Funcoids and Reloids*

a generalization of proximities and uniformities

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Abstract

It is a part of my Algebraic General Topology research.

In this article, I introduce the concepts of *funcoids*, which generalize proximity spaces and *reloids*, which generalize uniform spaces. The concept of funcoid is generalized concept of proximity, the concept of reloid is cleared from superfluous details (generalized) concept of uniformity. Also funcoids generalize pretopologies and preclosures. Also funcoids and reloids are generalizations of binary relations whose domains and ranges are filters (instead of sets).

Also funcoids and reloids can be considered as a generalization of (oriented) graphs, this provides us with a common generalization of analysis and discrete mathematics.

The concept of continuity is defined by an algebraic formula (instead of old messy epsilon-delta notation) for arbitrary morphisms (including funcoids and reloids) of a partially ordered category. In one formula are generalized continuity, proximity continuity, and uniform continuity.

Keywords: algebraic general topology, quasi-uniform spaces, generalizations of proximity spaces, generalizations of nearness spaces, generalizations of uniform spaces

A.M.S. subject classification: 54J05, 54A05, 54D99, 54E05, 54E15, 54E17, 54E99

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1 Common

1.1 Draft status

This article is a draft, an almost ready preprint.

This text refers to a preprint edition of [15]. Theorem number clashes may appear due editing both of these manuscripts.

1.2 Earlier works

Some mathematicians researched generalizations of proximities and uniformities before me but they have failed to reach the right degree of generalization which is presented in this work allowing to represent properties of spaces with algebraic (or categorical) formulas.

Proximity structures were introduced by Smirnov in [5].

Some references to predecessors:

- In [6], [7], [12], [2], [19] are studied generalized uniformities and proximities.
- Proximities and uniformities are also studied in [10], [11], [18], [20], [21].
- [8] and [9] contains recent progress in quasi-uniform spaces. [9] has a very long list of related literature.

Some works ([17]) about proximity spaces consider relationships of proximities and compact topological spaces. In this work is not done the attempt to define or research their generalization, compactness of funcoids or reloids. It seems potentially productive to attempt to borrow the definitions and procedures from the above mentioned works. I hope to do this study in a separate article.

[4] studies mappings between proximity structures. (In this work no attempt to research mappings between funcoids is done.) [13] researches relationships of quasi-uniform spaces and topological spaces. [1] studies how proximity structures can be treated as uniform structures and compactification regarding proximity and uniform spaces.

1.3 Used concepts, notation and statements

The set of functions from a set A to a set B is denoted as B^A.

I will often skip parentheses and write fx instead of $f(x)$ to denote the result of a function f acting on the argument x.

I will call *small* sets members of some Grothendieck universe. (Let us assume the axiom of existence of a Grothendieck universe.)

Let f is a small binary relation.

I will denote $\langle f\rangle X = \{f\alpha \mid \alpha \in X\}$ and $X[f]Y \Leftrightarrow \exists x\in X, y\in Y: x\, f\, y$ for small sets X, Y.

By just $\langle f\rangle$ and $[f]$ I will denote the corresponding function and relation on small sets.

$\lambda x\in D: f(x) = \{(x; f(x)) \mid x\in D\}$ for every formula $f(x)$ depended on a variable x and set D.

I will denote the least and the greatest element of a poset $\mathfrak{A}$ as $0^{\mathfrak{A}}$ and $1^{\mathfrak{A}}$ respectively.

For elements a and b of a lattice with a minimal element I will denote $a \asymp b$ when $a\cap b$ is the minimal element of the lattice and $a \not\asymp b$ otherwise. See [15] for a more general notion.

1.3.1 Filters

In this work the word *filter* will refer to a filter on a set (in contrast to [15] where filters are considered on arbitrary posets). Note that I do not require filters to be proper.

I will call the set of filters on a set A (*base set*) ordered reverse to set-theoretic inclusion of filters *the set of filter objects* on A and denote it $\mathfrak{F}(A)$ or just $\mathfrak{F}$ when the base set is implied and call its element *filter objects* (f.o. for short). I will denote $\operatorname{up}\mathcal{F}$ the filter corresponding to a filter object $\mathcal{F}$. So we have $\mathcal{A}\subseteq\mathcal{B}\Leftrightarrow \operatorname{up}\mathcal{A}\supseteq\operatorname{up}\mathcal{B}$ for every filter objects $\mathcal{A}$ and $\mathcal{B}$ on the same set.

In this particular manuscript, we will *not* equate principal filter objects with corresponding sets as it is done in [15]. Instead we will have $\operatorname{Base}(\mathcal{A})$ equal to the unique base of a f.o. $\mathcal{A}$. I will denote $\uparrow^A X$ (or just $\uparrow X$ when A is implied) the principal filter object on A corresponding to the set X.

Filters are studied in the work [15].

Every set $\mathfrak{F}(A)$ is a complete lattice and we will apply lattice operations to subsets of such sets without explicitly mentioning $\mathfrak{F}(A)$.

Prior reading of [15] is needed to fully understand this work.

Filter objects corresponding to ultrafilters are atoms of the lattice $\mathfrak{F}(A)$ and will be called *atomic filter objects* (on A).

Also we will need to introduce the concept of *generalized filter base*.

Definition 1. *Generalized filter base* is a set $S\in\mathscr{P}\mathfrak{F}\setminus\{0^{\mathfrak{F}}\}$ such that

$$\forall\mathcal{A},\mathcal{B}\in S\,\exists\mathcal{C}\in S: \mathcal{C}\subseteq\mathcal{A}\cap\mathcal{B}.$$

Proposition 2. Let S is a generalized filter base. If $\mathcal{A}_1,\dots,\mathcal{A}_n\in S$ $(n\in\mathbb{N})$, then

$$\exists\mathcal{C}\in S: \mathcal{C}\subseteq\mathcal{A}_1\cap\dots\cap\mathcal{A}_n.$$

Proof. Can be easily proved by induction. □

Theorem 3. If S is a generalized filter base, then $\operatorname{up} \bigcap S = \bigcup \langle \operatorname{up} \rangle S$.

Proof. Obviously $\operatorname{up} \bigcap S \supseteq \bigcup \langle \operatorname{up} \rangle S$. Reversely, let $K \in \operatorname{up} \bigcap S$; then $K = A_1 \cap \ldots \cap A_n$ where $A_i \in \operatorname{up} \mathcal{A}_i$ where $\mathcal{A}_i \in S$, $i = 1, \ldots, n$, $n \in \mathbb{N}$; so exists $\mathcal{C} \in S$ such that $\mathcal{C} \subseteq \mathcal{A}_1 \cap \ldots \cap \mathcal{A}_n \subseteq \uparrow(A_1 \cap \ldots \cap A_n) = \uparrow K$, $K \in \operatorname{up} \mathcal{C}$, $K \in \bigcup \langle \operatorname{up} \rangle S$. □

Corollary 4. If S is a generalized filter base, then $\bigcap S = 0^{\mathfrak{F}} \Leftrightarrow 0^{\mathfrak{F}} \in S$.

Proof. $\bigcap S = 0^{\mathfrak{F}} \Leftrightarrow \emptyset \in \operatorname{up} \bigcap S \Leftrightarrow \emptyset \in \bigcup \langle \operatorname{up} \rangle S \Leftrightarrow \exists \mathcal{X} \in S: \emptyset \in \operatorname{up} \mathcal{X} \Leftrightarrow 0^{\mathfrak{F}} \in S$. □

Obvious 5. If S is a filter base on a set A then $\langle \uparrow^A \rangle S$ is a generalized filter base.

Definition 6. I will call *shifted filtrator* a triple $(\mathfrak{A}; \mathfrak{Z}; \uparrow)$ where $\mathfrak{A}$ and $\mathfrak{Z}$ are posets and $\uparrow$ is an order embedding from $\mathfrak{Z}$ to $\mathfrak{A}$.

Some concepts and notation can be defined for shifted filtrators through similar concepts for filtrators: $\langle \uparrow \rangle \operatorname{up} a = \operatorname{up}^{(\mathfrak{A}; \langle \uparrow \rangle \mathfrak{Z})} a$; $\langle \uparrow \rangle \operatorname{Cor} a = \operatorname{Cor}^{(\mathfrak{A}; \langle \uparrow \rangle \mathfrak{Z})} a$, etc.

For a set $\mathfrak{A}$ and the set of f.o. $\mathfrak{F}$ on this set we will consider the shifted filtrator $(\mathfrak{F}; \mathfrak{A}; \uparrow)$.

2 Partially ordered dagger categories

2.1 Partially ordered categories

Definition 7. I will call a *partially ordered (pre)category* a (pre)category together with partial order $\subseteq$ on each of its Hom-sets with the additional requirement that

$$f_1 \subseteq f_2 \wedge g_1 \subseteq g_2 \Rightarrow g_1 \circ f_1 \subseteq g_2 \circ f_2$$

for every morphisms f_1, g_1, f_2, g_2 such that $\operatorname{Src} f_1 = \operatorname{Src} f_2 \wedge \operatorname{Dst} f_1 = \operatorname{Dst} f_2 = \operatorname{Src} g_1 = \operatorname{Src} g_2 \wedge \operatorname{Dst} g_1 = \operatorname{Dst} g_2$.

2.2 Dagger categories

Definition 8. I will call a *dagger precategory* a precategory together with an involutive contravariant identity-on-objects prefunctor $x \mapsto x^\dagger$.

In other words, a *dagger precategory* is a precategory equipped with a function $x \mapsto x^\dagger$ on its set of morphisms which reverses the source and the destination and is subject to the following identities for every morphisms f and g:

1. $f^{\dagger\dagger} = f$;
2. $(g \circ f)^\dagger = f^\dagger \circ g^\dagger$.

Definition 9. I will call a *dagger category* a category together with an involutive contravariant identity-on-objects functor $x \mapsto x^\dagger$.

In other words, a *dagger category* is a category equipped with a function $x \mapsto x^\dagger$ on its set of morphisms which reverses the source and the destination and is subject to the following identities for every morphisms f and g and object A:

1. $f^{\dagger\dagger} = f$;
2. $(g \circ f)^\dagger = f^\dagger \circ g^\dagger$;
3. $(1_A)^\dagger = 1_A$.

Theorem 10. If a category is a dagger precategory then it is a dagger category.

Proof. We need to prove only that $(1_A)^\dagger = 1_A$. Really

$$(1_A)^\dagger = (1_A)^\dagger \circ 1_A = (1_A)^\dagger \circ (1_A)^{\dagger\dagger} = ((1_A)^\dagger \circ 1_A)^\dagger = (1_A)^{\dagger\dagger} = 1_A. \qquad \square$$

For a partially ordered dagger (pre)category I will additionally require (for every morphisms f and g)

$$f^\dagger \subseteq g^\dagger \Leftrightarrow f \subseteq g.$$

An example of dagger category is the category **Rel** whose objects are sets and whose morphisms are binary relations between these sets with usual composition of binary relations and with $f^\dagger = f^{-1}$.

Definition 11. A morphism f of a dagger category is called *unitary* when it is an isomorphism and $f^\dagger = f^{-1}$.

Definition 12. *Symmetric* (endo)morphism of a dagger precategory is such a morphism f that $f = f^\dagger$.

Definition 13. *Transitive* (endo)morphism of a precategory is such a morphism f that $f = f \circ f$.

Theorem 14. The following conditions are equivalent for a morphism f of a dagger precategory:

1. f is symmetric and transitive.
2. $f = f^\dagger \circ f$.

Proof.

(1)⇒(2). If f is symmetric and transitive then $f^\dagger \circ f = f \circ f = f$.

(2)⇒(1). $f^\dagger = (f^\dagger \circ f)^\dagger = f^\dagger \circ f^{\dagger\dagger} = f^\dagger \circ f = f$, so f is symmetric. $f = f^\dagger \circ f = f \circ f$, so f is transitive. $\square$

2.2.1 Some special classes of morphisms

Definition 15. For a partially ordered dagger category I will call *monovalued* morphism such a morphism f that $f \circ f^\dagger \subseteq 1_{\mathrm{Dst}\, f}$.

Definition 16. For a partially ordered dagger category I will call *entirely defined* morphism such a morphism f that $f^\dagger \circ f \supseteq 1_{\mathrm{Src}\, f}$.

Definition 17. For a partially ordered dagger category I will call *injective* morphism such a morphism f that $f^\dagger \circ f \subseteq 1_{\mathrm{Src}\, f}$.

Definition 18. For a partially ordered dagger category I will call *surjective* morphism such a morphism f that $f \circ f^\dagger \supseteq 1_{\mathrm{Dst}\, f}$.

Remark 19. It's easy to show that this is a generalization of monovalued, entirely defined, injective, and surjective binary relations as morphisms of the category **Rel**.

Obvious 20. "Injective morphism" is a dual of "monovalued morphism" and "surjective morphism" is a dual of "entirely defined morphism".

Definition 21. For a given partially ordered dagger category C the *category of monovalued (entirely defined, injective, surjective) morphisms* of C is the category with the same set of objects as of C and the set of morphisms being the set of monovalued (entirely defined, injective, surjective) morphisms of C with the composition of morphisms the same as in C.

We need to prove that these are really categories, that is that composition of monovalued (entirely defined) morphisms is monovalued (entirely defined) and that identity morphisms are monovalued and entirely defined.

Proof. We will prove only for monovalued morphisms and entirely defined morphisms, as injective and surjective morphisms are their duals.

Monovalued. Let f and g are monovalued morphisms, $\operatorname{Dst} f = \operatorname{Src} g$. $(g \circ f) \circ (g \circ f)^\dagger = g \circ f \circ f^\dagger \circ g^\dagger \subseteq g \circ 1_{\operatorname{Dst} f} \circ g^\dagger = g \circ 1_{\operatorname{Src} g} \circ g^\dagger = g \circ g^\dagger \subseteq 1_{\operatorname{Dst} g} = 1_{\operatorname{Dst}(g \circ f)}$. So $g \circ f$ is monovalued.

That identity morphisms are monovalued follows from the following: $1_A \circ (1_A)^\dagger = 1_A \circ 1_A = 1_A = 1_{\operatorname{Dst} 1_A} \subseteq 1_{\operatorname{Dst} 1_A}$.

Entirely defined. Let f and g are entirely defined morphisms, $\operatorname{Dst} f = \operatorname{Src} g$. $(g \circ f)^\dagger \circ (g \circ f) = f^\dagger \circ g^\dagger \circ g \circ f \supseteq f^\dagger \circ 1_{\operatorname{Src} g} \circ f = f^\dagger \circ 1_{\operatorname{Dst} f} \circ f = f^\dagger \circ f \supseteq 1_{\operatorname{Src} f} = 1_{\operatorname{Src}(g \circ f)}$. So $g \circ f$ is entirely defined.

That identity morphisms are entirely defined follows from the following: $(1_A)^\dagger \circ 1_A = 1_A \circ 1_A = 1_A = 1_{\operatorname{Src} 1_A} \supseteq 1_{\operatorname{Src} 1_A}$. □

Definition 22. I will call a *bijective* morphism a morphism which is entirely defined, monovalued, injective, and surjective.

Obvious 23. Bijective morphisms form a full subcategory.

Proposition 24. If a morphism is bijective then it is an isomorphism.

Proof. Let f is bijective. Then $f \circ f^\dagger \subseteq 1_{\operatorname{Dst} f}$, $f^\dagger \circ f \supseteq 1_{\operatorname{Src} f}$, $f^\dagger \circ f \subseteq 1_{\operatorname{Src} f}$, $f \circ f^\dagger \supseteq 1_{\operatorname{Dst} f}$. Thus $f \circ f^\dagger = 1_{\operatorname{Dst} f}$ and $f^\dagger \circ f = 1_{\operatorname{Src} f}$ that is $f^\dagger$ is an inverse of f. □

3 Funcoids

3.1 Informal introduction into funcoids

Funcoids are a generalization of proximity spaces and a generalization of pretopological spaces. Also funcoids are a generalization of binary relations.

That funcoids are a common generalization of “spaces” (proximity spaces, (pre)topological spaces) and binary relations (including monovalued functions) makes them smart for describing properties of functions in regard of spaces. For example the statement “f is a continuous function from a space μ to a space ν” can be described in terms of funcoids as the formula $f \circ \mu \subseteq \nu \circ f$ (see below for details).

Most naturally funcoids appear as a generalization of proximity spaces.

Let δ be a proximity that is certain binary relation so that $A \mathrel{\delta} B$ is defined for every sets A and B. We will extend it from sets to filter objects by the formula:

$$\mathcal{A} \mathrel{\delta'} \mathcal{B} \Leftrightarrow \forall A \in \operatorname{up} \mathcal{A}, B \in \operatorname{up} \mathcal{B}: A \mathrel{\delta} B.$$

Then (as it will be proved below) there exist two functions $\alpha, \beta \in \mathfrak{F}^{\mathfrak{F}}$ such that

$$\mathcal{A} \mathrel{\delta'} \mathcal{B} \Leftrightarrow \mathcal{B} \cap^{\mathfrak{F}} \alpha \mathcal{A} \neq 0^{\mathfrak{F}} \Leftrightarrow \mathcal{A} \cap^{\mathfrak{F}} \beta \mathcal{B} \neq 0^{\mathfrak{F}}.$$

The pair $(\alpha; \beta)$ is called *funcoid* when $\mathcal{B} \cap^{\mathfrak{F}} \alpha \mathcal{A} \neq 0^{\mathfrak{F}} \Leftrightarrow \mathcal{A} \cap^{\mathfrak{F}} \beta \mathcal{B} \neq 0^{\mathfrak{F}}$. So funcoids are a generalization of proximity spaces.

Funcoids consist of two components the first α and the second β. The first component of a funcoid f is denoted as $\langle f \rangle$ and the second component is denoted as $\langle f^{-1} \rangle$. (The similarity of this notation with the notation for the image of a set under a function is not a coincidence, we will see that in the case of discrete funcoids (see below) these coincide.)

One of the most important properties of a funcoid is that it is uniquely determined by just one of its components. That is a funcoid f is uniquely determined by the function $\langle f \rangle$. Moreover a funcoid f is uniquely determined by $\langle f \rangle|_{\mathscr{P} \bigcup \operatorname{dom} \langle f \rangle}$ that is by values of function $\langle f \rangle$ on sets.

Next we will consider some examples of funcoids determined by specified values of the first component on sets.

Funcoids as a generalization of pretopological spaces: Let α be a pretopological space that is a map $\alpha \in \mathfrak{F}^{\mho}$ for some set $\mho$. Then we define $\alpha' X \overset{\text{def}}{=} \bigcup^{\mathfrak{F}} \{\alpha x \mid x \in X\}$ for every set $X \in \mathscr{P}\mho$. We will prove that there exists a unique funcoid f such that $\alpha' = \langle f \rangle|_{\mathscr{P}\mho}$. So funcoids are a generalization of pretopological spaces. Funcoids are also a generalization of preclosure operators: For every preclosure operator p on a set $\mho$ exists a unique funcoid f such that $\langle f \rangle|_{\mathscr{P}\mho} = \uparrow \circ p$.

For every binary relation p on a set $\mho$ it exists unique funcoid f such that $\forall X \in \mathscr{P}\mho: \langle f \rangle \uparrow X = \uparrow \langle p \rangle X$ (where $\langle p \rangle$ is defined in the introduction), recall that a funcoid is uniquely determined by the values of its first component on sets. I will call such funcoids *discrete*. So funcoids are a generalization of binary relations.

Composition of binary relations (i.e. of discrete funcoids) complies with the formulas:

$$\langle g \circ f \rangle = \langle g \rangle \circ \langle f \rangle \quad \text{and} \quad \langle (g \circ f)^{-1} \rangle = \langle f^{-1} \rangle \circ \langle g^{-1} \rangle.$$

By the same formulas we can define composition of every two funcoids. Funcoids with this composition form a category (*the category of funcoids*).

Also funcoids can be reversed (like reversal of X and Y in a binary relation) by the formula $(\alpha; \beta)^{-1} = (\beta; \alpha)$. In particular case if μ is a proximity we have $\mu^{-1} = \mu$ because proximities are symmetric.

Funcoids behave similarly to (multivalued) functions but acting on filter objects instead of acting on sets. Below these will be defined domain and image of a funcoid (the domain and the image of a funcoid are filter objects).

3.2 Basic definitions

Definition 25. Let's call *a funcoid* from a set A to a set B a quadruple $(A; B; \alpha; \beta)$ where $\alpha \in \mathfrak{F}(B)^{\mathfrak{F}(A)}$, $\beta \in \mathfrak{F}(A)^{\mathfrak{F}(B)}$ such that

$$\forall \mathcal{X} \in \mathfrak{F}(A), \mathcal{Y} \in \mathfrak{F}(B): (\mathcal{Y} \not\asymp \alpha \mathcal{X} \Leftrightarrow \mathcal{X} \not\asymp \beta \mathcal{Y}).$$

Futher we will assume that all funcoids in consideration are small without mentioning it explicitly.

Definition 26. *Source* and *destination* of every funcoid $(A; B; \alpha; \beta)$ are defined as

$$\operatorname{Src}(A; B; \alpha; \beta) = A \quad \text{and} \quad \operatorname{Dst}(A; B; \alpha; \beta) = B.$$

I will denote $\mathsf{FCD}(A; B)$ the set of funcoids from A to B.

I will denote FCD the set of all funcoids (for small sets).

Definition 27. $\langle (A; B; \alpha; \beta) \rangle \overset{\text{def}}{=} \alpha$ for a funcoid $(A; B; \alpha; \beta)$.

Definition 28. $(A; B; \alpha; \beta)^{-1} = (B; A; \beta; \alpha)$ for a funcoid $(A; B; \alpha; \beta)$.

Proposition 29. If f is a funcoid then f^{-1} is also a funcoid.

Proof. It follows from symmetry in the definition of funcoid. □

Obvious 30. $(f^{-1})^{-1} = f$ for a funcoid f.

Definition 31. The relation $[f] \in \mathscr{P}(\mathfrak{F}(\operatorname{Src} f) \times \mathfrak{F}(\operatorname{Dst} f))$ is defined (for every funcoid f and $\mathcal{X} \in \mathfrak{F}(\operatorname{Src} f)$, $\mathcal{Y} \in \mathfrak{F}(\operatorname{Dst} f)$) by the formula $\mathcal{X} [f] \mathcal{Y} \overset{\text{def}}{=} \mathcal{Y} \not\asymp \langle f \rangle \mathcal{X}$.

Obvious 32. $\mathcal{X} [f] \mathcal{Y} = \mathcal{Y} \not\asymp \langle f \rangle \mathcal{X} \Leftrightarrow \mathcal{X} \not\asymp \langle f^{-1} \rangle \mathcal{Y}$ for every funcoid f and $\mathcal{X} \in \mathfrak{F}(\operatorname{Src} f)$, $\mathcal{Y} \in \mathfrak{F}(\operatorname{Dst} f)$.

Obvious 33. $[f^{-1}] = [f]^{-1}$ for a funcoid f.

Theorem 34. Let A, B are small sets.

1. For given value of $\langle f \rangle$ exists no more than one funcoid $f \in \mathsf{FCD}(A; B)$.

2. For given value of $[f]$ exists no more than one funcoid $f \in \mathsf{FCD}(A; B)$.

Proof. Let $f, g \in \mathsf{FCD}(A; B)$.

Obviously $\langle f \rangle = \langle g \rangle \Rightarrow [f] = [g]$ and $\langle f^{-1} \rangle = \langle g^{-1} \rangle \Rightarrow [f] = [g]$. So enough to prove that $[f] = [g] \Rightarrow \langle f \rangle = \langle g \rangle$.

Provided that $[f] = [g]$ we have $\mathcal{Y} \not\asymp \langle f \rangle \mathcal{X} \Leftrightarrow \mathcal{X} [f] \mathcal{Y} \Leftrightarrow \mathcal{X} [g] \mathcal{Y} \Leftrightarrow \mathcal{Y} \not\asymp \langle g \rangle \mathcal{X}$ and consequently $\langle f \rangle \mathcal{X} = \langle g \rangle \mathcal{X}$ for every $\mathcal{X} \in \mathfrak{F}(A)$ and $\mathcal{Y} \in \mathfrak{F}(B)$ because a set of filter objects is separable [15], thus $\langle f \rangle = \langle g \rangle$. ☐

Proposition 35. $\langle f \rangle 0^{\mathfrak{F}(\mathrm{Src}\, f)} = 0^{\mathfrak{F}(\mathrm{Dst}\, f)}$ for every funcoid f.

Proof. $\mathcal{Y} \not\asymp \langle f \rangle 0^{\mathfrak{F}(\mathrm{Src}\, f)} \Leftrightarrow 0^{\mathfrak{F}(\mathrm{Src}\, f)} \not\asymp \langle f^{-1} \rangle \mathcal{Y} \Leftrightarrow 0 \Leftrightarrow \mathcal{Y} \not\asymp 0^{\mathfrak{F}(\mathrm{Dst}\, f)}$. Thus $\langle f \rangle 0^{\mathfrak{F}(\mathrm{Src}\, f)} = 0^{\mathfrak{F}(\mathrm{Dst}\, f)}$ by separability of filter objects. ☐

Proposition 36. $\langle f \rangle (\mathcal{I} \cup \mathcal{J}) = \langle f \rangle \mathcal{I} \cup \langle f \rangle \mathcal{J}$ for every funcoid f and $\mathcal{I}, \mathcal{J} \in \mathfrak{F}(\mathrm{Src}\, f)$.

Proof.

$$\begin{aligned} \star \langle f \rangle (\mathcal{I} \cup \mathcal{J}) &= \\ \{ \mathcal{Y} \in \mathfrak{F} \mid \mathcal{Y} \not\asymp \langle f \rangle (\mathcal{I} \cup \mathcal{J}) \} &= \\ \{ \mathcal{Y} \in \mathfrak{F} \mid \mathcal{I} \cup \mathcal{J} \not\asymp \langle f^{-1} \rangle \mathcal{Y} \} &= \text{(by corollary 10 in [15])} \\ \{ \mathcal{Y} \in \mathfrak{F} \mid \mathcal{I} \not\asymp \langle f^{-1} \rangle \mathcal{Y} \vee \mathcal{J} \not\asymp \langle f^{-1} \rangle \mathcal{Y} \} &= \\ \{ \mathcal{Y} \in \mathfrak{F} \mid \mathcal{Y} \not\asymp \langle f \rangle \mathcal{I} \vee \mathcal{Y} \not\asymp \langle f \rangle \mathcal{J} \} &= \\ \{ \mathcal{Y} \in \mathfrak{F} \mid \mathcal{Y} \not\asymp \langle f \rangle \mathcal{I} \cup \langle f \rangle \mathcal{J} \} &= \\ \star (\langle f \rangle \mathcal{I} \cup \langle f \rangle \mathcal{J}). & \end{aligned}$$

Thus $\langle f \rangle (\mathcal{I} \cup \mathcal{J}) = \langle f \rangle \mathcal{I} \cup \langle f \rangle \mathcal{J}$ because $\mathfrak{F}(\mathrm{Dst}\, f)$ is separable. ☐

Proposition 37. For every $f \in \mathsf{FCD}(A; B)$ for every small sets A and B we have:

1. $\mathcal{K} [f] \mathcal{I} \cup \mathcal{J} \Leftrightarrow \mathcal{K} [f] \mathcal{I} \vee \mathcal{K} [f] \mathcal{J}$ for every $\mathcal{I}, \mathcal{J} \in \mathfrak{F}(B)$, $\mathcal{K} \in \mathfrak{F}(A)$.
2. $\mathcal{I} \cup \mathcal{J} [f] \mathcal{K} \Leftrightarrow \mathcal{I} [f] \mathcal{K} \vee \mathcal{J} [f] \mathcal{K}$ for every $\mathcal{I}, \mathcal{J} \in \mathfrak{F}(A)$, $\mathcal{K} \in \mathfrak{F}(B)$.

Proof. 1. $\mathcal{K} [f] \mathcal{I} \cup \mathcal{J} \Leftrightarrow (\mathcal{I} \cup \mathcal{J}) \cap \langle f \rangle \mathcal{K} \neq 0^{\mathfrak{F}(B)} \Leftrightarrow (\mathcal{I} \cap \langle f \rangle \mathcal{K}) \cup (\mathcal{J} \cap \langle f \rangle \mathcal{K}) \neq 0^{\mathfrak{F}(B)} \Leftrightarrow \mathcal{I} \cap \langle f \rangle \mathcal{K} \neq 0^{\mathfrak{F}(B)} \vee \mathcal{J} \cap \langle f \rangle \mathcal{K} \neq 0^{\mathfrak{F}(B)} \Leftrightarrow \mathcal{K} [f] \mathcal{I} \vee \mathcal{K} [f] \mathcal{J}$.

2. Similar. ☐

3.2.1 Composition of funcoids

Definition 38. Funcoids f and g are *composable* when $\mathrm{Dst}\, f = \mathrm{Src}\, g$.

Definition 39. *Composition* of composable funcoids is defined by the formula

$$(B; C; \alpha_2; \beta_2) \circ (A; B; \alpha_1; \beta_1) = (A; C; \alpha_2 \circ \alpha_1; \beta_1 \circ \beta_2).$$

Proposition 40. If f, g are composable funcoids then $g \circ f$ is a funcoid.

Proof. Let $f = (A; B; \alpha_1; \beta_1)$, $g = (B; C; \alpha_2; \beta_2)$. For every $\mathcal{X} \in \mathfrak{F}(A)$, $\mathcal{Y} \in \mathfrak{F}(C)$ we have

$$\mathcal{Y} \not\asymp (\alpha_2 \circ \alpha_1) \mathcal{X} \Leftrightarrow \mathcal{Y} \not\asymp \alpha_2 \alpha_1 \mathcal{X} \Leftrightarrow \alpha_1 \mathcal{X} \not\asymp \beta_2 \mathcal{Y} \Leftrightarrow \mathcal{X} \not\asymp \beta_1 \beta_2 \mathcal{Y} \Leftrightarrow \mathcal{X} \not\asymp (\beta_1 \circ \beta_2) \mathcal{Y}.$$

So $(A; C; \alpha_2 \circ \alpha_1; \beta_1 \circ \beta_2)$ is a funcoid. ☐

Obvious 41. $\langle g \circ f \rangle = \langle g \rangle \circ \langle f \rangle$ for every composable funcoids f and g.

Proposition 42. $(h \circ g) \circ f = h \circ (g \circ f)$ for every composable funcoids f, g, h.

Proof.

$$\langle (h\circ g)\circ f\rangle = \langle h\circ g\rangle \circ \langle f\rangle = (\langle h\rangle \circ \langle g\rangle)\circ \langle f\rangle = \langle h\rangle \circ (\langle g\rangle \circ \langle f\rangle) = \langle h\rangle \circ \langle g\circ f\rangle = \langle h\circ (g\circ f)\rangle. \quad \square$$

Theorem 43. $(g\circ f)^{-1} = f^{-1}\circ g^{-1}$ for every composable funcoids f and g.

Proof. $\langle (g\circ f)^{-1}\rangle = \langle f^{-1}\rangle \circ \langle g^{-1}\rangle = \langle f^{-1}\circ g^{-1}\rangle$. □

3.3 Funcoid as continuation

Let f is a funcoid.

Definition 44. $\langle f\rangle^*$ is the function $\mathscr{P}(\mathrm{Src}\, f)\rightarrow \mathfrak{F}(\mathrm{Dst}\, f)$ defined by the formula

$$\langle f\rangle^* X = \langle f\rangle {\uparrow^{\mathrm{Src}\, f}} X.$$

Definition 45. $[f]^*$ is the relation between $\mathscr{P}(\mathrm{Src}\, f)$ and $\mathscr{P}(\mathrm{Dst}\, f)$ defined by the formula

$$X [f]^* Y = {\uparrow^{\mathrm{Src}\, f}} X [f] {\uparrow^{\mathrm{Dst}\, f}} Y.$$

Obvious 46.

1. $\langle f\rangle^* = \langle f\rangle \circ {\uparrow^{\mathrm{Src}\, f}}$;
2. $[f]^* = ({\uparrow^{\mathrm{Dst}\, f}})^{-1}\circ [f]\circ {\uparrow^{\mathrm{Src}\, f}}$.

Theorem 47. For every funcoid f and $\mathcal{X}\in \mathfrak{F}(\mathrm{Src}\, f)$ and $\mathcal{Y}\in \mathfrak{F}(\mathrm{Dst}\, f)$

1. $\langle f\rangle \mathcal{X} = \bigcap \langle \langle f\rangle^*\rangle \mathrm{up}\, \mathcal{X}$;
2. $\mathcal{X} [f] \mathcal{Y} \Leftrightarrow \forall X\in \mathrm{up}\, \mathcal{X}, Y\in \mathrm{up}\, \mathcal{Y}\colon X [f]^* Y$.

Proof. 2. $\mathcal{X} [f] \mathcal{Y} \Leftrightarrow \mathcal{Y}\cap \langle f\rangle \mathcal{X} \neq 0^{\mathfrak{F}(\mathrm{Dst}\, f)} \Leftrightarrow \forall Y\in \mathrm{up}\, \mathcal{Y}\colon {\uparrow^{\mathrm{Dst}\, f}} Y\cap \langle f\rangle \mathcal{X} \neq 0^{\mathfrak{F}(\mathrm{Dst}\, f)} \Leftrightarrow \forall Y\in \mathrm{up}\, \mathcal{Y}\colon \mathcal{X} [f] {\uparrow^{\mathrm{Dst}\, f}} Y$.

Analogously $\mathcal{X} [f] \mathcal{Y} \Leftrightarrow \forall X\in \mathrm{up}\, \mathcal{X}\colon {\uparrow^{\mathrm{Src}\, f}} X [f] \mathcal{Y}$. Combining these two equivalences we get

$$\mathcal{X} [f] \mathcal{Y} \Leftrightarrow \forall X\in \mathrm{up}\, \mathcal{X}, Y\in \mathrm{up}\, \mathcal{Y}\colon {\uparrow^{\mathrm{Src}\, f}} X [f] {\uparrow^{\mathrm{Dst}\, f}} Y \Leftrightarrow \forall X\in \mathrm{up}\, \mathcal{X}, Y\in \mathrm{up}\, \mathcal{Y}\colon X [f]^* Y.$$

1. $\mathcal{Y}\cap \langle f\rangle \mathcal{X} \neq \emptyset \Leftrightarrow \mathcal{X} [f] \mathcal{Y} \Leftrightarrow \forall X\in \mathrm{up}\, \mathcal{X}\colon {\uparrow^{\mathrm{Src}\, f}} X [f] \mathcal{Y} \Leftrightarrow \forall X\in \mathrm{up}\, \mathcal{X}\colon \mathcal{Y}\cap \langle f\rangle^* X \neq 0^{\mathfrak{F}(\mathrm{Dst}\, f)}$.

Let's denote $W = \{\mathcal{Y}\cap \langle f\rangle^* X \mid X\in \mathrm{up}\, \mathcal{X}\}$. We will prove that W is a generalized filter base. To prove this enough to show that $V = \{\langle f\rangle^* X \mid X\in \mathrm{up}\, \mathcal{X}\}$ is a generalized filter base.

Let $\mathcal{P}, \mathcal{Q}\in V$. Then $\mathcal{P} = \langle f\rangle^* A$, $\mathcal{Q} = \langle f\rangle^* B$ where $A, B\in \mathrm{up}\, \mathcal{X}$; $A\cap B\in \mathrm{up}\, \mathcal{X}$ and $\mathcal{R}\subseteq \mathcal{P}\cap \mathcal{Q}$ for $\mathcal{R} = \langle f\rangle^* (A\cap B)\in V$. So V is a generalized filter base and thus W is a generalized filter base.

$0^{\mathfrak{F}(\mathrm{Dst}\, f)}\notin W \Leftrightarrow \bigcap W\neq 0^{\mathfrak{F}(\mathrm{Dst}\, f)}$ by the corollary 4 of the theorem 3. That is

$$\forall X\in \mathrm{up}\, \mathcal{X}\colon \mathcal{Y}\cap \langle f\rangle^* X \neq 0^{\mathfrak{F}(\mathrm{Dst}\, f)} \Leftrightarrow \mathcal{Y}\cap \bigcap \langle \langle f\rangle^*\rangle \mathrm{up}\, \mathcal{X} \neq 0^{\mathfrak{F}(\mathrm{Dst}\, f)}.$$

Comparing with the above, $\mathcal{Y}\cap \langle f\rangle \mathcal{X} \neq \emptyset \Leftrightarrow \mathcal{Y}\cap \bigcap \langle \langle f\rangle^*\rangle \mathrm{up}\, \mathcal{X} \neq 0^{\mathfrak{F}(\mathrm{Dst}\, f)}$. So $\langle f\rangle \mathcal{X} = \bigcap \langle \langle f\rangle^*\rangle \mathrm{up}\, \mathcal{X}$ because the lattice of filter objects is separable. □

Proposition 48. For every $f\in \mathsf{FCD}(A; B)$ we have (for every $I, J\in \mathscr{P}A$)

$$\langle f\rangle^* \emptyset = 0^{\mathfrak{F}(B)}, \quad \langle f\rangle^* (I\cup J) = \langle f\rangle^* I\cup \langle f\rangle^* J$$

and

$$\neg (\emptyset [f]^* I),\ \ I\cup J [f]^* K \Leftrightarrow I [f]^* K\vee J [f]^* K \quad (\text{for every } I, J\in \mathscr{P}A,\ K\in \mathscr{P}B),$$
$$\neg (I [f]^* \emptyset),\ \ K [f]^* I\cup J \Leftrightarrow K [f]^* I\vee K [f]^* J \quad (\text{for every } I, J\in \mathscr{P}B,\ K\in \mathscr{P}A).$$

Proof. $\langle f\rangle^* \emptyset = \langle f\rangle {\uparrow^A} \emptyset = \langle f\rangle 0^{\mathfrak{F}(A)} = 0^{\mathfrak{F}(B)}$; $\langle f\rangle^* (I\cup J) = \langle f\rangle {\uparrow^A} (I\cup J) = \langle f\rangle ({\uparrow^A} I\cup {\uparrow^A} J) = \langle f\rangle {\uparrow^A} I\cup \langle f\rangle {\uparrow^A} J = \langle f\rangle^* I\cup \langle f\rangle^* J$.

$I [f]^* \emptyset \Leftrightarrow 0^{\mathfrak{F}(B)} \not\asymp \langle f\rangle {\uparrow^A} I \Leftrightarrow 0$; $I\cup J [f]^* K \Leftrightarrow {\uparrow^A} (I\cup J) [f] {\uparrow^B} K \Leftrightarrow {\uparrow^B} K \not\asymp \langle f\rangle^* (I\cup J) \Leftrightarrow {\uparrow^B} K \not\asymp \langle f\rangle^* I\cup \langle f\rangle^* J \Leftrightarrow {\uparrow^B} K \not\asymp \langle f\rangle^* I\vee {\uparrow^B} K \not\asymp \langle f\rangle^* J \Leftrightarrow I [f]^* K\vee J [f]^* K$.

The rest follows from symmetry. □

Theorem 49. Fix small sets A and B. Let $L_F = \lambda f \in \mathsf{FCD}(A;B): \langle f\rangle^*$ and $L_R = \lambda f \in \mathsf{FCD}(A;B): [f]^*$.

1. L_F is a bijection from the set $\mathsf{FCD}(A;B)$ to the set of functions $\alpha \in \mathfrak{F}(B)^{\mathscr{P}A}$ that obey the conditions (for every $I, J \in \mathscr{P}A$)

$$\alpha\emptyset = 0^{\mathfrak{F}(B)}, \quad \alpha(I \cup J) = \alpha I \cup \alpha J. \tag{1}$$

For such α it holds (for every $\mathcal{X} \in \mathfrak{F}(A)$)

$$\langle {L_F}^{-1}\alpha\rangle\mathcal{X} = \bigcap \langle\alpha\rangle \mathrm{up}\,\mathcal{X} \tag{2}$$

2. L_R is a bijection from the set $\mathsf{FCD}(A;B)$ to the set of binary relations $\delta \in \mathscr{P}(\mathscr{P}A \times \mathscr{P}B)$ that obey the conditions

$$\begin{array}{ll}\neg(\emptyset\,\delta\,I), \;\; I \cup J\,\delta\,K \Leftrightarrow I\,\delta\,K \vee J\,\delta\,K & (\text{for every } I, J \in \mathscr{P}A,\ K \in \mathscr{P}B),\\ \neg(I\,\delta\,\emptyset), \;\; K\,\delta\,I \cup J \Leftrightarrow K\,\delta\,I \vee K\,\delta\,J & (\text{for every } I, J \in \mathscr{P}B,\ K \in \mathscr{P}A).\end{array} \tag{3}$$

For such δ it holds (for every $\mathcal{X} \in \mathfrak{F}(A)$, $\mathcal{Y} \in \mathfrak{F}(B)$)

$$\mathcal{X}[{L_R}^{-1}\delta]\mathcal{Y} \Leftrightarrow \forall X \in \mathrm{up}\,\mathcal{X}, Y \in \mathrm{up}\,\mathcal{Y}: X\,\delta\,Y. \tag{4}$$

Proof. Injectivity of L_F and L_R, formulas (2) (for $\alpha \in \mathrm{im}\,L_F$) and (4) (for $\delta \in \mathrm{im}\,L_R$), formulas (1) and (3) follow from two previous theorems. The only thing remained to prove is that for every α and δ that obey the above conditions exists a corresponding funcoid f.

2. Let define $\alpha \in \mathfrak{F}(B)^{\mathscr{P}A}$ by the formula $\partial(\alpha X) = \{Y \in \mathscr{P}B \mid X\,\delta\,Y\}$ for every $X \in \mathscr{P}A$. (It is obvious that $\{Y \in \mathscr{P}B \mid X\,\delta\,Y\}$ is a free star.) Analogously it can be defined $\beta \in \mathfrak{F}(A)^{\mathscr{P}B}$ by the formula $\partial(\beta X) = \{X \in \mathscr{P}A \mid X\,\delta\,Y\}$. Let's continue α and β to $\alpha' \in \mathfrak{F}(B)^{\mathfrak{F}(A)}$ and $\beta' \in \mathfrak{F}(A)^{\mathfrak{F}(B)}$ by the formulas

$$\alpha'\mathcal{X} = \bigcap \langle\alpha\rangle \mathrm{up}\,\mathcal{X} \quad \text{and} \quad \beta'\mathcal{X} = \bigcap \langle\beta\rangle \mathrm{up}\,\mathcal{X}$$

and δ to $\delta' \in \mathscr{P}(\mathfrak{F}(A) \times \mathfrak{F}(B))$ by the formula

$$\mathcal{X}\,\delta'\,\mathcal{Y} \Leftrightarrow \forall X \in \mathrm{up}\,\mathcal{X}, Y \in \mathrm{up}\,\mathcal{Y}: X\,\delta\,Y.$$

$\mathcal{Y} \cap \alpha'\mathcal{X} \neq 0^{\mathfrak{F}(B)} \Leftrightarrow \mathcal{Y} \cap \bigcap \langle\alpha\rangle \mathrm{up}\,\mathcal{X} \neq 0^{\mathfrak{F}(B)} \Leftrightarrow \bigcap \langle\mathcal{Y}\cap\rangle\langle\alpha\rangle \mathrm{up}\,\mathcal{X} \neq 0^{\mathfrak{F}(B)}$. Let's prove that

$$W = \langle\mathcal{Y}\cap\rangle\langle\alpha\rangle \mathrm{up}\,\mathcal{X}$$

is a generalized filter base: To prove it is enough to show that $\langle\alpha\rangle \mathrm{up}\,\mathcal{X}$ is a generalized filter base. If $\mathcal{A}, \mathcal{B} \in \langle\alpha\rangle \mathrm{up}\,\mathcal{X}$ then exist $X_1, X_2 \in \mathrm{up}\,\mathcal{X}$ such that $\mathcal{A} = \alpha X_1$ and $\mathcal{B} = \alpha X_2$.

Then $\alpha(X_1 \cap X_2) \in \langle\alpha\rangle \mathrm{up}\,\mathcal{X}$. So $\langle\alpha\rangle \mathrm{up}\,\mathcal{X}$ is a generalized filter base and thus W is a generalized filter base.

Accordingly the corollary 4 of the theorem 3, $\bigcap \langle\mathcal{Y}\cap\rangle\langle\alpha\rangle \mathrm{up}\,\mathcal{X} \neq 0^{\mathfrak{F}(B)}$ is equivalent to

$$\forall X \in \mathrm{up}\,\mathcal{X}: \mathcal{Y} \cap \alpha X \neq 0^{\mathfrak{F}(B)},$$

what is equivalent to $\forall X \in \mathrm{up}\,\mathcal{X}, Y \in \mathrm{up}\,\mathcal{Y}: \uparrow^B Y \cap \alpha X \neq 0^{\mathfrak{F}(B)} \Leftrightarrow \forall X \in \mathrm{up}\,\mathcal{X}, Y \in \mathrm{up}\,\mathcal{Y}: Y \in \partial(\alpha X) \Leftrightarrow \forall X \in \mathrm{up}\,\mathcal{X}, Y \in \mathrm{up}\,\mathcal{Y}: X\,\delta\,Y$. Combining the equivalencies we get $\mathcal{Y} \cap \alpha'\mathcal{X} \neq 0^{\mathfrak{F}(B)} \Leftrightarrow \mathcal{X}\,\delta'\,\mathcal{Y}$. Analogously $\mathcal{X} \cap \beta'\mathcal{Y} \neq 0^{\mathfrak{F}(A)} \Leftrightarrow X\,\delta'\,Y$. So $\mathcal{Y} \cap \alpha'\mathcal{X} \neq 0^{\mathfrak{F}(B)} \Leftrightarrow \mathcal{X} \cap \beta'\mathcal{Y} \neq 0^{\mathfrak{F}(A)}$, that is $(A; B; \alpha'; \beta')$ is a funcoid. From the formula $\mathcal{Y} \cap \alpha'\mathcal{X} \neq 0^{\mathfrak{F}(B)} \Leftrightarrow \mathcal{X}\,\delta'\,\mathcal{Y}$ it follows that

$$X[(A; B; \alpha'; \beta')]^* Y \Leftrightarrow \uparrow^B Y \cap \alpha' \uparrow^A X \neq 0^{\mathfrak{F}(B)} \Leftrightarrow \uparrow^B X\,\delta'\uparrow^A Y \Leftrightarrow X\,\delta\,Y.$$

1. Let define the relation $\delta \in \mathscr{P}(\mathscr{P}A \times \mathscr{P}B)$ by the formula $X\,\delta\,Y \Leftrightarrow \uparrow^B Y \cap \alpha X \neq 0^{\mathfrak{F}(B)}$.

That $\neg(\emptyset\,\delta\,I)$ and $\neg(I\,\delta\,\emptyset)$ is obvious. We have $I \cup J\,\delta\,K \Leftrightarrow \uparrow^B K \cap \alpha(I \cup J) \neq 0^{\mathfrak{F}(B)} \Leftrightarrow \uparrow^B K \cap (\alpha I \cup \alpha J) \neq 0^{\mathfrak{F}(B)} \Leftrightarrow \uparrow^B K \cap \alpha I \neq 0^{\mathfrak{F}(B)} \vee \uparrow^B K \cap \alpha I \neq 0^{\mathfrak{F}(B)} \Leftrightarrow I\,\delta\,K \vee J\,\delta\,K$ and

$K\,\delta\,I \cup J \Leftrightarrow \uparrow^B (I \cup J) \cap \alpha K \neq 0^{\mathfrak{F}(B)} \Leftrightarrow (\uparrow^B I \cup \uparrow^B J) \cap \alpha K \neq 0^{\mathfrak{F}(B)} \Leftrightarrow (\uparrow^B I \cap \alpha K) \cup (\uparrow^B J \cap \alpha K) \neq 0^{\mathfrak{F}(B)} \Leftrightarrow \uparrow^B I \cap \alpha K \neq 0^{\mathfrak{F}(B)} \vee \uparrow^B J \cap \alpha K \neq 0^{\mathfrak{F}(B)} \Leftrightarrow K\,\delta\,I \vee K\,\delta\,J$.

That is the formulas (3) are true.

Accordingly the above there exist a funcoid f such that

$$\mathcal{X}[f]\mathcal{Y}\Leftrightarrow\forall X\in\operatorname{up}\mathcal{X},Y\in\operatorname{up}\mathcal{Y}: X\,\delta\,Y.$$

$\forall X\in\mathscr{P}A,Y\in\mathscr{P}B: \left(\uparrow^B Y\cap\langle f\rangle\uparrow^A X\neq\emptyset\Leftrightarrow\uparrow^A X[f]\uparrow^B Y\Leftrightarrow X\,\delta\,Y\Leftrightarrow\uparrow^B Y\cap\alpha X\neq 0^{\mathfrak{F}(B)}\right)$, consequently $\forall X\in\mathscr{P}A: \alpha X=\langle f\rangle\uparrow^A X=\langle f\rangle^* X$. □

Note that by the last theorem to every proximity δ corresponds a unique funcoid. So funcoids are a generalization of (quasi-)proximity structures.

Reverse funcoids can be considered as a generalization of conjugate quasi-proximity.

Definition 50. Any small (multivalued) function $F: A\to B$ corresponds to a funcoid $\uparrow^{\mathsf{FCD}(A;B)}F\in\mathsf{FCD}(A;B)$, where by definition $\left\langle\uparrow^{\mathsf{FCD}(A;B)}F\right\rangle\mathcal{X}=\bigcap\,\langle\uparrow^B\rangle\langle\langle F\rangle\rangle\operatorname{up}\mathcal{X}$ for every $\mathcal{X}\in\mathfrak{F}(A)$.

Using the last theorem it is easy to show that this definition is monovalued and does not contradict to former stuff. (Take $\alpha=\uparrow^B\circ\langle F\rangle$.)

Definition 51. Funcoids corresponding to a binary relation (= multivalued function) are called *discrete funcoids*.

We may equate discrete funcoids with corresponding binary relations by the method of appendix B in [15]. This is useful for describing relationships of funcoids and binary relations, such as for the formulas of continuous functions and continuous funcoids (see below).

Theorem 52. If S is a generalized filter base on $\operatorname{Src} f$ then $\langle f\rangle\bigcap S=\bigcap\langle\langle f\rangle\rangle S$ for every funcoid f.

Proof. $\langle f\rangle\bigcap S\subseteq\langle f\rangle X$ for every $X\in S$ and thus $\langle f\rangle\bigcap S\subseteq\bigcap\langle\langle f\rangle\rangle S$.

By properties of generalized filter bases:

$\langle f\rangle\bigcap S=\bigcap\langle\langle f\rangle^*\rangle\operatorname{up}\bigcap S=\bigcap\langle\langle f\rangle^*\rangle\{X\mid\exists\mathcal{P}\in S: X\in\operatorname{up}\mathcal{P}\}=\bigcap\{\langle f\rangle^* X\mid\exists\mathcal{P}\in S: X\in\operatorname{up}\mathcal{P}\}\supseteq\bigcap\{\langle f\rangle\mathcal{P}\mid\mathcal{P}\in S\}=\bigcap\langle\langle f\rangle\rangle S$. □

3.4 Lattices of funcoids

Definition 53. $f\subseteq g\overset{\text{def}}{=}[f]\subseteq[g]$ for $f,g\in\mathsf{FCD}$.

Thus every $\mathsf{FCD}(A;B)$ is a poset. (Taken into account that $[f]\neq[g]$ if $f\neq g$.)

Definition 54. I will call a *shifted filtrator of funcoids* the shifted filtrator

$$(\mathsf{FCD}(A;B);\mathscr{P}(A\times B);\uparrow^{\mathsf{FCD}(A;B)})$$

for some small sets A, B.

$\operatorname{up} f\overset{\text{def}}{=}\operatorname{up}^{(\mathsf{FCD}(A;B);\mathscr{P}(A\times B);\uparrow^{\mathsf{FCD}(A;B)})} f$ for every funcoid $f\in\mathsf{FCD}(A;B)$.

Lemma 55. $\langle f\rangle^* X=\bigcap\{\uparrow^{\operatorname{Dst} f}\langle F\rangle X\mid F\in\operatorname{up} f\}$ for every funcoid f and set $X\in\mathscr{P}(\operatorname{Src} f)$.

Proof. Obviously $\langle f\rangle^* X\subseteq\bigcap\{\uparrow^{\operatorname{Dst} f}\langle F\rangle X\mid F\in\operatorname{up} f\}$.

Let $B\in\operatorname{up}\langle f\rangle^* X$. Let $F_B=X\times B\cup((\operatorname{Src} f)\setminus X)\times(\operatorname{Dst} f)$.

$\langle F_B\rangle X=B$.

We have $\emptyset\neq P\subseteq X\Rightarrow\langle F_B\rangle P=B\supseteq\langle f\rangle^* P$ and $P\nsubseteq X\Rightarrow\langle F_B\rangle P=\operatorname{Dst} f\supseteq\langle f\rangle^* P$. Thus $\langle F_B\rangle P\supseteq\langle f\rangle^* P$ for every set $P\in\mathscr{P}(\operatorname{Src} f)$ and so $\uparrow^{\mathsf{FCD}(\operatorname{Src} f;\operatorname{Dst} f)}F_B\supseteq f$ that is $F_B\in\operatorname{up} f$.

Thus $\forall B\in\operatorname{up}\langle f\rangle^* X: B\in\operatorname{up}\bigcap\{\uparrow^{\operatorname{Dst} f}\langle F\rangle X\mid F\in\operatorname{up} f\}$ because $B\in\operatorname{up}\uparrow^{\operatorname{Dst} f}\langle F_B\rangle X$.

So $\bigcap\{\uparrow^{\operatorname{Dst} f}\langle F\rangle X\mid F\in\operatorname{up} f\}\subseteq\langle f\rangle^* X$. □

Theorem 56. $\langle f\rangle\mathcal{X}=\bigcap\left\{\left\langle\uparrow^{\mathsf{FCD}(\operatorname{Src} f;\operatorname{Dst} f)}F\right\rangle\mathcal{X}\mid F\in\operatorname{up} f\right\}$ for every funcoid f and $\mathcal{X}\in\mathfrak{F}(\operatorname{Src} f)$.

Proof. $\bigcap\left\{\left\langle\uparrow^{\mathsf{FCD}(\operatorname{Src} f;\operatorname{Dst} f)}F\right\rangle\mathcal{X}\mid F\in\operatorname{up} f\right\}=\bigcap\left\{\bigcap\langle\uparrow^{\operatorname{Dst} f}\rangle\langle\langle F\rangle\rangle\operatorname{up}\mathcal{X}\mid F\in\operatorname{up} f\right\}=\bigcap\left\{\bigcap\{\uparrow^{\operatorname{Dst} f}\langle F\rangle X\mid X\in\operatorname{up}\mathcal{X}\}\mid F\in\operatorname{up} f\right\}=\bigcap\left\{\bigcap\{\uparrow^{\operatorname{Dst} f}\langle F\rangle X\mid F\in\operatorname{up} f\}\mid X\in\operatorname{up}\mathcal{X}\right\}=\bigcap\{\uparrow^{\operatorname{Dst} f}\langle f\rangle^* X\mid X\in\operatorname{up}\mathcal{X}\}=\langle f\rangle\mathcal{X}$ (the lemma used). □

Conjecture 57. Every filtrator of funcoids is:

1. with separable core;
2. with co-separable core.

Below it is shown that $\mathsf{FCD}(A;B)$ are complete lattices for every small sets A and B. We will apply lattice operations to subsets of such sets without explicitly mentioning $\mathsf{FCD}(A;B)$.

Theorem 58. $\mathsf{FCD}(A;B)$ is a complete lattice (for every small sets A and B). For every $R \in \mathscr{P}\mathsf{FCD}(A;B)$ and $X \in \mathscr{P}A$, $Y \in \mathscr{P}B$

1. $X[\bigcup R]^* Y \Leftrightarrow \exists f \in R: X[f]^* Y$;
2. $\langle \bigcup R\rangle^* X = \bigcup \{\langle f\rangle^* X \mid f \in R\}$.

Proof. Accordingly [14] to prove that it is a complete lattice enough to prove existence of all joins.

2. $\alpha X \overset{\text{def}}{=} \bigcup \{\langle f\rangle^* X \mid f \in R\}$. We have $\alpha\emptyset = \emptyset$;

$$\begin{aligned}\alpha(I \cup J) &= \bigcup \{\langle f\rangle^*(I \cup J) \mid f \in R\}\\ &= \bigcup \{\langle f\rangle^* I \cup \langle f\rangle^* J \mid f \in R\}\\ &= \bigcup \{\langle f\rangle^* I \mid f \in R\} \cup \bigcup \{\langle f\rangle^* J \mid f \in R\}\\ &= \alpha I \cup \alpha J.\end{aligned}$$

So $\langle h\rangle \circ \uparrow^A = \alpha$ for some funcoid h. Obviously

$$\forall f \in R: h \supseteq f. \tag{5}$$

And h is the least funcoid for which holds the condition (5). So $h = \bigcup R$.

1. $X[\bigcup R]^* Y \Leftrightarrow \uparrow^{\mathrm{Dst}\, f} Y \cap \langle \bigcup R\rangle^* X \neq 0^{\mathfrak{F}(\mathrm{Dst}\, f)} \Leftrightarrow \uparrow^{\mathrm{Dst}\, f} Y \cap \bigcup \{\langle f\rangle^* X \mid f \in R\} \neq 0^{\mathfrak{F}(\mathrm{Dst}\, f)} \Leftrightarrow \exists f \in R: \uparrow^{\mathrm{Dst}\, f} Y \cap \langle f\rangle^* X \neq 0^{\mathfrak{F}(\mathrm{Dst}\, f)} \Leftrightarrow \exists f \in R: X[f]^* Y$ (used the theorem 40 in [15]). □

In the next theorem, compared to the previous one, the class of infinite unions is replaced with lesser class of finite unions and simultaneously class of sets is changed to more wide class of filter objects.

Theorem 59. For every $f, g \in \mathsf{FCD}(A;B)$ and $\mathcal{X} \in \mathfrak{F}(A)$ (for every small sets A, B)

1. $\langle f \cup g\rangle \mathcal{X} = \langle f\rangle \mathcal{X} \cup \langle g\rangle \mathcal{X}$;
2. $[f \cup g] = [f] \cup [g]$.

Proof.

1. Let $\alpha \mathcal{X} \overset{\text{def}}{=} \langle f\rangle \mathcal{X} \cup \langle g\rangle \mathcal{X}$; $\beta \mathcal{Y} \overset{\text{def}}{=} \langle f^{-1}\rangle \mathcal{Y} \cup \langle g^{-1}\rangle \mathcal{Y}$ for every $\mathcal{X} \in \mathfrak{F}(A)$, $\mathcal{Y} \in \mathfrak{F}(B)$. Then

$$\begin{aligned}\mathcal{Y} \cap \alpha \mathcal{X} \neq \emptyset &\Leftrightarrow \mathcal{Y} \cap \langle f\rangle \mathcal{X} \neq 0^{\mathfrak{F}(B)} \vee \mathcal{Y} \cap \langle g\rangle \mathcal{X} \neq 0^{\mathfrak{F}(B)}\\ &\Leftrightarrow \mathcal{X} \cap \langle f^{-1}\rangle \mathcal{Y} \neq 0^{\mathfrak{F}(A)} \vee \mathcal{X} \cap \langle g^{-1}\rangle \mathcal{Y} \neq 0^{\mathfrak{F}(A)}\\ &\Leftrightarrow \mathcal{X} \cap \beta \mathcal{Y} \neq 0^{\mathfrak{F}(A)}.\end{aligned}$$

   So $h = (A; B; \alpha; \beta)$ is a funcoid. Obviously $h \supseteq f$ and $h \supseteq g$. If $p \supseteq f$ and $p \supseteq g$ for some funcoid p then $\langle p\rangle \mathcal{X} \supseteq \langle f\rangle \mathcal{X} \cup \langle g\rangle \mathcal{X} = \langle h\rangle \mathcal{X}$ that is $p \supseteq h$. So $f \cup g = h$.

2. $\mathcal{X}[f \cup g]\mathcal{Y} \Leftrightarrow \mathcal{Y} \cap \langle f \cup g\rangle \mathcal{X} \neq 0^{\mathfrak{F}(B)} \Leftrightarrow \mathcal{Y} \cap (\langle f\rangle \mathcal{X} \cup \langle g\rangle \mathcal{X}) \neq 0^{\mathfrak{F}(B)} \Leftrightarrow \mathcal{Y} \cap \langle f\rangle \mathcal{X} \neq 0^{\mathfrak{F}(B)} \vee \mathcal{Y} \cap \langle g\rangle \mathcal{X} \neq 0^{\mathfrak{F}(B)} \Leftrightarrow \mathcal{X}[f]\mathcal{Y} \vee \mathcal{X}[g]\mathcal{Y}$ for every $\mathcal{X} \in \mathfrak{F}(A)$, $\mathcal{Y} \in \mathfrak{F}(B)$. □

3.5 More on composition of funcoids

Proposition 60. $[g \circ f] = [g] \circ \langle f\rangle = \langle g^{-1}\rangle^{-1} \circ [f]$ for every composable funcoids f and g.

Proof. $\mathcal{X}[g \circ f]\mathcal{Y} \Leftrightarrow \mathcal{Y} \cap \langle g \circ f\rangle \mathcal{X} \neq 0^{\mathfrak{F}(\mathrm{Dst}\, g)} \Leftrightarrow \mathcal{Y} \cap \langle g\rangle \langle f\rangle \mathcal{X} \neq 0^{\mathfrak{F}(\mathrm{Dst}\, g)} \Leftrightarrow \langle f\rangle \mathcal{X}[g]\mathcal{Y} \Leftrightarrow \mathcal{X}([g] \circ \langle f\rangle)\mathcal{Y}$ for every $\mathcal{X} \in \mathfrak{F}(\mathrm{Src}\, f)$, $\mathcal{Y} \in \mathfrak{F}(\mathrm{Dst}\, g)$. $[g \circ f] = [(f^{-1} \circ g^{-1})^{-1}] = [f^{-1} \circ g^{-1}]^{-1} = ([f^{-1}] \circ \langle g^{-1}\rangle)^{-1} = \langle g^{-1}\rangle^{-1} \circ [f]$. □

The following theorem is a variant for funcoids of the statement (which defines compositions of relations) that $x(g\circ f)z \Leftrightarrow \exists y(x\, f\, y\wedge y\, g\, z)$ for every x and z and every binary relations f and g.

Theorem 61. For every small sets A, B, C and $f\in\mathsf{FCD}(A;B)$, $g\in\mathsf{FCD}(B;C)$ and $\mathcal{X}\in\mathfrak{F}(A)$, $\mathcal{Z}\in\mathfrak{F}(C)$.

$$\mathcal{X}[g\circ f]\mathcal{Z}\Leftrightarrow\exists y\in\operatorname{atoms}1^{\mathfrak{F}(B)}\colon(\mathcal{X}[f]y\wedge y[g]\mathcal{Z}).$$

Proof.

$$\begin{aligned}\exists y\in\operatorname{atoms}1^{\mathfrak{F}(B)}\colon(\mathcal{X}[f]y\wedge y[g]\mathcal{Z}) &\Leftrightarrow \exists y\in\operatorname{atoms}1^{\mathfrak{F}(B)}\colon\left(\mathcal{Z}\cap\langle g\rangle y\neq 0^{\mathfrak{F}(C)}\wedge y\cap\langle f\rangle\mathcal{X}\neq 0^{\mathfrak{F}(B)}\right)\\ &\Leftrightarrow \exists y\in\operatorname{atoms}1^{\mathfrak{F}(B)}\colon(\mathcal{Z}\cap\langle g\rangle y\neq 0^{\mathfrak{F}(C)}\wedge y\subseteq\langle f\rangle\mathcal{X})\\ &\Rightarrow \mathcal{Z}\cap\langle g\rangle\langle f\rangle\mathcal{X}\neq 0^{\mathfrak{F}(C)}\\ &\Leftrightarrow \mathcal{X}[g\circ f]\mathcal{Z}.\end{aligned}$$

Reversely, if $\mathcal{X}[g\circ f]\mathcal{Z}$ then $\langle f\rangle\mathcal{X}[g]\mathcal{Z}$, consequently exists $y\in\operatorname{atoms}\langle f\rangle\mathcal{X}$ such that $y[g]\mathcal{Z}$; we have $\mathcal{X}[f]y$. □

Theorem 62. For every small sets A, B, C

1. $f\circ(g\cup h)=f\circ g\cup f\circ h$ for $g,h\in\mathsf{FCD}(A;B)$ and $f\in\mathsf{FCD}(B;C)$;
2. $(g\cup h)\circ f=g\circ f\cup h\circ f$ for $g,h\in\mathsf{FCD}(B;C)$ and $f\in\mathsf{FCD}(A;B)$.

Proof. I will prove only the first equality because the other is analogous.
For every $\mathcal{X},\mathcal{Z}\in\mathfrak{F}$

$$\begin{aligned}\mathcal{X}[f\circ(g\cup h)]\mathcal{Z} &\Leftrightarrow \exists y\in\operatorname{atoms}1^{\mathfrak{F}(B)}\colon(\mathcal{X}[g\cup h]y\wedge y[f]\mathcal{Z})\\ &\Leftrightarrow \exists y\in\operatorname{atoms}1^{\mathfrak{F}(B)}\colon((\mathcal{X}[g]y\vee\mathcal{X}[h]y)\wedge y[f]\mathcal{Z})\\ &\Leftrightarrow \exists y\in\operatorname{atoms}1^{\mathfrak{F}(B)}\colon(\mathcal{X}[g]y\wedge y[f]\mathcal{Z}\vee\mathcal{X}[h]y\wedge y[f]\mathcal{Z})\\ &\Leftrightarrow \exists y\in\operatorname{atoms}1^{\mathfrak{F}(B)}\colon(\mathcal{X}[g]y\wedge y[f]\mathcal{Z})\vee\exists y\in\operatorname{atoms}1^{\mathfrak{F}(B)}\colon(\mathcal{X}[h]y\wedge y[f]\mathcal{Z})\\ &\Leftrightarrow \mathcal{X}[f\circ g]\mathcal{Z}\vee\mathcal{X}[f\circ h]\mathcal{Z}\\ &\Leftrightarrow \mathcal{X}[f\circ g\cup f\circ h]\mathcal{Z}.\end{aligned}$$

□

3.6 Domain and range of a funcoid

Definition 63. Let A is a small set. The *identity funcoid* $I^{\mathsf{FCD}(A)}=(A;A;(=)|_{\mathfrak{F}(A)};(=)|_{\mathfrak{F}(A)})$.

Obvious 64. The identity funcoid is a funcoid.

Definition 65. Let A is a small set, $\mathcal{A}\in\mathfrak{F}(A)$. The *restricted identity funcoid*

$$I^{\mathsf{FCD}}_{\mathcal{A}}=(A;A;\mathcal{A}\cap;\mathcal{A}\cap).$$

Proposition 66. The restricted identity funcoid is a funcoid.

Proof. We need to prove that $(\mathcal{A}\cap\mathcal{X})\cap\mathcal{Y}\neq\emptyset\Leftrightarrow(\mathcal{A}\cap\mathcal{Y})\cap\mathcal{X}\neq\emptyset$ what is obvious. □

Obvious 67.

1. $\left(I^{\mathsf{FCD}(A)}\right)^{-1}=I^{\mathsf{FCD}(A)}$;
2. $(I^{\mathsf{FCD}}_{\mathcal{A}})^{-1}=I^{\mathsf{FCD}}_{\mathcal{A}}$.

Obvious 68. For every $\mathcal{X},\mathcal{Y}\in\mathfrak{F}(A)$

1. $\mathcal{X}\left[I^{\mathsf{FCD}(A)}\right]\mathcal{Y}\Leftrightarrow\mathcal{X}\cap\mathcal{Y}\neq\emptyset$.
2. $\mathcal{X}[I^{\mathsf{FCD}}_{\mathcal{A}}]\mathcal{Y}\Leftrightarrow\mathcal{A}\cap\mathcal{X}\cap\mathcal{Y}\neq\emptyset$.

Definition 69. I will define *restricting* of a funcoid f to a filter object $\mathcal{A}\in\mathfrak{F}(\operatorname{Src} f)$ by the formula

$$f|_{\mathcal{A}}\overset{\mathrm{def}}{=} f\circ I^{\mathsf{FCD}}_{\mathcal{A}}.$$

Definition 70. *Image* of a funcoid f will be defined by the formula $\operatorname{im} f=\langle f\rangle 1^{\mathfrak{F}(\operatorname{Src} f)}$.

Domain of a funcoid f is defined by the formula $\operatorname{dom} f=\operatorname{im} f^{-1}$.

Proposition 71. $\langle f\rangle\mathcal{X}=\langle f\rangle(\mathcal{X}\cap\operatorname{dom} f)$ for every $f\in\mathsf{FCD}$, $\mathcal{X}\in\mathfrak{F}(\operatorname{Src} f)$.

Proof. For every $\mathcal{Y}\in\mathfrak{F}(\operatorname{Dst} f)$ we have $\mathcal{Y}\cap\langle f\rangle(\mathcal{X}\cap\operatorname{dom} f)\neq 0^{\mathfrak{F}(\operatorname{Dst} f)}\Leftrightarrow\mathcal{X}\cap\operatorname{dom} f\cap\langle f^{-1}\rangle\mathcal{Y}\neq 0^{\mathfrak{F}(\operatorname{Src} f)}\Leftrightarrow\mathcal{X}\cap\operatorname{im} f^{-1}\cap\langle f^{-1}\rangle\mathcal{Y}\neq 0^{\mathfrak{F}(\operatorname{Src} f)}\Leftrightarrow\mathcal{X}\cap\langle f^{-1}\rangle\mathcal{Y}\neq 0^{\mathfrak{F}(\operatorname{Src} f)}\Leftrightarrow\mathcal{Y}\cap\langle f\rangle\mathcal{X}\neq 0^{\mathfrak{F}(\operatorname{Dst} f)}$. Thus $\langle f\rangle\mathcal{X}=\langle f\rangle(\mathcal{X}\cap\operatorname{dom} f)$ because the lattice of filter objects is separable. □

Proposition 72. $\mathcal{X}\cap\operatorname{dom} f\neq 0^{\mathfrak{F}(\operatorname{Src} f)}\Leftrightarrow\langle f\rangle\mathcal{X}\neq 0^{\mathfrak{F}(\operatorname{Dst} f)}$ for every $f\in\mathsf{FCD}$, $\mathcal{X}\in\mathfrak{F}(\operatorname{Src} f)$.

Proof. $\mathcal{X}\cap\operatorname{dom} f\neq 0^{\mathfrak{F}(\operatorname{Src} f)}\Leftrightarrow\mathcal{X}\cap\langle f^{-1}\rangle 1^{\mathfrak{F}(\operatorname{Dst} f)}\neq 0^{\mathfrak{F}(\operatorname{Src} f)}\Leftrightarrow 1^{\mathfrak{F}(\operatorname{Dst} f)}\cap\langle f\rangle\mathcal{X}\neq 0^{\mathfrak{F}(\operatorname{Dst} f)}\Leftrightarrow\langle f\rangle\mathcal{X}\neq 0^{\mathfrak{F}(\operatorname{Dst} f)}$. □

Corollary 73. $\operatorname{dom} f=\bigcup\left\{a\in\operatorname{atoms} 1^{\mathfrak{F}(\operatorname{Src} f)}\mid\langle f\rangle a\neq 0^{\mathfrak{F}(\operatorname{Dst} f)}\right\}$.

Proof. This follows from the fact that $\mathfrak{F}(\operatorname{Src} f)$ is an atomistic lattice. □

Proposition 74. $\operatorname{dom} f|_{\mathcal{A}}=\mathcal{A}\cap\operatorname{dom} f$ for every funcoid f and $\mathcal{A}\in\mathfrak{F}(\operatorname{Src} f)$.

Proof. $\operatorname{dom} f|_{\mathcal{A}}=\operatorname{im}(I^{\mathsf{FCD}}_{\mathcal{A}}\circ f^{-1})=\langle I^{\mathsf{FCD}}_{\mathcal{A}}\rangle\langle f^{-1}\rangle 1^{(\operatorname{Dst} f)}=\mathcal{A}\cap\langle f^{-1}\rangle 1^{(\operatorname{Dst} f)}=\mathcal{A}\cap\operatorname{dom} f$. □

Theorem 75. $\operatorname{im} f=\bigcap\langle\uparrow^{\operatorname{Dst} f}\rangle\langle\operatorname{im}\rangle\operatorname{up} f$ and $\operatorname{dom} f=\bigcap\langle\uparrow^{\operatorname{Src} f}\rangle\langle\operatorname{dom}\rangle\operatorname{up} f$ for every funcoid f.

Proof. $\operatorname{im} f=\langle f\rangle 1^{\mathfrak{F}(\operatorname{Src} f)}=\bigcap\left\{\left\langle\uparrow^{\mathsf{FCD}(\operatorname{Src} f;\operatorname{Dst} f)}F\right\rangle 1^{\mathfrak{F}(\operatorname{Src} f)}\mid F\in\operatorname{up} f\right\}=\bigcap\{\uparrow^{\operatorname{Dst} f}\operatorname{im} F\mid F\in\operatorname{up} f\}=\bigcap\langle\uparrow^{\operatorname{Dst} f}\rangle\langle\operatorname{im}\rangle\operatorname{up} f$ (used the theorem 56).

The second formula follows from symmetry. □

Proposition 76. For every composable funcoids f, g:

1. If $\operatorname{im} f\supseteq\operatorname{dom} g$ then $\operatorname{im}(g\circ f)=\operatorname{im} g$.
2. If $\operatorname{im} f\subseteq\operatorname{dom} g$ then $\operatorname{dom}(g\circ f)=\operatorname{dom} f$.

Proof.

1. $\operatorname{im}(g\circ f)=\langle g\circ f\rangle 1^{\mathfrak{F}(\operatorname{Src} f)}=\langle g\rangle\langle f\rangle 1^{\mathfrak{F}(\operatorname{Src} f)}=\langle g\rangle\operatorname{im} f=\langle g\rangle(\operatorname{im} f\cap\operatorname{dom} g)=\langle g\rangle\operatorname{dom} g=\langle g\rangle 1^{\mathfrak{F}(\operatorname{Src} g)}=\operatorname{im} g$.
2. $\operatorname{dom}(g\circ f)=\operatorname{im}(f^{-1}\circ g^{-1})$ what by proved above is equal to $\operatorname{im} f^{-1}$ that is $\operatorname{dom} f$. □

3.7 Categories of funcoids

I will define two categories, the *category of funcoids* and the *category of funcoid triples*.

The *category of funcoids* is defined as follows:

- Objects are small sets.
- The set of morphisms from a set A to a set B is $\mathsf{FCD}(A;B)$.
- The composition is the composition of funcoids.
- Identity morphism for a set is the identity funcoid for that set.

To show it is really a category is trivial.

The *category of funcoid triples* is defined as follows:

- Objects are filter objects on small sets.
- The morphisms from a f.o. $\mathcal{A}$ to a f.o. $\mathcal{B}$ are triples $(f;\mathcal{A};\mathcal{B})$ where $f\in\mathsf{FCD}(\operatorname{Base}(\mathcal{A});\operatorname{Base}(\mathcal{B}))$ and $\operatorname{dom} f\subseteq\mathcal{A}\wedge\operatorname{im} f\subseteq\mathcal{B}$.

- The composition is defined by the formula $(g;\mathcal{B};\mathcal{C})\circ(f;\mathcal{A};\mathcal{B})=(g\circ f;\mathcal{A};\mathcal{C})$.
- Identity morphism for an f.o. $\mathcal{A}$ is $I^{\mathsf{FCD}}_{\mathcal{A}}$.

To prove that it is really a category is trivial.

3.8 Specifying funcoids by functions or relations on atomic filter objects

Theorem 77. For every funcoid f and $\mathcal{X}\in\mathfrak{F}(\mathrm{Src}\,f)$, $\mathcal{Y}\in\mathfrak{F}(\mathrm{Dst}\,f)$

1. $\langle f\rangle\mathcal{X}=\bigcup\,\langle\langle f\rangle\rangle\mathrm{atoms}\,\mathcal{X}$;
2. $\mathcal{X}[f]\mathcal{Y}\Leftrightarrow\exists x\in\mathrm{atoms}\,\mathcal{X},y\in\mathrm{atoms}\,\mathcal{Y}\colon x[f]y$.

Proof. 1.

$$\begin{aligned}\mathcal{Y}\cap\langle f\rangle\mathcal{X}\neq 0^{\mathfrak{F}(\mathrm{Dst}\,f)} &\Leftrightarrow \mathcal{X}\cap\langle f^{-1}\rangle\mathcal{Y}\neq 0^{\mathfrak{F}(\mathrm{Src}\,f)}\\ &\Leftrightarrow \exists x\in\mathrm{atoms}\,\mathcal{X}\colon x\cap\langle f^{-1}\rangle\mathcal{Y}\neq 0^{\mathfrak{F}(\mathrm{Src}\,f)}\\ &\Leftrightarrow \exists x\in\mathrm{atoms}\,\mathcal{X}\colon \mathcal{Y}\cap\langle f\rangle x\neq 0^{\mathfrak{F}(\mathrm{Dst}\,f)}.\end{aligned}$$

$\partial\langle f\rangle\mathcal{X}=\bigcup\,\langle\partial\rangle\,\langle\langle f\rangle\rangle\mathrm{atoms}\,\mathcal{X}=\partial\bigcup\,\langle\langle f\rangle\rangle\mathrm{atoms}\,\mathcal{X}$.

2. If $\mathcal{X}[f]\mathcal{Y}$, then $\mathcal{Y}\cap\langle f\rangle\mathcal{X}\neq 0^{\mathfrak{F}(\mathrm{Dst}\,f)}$, consequently exists $y\in\mathrm{atoms}\,\mathcal{Y}$ such that $y\cap\langle f\rangle\mathcal{X}\neq 0^{\mathfrak{F}(\mathrm{Dst}\,f)}$, $\mathcal{X}[f]y$. Repeating this second time we get that there exist $x\in\mathrm{atoms}\,\mathcal{X}$ such that $x[f]y$. From this follows

$$\exists x\in\mathrm{atoms}\,\mathcal{X},y\in\mathrm{atoms}\,\mathcal{Y}\colon x[f]y.$$

The reverse is obvious. □

Theorem 78. Let A and B be small sets.

1. A function $\alpha\in\mathfrak{F}(B)^{\mathrm{atoms}\,1^{\mathfrak{F}(A)}}$ such that (for every $a\in\mathrm{atoms}\,1^{\mathfrak{F}(A)}$)

$$\alpha a\subseteq\bigcap\,\left\langle\bigcup\circ\langle\alpha\rangle\circ\mathrm{atoms}\circ\uparrow^{A}\right\rangle\mathrm{up}\,a \tag{6}$$

can be continued to the function $\langle f\rangle$ for a unique $f\in\mathsf{FCD}(A;B)$;

$$\langle f\rangle\mathcal{X}=\bigcup\,\langle\alpha\rangle\mathrm{atoms}\,\mathcal{X} \tag{7}$$

for every $\mathcal{X}\in\mathfrak{F}(A)$.

2. A relation $\delta\in\mathscr{P}(\mathrm{atoms}\,A\times\mathrm{atoms}\,B)$ such that (for every $a\in\mathrm{atoms}\,A$, $b\in\mathrm{atoms}\,B$)

$$\forall X\in\mathrm{up}\,a,Y\in\mathrm{up}\,b\,\exists x\in\mathrm{atoms}\uparrow^{A}X,y\in\mathrm{atoms}\uparrow^{B}Y\colon x\,\delta\,y\Rightarrow a\,\delta\,b \tag{8}$$

can be continued to the relation $[f]$ for a unique $f\in\mathsf{FCD}(A;B)$;

$$\mathcal{X}[f]\mathcal{Y}\Leftrightarrow\exists x\in\mathrm{atoms}\,\mathcal{X},y\in\mathrm{atoms}\,\mathcal{Y}\colon x\,\delta\,y \tag{9}$$

for every $\mathcal{X}\in\mathfrak{F}(A)$, $\mathcal{Y}\in\mathfrak{F}(B)$.

Proof. Existence of no more than one such funcoids and formulas (7) and (9) follow from the previous theorem.

1. Consider the function $\alpha'\in\mathfrak{F}(B)^{\mathscr{P}A}$ defined by the formula (for every $X\in\mathscr{P}A$)

$$\alpha' X=\bigcup\,\langle\alpha\rangle\mathrm{atoms}\uparrow^{A}X.$$

Obviously $\alpha'\emptyset=0^{\mathfrak{F}(B)}$. For every $I,J\in\mathscr{P}A$

$$\begin{aligned}\alpha'(I\cup J) &= \bigcup\,\langle\alpha'\rangle\mathrm{atoms}\uparrow^{A}(I\cup J)\\ &= \bigcup\,\langle\alpha'\rangle(\mathrm{atoms}\uparrow^{A}I\cup\mathrm{atoms}\uparrow^{A}J)\\ &= \bigcup\,(\langle\alpha'\rangle\mathrm{atoms}\uparrow^{A}I\cup\langle\alpha'\rangle\mathrm{atoms}\uparrow^{A}J)\\ &= \bigcup\,\langle\alpha'\rangle\,\mathrm{atoms}\uparrow^{A}I\cup\bigcup\,\langle\alpha'\rangle\mathrm{atoms}\uparrow^{A}J.\\ &= \alpha' I\cup\alpha' J.\end{aligned}$$

Let continue α' till a funcoid f (by the theorem 49): $\langle f\rangle\mathcal{X}=\bigcap\langle\alpha'\rangle\mathrm{up}\,\mathcal{X}$.

Let's prove the reverse of (6):

$$\begin{aligned}\bigcap\left\langle\bigcup\circ\langle\alpha\rangle\circ\mathrm{atoms}\circ\uparrow^A\right\rangle\mathrm{up}\,a &= \bigcap\left\langle\bigcup\circ\langle\alpha\rangle\right\rangle\langle\mathrm{atoms}\rangle\langle\uparrow^A\rangle\mathrm{up}\,a\\ &\subseteq \bigcap\left\langle\bigcup\circ\langle\alpha\rangle\right\rangle\{\{a\}\}\\ &= \bigcap\left\{\left(\bigcup\circ\langle\alpha\rangle\right)\{a\}\right\}\\ &= \bigcap\left\{\bigcup\langle\alpha\rangle\{a\}\right\}\\ &= \bigcap\left\{\bigcup\{\alpha a\}\right\}=\bigcap\{\alpha a\}=\alpha a.\end{aligned}$$

Finally,

$$\alpha a=\bigcap\left\langle\bigcup\circ\langle\alpha\rangle\circ\mathrm{atoms}\circ\uparrow^A\right\rangle\mathrm{up}\,a=\bigcap\langle\alpha'\rangle\mathrm{up}\,a=\langle f\rangle a,$$

so $\langle f\rangle$ is a continuation of α.

2. Consider the relation $\delta'\in\mathscr{P}(\mathscr{P}A\times\mathscr{P}B)$ defined by the formula (for every $X\in\mathscr{P}A$, $Y\in\mathscr{P}B$)

$$X\,\delta'\,Y\Leftrightarrow\exists x\in\mathrm{atoms}\uparrow^A X,y\in\mathrm{atoms}\uparrow^B Y:x\,\delta\,y.$$

Obviously $\neg(X\,\delta'\,\emptyset)$ and $\neg(\emptyset\,\delta'\,Y)$.

For suitable I and J we have:

$$\begin{aligned}(I\cup J)\,\delta'\,Y &\Leftrightarrow \exists x\in\mathrm{atoms}\uparrow^A(I\cup J),y\in\mathrm{atoms}\uparrow^B Y:x\,\delta\,y\\ &\Leftrightarrow \exists x\in\mathrm{atoms}\uparrow^A I\cup\mathrm{atoms}\uparrow^A J,y\in\mathrm{atoms}\uparrow^B Y:x\,\delta\,y\\ &\Leftrightarrow \exists x\in\mathrm{atoms}\uparrow^A I,y\in\mathrm{atoms}\uparrow^B Y:x\,\delta\,y\vee\exists x\in\mathrm{atoms}\uparrow^A J,y\in\mathrm{atoms}\uparrow^B Y:x\,\delta\,y\\ &\Leftrightarrow I\,\delta'\,Y\vee J\,\delta'\,Y;\end{aligned}$$

analogously $X\,\delta'\,(I\cup J)\Leftrightarrow X\,\delta'\,I\vee X\,\delta'\,J$ for suitable I and J. Let's continue δ' till a funcoid f (by the theorem 49):

$$\mathcal{X}[f]\mathcal{Y}\Leftrightarrow\forall X\in\mathrm{up}\,\mathcal{X},Y\in\mathrm{up}\,\mathcal{Y}:X\,\delta'\,Y$$

The reverse of (8) implication is trivial, so

$$\forall X\in\mathrm{up}\,a,Y\in\mathrm{up}\,b\,\exists x\in\mathrm{atoms}\uparrow^A X,y\in\mathrm{atoms}\uparrow^B Y:x\,\delta\,y\Leftrightarrow a\,\delta\,b.$$

$\forall X\in\mathrm{up}\,a,Y\in\mathrm{up}\,b\,\exists x\in\mathrm{atoms}\uparrow^A X,y\in\mathrm{atoms}\uparrow^B Y:x\,\delta\,y\Leftrightarrow\forall X\in\mathrm{up}\,a,Y\in\mathrm{up}\,b:X\,\delta'\,Y\Leftrightarrow a[f]b$.

So $a\,\delta\,b\Leftrightarrow a[f]b$, that is $[f]$ is a continuation of δ. □

One of uses of the previous theorem is the proof of the following theorem:

Theorem 79. If A, B are small sets, $R\in\mathscr{P}\mathsf{FCD}(A;B)$, $x\in\mathrm{atoms}\,1^{\mathfrak{F}(A)}$, $y\in\mathrm{atoms}\,1^{\mathfrak{F}(B)}$, then

1. $\langle\bigcap R\rangle x=\bigcap\{\langle f\rangle x\mid f\in R\}$;
2. $x[\bigcap R]y\Leftrightarrow\forall f\in R:x[f]y$.

Proof. 2. Let denote $x\,\delta\,y\Leftrightarrow\forall f\in R:x[f]y$.

$$\begin{gathered}\forall X\in\mathrm{up}\,a,Y\in\mathrm{up}\,b\,\exists x\in\mathrm{atoms}\uparrow^A X,y\in\mathrm{atoms}\uparrow^B Y:x\,\delta\,y\Leftrightarrow\\ \forall f\in R,X\in\mathrm{up}\,a,Y\in\mathrm{up}\,b\,\exists x\in\mathrm{atoms}\uparrow^A X,y\in\mathrm{atoms}\uparrow^B Y:x[f]y\Rightarrow\\ \forall f\in R,X\in\mathrm{up}\,a,Y\in\mathrm{up}\,b:X[f]^*Y\Rightarrow\\ \forall f\in R:a[f]b\Leftrightarrow\\ a\,\delta\,b.\end{gathered}$$

So, by the theorem 78, δ can be continued till $[p]$ for some funcoid $p\in\mathsf{FCD}(A;B)$.

For every funcoid $q\in\mathsf{FCD}(A;B)$ such that $\forall f\in R:q\subseteq f$ we have $x[q]y\Rightarrow\forall f\in R:x[f]y\Leftrightarrow x\,\delta\,y\Leftrightarrow x[p]y$, so $q\subseteq p$. Consequently $p=\bigcap R$.

From this $x[\bigcap R]y\Leftrightarrow\forall f\in R:x[f]y$.

1. From the former $y\in\mathrm{atoms}\,\langle\bigcap R\rangle x\Leftrightarrow y\cap\langle\bigcap R\rangle x\neq 0^{\mathfrak{F}(B)}\Leftrightarrow\forall f\in R:y\cap\langle f\rangle x\neq\emptyset\Leftrightarrow y\in\bigcap\langle\mathrm{atoms}\rangle\{\langle f\rangle x\mid f\in R\}\Leftrightarrow y\in\mathrm{atoms}\bigcap\{\langle f\rangle x\mid f\in R\}$ for every $y\in\mathrm{atoms}\,1^{\mathfrak{F}(A)}$. From this follows $\langle\bigcap R\rangle x=\bigcap\{\langle f\rangle x\mid f\in R\}$. □

3.9 Direct product of filter objects

A generalization of direct (Cartesian) product of two sets is funcoidal product of two filter objects:

Definition 80. *Funcoidal product* of filter objects $\mathcal{A}$ and $\mathcal{B}$ is such a funcoid $\mathcal{A} \times^{\mathsf{FCD}} \mathcal{B} \in \mathsf{FCD}(\mathrm{Base}(\mathcal{A}); \mathrm{Base}(\mathcal{B}))$ that for every $\mathcal{X} \in \mathfrak{F}(\mathrm{Base}(\mathcal{A}))$, $\mathcal{Y} \in \mathfrak{F}(\mathrm{Base}(\mathcal{B}))$

$$\mathcal{X}[\mathcal{A} \times^{\mathsf{FCD}} \mathcal{B}]\mathcal{Y} \Leftrightarrow \mathcal{X} \cap \mathcal{A} \neq \emptyset \wedge \mathcal{Y} \cap \mathcal{B} \neq \emptyset.$$

Proposition 81. $\mathcal{A} \times^{\mathsf{FCD}} \mathcal{B}$ is really a funcoid and

$$\langle \mathcal{A} \times^{\mathsf{FCD}} \mathcal{B} \rangle \mathcal{X} = \begin{cases} \mathcal{B} & \text{if } \mathcal{X} \not\asymp \mathcal{A}; \\ \emptyset & \text{if } \mathcal{X} \asymp \mathcal{A}. \end{cases}$$

Proof. Obvious. □

Obvious 82. $\uparrow^{\mathsf{FCD}(U;V)}(A \times B) = \uparrow^U A \times^{\mathsf{FCD}} \uparrow^V B$ for sets $A \subseteq U$ and $B \subseteq V$ (for some small sets U and V).

Proposition 83. $f \subseteq \mathcal{A} \times^{\mathsf{FCD}} \mathcal{B} \Leftrightarrow \mathrm{dom}\, f \subseteq \mathcal{A} \wedge \mathrm{im}\, f \subseteq \mathcal{B}$ for every $f \in \mathsf{FCD}(A; B)$ and $\mathcal{A} \in \mathfrak{F}(A)$, $\mathcal{B} \in \mathfrak{F}(B)$.

Proof. If $f \subseteq \mathcal{A} \times^{\mathsf{FCD}} \mathcal{B}$ then $\mathrm{dom}\, f \subseteq \mathrm{dom}(\mathcal{A} \times^{\mathsf{FCD}} \mathcal{B}) \subseteq \mathcal{A}$, $\mathrm{im}\, f \subseteq \mathrm{im}(\mathcal{A} \times^{\mathsf{FCD}} \mathcal{B}) \subseteq \mathcal{B}$. If $\mathrm{dom}\, f \subseteq \mathcal{A} \wedge \mathrm{im}\, f \subseteq \mathcal{B}$ then

$$\forall \mathcal{X} \in \mathfrak{F}(A), \mathcal{Y} \in \mathfrak{F}(B) \colon (\mathcal{X}[f]\mathcal{Y} \Rightarrow \mathcal{X} \cap \mathcal{A} \neq 0^{\mathfrak{F}(A)} \wedge \mathcal{Y} \cap \mathcal{B} \neq 0^{\mathfrak{F}(B)});$$

consequently $f \subseteq \mathcal{A} \times^{\mathsf{FCD}} \mathcal{B}$. □

The following theorem gives a formula for calculating an important particular case of intersection on the lattice of funcoids:

Theorem 84. $f \cap (\mathcal{A} \times^{\mathsf{FCD}} \mathcal{B}) = I_{\mathcal{B}}^{\mathsf{FCD}} \circ f \circ I_{\mathcal{A}}^{\mathsf{FCD}}$ for every funcoid f and $\mathcal{A} \in \mathfrak{F}(\mathrm{Src}\, f)$, $\mathcal{B} \in \mathfrak{F}(\mathrm{Dst}\, f)$.

Proof. $h \stackrel{\text{def}}{=} I_{\mathcal{B}}^{\mathsf{FCD}} \circ f \circ I_{\mathcal{A}}^{\mathsf{FCD}}$. For every $\mathcal{X} \in \mathfrak{F}(\mathrm{Src}\, f)$

$$\langle h \rangle \mathcal{X} = \langle I_{\mathcal{B}}^{\mathsf{FCD}} \rangle \langle f \rangle \langle I_{\mathcal{A}}^{\mathsf{FCD}} \rangle \mathcal{X} = \mathcal{B} \cap \langle f \rangle (\mathcal{A} \cap \mathcal{X}).$$

From this, as easy to show, $h \subseteq f$ and $h \subseteq \mathcal{A} \times^{\mathsf{FCD}} \mathcal{B}$. If $g \subseteq f \wedge g \subseteq \mathcal{A} \times^{\mathsf{FCD}} \mathcal{B}$ for a $g \in \mathsf{FCD}(\mathrm{Src}\, f; \mathrm{Dst}\, f)$ then $\mathrm{dom}\, g \subseteq \mathcal{A}$, $\mathrm{im}\, g \subseteq \mathcal{B}$,

$$\langle g \rangle \mathcal{X} = \mathcal{B} \cap \langle g \rangle (\mathcal{A} \cap \mathcal{X}) \subseteq \mathcal{B} \cap \langle f \rangle (\mathcal{A} \cap \mathcal{X}) = \langle I_{\mathcal{B}}^{\mathsf{FCD}} \rangle \langle f \rangle \langle I_{\mathcal{A}}^{\mathsf{FCD}} \rangle\, \mathcal{X} = \langle h \rangle \mathcal{X},$$

$g \subseteq h$. So $h = f \cap^{\mathsf{FCD}} (\mathcal{A} \times^{\mathsf{FCD}} \mathcal{B})$. □

Corollary 85. $f|_{\mathcal{A}} = f \cap (\mathcal{A} \times^{\mathsf{FCD}} 1^{\mathfrak{F}(\mathrm{Dst}\, f)})$ for every $f \in \mathsf{FCD}$ and $\mathcal{A} \in \mathfrak{F}(\mathrm{Src}\, f)$.

Proof. $f \cap (\mathcal{A} \times^{\mathsf{FCD}} 1^{\mathfrak{F}(\mathrm{Dst}\, f)}) = I_{1^{\mathfrak{F}(\mathrm{Dst}\, f)}}^{\mathsf{FCD}} \circ f \circ I_{\mathcal{A}}^{\mathsf{FCD}} = f \circ I_{\mathcal{A}}^{\mathsf{FCD}} = f|_{\mathcal{A}}$. □

Corollary 86. $f \not\asymp (\mathcal{A} \times^{\mathsf{FCD}} \mathcal{B}) \Leftrightarrow \mathcal{A}[f]\mathcal{B}$ for every $f \in \mathsf{FCD}$, $\mathcal{A} \in \mathfrak{F}(\mathrm{Src}\, f)$, $\mathcal{B} \in \mathfrak{F}(\mathrm{Dst}\, f)$.

Proof. $f \not\asymp (\mathcal{A} \times^{\mathsf{FCD}} \mathcal{B}) \Leftrightarrow \langle f \cap (\mathcal{A} \times^{\mathsf{FCD}} \mathcal{B}) \rangle^* (\mathrm{Src}\, f) \neq 0^{\mathfrak{F}(\mathrm{Dst}\, f)} \Leftrightarrow \langle I_{\mathcal{B}}^{\mathsf{FCD}} \circ f \circ I_{\mathcal{A}} \rangle^* (\mathrm{Src}\, f) \neq 0^{\mathfrak{F}(\mathrm{Dst}\, f)} \Leftrightarrow \langle I_{\mathcal{B}}^{\mathsf{FCD}} \rangle \langle f \rangle \langle I_{\mathcal{A}}^{\mathsf{FCD}} \rangle 1^{\mathfrak{F}(\mathrm{Src}\, f)} \neq 0^{\mathfrak{F}(\mathrm{Dst}\, f)} \Leftrightarrow \mathcal{B} \cap \langle f \rangle (\mathcal{A} \cap 1^{\mathfrak{F}(\mathrm{Src}\, f)}) \neq 0^{\mathfrak{F}(\mathrm{Dst}\, f)} \Leftrightarrow \mathcal{B} \cap \langle f \rangle \mathcal{A} \neq 0^{\mathfrak{F}(\mathrm{Dst}\, f)} \Leftrightarrow \mathcal{A}[f]\mathcal{B}$. □

Corollary 87. Every filtrator of funcoids is star-separable.

Proof. The set of direct products of principal filter objects is a separation subset of the lattice of funcoids. □

Theorem 88. Let A, B are small sets. If $S \in \mathscr{P}(\mathfrak{F}(A) \times \mathfrak{F}(B))$ then

$$\bigcap \{\mathcal{A} \times^{\mathsf{FCD}} \mathcal{B} \mid (\mathcal{A}; \mathcal{B}) \in S\} = \bigcap \operatorname{dom} S \times^{\mathsf{FCD}} \bigcap \operatorname{im} S.$$

Proof. If $x \in \operatorname{atoms} 1^{\mathfrak{F}(A)}$ then by the theorem 79

$$\left\langle \bigcap \{\mathcal{A} \times^{\mathsf{FCD}} \mathcal{B} \mid (\mathcal{A}; \mathcal{B}) \in S\} \right\rangle x = \bigcap \{\langle \mathcal{A} \times^{\mathsf{FCD}} \mathcal{B} \rangle x \mid (\mathcal{A}; \mathcal{B}) \in S\}.$$

If $x \not\asymp \bigcap \operatorname{dom} S$ then

$$\forall (\mathcal{A}; \mathcal{B}) \in S: (x \cap \mathcal{A} \neq 0^{\mathfrak{F}(A)} \wedge \langle \mathcal{A} \times^{\mathsf{FCD}} \mathcal{B} \rangle x = \mathcal{B});$$
$$\{\langle \mathcal{A} \times^{\mathsf{FCD}} \mathcal{B} \rangle x \mid (\mathcal{A}; \mathcal{B}) \in S\} = \operatorname{im} S;$$

if $x \asymp \bigcap \operatorname{dom} S$ then

$$\exists (\mathcal{A}; \mathcal{B}) \in S: (x \cap \mathcal{A} = 0^{\mathfrak{F}(A)} \wedge \langle \mathcal{A} \times^{\mathsf{FCD}} \mathcal{B} \rangle x = 0^{\mathfrak{F}(B)});$$
$$\{\langle \mathcal{A} \times^{\mathsf{FCD}} \mathcal{B} \rangle x \mid (\mathcal{A}; \mathcal{B}) \in S\} \ni 0^{\mathfrak{F}(B)}.$$

So

$$\left\langle \bigcap \{\mathcal{A} \times^{\mathsf{FCD}} \mathcal{B} \mid (\mathcal{A}; \mathcal{B}) \in S\} \right\rangle x = \begin{cases} \bigcap \operatorname{im} S & \text{if } x \not\asymp \bigcap \operatorname{dom} S; \\ 0^{\mathfrak{F}(B)} & \text{if } x \asymp \bigcap \operatorname{dom} S. \end{cases}$$

From this follows the statement of the theorem. □

Corollary 89. For every $\mathcal{A}_0, \mathcal{A}_1 \in \mathfrak{F}(A)$, $\mathcal{B}_0, \mathcal{B}_1 \in \mathfrak{F}(B)$ (for every small sets A, B)

$$(\mathcal{A}_0 \times^{\mathsf{FCD}} \mathcal{B}_0) \cap (\mathcal{A}_1 \times^{\mathsf{FCD}} \mathcal{B}_1) = (\mathcal{A}_0 \cap \mathcal{A}_1) \times^{\mathsf{FCD}} (\mathcal{B}_0 \cap \mathcal{B}_1).$$

Proof. $(\mathcal{A}_0 \times^{\mathsf{FCD}} \mathcal{B}_0) \cap (\mathcal{A}_1 \times^{\mathsf{FCD}} \mathcal{B}_1) = \bigcap \{\mathcal{A}_0 \times^{\mathsf{FCD}} \mathcal{B}_0, \mathcal{A}_1 \times^{\mathsf{FCD}} \mathcal{B}_1\}$ what is by the last theorem equal to $(\mathcal{A}_0 \cap \mathcal{A}_1) \times^{\mathsf{FCD}} (\mathcal{B}_0 \cap \mathcal{B}_1)$. □

Theorem 90. If A, B are small sets and $\mathcal{A} \in \mathfrak{F}(A)$ then $\mathcal{A} \times^{\mathsf{FCD}}$ is a complete homomorphism of the lattice $\mathfrak{F}(B)$ to a complete sublattice of the lattice $\mathsf{FCD}(A; B)$, if also $\mathcal{A} \neq 0^{\mathfrak{F}(A)}$ then it is an isomorphism.

Proof. Let $S \in \mathscr{P}\mathfrak{F}(B)$, $X \in \mathscr{P}A$, $x \in \operatorname{atoms} 1^{\mathfrak{F}(A)}$.

$$\begin{aligned} \left\langle \bigcup \langle \mathcal{A} \times^{\mathsf{FCD}} \rangle S \right\rangle X &= \bigcup \{\langle \mathcal{A} \times^{\mathsf{FCD}} \mathcal{B} \rangle X \mid \mathcal{B} \in S\} \\ &= \begin{cases} \bigcup S & \text{if } X \in \star\partial\mathcal{A} \\ 0^{\mathfrak{F}(B)} & \text{if } X \notin \star\partial\mathcal{A} \end{cases} \\ &= \left\langle \mathcal{A} \times^{\mathsf{FCD}} \bigcup S \right\rangle X; \\ \left\langle \bigcap \langle \mathcal{A} \times^{\mathsf{FCD}} \rangle S \right\rangle x &= \bigcap \{\langle \mathcal{A} \times^{\mathsf{FCD}} \mathcal{B} \rangle x \mid \mathcal{B} \in S\} \\ &= \begin{cases} \bigcap S & \text{if } x \not\asymp \mathcal{A} \\ 0^{\mathfrak{F}(B)} & \text{if } x \asymp \mathcal{A} \end{cases} \\ &= \left\langle \mathcal{A} \times^{\mathsf{FCD}} \bigcap S \right\rangle x. \end{aligned}$$

If $\mathcal{A} \neq 0^{\mathfrak{F}(A)}$ then obviously the function $\mathcal{A} \times^{\mathsf{FCD}}$ is injective. □

The following proposition states that cutting a rectangle of atomic width from a funcoid always produces a rectangular (representable as a direct product of filter objects) funcoid (of atomic width).

Proposition 91. If $f \in \mathsf{FCD}$ and a is an atomic filter object on $\operatorname{Src} f$ then

$$f|_a = a \times^{\mathsf{FCD}} \langle f \rangle a.$$

Proof. Let $\mathcal{X} \in \mathfrak{F}(\mathrm{Src}\, f)$.

$$\mathcal{X} \not\asymp a \Rightarrow \langle f|_a \rangle \mathcal{X} = \langle f \rangle a, \quad \mathcal{X} \asymp a \Rightarrow \langle f|_a \rangle \mathcal{X} = 0^{\mathfrak{F}(\mathrm{Dst}\, f)}. \qquad \square$$

3.10 Atomic funcoids

Theorem 92. An $f \in \mathsf{FCD}(A; B)$ is an atom of the lattice $\mathsf{FCD}(A; B)$ (for small sets A, B) iff it is direct product of two atomic filter objects.

Proof.

- **⇒.** Let $f \in \mathsf{FCD}(A; B)$ is an atom of the lattice $\mathsf{FCD}(A; B)$. Let's get elements $a \in \mathrm{atoms}\, \mathrm{dom}\, f$ and $b \in \mathrm{atoms}\, \langle f \rangle a$. Then for every $\mathcal{X} \in \mathfrak{F}(A)$

$$\mathcal{X} \asymp a \Rightarrow \langle a \times^{\mathsf{FCD}} b \rangle \mathcal{X} = 0^{\mathfrak{F}(B)} \subseteq \langle f \rangle \mathcal{X}, \quad \mathcal{X} \not\asymp a \Rightarrow \langle a \times^{\mathsf{FCD}} b \rangle \mathcal{X} = b \subseteq \langle f \rangle \mathcal{X}.$$

  So $a \times^{\mathsf{FCD}} b \subseteq f$; because f is atomic we have $f = a \times^{\mathsf{FCD}} b$.

- **⇐.** Let $a \in \mathrm{atoms}\, 1^{\mathfrak{F}(A)}$, $b \in \mathrm{atoms}\, 1^{\mathfrak{F}(B)}$, $f \in \mathsf{FCD}(A; B)$. If $b \asymp \langle f \rangle a$ then $\neg(a [f] b)$, $f \asymp a \times^{\mathsf{FCD}} b$; if $b \subseteq \langle f \rangle a$ then $\forall \mathcal{X} \in \mathfrak{F}(A)$: $(\mathcal{X} \not\asymp a \Rightarrow \langle f \rangle \mathcal{X} \supseteq b)$, $f \supseteq a \times^{\mathsf{FCD}} b$. Consequently $f \asymp a \times^{\mathsf{FCD}} b \vee f \supseteq a \times^{\mathsf{FCD}} b$; that is $a \times^{\mathsf{FCD}} b$ is an atomic filter object. □

Theorem 93. The lattice $\mathsf{FCD}(A; B)$ is atomic (for every small sets A, B).

Proof. Let f is a non-empty funcoid. Then $\mathrm{dom}\, f \neq \emptyset$, thus by the theorem 46 in [15] exists $a \in \mathrm{atoms}\, \mathrm{dom}\, f$. So $\langle f \rangle a \neq 0^{\mathfrak{F}(B)}$ thus exists $b \in \mathrm{atoms}\, \langle f \rangle a$. Finally the atomic funcoid $a \times^{\mathsf{FCD}} b \subseteq f$. □

Theorem 94. The lattice $\mathsf{FCD}(A; B)$ is separable (for every small sets A, B).

Proof. Let $f, g \in \mathsf{FCD}(A; B)$, $f \subset g$. Then exists $a \in \mathrm{atoms}\, 1^{\mathfrak{F}(A)}$ such that $\langle f \rangle a \subset \langle g \rangle a$. So because the lattice $\mathfrak{F}(B)$ is atomically separable then exists $b \in \mathrm{atoms}\, 1^{\mathfrak{F}(B)}$ such that $\langle f \rangle a \cap b = 0^{\mathfrak{F}(B)}$ and $b \subseteq \langle g \rangle a$. For every $x \in \mathrm{atoms}\, 1^{\mathfrak{F}(A)}$

$$\langle f \rangle a \cap \langle a \times^{\mathsf{FCD}} b \rangle a = \langle f \rangle a \cap b = 0^{\mathfrak{F}(B)},$$
$$x \neq a \Rightarrow \langle f \rangle x \cap \langle a \times^{\mathsf{FCD}} b \rangle x = \langle f \rangle x \cap 0^{\mathfrak{F}(B)} = 0^{\mathfrak{F}(B)}.$$

Thus $\langle f \rangle x \cap \langle a \times^{\mathsf{FCD}} b \rangle x = 0^{\mathfrak{F}(B)}$ and consequently $f \asymp a \times^{\mathsf{FCD}} b$.

$$\langle a \times^{\mathsf{FCD}} b \rangle a = b \subseteq \langle g \rangle a,$$
$$x \neq a \Rightarrow \langle a \times^{\mathsf{FCD}} b \rangle x = 0^{\mathfrak{F}(B)} \subseteq \langle g \rangle a.$$

Thus $\langle a \times^{\mathsf{FCD}} b \rangle x \subseteq \langle g \rangle x$ and consequently $a \times^{\mathsf{FCD}} b \subseteq g$.

So the lattice $\mathsf{FCD}(A; B)$ is separable by the theorem 19 in [15]. □

Corollary 95. The lattice $\mathsf{FCD}(A; B)$ is:

1. separable;
2. atomically separable;
3. conforming to Wallman's disjunction property.

Proof. By the theorem 22 in [15]. □

Remark 96. For more ways to characterize (atomic) separability of the lattice of funcoids see [15], subsections "Separation subsets and full stars" and "Atomically separable lattices".

Corollary 97. The lattice $\mathsf{FCD}(A; B)$ is an atomistic lattice.

Proof. Let $f \in \mathsf{FCD}(A; B)$. Suppose contrary to the statement to be proved that $\bigcup \mathrm{atoms}\, f \subset f$. Then it exists $a \in \mathrm{atoms}\, f$ such that $a \cap \bigcup \mathrm{atoms}\, f = \emptyset$ what is impossible. □

Proposition 98. $\operatorname{atoms}(f \cup g) = \operatorname{atoms} f \cup \operatorname{atoms} g$ for every funcoids $f, g \in \mathsf{FCD}(A; B)$ (for every small sets A and B).

Proof. $a \times^{\mathsf{FCD}} b \not\asymp f \cup g \Leftrightarrow a[f \cup g]b \Leftrightarrow a[f]b \vee a[g]b \Leftrightarrow a \times^{\mathsf{FCD}} b \not\asymp f \vee a \times^{\mathsf{FCD}} b \not\asymp g$ for every atomic filter objects a and b. □

Theorem 99. For every $f, g, h \in \mathsf{FCD}(A; B)$, $R \in \mathscr{P}\mathsf{FCD}(A; B)$ (for every small sets A and B)

1. $f \cap (g \cup h) = (f \cap g) \cup (f \cap h)$;
2. $f \cup \bigcap R = \bigcap \langle f \cup \rangle R$.

Proof. We will take in account that the lattice of funcoids is an atomistic lattice.

1. $\operatorname{atoms}(f \cap (g \cup h)) = \operatorname{atoms} f \cap \operatorname{atoms}(g \cup h) = \operatorname{atoms} f \cap (\operatorname{atoms} g \cup \operatorname{atoms} h) = (\operatorname{atoms} f \cap \operatorname{atoms} g) \cup (\operatorname{atoms} f \cap \operatorname{atoms} h) = \operatorname{atoms}(f \cap g) \cup \operatorname{atoms}(f \cap h) = \operatorname{atoms}((f \cap g) \cup (f \cap h))$.
2. $\operatorname{atoms}(f \cup \bigcap R) = \operatorname{atoms} f \cup \operatorname{atoms} \bigcap R = \operatorname{atoms} f \cup \bigcap \langle \operatorname{atoms} \rangle R = \bigcap \langle (\operatorname{atoms} f) \cup \rangle \langle \operatorname{atoms} \rangle R = \bigcap \langle \operatorname{atoms} \rangle \langle f \cup \rangle R = \operatorname{atoms} \bigcap \langle f \cup \rangle R$. (Used the following equality.)

$$\begin{aligned}
\langle (\operatorname{atoms} f) \cup \rangle \langle \operatorname{atoms} \rangle R &= \\
\{(\operatorname{atoms} f) \cup A \mid A \in \langle \operatorname{atoms} \rangle R\} &= \\
\{(\operatorname{atoms} f) \cup A \mid \exists C \in R: A = \operatorname{atoms} C\} &= \\
\{(\operatorname{atoms} f) \cup (\operatorname{atoms} C) \mid C \in R\} &= \\
\{\operatorname{atoms}(f \cup C) \mid C \in R\} &= \\
\{\operatorname{atoms} B \mid \exists C \in R: B = f \cup C\} &= \\
\{\operatorname{atoms} B \mid B \in \langle f \cup \rangle R\} &= \\
\langle \operatorname{atoms} \rangle \langle f \cup \rangle R. &
\end{aligned}$$

□

Note that distributivity of the lattice of funcoids is proved through using atoms of this lattice. I have never seen such method of proving distributivity.

Corollary 100. The lattice $\mathsf{FCD}(A; B)$ is co-brouwerian (for every small sets A and B).

The next proposition is one more (among the theorem 61) generalization for funcoids of composition of relations.

Proposition 101. For every composable funcoids f, g

$\operatorname{atoms}(g \circ f) = \{x \times^{\mathsf{FCD}} z \mid x \in \operatorname{atoms} 1^{\mathfrak{F}(\operatorname{Src} f)}, z \in \operatorname{atoms} 1^{\mathfrak{F}(\operatorname{Dst} g)}, \exists y \in \operatorname{atoms} 1^{\mathfrak{F}(\operatorname{Dst} f)}: (x \times^{\mathsf{FCD}} y \in \operatorname{atoms} f \wedge y \times^{\mathsf{FCD}} z \in \operatorname{atoms} g)\}$.

Proof. $(x \times^{\mathsf{FCD}} z) \cap (g \circ f) \neq \emptyset \Leftrightarrow x[g \circ f]z \Leftrightarrow \exists y \in \operatorname{atoms} 1^{\mathfrak{F}(\operatorname{Dst} f)}: (x[f]y \wedge y[g]z) \Leftrightarrow \exists y \in \operatorname{atoms} 1^{\mathfrak{F}(\operatorname{Dst} f)}: ((x \times^{\mathsf{FCD}} y) \cap f \neq \emptyset \wedge (y \times^{\mathsf{FCD}} z) \cap g \neq \emptyset)$ (it was used the theorem 61). □

3.11 Complete funcoids

Definition 102. I will call *co-complete* such a funcoid f that $\langle f \rangle^* X$ is a principal f.o. for every $X \in \mathscr{P}(\operatorname{Src} f)$.

Remark 103. I will call *generalized closure* such a function $\alpha \in \mathscr{P}B^{\mathscr{P}A}$ (for some small sets A, B) that

1. $\alpha\emptyset = \emptyset$;
2. $\forall I, J \in \mathscr{P}A: \alpha(I \cup J) = \alpha I \cup \alpha J$.

Obvious 104. A funcoid f is co-complete iff $\langle f \rangle^* = \uparrow^{\operatorname{Dst} f} \circ \alpha$ for a generalized closure α.

Remark 105. Thus funcoids can be considered as a generalization of generalized closures. A topological space in Kuratowski sense is the same as reflexive and transitive generalized closure. So topological spaces can be considered as a special case of funcoids.

Definition 106. I will call a *complete funcoid* a funcoid whose reverse is co-complete.

Theorem 107. The following conditions are equivalent for every funcoid f:

1. funcoid f is complete;
2. $\forall S \in \mathscr{P}\mathfrak{F}(\mathrm{Src}\, f), J \in \mathscr{P}(\mathrm{Dst}\, f)$: $(\bigcup S [f] \uparrow^{\mathrm{Dst}\, f} J \Leftrightarrow \exists \mathcal{I} \in S: \mathcal{I} [f] \uparrow^{\mathrm{Dst}\, f} J)$;
3. $\forall S \in \mathscr{P}\mathscr{P}(\mathrm{Src}\, f), J \in \mathscr{P}(\mathrm{Dst}\, f)$: $(\bigcup S [f]^* J \Leftrightarrow \exists I \in S: I [f]^* J)$;
4. $\forall S \in \mathscr{P}\mathfrak{F}(\mathrm{Src}\, f)$: $\langle f \rangle \bigcup S = \bigcup \langle \langle f \rangle \rangle S$;
5. $\forall S \in \mathscr{P}\mathscr{P}(\mathrm{Src}\, f)$: $\langle f \rangle^* \bigcup S = \bigcup \langle \langle f \rangle^* \rangle S$;
6. $\forall A \in \mathscr{P}(\mathrm{Src}\, f)$: $\langle f \rangle^* A = \bigcup \{ \langle f \rangle^* \{a\} \mid a \in A \}$.

Proof.

(3)⇒(1). For every $S \in \mathscr{P}\mathscr{P}(\mathrm{Src}\, f)$, $J \in \mathscr{P}(\mathrm{Dst}\, f)$

$$\uparrow^{\mathrm{Src}\, f} \bigcup S \cap \langle f^{-1} \rangle^* J \neq 0^{\mathrm{Src}\, f} \Leftrightarrow \exists I \in S: \uparrow^{\mathrm{Src}\, f} I \cap \langle f^{-1} \rangle^* J \neq 0^{\mathrm{Src}\, f}, \tag{10}$$

consequently by the theorem 52 in [15] we have that $\langle f^{-1} \rangle^* J$ is a principal f.o.

(1)⇒(2). For every $S \in \mathscr{P}\mathfrak{F}(\mathrm{Src}\, f)$, $J \in \mathscr{P}(\mathrm{Dst}\, f)$ we have $\langle f^{-1} \rangle^* J$ a principal f.o., consequently

$$\bigcup S \cap \langle f^{-1} \rangle^* J \neq 0^{\mathrm{Src}\, f} \Leftrightarrow \exists \mathcal{I} \in S: \mathcal{I} \cap \langle f^{-1} \rangle^* J \neq 0^{\mathrm{Src}\, f}.$$

From this follows (2).

(6)⇒(5). $\langle f \rangle^* \bigcup S = \bigcup \{ \langle f \rangle^* \{a\} \mid a \in \bigcup S \} = \bigcup \{ \bigcup \{ \langle f \rangle^* \{a\} \mid a \in A \} \mid A \in S \} = \bigcup \{ \langle f \rangle^* A \mid A \in S \} = \bigcup \langle \langle f \rangle^* \rangle S$.

(2)⇒(4). $\uparrow^{\mathrm{Dst}\, f} J \not\asymp \langle f \rangle \bigcup S \Leftrightarrow \bigcup S [f] \uparrow^{\mathrm{Dst}\, f} J \Leftrightarrow \exists \mathcal{I} \in S: \mathcal{I} [f] \uparrow^{\mathrm{Dst}\, f} J \Leftrightarrow \exists \mathcal{I} \in S: \uparrow^{\mathrm{Dst}\, f} J \not\asymp \langle f \rangle \mathcal{I} \Leftrightarrow \uparrow^{\mathrm{Dst}\, f} J \not\asymp \bigcup \langle \langle f \rangle \rangle S$ (used the theorem 53 in [15]).

(2)⇒(3), (4)⇒(5), (5)⇒(3), (5)⇒(6). Obvious. □

The following proposition shows that complete funcoids are a direct generalization of pre-topological spaces.

Proposition 108. To specify a complete funcoid f it is enough to specify $\langle f \rangle^*$ on one-element sets, values of $\langle f \rangle^*$ on one element sets can be specified arbitrarily.

Proof. From the above theorem is clear that knowing $\langle f \rangle^*$ on one-element sets $\langle f \rangle^*$ can be found on every set and then its value can be inferred for every filter objects.

Choosing arbitrarily the values of $\langle f \rangle^*$ on one-element sets we can define a complete funcoid the following way: $\langle f \rangle^* X \stackrel{\mathrm{def}}{=} \bigcup \{ \langle f \rangle^* \{\alpha\} \mid \alpha \in X \}$ for every $X \in \mathscr{P}(\mathrm{Src}\, f)$. Obviously it is really a complete funcoid. □

Theorem 109. A funcoid is discrete iff it is both complete and co-complete.

Proof.

⇒. Obvious.

⇐. Let f is both a complete and co-complete funcoid. Consider the relation g defined by that $\uparrow^{\mathrm{Dst}\, f} \langle g \rangle \{\alpha\} = \langle f \rangle^* \{\alpha\}$ (g is correctly defined because f corresponds to a generalized closure). Because f is a complete funcoid f is the funcoid corresponding to g. □

Theorem 110. If $R \in \mathscr{P}\mathsf{FCD}(A; B)$ is a set of (co-)complete funcoids then $\bigcup R$ is a (co-)complete funcoid (for every small sets A and B).

Proof. It is enough to prove only for co-complete funcoids. Let $R \in \mathscr{P}\mathsf{FCD}(A;B)$ is a set of co-complete funcoids. Then for every $X \in \mathscr{P}(\operatorname{Src} f)$

$$\left\langle \bigcup R \right\rangle^* X = \bigcup \{\langle f \rangle^* X \mid f \in R\}$$

is a principal f.o. (used the theorem 58). $\square$

Corollary 111. If R is a set of binary relations between small sets A and B then $\bigcup \left\langle \uparrow^{\mathsf{FCD}(A;B)} \right\rangle R = \uparrow^{\mathsf{FCD}(A;B)} \bigcup R$.

Proof. From two last theorems. $\square$

Theorem 112. Filtrators of funcoids are filtered.

Proof. It's enough to prove that every funcoid is representable as (infinite) meet (on the lattice $\mathsf{FCD}(A;B)$) of some set of discrete funcoids.

Let $f \in \mathsf{FCD}(A;B)$, $X \in \mathscr{P}A$, $Y \in \operatorname{up}\langle f \rangle X$, $g(X;Y) \overset{\text{def}}{=} \uparrow^A X \times^{\mathsf{FCD}} \uparrow^B Y \cup \uparrow^A \overline{X} \times^{\mathsf{FCD}} 1^{\mathfrak{F}(B)}$. For every $K \in \mathscr{P}A$

$$\langle g(X;Y) \rangle^* K = \langle \uparrow^A X \times^{\mathsf{FCD}} \uparrow^B Y \rangle^* K \cup \left\langle \uparrow^A \overline{X} \times^{\mathsf{FCD}} 1^{\mathfrak{F}(B)} \right\rangle^* K = \left(\begin{cases} 0^{\mathfrak{F}(B)} & \text{if } K = \emptyset \\ Y & \text{if } \emptyset \neq K \subseteq X \\ 1^{\mathfrak{F}(B)} & \text{if } K \nsubseteq X \end{cases} \right) \supseteq \langle f \rangle^* K;$$

so $g(X;Y) \supseteq f$. For every $X \in \mathscr{P}A$

$$\bigcap \{\langle g(X;Y) \rangle^* X \mid Y \in \operatorname{up} \langle f \rangle X\} = \bigcap \{Y \mid Y \in \operatorname{up} \langle f \rangle^* X\} = \langle f \rangle^* X;$$

consequently

$$\left\langle \bigcap \{g(X;Y) \mid X \in \mathscr{P}A, Y \in \operatorname{up} \langle f \rangle^* X\} \right\rangle^* X \subseteq \langle f \rangle^* X$$

that is

$$\bigcap \{g(X;Y) \mid X \in \mathscr{P}A, Y \in \operatorname{up} \langle f \rangle^* X\} \subseteq f$$

and finally

$$f = \bigcap \{g(X;Y) \mid X \in \mathscr{P}A, Y \in \operatorname{up} \langle f \rangle^* X\}.$$ $\square$

Conjecture 113. If $f \in \mathsf{FCD}(B;C)$ is a complete funcoid and $R \in \mathscr{P}\mathsf{FCD}(A;B)$ then $f \circ \bigcup R = \bigcup \langle f \circ \rangle R$.

This conjecture can be weakened:

Conjecture 114. If f is a discrete funcoid from B to C and $R \in \mathscr{P}\mathsf{FCD}(A;B)$ then $f \circ \bigcup R = \bigcup \langle f \circ \rangle R$.

I will denote $\mathrm{Compl}\mathsf{FCD}$ and $\mathrm{CoCompl}\mathsf{FCD}$ the sets of complete and co-complete funcoids correspondingly. $\mathrm{Compl}\mathsf{FCD}(A;B)$ are complete funcoids from A to B and likewise with $\mathrm{CoCompl}\mathsf{FCD}(A;B)$.

Obvious 115. $\mathrm{Compl}\mathsf{FCD}$ and $\mathrm{CoCompl}\mathsf{FCD}$ are closed regarding composition of funcoids.

Proposition 116. $\mathrm{Compl}\mathsf{FCD}(A;B)$ and $\mathrm{CoCompl}\mathsf{FCD}(A;B)$ (with induced order) are complete lattices.

Proof. It follows from the theorem 110. $\square$

Theorem 117. Atoms of the lattice $\mathrm{Compl}\mathsf{FCD}(A;B)$ are exactly direct products of the form $\uparrow^A\{\alpha\} \times^{\mathsf{FCD}} b$ where $\alpha \in A$ and b is an atomic f.o. on B

Proof. First, it's easy to see that $\{\alpha\} \times^{\mathsf{FCD}} b$ are elements of $\mathrm{Compl}\mathsf{FCD}(A;B)$. Also $0^{\mathsf{FCD}(A;B)}$ is an element of $\mathrm{Compl}\mathsf{FCD}(A;B)$.

$\uparrow^A\{\alpha\} \times^{\mathsf{FCD}} b$ are atoms of $\mathrm{Compl}\mathsf{FCD}(A;B)$ because these are atoms of $\mathsf{FCD}(A;B)$.

It remains to prove that if f is an atom of $\mathrm{Compl}\mathsf{FCD}(A;B)$ then $f = \{\alpha\} \times^{\mathsf{FCD}} b$ for some $\alpha \in A$ and an atomic f.o. b on B.

Suppose $f \in \mathsf{FCD}(A;B)$ is a non-empty complete funcoid. Then exists $\alpha \in A$ such that $\langle f\rangle^*\{\alpha\} \neq 0^{\mathfrak{F}(B)}$. Thus $\uparrow^A\{\alpha\} \times^{\mathsf{FCD}} b \subseteq f$ for some atomic f.o. b on B. If f is an atom then $f = \uparrow^A\{\alpha\} \times^{\mathsf{FCD}} b$. □

Theorem 118.

1. A funcoid $f \in \mathsf{FCD}(A;B)$ is complete iff there exists a function $G: \mathrm{Src}\, f \to \mathfrak{F}(\mathrm{Dst}\, f)$ such that
$$f = \bigcup \{\uparrow^A\{\alpha\} \times^{\mathsf{FCD}} G(\alpha) \mid \alpha \in \mathrm{Src}\, f\}. \quad (11)$$
2. A funcoid $f \in \mathsf{FCD}(A;B)$ is co-complete iff there exists a function $G: \mathrm{Dst}\, f \to \mathfrak{F}(\mathrm{Src}\, f)$ such that
$$f = \bigcup \{G(\alpha) \times^{\mathsf{FCD}} \uparrow^A\{\alpha\} \mid \alpha \in \mathrm{Dst}\, f\}.$$

Proof. We will prove only the first as the second is symmetric.

⇒. Let f is complete. Then take
$$G(\alpha) = \bigcup \left\{b \in \operatorname{atoms} 1^{\mathfrak{F}(\mathrm{Dst}\, f)} \mid \exists \alpha \in \mathrm{Src}\, f: \uparrow^A\{\alpha\} \times^{\mathsf{FCD}} b \subseteq f\right\}$$
and we have (11) obviously.

⇐. Let (11) holds. Then $G(\alpha) = \bigcup \operatorname{atoms} G(\alpha)$ and thus
$$f = \bigcup \{\{\alpha\} \times^{\mathsf{FCD}} b \mid \alpha \in \mathrm{Src}\, f, b \in \operatorname{atoms} G(\alpha)\}$$
and so f is complete. □

Theorem 119.

1. For a complete funcoid f there exist exactly one function $F \in \mathfrak{F}(\mathrm{Dst}\, f)^{\mathrm{Src}\, f}$ such that
$$f = \bigcup \{\uparrow^{\mathrm{Src}\, f}\{\alpha\} \times^{\mathsf{FCD}} F(\alpha) \mid \alpha \in \mathrm{Src}\, f\}.$$
2. For a co-complete funcoid f there exist exactly one function $F \in \mathfrak{F}(\mathrm{Src}\, f)^{\mathrm{Dst}\, f}$ such that
$$f = \bigcup \{F(\alpha) \times^{\mathsf{FCD}} \uparrow^{\mathrm{Dst}\, f}\{\alpha\} \mid \alpha \in \mathrm{Dst}\, f\}.$$

Proof. We will prove only the first as the second is similar. Let
$$f = \bigcup \{\uparrow^{\mathrm{Src}\, f}\{\alpha\} \times^{\mathsf{FCD}} F(\alpha) \mid \alpha \in \mathrm{Src}\, f\} = \bigcup \{\uparrow^{\mathrm{Src}\, f}\{\alpha\} \times^{\mathsf{FCD}} G(\alpha) \mid \alpha \in \mathrm{Src}\, f\}$$
for some $F, G \in \mathfrak{F}(\mathrm{Dst}\, f)^{\mathrm{Src}\, f}$. We need to prove $F = G$. Let $\beta \in \mathrm{Src}\, f$.
$$\langle f\rangle\{\beta\} = \bigcup \{\langle \uparrow^{\mathrm{Src}\, f}\{\alpha\} \times^{\mathsf{FCD}} F(\alpha)\rangle\{\beta\} \mid \alpha \in \mathrm{Src}\, f\} = F(\beta).$$
Similarly $\langle f\rangle\{\beta\} = G(\beta)$. So $F(\beta) = G(\beta)$. □

3.12 Completion of funcoids

Theorem 120. $\mathrm{Cor}\, f = \mathrm{Cor}'\, f$ for an element f of a filtrator of funcoids. (Core part is taken for the shifted filtrator of funcoids.)

Proof. From the theorem 26 in [15] and the corollary 111 and theorem 112. □

Definition 121. *Completion* of a funcoid $f \in \mathsf{FCD}(A;B)$ is the complete funcoid $\mathrm{Compl}\, f \in \mathsf{FCD}(A;B)$ defined by the formula $\langle \mathrm{Compl}\, f\rangle^*\{\alpha\} = \langle f\rangle^*\{\alpha\}$ for $\alpha \in \mathrm{Src}\, f$.

Definition 122. *Co-completion* of a funcoid f is defined by the formula

$$\mathrm{CoCompl}\, f = (\mathrm{Compl}\, f^{-1})^{-1}.$$

Obvious 123. $\mathrm{Compl}\, f \subseteq f$ and $\mathrm{CoCompl}\, f \subseteq f$ for every funcoid f.

Proposition 124. The filtrator $(\mathsf{FCD}(A; B); \mathrm{Compl}\mathsf{FCD}(A; B))$ is filtered.

Proof. Because the shifted filtrator $(\mathsf{FCD}(A; B); \mathscr{P}(A \times B); \uparrow^{\mathsf{FCD}(A;B)})$ is filtered. □

Theorem 125. $\mathrm{Compl}\, f = \mathrm{Cor}^{(\mathsf{FCD}(A;B);\mathrm{Compl}\mathsf{FCD}(A;B))} f = \mathrm{Cor}'^{(\mathsf{FCD}(A;B);\mathrm{Compl}\mathsf{FCD}(A;B))} f$ for every funcoid $f \in \mathsf{FCD}(A; B)$.

Proof. $\mathrm{Cor}^{(\mathsf{FCD}(A;B);\mathrm{Compl}\mathsf{FCD}(A;B))} f = \mathrm{Cor}'^{(\mathsf{FCD}(A;B);\mathrm{Compl}\mathsf{FCD}(A;B))} f$ since (the theorem 26 in [15]) the filtrator $(\mathsf{FCD}(A; B); \mathrm{Compl}\mathsf{FCD}(A; B))$ is filtered and with join closed core (the theorem 110).

Let $g \in \mathrm{up}^{(\mathsf{FCD}(A;B);\mathrm{Compl}\mathsf{FCD}(A;B))} f$. Then $g \in \mathrm{Compl}\mathsf{FCD}(A; B)$ and $g \supseteq f$. Thus $g = \mathrm{Compl}\, g \supseteq \mathrm{Compl}\, f$.

Thus $\forall g \in \mathrm{up}^{(\mathsf{FCD}(A;B);\mathrm{Compl}\mathsf{FCD}(A;B))} f \colon g \supseteq \mathrm{Compl}\, f$.

Let $\forall g \in \mathrm{up}^{(\mathsf{FCD}(A;B);\mathrm{Compl}\mathsf{FCD}(A;B))} f \colon h \subseteq g$ for some $h \in \mathrm{Compl}\mathsf{FCD}(A; B)$.

Then $h \subseteq \bigcap^{\mathsf{FCD}(A;B)} \mathrm{up}^{(\mathsf{FCD}(A;B);\mathrm{Compl}\mathsf{FCD}(A;B))} f = f$ and consequently $h = \mathrm{Compl}\, h \subseteq \mathrm{Compl}\, f$.

Thus

$$\mathrm{Compl}\, f = \bigcap^{\mathrm{Compl}\mathsf{FCD}(A;B)} \mathrm{up}^{(\mathsf{FCD}(A;B);\mathrm{Compl}\mathsf{FCD}(A;B))} f = \mathrm{Cor}^{(\mathsf{FCD}(A;B);\mathrm{Compl}\mathsf{FCD}(A;B))} f.$$ □

Theorem 126. $\langle \mathrm{CoCompl}\, f\rangle^* X = \mathrm{Cor}\, \langle f\rangle^* X$ for every funcoid f and set $X \in \mathscr{P}(\mathrm{Src}\, f)$.

Proof. $\mathrm{CoCompl}\, f \subseteq f$ thus $\langle \mathrm{CoCompl}\, f\rangle^* X \subseteq \langle f\rangle^* X$, but $\langle \mathrm{CoCompl}\, f\rangle^* X$ is a principal f.o. thus $\langle \mathrm{CoCompl}\, f\rangle^* X \subseteq \mathrm{Cor}\, \langle f\rangle^* X$.

Let $\alpha X = \mathrm{Cor}\, \langle f\rangle^* X$. Then $\alpha \emptyset = 0^{\mathfrak{F}(\mathrm{Dst}\, f)}$ and

$$\alpha(X \cup Y) = \mathrm{Cor}\, \langle f\rangle^*(X \cup Y) = \mathrm{Cor}(\langle f\rangle^* X \cup \langle f\rangle^* Y) = \mathrm{Cor}\, \langle f\rangle^* X \cup \mathrm{Cor}\, \langle f\rangle^* Y = \alpha X \cup \alpha Y.$$

(used the theorem 64 from [15]). Thus α can be continued till $\langle g\rangle$ for some funcoid g. This funcoid is co-complete.

Evidently g is the greatest co-complete element of $\mathsf{FCD}(\mathrm{Src}\, f; \mathrm{Dst}\, f)$ which is lower than f.

Thus $g = \mathrm{CoCompl}\, f$ and so $\mathrm{Cor}\, \langle f\rangle^* X = \alpha X = \langle g\rangle^* X = \langle \mathrm{CoCompl}\, f\rangle^* X$. □

Theorem 127. $\mathrm{Compl}\mathsf{FCD}(A; B)$ is an atomistic lattice.

Proof. Let $f \in \mathrm{Compl}\mathsf{FCD}(A; B)$. $\langle f\rangle^* X = \bigcup \{\langle f\rangle^* \{x\} \mid x \in X\} = \bigcup \left\{\left\langle f|_{\uparrow^{\mathrm{Src}\, f}\{x\}}\right\rangle^* \{x\} \mid x \in X\right\} = \bigcup \left\{\left\langle f|_{\uparrow^{\mathrm{Src}\, f}\{x\}}\right\rangle^* X \mid x \in X\right\}$, thus $f = \bigcup \left\{f|_{\uparrow^{\mathrm{Src}\, f}\{x\}} \mid x \in X\right\}$. It is trivial that every $f|_{\uparrow^{\mathrm{Src}\, f}\{x\}}$ is a union of atoms of $\mathrm{Compl}\mathsf{FCD}(A; B)$. □

Theorem 128. A funcoid f is complete iff it is a join (on the lattice $\mathsf{FCD}(\mathrm{Src}\, f; \mathrm{Dst}\, f)$) of atomic complete funcoids.

Proof. Follows from the theorem 110 and the previous theorem. □

Corollary 129. $\mathrm{Compl}\mathsf{FCD}(A; B)$ is join-closed.

Theorem 130. $\mathrm{Compl}(\bigcup R) = \bigcup \langle \mathrm{Compl}\rangle R$ for every $R \in \mathscr{P}\mathsf{FCD}(A; B)$ (for every small sets A, B).

Proof. $\langle \mathrm{Compl}(\bigcup R)\rangle^* X = \bigcup \{\langle \bigcup R\rangle^* \{\alpha\} \mid \alpha \in X\} = \bigcup \{\bigcup \{\langle f\rangle^* \{\alpha\} \mid f \in R\} \mid \alpha \in X\} = \bigcup \{\bigcup \{\langle f\rangle^* \{\alpha\} \mid \alpha \in X\} \mid f \in R\} = \bigcup \{\langle \mathrm{Compl}\, f\rangle^* X \mid f \in R\} = \langle \bigcup \langle \mathrm{Compl}\rangle R\rangle^* X$ for every set X. □

Proposition 131. $\mathrm{Compl}\, f = \bigcup \left\{f|_{\uparrow^{\mathrm{Src}\, f}\{\alpha\}} \mid \alpha \in \mathrm{Src}\, f\right\}$ for every funcoid f.

Proof. Let denote R the right part of the equality to prove.

$\langle R\rangle^{\ast}\{\beta\} = \bigcup \left\{ \langle f|_{\uparrow^{\mathrm{Src}\, f}\{\alpha\}}\rangle^{\ast}\{\beta\} \mid \alpha \in \mathrm{Src}\, f \right\} = \langle f\rangle^{\ast}\{\beta\}$ for every $\beta \in A$ and R is complete as a join of complete funcoids.

Thus R is the completion of f. □

Corollary 132. Compl is a lower adjoint.

Conjecture 133. Compl is not an upper adjoint (in general).

Conjecture 134. $\mathrm{Compl}\, f = f \setminus^{\ast} (\Omega \times^{\mathsf{FCD}} \mho)$ for every funcoid f.

This conjecture may be proved by considerations similar to these in the section "Fréchet filter" in [15].

Lemma 135. Co-completion of a complete funcoid is complete.

Proof. Let f is a complete funcoid.

$\langle \mathrm{CoCompl}\, f\rangle^{\ast} X = \mathrm{Cor}\, \langle f\rangle^{\ast} X = \mathrm{Cor}\, \bigcup \{\langle f\rangle^{\ast}\{x\} \mid x \in X\} = \bigcup \{\mathrm{Cor}\, \langle f\rangle^{\ast}\{x\} \mid x \in X\} = \bigcup \{\langle \mathrm{CoCompl}\, f\rangle^{\ast}\{x\} \mid x \in X\}$ for every set X. Thus $\mathrm{CoCompl}\, f$ is complete. □

Theorem 136. $\mathrm{Compl}\,\mathrm{CoCompl}\, f = \mathrm{CoCompl}\,\mathrm{Compl}\, f = \mathrm{Cor}\, f$ for every funcoid f.

Proof. $\mathrm{Compl}\,\mathrm{CoCompl}\, f$ is co-complete since (used the lemma) $\mathrm{CoCompl}\, f$ is co-complete. Thus $\mathrm{Compl}\,\mathrm{CoCompl}\, f$ is a discrete funcoid. $\mathrm{CoCompl}\, f$ is the the greatest co-complete funcoid under f and $\mathrm{Compl}\,\mathrm{CoCompl}\, f$ is the greatest complete funcoid under $\mathrm{CoCompl}\, f$. So $\mathrm{Compl}\,\mathrm{CoCompl}\, f$ is greater than any discrete funcoid under $\mathrm{CoCompl}\, f$ which is greater than any discrete funcoid under f. Thus $\mathrm{Compl}\,\mathrm{CoCompl}\, f$ it is the greatest discrete funcoid under f. Thus $\mathrm{Compl}\,\mathrm{CoCompl}\, f = \mathrm{Cor}\, f$. Similarly $\mathrm{CoCompl}\,\mathrm{Compl}\, f = \mathrm{Cor}\, f$. □

Question 137. Is $\mathrm{Compl}\mathsf{FCD}(A; B)$ a co-brouwerian lattice for every small sets A, B?

3.13 Monovalued and injective funcoids

Following the idea of definition of monovalued morphism let's call *monovalued* such a funcoid f that $f \circ f^{-1} \subseteq I^{\mathsf{FCD}}_{\mathrm{im}\, f}$.

Similarly, I will call a funcoid *injective* when $f^{-1} \circ f \subseteq I^{\mathsf{FCD}}_{\mathrm{dom}\, f}$.

Obvious 138. A funcoid f is

- monovalued iff $f \circ f^{-1} \subseteq I^{\mathsf{FCD}(\mathrm{Dst}\, f)}$;
- injective iff $f^{-1} \circ f \subseteq I^{\mathsf{FCD}(\mathrm{Src}\, f)}$.

In other words, a funcoid is monovalued (injective) when it is a monovalued (injective) morphism of the category of funcoids.

Monovaluedness is dual of injectivity.

Obvious 139.

1. A morphism $(f; \mathcal{A}; \mathcal{B})$ of the category of funcoid triples is monovalued iff the funcoid f is monovalued.
2. A morphism $(f; \mathcal{A}; \mathcal{B})$ of the category of funcoid triples is injective iff the funcoid f is injective.

Theorem 140. The following statements are equivalent for a funcoid f:

1. f is monovalued.
2. $\forall a \in \mathrm{atoms}\, 1^{\mathfrak{F}(\mathrm{Src}\, f)}: \langle f\rangle a \in \mathrm{atoms}\, 1^{\mathfrak{F}(\mathrm{Dst}\, f)} \cup \left\{0^{\mathfrak{F}(\mathrm{Dst}\, f)}\right\}$.
3. $\forall \mathcal{I}, \mathcal{J} \in \mathfrak{F}(\mathrm{Dst}\, f): \langle f^{-1}\rangle(\mathcal{I} \cap \mathcal{J}) = \langle f^{-1}\rangle\mathcal{I} \cap \langle f^{-1}\rangle\mathcal{J}$.

4. $\forall I, J \in \mathscr{P}(\mathrm{Dst}\, f)$: $\langle f^{-1}\rangle^* (I \cap J) = \langle f^{-1}\rangle^* I \cap \langle f^{-1}\rangle^* J$.

Proof.

(2)⇒(3). Let $a \in \mathrm{atoms}\, 1^{\mathfrak{F}(\mathrm{Src}\, f)}$, $\langle f\rangle a = b$. Then because $b \in \mathrm{atoms}\, 1^{\mathfrak{F}(\mathrm{Dst}\, f)} \cup \{0^{\mathfrak{F}(\mathrm{Dst}\, f)}\}$

$$(\mathcal{I} \cap \mathcal{J}) \cap b \neq \emptyset \Leftrightarrow \mathcal{I} \cap b \neq \emptyset \wedge \mathcal{J} \cap b \neq \emptyset;$$
$$a [f] (\mathcal{I} \cap \mathcal{J}) \Leftrightarrow a [f] \mathcal{I} \wedge a [f] \mathcal{J};$$
$$(\mathcal{I} \cap \mathcal{J}) [f^{-1}] a \Leftrightarrow \mathcal{I} [f^{-1}] a \wedge \mathcal{J} [f^{-1}] a;$$
$$a \cap \langle f^{-1}\rangle (\mathcal{I} \cap \mathcal{J}) \neq \emptyset \Leftrightarrow a \cap \langle f^{-1}\rangle \mathcal{I} \neq \emptyset \wedge a \cap \langle f^{-1}\rangle \mathcal{J} \neq \emptyset;$$
$$\langle f^{-1}\rangle (\mathcal{I} \cap \mathcal{J}) = \langle f^{-1}\rangle \mathcal{I} \cap \langle f^{-1}\rangle \mathcal{J}.$$

(3)⇒(1). $\langle f^{-1}\rangle a \cap \langle f^{-1}\rangle b = \langle f^{-1}\rangle (a \cap b) = \langle f^{-1}\rangle 0^{\mathfrak{F}(\mathrm{Dst}\, f)} = 0^{\mathfrak{F}(\mathrm{Src}\, f)}$ for every two distinct atomic filter objects a and b on $\mathrm{Dst}\, f$. This is equivalent to $\neg(\langle f^{-1}\rangle a [f] b)$; $b \asymp \langle f\rangle \langle f^{-1}\rangle a$; $b \asymp \langle f \circ f^{-1}\rangle a$; $\neg(a [f \circ f^{-1}] b)$. So $a [f \circ f^{-1}] b \Rightarrow a = b$ for every atomic filter objects a and b. This is possible only when $f \circ f^{-1} \subseteq I^{\mathsf{FCD}(\mathrm{Dst}\, f)}$.

(4)⇒(3). $\langle f^{-1}\rangle (\mathcal{I} \cap \mathcal{J}) = \bigcap \langle\langle f^{-1}\rangle^*\rangle \mathrm{up}(\mathcal{I} \cap \mathcal{J}) = \bigcap \langle\langle f^{-1}\rangle^*\rangle \{I \cap J \mid I \in \mathrm{up}\, \mathcal{I}, J \in \mathrm{up}\, \mathcal{J}\} = \bigcap \{\langle f^{-1}\rangle^* (I \cap J) \mid I \in \mathrm{up}\, \mathcal{I}, J \in \mathrm{up}\, \mathcal{J}\} = \bigcap \{\langle f^{-1}\rangle^* I \cap \langle f^{-1}\rangle^* J \mid I \in \mathrm{up}\, \mathcal{I}, J \in \mathrm{up}\, \mathcal{J}\} = \bigcap \{\langle f^{-1}\rangle^* I \mid I \in \mathrm{up}\, \mathcal{I}\} \cap \bigcap \{\langle f^{-1}\rangle^* J \mid J \in \mathrm{up}\, \mathcal{J}\} = \langle f^{-1}\rangle \mathcal{I} \cap \langle f^{-1}\rangle \mathcal{J}$.

(3)⇒(4). Obvious.

¬(2)⇒¬(1). Suppose $\langle f\rangle a \notin \mathrm{atoms}\, 1^{\mathfrak{F}(\mathrm{Dst}\, f)} \cup \{0^{\mathfrak{F}(\mathrm{Dst}\, f)}\}$ for some $a \in \mathrm{atoms}\, \mathcal{A}$. Then there exist two atomic filter objects p and q on $\mathrm{Dst}\, f$ such that $p \neq q$ and $\langle f\rangle a \supseteq p \wedge \langle f\rangle a \supseteq q$. Consequently $p \not\asymp \langle f\rangle a$; $a \not\asymp \langle f^{-1}\rangle p$; $a \subseteq \langle f^{-1}\rangle p$; $\langle f \circ f^{-1}\rangle p = \langle f\rangle \langle f^{-1}\rangle p \supseteq \langle f\rangle a \supseteq q$; $\langle f \circ f^{-1}\rangle p \nsubseteq p$ and $\langle f \circ f^{-1}\rangle p \neq 0^{\mathfrak{F}(\mathrm{Dst}\, f)}$. So it cannot be $f \circ f^{-1} \subseteq I^{\mathsf{FCD}(\mathrm{Dst}\, f)}$. □

Corollary 141. A binary relation corresponds to a monovalued funcoid iff it is a function.

Proof. Because $\forall I, J \in \mathscr{P}(\mathrm{im}\, f)$: $\langle f^{-1}\rangle^* (I \cap J) = \langle f^{-1}\rangle^* I \cap \langle f^{-1}\rangle^* J$ is true for a funcoid f corresponding to a binary relation if and only if it is a function. □

Remark 142. This corollary can be reformulated as follows: For binary relations (discrete funcoids) the classic concept of monovaluedness and monovaluedness in the above defined sense of monovaluedness of a funcoid are the same.

3.14 T_0-, T_1- and T_2-separable funcoids

For funcoids it can be generalized T_0-, T_1- and T_2- separability. Worthwhile note that T_0 and T_2 separability is defined through T_1 separability.

Definition 143. Let call T_1-*separable* such funcoid f that for every $\alpha \in \mathrm{Src}\, f$, $\beta \in \mathrm{Dst}\, f$ is true

$$\alpha \neq \beta \Rightarrow \neg(\{\alpha\} [f]^* \{\beta\}).$$

Definition 144. Let call T_0-*separable* such funcoid $f \in \mathsf{FCD}(A; A)$ that $f \cap f^{-1}$ is T_1-separable.

Definition 145. Let call T_2-*separable* such funcoid f that the funcoid $f^{-1} \circ f$ is T_1-separable.

For symmetric transitive funcoids T_1- and T_2-separability are the same (see theorem 14).

Obvious 146. A funcoid f is T_2-*separable* iff $\alpha \neq \beta \Rightarrow \langle f\rangle^* \{\alpha\} \asymp \langle f\rangle^* \{\beta\}$ for every $\alpha, \beta \in \mathrm{Src}\, f$.

3.15 Filter objects closed regarding a funcoid

Definition 147. Let's call *closed* regarding a funcoid $f \in \mathsf{FCD}(A; A)$ such filter object $\mathcal{A} \in \mathfrak{F}(\mathrm{Src}\, f)$ that $\langle f\rangle \mathcal{A} \subseteq \mathcal{A}$.

This is a generalization of closedness of a set regarding an unary operation.

Proposition 148. If $\mathcal{I}$ and $\mathcal{J}$ are closed (regarding some funcoid f), S is a set of closed filter objects on $\operatorname{Src} f$, then

1. $\mathcal{I} \cup \mathcal{J}$ is a closed filter object;
2. $\bigcap S$ is a closed filter object.

Proof. Let denote the given funcoid as f. $\langle f\rangle(\mathcal{I} \cup \mathcal{J}) = \langle f\rangle\mathcal{I} \cup \langle f\rangle\mathcal{J} \subseteq \mathcal{I} \cup \mathcal{J}$, $\langle f\rangle \bigcap S \subseteq \bigcap \langle\langle f\rangle\rangle S \subseteq \bigcap S$. Consequently the filter objects $\mathcal{I} \cup \mathcal{J}$ and $\bigcap S$ are closed. □

Proposition 149. If S is a set of filter objects closed regarding a complete funcoid, then the filter object $\bigcup S$ is also closed regarding our funcoid.

Proof. $\langle f\rangle \bigcup S = \bigcup \langle\langle f\rangle\rangle S \subseteq \bigcup S$ where f is the given funcoid. □

4 Reloids

Definition 150. I will call a *reloid* from a small set A to a small set B a triple $(A; B; F)$ where $F \in \mathfrak{F}(A \times B)$.

Definition 151. *Source* and *destination* of every reloid $(A; B; F)$ are defined as

$$\operatorname{Src}(A; B; F) = A \quad \text{and} \quad \operatorname{Dst}(A; B; F) = B.$$

I will denote $\mathsf{RLD}(A; B)$ the set of reloids from A to B.

I will denote RLD the set of all reloids (for small sets).

Further we will assume that all reloids in consideration are small.

Reloids are a generalization of uniform spaces. Also reloids are generalization of binary relations (I will call a reloid $(A; B; F)$ *discrete* when F is a principal filter on $A \times B$.)

I will denote $\operatorname{up}(A; B; F) = \operatorname{up} F$ for every reloid $(A; B; F)$.

Definition 152. The *reverse* reloid of a reloid f is defined by the formula

$$(A; B; F)^{-1} = (B; A; \{F^{-1} \mid F \in \operatorname{up} f^{-1}\}).$$

Reverse reloid is a generalization of conjugate quasi-uniformity.

I will denote $\uparrow^{\mathsf{RLD}(A;B)} f = (A;\ B;\ \uparrow^{A \times B} f)$ for every small sets A, B and a binary relation $f \subseteq A \times B$.

The order (in fact a complete lattice) on $\mathsf{RLD}(A; B)$ is defined by the formula

$$(A; B; F) \subseteq (A; B; G) \Leftrightarrow F \subseteq G.$$

We will apply lattice operations to subsets of $\mathsf{RLD}(A; B)$ without explicitly mentioning $\mathsf{RLD}(A; B)$.

4.1 Composition of reloids

Definition 153. Reloids f and g are *composable* when $\operatorname{Dst} f = \operatorname{Src} g$.

Definition 154. Composition of (composable) reloids is defined by the formula

$$g \circ f = \bigcap \left\{ \uparrow^{\mathsf{RLD}(\operatorname{Src} f; \operatorname{Dst} g)} (G \circ F) \mid F \in \operatorname{up} f, G \in \operatorname{up} g \right\}.$$

Composition of reloids is a reloid.

Theorem 155. $(h \circ g) \circ f = h \circ (g \circ f)$ for every composable reloids f, g, h.

Proof. For two nonempty collections A and B of sets I will denote

$$A\sim B\Leftrightarrow(\forall K\in A\exists L\in B\colon L\subseteq K)\wedge(\forall K\in B\exists L\in A\colon L\subseteq K).$$

It is easy to see that $\sim$ is a transitive relation.

I will denote $B\circ A=\{L\circ K\,|\,K\in A,L\in B\}$.

Let first prove that for every nonempty collections of relations A, B, C

$$A\sim B\Rightarrow A\circ C\sim B\circ C.$$

Suppose $A\sim B$ and $P\in A\circ C$ that is $K\in A$ and $M\in C$ such that $P=K\circ M$. $\exists K'\in B\colon K'\subseteq K$ because $A\sim B$. We have $P'=K'\circ M\in B\circ C$. Obviously $P'\subseteq P$. So for every $P\in A\circ C$ exist $P'\in B\circ C$ such that $P'\subseteq P$; the vice versa is analogous. So $A\circ C\sim B\circ C$.

$\operatorname{up}((h\circ g)\circ f)\sim\operatorname{up}(h\circ g)\circ\operatorname{up}f$, $\operatorname{up}(h\circ g)\sim(\operatorname{up}h)\circ(\operatorname{up}g)$. By proven above $\operatorname{up}((h\circ g)\circ f)\sim(\operatorname{up}h)\circ(\operatorname{up}g)\circ(\operatorname{up}f)$.

Analogously $\operatorname{up}(h\circ(g\circ f))\sim(\operatorname{up}h)\circ(\operatorname{up}g)\circ(\operatorname{up}f)$.

So $\operatorname{up}((h\circ g)\circ f)\sim\operatorname{up}(h\circ(g\circ f))$ what is possible only if $\operatorname{up}((h\circ g)\circ f)=\operatorname{up}(h\circ(g\circ f))$. □

Theorem 156. For every reloid f:

1. $f\circ f=\bigcap\left\{\uparrow^{\mathsf{RLD}(\operatorname{Src}f;\operatorname{Dst}f)}(F\circ F)\mid F\in\operatorname{up}f\right\}$ if $\operatorname{Src}f=\operatorname{Dst}f$;
2. $f^{-1}\circ f=\bigcap\left\{\uparrow^{\mathsf{RLD}(\operatorname{Src}f;\operatorname{Src}f)}(F^{-1}\circ F)\mid F\in\operatorname{up}f\right\}$;
3. $f\circ f^{-1}=\bigcap\left\{\uparrow^{\mathsf{RLD}(\operatorname{Dst}f;\operatorname{Dst}f)}(F\circ F^{-1})\mid F\in\operatorname{up}f\right\}$.

Proof. I will prove only (1) and (2) because (3) is analogous to (2).

1. It's enough to show that $\forall F,G\in\operatorname{up}f\exists H\in\operatorname{up}f\colon H\circ H\subseteq G\circ F$. To prove it take $H=F\cap G$.
2. It's enough to show that $\forall F,G\in\operatorname{up}f\exists H\in\operatorname{up}f\colon H^{-1}\circ H\subseteq G^{-1}\circ F$. To prove it take $H=F\cap G$. Then $H^{-1}\circ H=(F\cap G)^{-1}\circ(F\cap G)\subseteq G^{-1}\circ F$. □

Theorem 157. For every small sets A, B, C if $g,h\in\mathsf{RLD}(A;B)$ then

1. $f\circ(g\cup h)=f\circ g\cup f\circ h$ for every $f\in\mathsf{RLD}(B;C)$;
2. $(g\cup h)\circ f=g\circ f\cup h\circ f$ for every $f\in\mathsf{RLD}(C;A)$.

Proof. We'll prove only the first as the second is dual.

By the infinite distributivity law for filters we have

$$\begin{aligned}f\circ g\cup f\circ h &= \bigcap\left\{\uparrow^{\mathsf{RLD}(A;C)}(F\circ G)\mid F\in\operatorname{up}f,G\in\operatorname{up}g\right\}\cup\bigcap\left\{\uparrow^{\mathsf{RLD}(A;C)}(F\circ H)\mid F\in\operatorname{up}f,\right.\\ &\left.\quad H\in\operatorname{up}h\right\}\\ &= \bigcap\left\{\uparrow^{\mathsf{RLD}(A;C)}(F_1\circ G)\cup\uparrow^{\mathsf{RLD}(A;C)}(F_2\circ H)\mid F_1,F_2\in\operatorname{up}f,G\in\operatorname{up}g,H\in\operatorname{up}h\right\}\\ &= \bigcap\left\{\uparrow^{\mathsf{RLD}(A;C)}((F_1\circ G)\cup(F_2\circ H))\mid F_1,F_2\in\operatorname{up}f,G\in\operatorname{up}g,H\in\operatorname{up}h\right\}.\end{aligned}$$

Obviously

$$\begin{aligned}\bigcap\left\{\uparrow^{\mathsf{RLD}(A;C)}((F_1\circ G)\cup(F_2\circ H))\mid F_1,F_2\in\operatorname{up}f,G\in\operatorname{up}g,H\in\operatorname{up}h\right\} &\supseteq\\ \bigcap\left\{\uparrow^{\mathsf{RLD}(A;C)}(((F_1\cap F_2)\circ G)\cup((F_1\cap F_2)\circ H))\mid F_1,F_2\in\operatorname{up}f,G\in\operatorname{up}g,H\in\operatorname{up}h\right\} &=\\ \bigcap\left\{\uparrow^{\mathsf{RLD}(A;C)}((F\circ G)\cup(F\circ H))\mid F\in\operatorname{up}f,G\in\operatorname{up}g,H\in\operatorname{up}h\right\} &=\\ \bigcap\left\{\uparrow^{\mathsf{RLD}(A;C)}(F\circ(G\cup H))\mid F\in\operatorname{up}f,G\in\operatorname{up}g,H\in\operatorname{up}h\right\}&.\end{aligned}$$

Because $G\in\operatorname{up}g\wedge H\in\operatorname{up}h\Rightarrow G\cup H\in\operatorname{up}(g\cup h)$ we have

$$\begin{aligned}\bigcap\left\{\uparrow^{\mathsf{RLD}(A;C)}(F\circ(G\cup H))\mid F\in\operatorname{up}f,G\in\operatorname{up}g,H\in\operatorname{up}h\right\} &\supseteq\\ \bigcap\left\{\uparrow^{\mathsf{RLD}(A;C)}(F\circ K)\mid F\in\operatorname{up}f,K\in\operatorname{up}(g\cup h)\right\} &=\\ f\circ(g\cup h)&.\end{aligned}$$

Thus we proved $f\circ g\cup f\circ h\supseteq f\circ(g\cup h)$. But obviously $f\circ(g\cup h)\supseteq f\circ g$ and $f\circ(g\cup h)\supseteq f\circ h$ and so $f\circ(g\cup h)\supseteq f\circ g\cup f\circ h$. Thus $f\circ(g\cup h)=f\circ g\cup f\circ h$. □

Conjecture 158. If f and g are reloids, then

$$g\circ f=\bigcup\{G\circ F\mid F\in\operatorname{atoms}f,G\in\operatorname{atoms}g\}.$$

4.2 Direct product of filter objects

Definition 159. *Reloidal product* of filter objects $\mathcal{A}$ and $\mathcal{B}$ is defined by the formula

$$\mathcal{A}\times^{\mathsf{RLD}}\mathcal{B}\overset{\text{def}}{=}\bigcap\left\{\uparrow^{\mathsf{RLD}(\operatorname{Base}(\mathcal{A});\operatorname{Base}(\mathcal{B}))}(A\times B)\mid A\in\operatorname{up}\mathcal{A},B\in\operatorname{up}\mathcal{B}\right\}.$$

Obvious 160. $\uparrow^U A\times^{\mathsf{RLD}}\uparrow^V B=\uparrow^{\mathsf{RLD}(U;V)}(A\times B)$ for every small sets $A\subseteq U$ and $B\subseteq V$.

Theorem 161. $\mathcal{A}\times^{\mathsf{RLD}}\mathcal{B}=\bigcup\{a\times^{\mathsf{RLD}}b\mid a\in\operatorname{atoms}\mathcal{A},b\in\operatorname{atoms}\mathcal{B}\}$ for every filter objects $\mathcal{A}$, $\mathcal{B}$.

Proof. Obviously

$$\mathcal{A}\times^{\mathsf{RLD}}\mathcal{B}\supseteq\bigcup\{a\times^{\mathsf{RLD}}b\mid a\in\operatorname{atoms}\mathcal{A},b\in\operatorname{atoms}\mathcal{B}\}$$

Reversely, let

$$K\in\operatorname{up}\bigcup\{a\times^{\mathsf{RLD}}b\mid a\in\operatorname{atoms}\mathcal{A},b\in\operatorname{atoms}\mathcal{B}\}.$$

Then $K\in\operatorname{up}(a\times^{\mathsf{RLD}}b)$ for every $a\in\operatorname{atoms}\mathcal{A}$, $b\in\operatorname{atoms}\mathcal{B}$; $K\supseteq X_a\times Y_b$ for some $X_a\in\operatorname{up}a$, $Y_b\in\operatorname{up}b$; $K\supseteq\bigcup\{X_a\times Y_b\mid a\in\operatorname{atoms}\mathcal{A},b\in\operatorname{atoms}\mathcal{B}\}=\bigcup\{X_a\mid a\in\operatorname{atoms}\mathcal{A}\}\times\bigcup\{Y_b\mid b\in\operatorname{atoms}\mathcal{A}\}\supseteq A\times B$ where $A\in\operatorname{up}\mathcal{A}$, $B\in\operatorname{up}\mathcal{B}$; $K\in\operatorname{up}(\mathcal{A}\times^{\mathsf{RLD}}\mathcal{B})$. □

Theorem 162. If $\mathcal{A}_0,\mathcal{A}_1\in\mathfrak{F}(A)$, $\mathcal{B}_0,\mathcal{B}_1\in\mathfrak{F}(B)$ for some small sets A, B then

$$(\mathcal{A}_0\times^{\mathsf{RLD}}\mathcal{B}_0)\cap(\mathcal{A}_1\times^{\mathsf{RLD}}\mathcal{B}_1)=(\mathcal{A}_0\cap\mathcal{A}_1)\times^{\mathsf{RLD}}(\mathcal{B}_0\cap\mathcal{B}_1).$$

Proof.

$$\begin{aligned}(\mathcal{A}_0\times^{\mathsf{RLD}}\mathcal{B}_0)\cap(\mathcal{A}_1\times^{\mathsf{RLD}}\mathcal{B}_1) &= \bigcap\left\{\uparrow^{\mathsf{RLD}(A;B)}(P\cap Q)\mid P\in\operatorname{up}(\mathcal{A}_0\times^{\mathsf{RLD}}\mathcal{B}_0),\ Q\in\operatorname{up}(\mathcal{A}_1\times^{\mathsf{RLD}}\mathcal{B}_1)\right\}\\ &= \bigcap\left\{\uparrow^{\mathsf{RLD}(A;B)}((A_0\times B_0)\cap(A_1\times B_1))\mid A_0\in\operatorname{up}\mathcal{A}_0,B_0\in\operatorname{up}\mathcal{B}_0,\ A_1\in\operatorname{up}\mathcal{A}_1,B_1\in\operatorname{up}\mathcal{B}_1\right\}\\ &= \bigcap\left\{\uparrow^{\mathsf{RLD}(A;B)}((A_0\cap A_1)\times(B_0\cap B_1))\mid A_0\in\operatorname{up}\mathcal{A}_0,B_0\in\operatorname{up}\mathcal{B}_0,\ A_1\in\operatorname{up}\mathcal{A}_1,B_1\in\operatorname{up}\mathcal{B}_1\right\}\\ &= \bigcap\left\{\uparrow^{\mathsf{RLD}(A;B)}(K\times L)\mid K\in\operatorname{up}(\mathcal{A}_0\cap\mathcal{A}_1),L\in\operatorname{up}(\mathcal{B}_0\cap\mathcal{B}_1)\right\}\\ &= (\mathcal{A}_0\cap\mathcal{A}_1)\times^{\mathsf{RLD}}(\mathcal{B}_0\cap\mathcal{B}_1).\end{aligned}$$

□

Theorem 163. If $S\in\mathscr{P}(\mathfrak{F}(A)\times\mathfrak{F}(B))$ for some small sets A, B then

$$\bigcap\{\mathcal{A}\times^{\mathsf{RLD}}\mathcal{B}\mid(\mathcal{A};\mathcal{B})\in S\}=\bigcap\operatorname{dom}S\times^{\mathsf{RLD}}\bigcap\operatorname{im}S.$$

Proof. Let $\mathcal{P}=\bigcap\operatorname{dom}S$, $\mathcal{Q}=\bigcap\operatorname{im}S$; $l=\bigcap\{\mathcal{A}\times^{\mathsf{RLD}}\mathcal{B}\mid(\mathcal{A};\mathcal{B})\in S\}$.

$\mathcal{P}\times^{\mathsf{RLD}}\mathcal{Q}\subseteq l$ is obvious.

Let $F\in\operatorname{up}(\mathcal{P}\times^{\mathsf{RLD}}\mathcal{Q})$. Then exist $P\in\operatorname{up}\mathcal{P}$ and $Q\in\operatorname{up}\mathcal{Q}$ such that $F\supseteq P\times Q$.

$P=P_1\cap\ldots\cap P_n$ where $P_i\in\langle\operatorname{up}\rangle\operatorname{dom}S$ and $Q=Q_1\cap\ldots\cap Q_m$ where $Q_i\in\langle\operatorname{up}\rangle\operatorname{im}S$.

$P\times Q=\bigcap_{i,j}(P_i\times Q_i)$.

$P_i\times Q_i\in\operatorname{up}(\mathcal{A}\times^{\mathsf{RLD}}\mathcal{B})$ for some $(\mathcal{A};\mathcal{B})\in S$. $P\times Q=\bigcap_{i,j}(P_i\times Q_i)\in\operatorname{up}l$. So $F\in\operatorname{up}l$. □

Conjecture 164. If $\mathcal{A}\in\mathfrak{F}$ then $\mathcal{A}\times^{\mathsf{RLD}}$ is a complete homomorphism of every lattice $\mathfrak{F}(B)$ to a complete sublattice of the lattice $\mathsf{RLD}(\operatorname{Base}(A);\operatorname{Base}(B))$, if also $\mathcal{A}\neq\emptyset$ then it is an isomorphism.

Definition 165. I will call a reloid *convex* iff it is a union of direct products.

Example 166. Non-convex reloids exist.

Proof. Let a is a non-trivial atomic f.o. Then $\uparrow^{\mathsf{RLD}(\mathrm{Base}(a);\mathrm{Base}(a))}(=)|_a$ is non-convex. This follows from the fact that only direct products which are below $(=)$ are direct products of atomic f.o. and $\uparrow^{\mathsf{RLD}(\mathrm{Base}(a);\mathrm{Base}(a))}(=)|_a$ is not their join. □

4.3 Restricting reloid to a filter object. Domain and image

Definition 167. *Identity reloid* for a small set A is defined by the formula $I^{\mathsf{RLD}(A)} = \uparrow^{\mathsf{RLD}(A;A)} I_A$.

Definition 168. I call *restricting* a reloid f to a filter object $\mathcal{A}$ as $f|_{\mathcal{A}} = f \cap \left(\mathcal{A} \times^{\mathsf{RLD}} 1^{\mathfrak{F}(\mathrm{Dst}\, f)}\right)$.

Definition 169. *Domain* and *image* of a reloid f are defined as follows:

$$\mathrm{dom}\, f = \bigcap \langle \uparrow^{\mathrm{Dst}\, f} \rangle \langle \mathrm{dom} \rangle \mathrm{up}\, f; \quad \mathrm{im}\, f = \bigcap \langle \uparrow^{\mathrm{Src}\, f} \rangle \langle \mathrm{im} \rangle \mathrm{up}\, f.$$

Proposition 170. $f \subseteq \mathcal{A} \times^{\mathsf{RLD}} \mathcal{B} \Leftrightarrow \mathrm{dom}\, f \subseteq \mathcal{A} \wedge \mathrm{im}\, f \subseteq \mathcal{B}$ for every reloid f and filter objects $\mathcal{A} \in \mathfrak{F}(\mathrm{Src}\, f)$, $\mathcal{B} \in \mathfrak{F}(\mathrm{Dst}\, f)$.

Proof.

⇒. Follows from $\mathrm{dom}(\mathcal{A} \times^{\mathsf{RLD}} \mathcal{B}) \subseteq \mathcal{A} \wedge \mathrm{im}(\mathcal{A} \times^{\mathsf{RLD}} \mathcal{B}) \subseteq \mathcal{B}$.

⇐. $\mathrm{dom}\, f \subseteq \mathcal{A} \Leftrightarrow \forall A \in \mathrm{up}\, \mathcal{A}\, \exists F \in \mathrm{up}\, f: \mathrm{dom}\, F \subseteq A$. Analogously

$$\mathrm{im}\, f \subseteq \mathcal{B} \Leftrightarrow \forall B \in \mathrm{up}\, \mathcal{B}\, \exists G \in \mathrm{up}\, f: \mathrm{im}\, G \subseteq B.$$

Let $\mathrm{dom}\, f \subseteq \mathcal{A} \wedge \mathrm{im}\, f \subseteq \mathcal{B}$, $A \in \mathrm{up}\, \mathcal{A}$, $B \in \mathrm{up}\, \mathcal{B}$. Then exist $F \in \mathrm{up}\, f, G \in \mathrm{up}\, f$ such that $\mathrm{dom}\, F \subseteq A \wedge \mathrm{im}\, G \subseteq B$. Consequently $F \cap G \in \mathrm{up}\, f$, $\mathrm{dom}(F \cap G) \subseteq A$, $\mathrm{im}(F \cap G) \subseteq B$ that is $F \cap G \subseteq A \times B$. So exists $H \in \mathrm{up}\, f$ such that $H \subseteq A \times B$ for every $A \in \mathrm{up}\, \mathcal{A}, B \in \mathrm{up}\, \mathcal{B}$. So $f \subseteq \mathcal{A} \times^{\mathsf{RLD}} \mathcal{B}$. □

Definition 171. I call *restricted identity reloid* for a filter object $\mathcal{A}$ the reloid

$$I_{\mathcal{A}}^{\mathsf{RLD}} \overset{\mathrm{def}}{=} \left(I^{\mathsf{RLD}(A)}\right)|_{\mathcal{A}}.$$

Theorem 172. $I_{\mathcal{A}}^{\mathsf{RLD}} = \bigcap \left\{\uparrow^{\mathsf{RLD}(\mathrm{Base}(\mathcal{A});\mathrm{Base}(\mathcal{A}))} I_A \mid A \in \mathrm{up}\, \mathcal{A}\right\}$ where I_A is the identity relation on a set A.

Proof. Let $K \in \mathrm{up} \bigcap \left\{\uparrow^{\mathsf{RLD}(\mathrm{Base}(\mathcal{A});\mathrm{Base}(\mathcal{A}))} I_A \mid A \in \mathrm{up}\, \mathcal{A}\right\}$, then exists $A \in \mathrm{up}\, \mathcal{A}$ such that $K \supseteq I_A$. Then

$I_{\mathcal{A}}^{\mathsf{RLD}} \subseteq \uparrow^{\mathsf{RLD}(\mathrm{Base}(\mathcal{A});\mathrm{Base}(\mathcal{A}))}(I_{\mathrm{Base}(\mathcal{A})}) \cap \left(\mathcal{A} \times^{\mathsf{RLD}} 1^{\mathfrak{F}(\mathrm{Base}(\mathcal{A}))}\right) \subseteq \uparrow^{\mathsf{RLD}(\mathrm{Base}(\mathcal{A});\mathrm{Base}(\mathcal{A}))}(I_{\mathrm{Base}(\mathcal{A})}) \cap \left(\uparrow^{\mathrm{Base}(\mathcal{A})} A \times^{\mathsf{RLD}} 1^{\mathfrak{F}(\mathrm{Base}(\mathcal{A}))}\right) = \uparrow^{\mathsf{RLD}(\mathrm{Base}(\mathcal{A});\mathrm{Base}(\mathcal{A}))}(I_{\mathrm{Base}(\mathcal{A})}) \cap \uparrow^{\mathsf{RLD}(\mathrm{Base}(\mathcal{A});\mathrm{Base}(\mathcal{A}))}(A \times \mathrm{Base}(\mathcal{A})) = \uparrow^{\mathsf{RLD}(\mathrm{Base}(\mathcal{A});\mathrm{Base}(\mathcal{A}))}(I_{\mathrm{Base}(\mathcal{A})} \cap (A \times \mathrm{Base}(\mathcal{A}))) = \uparrow^{\mathsf{RLD}(\mathrm{Base}(\mathcal{A});\mathrm{Base}(\mathcal{A}))} I_A \subseteq \uparrow^{\mathsf{RLD}(\mathrm{Base}(\mathcal{A});\mathrm{Base}(\mathcal{A}))} K$,

Thus $K \in \mathrm{up}\, I_{\mathcal{A}}^{\mathsf{RLD}}$.

Reversely let $K \in \mathrm{up}\, I_{\mathcal{A}}^{\mathsf{RLD}} = \mathrm{up}\left(I^{\mathsf{RLD}(A)} \cap \left(\mathcal{A} \times^{\mathsf{RLD}} 1^{\mathfrak{F}(\mathrm{Base}(\mathcal{A}))}\right)\right)$, then exists $A \in \mathrm{up}\, \mathcal{A}$ such that $K \in \mathrm{up}\left(I^{\mathsf{RLD}(A)} \cap \left(A \times^{\mathsf{RLD}} 1^{\mathfrak{F}(\mathrm{Base}(\mathcal{A}))}\right)\right) = \mathrm{up} \uparrow^{\mathsf{RLD}(\mathrm{Base}(\mathcal{A});\mathrm{Base}(\mathcal{A}))} I_A \subseteq \mathrm{up} \bigcap \left\{\uparrow^{\mathsf{RLD}(\mathrm{Base}(\mathcal{A});\mathrm{Base}(\mathcal{A}))} I_A \mid A \in \mathrm{up}\mathcal{A}\right\}$. □

Proposition 173. $(I_{\mathcal{A}}^{\mathsf{RLD}})^{-1} = I_{\mathcal{A}}^{\mathsf{RLD}}$.

Proof. It follows from the previous theorem. □

Theorem 174. $f|_{\mathcal{A}} = f \circ I_{\mathcal{A}}^{\mathsf{RLD}}$ for every reloid f and $\mathcal{A} \in \mathfrak{F}(\mathrm{Src}\, f)$.

Proof. We need to prove that $f \cap \left(\mathcal{A} \times^{\mathsf{RLD}} 1^{\mathfrak{F}(\operatorname{Dst} f)} \right) = f \circ \bigcap \left\{ \uparrow^{\mathsf{RLD}(\operatorname{Src} f; \operatorname{Src} f)} I_A \mid A \in \operatorname{up} \mathcal{A} \right\}$. We have $f \circ \bigcap \left\{ \uparrow^{\mathsf{RLD}(\operatorname{Src} f; \operatorname{Src} f)} I_A \mid A \in \operatorname{up} \mathcal{A} \right\} = \bigcap \left\{ \uparrow^{\mathsf{RLD}(\operatorname{Src} f; \operatorname{Dst} f)} (F \circ I_A) \mid F \in \operatorname{up} f, A \in \operatorname{up} \mathcal{A} \right\} = \bigcap \left\{ \uparrow^{\mathsf{RLD}(\operatorname{Src} f; \operatorname{Dst} f)} (F|_A) \mid F \in \operatorname{up} f, A \in \operatorname{up} \mathcal{A} \right\} = \bigcap \left\{ \uparrow^{\mathsf{RLD}(\operatorname{Src} f; \operatorname{Dst} f)} (F \cap (A \times \operatorname{Dst} f)) \mid F \in \operatorname{up} f, A \in \operatorname{up} \mathcal{A} \right\} = \bigcap \left\{ \uparrow^{\mathsf{RLD}(\operatorname{Src} f; \operatorname{Dst} f)} F \mid F \in \operatorname{up} f \right\} \cap \bigcap \left\{ \uparrow^{\mathsf{RLD}(\operatorname{Src} f; \operatorname{Dst} f)} (A \times \operatorname{Dst} f) \mid A \in \operatorname{up} \mathcal{A} \right\} = f \cap \left(\mathcal{A} \times^{\mathsf{RLD}} 1^{\mathfrak{F}(\operatorname{Dst} f)} \right)$. □

Theorem 175. $(g \circ f)|_{\mathcal{A}} = g \circ (f|_{\mathcal{A}})$ for every composable reloids f and g and $\mathcal{A} \in \mathfrak{F}(\operatorname{Src} f)$.

Proof. $(g \circ f)|_{\mathcal{A}} = (g \circ f) \circ I^{\mathsf{RLD}}_{\mathcal{A}} = g \circ (f \circ I^{\mathsf{RLD}}_{\mathcal{A}}) = g \circ (f|_{\mathcal{A}})$. □

Theorem 176. $f \cap (\mathcal{A} \times^{\mathsf{RLD}} \mathcal{B}) = I^{\mathsf{RLD}}_{\mathcal{B}} \circ f \circ I^{\mathsf{RLD}}_{\mathcal{A}}$ for every reloid f and $\mathcal{A} \in \mathfrak{F}(\operatorname{Src} f)$, $\mathcal{B} \in \mathfrak{F}(\operatorname{Dst} f)$.

Proof. $f \cap (\mathcal{A} \times^{\mathsf{RLD}} \mathcal{B}) = f \cap (\mathcal{A} \times^{\mathsf{RLD}} 1^{\mathfrak{F}(\operatorname{Dst} f)}) \cap (1^{\mathfrak{F}(\operatorname{Src} f)} \times^{\mathsf{RLD}} \mathcal{B}) = f|_{\mathcal{A}} \cap (1^{\mathfrak{F}(\operatorname{Src} f)} \times^{\mathsf{RLD}} \mathcal{B}) = (f \circ I^{\mathsf{RLD}}_{\mathcal{A}}) \cap (1^{\mathfrak{F}(\operatorname{Src} f)} \times^{\mathsf{RLD}} \mathcal{B}) = \left((f \circ I^{\mathsf{RLD}}_{\mathcal{A}})^{-1} \cap (1^{\mathfrak{F}(\operatorname{Src} f)} \times^{\mathsf{RLD}} \mathcal{B})^{-1} \right)^{-1} = \left((I^{\mathsf{RLD}}_{\mathcal{A}} \circ f^{-1}) \cap (\mathcal{B} \times^{\mathsf{RLD}} 1^{\mathfrak{F}(\operatorname{Src} f)}) \right)^{-1} = (I^{\mathsf{RLD}}_{\mathcal{A}} \circ f^{-1} \circ I^{\mathsf{RLD}}_{\mathcal{B}})^{-1} = I^{\mathsf{RLD}}_{\mathcal{B}} \circ f \circ I^{\mathsf{RLD}}_{\mathcal{A}}$. □

Theorem 177. $f|_{\uparrow^{\operatorname{Src} f}\{\alpha\}} = \uparrow^{\operatorname{Src} f} \{\alpha\} \times^{\mathsf{RLD}} \operatorname{im}\left(f|_{\uparrow^{\operatorname{Src} f}\{\alpha\}} \right)$ for every reloid f and $\alpha \in \operatorname{Src} f$.

Proof. First,

$$\begin{aligned}
\operatorname{im}\left(f|_{\uparrow^{\operatorname{Src} f}\{\alpha\}} \right) &= \\
\bigcap \langle \uparrow^{\operatorname{Dst} f} \rangle \langle \operatorname{im} \rangle \operatorname{up}\left(f|_{\uparrow^{\operatorname{Src} f}\{\alpha\}} \right) &= \\
\bigcap \langle \uparrow^{\operatorname{Dst} f} \rangle \langle \operatorname{im} \rangle \operatorname{up}\left(f \cap \left(\uparrow^{\operatorname{Src} f} \{\alpha\} \times^{\mathsf{RLD}} 1^{\mathfrak{F}(\operatorname{Dst} f)} \right) \right) &= \\
\bigcap \left\{ \uparrow^{\operatorname{Dst} f} \operatorname{im}(F \cap (\{\alpha\} \times \operatorname{Dst} f)) \mid F \in \operatorname{up} f \right\} &= \\
\bigcap \left\{ \uparrow^{\operatorname{Dst} f} \operatorname{im}(F|_{\{\alpha\}}) \mid F \in \operatorname{up} f \right\}. &
\end{aligned}$$

Taking this into account we have:

$$\begin{aligned}
\uparrow^{\operatorname{Src} f} \{\alpha\} \times^{\mathsf{RLD}} \operatorname{im}\left(f|_{\uparrow^{\operatorname{Src} f}\{\alpha\}} \right) &= \\
\bigcap \left\{ \uparrow^{\mathsf{RLD}(\operatorname{Src} f; \operatorname{Dst} f)} (\{\alpha\} \times K) \mid K \in \operatorname{up} \operatorname{im}\left(f|_{\uparrow^{\operatorname{Src} f}\{\alpha\}} \right) \right\} &= \\
\bigcap \left\{ \uparrow^{\mathsf{RLD}(\operatorname{Src} f; \operatorname{Dst} f)} (\{\alpha\} \times \operatorname{im}(F|_{\{\alpha\}})) \mid F \in \operatorname{up} f \right\} &= \\
\bigcap \left\{ \uparrow^{\mathsf{RLD}(\operatorname{Src} f; \operatorname{Dst} f)} (F|_{\{\alpha\}}) \mid F \in \operatorname{up} f \right\} &= \\
\bigcap \left\{ \uparrow^{\mathsf{RLD}(\operatorname{Src} f; \operatorname{Dst} f)} (F \cap (\{\alpha\} \times \operatorname{Dst} f)) \mid F \in \operatorname{up} f \right\} &= \\
\bigcap \left\{ \uparrow^{\mathsf{RLD}(\operatorname{Src} f; \operatorname{Dst} f)} F \mid F \in \operatorname{up} f \right\} \cap \uparrow^{\mathsf{RLD}(\operatorname{Src} f; \operatorname{Dst} f)} (\{\alpha\} \times \operatorname{Dst} f) &= \\
f \cap \uparrow^{\mathsf{RLD}(\operatorname{Src} f; \operatorname{Dst} f)} (\{\alpha\} \times \operatorname{Dst} f) &= \\
f|_{\uparrow^{\operatorname{Src} f}\{\alpha\}}. &
\end{aligned}$$

□

4.4 Categories of reloids

I will define two categories, the *category of reloids* and the *category of reloid triples*.

The *category of reloids* is defined as follows:

- Objects are small sets.
- The set of morphisms from a set A to a set B is $\mathsf{RLD}(A; B)$.
- The composition is the composition of reloids.
- Identity morphism for a set is the identity reloid for that set.

To show it is really a category is trivial.

The *category of reloid triples* is defined as follows:

- Objects are filter objects on small sets.
- The morphisms from a f.o. $\mathcal{A}$ to a f.o. $\mathcal{B}$ are triples $(f; \mathcal{A}; \mathcal{B})$ where $f \in \mathsf{RLD}(\operatorname{Base}(\mathcal{A}); \operatorname{Base}(\mathcal{B}))$ and $\operatorname{dom} f \subseteq \mathcal{A} \wedge \operatorname{im} f \subseteq \mathcal{B}$.

- The composition is defined by the formula $(g;\mathcal{B};\mathcal{C})\circ(f;\mathcal{A};\mathcal{B})=(g\circ f;\mathcal{A};\mathcal{C})$.
- Identity morphism for an f.o. $\mathcal{A}$ is $I^{\mathsf{RLD}}_{\mathcal{A}}$.

To prove that it is really a category is trivial.

4.5 Monovalued and injective reloids

Following the idea of definition of monovalued morphism let's call *monovalued* such a reloid f that $f\circ f^{-1}\subseteq I^{\mathsf{RLD}}_{\operatorname{im}f}$.

Similarly, I will call a reloid *injective* when $f^{-1}\circ f\subseteq I^{\mathsf{RLD}}_{\operatorname{dom}f}$.

Obvious 178. A reloid f is

- monovalued iff $f\circ f^{-1}\subseteq I^{\mathsf{RLD}(\operatorname{Dst}f)}$;
- injective iff $f^{-1}\circ f\subseteq I^{\mathsf{RLD}(\operatorname{Src}f)}$.

In other words, a funcoid is monovalued (injective) when it is a monovalued (injective) morphism of the category of funcoids.

Monovaluedness is dual of injectivity.

Obvious 179.

1. A morphism $(f;\ \mathcal{A};\ \mathcal{B})$ of the category of reloid triples is monovalued iff the reloid f is monovalued.
2. A morphism $(f;\mathcal{A};\mathcal{B})$ of the category of reloid triples is injective iff the reloid f is injective.

Theorem 180.

1. A reloid f is a monovalued iff it exists a function (monovalued binary relation) $F\in\operatorname{up}f$.
2. A reloid f is a injective iff it exists an injective binary relation $F\in\operatorname{up}f$.
3. A reloid f is a both monovalued and injective iff exists an injection (a monovalued and injective binary relation = injective function) $F\in\operatorname{up}f$.

Proof. The reverse implications are obvious. Let's prove the direct implications:

1. Let f is a monovalued reloid. Then $f\circ f^{-1}\subseteq I^{\mathsf{RLD}(\operatorname{Dst}f)}$. So exists
$$h\in\operatorname{up}(f\circ f^{-1})=\operatorname{up}\bigcap\left\{\uparrow^{\mathsf{RLD}(\operatorname{Dst}f;\operatorname{Dst}f)}(F\circ F^{-1})\mid F\in\operatorname{up}f\right\}$$
such that $h\subseteq I^{\mathsf{RLD}(\operatorname{Dst}f)}$. It's simple to show that $\{F\circ F^{-1}\mid F\in\operatorname{up}f\}$ is a filter base. Consequently it exists $F\in\operatorname{up}f$ such that $F\circ F^{-1}\subseteq I_{\operatorname{Dst}f}$ that is F is a function.
2. Similar.
3. Let f is a both monovalued and injective reloid. Then by proved above there exist F, $G\in\operatorname{up}f$ such that F is monovalued and G is injective. Thus $F\cap G\in\operatorname{up}f$ is both monovalued and injective. □

Conjecture 181. A reloid f is monovalued iff
$$\forall g\in\mathsf{RLD}(\operatorname{Src}f;\operatorname{Dst}f):(g\subseteq f\Rightarrow\exists\mathcal{A}\in\mathfrak{F}(\operatorname{Src}f):g=f|_{\mathcal{A}}).$$

4.6 Complete reloids and completion of reloids

Definition 182. A *complete* reloid is a reloid representable as join of direct products $\uparrow^A\{\alpha\}\times^{\mathsf{RLD}}b$ where $\alpha\in A$ and b is an atomic f.o. on B for some small sets A and B.

Definition 183. A *co-complete* reloid is a reloid representable as join of direct products $a\times^{\mathsf{RLD}}\uparrow^B\{\beta\}$ where $\beta\in B$ and a is an atomic f.o. on A for some small sets A and B.

I will denote the sets of complete and co-complete reloids correspondingly as $\mathrm{Compl}\mathsf{RLD}$ and $\mathrm{CoCompl}\mathsf{RLD}$.

Obvious 184. Complete and co-complete are dual.

Theorem 185.

1. A reloid f is complete iff there exists a function $G: \operatorname{Src} f \to \mathfrak{F}(\operatorname{Dst} f)$ such that
$$f = \bigcup \{\uparrow^{\operatorname{Src} f}\{\alpha\} \times^{\mathsf{RLD}} G(\alpha) \mid \alpha \in \operatorname{Src} f\}. \tag{12}$$
2. A reloid f is co-complete iff there exists a function $G: \operatorname{Dst} f \to \mathfrak{F}(\operatorname{Src} f)$ such that
$$f = \bigcup \{G(\alpha) \times^{\mathsf{RLD}} \uparrow^{\operatorname{Dst} f}\{\alpha\} \mid \alpha \in \operatorname{Dst} f\}.$$

Proof. We will prove only the first as the second is symmetric.

⇒. Let f is complete. Then take
$$G(\alpha) = \bigcup \left\{b \in \operatorname{atoms} 1^{\mathfrak{F}(\operatorname{Dst} f)} \mid \exists \alpha \in \operatorname{Src} f: \uparrow^{\operatorname{Src} f}\{\alpha\} \times^{\mathsf{RLD}} b \subseteq f\right\}$$
and we have (12) obviously.

⇐. Let (12) holds. Then $G(\alpha) = \bigcup \operatorname{atoms} G(\alpha)$ and thus
$$f = \bigcup \{\uparrow^{\operatorname{Src} f}\{\alpha\} \times^{\mathsf{RLD}} b \mid \alpha \in \operatorname{Src} f, b \in \operatorname{atoms} G(\alpha)\}$$
and so f is complete. □

Obvious 186. Complete and co-complete reloids are convex.

Obvious 187. Discrete reloids are complete and co-complete.

Obvious 188. Join (on the lattice of reloids) of complete reloids is complete.

Corollary 189. $\mathrm{Compl}\mathsf{RLD}$ (with the induced order) is a complete lattice.

Theorem 190. A reloid which is both complete and co-complete is discrete.

Proof. Let f is a complete and co-complete reloid. We have
$$f = \bigcup \{\uparrow^{\operatorname{Src} f}\{\alpha\} \times^{\mathsf{RLD}} G(\alpha) \mid \alpha \in \operatorname{Src} f\} \quad \text{and} \quad f = \bigcup \{H(\beta) \times^{\mathsf{RLD}} \uparrow^{\operatorname{Dst} f}\{\beta\} \mid \beta \in \operatorname{Dst} f\}$$
for some functions $G: \operatorname{Src} f \to \mathfrak{F}(\operatorname{Dst} f)$, $H: \operatorname{Dst} f \to \mathfrak{F}(\operatorname{Src} f)$. For every $\alpha \in \operatorname{Src} f$ we have
$$\begin{aligned}
G(\alpha) &= \\
\operatorname{im} f|_{\uparrow^{\operatorname{Src} f}\{\alpha\}} &= \\
\operatorname{im}\left(f \cap \left(\uparrow^{\operatorname{Src} f}\{\alpha\} \times^{\mathsf{RLD}} 1^{\mathfrak{F}(\operatorname{Dst} f)}\right)\right) &= (*) \\
\operatorname{im} \bigcup \left\{(H(\beta) \times^{\mathsf{RLD}} \uparrow^{\operatorname{Dst} f}\{\beta\}) \cap \left(\uparrow^{\operatorname{Src} f}\{\alpha\} \times^{\mathsf{RLD}} 1^{\mathfrak{F}(\operatorname{Dst} f)}\right) \mid \beta \in \operatorname{Dst} f\right\} &= \\
\operatorname{im} \bigcup \{(H(\beta) \cap \uparrow^{\operatorname{Src} f}\{\alpha\}) \times^{\mathsf{RLD}} \uparrow^{\operatorname{Dst} f}\{\beta\} \mid \beta \in \operatorname{Dst} f\} &= \\
\operatorname{im} \bigcup \left\{\left(\begin{cases} \uparrow^{\operatorname{Src} f}\{\alpha\} \times \uparrow^{\operatorname{Dst} f}\{\beta\} & \text{if } H(\beta) \not\asymp \uparrow^{\operatorname{Src} f}\{\alpha\} \\ 0^{\mathsf{RLD}(\operatorname{Src} f; \operatorname{Dst} f)} & \text{if } H(\beta) \asymp \uparrow^{\operatorname{Src} f}\{\alpha\} \end{cases}\right) \mid \beta \in \operatorname{Dst} f\right\} &= \\
\operatorname{im} \bigcup \{\uparrow^{\operatorname{Src} f}\{\alpha\} \times \uparrow^{\operatorname{Dst} f}\{\beta\} \mid \beta \in \operatorname{Dst} f, H(\beta) \not\asymp \uparrow^{\operatorname{Src} f}\{\alpha\}\} &= \\
\operatorname{im} \bigcup \{\uparrow^{\mathsf{RLD}(\operatorname{Src} f; \operatorname{Dst} f)}\{(\alpha; \beta)\} \mid \beta \in \operatorname{Dst} f, H(\beta) \not\asymp \uparrow^{\operatorname{Src} f}\{\alpha\}\} &= \\
\bigcup \{\uparrow^{\operatorname{Dst} f}\{\beta\} \mid \beta \in \operatorname{Dst} f, H(\beta) \not\asymp \uparrow^{\operatorname{Src} f}\{\alpha\}\}&.
\end{aligned}$$

* the theorem 40 from [15] was used.

Thus $G(\alpha)$ is a principal f.o. that is $G(\alpha) = \uparrow^{\operatorname{Dst} f} g(\alpha)$ for some g: $\operatorname{Src} f \to \operatorname{Dst} f$; $\uparrow^{\operatorname{Src} f}\{\alpha\} \times^{\mathsf{RLD}} G(\alpha) = \uparrow^{\mathsf{RLD}(\operatorname{Src} f; \operatorname{Dst} f)}(\{\alpha\} \times g(\alpha))$; f is discrete as a join of discrete reloids. □

Conjecture 191. Composition of complete reloids is complete.

Theorem 192.

1. For a complete reloid f there exist exactly one function $F \in \mathfrak{F}(\operatorname{Dst} f)^{\operatorname{Src} f}$ such that
$$f = \bigcup \{\uparrow^{\operatorname{Src} f}\{\alpha\} \times^{\mathsf{RLD}} F(\alpha) \mid \alpha \in \operatorname{Src} f\}.$$
2. For a co-complete reloid f there exist exactly one function $F \in \mathfrak{F}(\operatorname{Src} f)^{\operatorname{Dst} f}$ such that
$$f = \bigcup \{F(\alpha) \times^{\mathsf{RLD}} \uparrow^{\operatorname{Dst} f}\{\alpha\} \mid \alpha \in \operatorname{Dst} f\}.$$

Proof. We will prove only the first as the second is similar. Let
$$f = \bigcup \{\uparrow^{\operatorname{Src} f}\{\alpha\} \times^{\mathsf{RLD}} F(\alpha) \mid \alpha \in \operatorname{Src} f\} = \bigcup \{\uparrow^{\operatorname{Src} f}\{\alpha\} \times^{\mathsf{RLD}} G(\alpha) \mid \alpha \in \operatorname{Src} f\}$$
for some $F, G \in \mathfrak{F}(\operatorname{Dst} f)^{\operatorname{Src} f}$. We need to prove $F = G$. Let $\beta \in \operatorname{Src} f$.
$$\begin{array}{rcl} f \cap \left(\uparrow^{\operatorname{Src} f}\{\beta\} \times^{\mathsf{RLD}} 1^{\mathfrak{F}(\operatorname{Dst} f)}\right) & = & \text{(theorem 40 in [15])} \\ \bigcup^{\mathsf{RLD}}\left\{(\uparrow^{\operatorname{Src} f}\{\alpha\} \times^{\mathsf{RLD}} F(\alpha)) \cap^{\mathsf{RLD}} \left(\uparrow^{\operatorname{Src} f}\{\beta\} \times 1^{\mathfrak{F}(\operatorname{Dst} f)}\right) \mid \alpha \in \operatorname{Src} f\right\} & = & \\ \uparrow^{\operatorname{Src} f}\{\beta\} \times^{\mathsf{RLD}} F(\beta). & & \end{array}$$
Similarly $f \cap \left(\uparrow^{\operatorname{Src} f}\{\beta\} \times 1^{\mathfrak{F}(\operatorname{Dst} f)}\right) = \uparrow^{\operatorname{Src} f}\{\beta\} \times^{\mathsf{RLD}} G(\beta)$. Thus $\uparrow^{\operatorname{Src} f}\{\beta\} \times^{\mathsf{RLD}} F(\beta) = \uparrow^{\operatorname{Src} f}\{\beta\} \times^{\mathsf{RLD}} G(\beta)$ and so $F(\beta) = G(\beta)$. □

Definition 193. *Completion* and *co-completion* of a reloid $f \in \mathsf{RLD}(A; B)$ are defined by the formulas:
$$\operatorname{Compl} f = \operatorname{Cor}^{(\mathsf{RLD}(A;B);\operatorname{Compl}\mathsf{RLD}(A;B))} f \quad \text{and} \quad \operatorname{CoCompl} f = \operatorname{Cor}^{(\mathsf{RLD}(A;B);\operatorname{CoCompl}\mathsf{RLD}(A;B))} f.$$

Theorem 194. Atoms of the lattice $\operatorname{Compl}\mathsf{RLD}(A; B)$ are exactly direct products of the form $\uparrow^A\{\alpha\} \times^{\mathsf{RLD}} b$ where $\alpha \in A$ and b is an atomic f.o. on B.

Proof. First, it's easy to see that $\uparrow^A\{\alpha\} \times^{\mathsf{FCD}} b$ are elements of $\operatorname{Compl}\mathsf{RLD}(A; B)$. Also $0^{\mathsf{RLD}(A;B)}$ is an element of $\operatorname{Compl}\mathsf{RLD}$.

$\uparrow^A\{\alpha\} \times^{\mathsf{RLD}} b$ are atoms of $\operatorname{Compl}\mathsf{FCD}$ because these are atoms of RLD.

It remains to prove that if f is an atom of $\operatorname{Compl}\mathsf{RLD}(A; B)$ then $f = \uparrow^A\{\alpha\} \times^{\mathsf{RLD}} b$ for some $\alpha \in A$ and an atomic f.o. b on B.

Suppose f is a non-empty complete reloid. Then $\uparrow^A\{\alpha\} \times^{\mathsf{RLD}} b \subseteq f$ for some $\alpha \in A$ and atomic f.o. b on B. If f is an atom then $f = \uparrow^A\{\alpha\} \times^{\mathsf{FCD}} b$. □

Obvious 195. $\operatorname{Compl}\mathsf{RLD}$ is an atomistic lattice.

Proposition 196. $\operatorname{Compl} f = \bigcup \left\{ f|_{\uparrow^{\operatorname{Src} f}\{\alpha\}} \mid \alpha \in \operatorname{Src} f \right\}$ for every reloid f.

Proof. Let's denote R the right part of the equality to be proven.

That R is a complete reloid follows from the equality
$$f|_{\uparrow^{\operatorname{Src} f}\{\alpha\}} = \uparrow^{\operatorname{Src} f}\{\alpha\} \times^{\mathsf{RLD}} \operatorname{im}\left(f|_{\uparrow^{\operatorname{Src} f}\{\alpha\}}\right).$$
The only thing left to prove is that $g \subseteq R$ for every complete reloid g such that $g \subseteq f$.

Really let g is a complete reloid such that $g \subseteq f$. Then
$$g = \bigcup \{\uparrow^{\operatorname{Src} f}\{\alpha\} \times^{\mathsf{RLD}} G(\alpha) \mid \alpha \in \operatorname{Src} f\}$$
for some function $G: \operatorname{Src} f \to \mathfrak{F}(\operatorname{Dst} f)$.

We have $\uparrow^{\operatorname{Src} f}\{\alpha\} \times^{\mathsf{RLD}} G(\alpha) = g|_{\uparrow^{\operatorname{Src} f}\{\alpha\}} \subseteq f|_{\uparrow^{\operatorname{Src} f}\{\alpha\}}$. Thus $g \subseteq R$. □

Conjecture 197. $\operatorname{Compl} f \cap \operatorname{Compl} g = \operatorname{Compl}(f \cap g)$ for every reloids f and g.

Theorem 198. $\operatorname{Compl}(\bigcup R) = \bigcup \langle \operatorname{Compl} \rangle R$ for every set $R \in \mathsf{RLD}(A; B)$ for every small sets A, B.

Proof.

$$\begin{aligned}\operatorname{Compl}\left(\bigcup R\right) &= \\ \bigcup \left\{\left(\bigcup R\right)|_{\uparrow^A\{\alpha\}} \mid \alpha\in A\right\} &= \text{(theorem 40 in [15])} \\ \bigcup \left\{\bigcup \left\{f|_{\uparrow^A\{\alpha\}} \mid \alpha\in A\right\} \mid f\in R\right\} &= \\ \bigcup \langle \operatorname{Compl}\rangle R.\end{aligned}$$

□

Lemma 199. Completion of a co-complete reloid is discrete.

Proof. Let f is a co-complete reloid. Then there is a function $F\colon \operatorname{Dst} f\to \mathfrak{F}(\operatorname{Src} f)$ such that

$$f=\bigcup \left\{F(\alpha)\times^{\mathsf{RLD}}\uparrow^{\operatorname{Dst} f}\{\alpha\} \mid \alpha\in \operatorname{Dst} f\right\}.$$

So

$$\begin{aligned}\operatorname{Compl} f &= \\ \bigcup \left\{\left(\bigcup \left\{F(\alpha)\times^{\mathsf{RLD}}\uparrow^{\operatorname{Dst} f}\{\alpha\} \mid \alpha\in\operatorname{Dst} f\right\}\right)|_{\uparrow^{\operatorname{Src} f}\{\beta\}} \mid \beta\in\operatorname{Src} f\right\} &= \\ \bigcup \left\{\left(\bigcup \left\{F(\alpha)\times^{\mathsf{RLD}}\uparrow^{\operatorname{Dst} f}\{\alpha\} \mid \alpha\in\operatorname{Dst} f\right\}\right)\cap\left(\uparrow^{\operatorname{Src} f}\{\beta\}\times 1^{\mathfrak{F}(\operatorname{Dst} f)}\right) \mid \beta\in\operatorname{Src} f\right\} &= (*) \\ \bigcup \left\{\bigcup \left\{\left(F(\alpha)\times^{\mathsf{RLD}}\uparrow^{\operatorname{Dst} f}\{\alpha\}\right)\cap\left(\uparrow^{\operatorname{Src} f}\{\beta\}\times 1^{\mathfrak{F}(\operatorname{Dst} f)}\right) \mid \alpha\in\operatorname{Dst} f\right\} \mid \beta\in\operatorname{Src} f\right\} &= \\ \bigcup \left\{\bigcup \left\{\uparrow^{\operatorname{Src} f}\{\beta\}\times^{\mathsf{RLD}}\uparrow^{\operatorname{Dst} f}\{\alpha\} \mid \alpha\in\operatorname{Dst} f\right\} \mid \beta\in\operatorname{Src} f, \{\beta\}\subseteq F(\alpha)\right\}.\end{aligned}$$

* theorem 40 in [15].

Thus $\operatorname{Compl} f$ is discrete. □

Theorem 200. $\operatorname{Compl}\operatorname{CoCompl} f=\operatorname{CoCompl}\operatorname{Compl} f=\operatorname{Cor} f$ for every reloid f.

Proof. We will prove only $\operatorname{Compl}\operatorname{CoCompl} f=\operatorname{Cor} f$. The rest follows from symmetry.

From the lemma $\operatorname{Compl}\operatorname{CoCompl} f$ is discrete. It is obvious $\operatorname{Compl}\operatorname{CoCompl} f\subseteq f$. So to finish the proof we need to show only that for every discrete reloid $F\subseteq f$ we have $F\subseteq\operatorname{Compl}\operatorname{CoCompl} f$.

Really, obviously $F\subseteq\operatorname{CoCompl} f$ and thus $F=\operatorname{Compl} F\subseteq\operatorname{Compl}\operatorname{CoCompl} f$. □

Question 201. Is $\operatorname{Compl}\mathsf{RLD}(A;B)$ a distributive lattice? Is $\operatorname{Compl}\mathsf{RLD}(A;B)$ a co-brouwerian lattice?

Conjecture 202. Let A, B, C are small sets. If $f\in\mathsf{RLD}(B;C)$ is a complete reloid and $R\in\mathscr{P}\mathsf{RLD}(A;B)$ then

$$f\circ\bigcup R=\bigcup \langle f\circ\rangle R.$$

This conjecture can be weakened:

Conjecture 203. Let A, B, C are small sets. If $f\in\mathsf{RLD}(B;C)$ is a discrete reloid and $R\in\mathscr{P}\mathsf{RLD}(A;B)$ then

$$f\circ\bigcup R=\bigcup \langle f\circ\rangle R.$$

Conjecture 204. $\operatorname{Compl} f=f\setminus^*\left(\Omega^{\operatorname{Src} f}\times^{\mathsf{RLD}}1^{\mathfrak{F}(\operatorname{Dst} f)}\right)$ for every reloid f.

5 Relationships between funcoids and reloids

5.1 Funcoid induced by a reloid

Every reloid f induces a funcoid $(\mathsf{FCD})f\in\mathsf{FCD}(\operatorname{Src} f;\operatorname{Dst} f)$ by the following formulas (for every $\mathcal{X}\in\mathfrak{F}(\operatorname{Src} f)$, $\mathcal{Y}\in\mathfrak{F}(\operatorname{Dst} f)$):

$$\mathcal{X}\left[(\mathsf{FCD})f\right]\mathcal{Y}\Leftrightarrow\forall F\in\operatorname{up} f\colon\mathcal{X}\left[\uparrow^{\mathsf{FCD}(\operatorname{Src} f;\operatorname{Dst} f)}F\right]\mathcal{Y}$$
$$\langle(\mathsf{FCD})f\rangle\mathcal{X}=\bigcap \left\{\left\langle\uparrow^{\mathsf{FCD}(\operatorname{Src} f;\operatorname{Dst} f)}F\right\rangle\mathcal{X} \mid F\in\operatorname{up} f\right\}.$$

We should prove that $(\mathsf{FCD})f$ is really a funcoid.

Proof. We need to prove that

$$\mathcal{X}[(\mathsf{FCD})f]\mathcal{Y} \Leftrightarrow \mathcal{Y}\cap\langle(\mathsf{FCD})f\rangle\mathcal{X}\neq 0^{\mathfrak{F}(\mathrm{Dst}\,f)} \Leftrightarrow \mathcal{X}\cap\langle(\mathsf{FCD})f^{-1}\rangle\mathcal{Y}\neq 0^{\mathfrak{F}(\mathrm{Dst}\,f)}.$$

The above formula is equivalent to:

$$\begin{aligned}\forall F\in\mathrm{up}\,f\colon \mathcal{X}\left[\uparrow^{\mathsf{FCD}(\mathrm{Src}\,f;\mathrm{Dst}\,f)}F\right]\mathcal{Y} &\Leftrightarrow\\ \mathcal{Y}\cap\bigcap\left\{\left\langle\uparrow^{\mathsf{FCD}(\mathrm{Src}\,f;\mathrm{Dst}\,f)}F\right\rangle\mathcal{X}\mid F\in\mathrm{up}\,f\right\}\neq 0^{\mathfrak{F}(\mathrm{Dst}\,f)} &\Leftrightarrow\\ \mathcal{X}\cap\bigcap\left\{\left\langle\uparrow^{\mathsf{FCD}(\mathrm{Src}\,f;\mathrm{Dst}\,f)}F^{-1}\right\rangle\mathcal{Y}\mid F\in\mathrm{up}\,f\right\}\neq 0^{\mathfrak{F}(\mathrm{Src}\,f)}&\end{aligned}$$

We have $\mathcal{Y}\cap\bigcap\left\{\left\langle\uparrow^{\mathsf{FCD}(\mathrm{Src}\,f;\mathrm{Dst}\,f)}F\right\rangle\mathcal{X}\mid F\in\mathrm{up}\,f\right\}=\bigcap\left\{\mathcal{Y}\cap\left\langle\uparrow^{\mathsf{FCD}(\mathrm{Src}\,f;\mathrm{Dst}\,f)}F\right\rangle\mathcal{X}\mid F\in\mathrm{up}\,f\right\}$.

Let's denote $W=\left\{\mathcal{Y}\cap\left\langle\uparrow^{\mathsf{FCD}(\mathrm{Src}\,f;\mathrm{Dst}\,f)}F\right\rangle\mathcal{X}\mid F\in\mathrm{up}\,f\right\}$.

$\forall F\in\mathrm{up}\,f\colon \mathcal{X}\left[\uparrow^{\mathsf{FCD}(\mathrm{Src}\,f;\mathrm{Dst}\,f)}F\right]\mathcal{Y}\Leftrightarrow\forall F\in\mathrm{up}\,f\colon\mathcal{Y}\cap\left\langle\uparrow^{\mathsf{FCD}(\mathrm{Src}\,f;\mathrm{Dst}\,f)}F\right\rangle\mathcal{X}\neq 0^{\mathfrak{F}(\mathrm{Dst}\,f)}\Leftrightarrow 0^{\mathfrak{F}(\mathrm{Dst}\,f)}\notin W$.

We need to prove that $0^{\mathfrak{F}(\mathrm{Dst}\,f)}\notin W\Leftrightarrow\bigcap W\neq 0^{\mathfrak{F}(\mathrm{Dst}\,f)}$. (The rest follows from symmetry.)

This follows from the fact that W is a generalized filter base.

Let's prove that W is a generalized filter base. For this enough to prove that $V=\left\{\left\langle\uparrow^{\mathsf{FCD}(\mathrm{Src}\,f;\mathrm{Dst}\,f)}F\right\rangle\mathcal{X}\mid F\in\mathrm{up}\,f\right\}$ is a generalized filter base. Let $\mathcal{A},\mathcal{B}\in V$ that is $\mathcal{A}=\left\langle\uparrow^{\mathsf{FCD}(\mathrm{Src}\,f;\mathrm{Dst}\,f)}P\right\rangle\mathcal{X}$, $\mathcal{B}=\left\langle\uparrow^{\mathsf{FCD}(\mathrm{Src}\,f;\mathrm{Dst}\,f)}Q\right\rangle\mathcal{X}$ where $P,Q\in\mathrm{up}\,f$. Then for $\mathcal{C}=\left\langle\uparrow^{\mathsf{FCD}(\mathrm{Src}\,f;\mathrm{Dst}\,f)}(P\cap Q)\right\rangle\mathcal{X}$ is true both $\mathcal{C}\in V$ and $\mathcal{C}\subseteq\mathcal{A},\mathcal{B}$. So V is a generalized filter base and thus W is a generalized filter base. □

Proposition 205. $(\mathsf{FCD})\uparrow^{\mathsf{RLD}(A;B)}f=\uparrow^{\mathsf{FCD}(A;B)}f$ for every small sets A, B and binary relation $f\subseteq A\times B$.

Proof. $\mathcal{X}\left[(\mathsf{FCD})\uparrow^{\mathsf{RLD}(A;B)}f\right]\mathcal{Y}\Leftrightarrow\forall F\in\mathrm{up}\uparrow^{\mathsf{RLD}(A;B)}f\colon\mathcal{X}\left[\uparrow^{\mathsf{FCD}(A;B)}F\right]\mathcal{Y}\Leftrightarrow\mathcal{X}\left[\uparrow^{\mathsf{FCD}(A;B)}f\right]\mathcal{Y}$ (for every $\mathcal{X},\mathcal{Y}\in\mathfrak{F}$). □

Theorem 206. $\mathcal{X}[(\mathsf{FCD})f]\mathcal{Y}\Leftrightarrow(\mathcal{X}\times^{\mathsf{RLD}}\mathcal{Y})\not\asymp f$ for every $f\in\mathsf{RLD}$ and $\mathcal{X}\in\mathfrak{F}(\mathrm{Src}\,f)$, $\mathcal{Y}\in\mathfrak{F}(\mathrm{Dst}\,f)$.

Proof.

$$\begin{aligned}(\mathcal{X}\times^{\mathsf{RLD}}\mathcal{Y})\not\asymp f &\Leftrightarrow \forall F\in\mathrm{up}\,f,P\in\mathrm{up}(\mathcal{X}\times^{\mathsf{RLD}}\mathcal{Y})\colon P\not\asymp F\\ &\Leftrightarrow \forall F\in\mathrm{up}\,f,X\in\mathrm{up}\,\mathcal{X},Y\in\mathrm{up}\,\mathcal{Y}\colon(X\times Y)\not\asymp F\\ &\Leftrightarrow \forall F\in\mathrm{up}\,f,X\in\mathrm{up}\,\mathcal{X},Y\in\mathrm{up}\,\mathcal{Y}\colon\uparrow^{\mathrm{Src}\,f}X\left[\uparrow^{\mathsf{FCD}(\mathrm{Src}\,f;\mathrm{Dst}\,f)}F\right]\uparrow^{\mathrm{Dst}\,f}Y\\ &\Leftrightarrow \forall F\in\mathrm{up}\,f\colon\mathcal{X}\left[\uparrow^{\mathsf{FCD}(\mathrm{Src}\,f;\mathrm{Dst}\,f)}F\right]\mathcal{Y}\\ &\Leftrightarrow \mathcal{X}[(\mathsf{FCD})f]\mathcal{Y}.\end{aligned}$$

□

Theorem 207. $(\mathsf{FCD})f=\bigcap\left\langle\uparrow^{\mathsf{FCD}(\mathrm{Src}\,f;\mathrm{Dst}\,f)}\right\rangle\mathrm{up}\,f$ for every reloid f.

Proof. Let a is an atomic filter object.

$\langle(\mathsf{FCD})f\rangle a=\bigcap\left\{\left\langle\uparrow^{\mathsf{FCD}(\mathrm{Src}\,f;\mathrm{Dst}\,f)}F\right\rangle a\mid F\in\mathrm{up}\,f\right\}$ by the definition of (FCD).

$\left\langle\bigcap\left\langle\uparrow^{\mathsf{FCD}(\mathrm{Src}\,f;\mathrm{Dst}\,f)}\right\rangle\mathrm{up}\,f\right\rangle a=\bigcap\left\{\left\langle\uparrow^{\mathsf{FCD}(\mathrm{Src}\,f;\mathrm{Dst}\,f)}F\right\rangle a\mid F\in\mathrm{up}\,f\right\}$ by the theorem 79.

So $\langle(\mathsf{FCD})f\rangle a=\left\langle\bigcap\left\langle\uparrow^{\mathsf{FCD}(\mathrm{Src}\,f;\mathrm{Dst}\,f)}\right\rangle\mathrm{up}\,f\right\rangle a$ for every atomic filter object a. □

Lemma 208. For every two filter bases S and T of binary relations on $U\times V$ for some small sets U, V and every set $A\subseteq U$

$$\bigcap\uparrow^{\mathsf{RLD}(U;V)}S=\bigcap\uparrow^{\mathsf{RLD}(U;V)}T\Rightarrow\bigcap\{\uparrow^V\langle F\rangle A\mid F\in S\}=\bigcap\{\uparrow^V\langle G\rangle A\mid G\in T\}$$

Proof. Let $\bigcap\uparrow^{\mathsf{RLD}(U;V)}S=\bigcap\uparrow^{\mathsf{RLD}(U;V)}T$.

First let prove that $\{\langle F\rangle A\mid F\in S\}$ is a filter base. Let $X,Y\in\{\langle F\rangle A\mid F\in S\}$. Then $X=\langle F_X\rangle A$ and $Y=\langle F_Y\rangle A$ for some $F_X,F_Y\in S$. Because S is a filter base, we have $S\ni F_Z\subseteq F_X\cap F_Y$. So $\langle F_Z\rangle A\subseteq X\cap Y$ and $\langle F_Z\rangle A\in\{\langle F\rangle A\mid F\in S\}$. So $\{\langle F\rangle A\mid F\in S\}$ is a filter base.

Suppose $X \in \operatorname{up} \bigcap \{\uparrow^V \langle F \rangle A \mid F \in S\}$. Then exists $X' \in \{\langle F \rangle A \mid F \in S\}$ where $X \supseteq X'$ because $\{\langle F \rangle A \mid F \in S\}$ is a filter base. That is $X' = \langle F \rangle A$ for some $F \in S$. There exists $G \in T$ such that $G \subseteq F$ because T is a filter base. Let $Y' = \langle G \rangle A$. We have $Y' \subseteq X' \subseteq X$; $Y' \in \{\langle G \rangle A \mid G \in T\}$; $Y' \in \operatorname{up} \bigcap \{\uparrow^V \langle G \rangle A \mid G \in T\}$; $X \in \operatorname{up} \bigcap \{\uparrow^V \langle G \rangle A \mid G \in T\}$. The reverse is symmetric. □

Lemma 209. $\{G \circ F \mid F \in \operatorname{up} f, G \in \operatorname{up} g\}$ is a filter base for every reloids f and g.

Proof. Let denote $D = \{G \circ F \mid F \in \operatorname{up} f, G \in \operatorname{up} g\}$. Let $A \in D \wedge B \in D$. Then $A = G_A \circ F_A \wedge B = G_B \circ F_B$ for some $F_A, F_B \in \operatorname{up} f$ and $G_A, G_B \in \operatorname{up} g$. So $A \cap B \supseteq (G_A \cap G_B) \circ (F_A \cap F_B) \in D$ because $F_A \cap F_B \in \operatorname{up} f$ and $G_A \cap G_B \in \operatorname{up} g$. □

Theorem 210. $(\mathsf{FCD})(g \circ f) = ((\mathsf{FCD})g) \circ ((\mathsf{FCD})f)$ for every composable reloids f and g.

Proof.

$$\begin{aligned}\langle (\mathsf{FCD})(g \circ f)\rangle^* X &= \bigcap \{\uparrow^{\operatorname{Dst} g} \langle H \rangle X \mid H \in \operatorname{up}(g \circ f)\}\\ &= \bigcap \left\{\uparrow^{\operatorname{Dst} g} \langle H \rangle X \mid H \in \operatorname{up} \bigcap \left\{\uparrow^{\mathsf{RLD}(\operatorname{Src} f; \operatorname{Dst} g)}(G \circ F) \mid F \in \operatorname{up} f, G \in \operatorname{up} g\right\}\right\}.\end{aligned}$$

Obviously

$$\bigcap \left\{\uparrow^{\mathsf{RLD}(\operatorname{Src} f; \operatorname{Dst} g)}(G \circ F) \mid F \in \operatorname{up} f, G \in \operatorname{up} g\right\} = \bigcap \left\langle \uparrow^{\mathsf{RLD}(\operatorname{Src} f; \operatorname{Dst} g)}\right\rangle \operatorname{up} \bigcap \left\{\uparrow^{\mathsf{RLD}(\operatorname{Src} f; \operatorname{Dst} g)}(G \circ F) \mid F \in \operatorname{up} f, G \in \operatorname{up} g\right\};$$

from this by the lemma 208 (taking in account that $\{G \circ F \mid F \in \operatorname{up} f, G \in \operatorname{up} g\}$ and $\operatorname{up} \bigcap \left\{\uparrow^{\mathsf{RLD}(\operatorname{Src} f; \operatorname{Dst} g)}(G \circ F) \mid F \in \operatorname{up} f, G \in \operatorname{up} g\right\}$ are filter bases)

$$\bigcap \left\{\uparrow^{\operatorname{Dst} g} \langle H \rangle X \mid H \in \operatorname{up} \bigcap \left\{\uparrow^{\mathsf{RLD}(\operatorname{Src} f; \operatorname{Dst} g)}(G \circ F) \mid F \in \operatorname{up} f, G \in \operatorname{up} g\right\}\right\} = \bigcap \{\uparrow^{\operatorname{Dst} g} \langle G \circ F \rangle X \mid F \in \operatorname{up} f, G \in \operatorname{up} g\}.$$

On the other side

$$\begin{aligned}\langle ((\mathsf{FCD})g) \circ ((\mathsf{FCD})f)\rangle^* X &= \langle (\mathsf{FCD})g\rangle \langle (\mathsf{FCD})f\rangle^* X\\ &= \langle (\mathsf{FCD})g\rangle \bigcap \{\uparrow^{\operatorname{Dst} f} \langle F \rangle X \mid F \in \operatorname{up} f\}\\ &= \bigcap \left\{\left\langle \uparrow^{\mathsf{FCD}(\operatorname{Src} g; \operatorname{Dst} g)} G\right\rangle \bigcap \{\uparrow^{\operatorname{Dst} f} \langle F \rangle X \mid F \in \operatorname{up} f\} \mid G \in \operatorname{up} g\right\}.\end{aligned}$$

Let's prove that $\{\langle F \rangle X \mid F \in \operatorname{up} f\}$ is a filter base. If $A, B \in \{\langle F \rangle X \mid F \in \operatorname{up} f\}$ then $A = \langle F_1 \rangle X$ and $B = \langle F_2 \rangle X$ where $F_1, F_2 \in \operatorname{up} f$. $A \cap B \supseteq \langle F_1 \cap F_2 \rangle X \in \{\langle F \rangle X \mid F \in \operatorname{up} f\}$. So $\{\langle F \rangle X \mid F \in \operatorname{up} f\}$ is really a filter base.

By the theorem 52 we have

$$\left\langle \uparrow^{\mathsf{FCD}(\operatorname{Src} g; \operatorname{Dst} g)} G\right\rangle \bigcap \{\uparrow^{\operatorname{Dst} f} \langle F \rangle X \mid F \in \operatorname{up} f\} = \bigcap \{\uparrow^{\operatorname{Dst} g} \langle G \rangle \langle F \rangle X \mid F \in \operatorname{up} f\}.$$

So continuing the above equalities,

$$\begin{aligned}\langle ((\mathsf{FCD})g) \circ ((\mathsf{FCD})f)\rangle^* X &= \bigcap \left\{\bigcap \{\uparrow^{\operatorname{Dst} g} \langle G \rangle \langle F \rangle X \mid F \in \operatorname{up} f\} \mid G \in \operatorname{up} g\right\}\\ &= \bigcap \{\uparrow^{\operatorname{Dst} g} \langle G \rangle \langle F \rangle X \mid F \in \operatorname{up} f, G \in \operatorname{up} g\}\\ &= \bigcap \{\uparrow^{\operatorname{Dst} g} \langle G \circ F \rangle X \mid F \in \operatorname{up} f, G \in \operatorname{up} g\}.\end{aligned}$$

Combining these equalities we get $\langle (\mathsf{FCD})(g \circ f)\rangle^* X = \langle ((\mathsf{FCD})g) \circ ((\mathsf{FCD})f)\rangle^* X$ for every set X. □

Corollary 211.

1. $(\mathsf{FCD})f$ is a monovalued funcoid if f is a monovalued reloid.
2. $(\mathsf{FCD})f$ is an injective funcoid if f is an injective reloid.

Proof. We will prove only the first as the second is dual. Let f is a monovalued reloid. Then $f \circ f^{-1} \subseteq I^{\mathsf{RLD}(\operatorname{Dst} f)}$; $(\mathsf{FCD})(f \circ f^{-1}) \subseteq I^{\mathsf{FCD}(\operatorname{Dst} f)}$; $(\mathsf{FCD})f \circ ((\mathsf{FCD})f)^{-1} \subseteq I^{\mathsf{FCD}(\operatorname{Dst} f)}$ that is $(\mathsf{FCD})f$ is *a* monovalued funcoid. □

Proposition 212. $(\mathsf{FCD})I^{\mathsf{RLD}}_{\mathcal{A}} = I^{\mathsf{FCD}}_{\mathcal{A}}$ for every f.o. $\mathcal{A}$.

Proof. Recall that $I^{\mathsf{RLD}}_{\mathcal{A}} = \bigcap \left\{ \uparrow^{\mathrm{Base}(\mathcal{A})} I_A \mid A \in \mathrm{up}\mathcal{A} \right\}$. For every $\mathcal{X}, \mathcal{Y} \in \mathfrak{F}(\mathrm{Base}(\mathcal{A}))$ we have:

$\mathcal{X} [(\mathsf{FCD}) I^{\mathsf{RLD}}_{\mathcal{A}}] \mathcal{Y} \Leftrightarrow \mathcal{X} \times^{\mathsf{RLD}} \mathcal{Y} \not\asymp I^{\mathsf{RLD}}_{\mathcal{A}} \Leftrightarrow \forall A \in \mathrm{up}\,\mathcal{A}: \mathcal{X} \times^{\mathsf{RLD}} \mathcal{Y} \not\asymp \uparrow^{\mathsf{RLD}(\mathrm{Base}(\mathcal{A}); \mathrm{Base}(\mathcal{A}))} I_A \Leftrightarrow \forall A \in \mathrm{up}\,\mathcal{A}: \mathcal{X} \left[\uparrow^{\mathsf{RLD}(\mathrm{Base}(\mathcal{A}); \mathrm{Base}(\mathcal{A}))} I_A \right] \mathcal{Y} \Leftrightarrow \forall A \in \mathrm{up}\,\mathcal{A}: \mathcal{X} \cap \mathcal{Y} \not\asymp \uparrow^{\mathrm{Base}(\mathcal{A})} A \Leftrightarrow \mathcal{X} \cap \mathcal{Y} \not\asymp \mathcal{A} \Leftrightarrow \mathcal{X} [(\mathsf{FCD}) I^{\mathsf{FCD}}_{\mathcal{A}}] \mathcal{Y}$ (used properties of generalized filter bases). □

Proposition 213. $(\mathsf{FCD})(\mathcal{A} \times^{\mathsf{RLD}} \mathcal{B}) = \mathcal{A} \times^{\mathsf{FCD}} \mathcal{B}$ for every f.o. $\mathcal{A}$ and $\mathcal{B}$.

Proof. $\mathcal{X} [(\mathsf{FCD})(\mathcal{A} \times^{\mathsf{RLD}} \mathcal{B})] \mathcal{Y} \Leftrightarrow \forall F \in \mathrm{up}(\mathcal{A} \times^{\mathsf{RLD}} \mathcal{B}): \mathcal{X} \left[\uparrow^{\mathsf{FCD}(\mathrm{Base}(\mathcal{A}); \mathrm{Base}(\mathcal{B}))} F \right] \mathcal{Y}$ (for every $\mathcal{X}$, $\mathcal{Y} \in \mathfrak{F}$).

Evidently

$\forall F \in \mathrm{up}(\mathcal{A} \times^{\mathsf{RLD}} \mathcal{B}): \mathcal{X} \left[\uparrow^{\mathsf{FCD}(\mathrm{Base}(\mathcal{A}); \mathrm{Base}(\mathcal{B}))} F \right] \mathcal{Y} \Rightarrow \forall A \in \mathrm{up}\,\mathcal{A}, B \in \mathrm{up}\,\mathcal{B}: \mathcal{X} \left[\uparrow^{\mathsf{FCD}(\mathrm{Base}(\mathcal{A}); \mathrm{Base}(\mathcal{B}))} (A \times B) \right] \mathcal{Y}$.

Let $\forall A \in \mathrm{up}\,\mathcal{A}, B \in \mathrm{up}\,\mathcal{B}: \mathcal{X} \left[\uparrow^{\mathsf{FCD}(\mathrm{Base}(\mathcal{A}); \mathrm{Base}(\mathcal{B}))} (A \times B) \right] \mathcal{Y}$. Then if $F \in \mathrm{up}(\mathcal{A} \times^{\mathsf{RLD}} \mathcal{B})$ then there are $A \in \mathrm{up}\,\mathcal{A}$, $B \in \mathrm{up}\,\mathcal{B}$ such that $F \supseteq A \times B$. So $\mathcal{X} \left[\uparrow^{\mathsf{FCD}(\mathrm{Base}(\mathcal{A}); \mathrm{Base}(\mathcal{B}))} F \right] \mathcal{Y}$.

We proved $\forall F \in \mathrm{up}(\mathcal{A} \times^{\mathsf{RLD}} \mathcal{B}): \mathcal{X} \left[\uparrow^{\mathsf{FCD}(\mathrm{Base}(\mathcal{A}); \mathrm{Base}(\mathcal{B}))} F \right] \mathcal{Y} \Leftrightarrow \forall A \in \mathrm{up}\,\mathcal{A}, B \in \mathrm{up}\,\mathcal{B}: \mathcal{X} \left[\uparrow^{\mathsf{FCD}(\mathrm{Base}(\mathcal{A}); \mathrm{Base}(\mathcal{B}))} (A \times B) \right] \mathcal{Y}$.

Further $\forall A \in \mathrm{up}\,\mathcal{A}, B \in \mathrm{up}\,\mathcal{B}: \mathcal{X} \left[\uparrow^{\mathsf{FCD}(\mathrm{Base}(\mathcal{A}); \mathrm{Base}(\mathcal{B}))} (A \times B) \right] \mathcal{Y} \Leftrightarrow \forall A \in \mathrm{up}\,\mathcal{A}, B \in \mathrm{up}\,\mathcal{B}: \mathcal{X} \not\asymp \uparrow^{\mathrm{Base}(\mathcal{A})} A \wedge \mathcal{Y} \not\asymp \uparrow^{\mathrm{Base}(\mathcal{B})} B \Leftrightarrow \mathcal{X} \not\asymp \mathcal{A} \wedge \mathcal{Y} \not\asymp \mathcal{B} \Leftrightarrow \mathcal{X} [\mathcal{A} \times^{\mathsf{FCD}} \mathcal{B}] \mathcal{Y}$.

Thus $\mathcal{X} [(\mathsf{FCD})(\mathcal{A} \times^{\mathsf{RLD}} \mathcal{B})] \mathcal{Y} \Leftrightarrow \mathcal{X} [\mathcal{A} \times^{\mathsf{FCD}} \mathcal{B}] \mathcal{Y}$. □

Proposition 214. $\mathrm{dom}\,(\mathsf{FCD}) f = \mathrm{dom}\, f$ and $\mathrm{im}\,(\mathsf{FCD}) f = \mathrm{im}\, f$ for every reloid f.

Proof. $\mathrm{im}\,(\mathsf{FCD}) f = \langle (\mathsf{FCD}) f \rangle 1^{\mathfrak{F}(\mathrm{Src}\, f)} = \bigcap \{ \uparrow^{\mathrm{Dst}\, f} \langle F \rangle (\mathrm{Src}\, f) \mid F \in \mathrm{up}\, f \} = \bigcap \{ \uparrow^{\mathrm{Dst}\, f} \mathrm{im}\, F \mid F \in \mathrm{up}\, f \} = \bigcap \langle \uparrow^{\mathrm{Dst}\, f} \rangle \langle \mathrm{im} \rangle \mathrm{up}\, f = \mathrm{im}\, f$.

$\mathrm{dom}\,(\mathsf{FCD}) f = \mathrm{dom}\, f$ is similar. □

Proposition 215. $(\mathsf{FCD})(f \cap (\mathcal{A} \times^{\mathsf{RLD}} \mathcal{B})) = (\mathsf{FCD}) f \cap (\mathcal{A} \times^{\mathsf{FCD}} \mathcal{B})$ for every reloid f and $\mathcal{A} \in \mathfrak{F}(\mathrm{Src}\, f)$ and $\mathcal{B} \in \mathfrak{F}(\mathrm{Dst}\, f)$.

Proof. $(\mathsf{FCD})(f \cap (\mathcal{A} \times^{\mathsf{RLD}} \mathcal{B})) = (\mathsf{FCD})(I^{\mathsf{RLD}}_{\mathcal{B}} \circ f \circ I^{\mathsf{RLD}}_{\mathcal{A}}) = (\mathsf{FCD}) I^{\mathsf{RLD}}_{\mathcal{B}} \circ (\mathsf{FCD}) f \circ (\mathsf{FCD}) I^{\mathsf{RLD}}_{\mathcal{A}} = I^{\mathsf{FCD}}_{\mathcal{B}} \circ (\mathsf{FCD}) f \circ I^{\mathsf{FCD}}_{\mathcal{A}} = (\mathsf{FCD}) f \cap (\mathcal{A} \times^{\mathsf{FCD}} \mathcal{B})$. □

Corollary 216. $(\mathsf{FCD})(f|_{\mathcal{A}}) = ((\mathsf{FCD}) f)|_{\mathcal{A}}$ for every reloid f and f.o. $\mathcal{A}$.

Proposition 217. $\langle (\mathsf{FCD}) f \rangle \mathcal{X} = \mathrm{im}(f|_{\mathcal{X}})$ for every reloid f and f.o. $\mathcal{X}$.

Proof. $\mathrm{im}(f|_{\mathcal{X}}) = \mathrm{im}\,(\mathsf{FCD})(f|_{\mathcal{X}}) = \mathrm{im}\,((\mathsf{FCD}) f)|_{\mathcal{X}}) = \langle (\mathsf{FCD}) f \rangle \mathcal{X}$. □

5.2 Reloids induced by funcoid

Every funcoid $f \in \mathsf{FCD}(A; B)$ induces a reloid from A to B in two ways, intersection of *outward* relations and union of *inward* direct products of filter objects:

$$(\mathsf{RLD})_{\mathrm{out}} f \overset{\mathrm{def}}{=} \bigcap \langle \uparrow^{\mathsf{RLD}(A;B)} \rangle \mathrm{up}\, f;$$
$$(\mathsf{RLD})_{\mathrm{in}} f \overset{\mathrm{def}}{=} \bigcup \{ \mathcal{A} \times^{\mathsf{RLD}} \mathcal{B} \mid \mathcal{A} \in \mathfrak{F}(A), \mathcal{B} \in \mathfrak{F}(B), \mathcal{A} \times^{\mathsf{FCD}} \mathcal{B} \subseteq f \}$$

Theorem 218. $(\mathsf{RLD})_{\mathrm{in}} f = \bigcup^{\mathsf{RLD}} \left\{ a \times^{\mathsf{RLD}} b \mid a \in \mathrm{atoms}\, 1^{\mathfrak{F}(\mathrm{Src}\, f)}, b \in \mathrm{atoms}\, 1^{\mathfrak{F}(\mathrm{Dst}\, f)}, a \times^{\mathsf{FCD}} b \subseteq f \right\}$.

Proof. Follows from the theorem 161. □

Remark 219. In seems that $(\mathsf{RLD})_{\mathrm{in}}$ has smoother properties and is more important than $(\mathsf{RLD})_{\mathrm{out}}$. (However see also the exercize below for $(\mathsf{RLD})_{\mathrm{in}}$ not preserving identities.)

Proposition 220. $(\mathsf{RLD})_{\mathrm{out}} \uparrow^{\mathsf{FCD}(A;B)} f = \uparrow^{\mathsf{RLD}(A;B)} f$ for every small sets A, B and binary relation $f \subseteq A \times B$.

Proof. $(\mathsf{RLD})_{\mathrm{out}}\uparrow^{\mathsf{FCD}(A;B)}f = \bigcap \langle\uparrow^{\mathsf{RLD}(A;B)}\rangle \mathrm{up}\uparrow^{\mathsf{FCD}(A;B)}f = \uparrow^{\mathsf{RLD}(A;B)}\min\ \mathrm{up}\uparrow^{\mathsf{FCD}(A;B)}f = \uparrow^{\mathsf{RLD}(A;B)}f$. □

Lemma 221. $F\in \mathrm{up}\,(\mathsf{RLD})_{\mathrm{in}}f \Leftrightarrow \forall a\in \mathrm{atoms}\,1^{\mathfrak{F}(\mathrm{Src}\,f)}, b\in \mathrm{atoms}\,1^{\mathfrak{F}(\mathrm{Dst}\,f)} : (a[f]b \Rightarrow \uparrow^{\mathsf{RLD}(\mathrm{Src}\,f;\mathrm{Dst}\,f)}F \supseteq a\times^{\mathsf{RLD}} b)$ for a funcoid f.

Proof.

$$\begin{aligned} F\in \mathrm{up}\,(\mathsf{RLD})_{\mathrm{in}}f &\Leftrightarrow F\in \mathrm{up}\bigcup \{a\times^{\mathsf{RLD}}b \mid a\in \mathrm{atoms}\,1^{\mathfrak{F}(\mathrm{Src}\,f)}, b\in\mathrm{atoms}\,1^{\mathfrak{F}(\mathrm{Dst}\,f)}, a\times^{\mathsf{FCD}} b\subseteq f\}\\ &\Leftrightarrow \forall a\in \mathrm{atoms}\,1^{\mathfrak{F}(\mathrm{Src}\,f)}, b\in\mathrm{atoms}\,1^{\mathfrak{F}(\mathrm{Dst}\,f)}: (a\times^{\mathsf{FCD}}b\subseteq f\Rightarrow F\in \mathrm{up}(a\times^{\mathsf{RLD}}b))\\ &\Leftrightarrow \forall a\in \mathrm{atoms}\,1^{\mathfrak{F}(\mathrm{Src}\,f)}, b\in\mathrm{atoms}\,1^{\mathfrak{F}(\mathrm{Dst}\,f)}: ((a\times^{\mathsf{FCD}}b)\not\asymp f\Rightarrow \uparrow^{\mathsf{RLD}(\mathrm{Src}\,f;\mathrm{Dst}\,f)}F\supseteq a\times^{\mathsf{RLD}}b)\\ &\Leftrightarrow \forall a\in \mathrm{atoms}\,1^{\mathfrak{F}(\mathrm{Src}\,f)}, b\in\mathrm{atoms}\,1^{\mathfrak{F}(\mathrm{Dst}\,f)}: (a[f]b\Rightarrow \uparrow^{\mathsf{RLD}(\mathrm{Src}\,f;\mathrm{Dst}\,f)}F\supseteq a\times^{\mathsf{RLD}}b). \end{aligned}$$

□

Surprisingly a funcoid is greater inward than outward:

Theorem 222. $(\mathsf{RLD})_{\mathrm{out}}f\subseteq(\mathsf{RLD})_{\mathrm{in}}f$ for every funcoid f.

Proof. We need to prove

$$\bigcap \langle\uparrow^{\mathsf{RLD}(\mathrm{Src}\,f;\mathrm{Dst}\,f)}\rangle \mathrm{up}\,f\subseteq \bigcup\{\mathcal{A}\times^{\mathsf{RLD}}\mathcal{B}\mid \mathcal{A},\mathcal{B}\in\mathfrak{F},\mathcal{A}\times^{\mathsf{FCD}}\mathcal{B}\subseteq f\}.$$

Let

$$K\in \mathrm{up}\bigcup\{\mathcal{A}\times^{\mathsf{RLD}}\mathcal{B}\mid \mathcal{A},\mathcal{B}\in\mathfrak{F},\mathcal{A}\times^{\mathsf{FCD}}\mathcal{B}\subseteq f\}.$$

Then

$$\begin{aligned} K &= \uparrow^{\mathsf{RLD}(\mathrm{Src}\,f;\mathrm{Dst}\,f)}\bigcup\{X_{\mathcal{A}}\times Y_{\mathcal{B}}\mid \mathcal{A},\mathcal{B}\in\mathfrak{F},\mathcal{A}\times^{\mathsf{FCD}}\mathcal{B}\subseteq f\}\\ &= \bigcup\{\uparrow^{\mathsf{RLD}(\mathrm{Src}\,f;\mathrm{Dst}\,f)}(X_{\mathcal{A}}\times Y_{\mathcal{B}})\mid \mathcal{A},\mathcal{B}\in\mathfrak{F},\mathcal{A}\times^{\mathsf{FCD}}\mathcal{B}\subseteq f\}\\ &\supseteq f \end{aligned}$$

where $X_{\mathcal{A}}\in \mathrm{up}\,\mathcal{A}$, $Y_{\mathcal{B}}\in\mathrm{up}\,\mathcal{B}$. So $K\in\mathrm{up}\,f$; $K\in \mathrm{up}\bigcap\langle\uparrow^{\mathsf{RLD}(\mathrm{Src}\,f;\mathrm{Dst}\,f)}\rangle\mathrm{up}\,f$. □

Theorem 223. $(\mathsf{FCD})(\mathsf{RLD})_{\mathrm{in}}f=f$ for every funcoid f.

Proof. For every sets $X\in\mathscr{P}(\mathrm{Src}\,f)$ and $Y\in\mathscr{P}(\mathrm{Dst}\,f)$

$$\begin{aligned} &X[(\mathsf{FCD})(\mathsf{RLD})_{\mathrm{in}}f]^*Y \Leftrightarrow\\ &(\uparrow^{\mathrm{Src}\,f}X\times^{\mathsf{RLD}}\uparrow^{\mathrm{Dst}\,f}Y)\not\asymp(\mathsf{RLD})_{\mathrm{in}}f \Leftrightarrow\\ &\uparrow^{\mathsf{RLD}(\mathrm{Src}\,f;\mathrm{Dst}\,f)}(X\times Y)\not\asymp \bigcup\{a\times^{\mathsf{RLD}}b\mid a\in\mathrm{atoms}\,1^{\mathfrak{F}(\mathrm{Src}\,f)}, b\in\mathrm{atoms}\,1^{\mathfrak{F}(\mathrm{Dst}\,f)}, a\times^{\mathsf{FCD}}b\subseteq f\} \Leftrightarrow (*)\\ &\exists a\in\mathrm{atoms}\,1^{\mathfrak{F}(\mathrm{Src}\,f)}, b\in\mathrm{atoms}\,1^{\mathfrak{F}(\mathrm{Dst}\,f)}: (a\times^{\mathsf{FCD}}b\subseteq f\wedge\uparrow^{\mathsf{RLD}(\mathrm{Src}\,f;\mathrm{Dst}\,f)}(X\times Y)\not\asymp(a\times^{\mathsf{RLD}}b)) \Leftrightarrow\\ &\exists a\in\mathrm{atoms}\,1^{\mathfrak{F}(\mathrm{Src}\,f)}, b\in\mathrm{atoms}\,1^{\mathfrak{F}(\mathrm{Dst}\,f)}: (a[f]b\wedge a\subseteq\uparrow^{\mathrm{Src}\,f}X\wedge b\subseteq\uparrow^{\mathrm{Dst}\,f}Y) \Leftrightarrow\\ &X[f]^*Y. \end{aligned}$$

* theorem 53 in [15].

Thus $(\mathsf{FCD})(\mathsf{RLD})_{\mathrm{in}}f=f$. □

Remark 224. The above theorem allows to represent funcoids as reloids.

Obvious 225. $(\mathsf{RLD})_{\mathrm{in}}(\mathcal{A}\times^{\mathsf{FCD}}\mathcal{B})=\mathcal{A}\times^{\mathsf{RLD}}\mathcal{B}$ for every f.o. $\mathcal{A}$, $\mathcal{B}$.

Conjecture 226. $(\mathsf{RLD})_{\mathrm{out}}I^{\mathsf{FCD}}_{\mathcal{A}}=I^{\mathsf{RLD}}_{\mathcal{A}}$ for every f.o. $\mathcal{A}$.

Exercise 1. Prove that generally $(\mathsf{RLD})_{\mathrm{in}}I^{\mathsf{FCD}}_{\mathcal{A}}\neq I^{\mathsf{RLD}}_{\mathcal{A}}$.

Conjecture 227. $\mathrm{dom}\,(\mathsf{RLD})_{\mathrm{in}}f=\mathrm{dom}\,f$ and $\mathrm{im}\,(\mathsf{RLD})_{\mathrm{in}}f=\mathrm{im}\,f$ for every funcoid f.

Proposition 228. $\operatorname{dom}(f|_{\mathcal{A}}) = \mathcal{A} \cap \operatorname{dom} f$ for every reloid f and f.o. $\mathcal{A} \in \mathfrak{F}(\operatorname{Src} f)$.

Proof. $\operatorname{dom}(f|_{\mathcal{A}}) = \operatorname{dom}(\mathsf{FCD}) f|_{\mathcal{A}} = \operatorname{dom}((\mathsf{FCD}) f)|_{\mathcal{A}} = \mathcal{A} \cap \operatorname{dom}(\mathsf{FCD}) f = \mathcal{A} \cap \operatorname{dom} f$. □

Theorem 229. For every reloids f, g:

1. If $\operatorname{im} f \supseteq \operatorname{dom} g$ then $\operatorname{im}(g \circ f) = \operatorname{im} g$.
2. If $\operatorname{im} f \subseteq \operatorname{dom} g$ then $\operatorname{dom}(g \circ f) = \operatorname{dom} f$.

Proof.

1. $\operatorname{im}(g \circ f) = \operatorname{im}(\mathsf{FCD})(g \circ f) = \operatorname{im}((\mathsf{FCD}) g \circ (\mathsf{FCD}) f) = \operatorname{im}(\mathsf{FCD}) g = \operatorname{im} g$.
2. Similar. □

Corollary 230. $(\mathsf{RLD})_{\mathrm{in}}(f|_{\mathcal{A}}) = ((\mathsf{RLD})_{\mathrm{in}} f)|_{\mathcal{A}})$ for every funcoid f and f.o. $\mathcal{A}$.

5.3 Galois connections of funcoids and reloids

Theorem 231. (FCD): $\mathsf{RLD}(A; B) \rightarrow \mathsf{FCD}(A; B)$ is the lower adjoint of $(\mathsf{RLD})_{\mathrm{in}}$: $\mathsf{FCD}(A; B) \rightarrow \mathsf{RLD}(A; B)$ for every small sets A, B.

Proof. Because (FCD) and $(\mathsf{RLD})_{\mathrm{in}}$ are trivially monotone, it's enough to prove (for every $f \in \mathsf{RLD}(A; B)$, $g \in \mathsf{FCD}(A; B)$)

$$f \subseteq (\mathsf{RLD})_{\mathrm{in}}(\mathsf{FCD}) f \text{ and } (\mathsf{FCD})(\mathsf{RLD})_{\mathrm{in}} g \subseteq g.$$

The second formula follows from the fact that $(\mathsf{FCD})(\mathsf{RLD})_{\mathrm{in}} g = g$.

$$\begin{aligned}
(\mathsf{RLD})_{\mathrm{in}}(\mathsf{FCD}) f &= \\
\bigcup \left\{ a \times^{\mathsf{RLD}} b \mid a \in \operatorname{atoms} 1^{\mathfrak{F}(A)}, b \in \operatorname{atoms} 1^{\mathfrak{F}(B)}, a \times^{\mathsf{FCD}} b \subseteq (\mathsf{FCD}) f \right\} &= \\
\bigcup \left\{ a \times^{\mathsf{RLD}} b \mid a \in \operatorname{atoms} 1^{\mathfrak{F}(A)}, b \in \operatorname{atoms} 1^{\mathfrak{F}(B)}, a [(\mathsf{FCD}) f] b \right\} &= \\
\bigcup \left\{ a \times^{\mathsf{RLD}} b \mid a \in \operatorname{atoms} 1^{\mathfrak{F}(A)}, b \in \operatorname{atoms} 1^{\mathfrak{F}(B)}, (a \times^{\mathsf{RLD}} b) \not\asymp f \right\} &\supseteq \\
\bigcup \left\{ p \in \operatorname{atoms}(a \times^{\mathsf{RLD}} b) \mid a \in \operatorname{atoms} 1^{\mathfrak{F}(A)}, b \in \operatorname{atoms} 1^{\mathfrak{F}(B)}, p \not\asymp f \right\} &= \\
\bigcup \left\{ p \in \operatorname{atoms} 1^{\mathfrak{F}(A \times B)} \mid p \not\asymp f \right\} &= \\
\bigcup \left\{ p \mid p \in \operatorname{atoms} f \right\} = f.
\end{aligned}$$

□

Corollary 232.

1. $(\mathsf{FCD}) \bigcup S = \bigcup \langle (\mathsf{FCD}) \rangle S$ if $S \in \mathscr{P} \mathsf{RLD}(A; B)$.
2. $(\mathsf{RLD})_{\mathrm{in}} \bigcap S = \bigcap \langle (\mathsf{RLD})_{\mathrm{in}} \rangle S$ if $S \in \mathscr{P} \mathsf{FCD}(A; B)$.

Proposition 233. $(\mathsf{RLD})_{\mathrm{in}}(f \cap (\mathcal{A} \times^{\mathsf{FCD}} \mathcal{B})) = ((\mathsf{RLD})_{\mathrm{in}} f) \cap (\mathcal{A} \times^{\mathsf{RLD}} \mathcal{B})$ for every funcoid f and f.o. $\mathcal{A} \in \mathfrak{F}(\operatorname{Src} f)$ and $\mathcal{B} \in \mathfrak{F}(\operatorname{Dst} f)$.

Proof. $(\mathsf{RLD})_{\mathrm{in}}(f \cap (\mathcal{A} \times^{\mathsf{FCD}} \mathcal{B})) = ((\mathsf{RLD})_{\mathrm{in}} f) \cap (\mathsf{RLD})_{\mathrm{in}}(\mathcal{A} \times^{\mathsf{FCD}} \mathcal{B})) = ((\mathsf{RLD})_{\mathrm{in}} f) \cap (\mathcal{A} \times^{\mathsf{RLD}} \mathcal{B})$. □

Conjecture 234. $(\mathsf{RLD})_{\mathrm{in}}$ is not a lower adjoint (in general).

Conjecture 235. $(\mathsf{RLD})_{\mathrm{out}}$ is neither a lower adjoint nor an upper adjoint (in general).

See also the corollary 297 below.

6 Continuous morphisms

This section uses the apparatus from the section "Partially ordered dagger categories".

6.1 Traditional definitions of continuity

In this section we will show that having a funcoid or reloid $\uparrow f$ corresponding to a function f we can express continuity of it by the formula $\uparrow f \circ \mu \subseteq \nu \circ \uparrow f$ (or similar formulas) where μ and ν are some spaces.

6.1.1 Pre-topology

Let μ and ν are funcoids representing some pre-topologies. By definition a function f is continuous map from μ to ν in point a iff

$$\forall \epsilon \in \operatorname{up}\langle \nu \rangle f a \exists \delta \in \operatorname{up}\langle \mu \rangle^{*}\{a\} : \langle f \rangle \delta \subseteq \epsilon.$$

Equivalently transforming this formula we get:

$$\forall \epsilon \in \operatorname{up}\langle \nu \rangle f a : \left\langle \uparrow^{\mathsf{FCD}(\operatorname{Src}\mu;\operatorname{Dst}\nu)} f \right\rangle \langle \mu \rangle \uparrow^{\operatorname{Src}\mu}\{a\} \subseteq \epsilon;$$
$$\left\langle \uparrow^{\mathsf{FCD}(\operatorname{Src}\mu;\operatorname{Dst}\nu)} f \right\rangle \langle \mu \rangle \uparrow^{\operatorname{Src}\mu}\{a\} \subseteq \langle \nu \rangle f a;$$
$$\left\langle \uparrow^{\mathsf{FCD}(\operatorname{Src}\mu;\operatorname{Dst}\nu)} f \right\rangle \langle \mu \rangle \uparrow^{\operatorname{Src}\mu}\{a\} \subseteq \langle \nu \rangle \left\langle \uparrow^{\mathsf{FCD}(\operatorname{Src}\mu;\operatorname{Dst}\nu)} f \right\rangle \uparrow^{\operatorname{Src}\mu}\{a\};$$
$$\left\langle \uparrow^{\mathsf{FCD}(\operatorname{Src}\mu;\operatorname{Dst}\nu)} f \circ \mu \right\rangle \uparrow^{\operatorname{Src}\mu}\{a\} \subseteq \left\langle \nu \circ \uparrow^{\mathsf{FCD}(\operatorname{Src}\mu;\operatorname{Dst}\nu)} f \right\rangle \uparrow^{\operatorname{Src}\mu}\{a\}.$$

So f is a continuous map from μ to ν in every point of its domain iff

$$\uparrow^{\mathsf{FCD}(\operatorname{Src}\mu;\operatorname{Dst}\nu)} f \circ \mu \subseteq \nu \circ \uparrow^{\mathsf{FCD}(\operatorname{Src}\mu;\operatorname{Dst}\nu)} f.$$

6.1.2 Proximity spaces

Let μ and ν are proximity (nearness) spaces (which I consider a special case of funcoids). By definition a function f is a proximity-continuous map (also called equivicontinuous) from μ to ν iff

$$\forall X \in \mathscr{P}(\operatorname{Src}\mu), Y \in \mathscr{P}(\operatorname{Dst}\mu) : (X[\mu]^{*}Y \Rightarrow f X[\nu]^{*} f Y).$$

Equivalently transforming this formula we get:

$$\forall X, Y \in \mathscr{P}\mho : \left(X[\mu]^{*}Y \Rightarrow \left\langle \uparrow^{\mathsf{FCD}(\operatorname{Src}\mu;\operatorname{Dst}\nu)} f \right\rangle Y \cap \langle \nu \rangle \left\langle \uparrow^{\mathsf{FCD}(\operatorname{Src}\mu;\operatorname{Dst}\nu)} f \right\rangle X \neq 0^{\mathfrak{F}(\operatorname{Dst}\nu)}\right);$$
$$\forall X, Y \in \mathscr{P}\mho : \left(X[\mu]^{*}Y \Rightarrow \left\langle \uparrow^{\mathsf{FCD}(\operatorname{Src}\mu;\operatorname{Dst}\nu)} f \right\rangle Y \cap \left\langle \nu \circ \uparrow^{\mathsf{FCD}(\operatorname{Src}\mu;\operatorname{Dst}\nu)} f \right\rangle X \neq 0^{\mathfrak{F}(\operatorname{Dst}\nu)}\right);$$
$$\forall X, Y \in \mathscr{P}\mho : \left(X[\mu]^{*}Y \Rightarrow X\left[\nu \circ \uparrow^{\mathsf{FCD}(\operatorname{Src}\mu;\operatorname{Dst}\nu)} f\right]^{*} \langle f \rangle Y\right);$$
$$\forall X, Y \in \mathscr{P}\mho : \left(X[\mu]^{*}Y \Rightarrow \langle f \rangle Y\left[\left(\nu \circ \uparrow^{\mathsf{FCD}(\operatorname{Src}\mu;\operatorname{Dst}\nu)} f\right)^{-1}\right]^{*} X\right);$$
$$\forall X, Y \in \mathscr{P}\mho : \left(X[\mu]^{*}Y \Rightarrow \langle f \rangle Y\left[\left(\uparrow^{\mathsf{FCD}(\operatorname{Src}\mu;\operatorname{Dst}\nu)} f\right)^{-1} \circ \nu^{-1}\right]^{*} X\right);$$
$$\forall X, Y \in \mathscr{P}\mho : \left(X[\mu]^{*}Y \Rightarrow \uparrow^{\mathfrak{F}(\operatorname{Src}\mu)} X \cap \left\langle \left(\uparrow^{\mathsf{FCD}(\operatorname{Src}\mu;\operatorname{Dst}\nu)} f\right)^{-1} \circ \nu^{-1} \right\rangle \left\langle \uparrow^{\mathsf{FCD}(\operatorname{Src}\mu;\operatorname{Dst}\nu)} f \right\rangle Y \neq 0^{\mathfrak{F}(\operatorname{Src}\mu)}\right);$$
$$\forall X, Y \in \mathscr{P}\mho : \left(X[\mu]^{*}Y \Rightarrow \uparrow^{\mathfrak{F}(\operatorname{Src}\mu)} X \cap \left\langle \left(\uparrow^{\mathsf{FCD}(\operatorname{Src}\mu;\operatorname{Dst}\nu)} f\right)^{-1} \circ \nu^{-1} \circ \uparrow^{\mathsf{FCD}(\operatorname{Src}\mu;\operatorname{Dst}\nu)} f \right\rangle Y \neq 0^{\mathfrak{F}(\operatorname{Src}\mu)}\right);$$
$$\forall X, Y \in \mathscr{P}\mho : \left(X[\mu]^{*}Y \Rightarrow Y\left[\left(\uparrow^{\mathsf{FCD}(\operatorname{Src}\mu;\operatorname{Dst}\nu)} f\right)^{-1} \circ \nu^{-1} \circ \uparrow^{\mathsf{FCD}(\operatorname{Src}\mu;\operatorname{Dst}\nu)} f\right]^{*} X\right);$$
$$\forall X, Y \in \mathscr{P}\mho : \left(X[\mu]^{*}Y \Rightarrow X\left[\left(\uparrow^{\mathsf{FCD}(\operatorname{Src}\mu;\operatorname{Dst}\nu)} f\right)^{-1} \circ \nu \circ \uparrow^{\mathsf{FCD}(\operatorname{Src}\mu;\operatorname{Dst}\nu)} f\right]^{*} Y\right);$$
$$\mu \subseteq \left(\uparrow^{\mathsf{FCD}(\operatorname{Src}\mu;\operatorname{Dst}\nu)} f\right)^{-1} \circ \nu \circ \uparrow^{\mathsf{FCD}(\operatorname{Src}\mu;\operatorname{Dst}\nu)} f.$$

So a function f is proximity-continuous iff $\mu \subseteq \left(\uparrow^{\mathsf{FCD}(\operatorname{Src}\mu;\operatorname{Dst}\nu)} f\right)^{-1} \circ \nu \circ \uparrow^{\mathsf{FCD}(\operatorname{Src}\mu;\operatorname{Dst}\nu)} f$.

6.1.3 Uniform spaces

Uniform spaces are a special case of reloids.

Let μ and ν are uniform spaces. By definition a function f is a uniformly continuous map from μ to ν iff

$$\forall \epsilon \in \operatorname{up}\nu \exists \delta \in \operatorname{up}\mu \forall (x; y) \in \delta : (f x; f y) \in \epsilon.$$

Equivalently transforming this formula we get:

$$\forall \epsilon \in \operatorname{up}\nu \exists \delta \in \operatorname{up}\mu \forall (x; y) \in \delta : \{(f x; f y)\} \subseteq \epsilon$$
$$\forall \epsilon \in \operatorname{up}\nu \exists \delta \in \operatorname{up}\mu \forall (x; y) \in \delta : f \circ \{(x; y)\} \circ f^{-1} \subseteq \epsilon$$
$$\forall \epsilon \in \operatorname{up}\nu \exists \delta \in \operatorname{up}\mu : f \circ \delta \circ f^{-1} \subseteq \epsilon$$
$$\forall \epsilon \in \operatorname{up}\nu : \uparrow^{\mathsf{RLD}(\operatorname{Dst}\mu;\operatorname{Dst}\nu)} f \circ \mu \circ \left(\uparrow^{\mathsf{RLD}(\operatorname{Dst}\mu;\operatorname{Dst}\nu)} f\right)^{-1} \subseteq \epsilon$$
$$\uparrow^{\mathsf{RLD}(\operatorname{Dst}\mu;\operatorname{Dst}\nu)} f \circ \mu \circ \left(\uparrow^{\mathsf{RLD}(\operatorname{Dst}\mu;\operatorname{Dst}\nu)} f\right)^{-1} \subseteq \nu.$$

So a function f is uniformly continuous iff $\uparrow^{\mathsf{RLD}(\operatorname{Dst}\mu;\operatorname{Dst}\nu)} f\circ\mu\circ\left(\uparrow^{\mathsf{RLD}(\operatorname{Dst}\mu;\operatorname{Dst}\nu)} f\right)^{-1}\subseteq\nu$.

6.2 Our three definitions of continuity

I have expressed different kinds of continuity with simple algebraic formulas hiding the complexity of traditional epsilon-delta notation behind a smart algebra. Let's summarize these three algebraic formulas:

Let μ and ν are endomorphisms of some partially ordered precategory. Continuous functions can be defined as these morphisms f of this precategory which conform to the following formula:

$$f\in\mathrm{C}(\mu;\nu)\Leftrightarrow f\in\mathrm{Mor}(\operatorname{Ob}\mu;\operatorname{Ob}\nu)\wedge f\circ\mu\subseteq\nu\circ f.$$

If the precategory is a partially ordered dagger precategory then continuity also can be defined in two other ways:

$$f\in\mathrm{C}'(\mu;\nu)\quad\Leftrightarrow\quad f\in\mathrm{Mor}(\operatorname{Ob}\mu;\operatorname{Ob}\nu)\wedge\mu\subseteq f^{\dagger}\circ\nu\circ f;$$
$$f\in\mathrm{C}''(\mu;\nu)\quad\Leftrightarrow\quad f\in\mathrm{Mor}(\operatorname{Ob}\mu;\operatorname{Ob}\nu)\wedge f\circ\mu\circ f^{\dagger}\subseteq\nu.$$

Remark 236. In the examples (above) about funcoids and reloids the "dagger functor" is the inverse of a funcoid or reloid, that is $f^{\dagger}=f^{-1}$.

Proposition 237. Every of these three definitions of continuity forms a sub-precategory (subcategory if the original precategory is a category).

Proof.

C. Let $f\in\mathrm{C}(\mu;\nu)$, $g\in\mathrm{C}(\nu;\pi)$. Then $f\circ\mu\subseteq\nu\circ f$, $g\circ\nu\subseteq\pi\circ g$; $g\circ f\circ\mu\subseteq g\circ\nu\circ f\subseteq\pi\circ g\circ f$. So $g\circ f\in\mathrm{C}(\mu;\pi)$. $1_{\operatorname{Ob}\mu}\in\mathrm{C}(\mu;\mu)$ is obvious.

***C′*.** Let $f\in\mathrm{C}'(\mu;\nu)$, $g\in\mathrm{C}'(\nu;\pi)$. Then $\mu\subseteq f^{\dagger}\circ\nu\circ f$, $\nu\subseteq g^{\dagger}\circ\pi\circ g$;

$$\mu\subseteq f^{\dagger}\circ g^{\dagger}\circ\pi\circ g\circ f;\quad\mu\subseteq(g\circ f)^{\dagger}\circ\pi\circ(g\circ f).$$

So $g\circ f\in\mathrm{C}'(\mu;\pi)$. $1_{\operatorname{Ob}\mu}\in\mathrm{C}'(\mu;\mu)$ is obvious.

C″. Let $f\in\mathrm{C}''(\mu;\nu)$, $g\in\mathrm{C}''(\nu;\pi)$. Then $f\circ\mu\circ f^{\dagger}\subseteq\nu$, $g\circ\nu\circ g^{\dagger}\subseteq\pi$;

$$g\circ f\circ\mu\circ f^{\dagger}\circ g^{\dagger}\subseteq\pi;\quad(g\circ f)\circ\mu\circ(g\circ f)^{\dagger}\subseteq\pi.$$

So $g\circ f\in\mathrm{C}''(\mu;\pi)$. $1_{\operatorname{Ob}\mu}\in\mathrm{C}''(\mu;\mu)$ is obvious. □

Proposition 238. For a monovalued morphism f of a partially ordered dagger category and its endomorphisms μ and ν

$$f\in\mathrm{C}'(\mu;\nu)\Rightarrow f\in\mathrm{C}(\mu;\nu)\Rightarrow f\in\mathrm{C}''(\mu;\nu).$$

Proof. Let $f\in\mathrm{C}'(\mu;\nu)$. Then $\mu\subseteq f^{\dagger}\circ\nu\circ f$; $f\circ\mu\subseteq f\circ f^{\dagger}\circ\nu\circ f\subseteq 1_{\operatorname{Dst}f}\circ\nu\circ f=\nu\circ f$; $f\in\mathrm{C}(\mu;\nu)$.

Let $f\in\mathrm{C}(\mu;\nu)$. Then $f\circ\mu\subseteq\nu\circ f$; $f\circ\mu\circ f^{\dagger}\subseteq\nu\circ f\circ f^{\dagger}\subseteq\nu\circ 1_{\operatorname{Dst}f}=\nu$; $f\in\mathrm{C}''(\mu;\nu)$. □

Proposition 239. For an entirely defined morphism f of a partially ordered dagger category and its endomorphisms μ and ν

$$f\in\mathrm{C}''(\mu;\nu)\Rightarrow f\in\mathrm{C}(\mu;\nu)\Rightarrow f\in\mathrm{C}'(\mu;\nu).$$

Proof. Let $f\in\mathrm{C}''(\mu;\nu)$. Then $f\circ\mu\circ f^{\dagger}\subseteq\nu$; $f\circ\mu\circ f^{\dagger}\circ f\subseteq\nu\circ f$; $f\circ\mu\circ 1_{\operatorname{Src}f}\subseteq\nu\circ f$; $f\circ\mu\subseteq\nu\circ f$; $f\in\mathrm{C}(\mu;\nu)$.

Let $f\in\mathrm{C}(\mu;\nu)$. Then $f\circ\mu\subseteq\nu\circ f$; $f^{\dagger}\circ f\circ\mu\subseteq f^{\dagger}\circ\nu\circ f$; $1_{\operatorname{Src}f}\circ\mu\subseteq f^{\dagger}\circ\nu\circ f$; $\mu\subseteq f^{\dagger}\circ\nu\circ f$; $f\in\mathrm{C}'(\mu;\nu)$. □

For entirely defined monovalued morphisms our three definitions of continuity coincide:

Theorem 240. If f is a monovalued and entirely defined morphism then

$$f\in\mathrm{C}'(\mu;\nu)\Leftrightarrow f\in\mathrm{C}(\mu;\nu)\Leftrightarrow f\in\mathrm{C}''(\mu;\nu).$$

Proof. From two previous propositions. □

The classical general topology theorem that uniformly continuous function from a uniform space to an other uniform space is near-continuous regarding the proximities generated by the uniformities, generalized for reloids and funcoids takes the following form:

Theorem 241. If an entirely defined morphism of the category of reloids $f \in \mathrm{C}''(\mu; \nu)$ for some endomorphisms μ and ν of the category of reloids, then $(\mathsf{FCD}) f \in \mathrm{C}'((\mathsf{FCD})\mu; (\mathsf{FCD})\nu)$.

Exercise 2. I leave a simple exercise for the reader to prove the last theorem.

6.3 Continuousness of a restricted morphism

Consider some partially ordered semigroup. (For example it can be the semigroup of funcoids or semigroup of reloids regarding the composition.) Consider also some lattice (*lattice of objects*). (For example take the lattice of set theoretic filters.)

We will map every object A to *identity element* I_A of the semigroup (for example identity funcoid or identity reloid). For identity elements we will require

1. $I_A \circ I_B = I_{A \cap B}$;
2. $f \circ I_A \subseteq f$; $I_A \circ f \subseteq f$.

In the case when our semigroup is "dagger" (that is is a dagger precategory) we will require also $(I_A)^{\dagger} = I_A$.

We can define *restricting* an element f of our semigroup to an object A by the formula $f|_A = f \circ I_A$.

We can define *rectangular restricting* an element μ of our semigroup to objects A and B as $I_B \circ \mu \circ I_A$. Optionally we can define direct product $A \times B$ of two objects by the formula (true for funcoids and for reloids):

$$\mu \cap (A \times B) = I_B \circ \mu \circ I_A.$$

Square restricting of an element μ to an object A is a special case of rectangular restricting and is defined by the formula $I_A \circ \mu \circ I_A$ (or by the formula $\mu \cap (A \times A)$).

Theorem 242. For every elements f, μ, ν of our semigroup and an object A

1. $f \in \mathrm{C}(\mu; \nu) \Rightarrow f|_A \in \mathrm{C}(I_A \circ \mu \circ I_A; \nu)$;
2. $f \in \mathrm{C}'(\mu; \nu) \Rightarrow f|_A \in \mathrm{C}'(I_A \circ \mu \circ I_A; \nu)$;
3. $f \in \mathrm{C}''(\mu; \nu) \Rightarrow f|_A \in \mathrm{C}''(I_A \circ \mu \circ I_A; \nu)$.

(Two last items are true for the case when our semigroup is dagger.)

Proof.

1. $f|_A \in \mathrm{C}(I_A \circ \mu \circ I_A; \nu) \Leftrightarrow f|_A \circ I_A \circ \mu \circ I_A \subseteq \nu \circ f|_A \Leftrightarrow f \circ I_A \circ I_A \circ \mu \circ I_A \subseteq \nu \circ f|_A \Leftrightarrow f \circ I_A \circ \mu \circ I_A \subseteq \nu \circ f \circ I_A \Leftarrow f \circ I_A \circ \mu \subseteq \nu \circ f \Leftarrow f \circ \mu \subseteq \nu \circ f \Leftrightarrow f \in \mathrm{C}(\mu; \nu)$.
2. $f|_A \in \mathrm{C}'(I_A \circ \mu \circ I_A; \nu) \Leftrightarrow I_A \circ \mu \circ I_A \subseteq (f|_A)^{\dagger} \circ \nu \circ f|_A \Leftarrow I_A \circ \mu \circ I_A \subseteq (f \circ I_A)^{\dagger} \circ \nu \circ f \circ I_A \Leftrightarrow I_A \circ \mu \circ I_A \subseteq I_A \circ f^{\dagger} \circ \nu \circ f \circ I_A \Leftarrow \mu \subseteq f^{\dagger} \circ \nu \circ f \Leftrightarrow f \in \mathrm{C}'(\mu; \nu)$.
3. $f|_A \in \mathrm{C}''(I_A \circ \mu \circ I_A; \nu) \Leftrightarrow f|_A \circ I_A \circ \mu \circ I_A \circ (f|_A)^{\dagger} \subseteq \nu \Leftrightarrow f \circ I_A \circ I_A \circ \mu \circ I_A \circ I_A \circ f^{\dagger} \subseteq \nu \Leftrightarrow f \circ I_A \circ \mu \circ I_A \circ f^{\dagger} \subseteq \nu \Leftarrow f \circ \mu \circ f^{\dagger} \subseteq \nu \Leftrightarrow f \in \mathrm{C}''(\mu; \nu)$. □

7 Connectedness regarding funcoids and reloids

Definition 243. I will call *endo-reloids* and *endo-funcoids* reloids and funcoids with the same source and destination.

7.1 Some lemmas

Lemma 244. If $\neg(A[f]^*B)\wedge A\cup B\in\operatorname{up}(\operatorname{dom} f\cup\operatorname{im} f)$ then f is closed on $\uparrow^U A$ for a funcoid $f\in\mathsf{FCD}(U;U)$ and sets $A,B\in\mathscr{P}U$ (for every small set U).

Proof. $\neg(A[f]^*B)\Leftrightarrow\uparrow^U B\cap\langle f\rangle\uparrow^U A=0^{\mathfrak{F}(U)}\Rightarrow(\operatorname{dom} f\cup\operatorname{im} f)\cap\uparrow^U B\cap\langle f\rangle^*A=0^{\mathfrak{F}(U)}\Rightarrow((\operatorname{dom} f\cup\operatorname{im} f)\setminus\uparrow^U A)\cap\langle f\rangle^*A=0^{\mathfrak{F}(U)}\Leftrightarrow\langle f\rangle^*A\subseteq\uparrow^U A$. □

Corollary 245. If $\neg(A[f]^*B)\wedge A\cup B\in\operatorname{up}(\operatorname{dom} f\cup\operatorname{im} f)$ then f is closed on $\uparrow^U(A\setminus B)$ for a funcoid f and sets $A,B\in\mathscr{P}U$ (for every small set U).

Proof. Let $\neg(A[f]^*B)\wedge A\cup B\in\operatorname{up}(\operatorname{dom} f\cup\operatorname{im} f)$. Then $\neg((A\setminus B)[f]^*B)\wedge\uparrow^U((A\setminus B)\cup B)\in\operatorname{up}(\operatorname{dom} f\cup\operatorname{im} f)$. □

Lemma 246. If $\neg(A[f]^*B)\wedge A\cup B\in\operatorname{up}(\operatorname{dom} f\cup\operatorname{im} f)$ then $\neg(A[f^n]^*B)$ for every whole positive n.

Proof. Let $\neg(A[f]^*B)\wedge A\cup B\in\operatorname{up}(\operatorname{dom} f\cup\operatorname{im} f)$. From the above proposition $\langle f\rangle^*A\subseteq\uparrow^U A$. $\uparrow^U B\cap\langle f\rangle\uparrow^U A=0^{\mathfrak{F}(U)}$, consequently $\langle f\rangle^*A\subseteq\uparrow^U(A\setminus B)$. Because (by the above corollary) f is closed on $\uparrow^U(A\setminus B)$, then $\langle f\rangle\langle f\rangle\uparrow^U A\subseteq\uparrow^U(A\setminus B)$, $\langle f\rangle\langle f\rangle\langle f\rangle\uparrow^U A\subseteq\uparrow^U(A\setminus B)$, etc. So $\langle f^n\rangle\uparrow^U A\subseteq\uparrow^U(A\setminus B)$, $\uparrow^U B\asymp\langle f^n\rangle\uparrow^U A$, $\neg(A[f^n]^*B)$. □

7.2 Endomorphism series

Definition 247. $S_1(\mu)\overset{\text{def}}{=}\mu\cup\mu^2\cup\mu^3\cup\ldots$ for an endomorphism μ of a precategory with countable union of morphisms.

Definition 248. $S(\mu)\overset{\text{def}}{=}\mu^0\cup S_1(\mu)=\mu^0\cup\mu\cup\mu^2\cup\mu^3\cup\ldots$ where $\mu^0\overset{\text{def}}{=}I_{\operatorname{Ob}\mu}$ (identity morphism for the object $\operatorname{Ob}\mu$) where $\operatorname{Ob}\mu$ is the object of endomorphism μ for an endomorphism μ of a category with countable union of morphisms.

I call S_1 and S *endomorphism series*.

We will consider the collection of all binary relations (on a set $\mho$), as well as the collection of all funcoids and the collection of all reloids on a fixed set, as categories with single object $\mho$ and the identity morphisms $I_\mho$, $I^{\mathsf{FCD}(\Omega)}$, $I^{\mathsf{RLD}(\Omega)}$.

Proposition 249. The relation $S(\mu)$ is transitive for the category of binary relations.

Proof.

$$\begin{aligned}S(\mu)\circ S(\mu)&=\mu^0\circ S(\mu)\cup\mu\circ S(\mu)\cup\mu^2\circ S(\mu)\cup\ldots\\&=(\mu^0\cup\mu^1\cup\mu^2\cup\ldots)\cup(\mu^1\cup\mu^2\cup\mu^3\cup\ldots)\cup(\mu^2\cup\mu^3\cup\mu^4\cup\ldots)\\&=\mu^0\cup\mu^1\cup\mu^2\cup\ldots\\&=S(\mu).\end{aligned}$$

□

7.3 Connectedness regarding binary relations

Before going to research connectedness for funcoids and reloids we will excurse into the basic special case of connectedness regarding binary relations on a set $\mho$.

Definition 250. A set A is called *(strongly) connected* regarding a binary relation μ when

$$\forall X\in\mathscr{P}(\operatorname{dom}\mu)\setminus\{\emptyset\},Y\in\mathscr{P}(\operatorname{im}\mu)\setminus\{\emptyset\}:(X\cup Y=A\Rightarrow X[\mu]Y).$$

Let $\mho$ is a set.

Definition 251. *Path* between two elements $a, b \in \mho$ in a set $A \subseteq \mho$ through binary relation μ is the finite sequence $x_0 ... x_n$ where $x_0 = a$, $x_n = b$ for $n \in \mathbb{N}$ and $x_i (\mu \cap A \times A) x_{i+1}$ for every $i = 0, ..., n - 1$. n is called path *length*.

Proposition 252. There exists path between every element $a \in \mho$ and that element itself.

Proof. It is the path consisting of one vertex (of length 0). □

Proposition 253. There is a path from element a to element b in a set A through a binary relation μ iff $a (S(\mu \cap A \times A)) b$ (that is $(a, b) \in S(\mu \cap A \times A)$).

Proof.

⇒. If exists a path from a to b, then $\{b\} \subseteq \langle (\mu \cap A \times A)^n \rangle \{a\}$ where n is the path length. Consequently $\{b\} \subseteq \langle S(\mu \cap A \times A) \rangle \{a\}$; $a (S(\mu \cap A \times A)) b$.

⇐. If $a (S(\mu \cap A \times A)) b$ then exists $n \in \mathbb{N}$ such that $a (\mu \cap A \times A)^n b$. By definition of composition of binary relations this means that there exist finite sequence $x_0 ... x_n$ where $x_0 = a$, $x_n = b$ for $n \in \mathbb{N}$ and $x_i (\mu \cap A \times A) x_{i+1}$ for every $i = 0, ..., n - 1$. That is there is path from a to b. □

Theorem 254. The following statements are equivalent for a relation μ and a set A:

1. For every $a, b \in A$ there is a path between a and b in A through μ.
2. $S(\mu \cap A \times A) \supseteq A \times A$.
3. $S(\mu \cap A \times A) = A \times A$.
4. A is connected regarding μ.

Proof.

(1)⇒(2). Let for every a, $b \in A$ there is a path between a and b in A through μ. Then $a (S(\mu \cap A \times A)) b$ for every $a, b \in A$. It is possible only when $S(\mu \cap A \times A) \supseteq A \times A$.

(3)⇒(1). For every two vertices a and b we have $a\ (S(\mu \cap A \times A))\ b$. So (by the previous theorem) for every two vertices a and b exist path from a to b.

(3)⇒(4). Suppose that $\neg (X [\mu \cap A \times A] Y)$ for some $X, Y \in \mathscr{P}\mho \setminus \{\emptyset\}$ such that $X \cup Y = A$. Then by a lemma $\neg (X [(\mu \cap A \times A)^n] Y)$ for every $n \in \mathbb{N}$. Consequently $\neg (X [S(\mu \cap A \times A)] Y)$. So $S(\mu \cap A \times A) \neq A \times A$.

(4)⇒(3). If $\langle S(\mu \cap A \times A) \rangle \{v\} = A$ for every vertex v then $S(\mu \cap A \times A) = A \times A$. Consider the remaining case when $V \overset{\text{def}}{=} \langle S(\mu \cap A \times A) \rangle \{v\} \subset A$ for some vertex v. Let $W = A \setminus V$. If $\operatorname{card} A = 1$ then $S(\mu \cap A \times A) \supseteq (=) = A \times A$; otherwise $W \neq \emptyset$. Then $V \cup W = A$ and so $V [\mu] W$ what is equivalent to $V [\mu \cap A \times A] W$ that is $\langle \mu \cap A \times A \rangle V \cap W \neq \emptyset$. This is impossible because $\langle \mu \cap A \times A \rangle V = \langle \mu \cap A \times A \rangle \langle S(\mu \cap A \times A) \rangle V = \langle S_1(\mu \cap A \times A) \rangle V \subseteq \langle S(\mu \cap A \times A) \rangle V = V$.

(2)⇒(3). Because $S(\mu \cap A \times A) \subseteq A \times A$. □

Corollary 255. A set A is connected regarding a binary relation μ iff it is connected regarding $\mu \cap A \times A$.

Definition 256. A *connected component* of a set A regarding a binary relation F is a maximal connected subset of A.

Theorem 257. The set A is partitioned into connected components (regarding every binary relation F).

Proof. Consider the binary relation $a \sim b \Leftrightarrow a\, (S(F))\, b \wedge b\, (S(F))\, a$. $\sim$ is a symmetric, reflexive, and transitive relation. So all points of A are partitioned into a collection of sets Q. Obviously each component is (strongly) connected. If a set $R \subseteq A$ is greater than one of that connected components A then it contains a point $b \in B$ where B is some other connected component. Consequently R is disconnected. □

Proposition 258. A set is connected (regarding a binary relation) iff it has one connected component.

Proof. Direct implication is obvious. Reverse is proved by contradiction. □

7.4 Connectedness regarding funcoids and reloids

Definition 259. $S_1^*(\mu) = \bigcap \left\{ \uparrow^{\mathsf{RLD}(\operatorname{Ob}\mu;\operatorname{Ob}\mu)} S_1(M) \mid M \in \operatorname{up}\mu \right\}$ for an endo-reloid μ.

Definition 260. *Connectivity reloid* $S^*(\mu)$ for an endo-reloid μ is defined as follows:

$$S^*(\mu) = \bigcap \left\{ \uparrow^{\mathsf{RLD}(\operatorname{Ob}\mu;\operatorname{Ob}\mu)} S(M) \mid M \in \operatorname{up}\mu \right\}.$$

Remark 261. Do not mess the word *connectivity* with the word *connectedness* which means being connected.[1]

Proposition 262. $S^*(\mu) = I^{\mathsf{RLD}(\operatorname{Ob}\mu)} \cup S_1^*(\mu)$ for every endo-reloid μ.

Proof. Follows from the theorem about distributivity of $\cup$ regarding $\bigcap$ (see [15]). □

Proposition 263. $S^*(\mu) = S(\mu)$ if μ is a discrete reloid.

Proof. $S^*(\mu) = \bigcap \{S(\mu)\} = S(\mu)$. □

Definition 264. A filter object $\mathcal{A} \in \mathfrak{F}(\operatorname{Ob}\mu)$ is called *connected* regarding an endo-reloid μ when $S^*(\mu \cap (\mathcal{A} \times^{\mathsf{RLD}} \mathcal{A})) \supseteq \mathcal{A} \times^{\mathsf{RLD}} \mathcal{A}$.

Obvious 265. A filter object $\mathcal{A} \in \operatorname{Ob}\mu$ is connected regarding a reloid μ iff $S^*(\mu \cap (\mathcal{A} \times^{\mathsf{RLD}} \mathcal{A})) = \mathcal{A} \times^{\mathsf{RLD}} \mathcal{A}$.

Definition 266. A filter object $\mathcal{A}$ is called *connected* regarding an endo-funcoid μ when

$$\forall \mathcal{X}, \mathcal{Y} \in \mathfrak{F}(\operatorname{Ob}\mu) \setminus \left\{ 0^{\mathfrak{F}(\operatorname{Ob}\mu)} \right\} : (\mathcal{X} \cup \mathcal{Y} = \mathcal{A} \Rightarrow \mathcal{X} [\mu] \mathcal{Y}).$$

Proposition 267. Let A be a set. The f.o. $\uparrow^{\operatorname{Ob}\mu} A$ is connected regarding an endo-funcoid μ iff

$$\forall \mathcal{X}, \mathcal{Y} \in \mathscr{P}(\operatorname{Ob}\mu) \setminus \{\emptyset\} : (X \cup Y = A \Rightarrow X [\mu]^* Y).$$

Proof.

⇒. Obvious.

⇐. Follows from co-separability of filter objects. □

Theorem 268. The following are equivalent for every set A and binary relation μ:

1. A is connected regarding binary relation μ.
2. $\uparrow^{\operatorname{Ob}\mu} A$ is connected regarding $\uparrow^{\mathsf{RLD}(\operatorname{Ob}\mu;\operatorname{Ob}\mu)} \mu$.
3. $\uparrow^{\operatorname{Ob}\mu} A$ is connected regarding $\uparrow^{\mathsf{FCD}(\operatorname{Ob}\mu;\operatorname{Ob}\mu)} \mu$.

Proof.

(1)⇔(2). $S^*\left(\uparrow^{\mathsf{RLD}(\operatorname{Ob}\mu;\operatorname{Ob}\mu)} \mu \cap (\uparrow^{\operatorname{Ob}\mu} A \times^{\mathsf{RLD}} \uparrow^{\operatorname{Ob}\mu} A)\right) = S^*\left(\uparrow^{\mathsf{RLD}(\operatorname{Ob}\mu;\operatorname{Ob}\mu)} (\mu \cap A \times A)\right) = \uparrow^{\mathsf{RLD}(\operatorname{Ob}\mu;\operatorname{Ob}\mu)} S(\mu \cap A \times A)$. So $S^*\left(\uparrow^{\mathsf{RLD}(\operatorname{Ob}\mu;\operatorname{Ob}\mu)} \mu \cap (\uparrow^{\operatorname{Ob}\mu} A \times^{\mathsf{RLD}} \uparrow^{\operatorname{Ob}\mu} A)\right) \supseteq \uparrow^{\operatorname{Ob}\mu} A \times^{\mathsf{RLD}} \uparrow^{\operatorname{Ob}\mu} A \Leftrightarrow \uparrow^{\mathsf{RLD}(\operatorname{Ob}\mu;\operatorname{Ob}\mu)} S(\mu \cap A \times A) \supseteq \uparrow^{\mathsf{RLD}(\operatorname{Ob}\mu;\operatorname{Ob}\mu)} (A \times A) = \uparrow^{\operatorname{Ob}\mu} A \times^{\mathsf{RLD}} \uparrow^{\operatorname{Ob}\mu} A$.

(1)⇔(3). Follows from the previous proposition. □

1. In some math literature these two words are used interchangeably.

Next is conjectured a statement more strong than the above theorem:

Conjecture 269. Let $\mathcal{A}$ is an f.o. and F is a binary relation on $A \times B$ for some sets A, B.
$\mathcal{A}$ is connected regarding $\uparrow^{\mathsf{FCD}(A;B)} F$ iff $\mathcal{A}$ is connected regarding $\uparrow^{\mathsf{RLD}(A;B)} F$.

Obvious 270. A filter object $\mathcal{A}$ is connected regarding a reloid μ iff it is connected regarding the reloid $\mu \cap (\mathcal{A} \times^{\mathsf{RLD}} \mathcal{A})$.

Obvious 271. A filter object $\mathcal{A}$ is connected regarding a funcoid μ iff it is connected regarding the funcoid $\mu \cap (\mathcal{A} \times^{\mathsf{FCD}} \mathcal{A})$.

Theorem 272. A filter object $\mathcal{A}$ is connected regarding a reloid f iff $\uparrow^{\mathrm{Ob}\, f} \mathcal{A}$ is connected regarding every $F \in \langle \uparrow^{\mathsf{RLD}(\mathrm{Ob}\, f;\mathrm{Ob}\, f)} \rangle \mathrm{up}\, f$.

Proof.

$\Rightarrow$. Obvious.

$\Leftarrow$. $\uparrow^{\mathrm{Ob}\, f} \mathcal{A}$ is connected regarding $\uparrow^{\mathsf{RLD}(\mathrm{Ob}\, f;\mathrm{Ob}\, f)} F$ iff $S(F) = F^0 \cup F^1 \cup F^2 \cup \ldots \in \mathrm{up}(\mathcal{A} \times^{\mathsf{RLD}} \mathcal{A})$.
$S^*(f) = \bigcap \{ \uparrow^{\mathsf{RLD}(\mathrm{Ob}\, \mu;\mathrm{Ob}\, \mu)} S(F) \mid F \in \mathrm{up}\, f \} \supseteq \bigcap \{ \mathcal{A} \times^{\mathsf{RLD}} \mathcal{A} \mid F \in \mathrm{up}\, f \} = \mathcal{A} \times^{\mathsf{RLD}} \mathcal{A}$. □

Conjecture 273. A filter object $\mathcal{A}$ is connected regarding a funcoid μ iff $\mathcal{A}$ is connected for every $F \in \langle \uparrow^{\mathsf{FCD}(\mathrm{Ob}\, \mu;\mathrm{Ob}\, \mu)} \rangle \mathrm{up}\, \mu$.

The above conjecture is open even for the case when $\mathcal{A}$ is a principal f.o.

Conjecture 274. A filter object $\mathcal{A}$ is connected regarding a reloid f iff it is connected regarding the funcoid $(\mathsf{FCD}) f$.

The above conjecture is true in the special case of principal filters:

Proposition 275. A f.o. $\uparrow^{\mathrm{Ob}\, \mu} A$ (for a set A) is connected regarding an endo-reloid f iff it is connected regarding the endo-funcoid $(\mathsf{FCD}) f$.

Proof. $\uparrow^{\mathrm{Ob}\, f} A$ is connected regarding a reloid f iff A is connected regarding every $F \in \mathrm{up}\, f$ that is when (taken in account that connectedness for $\uparrow^{\mathsf{RLD}(\mathrm{Ob}\, f;\mathrm{Ob}\, f)} F$ is the same as connectedness of $\uparrow^{\mathsf{FCD}(\mathrm{Ob}\, f;\mathrm{Ob}\, f)} F$)

$$\begin{aligned}
\forall F \in \mathrm{up}\, f \forall \mathcal{X}, \mathcal{Y} \in \mathfrak{F}(\mathrm{Ob}\, f) \setminus \{0^{\mathfrak{F}(\mathrm{Ob}\, f)}\} : (\mathcal{X} \cup \mathcal{Y} = \uparrow^{\mathrm{Ob}\, f} A \Rightarrow \mathcal{X} [\uparrow^{\mathsf{FCD}(\mathrm{Ob}\, f;\mathrm{Ob}\, f)} F] \mathcal{Y}) &\Leftrightarrow \\
\forall \mathcal{X}, \mathcal{Y} \in \mathfrak{F}(\mathrm{Ob}\, f) \setminus \{0^{\mathfrak{F}(\mathrm{Ob}\, f)}\} \forall F \in \mathrm{up}\, f : (\mathcal{X} \cup \mathcal{Y} = \uparrow^{\mathrm{Ob}\, f} A \Rightarrow \mathcal{X} [\uparrow^{\mathsf{FCD}(\mathrm{Ob}\, f;\mathrm{Ob}\, f)} F] \mathcal{Y}) &\Leftrightarrow \\
\forall \mathcal{X}, \mathcal{Y} \in \mathfrak{F}(\mathrm{Ob}\, f) \setminus \{0^{\mathfrak{F}(\mathrm{Ob}\, f)}\} : (\mathcal{X} \cup \mathcal{Y} = \uparrow^{\mathrm{Ob}\, f} A \Rightarrow \forall F \in \mathrm{up}\, f : \mathcal{X} [\uparrow^{\mathsf{FCD}(\mathrm{Ob}\, f;\mathrm{Ob}\, f)} F] \mathcal{Y}) &\Leftrightarrow \\
\forall \mathcal{X}, \mathcal{Y} \in \mathfrak{F}(\mathrm{Ob}\, f) \setminus \{0^{\mathfrak{F}(\mathrm{Ob}\, f)}\} : (\mathcal{X} \cup \mathcal{Y} = \uparrow^{\mathrm{Ob}\, f} A \Rightarrow \mathcal{X} [(\mathsf{FCD}) f] \mathcal{Y})
\end{aligned}$$

that is when the set $\uparrow^{\mathrm{Ob}\, f} A$ is connected regarding the funcoid $(\mathsf{FCD}) f$. □

7.5 Algebraic properties of S and S^*

Theorem 276. $S^*(S^*(f)) = S^*(f)$ for every endo-reloid f.

Proof. $S^*(S^*(f)) = \bigcap \{ \uparrow^{\mathsf{RLD}(\mathrm{Ob}\, f;\mathrm{Ob}\, f)} S(R) \mid R \in \mathrm{up}\, S^*(f) \} \subseteq \bigcap \{ \uparrow^{\mathsf{RLD}(\mathrm{Ob}\, f;\mathrm{Ob}\, f)} S(R) \mid R \in \{ S(F) \mid F \in \mathrm{up}\, f \} \} = \bigcap \{ \uparrow^{\mathsf{RLD}(\mathrm{Ob}\, f;\mathrm{Ob}\, f)} S(S(F)) \mid F \in \mathrm{up}\, f \} = \bigcap \{ \uparrow^{\mathsf{RLD}(\mathrm{Ob}\, f;\mathrm{Ob}\, f)} S(F) \mid F \in \mathrm{up}\, f \} = S^*(f)$.

So $S^*(S^*(f)) \subseteq S^*(f)$. That $S^*(S^*(f)) \supseteq S^*(f)$ is obvious. □

Corollary 277. $S^*(S(f)) = S(S^*(f)) = S^*(f)$ for any endo-reloid f.

Proof. Obviously $S^*(S(f)) \supseteq S^*(f)$ and $S(S^*(f)) \supseteq S^*(f)$.
But $S^*(S(f)) \subseteq S^*(S^*(f)) = S^*(f)$ and $S(S^*(f)) \subseteq S^*(S^*(f)) = S^*(f)$. □

Conjecture 278. $S(S(f)) = S(f)$ for

1. every endo-reloid f;
2. every endo-funcoid f.

Conjecture 279. For every endo-reloid f

1. $S(f) \circ S(f) = S(f)$;
2. $S^*(f) \circ S^*(f) = S^*(f)$;
3. $S(f) \circ S^*(f) = S^*(f) \circ S(f) = S^*(f)$.

Conjecture 280. $S(f) \circ S(f) = S(f)$ for every endo-funcoid f.

8 Postface

8.1 Misc

See this Web page for my research plans: http://www.mathematics21.org/agt-plans.html

I deem that now the most important research topics in Algebraic General Topology are:

- to solve the open problems mentioned in this work;
- define and research compactness of funcoids.
- research are n-ary (where n is an ordinal, or more generally an index set) funcoids and reloids (plain funcoids and reloids are binary by analogy with binary relations).

We should also research relationships between complete funcoids and complete reloids.

All my research of funcoids and reloids is presented at http://www.mathematics21.org/algebraic-general-topology.html

Appendix A Some counter-examples

For further examples we will use the filter object Δ defined by the formula

$$\Delta = \bigcap \left\{ \uparrow^{\mathfrak{F}(\mathbb{R})} (-\varepsilon; \varepsilon) \mid \varepsilon \in \mathbb{R}, \varepsilon > 0 \right\}.$$

I also will denote $\Omega(A)$ the Fréchet f.o. on the set A.

Example 281. There exist a funcoid f and a set S of funcoids such that $f \cap \bigcup S \neq \bigcup \langle f \cap \rangle S$.

Proof. Let $f = \Delta \times^{\mathsf{FCD}} \uparrow^{\mathfrak{F}(\mathbb{R})} \{0\}$ and $S = \left\{ \uparrow^{\mathsf{FCD}(\mathbb{R};\mathbb{R})} ((\varepsilon; +\infty) \times \{0\}) \mid \varepsilon > 0 \right\}$. Then $f \cap \bigcup S = \uparrow^{\mathsf{FCD}(\mathbb{R};\mathbb{R})} (\Delta \times \uparrow^{\mathfrak{F}(\mathbb{R})} \{0\}) \cap \uparrow^{\mathsf{FCD}(\mathbb{R};\mathbb{R})} ((0; +\infty) \times \{0\}) = (\Delta \cap \uparrow^{\mathfrak{F}(\mathbb{R})} (0; +\infty)) \times^{\mathsf{FCD}} \uparrow^{\mathfrak{F}(\mathbb{R})} \{0\} \neq 0^{\mathsf{FCD}(\mathbb{R};\mathbb{R})}$ while $\bigcup \langle f \cap \rangle S = \bigcup \left\{ 0^{\mathsf{FCD}(\mathbb{R};\mathbb{R})} \right\} = 0^{\mathsf{FCD}(\mathbb{R};\mathbb{R})}$. □

Conjecture 282. There exist a set R of funcoids and a funcoid f such that $f \circ \bigcup R \neq \bigcup \langle f \circ \rangle R$.

Example 283. There exist a set R of funcoids and f.o. $\mathcal{X}$ and $\mathcal{Y}$ such that

1. $\mathcal{X} [\bigcup R] \mathcal{Y} \wedge \nexists f \in R: \mathcal{X} [f] \mathcal{Y}$;
2. $\langle \bigcup R \rangle \mathcal{X} \supset \bigcup \{ \langle f \rangle \mathcal{X} \mid f \in R \}$.

Proof.

1. Let $\mathcal{X} = \Delta$ and $\mathcal{Y} = 1^{\mathfrak{F}(\mathbb{R})}$. Let $R = \left\{ \uparrow^{\mathsf{FCD}(\mathbb{R};\mathbb{R})} ((\varepsilon; +\infty) \times \mathbb{R}) \mid \varepsilon \in \mathbb{R}, \varepsilon > 0 \right\}$. Then $\bigcup R = \uparrow^{\mathsf{FCD}(\mathbb{R};\mathbb{R})} ((0; +\infty) \times \mathbb{R})$. So $\mathcal{X} [\bigcup R] \mathcal{Y}$ and $\forall f \in R: \neg (\mathcal{X} [f] \mathcal{Y})$.
2. With the same $\mathcal{X}$ and R we have $\langle \bigcup R \rangle \mathcal{X} = \mathbb{R}$ and $\langle f \rangle \mathcal{X} = 0^{\mathfrak{F}(\mathbb{R})}$ for every $f \in R$, thus $\bigcup \{ \langle f \rangle \mathcal{X} \mid f \in R \} = 0^{\mathfrak{F}(\mathbb{R})}$. □

Theorem 284. For a f.o. a we have $a \times^{\mathsf{RLD}} a \subseteq I^{\mathsf{RLD}(\mathrm{Base}(a))}$ only in the case if $a = 0^{\mathfrak{F}(\mathrm{Base}(a))}$ or a is a trivial atomic f.o. (that is corresponds to an one-element set).

Proof. If $a \times^{\mathsf{RLD}} a \subseteq I^{\mathsf{RLD}(\mathrm{Base}(a))}$ then exists $m \in \mathrm{up}(a \times^{\mathsf{RLD}} a)$ such that $m \subseteq I_{\mathrm{Base}(a)}$. Consequently exist $A, B \in \mathrm{up}\, a$ such that $A \times B \subseteq I_{\mathrm{Base}(a)}$ what is possible only in the case when $A = B = a$ is an one-element set or empty set. □

Corollary 285. Reloidal product of a non-trivial atomic filter object with itself is non-atomic.

Proof. Obviously $(a \times^{\mathsf{RLD}} a) \cap I^{\mathsf{RLD}(\mathrm{Base}(a))} \neq 0^{\mathfrak{F}(\mathrm{Base}(a))}$ and $(a \times^{\mathsf{RLD}} a) \cap I^{\mathsf{RLD}(\mathrm{Base}(a))} \subset a \times^{\mathsf{RLD}} a$. □

Example 286. $(\mathsf{RLD})_{\mathrm{in}} f \neq (\mathsf{RLD})_{\mathrm{out}} f$ for a funcoid f.

Proof. Let $f = I^{\mathsf{FCD}(\mathbb{N})}$. Then $(\mathsf{RLD})_{\mathrm{in}} f = \bigcup \left\{ a \times^{\mathsf{RLD}} a \mid a \in \mathrm{atoms}\, 1^{\mathfrak{F}(\mathbb{N})} \right\}$ and $(\mathsf{RLD})_{\mathrm{out}} f = I^{\mathsf{RLD}(\mathbb{N})}$. But as we shown above $a \times^{\mathsf{RLD}} a \nsubseteq I^{\mathsf{RLD}(\mathbb{N})}$ for non-trivial f.o. a, and so $(\mathsf{RLD})_{\mathrm{in}} f \nsubseteq (\mathsf{RLD})_{\mathrm{out}} f$. □

Proposition 287. $I^{\mathsf{FCD}(\mathbb{N})} \cap \uparrow^{\mathsf{FCD}(\mathbb{N};\mathbb{N})}((\mathbb{N} \times \mathbb{N}) \setminus I_{\mathbb{N}}) = I^{\mathsf{FCD}}_{\Omega(\mathbb{N})} \neq 0^{\mathsf{FCD}(\mathbb{N};\mathbb{N})}$.

Proof. Note that $\left\langle I^{\mathsf{FCD}}_{\Omega(\mathbb{N})} \right\rangle \mathcal{X} = \mathcal{X} \cap \Omega(\mathbb{N})$.

Let $f = I^{\mathsf{FCD}(\mathbb{N})}$, $g = \uparrow^{\mathsf{FCD}(\mathbb{N};\mathbb{N})}((\mathbb{N} \times \mathbb{N}) \setminus I_{\mathbb{N}})$.

Let x is a non-trivial atomic f.o. If $X \in \mathrm{up}\, x$ then $\mathrm{card}\, X \geqslant 2$ (In fact, X is infinite but we don't need this.) and consequently $\langle g \rangle^* X = 1^{\mathfrak{F}(\mathbb{N})}$. Thus $\langle g \rangle x = 1^{\mathfrak{F}(\mathbb{N})}$. Consequently

$$\langle f \cap g \rangle x = \langle f \rangle x \cap \langle g \rangle x = x \cap 1^{\mathfrak{F}(\mho)} = x.$$

Also $\left\langle I^{\mathsf{FCD}}_{\Omega(\mathbb{N})} \right\rangle x = x \cap \Omega(\mathbb{N}) = x$.

Let now x is a trivial f.o. Then $\langle f \rangle x = x$ and $\langle g \rangle x = 1^{\mathfrak{F}(\mathbb{N})} \setminus x$. So

$$\langle f \cap g \rangle x = \langle f \rangle x \cap \langle g \rangle x = x \cap \left(1^{\mathfrak{F}(\mathbb{N})} \setminus x\right) = 0^{\mathfrak{F}(\mathbb{N})}.$$

Also $\left\langle I^{\mathsf{FCD}}_{\Omega(\mathbb{N})} \right\rangle x = x \cap \Omega(\mathbb{N}) = 0^{\mathfrak{F}(\mathbb{N})}$.

So $\langle f \cap g \rangle x = \left\langle I^{\mathsf{FCD}}_{\Omega(\mathbb{N})} \right\rangle x$ for every atomic f.o. x. Thus $f \cap g = I^{\mathsf{FCD}}_{\Omega(\mathbb{N})}$. □

Example 288. There exist binary relations f and g such that $\uparrow^{\mathsf{FCD}(A;B)} f \cap \uparrow^{\mathsf{FCD}(A;B)} g \neq f \cap g$ for some sets A, B such that $f, g \subseteq A \times B$.

Proof. From the proposition above. □

Example 289. There exists a discrete funcoid which is not a complemented element of the lattice of funcoids.

Proof. I will prove that quasi-complement (see [15] for the definition of quasi-complement) of the funcoid $I^{\mathsf{FCD}(\mathbb{N})}$ is not its complement. We have:

$$\begin{aligned}
\left(I^{\mathsf{FCD}(\mathbb{N})}\right)^* &= \bigcup \left\{ c \in \mathsf{FCD} \mid c \asymp I^{\mathsf{FCD}(\mathbb{N})} \right\} \\
&\supseteq \bigcup \left\{ \uparrow^{\mathbb{N}}\{\alpha\} \times^{\mathsf{FCD}} \uparrow^{\mathbb{N}}\{\beta\} \mid \alpha, \beta \in \mathbb{N}, \{\alpha\} \times^{\mathsf{FCD}} \{\beta\} \asymp I^{\mathsf{FCD}(\mathbb{N})} \right\} \\
&= \bigcup \left\{ \uparrow^{\mathbb{N}}\{\alpha\} \times^{\mathsf{FCD}} \uparrow^{\mathbb{N}}\{\beta\} \mid \alpha, \beta \in \mathbb{N}, \alpha \neq \beta \right\} \\
&= \uparrow^{\mathsf{FCD}(\mathbb{N};\mathbb{N})} \bigcup \left\{ \{\alpha\} \times \{\beta\} \mid \alpha, \beta \in \mathbb{N}, \alpha \neq \beta \right\} \\
&= \uparrow^{\mathsf{FCD}(\mathbb{N};\mathbb{N})}(\mathbb{N} \times \mathbb{N} \setminus I_{\mathbb{N}})
\end{aligned}$$

(used the corollary 111). But by proved above

$$\left(I^{\mathsf{FCD}(\mathbb{N})}\right)^* \cap I^{\mathsf{FCD}(\mathbb{N})} \neq 0^{\mathfrak{F}(\mathbb{N})}.$$ □

Example 290. There exists funcoid h such that $\mathrm{up}\, h$ is not a filter.

Proof. Consider the funcoid $h = I^{\mathsf{FCD}}_{\Omega(\mathbb{N})}$. We have (from the proposition) that $f \in \operatorname{up} h$ and $g \in \operatorname{up} f$, but $f \cap g = \emptyset \notin \operatorname{up} h$. □

Example 291. There exists a funcoid $h \neq 0^{\mathsf{FCD}(A;B)}$ such that $(\mathsf{RLD})_{\mathrm{out}} h = 0^{\mathsf{RLD}(A;B)}$.

Proof. Consider $h = I^{\mathsf{FCD}}_{\Omega(\mathbb{N})}$. By proved above $h = f \cap g$ where $f = I^{\mathsf{FCD}(\mathbb{N})}$, $g = \uparrow^{\mathsf{FCD}(\mathbb{N};\mathbb{N})}((\mathbb{N} \times \mathbb{N}) \setminus I_{\mathbb{N}})$.

We have $f, g \in \operatorname{up} h$.

So $(\mathsf{RLD})_{\mathrm{out}} h = \bigcap \langle \uparrow^{\mathsf{RLD}(\mathbb{N};\mathbb{N})} \rangle \operatorname{up} h \subseteq \uparrow^{\mathsf{RLD}(\mathbb{N};\mathbb{N})}(f \cap g) = 0^{\mathsf{RLD}(\mathbb{N};\mathbb{N})}$; and thus $(\mathsf{RLD})_{\mathrm{out}} h = 0^{\mathsf{RLD}(\mathbb{N};\mathbb{N})}$. □

Example 292. There exists a funcoid h such that $(\mathsf{FCD})(\mathsf{RLD})_{\mathrm{out}} h \neq h$.

Proof. Follows from the previous example. □

Example 293. $(\mathsf{RLD})_{\mathrm{in}}(\mathsf{FCD}) f \neq f$ for some convex reloid f.

Proof. Let $f = I^{\mathsf{RLD}(\mathbb{N})}$. Then $(\mathsf{FCD}) f = I^{\mathsf{FCD}(\mathbb{N})}$. Let a is some nontrivial atomic f.o. Then $(\mathsf{RLD})_{\mathrm{in}}(\mathsf{FCD}) f \supseteq a \times^{\mathsf{RLD}} a \nsubseteq I^{\mathsf{RLD}(\mathbb{N})}$ and thus $(\mathsf{RLD})_{\mathrm{in}}(\mathsf{FCD}) f \nsubseteq f$. □

Remark 294. Before I found the last counter-example, I thought that $(\mathsf{RLD})_{\mathrm{in}}$ is an isomorphism from the set of of funcoids to the set of convex reloids. As this conjecture failed, we need an other way to characterize the set of reloids isomorphic to funcoids.

Example 295. There exist funcoids f and g such that

$$(\mathsf{RLD})_{\mathrm{out}}(g \circ f) \neq (\mathsf{RLD})_{\mathrm{out}} g \circ (\mathsf{RLD})_{\mathrm{out}} f.$$

Proof. Take $f = I^{\mathsf{FCD}}_{\Omega(\mathbb{N})}$ and $g = 1^{\mathfrak{F}(\mathbb{N})} \times^{\mathsf{FCD}} \uparrow^{\mathbb{N}} \{\alpha\}$ for some $\alpha \in \mathbb{N}$. Then $(\mathsf{RLD})_{\mathrm{out}} f = 0^{\mathsf{RLD}(\mathbb{N};\mathbb{N})}$ and thus $(\mathsf{RLD})_{\mathrm{out}} g \circ (\mathsf{RLD})_{\mathrm{out}} f = 0^{\mathsf{RLD}(\mathbb{N};\mathbb{N})}$.

We have $g \circ f = \Omega(\mathbb{N}) \times^{\mathsf{FCD}} \uparrow^{\mathbb{N}} \{\alpha\}$.

Let's prove $(\mathsf{RLD})_{\mathrm{out}}(\Omega(\mathbb{N}) \times^{\mathsf{FCD}} \uparrow^{\mathbb{N}} \{\alpha\}) = \Omega(\mathbb{N}) \times^{\mathsf{RLD}} \uparrow^{\mathbb{N}} \{\alpha\}$.

Really: $(\mathsf{RLD})_{\mathrm{out}}(\Omega(\mathbb{N}) \times^{\mathsf{FCD}} \uparrow^{\mathbb{N}} \{\alpha\}) = \bigcap \uparrow^{\mathsf{RLD}(\mathbb{N};\mathbb{N})} \operatorname{up}(\Omega(\mathbb{N}) \times^{\mathsf{FCD}} \{\alpha\}) = \bigcap \{\uparrow^{\mathsf{RLD}(\mathbb{N};\mathbb{N})}(K \times \{\alpha\}) \mid K \in \operatorname{up} \Omega(\mathbb{N})\}$.

$F \in \operatorname{up} \bigcap \{\uparrow^{\mathsf{RLD}(\mathbb{N};\mathbb{N})}(K \times \{\alpha\}) \mid K \in \operatorname{up} \Omega(\mathbb{N})\} \Leftrightarrow F \in \operatorname{up}(\bigcap \{\uparrow^{\mathbb{N}} K \mid K \in \operatorname{up} \Omega(\mathbb{N})\} \times^{\mathsf{RLD}} \uparrow^{\mathbb{N}} \{\alpha\})$ for every $F \in \mathscr{P}(\mathbb{N} \times \mathbb{N})$. Thus

$$\bigcap \{\uparrow^{\mathsf{RLD}(\mathbb{N};\mathbb{N})}(K \times \{\alpha\}) \mid K \in \operatorname{up} \Omega(\mathbb{N})\} = \bigcap \{\uparrow^{\mathbb{N}} K \mid K \in \operatorname{up} \Omega(\mathbb{N})\} \times^{\mathsf{RLD}} \uparrow^{\mathbb{N}} \{\alpha\} = \Omega(\mathbb{N}) \times^{\mathsf{RLD}} \uparrow^{\mathbb{N}} \{\alpha\}.$$

So $(\mathsf{RLD})_{\mathrm{out}}(\Omega(\mathbb{N}) \times^{\mathsf{FCD}} \uparrow^{\mathbb{N}} \{\alpha\}) = \Omega(\mathbb{N}) \times^{\mathsf{RLD}} \uparrow^{\mathbb{N}} \{\alpha\}$.

Thus $(\mathsf{RLD})_{\mathrm{out}}(g \circ f) = \Omega(\mathbb{N}) \times^{\mathsf{RLD}} \{\alpha\} \neq 0^{\mathsf{RLD}(\mathbb{N};\mathbb{N})}$. □

Example 296. (FCD) does not preserve finite meets.

Proof. $(\mathsf{FCD})(I^{\mathsf{RLD}(\mathbb{N})} \cap (1^{\mathsf{RLD}(\mathbb{N};\mathbb{N})} \setminus I^{\mathsf{RLD}(\mathbb{N})})) = (\mathsf{FCD}) 0^{\mathsf{RLD}(\mathbb{N};\mathbb{N})} = 0^{\mathsf{FCD}(\mathbb{N};\mathbb{N})}$.

On the other hand

$$(\mathsf{FCD}) I^{\mathsf{RLD}(\mathbb{N})} \cap (\mathsf{FCD})(1^{\mathsf{RLD}(\mathbb{N};\mathbb{N})} \setminus I^{\mathsf{RLD}(\mathbb{N})})) = I^{\mathsf{FCD}(\mathbb{N})} \cap (1^{\mathsf{FCD}(\mathbb{N};\mathbb{N})} \setminus I^{\mathsf{FCD}(\mathbb{N})}) = I^{\mathsf{FCD}}_{\Omega(\mathbb{N})} \neq 0^{\mathsf{FCD}(\mathbb{N};\mathbb{N})}$$

(used the proposition 205). □

Corollary 297. (FCD) is not an upper adjoint (in general).

Considering restricting polynomials (considered as reloids) to atomic filter objects, it is simple to prove that each that restriction is injective if not restricting a constant polynomial. Does this hold in general? No, see the following example:

Example 298. There exists a monovalued reloid with atomic domain which is neither injective nor constant (that is not a restriction of a constant function).

Proof. (based on [16]) Consider the function $F \in \mathbb{N}^{\mathbb{N} \times \mathbb{N}}$ defined by the formula $(x; y) \mapsto x$.

Let ω_x is a non-principal atomic filter object on the vertical line $\{x\} \times \mathbb{N}$ for every $x \in \mathbb{N}$.

Let T is the collection of such sets Y that $Y \cap (\{x\} \times \mathbb{N}) \in \operatorname{up} \omega_x$ for all but finitely many vertical lines. Obviously T is a filter.

Let $\omega \in \operatorname{atoms} \operatorname{up}^{-1} T$.

For every $x \in \mathbb{N}$ we have some $Y \in T$ for which $(\{x\} \times \mathbb{N}) \cap Y = \emptyset$ and thus $(\{x\} \times \mathbb{N}) \cap \operatorname{up} \omega = \emptyset$.

Let $g = \left(\uparrow^{\mathsf{RLD}(\mathbb{N};\mathbb{N})} F\right)|_{\omega}$. If g is constant, then there exist a constant function $G \in \operatorname{up} g$ and $F \cap G$ is also constant. Obviously $\operatorname{dom} \uparrow^{\mathsf{RLD}(\mathbb{N};\mathbb{N})}(F \cap G) \supseteq \omega$. The function $F \cap G$ cannot be constant because otherwise $\omega \subseteq \operatorname{dom} \uparrow^{\mathsf{RLD}(\mathbb{N};\mathbb{N})}(F \cap G) \subseteq \uparrow^{\mathbb{N}}\{x\} \times \mathbb{N}$ for some $x \in \mathbb{N}$ what is impossible by proved above. So g is not constant.

Suppose there g is injective. Then there exists an injection $G \in \operatorname{up} g$. So $\operatorname{dom} G$ intersects each vertical line by atmost one element that is $\overline{\operatorname{dom} G}$ intersects every vertical line by the whole line or the line without one element. Thus $\overline{\operatorname{dom} G} \in T \subseteq \operatorname{up} \omega$ and consequently $\operatorname{dom} G \notin \operatorname{up} \omega$ what is impossible.

Thus g is neither injective nor constant. □

9 Second product. Oblique product

Definition 299. $\mathcal{A} \times^{\mathsf{RLD}}_F \mathcal{B} \overset{\text{def}}{=} (\mathsf{RLD})_{\text{out}}(\mathcal{A} \times^{\mathsf{FCD}} \mathcal{B})$ for every f.o. $\mathcal{A}$ and $\mathcal{B}$. I will call it *second direct product* of filter objects $\mathcal{A}$ and $\mathcal{B}$.

Remark 300. The letter F is the above definition is from the word "funcoid". It signifies that it seems to be impossible to define $\mathcal{A} \times^{\mathsf{RLD}}_F \mathcal{B}$ directly without refering to the direct product $\mathcal{A} \times^{\mathsf{FCD}} \mathcal{B}$ of funcoids.

Definition 301. *Oblique products* of f.o. $\mathcal{A}$ and $\mathcal{B}$ are defined as

$\mathcal{A} \ltimes \mathcal{B} = \bigcap \left\{\uparrow^{\mathsf{RLD}(\operatorname{Base}(\mathcal{A});\operatorname{Base}(\mathcal{B}))} f \mid f \in \mathscr{P}(\operatorname{Base}(\mathcal{A}) \times \operatorname{Base}(\mathcal{B})), \forall B \in \operatorname{up} \mathcal{B}: \uparrow^{\mathsf{FCD}(\operatorname{Base}(\mathcal{A});\operatorname{Base}(\mathcal{B}))} f \supseteq \mathcal{A} \times^{\mathsf{FCD}} \uparrow^{\operatorname{Base}(\mathcal{B})} B\right\}$;

$\mathcal{A} \rtimes \mathcal{B} = \bigcap \left\{\uparrow^{\mathsf{RLD}(\operatorname{Base}(\mathcal{A});\operatorname{Base}(\mathcal{B}))} f \mid f \in \mathscr{P}(\operatorname{Base}(\mathcal{A}) \times \operatorname{Base}(\mathcal{B})), \forall A \in \operatorname{up} \mathcal{A}: \uparrow^{\mathsf{FCD}(\operatorname{Base}(\mathcal{A});\operatorname{Base}(\mathcal{B}))} f \supseteq \uparrow^{\operatorname{Base}(\mathcal{A})} A \times^{\mathsf{FCD}} \mathcal{B}\right\}$.

Proposition 302. $\mathcal{A} \times^{\mathsf{RLD}}_F \mathcal{B} \subseteq \mathcal{A} \ltimes \mathcal{B} \subseteq \mathcal{A} \times^{\mathsf{RLD}} \mathcal{B}$ for every f.o. $\mathcal{A}$, $\mathcal{B}$.

Proof. $\mathcal{A} \ltimes \mathcal{B} \subseteq \bigcap \left\{\uparrow^{\mathsf{RLD}(\operatorname{Base}(\mathcal{A});\operatorname{Base}(\mathcal{B}))} f \mid f \in \mathscr{P}(\operatorname{Base}(\mathcal{A}) \times \operatorname{Base}(\mathcal{B})), \forall A \in \operatorname{up} \mathcal{A}, \forall B \in \operatorname{up} \mathcal{B}: \uparrow^{\mathsf{FCD}(\operatorname{Base}(\mathcal{A});\operatorname{Base}(\mathcal{B}))} f \supseteq \uparrow^{\operatorname{Base}(\mathcal{A})} A \times^{\mathsf{FCD}} \uparrow^{\operatorname{Base}(\mathcal{B})} B\right\} \subseteq \bigcap \left\{\uparrow^{\operatorname{Base}(\mathcal{A})} A \times^{\mathsf{FCD}} \uparrow^{\operatorname{Base}(\mathcal{B})} B \mid A \in \operatorname{up} \mathcal{A}, B \in \operatorname{up} \mathcal{B}\right\} = \mathcal{A} \times^{\mathsf{RLD}} \mathcal{B}$.

$\mathcal{A} \ltimes \mathcal{B} \supseteq \bigcap \left\{\uparrow^{\mathsf{RLD}(\operatorname{Base}(\mathcal{A});\operatorname{Base}(\mathcal{B}))} f \mid f \in \mathscr{P}(\operatorname{Base}(\mathcal{A}) \times \operatorname{Base}(\mathcal{B})), \uparrow^{\mathsf{FCD}(\operatorname{Base}(\mathcal{A});\operatorname{Base}(\mathcal{B}))} f \supseteq \mathcal{A} \times^{\mathsf{FCD}} \mathcal{B}\right\} = \bigcap \left\{\uparrow^{\mathsf{RLD}(\operatorname{Base}(\mathcal{A});\operatorname{Base}(\mathcal{B}))} f \mid f \in \operatorname{up}(\mathcal{A} \times^{\mathsf{FCD}} \mathcal{B})\right\} = (\mathsf{RLD})_{\text{out}}(\mathcal{A} \times^{\mathsf{FCD}} \mathcal{B}) = \mathcal{A} \times^{\mathsf{RLD}}_F \mathcal{B}$. □

Conjecture 303. $\mathcal{A} \times^{\mathsf{RLD}}_F \mathcal{B} \subset \mathcal{A} \ltimes \mathcal{B}$ for some f.o. $\mathcal{A}$, $\mathcal{B}$.

A stronger conjecture:

Conjecture 304. $\mathcal{A} \times^{\mathsf{RLD}}_F \mathcal{B} \subset \mathcal{A} \ltimes \mathcal{B} \subset \mathcal{A} \times^{\mathsf{RLD}} \mathcal{B}$ for some f.o. $\mathcal{A}$, $\mathcal{B}$. Particularly, is this formula true for $\mathcal{A} = \mathcal{B} = \Delta \cap \uparrow^{\mathbb{R}}(0; +\infty)$?

The above conjecture is similar to Fermat Last Theorem as having no value by itself but being somehow challenging to prove it.

Example 305. $\mathcal{A} \ltimes \mathcal{B} \subset \mathcal{A} \times^{\mathsf{RLD}} \mathcal{B}$ for some f.o. $\mathcal{A}$, $\mathcal{B}$.

Proof. It's enough to prove $\mathcal{A} \ltimes \mathcal{B} \neq \mathcal{A} \times^{\mathsf{RLD}} \mathcal{B}$.

Let $\Delta_+ = \Delta \cap (0; +\infty)$. Let $\mathcal{A} = \mathcal{B} = \Delta_+$.

Let $K = (\geqslant)|_{\mathbb{R} \times \mathbb{R}}$.

Obviously $K \notin \operatorname{up}(\mathcal{A} \times^{\mathsf{RLD}} \mathcal{B})$.

$\mathcal{A} \ltimes \mathcal{B} \subseteq \uparrow^{\mathsf{RLD}(\operatorname{Base}(\mathcal{A}); \operatorname{Base}(\mathcal{B}))} K$ and thus $K \in \operatorname{up}(\mathcal{A} \ltimes \mathcal{B})$ because $\uparrow^{\mathsf{FCD}(\operatorname{Base}(\mathcal{A}); \operatorname{Base}(\mathcal{B}))} K \supseteq \Delta_+ \times^{\mathsf{FCD}} \uparrow^{\operatorname{Base}(\mathcal{B})} B = \mathcal{A} \times^{\mathsf{FCD}} \uparrow^{\operatorname{Base}(\mathcal{B})} B$ for $B = (0; +\infty)$.

Thus $\mathcal{A} \ltimes \mathcal{B} \neq \mathcal{A} \times^{\mathsf{RLD}} \mathcal{B}$. □

Example 306. $\mathcal{A} \times^{\mathsf{RLD}}_F \mathcal{B} \subset \mathcal{A} \times^{\mathsf{RLD}} \mathcal{B}$ for some f.o. $\mathcal{A}$, $\mathcal{B}$.

Proof. This follows from the above example. □

Proposition 307. $(\mathcal{A} \ltimes \mathcal{B}) \cap (\mathcal{A} \rtimes \mathcal{B}) = \mathcal{A} \times^{\mathsf{RLD}}_F \mathcal{B}$ for every f.o. $\mathcal{A}$, $\mathcal{B}$.

Proof. $(\mathcal{A} \ltimes \mathcal{B}) \cap (\mathcal{A} \rtimes \mathcal{B}) \subseteq \bigcap \left\{ \uparrow^{\mathsf{RLD}(\operatorname{Base}(\mathcal{A}); \operatorname{Base}(\mathcal{B}))} f \mid f \in \mathscr{P}(\operatorname{Base}(\mathcal{A}) \times \operatorname{Base}(\mathcal{B})), \uparrow^{\mathsf{FCD}(\operatorname{Base}(\mathcal{A}); \operatorname{Base}(\mathcal{B}))} f \supseteq \mathcal{A} \times^{\mathsf{FCD}} \mathcal{B} \right\} = \bigcap \left\{ \uparrow^{\mathsf{RLD}(\operatorname{Base}(\mathcal{A}); \operatorname{Base}(\mathcal{B}))} f \mid f \in \operatorname{up}(\mathcal{A} \times^{\mathsf{FCD}} \mathcal{B}) \right\} = (\mathsf{RLD})_{\mathrm{out}}(\mathcal{A} \times^{\mathsf{FCD}} \mathcal{B}) = \mathcal{A} \times^{\mathsf{RLD}}_F \mathcal{B}$.

To finish the proof we need to show $\mathcal{A} \ltimes \mathcal{B} \supseteq \mathcal{A} \times^{\mathsf{RLD}}_F \mathcal{B}$ and $\mathcal{A} \rtimes \mathcal{B} \supseteq \mathcal{A} \times^{\mathsf{RLD}}_F \mathcal{B}$. By symmetry it's enough to show $\mathcal{A} \ltimes \mathcal{B} \supseteq \mathcal{A} \times^{\mathsf{RLD}}_F \mathcal{B}$ what is proved above. □

Example 308. $(\mathcal{A} \ltimes \mathcal{B}) \cup (\mathcal{A} \rtimes \mathcal{B}) \subset \mathcal{A} \times^{\mathsf{RLD}} \mathcal{B}$ for some f.o. $\mathcal{A}$, $\mathcal{B}$.

Proof. (based on [3]) Let $\mathcal{A} = \mathcal{B} = \Omega(\mathbb{N})$. It's enough to prove $(\mathcal{A} \ltimes \mathcal{B}) \cup (\mathcal{A} \rtimes \mathcal{B}) \neq \mathcal{A} \times^{\mathsf{RLD}} \mathcal{B}$.

Let $X \in \operatorname{up} \mathcal{A}$, $Y \in \operatorname{up} \mathcal{B}$ that is $X \in \Omega(\mathbb{N})$, $Y \in \Omega(\mathbb{N})$.

Removing one element x from X produces a set P. Removing one element y from Y produces a set Q. Obviously $P \in \Omega(\mathbb{N})$, $Q \in \Omega(\mathbb{N})$.

Obviously $(P \times \mathbb{N}) \cup (\mathbb{N} \times Q) \in \operatorname{up}((\mathcal{A} \ltimes \mathcal{B}) \cup (\mathcal{A} \rtimes \mathcal{B}))$.

$(P \times \mathbb{N}) \cup (\mathbb{N} \times Q) \nsupseteq X \times Y$ because $(x; y) \in X \times Y$ but $(x; y) \notin (P \times \mathbb{N}) \cup (\mathbb{N} \times Q)$.

Thus $(P \times \mathbb{N}) \cup (\mathbb{N} \times Q) \notin \operatorname{up}(\mathcal{A} \times^{\mathsf{RLD}} \mathcal{B})$ by properties of filter bases. □

Example 309. $(\mathsf{RLD})_{\mathrm{out}}(\mathsf{FCD}) f \neq f$ for some convex reloid f.

Proof. Let $f = \mathcal{A} \times^{\mathsf{RLD}} \mathcal{B}$ where $\mathcal{A}$ and $\mathcal{B}$ are from the previous example.

$(\mathsf{FCD})(\mathcal{A} \times^{\mathsf{RLD}} \mathcal{B}) = \mathcal{A} \times^{\mathsf{FCD}} \mathcal{B}$ by the proposition 213.

So $(\mathsf{RLD})_{\mathrm{out}}(\mathsf{FCD})(\mathcal{A} \times^{\mathsf{RLD}} \mathcal{B}) = (\mathsf{RLD})_{\mathrm{out}}(\mathcal{A} \times^{\mathsf{FCD}} \mathcal{B}) = \mathcal{A} \times^{\mathsf{RLD}}_F \mathcal{B} \neq \mathcal{A} \times^{\mathsf{RLD}} \mathcal{B}$. □

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