\operatorname{up}b\colon X[f]Y\Rightarrow\\ \forall f\in R\colon a[f]b\Leftrightarrow\\ a\,\delta\,b.\end{gathered}$$

So, by the theorem 60, δ can be continued till $[p]$ for some funcoid p.

For every funcoid q such that $\forall f\in R\colon q\subseteq f$ we have $x[q]y\Rightarrow\forall f\in R\colon x[f]y\Leftrightarrow x\,\delta\,y\Leftrightarrow x[p]y$, so $q\subseteq p$. Consequently $p=\bigcap^{\mathsf{FCD}}R$.

From this $x\left[\bigcap^{\mathsf{FCD}}R\right]y\Leftrightarrow\forall f\in R\colon x[f]y$.

1. From the former $y\in\operatorname{atoms}^{\mathfrak{F}}\langle\bigcap^{\mathsf{FCD}}R\rangle x\Leftrightarrow y\cap^{\mathfrak{F}}\langle\bigcap^{\mathsf{FCD}}R\rangle x\neq\emptyset\Leftrightarrow\forall f\in R\colon y\cap^{\mathfrak{F}}\langle f\rangle x\neq\emptyset\Leftrightarrow y\in\bigcap\langle\operatorname{atoms}^{\mathfrak{F}}\rangle\{\langle f\rangle x\mid f\in R\}\Leftrightarrow y\in\operatorname{atoms}^{\mathfrak{F}}\bigcap^{\mathfrak{F}}\{\langle f\rangle x\mid f\in R\}$ for every $y\in\operatorname{atoms}^{\mathfrak{F}}\mho$. From this follows $\langle\bigcap^{\mathsf{FCD}}R\rangle x=\bigcap^{\mathfrak{F}}\{\langle f\rangle x\mid f\in R\}$. □

3.9 Direct product of filter objects

A generalization of direct (Cartesian) product of two sets is direct product of two filter objects as defined in the theory of funcoids:

Definition 62. *Direct product* of filter objects $\mathcal{A}$ and $\mathcal{B}$ is such a funcoid $\mathcal{A}\times^{\mathsf{FCD}}\mathcal{B}$ that

$$\mathcal{X}\left[\mathcal{A}\times^{\mathsf{FCD}}\mathcal{B}\right]\mathcal{Y}\Leftrightarrow\mathcal{X}\cap^{\mathfrak{F}}\mathcal{A}\neq\emptyset\wedge\mathcal{Y}\cap^{\mathfrak{F}}\mathcal{B}\neq\emptyset.$$

Proposition 63. $\mathcal{A}\times^{\mathsf{FCD}}\mathcal{B}$ is really a funcoid and

$$\langle\mathcal{A}\times^{\mathsf{FCD}}\mathcal{B}\rangle\mathcal{X}=\begin{cases}\mathcal{B} & \text{if }\mathcal{X}\cap^{\mathfrak{F}}\mathcal{A}\neq\emptyset;\\ \emptyset & \text{if }\mathcal{X}\cap^{\mathfrak{F}}\mathcal{A}=\emptyset.\end{cases}$$

Proof. Obvious. □

Obvious 64. $A\times B=A\times^{\mathsf{FCD}}B$ for sets A and B.

Proposition 65. $f\subseteq\mathcal{A}\times^{\mathsf{FCD}}\mathcal{B}\Leftrightarrow\operatorname{dom}f\subseteq\mathcal{A}\wedge\operatorname{im}f\subseteq\mathcal{B}$ for every $f\in\mathsf{FCD}$ and $\mathcal{A},\mathcal{B}\in\mathfrak{F}$.

Proof. If $f\subseteq\mathcal{A}\times^{\mathsf{FCD}}\mathcal{B}$ then $\operatorname{dom}f\subseteq\operatorname{dom}(\mathcal{A}\times^{\mathsf{FCD}}\mathcal{B})\subseteq\mathcal{A}$, $\operatorname{im}f\subseteq\operatorname{im}(\mathcal{A}\times^{\mathsf{FCD}}\mathcal{B})\subseteq\mathcal{B}$. If $\operatorname{dom}f\subseteq\mathcal{A}\wedge\operatorname{im}f\subseteq\mathcal{B}$ then

$$\forall\mathcal{X},\mathcal{Y}\in\mathfrak{F}\colon(\mathcal{X}[f]\mathcal{Y}\Rightarrow\mathcal{X}\cap^{\mathfrak{F}}\mathcal{A}\neq\emptyset\wedge\mathcal{Y}\cap^{\mathfrak{F}}\mathcal{B}\neq\emptyset);$$

consequently $f\subseteq\mathcal{A}\times^{\mathsf{FCD}}\mathcal{B}$. □

The following theorem gives a formula for calculating an important particular case of intersection on the lattice of funcoids:

Theorem 66. $f \cap^{\mathsf{FCD}} (\mathcal{A} \times^{\mathsf{FCD}} \mathcal{B}) = I_{\mathcal{B}}^{\mathsf{FCD}} \circ f \circ I_{\mathcal{A}}^{\mathsf{FCD}}$ for every $f \in \mathsf{FCD}$ and $\mathcal{A}, \mathcal{B} \in \mathfrak{F}$.

Proof. $h \overset{\text{def}}{=} I_{\mathcal{B}}^{\mathsf{FCD}} \circ f \circ I_{\mathcal{A}}^{\mathsf{FCD}}$. For every $\mathcal{X} \in \mathfrak{F}$

$$\langle h \rangle \mathcal{X} = \left\langle I_{\mathcal{B}}^{\mathsf{FCD}} \right\rangle \langle f \rangle \left\langle I_{\mathcal{A}}^{\mathsf{FCD}} \right\rangle \mathcal{X} = \mathcal{B} \cap \langle f \rangle (\mathcal{A} \cap \mathcal{X}).$$

From this, as easy to show, $h \subseteq f$ and $h \subseteq \mathcal{A} \times \mathcal{B}$. If $g \subseteq f \wedge g \subseteq \mathcal{A} \times^{\mathsf{FCD}} \mathcal{B}$ for a funcoid g then $\operatorname{dom} g \subseteq \mathcal{A}$, $\operatorname{im} g \subseteq \mathcal{B}$,

$$\langle g \rangle \mathcal{X} = \mathcal{B} \cap^{\mathfrak{F}} \langle g \rangle (\mathcal{A} \cap^{\mathfrak{F}} \mathcal{X}) \subseteq \mathcal{B} \cap^{\mathfrak{F}} \langle f \rangle (\mathcal{A} \cap^{\mathfrak{F}} \mathcal{X}) = \left\langle I_{\mathcal{B}}^{\mathsf{FCD}} \right\rangle \langle f \rangle \left\langle I_{\mathcal{A}}^{\mathsf{FCD}} \right\rangle \mathcal{X} = \langle h \rangle \mathcal{X},$$

$g \subseteq h$. So $h = f \cap^{\mathsf{FCD}} (\mathcal{A} \times^{\mathsf{FCD}} \mathcal{B})$. □

Corollary 67. $f|_{\mathcal{A}}^{\mathsf{FCD}} = f \cap (\mathcal{A} \times^{\mathsf{FCD}} \mho)$ for every $f \in \mathsf{FCD}$ and $\mathcal{A} \in \mathfrak{F}$.

Proof. $f \cap^{\mathsf{FCD}} (\mathcal{A} \times^{\mathsf{FCD}} \mho) = I_{\mho}^{\mathsf{FCD}} \circ f \circ I_{\mathcal{A}}^{\mathsf{FCD}} = f \circ I_{\mathcal{A}}^{\mathsf{FCD}} = f|_{\mathcal{A}}^{\mathsf{FCD}}$. □

Corollary 68. $f \cap^{\mathsf{FCD}} (\mathcal{A} \times^{\mathsf{FCD}} \mathcal{B}) \neq \emptyset \Leftrightarrow \mathcal{A} [f] \mathcal{B}$ for every $f \in \mathsf{FCD}$, $\mathcal{A}, \mathcal{B} \in \mathfrak{F}$.

Proof. $f \cap^{\mathsf{FCD}} (\mathcal{A} \times^{\mathsf{FCD}} \mathcal{B}) \neq \emptyset \Leftrightarrow \left\langle f \cap^{\mathsf{FCD}} (\mathcal{A} \times^{\mathsf{FCD}} \mathcal{B}) \right\rangle \mho \neq \emptyset \Leftrightarrow \left\langle I_{\mathcal{B}}^{\mathsf{FCD}} \circ f \circ I_{\mathcal{A}}^{\mathsf{FCD}} \right\rangle \mho \neq \emptyset \Leftrightarrow \left\langle I_{\mathcal{B}}^{\mathsf{FCD}} \right\rangle \langle f \rangle \left\langle I_{\mathcal{A}}^{\mathsf{FCD}} \right\rangle \mho \neq \emptyset \Leftrightarrow \mathcal{B} \cap^{\mathsf{FCD}} \langle f \rangle (\mathcal{A} \cap^{\mathfrak{F}} \mho) \neq \emptyset \Leftrightarrow \mathcal{B} \cap^{\mathfrak{F}} \langle f \rangle \mathcal{A} \neq \emptyset \Leftrightarrow \mathcal{A} [f] \mathcal{B}$. □

Corollary 69. The filtrator of funcoids is star-separable.

Proof. The set of direct products of sets is a separation subset of the lattice of funcoids. □

Theorem 70. If $S \in \mathscr{P} \mathfrak{F}^2$ then

$$\bigcap^{\mathsf{FCD}} \left\{ \mathcal{A} \times^{\mathsf{FCD}} \mathcal{B} \mid (\mathcal{A}; \mathcal{B}) \in S \right\} = \bigcap^{\mathfrak{F}} \operatorname{dom} S \times^{\mathsf{FCD}} \bigcap^{\mathfrak{F}} \operatorname{im} S.$$

Proof. If $x \in \operatorname{atoms}^{\mathfrak{F}} \mho$ then by the theorem 61

$$\left\langle \bigcap^{\mathsf{FCD}} \left\{ \mathcal{A} \times^{\mathsf{FCD}} \mathcal{B} \mid (\mathcal{A}; \mathcal{B}) \in S \right\} \right\rangle x = \bigcap^{\mathfrak{F}} \left\{ \left\langle \mathcal{A} \times^{\mathsf{FCD}} \mathcal{B} \right\rangle x \mid (\mathcal{A}; \mathcal{B}) \in S \right\}.$$

If $x \cap^{\mathfrak{F}} \bigcap^{\mathfrak{F}} \operatorname{dom} S \neq \emptyset$ then

$$\forall (\mathcal{A}; \mathcal{B}) \in S: (x \cap^{\mathfrak{F}} \mathcal{A} \neq \emptyset \wedge \left\langle \mathcal{A} \times^{\mathsf{FCD}} \mathcal{B} \right\rangle x = \mathcal{B});$$
$$\left\{ \left\langle \mathcal{A} \times^{\mathsf{FCD}} \mathcal{B} \right\rangle x \mid (\mathcal{A}; \mathcal{B}) \in S \right\} = \operatorname{im} S;$$

if $x \cap^{\mathfrak{F}} \bigcap^{\mathfrak{F}} \operatorname{dom} S = \emptyset$ then

$$\exists (\mathcal{A}; \mathcal{B}) \in S: (x \cap^{\mathfrak{F}} \mathcal{A} = \emptyset \wedge \left\langle \mathcal{A} \times^{\mathsf{FCD}} \mathcal{B} \right\rangle x = \emptyset);$$
$$\left\{ \left\langle \mathcal{A} \times^{\mathsf{FCD}} \mathcal{B} \right\rangle x \mid (\mathcal{A}; \mathcal{B}) \in S \right\} \ni \emptyset.$$

So

$$\left\langle \bigcap^{\mathsf{FCD}} \left\{ \mathcal{A} \times^{\mathsf{FCD}} \mathcal{B} \mid (\mathcal{A}; \mathcal{B}) \in S \right\} \right\rangle x = \begin{cases} \bigcap^{\mathfrak{F}} \operatorname{im} S & \text{if } x \cap^{\mathfrak{F}} \bigcap^{\mathfrak{F}} \operatorname{dom} S \neq \emptyset; \\ \emptyset & \text{if } x \cap^{\mathfrak{F}} \bigcap^{\mathfrak{F}} \operatorname{dom} S = \emptyset. \end{cases}$$

From this follows the statement of the theorem. □

Corollary 71. $(\mathcal{A}_0 \times^{\mathsf{FCD}} \mathcal{B}_0) \cap^{\mathsf{FCD}} (\mathcal{A}_1 \times^{\mathsf{FCD}} \mathcal{B}_1) = (\mathcal{A}_0 \cap^{\mathfrak{F}} \mathcal{A}_1) \times^{\mathsf{FCD}} (\mathcal{B}_0 \cap^{\mathfrak{F}} \mathcal{B}_1)$ for every $\mathcal{A}_0, \mathcal{A}_1, \mathcal{B}_0, \mathcal{B}_1 \in \mathfrak{F}$.

Proof. $(\mathcal{A}_0 \times^{\mathsf{FCD}} \mathcal{B}_0) \cap^{\mathsf{FCD}} (\mathcal{A}_1 \times^{\mathsf{FCD}} \mathcal{B}_1) = \bigcap^{\mathsf{FCD}} \left\{ \mathcal{A}_0 \times^{\mathsf{FCD}} \mathcal{B}_0, \mathcal{A}_1 \times^{\mathsf{FCD}} \mathcal{B}_1 \right\}$ what is by the last theorem equal to $(\mathcal{A}_0 \cap^{\mathfrak{F}} \mathcal{A}_1) \times^{\mathsf{FCD}} (\mathcal{B}_0 \cap^{\mathfrak{F}} \mathcal{B}_1)$. □

Theorem 72. If $\mathcal{A} \in \mathfrak{F}$ then $\mathcal{A} \times^{\mathsf{FCD}}$ is a complete homomorphism of the lattice $\mathfrak{F}$ to a complete sublattice of the lattice FCD, if also $\mathcal{A} \neq \emptyset$ then it is an isomorphism.

Proof. Let $S \in \mathscr{P}\mathfrak{F}$, $X \in \mathscr{P}\mho$, $x \in \operatorname{atoms}^{\mathfrak{F}}\mho$.

$$\begin{aligned}
\left\langle \bigcup^{\mathsf{FCD}} \langle \mathcal{A} \times^{\mathsf{FCD}} \rangle S \right\rangle X &= \bigcup^{\mathfrak{F}} \{ \langle \mathcal{A} \times^{\mathsf{FCD}} \mathcal{B} \rangle X \mid \mathcal{B} \in S \} \\
&= \begin{cases} \bigcup^{\mathfrak{F}} S & \text{if } X \cap^{\mathfrak{F}} \mathcal{A} \neq \emptyset \\ \emptyset & \text{if } X \cap^{\mathfrak{F}} \mathcal{A} = \emptyset \end{cases} \\
&= \left\langle \mathcal{A} \times^{\mathsf{FCD}} \bigcup^{\mathfrak{F}} S \right\rangle X; \\
\left\langle \bigcap^{\mathsf{FCD}} \langle \mathcal{A} \times^{\mathsf{FCD}} \rangle S \right\rangle x &= \bigcap^{\mathfrak{F}} \{ \langle \mathcal{A} \times^{\mathsf{FCD}} \mathcal{B} \rangle x \mid \mathcal{B} \in S \} \\
&= \begin{cases} \bigcap^{\mathfrak{F}} S & \text{if } x \cap^{\mathfrak{F}} \mathcal{A} \neq \emptyset \\ \emptyset & \text{if } x \cap^{\mathfrak{F}} \mathcal{A} = \emptyset \end{cases} \\
&= \left\langle \mathcal{A} \times^{\mathsf{FCD}} \bigcap^{\mathfrak{F}} S \right\rangle x.
\end{aligned}$$

If $\mathcal{A} \neq \emptyset$ then obviously the function $\mathcal{A} \times^{\mathsf{FCD}}$ is injective. □

The following proposition states that cutting a rectangle of atomic width from a funcoid always produces a rectangular (representable as a direct product of filter objects) funcoid (of atomic width).

Proposition 73. If a is an atomic filter object, $f \in \mathsf{FCD}$ then $f|_a^{\mathsf{FCD}} = a \times^{\mathsf{FCD}} \langle f \rangle a$.

Proof. Let $\mathcal{X} \in \mathfrak{F}$.

$$\mathcal{X} \cap^{\mathfrak{F}} a \neq \emptyset \Rightarrow \langle f|_a^{\mathsf{FCD}} \rangle \mathcal{X} = \langle f \rangle a, \quad \mathcal{X} \cap^{\mathfrak{F}} a = \emptyset \Rightarrow \langle f|_a^{\mathsf{FCD}} \rangle \mathcal{X} = \emptyset.$$ □

3.10 Atomic funcoids

Theorem 74. A funcoid is an atom of the lattice of funcoids iff it is direct product of two atomic filter objects.

Proof.

⇒. Let f is an atomic funcoid. Let's get elements $a \in \operatorname{atoms}^{\mathfrak{F}} \operatorname{dom} f$ and $b \in \operatorname{atoms}^{\mathfrak{F}} \langle f \rangle a$. Then for every $\mathcal{X} \in \mathfrak{F}$

$$\mathcal{X} \cap^{\mathfrak{F}} a = \emptyset \Rightarrow \langle a \times^{\mathsf{FCD}} b \rangle \mathcal{X} = \emptyset \subseteq \langle f \rangle \mathcal{X}, \quad \mathcal{X} \cap^{\mathfrak{F}} a \neq \emptyset \Rightarrow \langle a \times^{\mathsf{FCD}} b \rangle \mathcal{X} = b \subseteq \langle f \rangle \mathcal{X}.$$

So $a \times^{\mathsf{FCD}} b \subseteq f$; because f is atomic we have $f = a \times^{\mathsf{FCD}} b$.

⇐. Let $a, b \in \operatorname{atoms}^{\mathfrak{F}}\mho$, $f \in \mathsf{FCD}$. If $b \cap^{\mathfrak{F}} \langle f \rangle a = \emptyset$ then $\neg(a[f]b)$, $f \cap^{\mathsf{FCD}} (a \times^{\mathsf{FCD}} b) = \emptyset$; if $b \subseteq \langle f \rangle a$ then $\forall \mathcal{X} \in \mathfrak{F}$: $(\mathcal{X} \cap^{\mathfrak{F}} a \neq \emptyset \Rightarrow \langle f \rangle \mathcal{X} \supseteq b)$, $f \supseteq a \times^{\mathsf{FCD}} b$. Consequently $f \cap^{\mathsf{FCD}} (a \times^{\mathsf{FCD}} b) = \emptyset \vee f \supseteq a \times^{\mathsf{FCD}} b$; that is $a \times^{\mathsf{FCD}} b$ is an atomic filter object. □

Theorem 75. The lattice of funcoids is atomic.

Proof. Let f is a non-empty funcoid. Then $\operatorname{dom} f \neq \emptyset$, thus by the theorem 46 in [14] exists $a \in \operatorname{atoms}^{\mathfrak{F}} \operatorname{dom} f$. So $\langle f \rangle a \neq \emptyset$ thus exists $b \in \operatorname{atoms} \langle f \rangle a$. Finally the atomic funcoid $a \times^{\mathsf{FCD}} b \subseteq f$. □

Theorem 76. The lattice of funcoids is separable.

Proof. Let $f, g \in \mathsf{FCD}$, $f \subset g$. Then exists $a \in \operatorname{atoms}^{\mathfrak{F}}\mho$ such that $\langle f \rangle a \subset \langle g \rangle a$. So because the lattice $\mathfrak{F}$ is atomically separable then exists $b \in \operatorname{atoms}^{\mathfrak{F}}\mho$ such that $\langle f \rangle a \cap^{\mathfrak{F}} b = \emptyset$ and $b \subseteq \langle g \rangle a$. For every $x \in \operatorname{atoms}^{\mathfrak{F}}\mho$

$$\langle f \rangle a \cap^{\mathfrak{F}} \langle a \times^{\mathsf{FCD}} b \rangle a = \langle f \rangle a \cap^{\mathfrak{F}} b = \emptyset,$$
$$x \neq a \Rightarrow \langle f \rangle x \cap^{\mathfrak{F}} \langle a \times^{\mathsf{FCD}} b \rangle x = \langle f \rangle x \cap^{\mathfrak{F}} \emptyset = \emptyset$$

Thus $\langle f\rangle x\cap^{\mathfrak{F}}\langle a\times b\rangle x=\emptyset$ and consequently $f\cap^{\mathsf{FCD}}(a\times^{\mathsf{FCD}}b)=\emptyset$.

$$\left\langle a\times^{\mathsf{FCD}} b\right\rangle a=b\subseteq\langle g\rangle a,$$
$$x\neq a\Rightarrow\left\langle a\times^{\mathsf{FCD}} b\right\rangle x=\emptyset\subseteq\langle g\rangle a.$$

Thus $\left\langle a\times^{\mathsf{FCD}} b\right\rangle x\subseteq\langle g\rangle x$ and consequently $a\times^{\mathsf{FCD}}b\subseteq g$.

So the lattice of funcoids is separable by the theorem 19 in [14]. □

Corollary 77. The lattice of funcoids is:

1. separable;
2. atomically separable;
3. conforming to Wallman's disjunction property.

Proof. By the theorem 22 in [14]. □

Remark 78. For more ways to characterize (atomic) separability of the lattice of funcoids see [14], subsections "Separation subsets and full stars" and "Atomically separable lattices".

Corollary 79. The lattice of funcoids is an atomistic lattice.

Proof. Let f is a funcoid. Suppose contrary to the statement to be proved that $\bigcup^{\mathsf{FCD}}\mathrm{atoms}^{\mathsf{FCD}}f\subset f$. Then exists $a\in\mathrm{atoms}^{\mathsf{FCD}}f$ such that $a\cap^{\mathsf{FCD}}\bigcup^{\mathsf{FCD}}\mathrm{atoms}^{\mathsf{FCD}}f=\emptyset$ what is impossible. □

Proposition 80. $\mathrm{atoms}^{\mathsf{FCD}}(f\cup^{\mathsf{FCD}}g)=\mathrm{atoms}^{\mathsf{FCD}}f\cup\mathrm{atoms}^{\mathsf{FCD}}g$ for every funcoids f and g.

Proof. $(a\times^{\mathsf{FCD}}b)\cap^{\mathsf{FCD}}(f\cup^{\mathsf{FCD}}g)\neq\emptyset\Leftrightarrow a\left[f\cup^{\mathsf{FCD}}g\right]b\Leftrightarrow a[f]b\vee a[g]b\Leftrightarrow(a\times^{\mathsf{FCD}}b)\cap^{\mathsf{FCD}}f\neq\emptyset\vee(a\times^{\mathsf{FCD}}b)\cap^{\mathsf{FCD}}g\neq\emptyset$ for every atomic filter objects a and b. □

Corollary 81. For every $f,g,h\in\mathsf{FCD}$, $R\in\mathscr{P}\mathsf{FCD}$

1. $f\cap^{\mathsf{FCD}}(g\cup^{\mathsf{FCD}}h)=(f\cap^{\mathsf{FCD}}g)\cup^{\mathsf{FCD}}(f\cap^{\mathsf{FCD}}h)$;
2. $f\cup^{\mathsf{FCD}}\bigcap^{\mathsf{FCD}}R=\bigcap^{\mathsf{FCD}}\left\langle f\cup^{\mathsf{FCD}}\right\rangle R$.

Proof. We will take in account that the lattice of funcoids is an atomistic lattice. To be concise I will write atoms instead of $\mathrm{atoms}^{\mathsf{FCD}}$ and $\cap$ and $\cup$ instead of $\cap^{\mathsf{FCD}}$ and $\cup^{\mathsf{FCD}}$.

1. $\mathrm{atoms}(f\cap(g\cup h))=\mathrm{atoms}\,f\cap\mathrm{atoms}(g\cup h)=\mathrm{atoms}\,f\cap(\mathrm{atoms}\,g\cup\mathrm{atoms}\,h)=(\mathrm{atoms}\,f\cap\mathrm{atoms}\,g)\cup(\mathrm{atoms}\,f\cap\mathrm{atoms}\,h)=\mathrm{atoms}(f\cap g)\cup\mathrm{atoms}(f\cap h)=\mathrm{atoms}((f\cap g)\cup(f\cap h))$.
2. $\mathrm{atoms}\left(f\cup\bigcap^{\mathsf{FCD}}R\right)=\mathrm{atoms}\,f\cup\mathrm{atoms}\bigcap^{\mathsf{FCD}}R=\mathrm{atoms}\,f\cup\bigcap^{\mathsf{FCD}}\langle\mathrm{atoms}\rangle R=\bigcap^{\mathsf{FCD}}\langle(\mathrm{atoms}\,f)\cup\rangle\langle\mathrm{atoms}\rangle R=\bigcap^{\mathsf{FCD}}\langle\mathrm{atoms}\rangle\langle f\cup\rangle R=\mathrm{atoms}\bigcap^{\mathsf{FCD}}\langle f\cup\rangle R$. (Used the following equality.)

$$\begin{aligned}\langle(\mathrm{atoms}\,f)\cup\rangle\langle\mathrm{atoms}\rangle R &=\\ \{(\mathrm{atoms}\,f)\cup A\mid A\in\langle\mathrm{atoms}\rangle R\} &=\\ \{(\mathrm{atoms}\,f)\cup A\mid\exists C\in R\colon A=\mathrm{atoms}\,C\} &=\\ \{(\mathrm{atoms}\,f)\cup(\mathrm{atoms}\,C)\mid C\in R\} &=\\ \{\mathrm{atoms}(f\cup C)\mid C\in R\} &=\\ \{\mathrm{atoms}\,B\mid\exists C\in R\colon B=f\cup C\} &=\\ \{\mathrm{atoms}\,B\mid B\in\langle f\cup\rangle R\} &=\\ \langle\mathrm{atoms}\rangle\langle f\cup\rangle.\end{aligned}$$

□

Note that distributivity of the lattice of funcoids is proved through using atoms of this lattice. I have never seen such method of proving distributivity.

Corollary 82. The lattice of funcoids is co-brouwerian.

The next proposition is one more (among the theorem 48) generalization for funcoids of composition of relations.

Proposition 83. For every $f, g \in \mathsf{FCD}$

$\operatorname{atoms}^{\mathsf{FCD}}(g \circ f) = \left\{ x \times^{\mathsf{FCD}} z \mid x, z \in \operatorname{atoms}^{\mathfrak{F}}\mho, \exists y \in \operatorname{atoms}^{\mathfrak{F}}\mho : (x \times^{\mathsf{FCD}} y \in \operatorname{atoms}^{\mathsf{FCD}} f \wedge y \times^{\mathsf{FCD}} z \in \operatorname{atoms}^{\mathsf{FCD}} g) \right\}$.

Proof. $(x \times^{\mathsf{FCD}} z) \cap^{\mathsf{FCD}} (g \circ f) \neq \emptyset \Leftrightarrow x[g \circ f]z \Leftrightarrow \exists y \in \operatorname{atoms}^{\mathfrak{F}}\mho : (x[f]y \wedge y[g]z) \Leftrightarrow \exists y \in \operatorname{atoms}^{\mathfrak{F}}\mho : ((x \times^{\mathsf{FCD}} y) \cap^{\mathsf{FCD}} f \neq \emptyset \wedge (y \times^{\mathsf{FCD}} z) \cap^{\mathsf{FCD}} g \neq \emptyset)$ (were used the theorem 48). □

Conjecture 84. The set of discrete funcoids is the center of the lattice of funcoids.

3.11 Complete funcoids

Definition 85. I will call *co-complete* such a funcoid f that $\forall X \in \mathscr{P}\mho : \langle f \rangle X \in \mathscr{P}\mho$.

Remark 86. I will call *generalized closure* such a function $\alpha \in \mathscr{P}\mho^{\mathscr{P}\mho}$ that

1. $\alpha\emptyset = \emptyset$;
2. $\forall I, J \in \mathscr{P}\mho : \alpha(I \cup J) = \alpha I \cup \alpha J$.

Obvious 87. A funcoid f is co-complete iff $\langle f \rangle|_{\mathscr{P}\mho}$ is a generalized closure.

Remark 88. Thus funcoids can be considered as a generalization of generalized closures. A topological space in Kuratowski sense is the same as reflexive and transitive generalized closure. So topological spaces can be considered as a special case of funcoids.

Definition 89. I will call a *complete funcoid* a funcoid whose reverse is co-complete.

Theorem 90. The following conditions are equivalent for every funcoid f:

1. funcoid f is complete;
2. $\forall S \in \mathscr{P}\mathfrak{F}, J \in \mathscr{P}\mho : (\bigcup^{\mathfrak{F}} S[f]J \Leftrightarrow \exists \mathcal{I} \in S : \mathcal{I}[f]J)$;
3. $\forall S \in \mathscr{P}\mathscr{P}\mho, J \in \mathscr{P}\mho : (\bigcup S[f]J \Leftrightarrow \exists I \in S : I[f]J)$;
4. $\forall S \in \mathscr{P}\mathfrak{F} : \langle f \rangle \bigcup^{\mathfrak{F}} S = \bigcup^{\mathfrak{F}} \langle\langle f \rangle\rangle S$;
5. $\forall S \in \mathscr{P}\mathscr{P}\mho : \langle f \rangle \bigcup S = \bigcup^{\mathfrak{F}} \langle\langle f \rangle\rangle S$;
6. $\forall A \in \mathscr{P}\mho : \langle f \rangle A = \bigcup^{\mathfrak{F}} \{ \langle f \rangle \{a\} \mid a \in A \}$.

Proof.

(3) ⇒ (1). For every $S \in \mathscr{P}\mathscr{P}\mho$, $J \in \mathscr{P}\mho$

$$\bigcup S \cap^{\mathfrak{F}} \left\langle f^{-1} \right\rangle J \neq \emptyset \Leftrightarrow \exists I \in S : I \cap^{\mathfrak{F}} \left\langle f^{-1} \right\rangle J \neq \emptyset, \tag{9}$$

consequently by the theorem 52 in [14] we have $\left\langle f^{-1} \right\rangle J \in \mathscr{P}\mho$.

(1) ⇒ (2). For every $S \in \mathscr{P}\mathfrak{F}$, $J \in \mathscr{P}\mho$ we have $\left\langle f^{-1} \right\rangle J \in \mathscr{P}\mho$, consequently the formula (9) is true. From this follows (2).

(6) ⇒ (5). $\langle f \rangle \bigcup S = \bigcup^{\mathfrak{F}} \{ \langle f \rangle \{a\} \mid a \in \bigcup S \} = \bigcup^{\mathfrak{F}} \{ \bigcup^{\mathfrak{F}} \{ \langle f \rangle \{a\} \mid a \in A \} \mid A \in S \} = \bigcup^{\mathfrak{F}} \{ \langle f \rangle A \mid A \in S \} = \bigcup^{\mathfrak{F}} \langle\langle f \rangle\rangle S$.

(2) ⇒ (4). $J \cap^{\mathfrak{F}} \langle f \rangle \bigcup^{\mathfrak{F}} S \neq 0 \Leftrightarrow \bigcup^{\mathfrak{F}} S[f]J \Leftrightarrow \exists \mathcal{I} \in S : \mathcal{I}[f]J \Leftrightarrow \exists \mathcal{I} \in S : J \cap^{\mathfrak{F}} \langle f \rangle \mathcal{I} \neq \emptyset \Leftrightarrow J \cap^{\mathfrak{F}} \bigcup^{\mathfrak{F}} \langle\langle f \rangle\rangle S \neq \emptyset$ (used the theorem 53 in [14]).

(2) ⇒ (3), (4) ⇒ (5), (5) ⇒ (3), (5) ⇒ (6). Obvious. □

The following proposition shows that complete funcoids are a direct generalization of pre-topological spaces.

Proposition 91. To specify a complete funcoid f it is enough to specify $\langle f\rangle$ on one-element sets, values of $\langle f\rangle$ on one element sets can be specified arbitrarily.

Proof. From the above theorem is clear that knowing $\langle f\rangle$ on one-element sets $\langle f\rangle$ can be found on every set and then its value can be inferred for every filter objects.

Choosing arbitrarily the values of $\langle f\rangle$ on one-element sets we can define a complete funcoid the following way: $\langle f\rangle X \overset{\text{def}}{=} \bigcup^{\mathfrak{F}}\{\langle f\rangle\{\alpha\} \mid \alpha\in X\}$ for every $X\in\mathscr{P}\mho$. Obviously it is really a complete funcoid. □

Theorem 92. A funcoid is discrete iff it is both complete and co-complete.

Proof.

- **⇒.** Obvious.
- **⇐.** Let f is both a complete and co-complete funcoid. Consider the relation g defined by that $\langle g\rangle\{\alpha\} = \langle f\rangle\{\alpha\}$ (g is correctly defined because f is a generalized closure). Because f is a complete funcoid $f = g$. □

Theorem 93. If R is a set of (co-)complete funcoids then $\bigcup^{\mathsf{FCD}} R$ is a (co-)complete funcoid.

Proof. It is enough to prove only for co-complete funcoids. Let R is a set of co-complete funcoids. Then for every $X\in\mathscr{P}\mho$

$$\left\langle \bigcup^{\mathsf{FCD}} R\right\rangle X = \bigcup\;\{\langle f\rangle X \mid f\in R\}\in\mathscr{P}\mho$$

(used the theorem 45). □

Corollary 94. If R is a set of binary relations then $\bigcup^{\mathsf{FCD}} R = \bigcup\; R$.

Proof. From two last theorems. □

Theorem 95. The filtrator of funcoids is filtered.

Proof. It's enough to prove that every funcoid is representable as (infinite) meet (on the lattice of funcoids) of some set of discrete funcoids.

Let $f\in\mathsf{FCD}$, $A\in\mathscr{P}\mho$, $B\in\operatorname{up}\langle f\rangle A$, $g(A;B) \overset{\text{def}}{=} A\times^{\mathsf{FCD}} B\cup^{\mathsf{FCD}} \overline{A}\times^{\mathsf{FCD}}\mho$. For every $X\in\mathscr{P}\mho$

$$\langle g(A;B)\rangle X = \left\langle A\times^{\mathsf{FCD}} B\right\rangle X\cup\left\langle \overline{A}\times^{\mathsf{FCD}}\mho\right\rangle X = \left(\begin{cases}\emptyset & \text{if } X=\emptyset\\ B & \text{if } \emptyset\neq X\subseteq A\\ \mho & \text{if } X\nsubseteq A\end{cases}\right)\supseteq\langle f\rangle X;$$

so $g(A;B)\supseteq f$. For every $A\in\mathscr{P}\mho$

$$\bigcap^{\mathfrak{F}}\{\langle g(A;B)\rangle A \mid B\in\operatorname{up}\langle f\rangle A\} = \bigcap^{\mathfrak{F}}\{B \mid B\in\operatorname{up}\langle f\rangle A\} = \langle f\rangle A;$$

consequently

$$\bigcap^{\mathsf{FCD}}\{g(A;B) \mid A\in\mathscr{P}\mho, B\in\operatorname{up}\langle f\rangle A\} = f. \qquad \square$$

Conjecture 96. If f is a complete funcoid and R is a set of funcoids then $f\circ\bigcup^{\mathsf{FCD}} R = \bigcup^{\mathsf{FCD}}\langle f\circ\rangle R$.

This conjecture can be weakened:

Conjecture 97. If f is a discrete funcoid and R is a set of funcoids then $f\circ\bigcup^{\mathsf{FCD}} R = \bigcup^{\mathsf{FCD}}\langle f\circ\rangle R$.

I will denote $\mathrm{Compl}\mathsf{FCD}$ and $\mathrm{CoCompl}\mathsf{FCD}$ the sets of complete and co-complete funcoids correspondingly.

Obvious 98. $\mathrm{Compl}\mathsf{FCD}$ and $\mathrm{CoCompl}\mathsf{FCD}$ are closed regarding composition of funcoids.

Proposition 99. $\mathrm{Compl}\mathsf{FCD}$ and $\mathrm{CoCompl}\mathsf{FCD}$ (with induced order) are complete lattices.

Proof. Follows from the theorem 93. □

3.12 Completion of funcoids

Theorem 100. $\mathrm{Cor}\, f = \mathrm{Cor}'\, f$ for an element f of the filtrator of funcoids. (Core part is taken for the filtrator of funcoids.)

Proof. From the theorem 26 in [14] and the corollary 94 and theorem 95. □

Definition 101. *Completion* of a funcoid f is the complete funcoid $\mathrm{Compl}\, f$ defined by the formula $\langle \mathrm{Compl}\, f\rangle\{\alpha\} = \langle f\rangle\{\alpha\}$ for $\alpha \in \mho$.

Definition 102. *Co-completion* of a funcoid f is defined by the formula

$$\mathrm{CoCompl}\, f = (\mathrm{Compl}\, f^{-1})^{-1}.$$

Obvious 103. $\mathrm{Compl}\, f \subseteq f$ and $\mathrm{CoCompl}\, f \subseteq f$ for every funcoid f.

Proposition 104. The filtrator $(\mathsf{FCD}; \mathrm{Compl}\mathsf{FCD})$ is filtered.

Proof. Because the filtrator $(\mathsf{FCD}; \mathscr{P}\mho^2)$ is filtered. □

Theorem 105. $\mathrm{Compl}\, f = \mathrm{Cor}^{(\mathsf{FCD};\mathrm{Compl}\mathsf{FCD})} f = \mathrm{Cor}'^{(\mathsf{FCD};\mathrm{Compl}\mathsf{FCD})} f$.

Proof. $\mathrm{Cor}^{(\mathsf{FCD};\mathrm{Compl}\mathsf{FCD})} f = \mathrm{Cor}'^{(\mathsf{FCD};\mathrm{Compl}\mathsf{FCD})} f$ since (the theorem 26 in [14]) the filtrator $(\mathsf{FCD}; \mathrm{Compl}\mathsf{FCD})$ is filtered and with join closed core (the theorem 93).

Let $g \in \mathrm{up}^{(\mathsf{FCD};\mathrm{Compl}\mathsf{FCD})} f$. Then $g \in \mathrm{Compl}\mathsf{FCD}$ and $g \supseteq f$. Thus $g = \mathrm{Compl}\, g \supseteq \mathrm{Compl}\, f$.

Thus $\forall g \in \mathrm{up}^{(\mathsf{FCD};\mathrm{Compl}\mathsf{FCD})} f : g \supseteq \mathrm{Compl}\, f$.

Let $\forall g \in \mathrm{up}^{(\mathsf{FCD};\mathrm{Compl}\mathsf{FCD})} f : h \subseteq g$ for some $h \in \mathrm{Compl}\mathsf{FCD}$.

Then $h \subseteq \bigcap^{\mathsf{FCD}} \mathrm{up}^{(\mathsf{FCD};\mathrm{Compl}\mathsf{FCD})} f = f$ and consequently $h = \mathrm{Compl}\, h \subseteq \mathrm{Compl}\, f$.

Thus $\mathrm{Compl}\, f = \bigcap^{\mathrm{Compl}\mathsf{FCD}} \mathrm{up}^{(\mathsf{FCD};\mathrm{Compl}\mathsf{FCD})} f = \mathrm{Cor}^{(\mathsf{FCD};\mathrm{Compl}\mathsf{FCD})} f$. □

Theorem 106. Atoms of the lattice $\mathrm{Compl}\mathsf{FCD}$ are exactly direct products of the form $\{\alpha\} \times^{\mathsf{FCD}} b$ where $\alpha \in \mho$ and b is an atomic f.o.

Proof. First, easy to see that $\{\alpha\} \times^{\mathsf{FCD}} b$ are elements of $\mathrm{Compl}\mathsf{FCD}$. Also $\emptyset$ is an element of $\mathrm{Compl}\mathsf{FCD}$.

$\{\alpha\} \times^{\mathsf{FCD}} b$ are atoms of $\mathrm{Compl}\mathsf{FCD}$ because these are atoms of FCD.

Remain to prove that if f is an atom of $\mathrm{Compl}\mathsf{FCD}$ then $f = \{\alpha\} \times^{\mathsf{FCD}} b$ for some $\alpha \in \mho$ and an atomic f.o. b.

Suppose f is a non-empty complete funcoid. Then exists $\alpha \in \mho$ such that $\langle f\rangle\{\alpha\} \neq \emptyset$. Thus $\{\alpha\} \times^{\mathsf{FCD}} b \subseteq f$ for some atomic f.o. b. If f is an atom then $f = \{\alpha\} \times^{\mathsf{FCD}} b$. □

Theorem 107. $\langle \mathrm{CoCompl}\, f\rangle X = \mathrm{Cor}\, \langle f\rangle X$ for every funcoid f and set X.

Proof. $\mathrm{CoCompl}\ f \subseteq f$ thus $\langle \mathrm{CoCompl}\ f\rangle X \subseteq \langle f\rangle X$, but $\langle \mathrm{CoCompl}\ f\rangle X \in \mathscr{P}\mho$ thus $\langle \mathrm{CoCompl}\, f\rangle X \subseteq \mathrm{Cor}\, \langle f\rangle X$.

Let $\alpha X = \mathrm{Cor}\, \langle f\rangle X$. Then $\alpha\emptyset = \emptyset$ and

$$\alpha(X \cup Y) = \mathrm{Cor}\, \langle f\rangle(X \cup Y) = \mathrm{Cor}(\langle f\rangle X \cup \langle f\rangle Y) = \mathrm{Cor}\, \langle f\rangle X \cup \mathrm{Cor}\, \langle f\rangle Y = \alpha X \cup \alpha Y.$$

(used the theorem 64 from [14]). Thus α can be continued till $\langle g\rangle$ for some funcoid g. This funcoid is co-complete.

Evidently g is the greatest co-complete funcoid which is lower than f.

Thus $g=\mathrm{CoCompl}\, f$ and so $\mathrm{Cor}\,\langle f\rangle X=\alpha X=\langle g\rangle X=\langle \mathrm{CoCompl}\, f\rangle X$. □

Theorem 108. $\mathrm{Compl}\mathsf{FCD}$ is an atomistic lattice.

Proof. Let $f\in \mathrm{Compl}\mathsf{FCD}$. $\langle f\rangle X=\bigcup^{\mathfrak{F}}\{\langle f\rangle\{x\}\mid x\in X\}=\bigcup^{\mathfrak{F}}\left\{\left\langle f|^{\mathsf{FCD}}_{\{x\}}\right\rangle\{x\}\mid x\in X\right\}=\bigcup^{\mathfrak{F}}\left\{\left\langle f|^{\mathsf{FCD}}_{\{x\}}\right\rangle X\mid x\in X\right\}$, thus $f=\bigcup^{\mathsf{FCD}}\left\{f|^{\mathsf{FCD}}_{\{x\}}\mid x\in X\right\}$. It is trivial that every $f|^{\mathsf{FCD}}_{\{x\}}$ is a union of atoms of $\mathrm{Compl}\mathsf{FCD}$. □

Theorem 109. A funcoid is complete iff it is a join (on the lattice FCD) of atomic complete funcoids.

Proof. Follows from the theorem 93 and the previous theorem. □

Corollary 110. $\mathrm{Compl}\mathsf{FCD}$ is join-closed.

Theorem 111. $\mathrm{Compl}(\bigcup^{\mathsf{FCD}}R)=\bigcup^{\mathsf{FCD}}\langle\mathrm{Compl}\rangle R$ for every set R of funcoids.

Proof. $\left\langle\mathrm{Compl}(\bigcup^{\mathsf{FCD}}R)\right\rangle X=\bigcup^{\mathfrak{F}}\left\{\left\langle\bigcup^{\mathsf{FCD}}R\right\rangle\{\alpha\}\mid\alpha\in X\right\}=\bigcup^{\mathfrak{F}}\left\{\bigcup^{\mathfrak{F}}\{\langle f\rangle\{\alpha\}\mid f\in R\}\mid\alpha\in X\right\}=\bigcup^{\mathfrak{F}}\left\{\bigcup^{\mathfrak{F}}\{\langle f\rangle\{\alpha\}\mid\alpha\in X\}\mid f\in R\right\}=\bigcup^{\mathfrak{F}}\{\langle\mathrm{Compl}\, f\rangle X\mid f\in R\}=\left\langle\bigcup^{\mathsf{FCD}}\langle\mathrm{Compl}\rangle R\right\rangle X$ for every set X. □

Corollary 112. Compl is a lower adjoint.

Conjecture 113. Compl is not an upper adjoint (in general).

Conjecture 114. $\mathrm{Compl}\, f=f\setminus^{*\,\mathsf{FCD}}(\Omega\times^{\mathsf{FCD}}\mho)$ for every funcoid f.

This conjecture may be proved by considerations similar to these in the section “Fréchet filter” in [14].

Lemma 115. Co-completion of a complete funcoid is complete.

Proof. Let f is a complete funcoid.

$\langle\mathrm{CoCompl}\ f\rangle X=\mathrm{Cor}\ \langle f\rangle X=\mathrm{Cor}\ \bigcup^{\mathfrak{F}}\{\langle f\rangle\{x\}\mid x\in X\}=\bigcup\ \{\mathrm{Cor}\ \langle f\rangle\{x\}\mid x\in X\}=\bigcup\ \{\langle\mathrm{CoCompl}\, f\rangle\,\{x\}\mid x\in X\}$ for every set X. Thus $\mathrm{CoCompl}\, f$ is complete. □

Theorem 116. $\mathrm{Compl}\,\mathrm{CoCompl}\, f=\mathrm{CoCompl}\,\mathrm{Compl}\, f=\mathrm{Cor}\, f$ for every funcoid f.

Proof. Compl CoCompl f is co-complete since (used the lemma) CoCompl f is co-complete. Thus Compl CoCompl f is a discrete funcoid. CoCompl f is the the greatest co-complete funcoid under f and Compl CoCompl f is the greatest complete funcoid under CoCompl f. So Compl CoCompl f is greater than any discrete funcoid under CoCompl f which is greater than any discrete funcoid under f. Thus Compl CoCompl f it is the greatest discrete funcoid under f. Thus $\mathrm{Compl}\,\mathrm{CoCompl}\, f=\mathrm{Cor}\, f$. Similarly $\mathrm{CoCompl}\,\mathrm{Compl}\, f=\mathrm{Cor}\, f$. □

Question 117. Is $\mathrm{Compl}\mathsf{FCD}$ a co-brouwerian lattice?

3.13 Monovalued funcoids

Following the idea of definition of monovalued morphism let's call *monovalued* such a funcoid f that $f\circ f^{-1}\subseteq I^{\mathsf{FCD}}_{\mathrm{im}\, f}$.

Similarly, I will call a reloid *injective* when $f^{-1}\circ f\subseteq I^{\mathsf{FCD}}_{\mathrm{dom}\, f}$.

Obvious 118. A funcoid f is

- monovalued iff $f\circ f^{-1}\subseteq(=)|_{\mho}$;

- injective iff $f^{-1}\circ f\subseteq(=)|_{\mho}$.

Monovaluedness is dual of injectivity.

Obvious 119.

1. A morphism $(f;\mathcal{A};\mathcal{B})$ of the category of funcoids is monovalued iff the funcoid f is monovalued.
2. A morphism $(f;\mathcal{A};\mathcal{B})$ of the category of funcoids is injective iff the funcoid f is injective.

Theorem 120. The following statements are equivalent for a funcoid f:

1. f is monovalued.
2. $\forall a\in\operatorname{atoms}^{\mathfrak{F}}\mathcal{A}$: $\langle f\rangle a\in\operatorname{atoms}^{\mathfrak{F}}\mho\cup\{\emptyset\}$.
3. $\forall\mathcal{I},\mathcal{J}\in\mathfrak{F}$: $\langle f^{-1}\rangle(\mathcal{I}\cap^{\mathfrak{F}}\mathcal{J})=\langle f^{-1}\rangle\mathcal{I}\cap^{\mathfrak{F}}\langle f^{-1}\rangle\mathcal{J}$.
4. $\forall I,J\in\mathscr{P}\mho$: $\langle f^{-1}\rangle(I\cap J)=\langle f^{-1}\rangle I\cap^{\mathfrak{F}}\langle f^{-1}\rangle J$.

Proof.

(2) ⇒ (3). Let $a\in\operatorname{atoms}^{\mathfrak{F}}\mho$, $\langle f\rangle a=b$. Then because $b\in\operatorname{atoms}^{\mathfrak{F}}\mho\cup\{\emptyset\}$

$$(\mathcal{I}\cap^{\mathfrak{F}}\mathcal{J})\cap^{\mathfrak{F}}b\neq\emptyset\Leftrightarrow\mathcal{I}\cap^{\mathfrak{F}}b\neq\emptyset\wedge\mathcal{J}\cap^{\mathfrak{F}}b\neq\emptyset;$$
$$a[f](\mathcal{I}\cap^{\mathfrak{F}}\mathcal{J})\Leftrightarrow a[f]\mathcal{I}\wedge a[f]\mathcal{J};$$
$$(\mathcal{I}\cap^{\mathfrak{F}}\mathcal{J})\left[f^{-1}\right]a\Leftrightarrow\mathcal{I}\left[f^{-1}\right]a\wedge\mathcal{J}\left[f^{-1}\right]a;$$
$$a\cap^{\mathfrak{F}}\langle f^{-1}\rangle(\mathcal{I}\cap^{\mathfrak{F}}\mathcal{J})\neq\emptyset\Leftrightarrow a\cap^{\mathfrak{F}}\langle f^{-1}\rangle\mathcal{I}\neq\emptyset\wedge a\cap^{\mathfrak{F}}\langle f^{-1}\rangle\mathcal{J}\neq\emptyset;$$
$$\langle f^{-1}\rangle(\mathcal{I}\cap^{\mathfrak{F}}\mathcal{J})=\langle f^{-1}\rangle\mathcal{I}\cap^{\mathfrak{F}}\langle f^{-1}\rangle\mathcal{J}.$$

(3) ⇒ (1). $\langle f^{-1}\rangle a\cap^{\mathfrak{F}}\langle f^{-1}\rangle b=\langle f^{-1}\rangle(a\cap^{\mathfrak{F}}b)=\langle f^{-1}\rangle\emptyset=\emptyset$ for every two distinct atomic filter objects a and b. This is equivalent to $\neg(\langle f^{-1}\rangle a[f]b)$; $b\cap^{\mathfrak{F}}\langle f\rangle\langle f^{-1}\rangle a=\emptyset$; $b\cap^{\mathfrak{F}}\langle f\circ f^{-1}\rangle a=\emptyset$; $\neg(a\left[f\circ f^{-1}\right]b)$. So $a\left[f\circ f^{-1}\right]b\Rightarrow a=b$ for every atomic filter objects a and b. This is possible only when $f\circ f^{-1}\subseteq I^{\mathsf{FCD}}_{\operatorname{im}f}$.

(4) ⇒ (3). $\langle f^{-1}\rangle(\mathcal{I}\cap^{\mathfrak{F}}\mathcal{J})=\bigcap^{\mathfrak{F}}\langle\langle f^{-1}\rangle\rangle\operatorname{up}(\mathcal{I}\cap^{\mathfrak{F}}\mathcal{J})=\bigcap^{\mathfrak{F}}\langle\langle f^{-1}\rangle\rangle\{I\cap J\mid I\in\operatorname{up}\mathcal{I},J\in\operatorname{up}\mathcal{J}\}=\bigcap^{\mathfrak{F}}\{\langle f^{-1}\rangle(I\cap J)\mid I\in\operatorname{up}\mathcal{I},J\in\operatorname{up}\mathcal{J}\}=\bigcap^{\mathfrak{F}}\{\langle f^{-1}\rangle I\cap^{\mathfrak{F}}\langle f^{-1}\rangle J\mid I\in\operatorname{up}\mathcal{I},J\in\operatorname{up}\mathcal{J}\}=\bigcap^{\mathfrak{F}}\{\langle f^{-1}\rangle I\mid I\in\operatorname{up}\mathcal{I}\}\cap^{\mathfrak{F}}\bigcap^{\mathfrak{F}}\{\langle f^{-1}\rangle J\mid J\in\operatorname{up}\mathcal{J}\}=\langle f^{-1}\rangle\mathcal{I}\cap^{\mathfrak{F}}\langle f^{-1}\rangle\mathcal{J}$.

(3) ⇒ (4). Obvious.

¬(2) ⇒ ¬(1). Suppose $\langle f\rangle a\notin\operatorname{atoms}^{\mathfrak{F}}\mathcal{B}\cup\{\emptyset\}$ for some $a\in\operatorname{atoms}^{\mathfrak{F}}\mathcal{A}$. Then there exist two atomic filter objects $p\neq q$ such that $\langle f\rangle a\supseteq p\wedge\langle f\rangle a\supseteq q$. Consequently $p\cap^{\mathfrak{F}}\langle f\rangle a\neq\emptyset$; $a\cap^{\mathfrak{F}}\langle f^{-1}\rangle p\neq\emptyset$; $a\subseteq\langle f^{-1}\rangle p$; $\langle f\circ f^{-1}\rangle p=\langle f\rangle\langle f^{-1}\rangle p\supseteq\langle f\rangle a\supseteq q$; $\langle f\circ f^{-1}\rangle p\nsubseteq p$ and $\langle f\circ f^{-1}\rangle p\neq\emptyset$. So it cannot be $f\circ f^{-1}\subseteq I^{\mathsf{FCD}}_{\operatorname{im}f}$. □

Corollary 121. A binary relation is a monovalued funcoid iff it is a function.

Proof. Because $\forall I,J\in\mathscr{P}\mho$: $\langle f^{-1}\rangle(I\cap J)=\langle f^{-1}\rangle I\cap^{\mathfrak{F}}\langle f^{-1}\rangle J$ is true for a binary relation f if and only if it is a function. □

Remark 122. This corollary can be reformulated as follows: For binary relations the classic concept of monovaluedness and monovaluedness in the above defined sense of monovaluedness of a funcoid are the same.

3.14 T_0-, T_1- and T_2-separable funcoids

For funcoids can be generalized T_0-, T_1- and T_2- separability. Worthwhile note that T_0 and T_2 separability is defined through T_1 separability.

Definition 123. Let call T_1-*separable* such funcoid f that for every $\alpha, \beta \in \mho$ is true

$$\alpha \neq \beta \Rightarrow \neg(\{\alpha\}[f]\{\beta\}).$$

Definition 124. Let call T_0-*separable* such funcoid f that $f \cap^{\mathsf{FCD}} f^{-1}$ is T_1-separable.

Definition 125. Let call T_2-*separable* such funcoid f that the funcoid $f^{-1} \circ f$ is T_1-separable.

For symmetric transitive funcoids T_1- and T_2-separability are the same (see theorem 12).

Obvious 126. A funcoid f is T_2-*separable* iff $\alpha \neq \beta \Rightarrow \langle f \rangle \{\alpha\} \cap^{\mathfrak{F}} \langle f \rangle \{\beta\} = \emptyset$ for every $\alpha, \beta \in \mho$.

3.15 Filter objects closed regarding a funcoid

Definition 127. Let's call *closed* regarding a funcoid f such filter object $\mathcal{A}$ that $\langle f \rangle \mathcal{A} \subseteq \mathcal{A}$.

This is a generalization of closedness of a set regarding an unary operation.

Proposition 128. If $\mathcal{I}$ and $\mathcal{J}$ are closed (regarding some funcoid), S is a set of closed filter objects, then

1. $\mathcal{I} \cup^{\mathfrak{F}} \mathcal{J}$ is a closed filter object;
2. $\bigcap^{\mathfrak{F}} S$ is a closed filter object.

Proof. Let denote the given funcoid as f. $\langle f \rangle (\mathcal{I} \cup^{\mathfrak{F}} \mathcal{J}) = \langle f \rangle \mathcal{I} \cup^{\mathfrak{F}} \langle f \rangle \mathcal{J} \subseteq \mathcal{I} \cup^{\mathfrak{F}} \mathcal{J}$, $\langle f \rangle \bigcap^{\mathfrak{F}} S \subseteq \bigcap^{\mathfrak{F}} \langle \langle f \rangle \rangle S \subseteq \bigcap^{\mathfrak{F}} S$. Consequently the filter objects $\mathcal{I} \cup^{\mathfrak{F}} \mathcal{J}$ and $\bigcap^{\mathfrak{F}} S$ are closed. □

Proposition 129. If S is a set of filter objects closed regarding a complete funcoid, then the filter object $\bigcup^{\mathfrak{F}} S$ is also closed regarding our funcoid.

Proof. $\langle f \rangle \bigcup^{\mathfrak{F}} S = \bigcup^{\mathfrak{F}} \langle \langle f \rangle \rangle S \subseteq \bigcup^{\mathfrak{F}} S$ where f is the given funcoid. □

4 Reloids

Definition 130. I will call a *reloid* a filter object on the set of binary relations.

Reloids are a generalization of uniform spaces. Also reloids are generalization of binary relations (the set of binary relations is a subset of the set of reloids, I will call *discrete* these reloids which are binary relations).

Definition 131. The *reverse* reloid of a reloid f is defined by the formula

$$\mathrm{up}\, f^{-1} = \left\{ F^{-1} \mid F \in \mathrm{up}\, f^{-1} \right\}.$$

Reverse reloid is a generalization of conjugate quasi-uniformity.

I will denote RLD either the set of reloids or the category of reloids (defined below), dependently on context.

4.1 Composition of reloids

Definition 132. Composition of reloids is defined by the formula

$$g \circ f = \bigcap^{\mathsf{RLD}} \{ G \circ F \mid F \in \mathrm{up}\, f, G \in \mathrm{up}\, g \}.$$

Composition of reloids is a reloid.

Theorem 133. $(h \circ g) \circ f = h \circ (g \circ f)$ for every reloids f, g, h.

Proof. For two nonempty collections A and B of sets I will denote

$$A \sim B \Leftrightarrow (\forall K \in A \exists L \in B: L \subseteq K) \wedge (\forall K \in B \exists L \in A: L \subseteq K).$$

It is easy to see that $\sim$ is a transitive relation.

I will denote $B \circ A = \{L \circ K | K \in A, L \in B\}$.

Let first prove that for every nonempty collections of relations A, B, C

$$A \sim B \Rightarrow A \circ C \sim B \circ C.$$

Suppose $A \sim B$ and $P \in A \circ C$ that is $K \in A$ and $M \in C$ such that $P = K \circ M$. $\exists K' \in B: K' \subseteq K$ because $A \sim B$. We have $P' = K' \circ M \in B \circ C$. Obviously $P' \subseteq P$. So for every $P \in A \circ C$ exist $P' \in B \circ C$ such that $P' \subseteq P$; vice versa is analogous. So $A \circ C \sim B \circ C$.

$\operatorname{up}((h \circ g) \circ f) \sim \operatorname{up}(h \circ g) \circ \operatorname{up} f$, $\operatorname{up}(h \circ g) \sim (\operatorname{up} h) \circ (\operatorname{up} g)$. By proven above $\operatorname{up}((h \circ g) \circ f) \sim (\operatorname{up} h) \circ (\operatorname{up} g) \circ (\operatorname{up} f)$.

Analogously $\operatorname{up}(h \circ (g \circ f)) \sim (\operatorname{up} h) \circ (\operatorname{up} g) \circ (\operatorname{up} f)$.

So $\operatorname{up}((h \circ g) \circ f) \sim \operatorname{up}(h \circ (g \circ f))$ what is possible only if $\operatorname{up}((h \circ g) \circ f) = \operatorname{up}(h \circ (g \circ f))$. □

Theorem 134.

1. $f \circ f = \bigcap^{\mathsf{RLD}} \{F \circ F \mid F \in \operatorname{up} f\}$;
2. $f^{-1} \circ f = \bigcap^{\mathsf{RLD}} \left\{F^{-1} \circ F \mid F \in \operatorname{up} f\right\}$;
3. $f \circ f^{-1} = \bigcap^{\mathsf{RLD}} \left\{F \circ F^{-1} \mid F \in \operatorname{up} f\right\}$.

Proof. I will prove only (1) and (2) because (3) is analogous to (2).

1. Enough to show that $\forall F, G \in \operatorname{up} f \exists H \in \operatorname{up} f: H \circ H \subseteq G \circ F$. To prove it take $H = F \cap G$.
2. Enough to show that $\forall F, G \in \operatorname{up} f \exists H \in \operatorname{up} f: H^{-1} \circ H \subseteq G^{-1} \circ F$. To prove it take $H = F \cap G$. Then $H^{-1} \circ H = (F \cap G)^{-1} \circ (F \cap G) \subseteq G^{-1} \circ F$. □

Conjecture 135. If f, g, h are reloids then

1. $f \circ (g \cup^{\mathsf{RLD}} h) = f \circ g \cup^{\mathsf{RLD}} f \circ h$;
2. $(g \cup^{\mathsf{RLD}} h) \circ f = g \circ f \cup^{\mathsf{RLD}} h \circ f$.

Conjecture 136. If f and g are reloids, then

$$g \circ f = \bigcup^{\mathsf{RLD}} \left\{G \circ F \mid F \in \operatorname{atoms}^{\mathsf{RLD}} f, G \in \operatorname{atoms}^{\mathsf{RLD}} g\right\}.$$

4.2 Direct product of filter objects

In theory of reloids direct product of filter objects $\mathcal{A}$ and $\mathcal{B}$ is defined by the formula

$$\mathcal{A} \times^{\mathsf{RLD}} \mathcal{B} \overset{\text{def}}{=} \bigcap^{\mathfrak{F}} \{A \times B \mid A \in \operatorname{up} \mathcal{A}, B \in \operatorname{up} \mathcal{B}\}.$$

Obvious 137. $A \times^{\mathsf{RLD}} B = A \times B$ for every sets A and B.

Theorem 138. $\mathcal{A} \times^{\mathsf{RLD}} \mathcal{B} = \bigcup^{\mathfrak{F}} \left\{a \times^{\mathsf{RLD}} b \mid a \in \operatorname{atoms}^{\mathfrak{F}} \mathcal{A}, b \in \operatorname{atoms}^{\mathfrak{F}} \mathcal{B}\right\}$ for every $\mathcal{A}, \mathcal{B} \in \mathfrak{F}$.

Proof. Obviously

$$\mathcal{A} \times^{\mathsf{RLD}} \mathcal{B} \supseteq \bigcup^{\mathfrak{F}} \left\{a \times^{\mathsf{RLD}} b \mid a \in \operatorname{atoms}^{\mathfrak{F}} \mathcal{A}, b \in \operatorname{atoms}^{\mathfrak{F}} \mathcal{B}\right\}$$

Reversely, let $K \in \operatorname{up} \bigcup^{\mathfrak{F}} \left\{a \times^{\mathsf{RLD}} b \mid a \in \operatorname{atoms}^{\mathfrak{F}} \mathcal{A}, b \in \operatorname{atoms}^{\mathfrak{F}} \mathcal{B}\right\}$. Then $K \in \operatorname{up}(a \times^{\mathsf{RLD}} b)$ for every $a \in \operatorname{atoms}^{\mathfrak{F}} \mathcal{A}$, $b \in \operatorname{atoms}^{\mathfrak{F}} \mathcal{B}$; $K \supseteq X_a \times^{\mathsf{RLD}} Y_b$ for some $X_a \in \operatorname{up} a$, $Y_b \in \operatorname{up} b$; $K \supseteq \bigcup \left\{X_a \times Y_b \mid a \in \operatorname{atoms}^{\mathfrak{F}} \mathcal{A}, b \in \operatorname{atoms}^{\mathfrak{F}} \mathcal{B}\right\} = \bigcup \left\{X_a \mid a \in \operatorname{atoms}^{\mathfrak{F}} \mathcal{A}\right\} \times \bigcup \left\{Y_b \mid b \in \operatorname{atoms}^{\mathfrak{F}} \mathcal{A}\right\} \supseteq A \times B$ where $A \in \operatorname{up} \mathcal{A}$, $B \in \operatorname{up} \mathcal{B}$; $K \in \operatorname{up}(\mathcal{A} \times^{\mathsf{RLD}} \mathcal{B})$. □

Theorem 139. $(\mathcal{A}_0 \times^{\mathsf{RLD}} \mathcal{B}_0) \cap^{\mathsf{RLD}} (\mathcal{A}_1 \times^{\mathsf{RLD}} \mathcal{B}_1) = (\mathcal{A}_0 \cap^{\mathsf{RLD}} \mathcal{A}_1) \times^{\mathsf{RLD}} (\mathcal{B}_0 \cap^{\mathsf{RLD}} \mathcal{B}_1)$ for every $\mathcal{A}_0, \mathcal{A}_1, \mathcal{B}_0, \mathcal{B}_1 \in \mathfrak{F}$.

Proof.

$$\begin{aligned}(\mathcal{A}_0 \times^{\mathsf{RLD}} \mathcal{B}_0) \cap^{\mathsf{RLD}} (\mathcal{A}_1 \times^{\mathsf{RLD}} \mathcal{B}_1) &= \bigcap^{\mathsf{RLD}} \{P \cap Q \mid P \in \mathrm{up}(\mathcal{A}_0 \times^{\mathsf{RLD}} \mathcal{B}_0), Q \in \mathrm{up}(\mathcal{A}_1 \times^{\mathsf{RLD}} \mathcal{B}_1)\} \\ &= \bigcap^{\mathsf{RLD}} \{(A_0 \times B_0) \cap (A_1 \times B_1) \mid A_0 \in \mathrm{up}\mathcal{A}_0, B_0 \in \mathrm{up}\mathcal{B}_0, A_1 \in \mathrm{up}\mathcal{A}_1, B_1 \in \mathrm{up}\mathcal{B}_1\} \\ &= \bigcap^{\mathsf{RLD}} \{(A_0 \cap A_1) \times (B_0 \cap B_1) \mid A_0 \in \mathrm{up}\mathcal{A}_0, B_0 \in \mathrm{up}\mathcal{B}_0, A_1 \in \mathrm{up}\mathcal{A}_1, B_1 \in \mathrm{up}\mathcal{B}_1\} \\ &= \bigcap^{\mathsf{RLD}} \{K \times L \mid K \in \mathrm{up}(\mathcal{A}_0 \cap \mathcal{A}_1), L \in \mathrm{up}(\mathcal{B}_0 \cap \mathcal{B}_1)\} \\ &= (\mathcal{A}_0 \cap^{\mathsf{RLD}} \mathcal{A}_1) \times^{\mathsf{RLD}} (\mathcal{B}_0 \cap^{\mathsf{RLD}} \mathcal{B}_1).\end{aligned}$$

□

Theorem 140. If $S \in \mathscr{P}\mathfrak{F}^2$ then

$$\bigcap^{\mathsf{RLD}} \{\mathcal{A} \times^{\mathsf{RLD}} \mathcal{B} \mid (\mathcal{A}; \mathcal{B}) \in S\} = \bigcap^{\mathfrak{F}} \mathrm{dom}\, S \times^{\mathsf{RLD}} \bigcap^{\mathfrak{F}} \mathrm{im}\, S.$$

Proof. Let $\mathcal{P} = \bigcap^{\mathfrak{F}} \mathrm{dom}\, S$, $\mathcal{Q} = \bigcap^{\mathfrak{F}} \mathrm{im}\, S$; $l = \bigcap^{\mathsf{RLD}} \{\mathcal{A} \times^{\mathsf{RLD}} \mathcal{B} \mid (\mathcal{A}; \mathcal{B}) \in S\}$.

$\mathcal{P} \times^{\mathsf{RLD}} \mathcal{Q} \subseteq l$ is obvious.

Let $F \in \mathrm{up}(\mathcal{P} \times^{\mathsf{RLD}} \mathcal{Q})$. Then exist $P \in \mathrm{up}\, \mathcal{P}$ and $Q \in \mathrm{up}\, \mathcal{Q}$ such that $F \supseteq P \times Q$.

$P = P_1 \cap \ldots \cap P_n$ where $P_i \in \langle \mathrm{up} \rangle \mathrm{dom}\, S$ and $Q = Q_1 \cap \ldots \cap Q_m$ where $Q_i \in \langle \mathrm{up} \rangle \mathrm{im}\, S$.

$P \times Q = \bigcap_{i,j} (P_i \times Q_i)$.

$P_i \times Q_i \supseteq \mathcal{A} \times^{\mathsf{RLD}} \mathcal{B}$ for some $(\mathcal{A}; \mathcal{B}) \in S$. $P \times Q = \bigcap_{i,j} (P_i \times Q_i) \supseteq l$. So $F \in \mathrm{up}\, l$. □

Conjecture 141. If $\mathcal{A} \in \mathfrak{F}$ then $\mathcal{A} \times^{\mathsf{RLD}}$ is a complete homomorphism of the lattice $\mathfrak{F}$ to a complete sublattice of the lattice RLD, if also $\mathcal{A} \neq \emptyset$ then it is an isomorphism.

Definition 142. I will call a reloid *convex* iff it is a union of direct products.

Example 143. Non-convex reloids exist.

Proof. Let a is a non-trivial atomic f.o. Then $(=)|_a$ is non-convex. This follows from the fact that only direct products which are below $(=)$ are direct products of atomic f.o. and $(=)|_a$ is not their join. □

I will call two filter objects *isomorphic* when the corresponding filters are isomorphic (in the sense defined in [14]).

Theorem 144. The reloid $\{a\} \times^{\mathsf{RLD}} \mathcal{F}$ is isomorphic to the filter object $\mathcal{F}$ for every $a \in \mho$.

Proof. Consider $B = \{a\} \times \mho$ and $f = \{(x; (a; x)) \mid x \in \mho\}$. Then f is a bijection from $\mho$ to B.

If $X \in \mathrm{up}\, \mathcal{F}$ then $\langle f \rangle X \subseteq B$ and $\langle f \rangle X = \{a\} \times X \in \mathrm{up}(\{a\} \times^{\mathsf{RLD}} \mathcal{F})$.

For every $Y \in \mathrm{up}(\{a\} \times^{\mathsf{RLD}} \mathcal{F}) \cap \mathscr{P}B$ we have $Y = \{a\} \times X$ for some $X \in \mathrm{up}\, \mathcal{F}$ and thus $Y = \langle f \rangle X$.

So $\langle f \rangle|_{\mathrm{up}\, \mathcal{F} \cap \mathscr{P}\mho} = \langle f \rangle|_{\mathrm{up}\, \mathcal{F}}$ is a bijection from $\mathrm{up}\, \mathcal{F} \cap \mathscr{P}\mho$ to $\mathrm{up}(\{a\} \times^{\mathsf{RLD}} \mathcal{F}) \cap \mathscr{P}B$.

We have $\mathrm{up}\, \mathcal{F} \cap \mathscr{P}\mho$ and $\mathrm{up}(\{a\} \times^{\mathsf{RLD}} \mathcal{F}) \cap \mathscr{P}B$ directly isomorphic and thus $\mathrm{up}\, \mathcal{F}$ is isomorphic to $\mathrm{up}(\{a\} \times^{\mathsf{RLD}} \mathcal{F})$. □

4.3 Restricting reloid to a filter object. Domain and image

Definition 145. I call *restricting* a reloid f to a filter object $\mathcal{A}$ as $f|^{\mathsf{RLD}}_{\mathcal{A}} = f \cap^{\mathsf{RLD}} (\mathcal{A} \times^{\mathsf{RLD}} \mho)$.

Definition 146. *Domain* and *image* of a reloid f are defined as follows:

$$\mathrm{dom}\, f = \bigcap^{\mathfrak{F}} \langle \mathrm{dom} \rangle \mathrm{up}\, f; \quad \mathrm{im}\, f = \bigcap^{\mathfrak{F}} \langle \mathrm{im} \rangle \mathrm{up}\, f.$$

Proposition 147. $f \subseteq \mathcal{A} \times^{\mathsf{RLD}} \mathcal{B} \Leftrightarrow \operatorname{dom} f \subseteq \mathcal{A} \wedge \operatorname{im} f \subseteq \mathcal{B}$.

Proof.

- $\Rightarrow$. Follows from $\operatorname{dom}(\mathcal{A} \times^{\mathsf{RLD}} \mathcal{B}) \subseteq \mathcal{A} \wedge \operatorname{im}(\mathcal{A} \times^{\mathsf{RLD}} \mathcal{B}) \subseteq \mathcal{B}$.
- $\Leftarrow$. $\operatorname{dom} f \subseteq \mathcal{A} \Leftrightarrow \forall A \in \operatorname{up}\mathcal{A}\, \exists F \in \operatorname{up} f : \operatorname{dom} F \subseteq A$. Analogously

$$\operatorname{im} f \subseteq \mathcal{B} \Leftrightarrow \forall B \in \operatorname{up}\mathcal{B}\, \exists G \in \operatorname{up} f : \operatorname{im} G \subseteq B.$$

Let $\operatorname{dom} f \subseteq \mathcal{A} \wedge \operatorname{im} f \subseteq \mathcal{B}$, $A \in \operatorname{up}\mathcal{A}$, $B \in \operatorname{up}\mathcal{B}$. Then exist $F \in \operatorname{up} f$, $G \in \operatorname{up} f$ such that $\operatorname{dom} F \subseteq A \wedge \operatorname{im} G \subseteq B$. Consequently $F \cap G \in \operatorname{up} f$, $\operatorname{dom}(F \cap G) \subseteq A$, $\operatorname{im}(F \cap G) \subseteq B$ that is $F \cap G \subseteq A \times B$. So exists $H \in \operatorname{up} f$ such that $H \subseteq A \times B$ for every $A \in \operatorname{up}\mathcal{A}$, $B \in \operatorname{up}\mathcal{B}$. So $f \subseteq \mathcal{A} \times^{\mathsf{RLD}} \mathcal{B}$. □

Definition 148. I call *identity reloid* for a filter object $\mathcal{A}$ the reloid $I_{\mathcal{A}}^{\mathsf{RLD}} \overset{\text{def}}{=} (=)|_{\mathcal{A}}^{\mathsf{RLD}}$.

Theorem 149. $I_{\mathcal{A}}^{\mathsf{RLD}} = \bigcap^{\mathfrak{F}} \{ I_A \mid A \in \operatorname{up}\mathcal{A} \}$ where I_A is the identity relation on a set A.

Proof. Let $K \in \operatorname{up} \bigcap^{\mathfrak{F}} \{ I_A \mid A \in \operatorname{up} \mathcal{A} \}$, then exists $A \in \operatorname{up}\mathcal{A}$ such that $K \supseteq I_A$. Then $I_{\mathcal{A}} = (=)|_{\mathcal{A}}^{\mathsf{RLD}} = (=) \cap^{\mathsf{RLD}} (\mathcal{A} \times \mho) \subseteq (=) \cap (A \times \mho) = I_A \subseteq K$; $K \in \operatorname{up} I_{\mathcal{A}}$.

Reversely let $K \in \operatorname{up} I_{\mathcal{A}}^{\mathsf{RLD}} = \operatorname{up}((=) \cap^{\mathsf{RLD}} (\mathcal{A} \times^{\mathsf{RLD}} \mho))$, then exists $A \in \operatorname{up} \mathcal{A}$ such that $K \in \operatorname{up}((=) \cap (A \times \mho)) = \operatorname{up} I_A \subseteq \operatorname{up} \bigcap^{\mathfrak{F}} \{ I_A \mid A \in \operatorname{up}\mathcal{A} \}$. □

Proposition 150. $(I_{\mathcal{A}}^{\mathsf{RLD}})^{-1} = I_{\mathcal{A}}^{\mathsf{RLD}}$.

Proof. Follows from the previous theorem. □

Theorem 151. $f|_{\mathcal{A}}^{\mathsf{RLD}} = f \circ I_{\mathcal{A}}^{\mathsf{RLD}}$ for every reloid f and filter object $\mathcal{A}$.

Proof. We need to prove that $f \cap^{\mathsf{RLD}} (\mathcal{A} \times \mho) = f \circ \bigcap^{\mathsf{RLD}} \{ I_A \mid A \in \operatorname{up} \mathcal{A} \}$. $f \circ \bigcap^{\mathsf{RLD}} \{ I_A \mid A \in \operatorname{up} \mathcal{A} \} = \bigcap^{\mathsf{RLD}} \{ F \circ I_A \mid F \in \operatorname{up} f, A \in \operatorname{up}\mathcal{A} \} = \bigcap^{\mathsf{RLD}} \{ F|_A \mid F \in \operatorname{up} f, A \in \operatorname{up}\mathcal{A} \} = \bigcap^{\mathsf{RLD}} \{ F \cap (A \times \mho) \mid F \in \operatorname{up} f, A \in \operatorname{up}\mathcal{A} \} = \bigcap^{\mathsf{RLD}} \{ F \mid F \in \operatorname{up} f \} \cap \bigcap^{\mathsf{RLD}} \{ A \times \mho \mid A \in \operatorname{up}\mathcal{A} \} = f \cap^{\mathsf{RLD}} (\mathcal{A} \times^{\mathsf{RLD}} \mho)$. □

Theorem 152. $(g \circ f)|_{\mathcal{A}}^{\mathsf{RLD}} = g \circ (f|_{\mathcal{A}}^{\mathsf{RLD}})$ for every reloids f and g and filter object $\mathcal{A}$.

Proof. $(g \circ f)|_{\mathcal{A}}^{\mathsf{RLD}} = (g \circ f) \circ I_{\mathcal{A}}^{\mathsf{RLD}} = g \circ (f \circ I_{\mathcal{A}}^{\mathsf{RLD}}) = g \circ (f|_{\mathcal{A}}^{\mathsf{RLD}})$. □

Theorem 153. $f \cap^{\mathsf{RLD}} (\mathcal{A} \times^{\mathsf{RLD}} \mathcal{B}) = I_{\mathcal{B}}^{\mathsf{RLD}} \circ f \circ I_{\mathcal{A}}^{\mathsf{RLD}}$ for every reloid f and filter objects $\mathcal{A}$ and $\mathcal{B}$.

Proof. $f \cap^{\mathsf{RLD}} (\mathcal{A} \times^{\mathsf{RLD}} \mathcal{B}) = f \cap^{\mathsf{RLD}} (\mathcal{A} \times^{\mathsf{RLD}} \mho) \cap^{\mathsf{RLD}} (\mho \times^{\mathsf{RLD}} \mathcal{B}) = f|_{\mathcal{A}}^{\mathsf{RLD}} \cap^{\mathsf{RLD}} (\mho \times \mathcal{B}) = (f \circ I_{\mathcal{A}}^{\mathsf{RLD}}) \cap^{\mathsf{RLD}} (\mho \times \mathcal{B}) = ((f \circ I_{\mathcal{A}}^{\mathsf{RLD}})^{-1} \cap^{\mathsf{RLD}} (\mho \times^{\mathsf{RLD}} \mathcal{B})^{-1})^{-1} = ((I_{\mathcal{A}}^{\mathsf{RLD}} \circ f^{-1}) \cap^{\mathsf{RLD}} (\mathcal{B} \times^{\mathsf{RLD}} \mho))^{-1} = (I_{\mathcal{A}}^{\mathsf{RLD}} \circ f^{-1} \circ I_{\mathcal{B}}^{\mathsf{RLD}})^{-1} = I_{\mathcal{B}}^{\mathsf{RLD}} \circ f \circ I_{\mathcal{A}}^{\mathsf{RLD}}$. □

4.4 Category of reloids

I will define the category RLD of reloids:

- The set of objects is $\mathfrak{F}$.
- The set of morphisms from a filter object $\mathcal{A}$ to a filter object $\mathcal{B}$ is the set of triples $(f; \mathcal{A}; \mathcal{B})$ where f is a reloid such that $\operatorname{dom} f \subseteq \mathcal{A}$, $\operatorname{im} f \subseteq \mathcal{B}$.
- Composition of morphisms is defined in the natural way.
- Identity morphism of a filter object $\mathcal{A}$ is $(I_{\mathcal{A}}^{\mathsf{RLD}}; \mathcal{A}; \mathcal{A})$.

To prove that it is really a category is trivial.

4.5 Monovalued and injective reloids

Following the idea of definition of monovalued morphism let's call *monovalued* such a reloid f that $f \circ f^{-1} \subseteq I^{\mathsf{RLD}}_{\operatorname{im} f}$.

Similarly, I will call a reloid *injective* when $f^{-1} \circ f \subseteq I^{\mathsf{RLD}}_{\operatorname{dom} f}$.

Obvious 154. A reloid f is

- monovalued iff $f \circ f^{-1} \subseteq (=)|_{\mho}$;
- injective iff $f^{-1} \circ f \subseteq (=)|_{\mho}$.

Obvious 155.

1. A morphism $(f; \mathcal{A}; \mathcal{B})$ of the category of reloids is monovalued iff the reloid f is monovalued.
2. A morphism $(f; \mathcal{A}; \mathcal{B})$ of the category of reloids is injective iff the reloid f is injective.

Theorem 156.

1. A reloid f is a monovalued iff exists a function (monovalued binary relation) $F \in \operatorname{up} f$.
2. A reloid f is a injective iff exists an injective binary relation $F \in \operatorname{up} f$.
3. A reloid f is a both monovalued and injective iff exists an injection (a monovalued and injective binary relation = injective function) $F \in \operatorname{up} f$.

Proof. The reverse implications are obvious. Let's prove the direct implications:

1. Let f is a monovalued reloid. Then $f \circ f^{-1} \subseteq (=)|_{\mho}$. So exists
$$h \in \operatorname{up}(f \circ f^{-1}) = \operatorname{up} \bigcap^{\mathsf{RLD}} \left\{ F \circ F^{-1} \mid F \in \operatorname{up} f \right\}$$
such that $h \subseteq (=)|_{\mho}$. It's simple to show that $\left\{ F \circ F^{-1} \mid F \in \operatorname{up} f \right\}$ is a filter base. Consequently exists $F \in \operatorname{up} f$ such that $F \circ F^{-1} \subseteq (=)|_{\mho}$ that is F is a function.
2. Similar.
3. Let f is a both monovalued and injective reloid. Then by proved above there exist $F, G \in \operatorname{up} f$ such that F is monovalued and G is injective. Thus $F \cap G \in \operatorname{up} f$ is both monovalued and injective. □

Conjecture 157. If a reloid is monovalued then it is a monovalued function restricted to some filter object.

Conjecture 158. A reloid f is monovalued iff $\forall g \in \mathsf{RLD}: (g \subseteq f \Rightarrow \exists \mathcal{A} \in \mathfrak{F}: g = f|^{\mathsf{RLD}}_{\mathcal{A}})$.

Conjecture 159. A monovalued reloid restricted to an atomic filter object is atomic or empty.

A weaker conjecture:

Conjecture 160. A (monovalued) function restricted to an atomic filter object is atomic or empty.

4.6 Complete reloids and completion of reloids

Definition 161. A *complete* reloid is a reloid representable as join of direct products $\{\alpha\} \times^{\mathsf{RLD}} b$ where $\alpha \in \mho$ and b is an atomic f.o.

Definition 162. A *co-complete* reloid is a reloid representable as join of direct products $a \times^{\mathsf{RLD}} \{\beta\}$ where $\beta \in \mho$ and a is an atomic f.o.

I will denote the sets of complete and co-complete reloids correspondingly as ComplRLD and CoComplRLD.

Obvious 163. Complete and co-complete are dual.

Obvious 164. Complete and co-complete reloids are convex.

Obvious 165. Discrete reloids are complete and co-complete.

Conjecture 166. If a reloid is both complete and co-complete then it is discrete.

Conjecture 167. Composition of complete reloids is complete.

Obvious 168. Join (on the lattice of reloids) of complete reloids is complete.

Corollary 169. ComplRLD (with the induced order) is a complete lattice.

Definition 170. *Completion* and *co-completion* of a reloid f are defined by the formulas:

$$\operatorname{Compl} f = \operatorname{Cor}^{(\mathsf{RLD};\mathrm{Compl}\mathsf{RLD})} f \quad \text{and} \quad \operatorname{CoCompl} f = \operatorname{Cor}^{(\mathsf{RLD};\mathrm{CoCompl}\mathsf{RLD})} f.$$

Theorem 171. Atoms of the lattice ComplRLD are exactly direct products of the form $\{\alpha\} \times^{\mathsf{RLD}} b$ where $\alpha \in \mho$ and b is an atomic f.o.

Proof. First, easy to see that $\{\alpha\} \times^{\mathsf{FCD}} b$ are elements of ComplRLD. Also $\emptyset$ is an element of ComplRLD.

$\{\alpha\} \times^{\mathsf{RLD}} b$ are atoms of ComplFCD because these are atoms of RLD.

Remain to prove that if f is an atom of ComplRLD then $f = \{\alpha\} \times^{\mathsf{RLD}} b$ for some $\alpha \in \mho$ and an atomic f.o. b.

Suppose f is a non-empty complete reloid. Then $\{\alpha\} \times^{\mathsf{RLD}} b \subseteq f$ for some $\alpha \in \mho$ and atomic f.o. b. If f is an atom then $f = \{\alpha\} \times^{\mathsf{FCD}} b$. □

Obvious 172. ComplRLD is an atomistic lattice.

Conjecture 173. $\operatorname{Compl} f \cap^{\mathsf{RLD}} \operatorname{Compl} g = \operatorname{Compl}(f \cap^{\mathsf{RLD}} g)$ for every reloids f and g.

Conjecture 174. $\operatorname{Compl}(\bigcup^{\mathsf{RLD}} R) = \bigcup^{\mathsf{RLD}} \langle \operatorname{Compl} \rangle R$ for every set R of reloids.

Conjecture 175. $\operatorname{Compl} \operatorname{CoCompl} f = \operatorname{CoCompl} \operatorname{Compl} f = \operatorname{Cor} f$ for every reloid f.

Question 176. Is ComplRLD a distributive lattice? Is ComplRLD a co-brouwerian lattice?

Conjecture 177. If f is a complete reloid and R is a set of reloids then

$$f \circ \bigcup^{\mathsf{RLD}} R = \bigcup^{\mathsf{RLD}} \langle f \circ \rangle R.$$

This conjecture can be weakened:

Conjecture 178. If f is a discrete reloid and R is a set of reloids then

$$f \circ \bigcup^{\mathsf{RLD}} R = \bigcup^{\mathsf{RLD}} \langle f \circ \rangle R.$$

Conjecture 179. $\operatorname{Compl} f = f \setminus^{*\,\mathsf{RLD}} (\Omega \times^{\mathsf{RLD}} \mho)$ for every reloid f.

5 Relationships of funcoids and reloids

5.1 Funcoid induced by a reloid

Every reloid f induces a funcoid $(\mathsf{FCD})f$ by the following formulas (for every $\mathcal{X}, \mathcal{Y} \in \mathfrak{F}$):

$$\mathcal{X} [(\mathsf{FCD}) f] \mathcal{Y} \Leftrightarrow \forall F \in \operatorname{up} f : \mathcal{X} [F] \mathcal{Y}$$
$$\langle (\mathsf{FCD}) f \rangle \mathcal{X} = \bigcap\nolimits^{\mathfrak{F}} \{ \langle F \rangle \mathcal{X} \mid F \in \operatorname{up} f \}.$$

We should prove that $(\mathsf{FCD})f$ is really a funcoid.

Proof. We need to prove that

$$\mathcal{X}[(\mathsf{FCD})f]\mathcal{Y} \Leftrightarrow \mathcal{Y} \cap^{\mathfrak{F}} \langle(\mathsf{FCD})f\rangle\mathcal{X} \neq \emptyset \Leftrightarrow \mathcal{X} \cap^{\mathfrak{F}} \left\langle(\mathsf{FCD})f^{-1}\right\rangle\mathcal{Y} \neq \emptyset.$$

The above formula is equivalent to:

$$\forall F \in \operatorname{up} f: \mathcal{X}[F]\mathcal{Y} \Leftrightarrow \mathcal{Y} \cap^{\mathfrak{F}} \bigcap^{\mathfrak{F}} \{\langle F\rangle\mathcal{X} \mid F \in \operatorname{up} f\} \neq \emptyset \Leftrightarrow \mathcal{X} \cap^{\mathfrak{F}} \bigcap^{\mathfrak{F}} \left\{\left\langle F^{-1}\right\rangle\mathcal{Y} \mid F \in \operatorname{up} f\right\} \neq \emptyset.$$

We have $\mathcal{Y} \cap^{\mathfrak{F}} \bigcap^{\mathfrak{F}} \{\langle F\rangle\mathcal{X} \mid F \in \operatorname{up} f\} = \bigcap^{\mathfrak{F}} \left\{\mathcal{Y} \cap^{\mathfrak{F}} \langle F\rangle\mathcal{X} \mid F \in \operatorname{up} f\right\}$.

Let's denote $W = \left\{\mathcal{Y} \cap^{\mathfrak{F}} \langle F\rangle\mathcal{X} \mid F \in \operatorname{up} f\right\}$.

$\forall F \in \operatorname{up} f: \mathcal{X}[F]\mathcal{Y} \Leftrightarrow \forall F \in \operatorname{up} f: \mathcal{Y} \cap^{\mathfrak{F}} \langle F\rangle\mathcal{X} \neq \emptyset \Leftrightarrow \emptyset \notin W$.

We need to prove that $\emptyset \notin W \Leftrightarrow \bigcap^{\mathfrak{F}} W \neq \emptyset$. (The rest follows from symmetry.)

This follows from the fact that W is a generalized filter base.

Let's prove that W is a generalized filter base. For this enough to prove that $V = \{\langle F\rangle\mathcal{X} \mid F \in \operatorname{up} f\}$ is a generalized filter base. Let $\mathcal{A}, \mathcal{B} \in V$ that is $\mathcal{A} = \langle P\rangle\mathcal{X}$, $\mathcal{B} = \langle Q\rangle\mathcal{X}$ where $P, Q \in \operatorname{up} f$. Then for $\mathcal{C} = \langle P \cap Q\rangle\mathcal{X}$ is true both $\mathcal{C} \in V$ and $\mathcal{C} \subseteq \mathcal{A}, \mathcal{B}$. So V is a generalized filter base and thus W is a generalized filter base. □

Proposition 180. $(\mathsf{FCD})f = f$ for every binary relation f.

Proof. $\mathcal{X}[(\mathsf{FCD})f]\mathcal{Y} \Leftrightarrow \forall F \in \operatorname{up} f: \mathcal{X}[F]\mathcal{Y} \Leftrightarrow \mathcal{X}[f]\mathcal{Y}$ (for every $\mathcal{X}, \mathcal{Y} \in \mathfrak{F}$). □

Theorem 181. $\mathcal{X}[(\mathsf{FCD})f]\mathcal{Y} \Leftrightarrow (\mathcal{X} \times^{\mathsf{RLD}} \mathcal{Y}) \cap^{\mathsf{RLD}} f \neq \emptyset$ for every $\mathcal{X}, \mathcal{Y} \in \mathfrak{F}$ and $f \in \mathsf{RLD}$.

Proof.

$$\begin{aligned}
(\mathcal{X} \times^{\mathsf{RLD}} \mathcal{Y}) \cap^{\mathsf{RLD}} f \neq \emptyset &\Leftrightarrow \forall F \in \operatorname{up} f, P \in \operatorname{up}(\mathcal{X} \times^{\mathsf{RLD}} \mathcal{Y}): P \cap F \neq \emptyset \\
&\Leftrightarrow \forall F \in \operatorname{up} f, X \in \operatorname{up} \mathcal{X}, Y \in \operatorname{up} \mathcal{Y}: (X \times^{\mathsf{RLD}} Y) \cap^{\mathsf{RLD}} F \neq \emptyset \\
&\Leftrightarrow \forall F \in \operatorname{up} f, X \in \operatorname{up} \mathcal{X}, Y \in \operatorname{up} \mathcal{Y}: (X \times Y) \cap F \neq \emptyset \\
&\Leftrightarrow \forall F \in \operatorname{up} f, X \in \operatorname{up} \mathcal{X}, Y \in \operatorname{up} \mathcal{Y}: X[F]Y \\
&\Leftrightarrow \forall F \in \operatorname{up} f: \mathcal{X}[F]\mathcal{Y} \\
&\Leftrightarrow \mathcal{X}[(\mathsf{FCD})f]\mathcal{Y}.
\end{aligned}$$

□

Theorem 182. $(\mathsf{FCD})f = \bigcap^{\mathsf{FCD}} \operatorname{up} f$ for every reloid f.

Proof. Let a is an atomic filter object.

$((\mathsf{FCD})f)a = \bigcap^{\mathfrak{F}} \{\langle F\rangle a \mid F \in \operatorname{up} f\}$ by the definition of (FCD).

$\left\langle\bigcap^{\mathsf{FCD}} \operatorname{up} f\right\rangle a = \bigcap^{\mathfrak{F}} \{\langle F\rangle a \mid F \in \operatorname{up} f\}$ by the theorem 61.

So $\langle(\mathsf{FCD})f\rangle a = \left\langle\bigcap^{\mathsf{FCD}} \operatorname{up} f\right\rangle a$ for every atomic filter object a. □

Lemma 183. $\langle g\rangle \bigcap^{\mathfrak{F}} S = \bigcap^{\mathfrak{F}} \langle\langle g\rangle\rangle S$ if g is a funcoid and S is a filter base.

Proof. $\operatorname{up} \bigcap^{\mathfrak{F}} S = \bigcup \langle\operatorname{up}\rangle S$ by the theorem 3.

$\langle g\rangle \bigcap^{\mathfrak{F}} S = \bigcap^{\mathfrak{F}} \langle\langle g\rangle\rangle \operatorname{up} \bigcap^{\mathfrak{F}} S$ by the theorem 38.

$\bigcap^{\mathfrak{F}} \langle\langle g\rangle\rangle \operatorname{up} \bigcap^{\mathfrak{F}} S = \bigcap^{\mathfrak{F}} \langle\langle g\rangle\rangle \bigcup \langle\operatorname{up}\rangle S$.

$\bigcap^{\mathfrak{F}} \langle\langle g\rangle\rangle \bigcup \langle\operatorname{up}\rangle S \subseteq \bigcap^{\mathfrak{F}} \langle\langle g\rangle\rangle S$ because $S \subseteq \bigcup \langle\operatorname{up}\rangle S$.

$\bigcap^{\mathfrak{F}} \langle\langle g\rangle\rangle \bigcup \langle\operatorname{up}\rangle S \supseteq \bigcap^{\mathfrak{F}} \langle\langle g\rangle\rangle S$ because every element of $\bigcup \langle\operatorname{up}\rangle S$ is greater than some element of S.

So $\bigcap^{\mathfrak{F}} \langle\langle g\rangle\rangle \bigcup \langle\operatorname{up}\rangle S = \bigcap^{\mathfrak{F}} \langle\langle g\rangle\rangle S$.

Combining these equalities we produce $\langle g\rangle \bigcap^{\mathfrak{F}} S = \bigcap^{\mathfrak{F}} \langle\langle g\rangle\rangle S$. □

Lemma 184. For every two filter bases S and T of binary relations and every set A

$$\bigcap^{\mathfrak{F}} S = \bigcap^{\mathfrak{F}} T \Rightarrow \bigcap^{\mathfrak{F}} \{\langle F\rangle A \mid F \in S\} = \bigcap^{\mathfrak{F}} \{\langle G\rangle A \mid G \in T\}$$

Proof. Let $\bigcap^{\mathfrak{F}} S = \bigcap^{\mathfrak{F}} T$.

First let prove that $\{\langle F\rangle A \mid F \in S\}$ is a filter base. Let $X, Y \in \{\langle F\rangle A \mid F \in S\}$. Then $X = \langle F_X\rangle A$ and $Y = \langle F_Y\rangle A$ for some $F_X, F_Y \in S$. Because S is a filter base, we have $S \ni F_Z \subseteq F_X \cap F_Y$. So $\langle F_Z\rangle A \subseteq X \cap Y$ and $\langle F_Z\rangle A \in \{\langle F\rangle A \mid F \in S\}$. So $\{\langle F\rangle A \mid F \in S\}$ is a filter base.

Suppose $X \in \operatorname{up} \bigcap^{\mathfrak{F}} \{\langle F\rangle A \mid F \in S\}$. Then exists $X' \in \{\langle F\rangle A \mid F \in S\}$ where $X \supseteq X'$ because $\{\langle F\rangle A \mid F \in S\}$ is a filter base. That is $X' = \langle F\rangle A$ for some $F \in S$. There exists $G \in T$ such that $G \subseteq F$ because T is a filter base. Let $Y' = \langle G\rangle A$. We have $Y' \subseteq X' \subseteq X$; $Y' \in \{\langle G\rangle A \mid G \in T\}$; $Y' \in \operatorname{up} \bigcap^{\mathfrak{F}} \{\langle G\rangle A \mid G \in T\}$; $X \in \operatorname{up} \bigcap^{\mathfrak{F}} \{\langle G\rangle A \mid G \in T\}$. The reverse is symmetric. □

Lemma 185. $\{G \circ F \mid F \in \operatorname{up} f, G \in \operatorname{up} g\}$ is a filter base for every reloids f and g.

Proof. Let denote $D = \{G \circ F \mid F \in \operatorname{up} f, G \in \operatorname{up} g\}$. Let $A \in D \wedge B \in D$. Then $A = G_A \circ F_A \wedge B = G_B \circ F_B$ for some $F_A, F_B \in \operatorname{up} f$ and $G_A, G_B \in \operatorname{up} g$. So $A \cap B \supseteq (G_A \cap G_B) \circ (F_A \cap F_B) \in D$ because $F_A \cap F_B \in \operatorname{up} f$ and $G_A \cap G_B \in \operatorname{up} g$. □

Theorem 186. $(\mathsf{FCD})(g \circ f) = ((\mathsf{FCD}) g) \circ ((\mathsf{FCD}) f)$ for every reloids f and g.

Proof.

$$\begin{aligned}\langle (\mathsf{FCD})(g \circ f)\rangle X &= \bigcap^{\mathfrak{F}} \{\langle H\rangle X \mid H \in \operatorname{up}(g \circ f)\}\\ &= \bigcap^{\mathfrak{F}} \left\{\langle H\rangle X \mid H \in \operatorname{up} \bigcap^{\mathsf{RLD}} \{G \circ F \mid F \in \operatorname{up} f, G \in \operatorname{up} g\}\right\}.\end{aligned}$$

Obviously

$$\bigcap^{\mathsf{RLD}} \{G \circ F \mid F \in \operatorname{up} f, G \in \operatorname{up} g\} = \bigcap^{\mathsf{RLD}} \operatorname{up} \bigcap^{\mathsf{RLD}} \{G \circ F \mid F \in \operatorname{up} f, G \in \operatorname{up} g\};$$

from this by the lemma 184 (taking in account that $\{G \circ F \mid F \in \operatorname{up} f, G \in \operatorname{up} g\}$ and $\operatorname{up} \bigcap^{\mathsf{RLD}} \{G \circ F \mid F \in \operatorname{up} f, G \in \operatorname{up} g\}$ are filter bases)

$$\bigcap^{\mathfrak{F}} \left\{\langle H\rangle X \mid H \in \operatorname{up} \bigcap^{\mathsf{RLD}} \{G \circ F \mid F \in \operatorname{up} f, G \in \operatorname{up} g\}\right\} = \bigcap^{\mathfrak{F}} \{\langle G \circ F\rangle X \mid F \in \operatorname{up} f, G \in \operatorname{up} g\}.$$

On the other side

$$\begin{aligned}\langle ((\mathsf{FCD}) g) \circ ((\mathsf{FCD}) f)\rangle X &= \langle (\mathsf{FCD}) g\rangle \langle (\mathsf{FCD}) f\rangle X\\ &= \langle (\mathsf{FCD}) g\rangle \bigcap^{\mathfrak{F}} \{\langle F\rangle X \mid F \in \operatorname{up} f\}\\ &= \bigcap^{\mathfrak{F}} \left\{\langle G\rangle \bigcap^{\mathfrak{F}} \{\langle F\rangle X \mid F \in \operatorname{up} f\} \mid G \in \operatorname{up} g\right\}.\end{aligned}$$

Let's prove that $\{\langle F\rangle X \mid F \in \operatorname{up} f\}$ is a filter base. If $A, B \in \{\langle F\rangle X \mid F \in \operatorname{up} f\}$ then $A = \langle F_1\rangle X$ and $B = \langle F_2\rangle X$ where $F_1, F_2 \in \operatorname{up} f$. $A \cap B \supseteq \langle F_1 \cap F_2\rangle X \in \{\langle F\rangle X \mid F \in \operatorname{up} f\}$. So $\{\langle F\rangle X \mid F \in \operatorname{up} f\}$ is really a filter base.

By the lemma 183 $\langle G\rangle \bigcap^{\mathfrak{F}} \{\langle F\rangle X \mid F \in \operatorname{up} f\} = \bigcap^{\mathfrak{F}} \{\langle G\rangle \langle F\rangle X \mid F \in \operatorname{up} f\}$. So continuing the above equalities,

$$\begin{aligned}\langle ((\mathsf{FCD}) g) \circ ((\mathsf{FCD}) f)\rangle X &= \bigcap^{\mathfrak{F}} \left\{\bigcap^{\mathfrak{F}} \{\langle G\rangle \langle F\rangle X \mid F \in \operatorname{up} f\} \mid G \in \operatorname{up} g\right\}\\ &= \bigcap^{\mathfrak{F}} \{\langle G\rangle \langle F\rangle X \mid F \in \operatorname{up} f, G \in \operatorname{up} g\}\\ &= \bigcap^{\mathfrak{F}} \{\langle G \circ F\rangle X \mid F \in \operatorname{up} f, G \in \operatorname{up} g\}.\end{aligned}$$

Combining these equalities we get $\langle (\mathsf{FCD})(g \circ f)\rangle X = \langle ((\mathsf{FCD}) g) \circ ((\mathsf{FCD}) f)\rangle X$ for every set X. □

Proposition 187. $(\mathsf{FCD})(\mathcal{A} \times^{\mathsf{RLD}} \mathcal{B}) = \mathcal{A} \times^{\mathsf{FCD}} \mathcal{B}$ for every f.o. $\mathcal{A}$ and $\mathcal{B}$.

Proof. $\mathcal{X} \left[(\mathsf{FCD})(\mathcal{A} \times^{\mathsf{RLD}} \mathcal{B})\right] \mathcal{Y} \Leftrightarrow \forall F \in \operatorname{up}(\mathcal{A} \times^{\mathsf{RLD}} \mathcal{B}) \colon \mathcal{X} [F] \mathcal{Y}$ (for every $\mathcal{X}, \mathcal{Y} \in \mathfrak{F}$).

Evidently $\forall F \in \operatorname{up}(\mathcal{A} \times^{\mathsf{RLD}} \mathcal{B}) \colon \mathcal{X}[F]\mathcal{Y} \Rightarrow \forall A \in \operatorname{up} \mathcal{A}, B \in \operatorname{up} \mathcal{B} \colon \mathcal{X}[A \times B]\mathcal{Y}$.

Let $\forall A \in \operatorname{up} \mathcal{A}, B \in \operatorname{up} \mathcal{B} \colon \mathcal{X}[A \times B]\mathcal{Y}$. Then if $F \in \operatorname{up}(\mathcal{A} \times^{\mathsf{RLD}} \mathcal{B})$ then there are $A \in \operatorname{up} \mathcal{A}$, $B \in \operatorname{up} \mathcal{B}$ such that $F \supseteq A \times B$. So $\mathcal{X}[F]\mathcal{Y}$.

We proved $\forall F \in \operatorname{up}(\mathcal{A} \times^{\mathsf{RLD}} \mathcal{B}) \colon \mathcal{X}[F]\mathcal{Y} \Leftrightarrow \forall A \in \operatorname{up} \mathcal{A}, B \in \operatorname{up} \mathcal{B} \colon \mathcal{X}[A \times B]\mathcal{Y}$.

Further $\forall A \in \operatorname{up} \mathcal{A}, B \in \operatorname{up} \mathcal{B} \colon \mathcal{X}[A \times B]\mathcal{Y} \Leftrightarrow \forall A \in \operatorname{up} \mathcal{A}, B \in \operatorname{up} \mathcal{B} \colon \mathcal{X} \cap^{\mathfrak{F}} A \neq \emptyset \wedge \mathcal{Y} \cap^{\mathfrak{F}} B \neq \emptyset \Leftrightarrow \mathcal{X} \cap^{\mathfrak{F}} \mathcal{A} \neq \emptyset \wedge \mathcal{Y} \cap^{\mathfrak{F}} \mathcal{B} \neq \emptyset \Leftrightarrow \mathcal{X}\left[\mathcal{A} \times^{\mathsf{FCD}} \mathcal{B}\right]\mathcal{Y}$.

Thus $\mathcal{X}\left[(\mathsf{FCD})(\mathcal{A} \times^{\mathsf{RLD}} \mathcal{B})\right]\mathcal{Y} \Leftrightarrow \mathcal{X}\left[\mathcal{A} \times^{\mathsf{FCD}} \mathcal{B}\right]\mathcal{Y}$. □

5.2 Reloids induced by funcoid

Every funcoid f induces a reloid in two ways, intersection of *outward* relations and union of *inward* direct products of filter objects:

$$(\mathsf{RLD})_{\mathrm{out}} f \overset{\mathrm{def}}{=} \bigcap^{\mathsf{RLD}} \operatorname{up} f;$$
$$(\mathsf{RLD})_{\mathrm{in}} f \overset{\mathrm{def}}{=} \bigcup^{\mathsf{RLD}} \left\{ \mathcal{A} \times^{\mathsf{RLD}} \mathcal{B} \mid \mathcal{A}, \mathcal{B} \in \mathfrak{F}, \mathcal{A} \times^{\mathsf{FCD}} \mathcal{B} \subseteq f \right\}$$

Theorem 188. $(\mathsf{RLD})_{\mathrm{in}} f = \bigcup^{\mathsf{RLD}} \left\{ a \times^{\mathsf{RLD}} b \mid a, b \in \operatorname{atoms}^{\mathfrak{F}} \mho, a \times^{\mathsf{FCD}} b \subseteq f \right\}$.

Proof. Follows from the theorem 138. □

Remark 189. In seems that $(\mathsf{RLD})_{\mathrm{in}}$ has smoother properties and is more important than $(\mathsf{RLD})_{\mathrm{out}}$. (However see also the exercize below for $(\mathsf{RLD})_{\mathrm{in}}$ not preserving identities.)

Proposition 190. $(\mathsf{RLD})_{\mathrm{out}} f = f$ for every binary relation f.

Proof. $(\mathsf{RLD})_{\mathrm{out}} f = \bigcap^{\mathsf{RLD}} \operatorname{up} f = \min \operatorname{up} f = f$. □

Lemma 191. $F \in \operatorname{up} (\mathsf{RLD})_{\mathrm{in}} f \Leftrightarrow \forall a, b \in \operatorname{atoms}^{\mathfrak{F}} \mho \colon (a[f]b \Rightarrow F \supseteq a \times^{\mathsf{RLD}} b)$ for a funcoid f.

Proof.

$$\begin{aligned}
F \in \operatorname{up} (\mathsf{RLD})_{\mathrm{in}} f &\Leftrightarrow F \in \operatorname{up} \bigcup^{\mathfrak{F}} \left\{ a \times^{\mathsf{RLD}} b \mid a, b \in \operatorname{atoms}^{\mathfrak{F}} \mho, a \times^{\mathsf{FCD}} b \subseteq f \right\} \\
&\Leftrightarrow \forall a, b \in \operatorname{atoms}^{\mathfrak{F}} \mho \colon (a \times^{\mathsf{FCD}} b \subseteq f \Rightarrow F \in \operatorname{up}(a \times^{\mathsf{RLD}} b)) \\
&\Leftrightarrow \forall a, b \in \operatorname{atoms}^{\mathfrak{F}} \mho \colon ((a \times^{\mathsf{FCD}} b) \cap^{\mathsf{FCD}} f \neq \emptyset \Rightarrow F \supseteq a \times^{\mathsf{RLD}} b) \\
&\Leftrightarrow \forall a, b \in \operatorname{atoms}^{\mathfrak{F}} \mho \colon (a[f]b \Rightarrow F \supseteq a \times^{\mathsf{RLD}} b).
\end{aligned}$$

□

Surprisingly a funcoid is greater inward than outward:

Theorem 192. $(\mathsf{RLD})_{\mathrm{out}} f \subseteq (\mathsf{RLD})_{\mathrm{in}} f$ for a funcoid f.

Proof. We need to prove

$$\bigcap^{\mathsf{RLD}} \operatorname{up} f \subseteq \bigcup^{\mathsf{RLD}} \left\{ \mathcal{A} \times^{\mathsf{RLD}} \mathcal{B} \mid \mathcal{A}, \mathcal{B} \in \mathfrak{F}, \mathcal{A} \times^{\mathsf{FCD}} \mathcal{B} \subseteq f \right\}.$$

Let

$$K \in \operatorname{up} \bigcup^{\mathfrak{F}} \left\{ \mathcal{A} \times^{\mathsf{RLD}} \mathcal{B} \mid \mathcal{A}, \mathcal{B} \in \mathfrak{F}, \mathcal{A} \times^{\mathsf{FCD}} \mathcal{B} \subseteq f \right\}.$$

Then

$$\begin{aligned}
K &= \bigcup \left\{ X_{\mathcal{A}} \times Y_{\mathcal{B}} \mid \mathcal{A}, \mathcal{B} \in \mathfrak{F}, \mathcal{A} \times^{\mathsf{FCD}} \mathcal{B} \subseteq f \right\} \\
&= \bigcup^{\mathsf{RLD}} \left\{ X_{\mathcal{A}} \times Y_{\mathcal{B}} \mid \mathcal{A}, \mathcal{B} \in \mathfrak{F}, \mathcal{A} \times^{\mathsf{FCD}} \mathcal{B} \subseteq f \right\} \\
&\supseteq f
\end{aligned}$$

where $X_{\mathcal{A}} \in \operatorname{up} \mathcal{A}$, $Y_{\mathcal{B}} \in \operatorname{up} \mathcal{B}$. So $K \in \operatorname{up} f$; $K \supseteq \bigcap^{\mathsf{RLD}} \operatorname{up} f$; $K \in \operatorname{up} \bigcap^{\mathsf{RLD}} \operatorname{up} f$. □

Theorem 193. $(\mathsf{FCD})(\mathsf{RLD})_{\mathrm{in}} f = f$ for every funcoid f.

Proof. For every sets X and Y

$$\begin{array}{rcl} X[(\mathsf{FCD})(\mathsf{RLD})_{\mathrm{in}} f]Y & \Leftrightarrow & \\ (X \times^{\mathsf{RLD}} Y) \cap^{\mathsf{RLD}} (\mathsf{RLD})_{\mathrm{in}} f \neq \emptyset & \Leftrightarrow & \\ (X \times Y) \cap^{\mathsf{RLD}} \bigcup^{\mathsf{RLD}} \{a \times^{\mathsf{RLD}} b \mid a, b \in \operatorname{atoms}^{\mathfrak{F}} \mho, a \times^{\mathsf{FCD}} b \subseteq f\} & \Leftrightarrow & \text{(theorem 53 in [14])} \\ \exists a, b \in \operatorname{atoms}^{\mathfrak{F}} \mho: (a \times^{\mathsf{FCD}} b \subseteq f \wedge (X \times Y) \cap^{\mathsf{RLD}} (a \times^{\mathsf{RLD}} b) \neq \emptyset) & \Leftrightarrow & \\ \exists a, b \in \operatorname{atoms}^{\mathfrak{F}} \mho: (a[f]b \subseteq f \wedge a \subseteq X \wedge b \subseteq Y) & \Leftrightarrow & \\ \exists a \in \operatorname{atoms}^{\mathfrak{F}} X, b \in \operatorname{atoms}^{\mathfrak{F}} Y: a[f]b & \Leftrightarrow & \\ X[f]Y. & & \end{array}$$

Thus $(\mathsf{FCD})(\mathsf{RLD})_{\mathrm{in}} f = f$. □

Remark 194. The above theorem allows to represent funcoids as reloids.

Obvious 195. $(\mathsf{RLD})_{\mathrm{in}}(\mathcal{A} \times^{\mathsf{FCD}} \mathcal{B}) = \mathcal{A} \times^{\mathsf{RLD}} \mathcal{B}$ for every f.o. $\mathcal{A}$, $\mathcal{B}$.

Question 196. $(\mathsf{RLD})_{\mathrm{out}}(\mathcal{A} \times^{\mathsf{FCD}} \mathcal{B}) = \mathcal{A} \times^{\mathsf{RLD}} \mathcal{B}$ for every f.o. $\mathcal{A}$, $\mathcal{B}$?

Conjecture 197. $(\mathsf{RLD})_{\mathrm{out}} I_{\mathcal{A}}^{\mathsf{FCD}} = I_{\mathcal{A}}^{\mathsf{RLD}}$ for every f.o. $\mathcal{A}$.

Exercise 1. Prove that generally $(\mathsf{RLD})_{\mathrm{in}} I_{\mathcal{A}}^{\mathsf{FCD}} \neq I_{\mathcal{A}}^{\mathsf{RLD}}$.

Conjecture 198. For a convex reloid f

1. $(\mathsf{RLD})_{\mathrm{out}}(\mathsf{FCD}) f = f$;
2. $(\mathsf{RLD})_{\mathrm{in}}(\mathsf{FCD}) f = f$.

6 Galois connections of funcoids and reloids

Theorem 199. (FCD) is the lower adjoint of $(\mathsf{RLD})_{\mathrm{in}}$.

Proof. Because (FCD) and $(\mathsf{RLD})_{\mathrm{in}}$ are trivially monotone, it's enough to prove

$$f \subseteq (\mathsf{RLD})_{\mathrm{in}}(\mathsf{FCD}) f \text{ and } (\mathsf{FCD})(\mathsf{RLD})_{\mathrm{in}} g \subseteq g.$$

The second formula follows from the fact that $(\mathsf{FCD})(\mathsf{RLD})_{\mathrm{in}} g = g$.

$$\begin{array}{rl} (\mathsf{RLD})_{\mathrm{in}}(\mathsf{FCD}) f & = \\ \bigcup^{\mathsf{RLD}} \{a \times^{\mathsf{RLD}} b \mid a, b \in \operatorname{atoms}^{\mathfrak{F}} \mho, a \times^{\mathsf{FCD}} b \subseteq (\mathsf{FCD}) f\} & = \\ \bigcup^{\mathsf{RLD}} \{a \times^{\mathsf{RLD}} b \mid a, b \in \operatorname{atoms}^{\mathfrak{F}} \mho, a[(\mathsf{FCD}) f]b\} & = \\ \bigcup^{\mathsf{RLD}} \{a \times^{\mathsf{RLD}} b \mid a, b \in \operatorname{atoms}^{\mathfrak{F}} \mho, (a \times^{\mathsf{RLD}} b) \cap^{\mathsf{RLD}} f \neq \emptyset\} & \supseteq \\ \bigcup^{\mathsf{RLD}} \{p \in \operatorname{atoms}^{\mathsf{RLD}} (a \times^{\mathsf{RLD}} b) \mid a, b \in \operatorname{atoms}^{\mathfrak{F}} \mho, p \cap^{\mathsf{RLD}} f \neq \emptyset\} & = \\ \bigcup^{\mathsf{RLD}} \{p \in \operatorname{atoms}^{\mathsf{RLD}} (\mho \times \mho) \mid p \cap^{\mathsf{RLD}} f \neq \emptyset\} & = \\ \bigcup^{\mathsf{RLD}} \{p \mid p \in \operatorname{atoms}^{\mathsf{RLD}} f\} = f. & \end{array}$$

□

Corollary 200. gedit my

1. $(\mathsf{FCD}) \bigcup^{\mathsf{RLD}} S = \bigcup^{\mathsf{FCD}} \langle (\mathsf{FCD}) \rangle S$ if S is a set of reloids.
2. $(\mathsf{RLD})_{\mathrm{in}} \bigcap^{\mathsf{FCD}} S = \bigcap^{\mathsf{RLD}} \langle (\mathsf{RLD})_{\mathrm{in}} \rangle S$ if S is a set of funcoids.

Conjecture 201. $(\mathsf{RLD})_{\mathrm{in}}$ is not a lower adjoint (in general).

Conjecture 202. $(\mathsf{RLD})_{\mathrm{out}}$ is neither a lower adjoint nor an upper adjoint (in general).

See also the corollary 262 below.

7 Continuous morphisms

This section will use the apparatus from the section "Partially ordered dagger categories".

7.1 Traditional definitions of continuity

7.1.1 Pre-topology

Let μ and ν are funcoids representing some pre-topologies. By definition a function f is continuous map from μ to ν in point a iff

$$\forall \epsilon \in \operatorname{up}\langle \nu \rangle f a \exists \delta \in \operatorname{up}\langle \mu \rangle \{a\} : \langle f \rangle \delta \subseteq \epsilon.$$

Equivalently transforming this formula we get:

$$\forall \epsilon \in \operatorname{up}\langle \nu \rangle f a : \langle f \rangle \langle \mu \rangle \{a\} \subseteq \epsilon;$$
$$\langle f \rangle \langle \mu \rangle \{a\} \subseteq \langle \nu \rangle f a;$$
$$\langle f \rangle \langle \mu \rangle \{a\} \subseteq \langle \nu \rangle \langle f \rangle \{a\};$$
$$\langle f \circ \mu \rangle \{a\} \subseteq \langle \nu \circ f \rangle \{a\}.$$

So f is a continuous map from μ to ν in every point of its domain iff $f \circ \mu \subseteq \nu \circ f$.

7.1.2 Proximity spaces

Let μ and ν are proximity (nearness) spaces (which I consider a special case of funcoids). By definition a function f is a proximity-continuous map (also called equivicontinuous) from μ to ν iff

$$\forall X, Y \in \mathscr{P}\mho : (X [\mu] Y \Rightarrow \langle f \rangle X [\nu] \langle f \rangle Y).$$

Equivalently transforming this formula we get:

$$\forall X, Y \in \mathscr{P}\mho : (X [\mu] Y \Rightarrow \langle f \rangle Y \cap \langle \nu \rangle \langle f \rangle X \neq \emptyset);$$
$$\forall X, Y \in \mathscr{P}\mho : (X [\mu] Y \Rightarrow \langle f \rangle Y \cap \langle \nu \circ f \rangle X \neq \emptyset);$$
$$\forall X, Y \in \mathscr{P}\mho : (X [\mu] Y \Rightarrow X [\nu \circ f] \langle f \rangle Y);$$
$$\forall X, Y \in \mathscr{P}\mho : (X [\mu] Y \Rightarrow \langle f \rangle Y \left[(\nu \circ f)^{-1} \right] X);$$
$$\forall X, Y \in \mathscr{P}\mho : (X [\mu] Y \Rightarrow \langle f \rangle Y \left[f^{-1} \circ \nu^{-1} \right] X);$$
$$\forall X, Y \in \mathscr{P}\mho : (X [\mu] Y \Rightarrow X \cap \left\langle f^{-1} \circ \nu^{-1} \right\rangle \langle f \rangle Y \neq \emptyset);$$
$$\forall X, Y \in \mathscr{P}\mho : (X [\mu] Y \Rightarrow X \cap \left\langle f^{-1} \circ \nu^{-1} \circ f \right\rangle Y \neq \emptyset);$$
$$\forall X, Y \in \mathscr{P}\mho : (X [\mu] Y \Rightarrow Y \left[f^{-1} \circ \nu^{-1} \circ f \right] X);$$
$$\forall X, Y \in \mathscr{P}\mho : (X [\mu] Y \Rightarrow X \left[f^{-1} \circ \nu \circ f \right] Y);$$
$$\mu \subseteq f^{-1} \circ \nu \circ f.$$

So a function f is proximity-continuous iff $\mu \subseteq f^{-1} \circ \nu \circ f$.

7.1.3 Uniform spaces

Uniform spaces are a special case of reloids.

Let μ and ν are uniform spaces. By definition a function f is a uniformly continuous map from μ to ν iff

$$\forall \epsilon \in \operatorname{up} \nu \exists \delta \in \operatorname{up} \mu \forall (x; y) \in \delta : (f x; f y) \in \epsilon.$$

Equivalently transforming this formula we get:

$$\forall \epsilon \in \operatorname{up} \nu \exists \delta \in \operatorname{up} \mu \forall (x; y) \in \delta : \{(f x; f y)\} \subseteq \epsilon$$
$$\forall \epsilon \in \operatorname{up} \nu \exists \delta \in \operatorname{up} \mu \forall (x; y) \in \delta : f \circ \{(x; y)\} \circ f^{-1} \subseteq \epsilon$$
$$\forall \epsilon \in \operatorname{up} \nu \exists \delta \in \operatorname{up} \mu : f \circ \delta \circ f^{-1} \subseteq \epsilon$$
$$\forall \epsilon \in \operatorname{up} \nu : f \circ \mu \circ f^{-1} \subseteq \epsilon$$
$$f \circ \mu \circ f^{-1} \subseteq \nu.$$

So a function f is uniformly continuous iff $f \circ \mu \circ f^{-1} \subseteq \nu$.

7.2 Our three definitions of continuity

I have expressed different kinds of continuity with simple algebraic formulas hiding the complexity of traditional epsilon-delta notation behind a smart algebra. Let's summarize these three algebraic formulas:

Let μ and ν are endomorphisms of some partially ordered precategory. Continuous functions can be defined as these morphisms f of this precategory which conform to the following formula:

$$f \in \mathrm{C}(\mu; \nu) \Leftrightarrow f \in \mathrm{Mor}(\mathrm{Ob}\,\mu; \mathrm{Ob}\,\nu) \wedge f \circ \mu \subseteq \nu \circ f.$$

If the precategory is a partially ordered dagger precategory then continuity also can be defined in two other ways:

$$f \in \mathrm{C}'(\mu; \nu) \Leftrightarrow f \in \mathrm{Mor}(\mathrm{Ob}\,\mu; \mathrm{Ob}\,\nu) \wedge \mu \subseteq f^\dagger \circ \nu \circ f;$$
$$f \in \mathrm{C}''(\mu; \nu) \Leftrightarrow f \in \mathrm{Mor}(\mathrm{Ob}\,\mu; \mathrm{Ob}\,\nu) \wedge f \circ \mu \circ f^\dagger \subseteq \nu.$$

Remark 203. In the examples (above) about funcoids and reloids the "dagger functor" is the inverse of a funcoid or reloid, that is $f^\dagger = f^{-1}$.

Proposition 204. Every of these three definitions of continuity forms a sub-precategory (subcategory if the original precategory is a category).

Proof.

C. Let $f \in \mathrm{C}(\mu; \nu)$, $g \in \mathrm{C}(\nu; \pi)$. Then $f \circ \mu \subseteq \nu \circ f$, $g \circ \nu \subseteq \pi \circ g$; $g \circ f \circ \mu \subseteq g \circ \nu \circ f \subseteq \pi \circ g \circ f$. So $g \circ f \in \mathrm{C}(\mu; \pi)$. $1_{\mathrm{Ob}\,\mu} \in \mathrm{C}(\mu; \mu)$ is obvious.

***C′*.** Let $f \in \mathrm{C}'(\mu; \nu)$, $g \in \mathrm{C}'(\nu; \pi)$. Then $\mu \subseteq f^\dagger \circ \nu \circ f$, $\nu \subseteq g^\dagger \circ \pi \circ g$;

$$\mu \subseteq f^\dagger \circ g^\dagger \circ \pi \circ g \circ f; \quad \mu \subseteq (g \circ f)^\dagger \circ \pi \circ (g \circ f).$$

So $g \circ f \in \mathrm{C}'(\mu; \pi)$. $1_{\mathrm{Ob}\,\mu} \in \mathrm{C}'(\mu; \mu)$ is obvious.

C″. Let $f \in \mathrm{C}''(\mu; \nu)$, $g \in \mathrm{C}''(\nu; \pi)$. Then $f \circ \mu \circ f^\dagger \subseteq \nu$, $g \circ \nu \circ g^\dagger \subseteq \pi$;

$$g \circ f \circ \mu \circ f^\dagger \circ g^\dagger \subseteq \pi; \quad (g \circ f) \circ \mu \circ (g \circ f)^\dagger \subseteq \pi.$$

So $g \circ f \in \mathrm{C}''(\mu; \pi)$. $1_{\mathrm{Ob}\,\mu} \in \mathrm{C}''(\mu; \mu)$ is obvious. □

Proposition 205. For a monovalued morphism f of a partially ordered dagger category and its endomorphisms μ and ν

$$f \in \mathrm{C}'(\mu; \nu) \Rightarrow f \in \mathrm{C}(\mu; \nu) \Rightarrow f \in \mathrm{C}''(\mu; \nu).$$

Proof. Let $f \in \mathrm{C}'(\mu; \nu)$. Then $\mu \subseteq f^\dagger \circ \nu \circ f$; $f \circ \mu \subseteq f \circ f^\dagger \circ \nu \circ f \subseteq 1_{\mathrm{Dst}\,f} \circ \nu \circ f = \nu \circ f$; $f \in \mathrm{C}(\mu; \nu)$.

Let $f \in \mathrm{C}(\mu; \nu)$. Then $f \circ \mu \subseteq \nu \circ f$; $f \circ \mu \circ f^\dagger \subseteq \nu \circ f \circ f^\dagger \subseteq \nu \circ 1_{\mathrm{Dst}\,f} = \nu$; $f \in \mathrm{C}''(\mu; \nu)$. □

Proposition 206. For an entirely defined morphism f of a partially ordered dagger category and its endomorphisms μ and ν

$$f \in \mathrm{C}''(\mu; \nu) \Rightarrow f \in \mathrm{C}(\mu; \nu) \Rightarrow f \in \mathrm{C}'(\mu; \nu).$$

Proof. Let $f \in \mathrm{C}''(\mu; \nu)$. Then $f \circ \mu \circ f^\dagger \subseteq \nu$; $f \circ \mu \circ f^\dagger \circ f \subseteq \nu \circ f$; $f \circ \mu \circ 1_{\mathrm{Src}\,f} \subseteq \nu \circ f$; $f \circ \mu \subseteq \nu \circ f$; $f \in \mathrm{C}(\mu; \nu)$.

Let $f \in \mathrm{C}(\mu; \nu)$. Then $f \circ \mu \subseteq \nu \circ f$; $f^\dagger \circ f \circ \mu \subseteq f^\dagger \circ \nu \circ f$; $1_{\mathrm{Src}\,f} \circ \mu \subseteq f^\dagger \circ \nu \circ f$; $\mu \subseteq f^\dagger \circ \nu \circ f$; $f \in \mathrm{C}'(\mu; \nu)$. □

For entirely defined monovalued morphisms our three definitions of continuity coincide:

Theorem 207. If f is a monovalued and entirely defined morphism then

$$f \in \mathrm{C}'(\mu; \nu) \Leftrightarrow f \in \mathrm{C}(\mu; \nu) \Leftrightarrow f \in \mathrm{C}''(\mu; \nu).$$

Proof. From two previous propositions. □

The classical general topology theorem that uniformly continuous function from a uniform space to an other uniform space is near-continuous regarding the proximities generated by the uniformities, generalized for reloids and funcoids takes the following form:

Theorem 208. If an entirely defined morphism of the category of reloids $f \in \mathrm{C}''(\mu;\nu)$ for some endomorphisms μ and ν of the category of reloids, then $(\mathsf{FCD}) f \in \mathrm{C}'((\mathsf{FCD})\mu;(\mathsf{FCD})\nu)$.

Exercise 2. I leave a simple exercise for the reader to prove the last theorem.

7.3 Continuousness of a restricted morphism

Consider some partially ordered semigroup. (For example it can be the semigroup of funcoids or semigroup of reloids regarding the composition.) Consider also some lattice (*lattice of objects*). (For example take the lattice of set theoretic filters.)

We will map every object A to *identity element* I_A of the semigroup (for example identity funcoid or identity reloid). For identity elements we will require

1. $I_A \circ I_B = I_{A \cap B}$;
2. $f \circ I_A \subseteq f$; $I_A \circ f \subseteq f$.

In the case when our semigroup is "dagger" (that is is a dagger precategory) we will require also $(I_A)^\dagger = I_A$.

We can define *restricting* an element f of our semigroup to an object A by the formula $f|_A = f \circ I_A$.

We can define *rectangular restricting* an element μ of our semigroup to objects A and B as $I_B \circ \mu \circ I_A$. Optionally we can define direct product $A \times B$ of two objects by the formula (true for funcoids and for reloids):

$$\mu \cap (A \times B) = I_B \circ \mu \circ I_A.$$

Square restricting of an element μ to an object A is a special case of rectangular restricting and is defined by the formula $I_A \circ \mu \circ I_A$ (or by the formula $\mu \cap (A \times A)$).

Theorem 209. For every elements f, μ, ν of our semigroup and an object A

1. $f \in \mathrm{C}(\mu;\nu) \Rightarrow f|_A \in \mathrm{C}(I_A \circ \mu \circ I_A;\nu)$;
2. $f \in \mathrm{C}'(\mu;\nu) \Rightarrow f|_A \in \mathrm{C}'(I_A \circ \mu \circ I_A;\nu)$;
3. $f \in \mathrm{C}''(\mu;\nu) \Rightarrow f|_A \in \mathrm{C}''(I_A \circ \mu \circ I_A;\nu)$.

(Two last items are true for the case when our semigroup is dagger.)

Proof.

1. $f|_A \in \mathrm{C}(I_A \circ \mu \circ I_A;\nu) \Leftrightarrow f|_A \circ I_A \circ \mu \circ I_A \subseteq \nu \circ f|_A \Leftrightarrow f \circ I_A \circ I_A \circ \mu \circ I_A \subseteq \nu \circ f|_A \Leftrightarrow f \circ I_A \circ \mu \circ I_A \subseteq \nu \circ f \circ I_A \Leftarrow f \circ I_A \circ \mu \subseteq \nu \circ f \Leftarrow f \circ \mu \subseteq \nu \circ f \Leftrightarrow f \in \mathrm{C}(\mu;\nu)$.
2. $f|_A \in \mathrm{C}'(I_A \circ \mu \circ I_A;\nu) \Leftrightarrow I_A \circ \mu \circ I_A \subseteq (f|_A)^\dagger \circ \nu \circ f|_A \Leftarrow I_A \circ \mu \circ I_A \subseteq (f \circ I_A)^\dagger \circ \nu \circ f \circ I_A \Leftrightarrow I_A \circ \mu \circ I_A \subseteq I_A \circ f^\dagger \circ \nu \circ f \circ I_A \Leftarrow \mu \subseteq f^\dagger \circ \nu \circ f \Leftrightarrow f \in \mathrm{C}'(\mu;\nu)$.
3. $f|_A \in \mathrm{C}''(I_A \circ \mu \circ I_A;\nu) \Leftrightarrow f|_A \circ I_A \circ \mu \circ I_A \circ (f|_A)^\dagger \subseteq \nu \Leftrightarrow f \circ I_A \circ I_A \circ \mu \circ I_A \circ I_A \circ f^\dagger \subseteq \nu \Leftrightarrow f \circ I_A \circ \mu \circ I_A \circ f^\dagger \subseteq \nu \Leftarrow f \circ \mu \circ f^\dagger \subseteq \nu \Leftrightarrow f \in \mathrm{C}''(\mu;\nu)$. □

8 Connectedness regarding funcoids and reloids

8.1 Some lemmas

Lemma 210. If $\neg(A[f]B) \wedge A \cup B \supseteq \operatorname{dom} f \cup \operatorname{im} f$ then f is closed on A for a funcoid f and sets A and B.

Proof. $\neg(A[f]B) \Leftrightarrow B \cap \langle f \rangle A = \emptyset \Leftrightarrow (\mathrm{dom}\, f \cup \mathrm{im}\, f) \cap B \cap \langle f \rangle A = \emptyset \Rightarrow ((\mathrm{dom}\, f \cup \mathrm{im}\, f) \setminus A) \cap \langle f \rangle A = \emptyset \Leftrightarrow \langle f \rangle A \subseteq A$. □

Corollary 211. If $\neg(A[f]B) \wedge A \cup B \supseteq \mathrm{dom}\, f \cup \mathrm{im}\, f$ then f is closed on $A \setminus B$ for a funcoid f and sets A and B.

Proof. Let $\neg(A[f]B) \wedge A \cup B \supseteq \mathrm{dom}\, f \cup \mathrm{im}\, f$. Then $\neg((A \setminus B)[f]B) \wedge (A \setminus B) \cup B \supseteq \mathrm{dom}\, f \cup \mathrm{im}\, f$. □

Lemma 212. If $\neg(A[f]B) \wedge A \cup B \supseteq \mathrm{dom}\, f \cup \mathrm{im}\, f$ then $\neg(A[f^n]B)$ for every whole positive n.

Proof. Let $\neg(A[f]B) \wedge A \cup B \supseteq \mathrm{dom}\, f \cup \mathrm{im}\, f$. From the above proposition $\langle f \rangle A \subseteq A$. $B \cap \langle f \rangle A = \emptyset$, consequently $\langle f \rangle A \subseteq A \setminus B$. Because (by the above corollary) f is closed on $A \setminus B$, then $\langle f \rangle \langle f \rangle A \subseteq A \setminus B$, $\langle f \rangle \langle f \rangle \langle f \rangle A \subseteq A \setminus B$, etc. So $\langle f^n \rangle A \subseteq A \setminus B$, $B \cap \langle f^n \rangle A = \emptyset$, $\neg(A[f^n]B)$. □

8.2 Endomorphism series

Definition 213. $S_1(\mu) \overset{\mathrm{def}}{=} \mu \cup \mu^2 \cup \mu^3 \cup \ldots$ for an endomorphism μ of a precategory with countable union of morphisms.

Definition 214. $S(\mu) \overset{\mathrm{def}}{=} \mu^0 \cup S_1(\mu)$ where $\mu^0 \overset{\mathrm{def}}{=} I_{\mathrm{Ob}\, \mu}$ (identity morphism for the object $\mathrm{Ob}\, \mu$) where $\mathrm{Ob}\, \mu$ is the object of endomorphism μ for an endomorphism μ of a category with countable union of morphisms.

I call S_1 and S *endomorphism series*.

We will consider the collection of all binary relations (on a set $\mho$), as well as the collection of all funcoids and the collection of all reloids, as categories with single object $\mho$ and the identity morphism $(=)$ or $(=)|_{\mho}$.

So if μ is a binary relation or a funcoid or a reloid we have

$S_1(\mu) = \mu \cup \mu^2 \cup \mu^3 \cup \ldots$ and $S(\mu) = (=) \cup \mu \cup \mu^2 \cup \mu^3 \cup \ldots$

Proposition 215. $S(\mu)$ is transitive for the category of binary relations.

Proof.

$$\begin{aligned} S(\mu) \circ S(\mu) &= \mu^0 \circ S(\mu) \cup \mu \circ S(\mu) \cup \mu^2 \circ S(\mu) \cup \ldots \\ &= (\mu^0 \cup \mu^1 \cup \mu^2 \cup \ldots) \cup (\mu^1 \cup \mu^2 \cup \mu^3 \cup \ldots) \cup (\mu^2 \cup \mu^3 \cup \mu^4 \cup \ldots) \\ &= \mu^0 \cup \mu^1 \cup \mu^2 \cup \ldots \\ &= S(\mu). \end{aligned}$$

□

8.3 Connectedness regarding binary relations

Before going to research connectedness for funcoids and reloids we will excurse into the basic special case of connectedness regarding binary relations.

Definition 216. A set A is called *(strongly) connected* regarding a binary relation μ when

$$\forall X, Y \in \mathscr{P}\mho \setminus \{\emptyset\}: (X \cup Y = A \Rightarrow X[\mu]Y).$$

Definition 217. *Path* between two elements $a, b \in \mho$ in a set A through binary relation μ is the finite sequence $x_0 \ldots x_n$ where $x_0 = a$, $x_n = b$ for $n \in \mathbb{N}$ and $x_i(\mu \cap A \times A)x_{i+1}$ for every $i = 0, \ldots, n-1$. n is called path *length*.

Proposition 218. There exists path between every element $a \in \mho$ and that element itself.

Proof. It is the path consisting of one vertex (of length 0). □

Proposition 219. There is a path from element a to element b in a set A through a binary relation μ iff $a\,(S(\mu\cap A\times A))\,b$ (that is $(a,b)\in S(\mu\cap A\times A)$).

Proof.

$\Rightarrow$. If exists a path from a to b, then $\{b\}\subseteq\langle(\mu\cap A\times A)^n\rangle\{a\}$ where n is the path length. Consequently $\{b\}\subseteq\langle S(\mu\cap A\times A)\rangle\{a\}$; $a\,(S(\mu\cap A\times A))\,b$.

$\Leftarrow$. If $a\,(S(\mu\cap A\times A))\,b$ then exists $n\in\mathbb{N}$ such that $a\,(\mu\cap A\times A)^n\,b$. By definition of composition of binary relations this means that there exist finite sequence $x_0 \ldots x_n$ where $x_0=a$, $x_n=b$ for $n\in\mathbb{N}$ and $x_i\,(\mu\cap A\times A)\,x_{i+1}$ for every $i=0,\ldots,n-1$. That is there is path from a to b. □

Theorem 220. The following statements are equivalent for a relation μ and a set A:

1. For every $a,b\in A$ there is a path between a and b in A through μ.
2. $S(\mu\cap A\times A)\supseteq A\times A$.
3. $S(\mu\cap A\times A)=A\times A$.
4. A is connected regarding μ.

Proof.

(1) $\Rightarrow$ (2). Let for every $a,\ b\in A$ there is a path between a and b in A through μ. Then $a\,(S(\mu\cap A\times A))\,b$ for every $a,b\in A$. It is possible only when $S(\mu\cap A\times A)\supseteq A\times A$.

(3) $\Rightarrow$ (1). For every two vertices a and b we have $a\,(S(\mu\cap A\times A))\,b$. So (by the previous theorem) for every two vertices a and b exist path from a to b.

(3) $\Rightarrow$ (4). Suppose that $\neg(X[\mu\cap A\times A]Y)$ for some $X,Y\in\mathscr{P}\mho\setminus\{\emptyset\}$ such that $X\cup Y=A$. Then by a lemma $\neg(X[(\mu\cap A\times A)^n]Y)$ for every $n\in\mathbb{N}$. Consequently $\neg(X[S(\mu\cap A\times A)]Y)$. So $S(\mu\cap A\times A)\neq A\times A$.

(4) $\Rightarrow$ (3). If $\langle S(\mu\cap A\times A)\rangle\{v\}=A$ for every vertex v then $S(\mu\cap A\times A)=A\times A$. Consider the remaining case when $V\overset{\mathrm{def}}{=}\langle S(\mu\cap A\times A)\rangle\{v\}\subset A$ for some vertex v. Let $W=A\setminus V$. If $\operatorname{card}A=1$ then $S(\mu\cap A\times A)\supseteq(=)=A\times A$; otherwise $W\neq\emptyset$. Then $V\cup W=A$ and so $V[\mu]W$ what is equivalent to $V[\mu\cap A\times A]W$ that is $\langle\mu\cap A\times A\rangle V\cap W\neq\emptyset$. This is impossible because $\langle\mu\cap A\times A\rangle V=\langle\mu\cap A\times A\rangle\langle S(\mu\cap A\times A)\rangle V=\langle S_1(\mu\cap A\times A)\rangle V\subseteq\langle S(\mu\cap A\times A)\rangle V=V$.

(2) $\Rightarrow$ (3). Because $S(\mu\cap A\times A)\subseteq A\times A$. □

Corollary 221. A set A is connected regarding a binary relation μ iff it is connected regarding $\mu\cap A\times A$.

Definition 222. A *connected component* of a set A regarding a binary relation F is a maximal connected subset of A.

Theorem 223. The set A is partitioned into connected components (regarding every binary relation F).

Proof. Consider the binary relation $a\sim b\Leftrightarrow a\,(S(F))\,b\wedge b\,(S(F))\,a$. $\sim$ is a symmetric, reflexive, and transitive relation. So all points of A are partitioned into a collection of sets Q. Obviously each component is (strongly) connected. If a set $R\subseteq A$ is greater than one of that connected components A then it contains a point $b\in B$ where B is some other connected component. Consequently R is disconnected. □

Proposition 224. A set is connected (regarding a binary relation) iff it has one connected component.

Proof. Direct implication is obvious. Reverse is proved by contradiction. □

8.4 Connectedness regarding funcoids and reloids

Definition 225. $S_1^\ast(\mu)=\bigcap^{\mathfrak{F}}\{S_1(M) \mid M\in\operatorname{up}\mu\}$ for a reloid μ.

Definition 226. *Connectivity reloid* $S^\ast(\mu)$ for a reloid μ is defined as follows:

$$S^\ast(\mu)=\bigcap^{\mathfrak{F}}\{S(M) \mid M\in\operatorname{up}\mu\}.$$

Remark 227. Do not mess the word *connectivity* with the word *connectedness* which means being connected.[1]

Proposition 228. $S^\ast(\mu)=(=)\cup^{\mathsf{RLD}} S_1^\ast(\mu)$ for every reloid μ.

Proof. Follows from the theorem about distributivity of $\cup^{\mathsf{RLD}}$ regarding $\bigcap^{\mathfrak{F}}$ (see [14]). □

Proposition 229. $S^\ast(\mu)=S(\mu)$ if μ is a binary relation.

Proof. $S^\ast(\mu)=\bigcap^{\mathfrak{F}}\{S(\mu)\}=S(\mu)$. □

Definition 230. A filter $\mathcal{A}$ is called *connected* regarding a reloid μ when $S^\ast(\mu\cap^{\mathsf{RLD}}(\mathcal{A}\times^{\mathsf{RLD}}\mathcal{A}))\supseteq\mathcal{A}\times^{\mathsf{RLD}}\mathcal{A}$.

Obvious 231. A filter object $\mathcal{A}$ is connected regarding a reloid μ when $S^\ast(\mu\cap^{\mathsf{RLD}}(\mathcal{A}\times^{\mathsf{RLD}}\mathcal{A}))=\mathcal{A}\times^{\mathsf{RLD}}\mathcal{A}$.

Definition 232. A filter object $\mathcal{A}$ is called *connected* regarding a funcoid μ when

$$\forall\mathcal{X},\mathcal{Y}\in\mathfrak{F}\setminus\{\emptyset\}:(\mathcal{X}\cup^{\mathfrak{F}}\mathcal{Y}=\mathcal{A}\Rightarrow\mathcal{X}[\mu]\mathcal{Y}).$$

Proposition 233. A set A is connected regarding a funcoid μ iff

$$\forall\mathcal{X},\mathcal{Y}\in\mathscr{P}\mho\setminus\{\emptyset\}:(X\cup Y=A\Rightarrow X[\mu]Y).$$

Proof.

⇒. Obvious.

⇐. Follows from co-separability of filter objects. □

Theorem 234. The following are equivalent for every set A and binary relation μ:

1. A is connected regarding binary relation μ.
2. A is connected regarding μ considered as a reloid.
3. A is connected regarding μ considered as a funcoid.

Proof.

(1) ⇔ (2). $S^\ast(\mu\cap^{\mathsf{RLD}}(A\times^{\mathsf{RLD}}A))=S^\ast(\mu\cap A\times A)=S(\mu\cap A\times A)$. So $S^\ast(\mu\cap^{\mathsf{RLD}}A\times^{\mathsf{RLD}}A)\supseteq A\times^{\mathsf{RLD}}A\Leftrightarrow S(\mu\cap A\times A)\supseteq A\times A$.

(1) ⇔ (3). Follows from the previous proposition. □

Next is conjectured a statement more strong than the above theorem:

Conjecture 235. A filter object is connected regarding a binary relation considered as a funcoid iff it is connected regarding this binary relation considered as a reloid.

1. In some math literature these two words are used interchangeably.

Obvious 236. A filter object is connected regarding a reloid μ iff it is connected regarding the reloid $\mu \cap^{\mathsf{RLD}} (\mathcal{A} \times^{\mathsf{RLD}} \mathcal{A})$.

Obvious 237. A filter object is connected regarding a funcoid μ iff it is connected regarding the funcoid $\mu \cap^{\mathsf{FCD}} (\mathcal{A} \times^{\mathsf{FCD}} \mathcal{A})$.

Theorem 238. A filter object $\mathcal{A}$ is connected regarding a reloid f iff it is connected regarding every $F \in \operatorname{up} f$ (considered as reloid).

Proof.

$\Rightarrow$. Obvious.

$\Leftarrow$. F is connected iff $S(F) = F^0 \cup F^1 \cup F^2 \cup \ldots \supseteq \mathcal{A} \times^{\mathsf{RLD}} \mathcal{A}$.
$S^*(f) = \bigcap^{\mathfrak{F}} \{S(F) \mid F \in \operatorname{up} f\} \supseteq \bigcap^{\mathfrak{F}} \{\mathcal{A} \times^{\mathsf{RLD}} \mathcal{A} \mid F \in \operatorname{up} f\} = \mathcal{A} \times^{\mathsf{RLD}} \mathcal{A}$. □

Conjecture 239. A filter object $\mathcal{A}$ is connected regarding a funcoid μ iff $\mathcal{A}$ is connected for every $F \in \operatorname{up} \mu$ (considered as funcoid).

The above conjecture is open even for the case when $\mathcal{A}$ is a set.

Conjecture 240. A filter object $\mathcal{A}$ is connected regarding a reloid f iff it is connected regarding the funcoid $(\mathsf{FCD}) f$.

The above conjecture is true in the special case of principal filters:

Proposition 241. A set A is connected regarding a reloid f iff it is connected regarding the funcoid $(\mathsf{FCD}) f$.

Proof. The set A is connected regarding a reloid f iff it is connected regarding every $F \in \operatorname{up} f$ that is when (taken in account that connectedness for F regarded as a reloid is the same as connectedness for F regarded as a funcoid)

$$\begin{aligned}
\forall F \in \operatorname{up} f \forall \mathcal{X}, \mathcal{Y} \in \mathfrak{F} \setminus \{\emptyset\} : (\mathcal{X} \cup^{\mathfrak{F}} \mathcal{Y} = A \Rightarrow \mathcal{X} [F] \mathcal{Y}) &\Leftrightarrow \\
\forall \mathcal{X}, \mathcal{Y} \in \mathfrak{F} \setminus \{\emptyset\} \forall F \in \operatorname{up} f : (\mathcal{X} \cup^{\mathfrak{F}} \mathcal{Y} = A \Rightarrow \mathcal{X} [F] \mathcal{Y}) &\Leftrightarrow \\
\forall \mathcal{X}, \mathcal{Y} \in \mathfrak{F} \setminus \{\emptyset\} (\mathcal{X} \cup^{\mathfrak{F}} \mathcal{Y} = A \Rightarrow \forall F \in \operatorname{up} f : \mathcal{X} [F] \mathcal{Y}) &\Leftrightarrow \\
\forall \mathcal{X}, \mathcal{Y} \in \mathfrak{F} \setminus \{\emptyset\} (\mathcal{X} \cup^{\mathfrak{F}} \mathcal{Y} = A \Rightarrow \mathcal{X} [(\mathsf{FCD}) f] \mathcal{Y})&
\end{aligned}$$

that is when the set A is connected regarding the funcoid $(\mathsf{FCD}) f$. □

8.5 Algebraic properties of S and S^*

Theorem 242. $S^*(S^*(f)) = S^*(f)$ for every reloid f.

Proof. $S^*(S^*(f)) = \bigcap^{\mathfrak{F}} \{S(R) \mid R \in \operatorname{up} S^*(f)\} \subseteq \bigcap^{\mathfrak{F}} \{S(R) \mid R \in \{S(F) \mid F \in \operatorname{up} f\}\} = \bigcap^{\mathfrak{F}} \{S(S(F)) \mid F \in \operatorname{up} f\} = \bigcap^{\mathfrak{F}} \{S(F) \mid F \in \operatorname{up} f\} = S^*(f)$.

So $S^*(S^*(f)) \subseteq S^*(f)$. That $S^*(S^*(f)) \supseteq S^*(f)$ is obvious. □

Corollary 243. $S^*(S(f)) = S(S^*(f)) = S^*(f)$ for any reloid f.

Proof. Obviously $S^*(S(f)) \supseteq S^*(f)$ and $S(S^*(f)) \supseteq S^*(f)$.

But $S^*(S(f)) \subseteq S^*(S^*(f)) = S^*(f)$ and $S(S^*(f)) \subseteq S^*(S^*(f)) = S^*(f)$. □

Conjecture 244. $S(S(f)) = S(f)$ for

1. every reloid f;
2. every funcoid f.

Conjecture 245. For every reloid f

1. $S(f) \circ S(f) = S(f)$;

2. $S^*(f) \circ S^*(f) = S^*(f)$;
3. $S(f) \circ S^*(f) = S^*(f) \circ S(f) = S^*(f)$.

Conjecture 246. $S(f) \circ S(f) = S(f)$ for every funcoid f.

9 Postface

9.1 Misc

See this Web page for my research plans: http://www.mathematics21.org/agt-plans.html

I deem that now two most important research topics in Algebraic General Topology are:

- to solve the open problems mentioned in this work;
- define and research compactness of funcoids.

Also a future research topic are n-ary (where n is an ordinal, or more generally an index set) funcoids and reloids (plain funcoids and reloids are binary by analogy with binary relations).

We should also research relationships between complete funcoids and complete reloids.

9.2 Pointfree funcoids and reloids

I have set wiki site http://funcoids.wikidot.com to write on that site the pointfree variant of the theory of funcoids and reloids (that is generalized funcoids on arbitrary lattices rather than funcoids on a lattice of sets as in this work).

Pointfree funcoids have use in the theory of (generalized) limits of discontinuous functions.

Pointfree theory of funcoids and reloids seems to be a trivial generalization of the theory of point-set funcoids and reloids. It is not similar to the traditional pointfree topology which is not an obvious generalization of point-set topology.

Appendix A Some counter-examples

For further examples we will use the filter object Δ defined by the formula

$$\Delta = \bigcap^{\mathfrak{F}} \{(-\varepsilon; \varepsilon) \mid \varepsilon \in \mathbb{R}, \varepsilon > 0\}.$$

Example 247. There exist a funcoid f and a set S of funcoids such that $f \cap^{\mathsf{FCD}} \bigcup^{\mathsf{FCD}} S \neq \bigcup^{\mathsf{FCD}} \langle f \cap^{\mathsf{FCD}} \rangle S$.

Proof. Let $f = \Delta \times^{\mathsf{FCD}} \{0\}$ and $S = \left\{(\varepsilon; +\infty) \times^{\mathsf{FCD}} \{0\} \mid \varepsilon > 0\right\}$. Then $f \cap^{\mathsf{FCD}} \bigcup^{\mathsf{FCD}} S = (\Delta \times^{\mathsf{FCD}} \{0\}) \cap^{\mathsf{FCD}} ((0; +\infty) \times^{\mathsf{FCD}} \{0\}) = (\Delta \cap^{\mathsf{FCD}} (0; +\infty)) \times^{\mathsf{FCD}} \{0\} \neq \emptyset$ while $\bigcup^{\mathsf{FCD}} \langle f \cap^{\mathsf{FCD}} \rangle S = \bigcup^{\mathsf{FCD}} \{\emptyset\} = \emptyset$. □

Conjecture 248. There exist a set R of funcoids and a funcoid f such that $f \circ \bigcup^{\mathsf{FCD}} R \neq \bigcup^{\mathsf{FCD}} \langle f \circ \rangle R$.

Example 249. There exist a set R of funcoids and f.o. $\mathcal{X}$ and $\mathcal{Y}$ such that

1. $\mathcal{X} \left[\bigcup^{\mathsf{FCD}} R \right] \mathcal{Y} \wedge \nexists f \in R: \mathcal{X} [f] \mathcal{Y}$;
2. $\left\langle \bigcup^{\mathsf{FCD}} R \right\rangle \mathcal{X} \supset \bigcup^{\mathfrak{F}} \{\langle f \rangle \mathcal{X} \mid f \in R\}$.

Proof.

1. Let $\mathcal{X} = \Delta$ and $\mathcal{Y} = \mathbb{R}$. Let $R = \left\{(\varepsilon; +\infty) \times^{\mathsf{FCD}} \mathbb{R} \mid \varepsilon \in \mathbb{R}, \varepsilon > 0\right\}$. Then $\bigcup^{\mathsf{FCD}} R = (0; +\infty) \times^{\mathsf{FCD}} \mathbb{R}$. So $\mathcal{X} \left[\bigcup^{\mathsf{FCD}} R \right] \mathcal{Y}$ and $\forall f \in R: \neg (\mathcal{X} [f] \mathcal{Y})$.

2. With the same $\mathcal{X}$ and R we have $\left\langle \bigcup^{\mathsf{FCD}} R \right\rangle \mathcal{X} = \mathbb{R}$ and $\langle f \rangle \mathcal{X} = \emptyset$ for every $f \in R$, thus $\bigcup^{\mathfrak{F}} \{ \langle f \rangle \mathcal{X} \mid f \in R \} = \emptyset$. □

Theorem 250. For a f.o. a we have $a \times^{\mathsf{RLD}} a \subseteq (=)|_{\mho}$ only in the case if $a = \emptyset$ or a is a trivial atomic f.o. (that is an one-element set).

Proof. If $a \times^{\mathsf{RLD}} a \subseteq (=)|_{\mho}$ then exists $m \in \operatorname{up}(a \times^{\mathsf{RLD}} a)$ such that $m \subseteq (=)|_{\mho}$. Consequently exist $A, B \in \operatorname{up} a$ such that $A \times B \subseteq (=)|_{\mho}$ what is possible only in the case when $A = B = a$ is an one-element set or empty set. □

Corollary 251. Direct product (in the sense of reloids) of non-trivial atomic filter objects is non-atomic.

Proof. Obviously $(a \times^{\mathsf{RLD}} a) \cap^{\mathsf{RLD}} (=)|_{\mho} \neq \emptyset$ and $(a \times^{\mathsf{RLD}} a) \cap^{\mathsf{RLD}} (=)|_{\mho} \subset a \times^{\mathsf{RLD}} a$. □

Example 252. There exist two atomic reloids whose composition is non-atomic and non-empty.

Proof. Let a is a non-trivial atomic filter object and $x \in \mho$. Then

$$(a \times \{x\}) \circ (\{x\} \times a) = \bigcap\nolimits^{\mathfrak{F}} \{ (A \times \{x\}) \circ (\{x\} \times A) \mid A \in \operatorname{up} a \} = \bigcap\nolimits^{\mathfrak{F}} \{ A \times A \mid A \in \operatorname{up} a \} = a \times a$$

is non-atomic despite of $a \times \{x\}$ and $\{x\} \times a$ are atomic. □

Example 253. There exists non-monovalued atomic reloid.

Proof. From the previous example follows that the atomic reloid $\{x\} \times a$ is not monovalued. □

Example 254. $(\mathsf{RLD})_{\mathrm{in}} f \neq (\mathsf{RLD})_{\mathrm{out}} f$ for a funcoid f.

Proof. Let $f = (=)|_{\mho}$. Then $(\mathsf{RLD})_{\mathrm{in}} f = \bigcup^{\mathsf{RLD}} \left\{ a \times^{\mathsf{RLD}} a \mid a \in \operatorname{atoms}^{\mathfrak{F}} \mho \right\}$ and $(\mathsf{RLD})_{\mathrm{out}} f = (=)|_{\mho}$. But as we shown above $a \times^{\mathsf{RLD}} a \nsubseteq (=)|_{\mho}$ for non-trivial f.o. a, and so $(\mathsf{RLD})_{\mathrm{in}} f \nsubseteq (\mathsf{RLD})_{\mathrm{out}} f$. □

Proposition 255. $(=)|_{\mho} \cap^{\mathsf{FCD}} (\mho \times \mho \setminus (=)|_{\mho}) = (=)|_{\Omega} \neq \emptyset$ (where Ω is the Fréchet filter object).

Proof. Note that $\langle (=)|_{\Omega} \rangle \mathcal{X} = \mathcal{X} \cap^{\mathfrak{F}} \Omega$.

Let $f = (=)|_{\mho}$, $g = \mho \times \mho \setminus f$.

Let x is a non-trivial atomic f.o. If $X \in \operatorname{up} x$ then $\operatorname{card} X \geqslant 2$ (In fact, X is infinite but we don't need this.) and consequently $\langle g \rangle X = \mho$. Thus $\langle g \rangle x = \mho$. Consequently

$$\left\langle f \cap^{\mathsf{FCD}} g \right\rangle x = \langle f \rangle x \cap^{\mathfrak{F}} \langle g \rangle x = x \cap^{\mathfrak{F}} \mho = x.$$

Also $\langle (=)|_{\Omega} \rangle x = x \cap^{\mathfrak{F}} \Omega = x$.

Let now x is a trivial f.o. Then $\langle f \rangle x = x$ and $\langle g \rangle x = \mho \setminus x$. So

$$\left\langle f \cap^{\mathsf{FCD}} g \right\rangle x = \langle f \rangle x \cap^{\mathfrak{F}} \langle g \rangle x = x \cap^{\mathfrak{F}} (\mho \setminus x) = x \cap (\mho \setminus x) = \emptyset.$$

Also $\langle (=)|_{\Omega} \rangle x = x \cap^{\mathfrak{F}} \Omega = \emptyset$.

So $\left\langle f \cap^{\mathsf{FCD}} g \right\rangle x = \langle (=)|_{\Omega} \rangle x$ for every atomic f.o. x. Thus $f \cap^{\mathsf{FCD}} g = (=)|_{\Omega}$. □

Example 256. There exist discrete funcoids f and g such that $f \cap^{\mathsf{FCD}} g \neq f \cap g$.

Proof. From the proposition above. □

Example 257. There exists funcoid h such that $\operatorname{up} h$ is not a filter.

Proof. Consider the funcoid $h = (=)|_{\Omega}$. We have (from the proposition) that $f \in \operatorname{up} h$ and $g \in \operatorname{up} f$, but $f \cap g = \emptyset \notin \operatorname{up} h$. □

Example 258. There exists a funcoid $h \neq \emptyset$ such that $(\mathsf{RLD})_{\mathrm{out}} h = \emptyset$.

Proof. Consider $h=(=)|_{\Omega}$. By proved above $h=f\sqcap^{\mathsf{FCD}} g$ where $f=(=)|_{\mho}$ and $g=\mho\times\mho\setminus f$.

We have $f,g\in\operatorname{up}h$.

So $(\mathsf{RLD})_{\mathrm{out}}h=\bigcap^{\mathsf{RLD}}\operatorname{up}h\subseteq f\sqcap^{\mathsf{RLD}} g=f\cap g=\emptyset$; and thus $(\mathsf{RLD})_{\mathrm{out}}h=\emptyset$. □

Example 259. There exists a funcoid h such that $(\mathsf{FCD})(\mathsf{RLD})_{\mathrm{out}}h\neq h$.

Proof. Follows from the previous example. □

Example 260. There exist funcoids f and g such that

$$(\mathsf{RLD})_{\mathrm{out}}(g\circ f)\neq(\mathsf{RLD})_{\mathrm{out}}g\circ(\mathsf{RLD})_{\mathrm{out}}f.$$

Proof. Take $f=(=)|_{\Omega}$ and $g=\mho\times^{\mathsf{FCD}}\{\alpha\}$ for some $\alpha\in\mho$. Then $(\mathsf{RLD})_{\mathrm{out}}f=\emptyset$ and thus $(\mathsf{RLD})_{\mathrm{out}}g\circ(\mathsf{RLD})_{\mathrm{out}}f=\emptyset$.

We have $g\circ f=\Omega\times^{\mathsf{FCD}}\{\alpha\}$.

Let's prove $(\mathsf{RLD})_{\mathrm{out}}(\Omega\times^{\mathsf{FCD}}\{\alpha\})=\Omega\times^{\mathsf{RLD}}\{\alpha\}$.

Really: $(\mathsf{RLD})_{\mathrm{out}}(\Omega\times^{\mathsf{FCD}}\{\alpha\})=\bigcap^{\mathsf{RLD}}\operatorname{up}(\Omega\times^{\mathsf{FCD}}\{\alpha\})=\bigcap^{\mathsf{RLD}}\{K\times\{\alpha\}\mid K\in\operatorname{up}\Omega\}$.

$F\in\operatorname{up}\bigcap^{\mathsf{RLD}}\{K\times\{\alpha\}\mid K\in\operatorname{up}\Omega\}\Leftrightarrow F\in\operatorname{up}(\bigcap^{\mathfrak{F}}\{K\mid K\in\operatorname{up}\Omega\}\times^{\mathsf{RLD}}\{\alpha\})$ for every $F\in\mathscr{P}\mho^2$. Thus

$$\bigcap^{\mathsf{RLD}}\{K\times\{\alpha\}\mid K\in\operatorname{up}\Omega\}=\bigcap^{\mathfrak{F}}\{K\mid K\in\operatorname{up}\Omega\}\times^{\mathsf{RLD}}\{\alpha\}=\Omega\times^{\mathsf{RLD}}\{\alpha\}.$$

So $(\mathsf{RLD})_{\mathrm{out}}(\Omega\times^{\mathsf{FCD}}\{\alpha\})=\Omega\times^{\mathsf{RLD}}\{\alpha\}$.

Thus $(\mathsf{RLD})_{\mathrm{out}}(g\circ f)=\Omega\times^{\mathsf{RLD}}\{\alpha\}\neq\emptyset$. □

Example 261. (FCD) does not preserve finite meets.

Proof. $(\mathsf{FCD})((=)|_{\mho}\sqcap^{\mathsf{RLD}}(\mho\times\mho\setminus(=)|_{\mho}))=(\mathsf{FCD})\emptyset=\emptyset$.

On the other hand

$$(\mathsf{FCD})(=)|_{\mho}\sqcap^{\mathsf{FCD}}(\mathsf{FCD})(\mho\times\mho\setminus(=)|_{\mho}))=(=)|_{\mho}\sqcap^{\mathsf{FCD}}(\mho\times\mho\setminus(=)|_{\mho})=(=)|_{\Omega}\neq\emptyset$$

(used the proposition 180). □

Corollary 262. (FCD) is not an upper adjoint (in general).

Considering restricting polynomials (considered as reloids) to atomic filter objects, it is simple to prove that each that restriction is injective if not restricting a constant polynomial. Does this hold in general? No, see the following example:

Example 263. There exists a monovalued reloid with atomic domain which is neither injective neither constant (that is not a restriction of a constant function).

Proof. Consider the function $F\in\mathbb{N}^{\mathbb{N}\times\mathbb{N}}$ defined by the formula $(x;y)\mapsto x$.

Let ω_x is a non-principal atomic filter object on the vertical line $\{x\}\times\mathbb{N}$ for every $x\in\mathbb{N}$.

Let T is the collection of such sets Y that $Y\cap(\{x\}\times\mathbb{N})\in\omega_x$ for all but finitely many vertical lines. Obviously T is a filter.

Let $\omega\in\operatorname{atoms}^{\mathfrak{F}}\operatorname{up}^{-1}T$.

For every $x\in\mathbb{N}$ we have some $Y\in T$ for which $(\{x\}\times\mathbb{N})\cap Y=\emptyset$ and thus $(\{x\}\times\mathbb{N})\cap\omega=\emptyset$.

Let $g=F|_{\omega}^{\mathsf{RLD}}$. If g is constant, then there exist a constant function $G\in\operatorname{up}g$ and $F\cap G$ is also constant. Obviously $\operatorname{dom}(F\cap G)\supseteq\omega$. The function $F\cap G$ cannot be constant because otherwise $\omega\subseteq\operatorname{dom}(F\cap G)\subseteq\{x\}\times\mathbb{N}$ for some $x\in\mathbb{N}$ what is impossible by proved above. So g is not constant.

Suppose there g is injective. Then there exists an injection $G\in\operatorname{up}g$. So $\operatorname{dom}G$ intersects each vertical line by atmost one element that is $\overline{\operatorname{dom}G}$ intersects every vertical line by the whole line or the line without one element. Thus $\overline{\operatorname{dom}G}\in T\subseteq\operatorname{up}\omega$ and consequently $\operatorname{dom}G\notin\operatorname{up}\omega$ what is impossible.

Thus g is neither injective neither constant. □

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