FOUNDATIONS OF SANTILLI'S ISONUMBER THEORY

With Applications to New Cryptograms, Fermat's Theorem and Goldbach's Conjecture

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With Applications to New Cryptograms, Fermat's Theorem and Goldbach's Conjecture

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Foreword

In my works (see the bibliography at the end of the Preface) I often expressed the view that the protracted lack of resolution of fundamental problems in science signals the needs of basically new mathematics. This is the case, for example, for: quantitative representations of biological structures; resolution of the vexing problem of grand-unification; invariant treatment of irreversibility at the classical and operator levels; identification of hadronic constituents definable in our spacetime; achievement of a classical representation of antimatter; and other basic open problems.

I have then shown that each of the above open problems admits basically novel resolutions if a new mathematics specifically conceived for the task at hand is built. I have moreover shown that no new mathematics is actually possible without new numbers. For this reason, as a physicist, I have dedicated my primary attention throughout all of my research life to the search of new numbers, because from new numbers new mathematical and physical theories can be constructed via mere compatibility arguments.

In view of the above, I would like to express my utmost appreciation to Professor **Chun-Xuan Jiang** for having understood the significance of the new iso-, geno-, hyper-numbers and their isoduals I identified for a resolution of the above problems. The significance of the new numbers had escaped other scholars in number theory in the past two decades since their original formulation.

I would like also to congratulate Professor Jiang for the simply monumental work he has done in this monograph, work that, to my best knowledge, has no prior occurrence in the history of number theory in regard to joint novelty, dimension, diversification, articulation and implications.

I have no doubt that Professor Jiang's monograph creates a new era in number theory which encompasses and includes as particular case all preceding work in the field.

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Preface

> There cannot be really new physical theories without really new mathematics, and there cannot be really new mathematics without new numbers.
>
> *Ruggero Maria Santilli*

It is shown by a number of major developments in the last century that the standard set of mathematical tools and physical models are not sufficient to allow us to faithfully describe the divers complex systems that arise in all branches of science and engineering.

This monograph is dedicated to the numerical foundations of a new important mathematics called *isomathematics* discovered by the Italian-American physicist **Ruggero Maria Santilli**, President of The Institute for Basic Research, Palm Harbor, Florida, USA (e-mail: ibr@gte.net, web site: http://www.i-b-r.org).

On August 23, 1997, Prof. Santilli was invited by the Chinese Academy of Sciences to conv

The reader should be informed that the pioneering research conducted by Prof. Santilli and his associates in various countries covers over 10,000 pages of publications including over 1,000 papers, some 25 post Ph.D. level monographs and some 50 volume of conference proceedings. It is evident that I cannot possibly review this vast literature to any extent. Therefore, in this Preface I shall only outline the original contributions by Prof. Santilli as listed at the end of this Preface, and apologize to his collaborators for the impossibility of quoting their subsequent work (for the latter, one can inspect the comprehensive bibliography of Prof. Santilli's recent monograph [16]).

To outline the main ideas, I use a simple mathematical language understandable to a broader audience.

1. Ordinary Numbers. Suppose that

$$a^0 = 1, \tag{1}$$

is the multiplicative unit and 0 the additive unit.

From (1) we define the multiplication ($\times$) and division ($\div$)

$$a \times b = ab = c \qquad\qquad a \div b = \frac{a}{b} = d. \tag{2}$$

The addition ($+$), substraction ($-$), multiplication ($\times$) and division ($\div$) form four arithmetic operations of ordinary numbers. In particular, the multiplicative unit (hereinafter simply referred to as the "unit") verifies the properties

$$1 \times a = a \times 1 = a, \quad 1^n = 1. \tag{3}$$

When the set of numbers a, b, c, ... is closed under the above operations it characterizes a field.

2. Santilli's Isodual Numbers. In papers [92, 93] of 1985 (see paper [129] for a technical treatment), Santilli introduced, apparently for the first time in the history of mathematics, the *negative-definite unit*

$$a^{\bar{0}} = -1, \tag{4}$$

which he called *isodual unit,* then lifted the conventional multiplication and division of numbers into the forms

$$a \bar{\times} b = a(-1)b = -ab, \qquad a \bar{\div} b = -\frac{a}{b}, \tag{5}$$

$$a \bar{\times} a \bar{\times} \cdots \bar{\times} a = a^{\bar{n}} = a^n(-1)^{n-1}, \qquad a^{-\bar{n}} = -1 \bar{\div} a^{\bar{n}} = a^{-n}(-1)^{-n-1}, \tag{6}$$

and proved that, if the original set of numbers forms a field, all axioms of a field remain verified under the above reformulation, resulting in a new field today called *Santilli isodual field.*

In particular, the isodual unit remains the correct left and right unit under the isodual product

$$(-1)\bar{\times}a = a\bar{\times}(-1) = a, \quad (-1)^{\bar{n}} = -1, \quad n = 0, 1, 2, \ldots \tag{7}$$

The reader should meditate a moment on the novelty of Santilli isodual numbers. In fact, these numbers have a *negative norm;* they imply the reversal of the sign of *all* conventional numbers; and *negative* isodual numbers have a meaning equivalent to that of conventional *positive* numbers (because the former are referred to a negative unit while the latter are referred to a positive unit).

Santilli also introduced a simple method for the construction of his isodual formulations [16] which consists in applying the following map (which is anti-canonical for classical treatments or anti-unitary for operator treatments)

$$A(x, \psi, ...) \rightarrow \bar{A}(\bar{x}, \bar{\psi}, ...) = U \left[A(UxU^{\dagger}, U\psi U^{\dagger}, ...) \right] U^{\dagger}, \quad UU^{\dagger} = -1, \tag{8}$$

to the *totality* of quantities and their operations of conventional formulations.

Santilli then reformulated most of conventional mathematics in terms of his isodual fields (see, e.g., Ref. [222]), and used the resulting new *isodual mathematics* for the first known *classical* formulation of antimatter.

It should be indicated that, prior to Santilli's studies, *a mathematics for the consistent classical description of antimatter simply did not exist.* This is due to the fact that *the correct treatment of antimatter requires a mathematics which is anti-isomorphic to that used for the treatment of matter.* Such a mathematics can indeed be constructed via *charge conjugation.* However, such a conjugation is only applicable in second quantization. Its application at the classical level implies the mere change of the sign of the charge resulting in a host of inconsistencies, e.g., the fact that quantization would lead to particles with the wrong sign of the charge and *not* to antiparticles.

In essence, Santilli identified the existence of one of the largest scientific unbalances of the 20-th century which consisted in the treatment of matter at all levels of study, from Newton to second quantization, while antimatter was solely studied at the level of second quantization. By contrast, matter and antimatter are equivalent in the universe to such an extent that we expect the existence of entire galaxies made up of antimatter.

Santilli then understood that such an unbalance could only be resolved via the *construction of new mathematics specifically built for the classical treatment of antimatter.* In this way he discovered his isodual mathematics which is manifestly anti-isomorphic to the conventional mathematics (that with unit +1) at all levels, beginning with the units and then passing to numbers, vector fields, Hilbert spaces, geometries, symmetries, etc.

He then proved that his isodual treatment of antimatter is consistent not only at the classical level, but also at the quantum level, thanks to his new *isodual* *quantization* for which classical antiparticles are correctly mapped into operator antiparticles. He also proved that the operator version of his isodual treatment of antimatter is equivalent to that via charge conjugation.

Via the use of these novel mathematical and physical theories, Santilli formulated potentially historical predictions, such as: the first capability in the history of science to conduct consistent quantitative studies as to whether a far away galaxy is made up of matter or of antimatter; the experimentally testable prediction that antimatter emits a new light which is different than the ordinary light emitted by matter; and the prediction of antigravity (negative curvature tensor) for antimatter in the field of matter or viceversa (see, e.g., Refs. [141, 189, 187]).

3. Santilli's Isonumbers. In two seminal memoirs, Refs. [64, 65], written at Harvard University in 1978 (see again Ref. [129] for a technical presentation), Santilli assumed, apparently for the first time in mathematics, a generalized unit which is real valued and invertible, yet otherwise arbitrary

$$a^{\hat{0}} = \hat{I} = 1/\hat{T}, \tag{9}$$

which he called *isounit.* He then generalized the multiplication and division in the forms

$$a\hat{\times}b = a\hat{T}b, \quad a\hat{\div}b = \hat{I}\frac{a}{b}, \tag{10}$$

$$a\hat{\times}a\hat{\times}\cdots a\hat{\times}a = a^{\hat{n}} = a^n(\hat{T})^{n-1}, \; a^{-\hat{n}} = \hat{I}\hat{\div}a^{\hat{n}} = a^{-n}(\hat{T})^{-n-1} = a^{-n}(\hat{I})^{n+1}, \tag{11}$$

which he called *isomultiplication* and *isodivision,* respectively. Santilli then proved that, if the original set of number forms a field, all axioms of a field remain verified under the above reformulation, resulting in a new field today called *Santilli isofield.* In particular, the isounit remains the correct left and right unit under the isoproduct

$$\hat{I}\hat{\times}a = a\hat{\times}\hat{I} = a, \quad (\hat{I})^{\hat{n}} = \hat{I}, \quad n = 0, 1, 2, \ldots \tag{12}$$

Keeping unchanged addition and subtraction, $(+, -, \hat{\times}, \hat{\div})$ form four arithmetic operations of isonumbers. When $\hat{I} = 1$, it is the operations of the ordinary numbers.

Santilli then conducted numerous studies for a second lifting of most of contemporary mathematics into a form admitting $\hat{I}$, rather than the trivial number 1, as the left and right unit at all levels, resulting in a new mathematics today called *Santilli isomathematics,* which includes: *isonumbers* and *isofields*, *isovector* and *isometric isospaces*, *isogeometries* and *isotopologies*, *isofunctional analysis* and *isodifferential calculus*, *isoalgebras* and *isogroups*, etc. (see most of the papers [66-230], the mathematical presentation [175] being the most important for this book).

Santilli also indicated a simple method for the construction of isotopic theories consisting of the noncanonical transform at the classical level and the nonunitary transform at the operator level

$$A \to UAU^{\dagger}, \quad UU^{\dagger} = \hat{I} \neq 1, \tag{13}$$

which must be applied to the totality of quantities and their operations of the conventional theory [16, 175].

By using his new isomathematics, Santilli achieved the first known structural generalization of Newton's equation since Newton's time capable of an invariant treatment of extended, nonspherical and deformable particles under the most general known nonlinear, nonlocal and nonhamiltonian forces [175].

Subsequently, he achieved: the isotopies of classical Hamiltonian mechanics, today known as *Hamilton-Santilli isomechanics* [3, 6, 10, 11, 12]; the isotopies of quantum mechanics today known as *hadronic mechanics* (see, again, Refs. [66-230], the recent monograph [16] providing an excellent outline with various applications and experimental verifications); the isotopies of special relativity, including those of the Lorentz and Poincaré symmetries [85, 127]; the isotopies of the Minkowskian geometry with a geometric unification of the Minkowskian, Riemannian and other geometries and a consequential unification of the special and general relativities [216]; the only known *axiomatically consistent and invariant grand-unification* of electroweak and gravitational interactions [196, 218]; a new cosmological conception of the universe characterized by one fundamental *symmetry* (rather than the usual covariance), the Poincaré-Santilli isosymmetry [210]; and reached other very innovative advances.

By using the above novel mathematical and physical theories, Santilli was finally able to develop novel industrial applications consisting of new energies and fuels, as reviewed in the recent monograph [16]. As a matter of fact, Santilli conceived his new isomathematics and isophysical theories precisely for quantitative studies on new energies and fuels.

The main physical difference between conventional and isotopic theories is that the former can only describe point particles under local differential interactions derivable from a potential (locally Hamiltonian systems), as one can see from the basic Euclidean topology. By comparison, Santilli's isotopic formulations can describe extended, nonspherical and deformable particles under linear and nonlinear, local and nonlocal and potential-Hamiltonian as well as nonpotential-nonhamiltonian interactions.

The new energies and fuels are based precisely on the new interactions outside the descriptive capacity of contemporary mathematical and physical theories in the fixed frame of the experimenter. The need for the new mathematics was motivated by the fact that, in its absence, the theories are *noninvariant,* thus being afflicted by catastrophic inconsistencies studies in memoir [219]. For the case of the isotopic theories, the Hamiltonian describes conventional local-differential interactions, while all nonhamiltonian effects are represented by the isounit, with the consequential assurance of their invariant treatment since the unit is the basic invariant of all theories, whether conventional or generalized.

To my best knowledge, Santilli is the only scientist who succeeded in achieving a *structural generalization of both pre-existing mathematical and physical theories*, since all previous discoveries dealt with advances in mathematics *or* in physics, and, more particularly, with advances in a specific branch of mathematics or physics rather than their entire generalization.

In view of these results, Santilli received various honors, including the listing by the Estonia Academy of Sciences among the most illustrious applied mathematicians of all times, jointly with Gauss, Weierstrass, Lie, Hamilton, etc. Also, various scholars have praised the new isomathematics as an epoch-making contribution to mathematics and physics. They believe that the theory is leading to a great change in mathematics itself, with entire new branches of science, such as iso-physics, iso-chemistry, iso-biology, etc. What I have done in this monograph is limited to the study of only the most fundamental mathematical entities, Santilli isonumbers, from which all mathematical and physical isotheories can be uniquely and unambiguously derived.

4. Santilli's Isodual Isonumbers. Despite their generality, isonumbers as defined above (those characterized by a positive-definite invertible isounit) resulted to be unable to reach a consistent *classical and operator* description of extended, deformable and nonspherical antiparticles under unrestricted interactions, as it is the case of the isotopic treatment of matter. To avoid the repetition of an unbalance between the treatment of matter and antimatter, Santilli worked out a yet new mathematics, today called *Santilli isodual isomathemartics* which is characterized by arbitrary, negative-definite and invertible *isodual isounits*

$$a^{\bar{0}} = \bar{I} = -\hat{I}^{\dagger} = -\hat{I} = 1/\bar{T} = -1/\hat{T}, \tag{14}$$

and related operations

$$a\bar{\times}b = a\bar{\times}b = -a\hat{T}b, \quad a\bar{\div}b = -\hat{I}\frac{a}{b},$$

$$a\bar{\times}a\bar{\times}\cdots\bar{\times}a = a^{\bar{n}}(-\hat{T})^{n-1},$$

$$a^{-\bar{n}} = -\hat{I}\bar{\div}a^{\bar{n}} = a^{-n}(-\hat{T})^{-n-1} = a^{-n}(-\hat{I})^{n+1}. \tag{15}$$

in which case $\bar{I}$ is indeed the correct left and right unit,

$$\bar{I}\bar{\times}a = a\bar{\times}\bar{I} = a, \quad (-\hat{I})^{\bar{n}} = -\hat{I}. \tag{16}$$

Keeping unchanged addition and subtraction, $(+, -, \bar{\times}, \bar{\div})$ form four arithmetic operations of isodual isonumbers.

Santilli then constructed the isodualities of his isotopic mathematics at all essential levels, including numbers and fields, vector and metric spaces, algebras and symmetries, etc. In this way he extended is antimatter theory to much broader physical conditions.

Santilli also introduced a simple method for the construction of his isodual isomathematics via the the anti-isomorphic map,

$$A(x, \psi, ...) \rightarrow -A^\dagger(-x^\dagger, -\phi^\dagger, ...), \tag{17}$$

which must be applied to *all* quantities and their operations of the isotopic theory with no exclusion.

5. Santilli's Genonumbers and their Isoduals. Despite the achievement of the above epoch making generalizations of all pre-existing mathematical and physical theories, Santilli remained dissatisfied because of their inability to permit an *invariant representation of irreversibility at the classical and operator levels.* In fact, both the conventional and isotopic theories are structurally reversible in time, in the sense that they admit no map producing an *inequivalent* time reversal version, which inequivalence is a necessary condition to represent irreversibility (the isodual map is inapplicable since it produces conjugation to antimatter, and not time reversal).

Santilli argued that *the historical inability of the 20-th century to achieve a significant representation of the irreversibility originates from the Hermiticity of the basic unit of the theory, whether conventional or isotopic.* At any rate, all known interactions (electric, magnetic, gravitational, etc.) are *reversible in time,* that is, their time reversal image is as causal as the conventional version. Therefore, *locally Hamiltonian methods are intrinsically insufficient to represent irreversibility,* thus leaving no other choice than that of *representing irreversibility via the construction of another structural generalization of mathematics which is intrinsically irreversible, namely, irreversible for all possible Hamiltonians.*

For this purpose Santilli introduced a new mathematics based on a generalized unit $\hat{I}$ which is invertible, but not Hermitean, as it is the case for complex numbers or real-valued nonsymmetric matrices. This implies the consequential existence of two generalized nonhermitean units with a ne interconnecting map [129],

$$\hat{I}^> = 1/\hat{T}^>, \quad {}^<\hat{I} = 1/{}^<\hat{T}, \tag{18}$$

$$\hat{I}^> = ({}^<\hat{I})^\dagger, \tag{19}$$

which he called *forward and backward genounits.*

He then assumed $\hat{I}^>$ as the unit for theories with motion forward in time and ${}^<\hat{I}$ as the unit for theories with motion backward in time, conjugation (19) characterizing their interconnecting map.

The selection of two different units required the construction of two different new mathematics, today known as *Santilli genomathematics,* because they require two different products, evidently one per each genounit. This latter goal was brilliantly achieved via the *introduction of an order in the multiplication.* In this way, Santilli introduced the following *genoproduct to the right and to the left* [*loc. cit.*]

$$a > b = a\hat{T}^> b, \quad a < b = a{}^<\hat{T} b, \tag{20}$$

$$a > \hat{I}^> = \hat{I}^> > a = a, \quad a <^< \hat{I} =^< \hat{I} < a = a, \tag{21}$$

and the products are correspondingly restricted to be either to the right or to the left. Santilli also indicated a simple method for the construction of genotopic theories consisting of the noncanonical transform at the classical level and the nonunitary transform at the operator level,

$$A \to UAW^\dagger, \quad B \to WBU^\dagger, \tag{22}$$

$$UW^\dagger = \hat{I}^>, \quad WU^\dagger =^< \hat{I}, \quad UU^\dagger \neq 1, \quad WW^\dagger \neq 1, \tag{23}$$

which maps must be again applied to the totality of the quantities and their operations of conventional or isotopic theories with no known exclusion (to avoid inconsistencies [219]).

In this way, Santilli resolved another historical unbalance of the 20-th century physics, the dichotomy between the evident reversibility of all mathematical and physical theories used in the 20-th century as compared to the evident irreversibility of the real world.

Moreover, Santilli achieved for the first time an invariant formulations of irreversibility not only at the classical level, but also and most importantly at the operator level. As a matter of fact, irreversibility emerged as originating at the *particle* level, and then propagate at the macroscopic level of our environment, contrary to a popular view of the 20-th century physics.

To elaborate on this historical point, recall that a rather general belief of the 20-th century was that the particle world is reversible and irreversibility is a mere macroscopic aspect. This position was dictated by the fact that *quantum mechanics is strictly and solely reversible in time.*

Santilli proved the following

Theorem [94]: *A classical macroscopic irreversible system cannot be consistently reduced to a finite number of elementary particles all in reversible conditions and, vice-versa, a finite number of elementary particles all in reversible conditions cannot consistently yield an irreversible macroscopic system.*

It is evident that the above theorem discredited as nonscientific any attempt to reduce the irreversibility of our macroscopic world to purely reversible quantum formulations. For this purpose Santilli and his associates constructed the *genotopic branch of hadronic mechanics* which is intrinsically irreversible, that is, irreversible for whatever selection of a reversible Hamiltonian, thus permitting for the first time in the history of science to reduce macroscopic irreversibility to the most elementary layers of nature.

In particular, as expected, the origin of irreversibility emerged precisely in the *nonhamiltonian interactions represented with the genounits,* such as the contact interactions among extended particles which are dramatically outside any representational capability of quantum mechanics.

In different words, quantum mechanics applies exactly to the system for which it was constructed, the orbits of the electron in the Hydrogen atom, which system is indeed reversible in time. However, the idea that a mechanics so effective in describing electron orbits must also be exact for all other conditions in nature is purely political-nonscientific. The genotopic branch of hadronic mechanics was constructed precisely for the representation of electrons in irreversible conditions, such as an electron in the core of a star which experiences Hamiltonian interactions as well as contact-zero range (thus non-Hamiltonian) interactions dramatically outside the representational capabilities of quantum mechanics. In this way irreversibility emerged in the very ultimate layer of interior particle problems, exactly as it occurs in nature.

The resolution of the vexing problem of irreversibility via the prior construction of a new mathematics specifically conceived for the task is, perhaps the most historical contribution made by Santilli to human knowledge to date, because it implies a profound revision of our entire mathematical and physical knowledge with direct applications to new clean energies and fuels so much needed by mankind. An excellent presentation of the genotopic treatment of irreversibility at the classical and operator level, including their unique interconnecting genoquantization, can be found in the recent monograph [16].

6. Santilli's Hypernumbers and their Isoduals. Despite the above historical discoveries, Santilli remained still dissatisfied because of the insufficiency of his genotheories to achieve a quantitative representation of biological structures. In fact, all biological entities are manifestly irreversible because they are born, age and die. As such, quantum mechanics is dramatically inapplicable in biology since it would imply that all biological structures are eternal (besides being perfectly rigid and have other catastrophic inconsistencies). It is evident that Santilli's genotheories provide a resolution of the excessive limitations of conventional mathematical and physical theories.

However, studies conducted by Illert and Santilli [13] have revealed that *quantitative representations of biological structures require formulations which are not only irreversible but also multi-dimensional,* thus implying the insufficiency of genotheories because they are irreversible but *single-valued.*

This occurrence, which illustrates the complexity of the biological world, was discovered via quantitative studies of *sea shells.* It is well know since long time that the *shape* of sea shells can indeed be reproduced in our three-dimensional Euclidean geometry. The novelty discovered by Illert and Santilli deals with quantitative representations of the *time evolution* of sea shells. In fact, it was shown via computer simulations that, if the axioms of the Euclidean geometry are rigidly imposed during its evolution in time, sea shells first grow in a deformed way and then they crack.

Moreover, Illert and Santilli [13] showed that *a quantitative representation of the growth of sea shells requires at least six dimensions, such as the doubling of each* *of the three Euclidean axes.* This created a very intriguing problem that occupied Santilli's mind for some time because the growth of sea shells is fully identified by our three-dimensional sensory perception (via our three Eustachian lobes). Therefore, scientific treatments of sea shells growth require a geometry which is: 1) structurally irreversible; 2) multi-dimensional; and 3) compatible with our three-dimensional sensory perception.

The brilliant solution found by Santilli for this quite intriguing geometric problem is the construction of yet a more general mathematics based on *invertible, nonhermitean and multi-valued generalized units* [14]

$$\hat{I}^> = \{\hat{I}_1^>, \quad \hat{I}_2^>, \hat{I}_3^>, ...\} = 1/\hat{T}^>, \quad {}^<\hat{I} = \{{}^<\hat{I}_1, \quad {}^<\hat{I}_2, {}^<\hat{I}_3, ...\} = 1/{}^<\hat{T}, \tag{24}$$

$$\hat{I}^> = ({}^<\hat{I})^\dagger, \tag{25}$$

which he called *Santilli forward and backward hyperunits,* where multi-valuedness is represented via sets. In this way we have the transition from the single-valuedness of the conventional, iso- and geno-units, to a set of values for the hyperunits.

Again, the selection of two different units required the following two different products [14]

$$a > b = a\hat{T}^> b = \{a\hat{T}_1^> b, a\hat{T}_2^> b, a\hat{T}_3^> b, ...\}, \tag{26}$$

$$a < b = a^<\hat{T} b = \{a^<\hat{T}_1 b, a^<\hat{T}_2 b, a^<\hat{T}_3 b, ...\}, \tag{27}$$

Chapter 1. Introduced first are the foundations of Santilli's isonumber theory of the first kind, which is characterized by the axiom-preserving isotopic lifting of the multiplicative unit, product and elements. And then the isogroup, the isodivisibility, the unique isofactorization theorem, the isoprime number theorem, the isocongruences, etc. are studied.

In this chapter, a new branch of number theory: Santilli's additive isoprime theory is introduced. By using the arithmetic function $J_n(\omega)$ the following theorems have been proved.

1 There exist infinitely many twin isoprimes.

2 Every isoeven number greater than $\hat{4}$ is the sum of two isoprimes, including the Yu's Mathematical Problem.

3 There exist infinitely many isoprimes of the form: $\hat{x}^{\hat{2}} + \hat{b}$.

4 The Forbes theorems have infinitely many isoprime solutions.

5 There exist infinitely many k-tuples of isoprimes.

6 Santilli's isoprime m-chains, $p_{j+1} = mp_j \pm (m-1), m = 2, 3, 4, \cdots$, including the Cunnigham chains.

7 Santilli's chains of isoprimes in arithmetic progressions.

8 Santilli's chains of prime-producing quadratics.

We study properties of the arithmetic function $J(\omega)$, and take the isogoldbach theorem as an example to study the relationships between the arithmetic function and subequations in more detail.

We study an application that the limit for the periodic table of the stable elements is Uranium with an atomic number of 92. We suggest the Pauli-Santilli principle which can apply to natural sciences and social sciences.

Chapter 2. Introduced are the foundations of Santilli's isonumber theory of the second kind, which characterized by the axiom-preserving isotopic lifting of the unit via an element of original field, with compatible lifting of multiplication, while the element of the original field remains unchanged. We study the isogroup, the isodivibility, the prime number theorem of isoarithmetic progression, the isocongruences, etc. In this chapter, a new branch of number theory: Santilli's isoadditive prime theory is introduced. By using the arithmetic function $J_n(\omega)$ the following theorems are proved.

1 There exist infinitely many twin primes. This is the simplest theorem.

2 The Goldbach theorem. Every even number greater than 4 is the sum of two primes. It is the simplest theorem in Santilli's isoadditive prime theory.

3 There exist infinitely many primes of the forms: $x^2+1, x^4+1, x^8+1, x^{16}+1$.

4 There exist infinitely many primes of the forms: ap^2+bp+c.

5 There exist infinitely many primes of the forms: $p_1^3+2, p_1^5+2, p_1^7+2, \cdots$.

6 There exist infinitely many primes of the forms: $(p_1+1)(p_2+1)+1, (p_1+1)(p_2+1)(p_3+1)+1, \cdots$.

7 There exist infinitely many triplets of consecutive integers, each being the product of k distinct primes. Smallest triplets: $33=3\times 11, 34=2\times 17, 35=5\times 7; 1727913=3\times 11\times 52361, 1727914=2\times 17\times 50821, 1727915=5\times 7\times 49369; \cdots$.

8 There exist infinitely many Carmichael numbers, which are the product of three primes, four primes, and five primes.

9 Every integer m may be written in infinitely many ways in the forms:

$$m=\frac{p_2+1}{p_1^k-1},$$

where $k=1,2,3,\cdots; p_1$ and p_2 are primes.

10 In a table of prime numbers there exist infinitely many k-tuples of primes, where $k=2,3,4,\ldots,10^5$. The theory of the table of prime numbers is the queen of

number theory. An indecomposable prime can structure a stable system. It can apply to the study of the stable structure of DNA and the stable genomic sequences. A table of prime numbers is the genbank of biology. It must be a mathematical problem of the 21st century.

In this chapter, the Riemann's hypothesis is disproved. $\min|\zeta(\frac{1}{2}+t_i)| \approx 0$ but $\neq 0$. The computation of all nontrivial zeros of $\zeta(\frac{1}{2}+ti)$ is error, which satisfies the Riemann's error hypothesis.

Chapter 3. From Fermat's mathematics six methods for proving the Fermat's last theorem can be obtained. Using $n = 3$ or 4 we prove that all the Fermat's equations in the cyclic determinants have no rational solutions. By lifting $F(a,+,\times) \to \hat{F}(\hat{a},+,\hat{\times})$ and $F(a,+,\times) \to \hat{F}(a,+,\hat{\times})$ we obtain the Fermat-Santilli isotheorems. Every positive hypercomplex number has the complex hyperbolic function. Every negative hypercomplex number has the complex trigonometric functions. Every hypercomplex number has an exponential formula. Every hypercomplex function has the Cauchy-Riemann equations. We found the relationships between the hypercomplex numbers and differential equations. From Fermat's mathematics we suggest the chaotic mathematics. It is shown that the study of the stability of nonlinear equations is reduced to the study of the stability of the equation $dA/dt = \sum dN_i/dt$, which is much simpler.

Chapter 4. In 1994 we discovered the new arithmetic function $J_2(\omega)$. Using it we proved the binary Goldbach's theorem. Since $J_2(\omega) \to \infty$ as $\omega \to \infty$ every even number N greater 4 is the sum of two primes. It is a generalization of Euler proof of the existence of the infinitely many primes. Since $J_2(\omega > \omega_g) \to \infty$ as $\omega \to \infty$ every even number N from some point onward can be expressed as the sum of two primes using only partial primes. It is $2/J_2(\omega_g)$ of the number of total solutions. We shall establish the additive prime theory with partial primes.

Chapter 5. In this chapter we present another important application of Santilli's isonumber theory. It deals with the isotopies of conventional cryptograms, called *isocryptograms,* which were first introduced by Santilli in monograph [11]. The importance of isocryptograms is that *they admit an infinite number of possible isounits,* thus rendering their resolution dramatically more difficult than that of conventional cryptograms, assuming that it is possible in a finite period of time. Within the context of contemporary society and its growing need for security, it is evident that Santilli's isocryptograms constitute an application of the new isonumber theory of direct societal relevance.

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Chapter 1

FOUNDATIONS OF SANTILLI'S ISONUMBER THEORY
PART I: ISONUMBER THEORY OF THE FIRST KIND

Mathematics is the queen of the sciences, and the theory of numbers is the queen of mathematics.

Carl Friedrich Gauss

1. Introduction

In the seminal works [1, 2] Santilli has introduced a generalization of real, complex and quaternionic numbers $a = n, c, q$ based on the lifting of the unit 1 of conventional numbers into an invertible and well behaved quantity with arbitrary functional dependences on local variables

$$1 \rightarrow \hat{I}(t, x, \dot{x}, \cdots) = 1/\hat{T} \neq 1 \tag{1.1}$$

while jointly lifting the product $ab = a \times b$ of conventional numbers into the form

$$ab \rightarrow a\hat{\times}b = a\hat{T}b \tag{1.2}$$

under which $\hat{I} = 1/\hat{T}$ is the correct left and right new unit

$$\hat{I}\hat{\times}a = \hat{T}^{-1}\hat{T}a = a\hat{\times}\hat{I} = a\hat{T}\hat{T}^{-1} = a \tag{1.3}$$

for all possible $a = n, c, q$.

Since the new multiplication $a\hat{\times}b$ is associative, Santilli [1, 2] has then proved that the new numbers verify all axioms of a field. The above liftings were then called *isotopic* in the Greek sense of being axiom-preserving. The prefix *iso* is then used whenever the original axioms are preserved.

Let $F(a,+,\times)$ be a conventional field with numbers $a=n,c,q$ equipped with the conventional sum $a+b\in F$, product $ab=a\times b\in F$, additive unit $0\in F$ and their multiplicative unit $1\in F$.

Definition 1.1. Santilli's isofields of the first kind $\hat{F}=\hat{F}(\hat{a},+,\hat{\times})$ are the rings with elements

$$\hat{a}=a\hat{I} \tag{1.4}$$

called *isonumbers*, where $a=n,c,q\in F, \hat{I}=1/\hat{T}$ is a well behaved, invertible and Hermitean quantity outside the original field $\hat{I}=1/\hat{T}\notin F$ and $a\hat{I}$ is the multiplication in F equipped with the isosum

$$\hat{a}+\hat{b}=(a+b)\hat{I} \tag{1.5}$$

with conventional additive unit $0=0\hat{I}=0,\ \hat{a}+0=\hat{a}+0=\hat{a}, \quad \forall\hat{a}\in\hat{F}$ and the isoproduct

$$\hat{a}\hat{\times}\hat{b}=\hat{a}\hat{T}\hat{b}=a\hat{I}\hat{T}b\hat{I}=(ab)\hat{I} \tag{1.6}$$

under which $\hat{I}=1/\hat{T}$ is the correct left and right new unit ($\hat{I}\hat{\times}\hat{a}=\hat{a}\hat{\times}\hat{I}=\hat{a}, \quad \forall\hat{a}\in\hat{F}$) called isounit.

Lemma 1.1 [1]. The isofields $\hat{F}(\hat{a},+,\hat{\times})$ of Definition 1.1 verify all axioms of a field. The lifting $F\rightarrow\hat{F}$ is then an isotopy. All operations depending on the product must then be lifted in $\hat{F}$ for consistency.

The Santilli's commutative isogroup of the first kind

$$\hat{a}^{\hat{I}}=a\hat{I},\quad \hat{a}^{-\hat{I}}=a^{-1}\hat{I},\quad \hat{a}^{\hat{I}}\hat{\times}\hat{a}^{-\hat{I}}=\hat{a}^{\hat{0}}=\hat{I}=\hat{T}^{-1}\neq 1.$$

$$\hat{a}^{\hat{b}}=a^{b}\hat{I},\quad \hat{a}^{-\hat{b}}=a^{-b}\hat{I},\quad \hat{a}^{\hat{b}}\hat{\times}\hat{a}^{-\hat{b}}=\hat{a}^{\hat{0}}=\hat{I}=\hat{T}^{-1}\neq 1.$$

$$\hat{a}^{\widehat{c/b}}=a^{\frac{c}{b}}\hat{I},\quad \hat{a}^{-\widehat{c/b}}=a^{-\frac{c}{b}}\hat{I},\quad \hat{a}^{\widehat{c/b}}\hat{\times}\hat{a}^{-\widehat{c/b}}=\hat{a}^{\hat{0}}=\hat{I}=\hat{T}^{-1}\neq 1.$$

$$\hat{a}\hat{\times}\hat{I}=\hat{a}\hat{T}\hat{I}=\hat{a}.\quad \hat{a}\hat{\times}\hat{b}=a\hat{I}\hat{T}b\hat{I}=ab\hat{I}=\widehat{ab}.$$

$$\hat{a}\hat{\times}\hat{b}^{-\hat{2}}=a\hat{I}\hat{T}b^{-2}\hat{I}=a/b^{2}\hat{I}.$$

where $\hat{I}$ is called an isounit, $\hat{T}$ is called an isoinverse of $\hat{I}$; $\hat{a}^{-\hat{b}}$ is called an isoinverse of $\hat{a}^{\hat{b}}$; $(\hat{a}^{\hat{b}},\hat{\times})$ is called the Santilli's commutative isogroup of the first kind.

Lemma 1.2. Santilli's isofields of the second kind $\hat{F} = \hat{F}(a, +, \hat{\times})$ (that is, when $a \in F$ is not lifted to $\hat{a} = a\hat{I}$) also verify all axioms of a field, if and only if the isounit is an element of the original field

$$\hat{I} = 1/\hat{T} \in F \tag{1.7}$$

The isoproduct is defined by

$$a\hat{\times}b = a\hat{T}b \in \hat{F} \tag{1.8}$$

The Santilli's commutative isogroup of the second kind

$$a^{\hat{I}} = a, \quad a^{-\hat{I}} = a^{-1}\hat{I}^2, \quad a^{\hat{I}}\hat{\times}a^{-\hat{I}} = a^{\hat{0}} = \hat{I} = \hat{T}^{-1} \neq 1.$$

$$a^{\hat{2}} = a^2\hat{T}, \quad a^{-\hat{2}} = a^{-2}\hat{I}^3, \quad a^{\hat{2}}\hat{\times}a^{-\hat{2}} = a^{\hat{0}} = \hat{I} = \hat{T}^{-1} \neq 1.$$

$$a^{\hat{3}} = a^3\hat{T}^2, \quad a^{-\hat{3}} = a^{-3}\hat{I}^4, \quad a^{\hat{3}}\hat{\times}a^{-\hat{3}} = a^{\hat{0}} = \hat{I} = \hat{T}^{-1} \neq 1.$$

$$\cdots\cdots$$

$$a^{\hat{n}} = a^n\hat{T}^{n-1}, \quad a^{-\hat{n}} = a^{-n}\hat{I}^{n+1}, \quad a^{\hat{n}}\hat{\times}a^{-\hat{n}} = a^{\hat{0}} = \hat{I} = \hat{T}^{-1} \neq 1.$$

$$a^{\widehat{1/2}} = a^{\frac{1}{2}}(\hat{I})^{\frac{1}{2}}, \quad a^{-\widehat{1/2}} = a^{-\frac{1}{2}}(\hat{I})^{\frac{3}{2}}, \quad a^{\widehat{1/2}}\hat{\times}a^{-\widehat{1/2}} = a^{\hat{0}} = \hat{I} = \hat{T}^{-1} \neq 1.$$

$$a^{\widehat{1/3}} = a^{\frac{1}{3}}(\hat{I})^{\frac{2}{3}}, \quad a^{-\widehat{1/3}} = a^{-\frac{1}{3}}(\hat{I})^{\frac{4}{3}}, \quad a^{\widehat{1/3}}\hat{\times}a^{-\widehat{1/3}} = a^{\hat{0}} = \hat{I} = \hat{T}^{-1} \neq 1.$$

$$a^{\widehat{1/n}} = a^{\frac{1}{n}}(\hat{I})^{1-\frac{1}{n}}, \quad a^{-\widehat{1/n}} = a^{-\frac{1}{n}}(\hat{I})^{1+\frac{1}{n}}, \quad a^{\widehat{1/n}}\hat{\times}a^{-\widehat{1/n}} = a^{\hat{0}} = \hat{I} = \hat{T}^{-1} \neq 1.$$

$$a^{\widehat{c/b}} = a^{\frac{c}{b}}(\hat{I})^{1-\frac{c}{b}}, \quad a^{-\widehat{c/b}} = a^{-\frac{c}{b}}(\hat{I})^{1+\frac{c}{b}}, \quad a^{\widehat{c/b}}\hat{\times}a^{-\widehat{c/b}} = a^{\hat{0}} = \hat{I} = \hat{T}^{-1} \neq 1.$$

$$a^{\hat{b}} = a^b(\hat{I})^{1-b} = a^b\hat{T}^{b-1}, \quad a^{-\hat{b}} = a^{-b}(\hat{I})^{1+b}, \quad a^{\hat{b}}\hat{\times}a^{-\hat{b}} = a^{\hat{0}} = \hat{I} = \hat{T}^{-1} \neq 1.$$

$$a^{\hat{I}}\hat{\times}b^{\hat{I}} = a\hat{T}b, \quad a^{\hat{I}}\hat{\times}b^{-\hat{I}} = ab^{-1}\hat{I}.$$

where $\hat{I}$ is called an isounit; $\hat{T}$ an isoinverse of $\hat{I}$; $a^{-\hat{b}}$ an isoinverse of $a^{\hat{b}}$; $(a^{\hat{b}}, \hat{\times})$ the Santilli's commutative isogroup of the second kind.

In this Chapter we study Santilli's isonumber theory of the first kind, that based on isofields $\hat{F} = \hat{F}(\hat{a}, +, \hat{\times})$ [3].

2. Foundations of Santilli's Isonumber Theory

By lifting $F(a,+,\times) \to \hat{F}(\hat{a},+,\hat{\times})$ we characterize Santilli's isonumber theory of the first kind. We can partition the positive isointegers into three classes:

(1) The isounit: $\hat{I}$;
(2) The isoprime numbers: $\hat{2}, \hat{3}, \hat{5}, \hat{7}, \cdots$;
(3) The isocomposite numbers: $\hat{4}, \hat{6}, \hat{8}, \hat{9}, \cdots$.

The Santilli's isonumber theory of the first kind is concerned primarily with isodivisibility properties of isointegers.

Definition 2.1: isodivisibility. We say $\hat{d}$ isodivides $\hat{n}$ and we write $\hat{d} \hat{|} \hat{n}$ whenever $\hat{n} = \hat{c}\hat{\times}\hat{d}$ for some $\hat{c}$. We also say that $\hat{n}$ is an isomultiple of $\hat{d}$, that $\hat{d}$ is an isodivisor of $\hat{n}$. If $\hat{d}$ does not isodivide $\hat{n}$ we write $\hat{d} \hat{\not|} \hat{n}$.

Theorem 2.1. Isodivisibility has the following properties:

(1) $\hat{n} \hat{|} \hat{n}$ (reflexive property)
(2) $\hat{d} \hat{|} \hat{n}$ and $\hat{n} \hat{|} \hat{m}$ implies $\hat{d} \hat{|} \hat{m}$ (transitive property)
(3) $\hat{d} \hat{|} \hat{n}$ implies $\hat{a}\hat{\times}\hat{d} \hat{|} \hat{a}\hat{\times}\hat{n}$ (multiplicative property)
(4) $\hat{d} \hat{|} \hat{n}$ and $\hat{d} \hat{|} \hat{m}$ implies $\hat{d} \hat{|} (\hat{a}\hat{\times}\hat{n} + \hat{b}\hat{\times}\hat{m})$ (linearity property)
(5) $\hat{a}\hat{\times}\hat{d} \hat{|} \hat{a}\hat{\times}\hat{n}$ and $\hat{a} \neq \hat{0}$ implies $\hat{d} \hat{|} \hat{n}$ (cancellation law)
(6) $\hat{I} \hat{|} \hat{n}$ ($\hat{I}$ isodivides every isointeger)

Definition 2.2. If $\hat{d}$ isodivides two isointegers $\hat{a}$ and $\hat{b}$, then $\hat{d}$ is called a common isodivisor of $\hat{a}$ and $\hat{b}$.

Theorem 2.2. Given any two isointegers $\hat{a}$ and $\hat{b}$, there is a common isodivisor $\hat{d}$ of $\hat{a}$ and $\hat{b}$ of the form

$$\hat{d} = \hat{a}\hat{\times}\hat{x} + \hat{b}\hat{\times}\hat{y} \tag{2.1}$$

where $\hat{x}$ and $\hat{y}$ are isointegers, moreover every common isodivisor of $\hat{a}$ and $\hat{b}$ isodivides this $\hat{d}$.

The isonumber $\hat{d}$ is called the greatest common isodivisor (gcid) of $\hat{a}$ and $\hat{b}$ and is denoted by $(\hat{a}, \hat{b})$. If $(\hat{a}, \hat{b}) = \hat{I}$, then $\hat{a}$ and $\hat{b}$ are said to be relatively isoprime.

Theorem 2.3. The gcid has the following properties:

(1) $(\hat{a}, \hat{b}) = (\hat{b}, \hat{a})$ (commutative law)
(2) $(\hat{a}, (\hat{b}, \hat{c})) = ((\hat{a}, \hat{b}), \hat{c})$ (associative law)
(3) $(\hat{a}\hat{\times}\hat{c}, \hat{b}\hat{\times}\hat{c}) = \hat{c}\hat{\times}(\hat{a}, \hat{b})$ (distributive law)

(4) $(\hat{a}, \hat{I}) = \hat{I}$

Definition 2.3. An isointeger $\hat{n}$ is called isoprime if $\hat{n} > \hat{I}$ and if the only positive isodivisors of $\hat{n}$ are $\hat{I}$ and $\hat{n}$. If $\hat{n} > \hat{I}$ and if $\hat{n}$ is not isoprime, then $\hat{n}$ is called isocomposite.

Theorem 2.4. The unique isofactorization theorem. Every isointeger $\hat{n} > \hat{I}$ can be represented as a isoproduct of isoprime factors in only one way, apart from the order of isodivisors. We can write

$$\hat{n} = (\hat{p}_1)^{\hat{a}_1} \hat{\times} \cdots \hat{\times} (\hat{p}_r)^{\hat{a}_r} = n\hat{I}, \tag{2.2}$$

e.g., $\widehat{20} = \hat{2}^{\hat{2}} \hat{\times} \hat{5} = \hat{2} \hat{\times} \hat{2} \hat{\times} \hat{5} = 20\hat{I}$.

Theorem 2.5. The isoprime number theorem. By lifting every $p \to \hat{p}$ we have the isoprime number theorem

$$\pi(N) = \frac{N}{\log N}(1 + O(1)), \tag{2.3}$$

where $\pi(N)$ is the number of isoprimes $\hat{p} \leq \hat{N}$.

Theorem 2.6. The isoprime number theorem for isoarithmetic progressions:

$$\hat{E}_a(k) = \hat{n} \hat{\times} \hat{k} + \hat{a}, \ (\hat{k}, \hat{a}) = \hat{I}, \ \hat{n} = \hat{0}, \hat{1}, \hat{2}, \cdots \tag{2.4}$$

which states that

$$\pi_a(N) = \frac{1}{\phi(k)} \frac{N}{\log N}(1 + O(1)), \tag{2.5}$$

where $\pi_a(x)$ denotes the number of isoprimes in $\hat{E}_a(k) \leq \hat{N}$ and $\phi(k)$ denotes Euler's function.

The isogoldbach problem: $\hat{N} = \hat{p}_1 + \hat{p}_2$. Every isoeven number $\hat{N}$ greater than $\hat{4}$ is the sum of two isoprimes, e.g., $\widehat{10} = \hat{5} + \hat{5} = \hat{3} + \hat{7}$. The twin isoprime problem: $\hat{p}_1 = \hat{p} + 2$, e.g., $\widehat{13} = \widehat{11} + \hat{2}$. Three isoprimes problem: $\hat{N} = \hat{p}_1 + \hat{p}_2 + \hat{p}_3$, $\hat{N} = \hat{p}_1 + \hat{p}_2 \hat{\times} \hat{p}_3$, e.g., $\widehat{11} = \hat{3} + \hat{3} + \hat{5}$, $\widehat{20} = \hat{5} + \hat{3} \hat{\times} \hat{5}$. Four isoprimes problem: $\hat{N} = \hat{p}_1 + \hat{p}_2 + \hat{p}_3 + \hat{p}_4$ and $\hat{N} = \hat{p}_1 + \hat{p}_2 \hat{\times} \hat{p}_3 \hat{\times} \hat{p}_4$. Five isoprimes problems: $\hat{N} = \hat{p}_1 + \hat{p}_2 + \hat{p}_3 + \hat{p}_4 + \hat{p}_5$ and $\hat{N} = \hat{p}_1 + \hat{p}_2 \hat{\times} \hat{p}_3 \hat{\times} \hat{p}_4 \hat{\times} \hat{p}_5$.

Theorem 2.7. The stable isoprime theorem. We prove that $\hat{1}, \hat{3}, \hat{5}, \hat{7}, \widehat{11}, \widehat{23}, \widehat{47}, \cdots$ are the stable isoprimes.

Theorem 2.8. The stable isoeven number theorem. We prove that $\hat{2}, \hat{4}, \hat{6}, \widehat{10}, \widehat{14}, \widehat{22}, \widehat{46}, \widehat{94}, \cdots$ are the stable isoeven numbers.

Any isoprime $\hat{p}$ can factorize the conventional composites with p factor, e.g., $\hat{5} = 10$, where $\hat{I} = 2$.

Theorem 2.9. Any isocomposite $\hat{n}$ can factorize $t(n)$ conventional primes, where

$$t(n) = \sum_{p|n} 1, \tag{2.6}$$

e.g., $\widehat{10} = 5$, where $\hat{I} = \frac{1}{2}$; $\widehat{10} = 2$, where $\hat{I} = 1/5$.

Definition 2.4. Given isointegers $\hat{a}, \hat{b}, \hat{m}$ with $\hat{m} > \hat{0}$. We say that $\hat{a}$ is isocongruent to $\hat{b}$ module $\hat{m}$ and we write

$$\hat{a} \hat{\equiv} \hat{b} \pmod{\hat{m}}. \tag{2.7}$$

If $\hat{m}$ isodivides the difference $\hat{a} - \hat{b}$, the isonumber $\hat{m}$ is called the modulus of isocongruence. The isocongruence (2.7) is equivalent to the isodivisibility relation

$$\hat{m} \;\hat{|}\; (\hat{a} - \hat{b}). \tag{2.8}$$

If $\hat{m} \;\hat{\not|}\; (\hat{a} - \hat{b})$ we write $\hat{a} \hat{\not\equiv} \hat{b} \pmod{\hat{m}}$ and say that $\hat{a}$ and $\hat{b}$ are nonisocongruent mod $\hat{m}$.

Theorem 2.10. The isocongruence is an equivalence relation:

(1) $\hat{a} \hat{\equiv} \hat{a} \pmod{\hat{m}}$ (reflexivity)
(2) $\hat{a} \hat{\equiv} \hat{b} \pmod{\hat{m}}$ implies $\hat{b} \hat{\equiv} \hat{a} (\mathrm{mod}\ \hat{m})$ (symmetry)
(3) $\hat{a} \hat{\equiv} \hat{b} \pmod{\hat{m}}$ and $\hat{b} \hat{\equiv} \hat{c} \pmod{\hat{m}}$ implies $\hat{a} \hat{\equiv} \hat{c} \pmod{\hat{m}}$ (transitivity)

Theorem 2.11. Assume $(\hat{a}, \hat{m}) = \hat{I}$. Then the linear isocongruence

$$\hat{a} \hat{\times} \hat{x} \hat{\equiv} \hat{b} \pmod{\hat{m}} \tag{2.9}$$

has exactly one solution.

Definition 2.5. The quadratic isocongruence

$$\hat{x}^{\hat{2}} \hat{\equiv} \hat{n} \pmod{\hat{p}}, \tag{2.10}$$

where $\hat{p}$ is an odd isoprime and $\hat{n} \hat{\not\equiv} \hat{0} \pmod{\hat{p}}$. (2.10) has at most two solutions. If $\hat{x}$ is a solution so is $-\hat{x}$, hence the number of solution is either 0 or 2.

If isocongruence (2.10) has a solution and we say that $\hat{n}$ is a quadratic residue mod $\hat{p}$ and we write $(\frac{\hat{n}}{\hat{p}}) = 1$, where $(\frac{\hat{n}}{\hat{p}})$ is isolegendre's symbol. If (2.10) has no solution we say that $\hat{n}$ is a quadratic nonresidue mod $\hat{p}$ and we write $(\frac{\hat{n}}{\hat{p}}) = -1$.

Theorem 2.12.

$$\sum_{i=1}^{n} \hat{x}_i^{\hat{2}} \hat{\equiv} \hat{a} \pmod{\hat{p}}. \tag{2.11}$$

Let $W_{n0}(\ \hat{p} \hat{|} \hat{a}), W_{n+}((\frac{\hat{a}}{\hat{p}}) = 1)$ and $W_{n-}((\frac{\hat{a}}{\hat{p}}) = -1)$ denote the number of solutions of (2.11).

If $\hat{p} \hat{\equiv} \hat{I}$ (mod $\hat{4}$) we have the recurrence formulas:

$$W_{n0} = (p-1)W_{(n-1)+},$$

$$W_{n+} = \frac{p-1}{2}(W_{(n-1)+} + W_{(n-1)-}) + 2W_{(n-1)0} - 2W_{(n-1)+},$$

$$W_{n-} = \frac{p-1}{2}(W_{(n-1)+} + W_{(n-1)-}). \tag{2.12}$$

If $\hat{p} \hat{\equiv} \hat{3}$ (mod $\hat{4}$) we have the recurrence formulas:

$$W_{n0} = (p-1)W_{(n-1)-},$$

$$W_{n+} = \frac{p-3}{2}(W_{(n-1)+} + W_{(n-1)-}) + 2W_{(n-1)0},$$

$$W_{n-} = \frac{p-1}{2}(W_{(n-1)+} + W_{(n-1)-}) + (W_{(n-1)+} - W_{(n-1)-}). \tag{2.13}$$

Theorem 2.13.

$$\sum_{i=1}^{n} \hat{x}_i \hat{\equiv} \hat{a} \text{ (mod } \hat{p}), \tag{2.14}$$

where $\hat{p}$ is an odd isoprime. (2.14) has exactly $J_n(p) + (-1)^n$ solutions, where $J_n(p) = \frac{(p-1)^n - (-1)^n}{p}$, if $\hat{p} \hat{|} \hat{a}$, and (2.14) $J_n(p)$ solutions if $\hat{p} \hat{\not|} \hat{a}$.

Theorem 2.14.

$$\hat{x}_1^{\hat{2}} + \hat{x}_2 + \cdots + \hat{x}_n \hat{\equiv} \hat{a} \text{ (mod } \hat{p}), \tag{2.15}$$

(2.15) has exactly $J_n(p) - (-1)^n$ solutions if $(\frac{\hat{a}}{\hat{p}}) = 1$ and (2.15) $J_n(p) + (-1)^n$ solutions if $(\frac{\hat{a}}{\hat{p}}) = -1$ and $\hat{p} \hat{|} \hat{a}$.

Theorem 2.15.

$$\hat{x}_1^{\hat{2}} + \hat{x}_2^{\hat{2}} + \hat{x}_3 + \cdots + \hat{x}_n \hat{\equiv} \hat{a} \text{ (mod } \hat{p}),\ n \geq 2. \tag{2.16}$$

For $\hat{p} \hat{\equiv} \hat{I}$ (mod $\hat{4}$), (2.16) has exactly $J_n(p) - 3(-1)^n$ solutions if $(\frac{\hat{a}}{\hat{p}}) = 1$; (2.16) $J_n(p) + p(-1)^n$ solutions if $\hat{p} \hat{|} \hat{a}$; (2.16) $J_n(p) + (-1)^n$ solutions if $(\frac{\hat{a}}{\hat{p}}) = -1$. For $\hat{p} \hat{\equiv} \hat{3}$ (mod $\hat{4}$), (2.16) has exactly $J_n(p) - J_2(p)(-1)^n$ solutions if $\hat{p} \hat{|} \hat{a}$; (2.16) $J_n(p) - (-1)^n$ solutions if $(\frac{\hat{a}}{\hat{p}}) = 1$; (2.16) $J_n(p) + 3(-1)^n$ solutions if $(\frac{\hat{a}}{\hat{p}}) = -1$.

Theorem 2.16.

$$\hat{x}_1^{\hat{2}} + \hat{x}_2^{\hat{2}} + \hat{x}_3^{\hat{2}} + \hat{x}_4^{\hat{2}} + \hat{x}_5 + \cdots + \hat{x}_n \hat{\equiv} \hat{a} \text{ (mod } \hat{p}) \ n \geq 4. \tag{2.17}$$

For $\hat{p} \hat{\equiv} \hat{I}$ (mod $\hat{4}$), (2.17) has exactly $J_n(p) + (p^2 + 5p - 5)(-1)^n$ solutions if $\hat{p} \hat{|} \hat{a}$; (2.17) $J_n(p) - (5p + 10)(-1)^n$ solutions if $(\frac{\hat{a}}{\hat{b}}) = 1$; (2.17) $J_n(p) + (3p - 2)(-1)^n$ solutions if $(\frac{\hat{a}}{\hat{b}}) = -1$. For $\hat{p} \hat{\equiv} \hat{3}$ (mod $\hat{4}$), (2.17) has exactly $J_n(p) + (p^2 - 7p - 7)(-1)^n$ solutions if $\hat{p} \hat{|} \hat{a}$; (2.17) $J_n(p) + (3p + 2)(-1)^n$ solutions if $(\frac{\hat{a}}{\hat{b}}) = 1$; (2.17) $J_n(p) - (5p - 10)(-1)^n$ solutions if $(\frac{\hat{a}}{\hat{b}}) = -1$.

Theorem 2.17.

$$\hat{x}_1^{\hat{3}} + \hat{x}_2 + \cdots + \hat{x}_n \hat{\equiv} \hat{a} (\text{mod } \hat{p}). \tag{2.18}$$

(2.18) has exactly $J_n(p) - 2(-1)^n$ solutions if iso $a^{\frac{p-1}{3}} \hat{\equiv} \hat{I}$ (mod $\hat{p}$); (2.18) $J_n(p)$ solutions if $\hat{p} \hat{|} \hat{a}$ and $\hat{3} \ \hat{\not|} \ (\hat{p} - \hat{I})$; $J_n(p) + (-1)^n$ solutions otherwise.

Theorem 2.18.

$$\hat{x}_1 \hat{\times} \cdots \hat{\times} \hat{x}_{n+1-r} + \sum_{i=n+2-r}^{n} \hat{x}_i \hat{\equiv} \hat{a} \text{ (mod } \hat{p}). \tag{2.19}$$

(2.19) has exactly $J_n(p) + (-1)^r J_{n+1-r}(p)$ solutions if $\hat{p} \hat{|} \hat{a}$;
(2.19) $J_n(p) + (-1)^{r+1} J_{n-r}(p)$ solutions if $\hat{p} \ \hat{\not|} \ \hat{a}$.

Theorem 2.19.

$$\hat{x}_1 \hat{\times} \cdots \hat{\times} \hat{x}_{k-1} \hat{\times} (\hat{x}_k + \cdots + \hat{x}_n) \hat{\equiv} \hat{a} \text{ (mod } \hat{p}). \tag{2.20}$$

(2.20) has exactly $J_n(p) - (-1)^k J_k(p)(-1)^n$ solutions if $\hat{p} \hat{|} \hat{a}$;
(2.20) $J_n(p) + (-1)^k J_{k-1}(p)(-1)^n$ solutions if $\hat{p} \hat{\not|} \hat{a}$.

Theorem 2.20.

$$(\hat{x}_1 + \cdots + \hat{x}_k) \hat{\times} \hat{x}_{k+1} \hat{\times} \cdots \hat{\times} \hat{x}_n \hat{\equiv} \hat{a} \text{ (mod } \hat{p}). \tag{2.21}$$

(2.21) has exactly $(J_k(p) + (-1)^k)(p-1)^{n-k}$ solutions if $\hat{p} \hat{|} \hat{a}$; (2.21) $J_k(p)$ $(p-1)^{n-k}$ solutions if $\hat{p} \hat{\not|} \hat{a}$.

Theorem 2.21.

$$\hat{f}(\hat{x}_1, \cdots, \hat{x}_n) \hat{\equiv} \hat{a} \text{ (mod } \hat{p}). \tag{2.22}$$

Let W_n denote the number of solutions of (2.22), we has

$$W_n = J_n(p)(1 + O(1)). \tag{2.23}$$

3. Additive Isoprime Theory

Definition 3.1. We define the arithmetic progressions[4]

$$E_{p_\alpha}(K) = \omega K + p_\alpha, \tag{3.1}$$

where $K = 0, 1, 2, \cdots;$

$$\omega = \prod_{2 \le p \le p_i} p, \ \ (p_\alpha, \omega) = 1, p_i p_i$$

can be expressed as the form $E_{p_\alpha}(K)$. We define the primes and composites by K.

By lifting $F(a, +, \times) \to \hat{F}(\hat{a}, +, \hat{\times})$ we have isoarithmetic progressions from (3.1)

$$\hat{E}_{p_\alpha}(K) = (\omega K + p_\alpha)\hat{I} = E_{p_\alpha}(K)\hat{I}. \tag{3.2}$$

By using isofields $\hat{F}(\hat{E}_{p_\alpha}(K), +, \hat{\times})$ we characterize a new branch of number theory called the Santilli's additive isoprime theory.

Theorem 3.1. Let

$$\hat{f}(\hat{p}) = \hat{a}_n \hat{\times} (\hat{p})^{\hat{n}} + \cdots + \hat{a}_0 = (a_n p^n + \cdots + a_0)\hat{I} = f(p)\hat{I} \tag{3.3}$$

be an isopolynomial.

If there exist infinitely many isoprimes $\hat{p}$ such that each of $\hat{f}_i(\hat{p})$ (for $i = 1, \cdots, k-1$) is also an isoprime, then $\hat{f}_i(\hat{p})$ must satisfy two necessary and sufficient conditions:

(I) Let $\hat{f}_i(\hat{p})$ be $k-1$ distinct irreducible isopolynomials with isointegral coefficients and one variable $\hat{p}$.

(II) There exists an arithmetic function $J_2(\omega)$ that is to separate the number of subequation k-tuples from (3.2). It is also the number of solutions

$$\left(\prod_{i=1}^{k-1} \hat{f}_i(\hat{p}_\alpha), \hat{\omega} \right) = \hat{I}. \tag{3.4}$$

Since $J_2(\omega) \le \phi(\omega)$, $J_2(\omega)$ can be expressed as the form

$$J_2(\omega) = \prod_{3 \le p \le p_i} (p - 1 - H(p)). \tag{3.5}$$

Now we determinate $H(p)$, every $\hat{p}_\alpha > \hat{p}$ can be expressed as the form

$$\hat{p}_\alpha = \hat{p} \hat{\times} \hat{g} + \hat{q}, \tag{3.6}$$

where $\hat{q} = \hat{1}, \hat{2}, \ldots, \hat{p} - \hat{I}$.

Substituting (3.6) into (3.4) we have isocongruence

$$\prod_{i=1}^{k-1} \hat{f}_i(\hat{q}) \hat{\equiv} \hat{0} \ (\text{mod } \hat{p}). \tag{3.7}$$

Let $H(p)$ denote the number of solutions of (3.7) for every isointeger $\hat{q}$. If (3.7) has no solutions for every isointeger, we have $H(p) = 0$ and $J_2(p) = \phi(p) = p - 1$. (3.7) has at most $H(p) = p - 1$ solutions. We have $J_2(p) = 0$. There can be no more than one k-tuples of isoprimes.

Example 3.1. 3-tuples: $\hat{p}, \hat{p}_1 = \hat{p} + \hat{2},\ \hat{p}_2 = \hat{p} + \hat{4}$.
From (3.5) we have

$$J_2(\omega) = \prod_{3 \leq p \leq p_i} (p - 1 - H(p)). \tag{3.8}$$

From (3.7) we have $H(3) = 2$ and $J_2(3) = 0$. There does not exist any 3-tuple of isoprimes except one 3-tuple: $\hat{3}, \hat{5}, \hat{7}$.

We deal with $\hat{p}_i = \hat{a}_i \hat{\times} \hat{p} + \hat{b}_i$ (for $i = 1, \cdots, k-1$), $\hat{2}\ \hat{|}\ \hat{a}_i \hat{\times} \hat{b}_i, (\hat{a}_i, \hat{b}_i) = \hat{I}, J_2(\omega)$ can be written in the form

$$J_2(\omega) = \prod_{3 \leq p \leq p_i} (p - k - \chi_1(p) - \cdots - \chi_{k-1}(p)), \tag{3.9}$$

where

$$\chi_i(p) = \begin{cases} -1 & \text{if } \hat{p}\ \hat{|}\ \hat{a} \hat{\times} \hat{b} \\ 0 & \text{if } \hat{p}\ \hat{\not|}\ \hat{a} \hat{\times} \hat{b} \end{cases}$$

$$\chi_i(p) = \begin{cases} -1 & \text{if } \hat{p}\ \hat{|}\ (\hat{a}_i \hat{\times} \hat{q} + \hat{b}_i) \text{ and } \hat{p} \hat{|} (\hat{a}_j \hat{\times} \hat{q} + \hat{b}_j), i \neq j \\ 0 & \text{if } \hat{p}\ \ \hat{\not|}\ (\hat{a}_i \hat{\times} \hat{q} + \hat{b}_i) \end{cases}$$

The arithmetic function $J_2(\omega)$ is a generalization of Euler's function. It plays an important role in the study of Santilli's additive isoprime theory.

By using the $J_2(\omega)$ from (3.2) it is possible to obtain the best asymptotic formulas of Santilli's additive isoprime theory.

Suppose that t_α is independent of p_α,where t_α denotes the number of primes K_p less than m in $\hat{f}_i(\hat{E}_{p_\alpha}(K_p)) =$ isoprime. We take $t_1 = t_\alpha$,where $\alpha = 2, \cdots, J_2(\omega)$, and have asymptotic formula

$$\pi_k(N, 2) = \left| \left\{ \hat{p} : \hat{p} \leq \hat{N}, \hat{f}_i(\hat{p}) = \text{isoprime} \right\} \right| = \sum_{\alpha=1}^{J_2(\omega)} t_\alpha \sim J_2(\omega) t_1. \tag{3.10}$$

We deal with one subequation k-tuple: $\hat{E}_{p_1}(K), \hat{f}_i(\hat{E}_{p_1}(K))$. We define the sequence:

$$K = 0, 1, 2, \cdots, m. \tag{3.11}$$

We take the average value formula

$$t_1 = \left| \left\{ K_p : K_p \leq m, \hat{f}_i(\hat{E}_{p_1}(K_p)) = \text{isoprime} \right\} \right| \sim \frac{C[\pi_1(\omega m)]^k}{m^{k-1}}, \tag{3.12}$$

where $\pi_1(\omega m)$ denotes the number of primes K less than m in $\hat{E}_{p_1}(K), C$ is a constant. t_1 is independent of p_1. The primes K_p seem to be equally distributed among $J_2(\omega)$. (3.10) is precise statement of this fact.

Isolagrange theorem. The isopolynomial isocongruence

$$\hat{f}_i \hat{\equiv} 0 \ (\bmod \hat{p}) \tag{3.13}$$

has at most deg$\hat{f}_i$ solutions. The large the deg$\hat{f}_i$, the less the number of isoprimes representable by $\hat{f}_i$. We take

$$C = \prod_{i=1}^{k-1} (\deg \hat{f}_i)^{-1}. \tag{3.14}$$

Let $N = \omega m$ and $\pi_1(N) \sim \frac{N}{\phi(\omega) \log N}$. Substituting it into (3.12), we have

$$t_1 = \frac{C\omega^{k-1}}{\phi^k(\omega)} \frac{N}{\log^k N}(1 + O(1)). \tag{3.15}$$

$t_1 = 0$ if $N < \omega$; $t_1 \neq 0$ if $N > \omega$ similar to (3.1). (3.15) can be applied to any k-tuple of subequations and t_1 is called the common factor in Santilli's additive isoprime theory. Substituting (3.14) and (3.15) into (3.10) we have

$$\pi_k(N, 2) = \prod_{i=1}^{k-1} (\deg \hat{f}_i)^{-1} \times \frac{J_2(\omega)\omega^{k-1}}{\phi^k(\omega)} \frac{N}{\log^k N}(1 + O(1)). \tag{3.16}$$

$\pi_k(N, 2)$ depends on $J_2(\omega)$ only. $\pi_k(N, 2) = 0$ if $J_2(\omega) = 0$; $\pi_k(N, 2) \neq 0$ if $J_2(\omega) \neq 0$. Since $\pi_k(N, 2) \to \infty$ as $J_2(\omega) \to \infty$. This relation implies there exist infinitely many k-tuples of isoprimes.
The prove of the theorems is transformed into studying the arithmetic functions $J_2(\omega)$. By using the $J_2(\omega)$ we prove the following theorems:

Theorem 3.1.1. $\hat{p}_1 = \hat{a} \hat{\times} \hat{p} + \hat{b},\ \hat{2} \hat{\mid} \hat{a} \hat{\times} \hat{b},\ (\hat{a}, \hat{b}) = \hat{I}$. From (3.9) we have

$$J_2(\omega) = \prod_{3 \le p \le p_i} (p-2) \prod_{p|ab} \frac{p-1}{p-2} \neq 0.$$

Since $J_2(\omega) \to \infty$ as $\omega \to 0$, there exist infinitely many isoprimes $\hat{p}$ such that $\hat{p}_1$ is also an isoprime.

Theorem 3.1.2. Isogoldbach theorem: $\hat{p}_1 = \hat{N} - \hat{p}, \hat{2} \hat{\mid} \hat{N}$,

$$J_2(\omega) = \prod_{3 \le p \le p_i} (p-2) \prod_{p|N} \frac{p-1}{p-2} \neq 0.$$

Since $J_2(\omega) \to \infty$ as $\omega \to \infty$, every isoeven number $\hat{N}$ greater than $\hat{4}$ is the sum of two isoprimes. It is the simplest theorem[3].

Theorem 3.1.3. Isoprime twins: $\hat{p}_1 = \hat{p} + \hat{b}, \hat{2} \hat{|} \hat{b}$,

$$J_2(\omega) = \prod_{3 \le p \le p_i} (p-2) \prod_{p|b} \frac{p-1}{p-2} \neq 0.$$

Since $J_2(\omega) \to \infty$ as $\omega \to \infty$, there exist infinitely many isoprimes $\hat{p}$ such that $\hat{p}_1$ is also an isoprime.

Theorem 3.1.4. $\hat{p}_1 = (\hat{p} \pm \hat{a})^{\hat{2}} + \hat{I}$, where $\hat{a}$ is an isoodd.

$$J_2(\omega) = \prod_{3 \le p \le p_i} (p - 2 - \chi(p)) \neq 0,$$

where $\chi(p) = 0$ if $\hat{p} \hat{|} (\hat{a}^{\hat{2}} + \hat{I})$; $\chi(p) = (-1)^{\frac{p-1}{2}}$ otherwise.

Since $J_2(\omega) \neq 0$, there exist infinitely many isoprimes $\hat{p}$ such that $\hat{p}_1$ is also an isoprime.

Theorem 3.1.5. $\hat{p}_1 = (\hat{p} + \hat{a})^{\hat{2}} + \hat{a}, \hat{a} \neq -\hat{b}^{\hat{2}}$,

$$J_2(\omega) = \prod_{3 \le p \le p_i} (p - 2 - \chi(p)) \neq 0.$$

where $\chi(p) = 0$ if $\hat{p} \hat{|} (\hat{a}^{\hat{2}} + \hat{a})$; $x(p) = (-\frac{\hat{a}}{\hat{p}})$ otherwise.

Since $J_2(\omega) \neq 0$, there exist infinitely many isoprime $\hat{p}$ such that $\hat{p}_1$ is also an isoprime.

Theorem 3.1.6. $\hat{p}_1 = \hat{p}^{\hat{2}} + \hat{p} + \hat{b}$, where $\hat{b}$ is an isoodd.

$$J_2(\omega) = \prod_{3 \le p \le p_i} (p - 2 - \chi(p)) \neq 0,$$

where $\chi(p) = 0$ if $\hat{p} \hat{|} \hat{b}$ and $\hat{p} \hat{|} (\hat{I} - \hat{4} \hat{\times} \hat{b})$; $\chi(p) = (\frac{\hat{I} - \hat{4} \hat{\times} \hat{b}}{\hat{p}})$ otherwise.

Theorem 3.1.7. $\hat{p}_i = \hat{p} + \hat{\omega}_g \hat{\times} \hat{I}$, where $\hat{I} = \hat{1}, \hat{2}, \cdots, \hat{k} - \hat{1}$;
$\omega_g = \prod_{2 \le p \le p_g} p$,

$$J_2(\omega) = \prod_{3 \le p \le p_g} (p-1) \prod_{p_{g+1} \le p \le p_i} (p-k).$$

$J_2(\omega) \neq 0$ if $k < P_{g+1}$; $J_2(p_{g+1}) = 0$ if $k = p_{g+1}$.

Theorem 3.1.8. $\hat{p}_1 = \hat{p} + \hat{2}$ and $\hat{p}_2 = \hat{p} + \hat{6}$. From (3.9) and (3.16) we have

$$J_2(\omega) = \prod_{5 \leq p \leq p_i} (p-3) \neq 0.$$

$$\pi_3(N,2) = \frac{J_2(\omega)\omega^2}{\phi^3(\omega)} \frac{N}{\log^3 N}(1+O(1))$$
$$= \frac{1}{4}\left(\frac{15}{4}\right)^2 \prod_{7 \leq p \leq p_i} \frac{p^2(p-3)}{(p-1)^3} \frac{N}{\log^3 N}(1+O(1)).$$

Let $\frac{N}{\log^3 N} \sim \frac{\pi^3(N)}{N^2}$. We have $\pi_3(N,2) = 57, 267, 1443, 8672, 56506, 9552915553$ if $N = 10^4, 10^5, 10^6, 10^7, 10^8, 10^{14}$.

Theorem 3.1.9. $\hat{p}_1 = \hat{p} + \hat{2}$, $\hat{p}_2 = \hat{p} + \hat{6}$ and $\hat{p}_3 = \hat{p} + \hat{8}$.

$$J_2(\omega) = \prod_{5 \leq p \leq p_i} (p-4) \neq 0.$$

$$\pi_4(N,2) = \frac{J_2(\omega)\omega^3}{\phi^4(\omega)} \frac{N}{\log^4 N}(1+O(1))$$
$$= \frac{1}{8}\left(\frac{15}{4}\right)^3 \prod_{7 \leq p \leq p_i} \frac{p^3(p-4)}{(p-1)^4} \frac{N}{\log^4 N}(1+O(1)).$$

Let $\frac{N}{\log^4 N} \sim \frac{\pi^4(N)}{N^3}$. We have $\pi_4(N,2) = 11, 39, 172, 866, 4894, 451570107$ if $N = 10^4, 10^5, 10^6, 10^7, 10^8, 10^{14}$.

Theorem 3.1.10. $\hat{p}_1 = \hat{p} + \hat{2}$, $\hat{p}_2 = \hat{p} + \hat{6}$, $\hat{p}_3 = \hat{p} + \hat{8}$, $\hat{p}_4 = \hat{p} + \widehat{12}$.

$$J_2(\omega) = \prod_{7 \leq p \leq p_i} (p-5) \neq 0.$$

$$\pi_5(N,2) = \frac{J_2(\omega)\omega^4}{\phi^5(\omega)} \frac{N}{\log^5 N}(1+O(1))$$
$$= \frac{1}{8}\left(\frac{15}{4}\right)^4 \prod_{7 \leq p \leq p_i} \frac{p^4(p-5)}{(p-1)^5} \frac{N}{\log^5 N}(1+O(1))$$

We have $\pi_5(N,2) = 10, 35, 147, 705$ if $N = 10^5, 10^6, 10^7, 10^8$.

Theorem 3.1.11. $\hat{p}_1 = \hat{p} + \hat{4},\ \hat{p}_2 = \hat{p} + \hat{6},\ \hat{p}_3 = \hat{p} + \widehat{10},\ \hat{p}_4 = \hat{p} + \widehat{12},\ \hat{p}_5 = \hat{p} + \widehat{16}.$

$$J_2(\omega) = \prod_{7 \le p \le p_i} (p-6) \ne 0,$$

$$\pi_6(N,2) = \frac{J_2(\omega)\omega^5}{\phi^6(\omega)} \frac{N}{\log^6 N}(1+O(1))$$

$$= \frac{1}{8}(\frac{15}{4})^5 \prod_{7 \le p \le p_i} \frac{p^5(p-6)}{(p-1)^6} \frac{N}{\log^6 N}(1+O(1)).$$

We have $\pi_6(N,2) = 18, 73, 337, 1730, 9328$ if $N = 10^7, 10^8, 10^9, 10^{10}, 10^{11}$.

Theorem 3.1.12. $\hat{p}_1 = \hat{p} + \hat{2},\ \hat{p}_2 = \hat{p} + \hat{6},\ \hat{p}_3 = \hat{p} + \hat{8},\ \hat{p}_4 = \hat{p} + \widehat{12},\ \hat{p}_5 = \hat{p} + \widehat{18},\ \hat{p}_6 = \hat{p} + \widehat{20}.$

$$J_2(\omega) = \prod_{11 \le p \le p_i} (p-7) \ne 0,$$

$$\pi_7(N,2) = \frac{J_2(\omega)\omega^6}{\phi^7(\omega)} \frac{N}{\log^7 N}(1+O(1))$$

$$= \frac{1}{48}\left(\frac{35}{8}\right)^6 \prod_{11 \le p \le p_i} \frac{p^6(p-7)}{(p-1)^7} \frac{N}{\log^7 N}(1+O(1)).$$

We have $\pi_7(N,2) = 56, 258, 1250, 6540$ if $N = 10^9, 10^{10}, 10^{11}, 10^{12}$.

Theorem 3.1.13. $\hat{p}_1 = \hat{p} + \hat{2},\ \hat{p}_2 = \hat{p} + \hat{6},\ \hat{p}_3 = \hat{p} + \widehat{12},\ \hat{p}_4 = \hat{p} + \widehat{14},\ \hat{p}_5 = \hat{p} + \widehat{20},\ \hat{p}_6 = \hat{p} + \widehat{24},\ \hat{p}_7 = \hat{p} + \widehat{26}.$

$$J_2(\omega) = 48 \prod_{17 \le p \le p_i} (p-8) \ne 0,$$

$$\pi_8(N,2) = \frac{J_2(\omega)\omega^7}{\phi^8(\omega)} \frac{N}{\log^8 N}(1+O(1))$$

$$= \frac{1}{120}\left(\frac{1001}{192}\right)^7 \prod_{17 \le p \le p_i} \frac{p^7(p-8)}{(p-1)^8} \frac{N}{\log^8 N}(1+O(1)).$$

We have $\pi_8(N,2) = 476, 2250$ if $N = 10^{11}, 10^{12}$.

Theorem 3.1.14. $\hat{p}_1 = \hat{p} + \hat{2},\ \hat{p}_2 = \hat{p} + \hat{6},\ \hat{p}_3 = \hat{p} + \hat{8},\ \hat{p}_4 = \hat{p} + \widehat{12},\ \hat{p}_5 = \hat{p} + \widehat{18},\ \hat{p}_6 = \hat{p} + \widehat{20},\ \hat{p}_7 = \hat{p} + \widehat{26},\ \hat{p}_8 = \hat{p} + \widehat{30}.$

$$J_2(\omega) = 15 \prod_{17 \le p \le p_i} (p-9) \ne 0.$$

$$\pi_9(N,2) = \frac{J_2(\omega)\omega^8}{\phi^9(\omega)}\frac{N}{\log^9 N}(1+O(1))$$

$$= \frac{1}{384}\left(\frac{1001}{192}\right)^8 \prod_{17\le p\le p_i} \frac{p^8(p-9)}{(p-1)^9}\frac{N}{\log^9 N}(1+O(1)).$$

We have $\pi_9(N,2) = 28, 118, 544, 2725$ if $N = 10^{11}, 10^{12}, 10^{13}, 10^{14}$.

Theorem 3.1.15. $\hat{p}_1 = \hat{p} + \hat{2},\ \hat{p}_2 = \hat{p} + \hat{6},\ \hat{p}_3 = \hat{p} + \hat{8},\ \hat{p}_4 = \hat{p} + \widehat{12},\ \hat{p}_5 = \hat{p} + \widehat{18},\ \hat{p}_6 = \hat{p} + \widehat{20},\ \hat{p}_7 = \hat{p} + \widehat{26},\ \hat{p}_8 = \hat{p} + \widehat{30},\ \hat{p}_9 = \hat{p} + \widehat{32}.$

$$J_2(\omega) = 10 \prod_{17\le p\le p_i} (p-10) \ne 0.$$

$$\pi_{10}(N,2) = \frac{J_2(\omega)\omega^9}{\phi^{10}(\omega)}\frac{N}{\log^{10} N}(1+O(1))$$

$$= \frac{1}{576}\left(\frac{1001}{192}\right)^9 \prod_{17\le p\le p_i} \frac{p^9(p-10)}{(p-1)^{10}}\frac{N}{\log^{10} N}(1+O(1)).$$

We have $\pi_{10}(N,2) = 13, 52, 241$ if $N = 10^{12}, 10^{13}, 10^{14}$.

Theorem 3.1.16. $\hat{p}_1 = \hat{p} + \hat{4},\ \hat{p}_2 = \hat{p} + \hat{6},\ \hat{p}_3 = \hat{p} + \widehat{10},\ \hat{p}_4 = \hat{p} + \widehat{16},\ \hat{p}_5 = \hat{p} + \widehat{18},\ \hat{p}_6 = \hat{p} + \widehat{24},\ \hat{p}_7 = \hat{p} + \widehat{28},\ \hat{p}_8 = \hat{p} + \widehat{30},\ \hat{p}_9 = \hat{p} + \widehat{34},\ \hat{p}_{10} = \hat{p} + \widehat{36}.$

$$J_2(\omega) = 28 \prod_{19\le p\le p_i} (p-11) \ne 0.$$

$$\pi_{11}(N,2) = \frac{J_2(\omega)\omega^{10}}{\phi^{11}(\omega)}\frac{N}{\log^{11} N}(1+O(1)).$$

$$= \frac{7}{23040}\left(\frac{17017}{3072}\right)^{10} \prod_{19\le p\le p_i} \frac{p^{10}(p-11)}{(p-1)^{11}}\frac{N}{\log^{11} N}(1+O(1)).$$

We have $\pi_{11}(N,2) = 15, 69, 321$ if $N = 10^{14}, 10^{15}, 10^{16}$.

Theorem 3.1.17. $\hat{p}_1 = \hat{p} + \hat{6},\ \hat{p}_2 = \hat{p} + \widehat{10},\ \hat{p}_3 = \hat{p} + \widehat{12},\ \hat{p}_4 = \hat{p} + \widehat{16},\ \hat{p}_5 = \hat{p} + \widehat{22},\ \hat{p}_6 = \hat{p} + \widehat{24},\ \hat{p}_7 = \hat{p} + \widehat{30},\ \hat{p}_8 = \hat{p} + \widehat{34},\ \hat{p}_9 = \hat{p} + \widehat{36},\ \hat{p}_{10} = \hat{p} + \widehat{40},\ \hat{p}_{11} = \hat{p} + \widehat{42}.$

$$J_2(\omega) = 21 \prod_{19\le p\le p_i} (p-12) \ne 0.$$

$$\pi_{12}(N,2) = \frac{J_2(\omega)\omega^{11}}{\phi^{12}(\omega)}\frac{N}{\log^{12} N}(1+O(1)).$$

$$= \frac{7}{30720}\left(\frac{17017}{3072}\right)^{11}\prod_{19\le p\le p_i}\frac{p^{11}(p-12)}{(p-1)^{12}}\frac{N}{\log^{12}N}(1+O(1))$$

We have $\pi_{12}(N,2)=7,31,140,693$ if $N=10^{15},10^{16},10^{17},10^{18}$.

Theorem 3.1.18. $\hat{p}_1=\hat{p}+\hat{2}$, $\hat{p}_2=\hat{p}+\hat{8}$, $\hat{p}_3=\hat{p}+\widehat{14}$, $\hat{p}_4=\hat{p}+\widehat{18}$, $\hat{p}_5=\hat{p}+\widehat{20}$, $\hat{p}_6=\hat{p}+\widehat{24}$, $\hat{p}_7=\hat{p}+\widehat{30}$, $\hat{p}_8=\hat{p}+\widehat{32}$, $\hat{p}_9=\hat{p}+\widehat{38}$, $\hat{p}_{10}=\hat{p}+\widehat{42}$, $\hat{p}_{11}=\hat{p}+\widehat{44}$, $\hat{p}_{12}=\hat{p}+\widehat{48}$.

$$J_2(\omega)=924\prod_{29\le p\le p_i}(p-13)\neq 0.$$

$$\pi_{13}(N,2)=\frac{J_2(\omega)\omega^{12}}{\phi^{13}(\omega)}\frac{N}{\log^{13}N}(1+O(1))$$

$$= \frac{7}{276480}\left(\frac{676039}{110592}\right)^{12}\prod_{29\le p\le p_i}\frac{p^{12}(p-13)}{(p-1)^{13}}\frac{N}{\log^{13}N}(1+O(1)).$$

We have $\pi_{13}(N,2)=12,55$ if $N=10^{17},10^{18}$.

Theorem 3.1.19. $\hat{p}_1=\hat{p}+\hat{2}$, $\hat{p}_2=\hat{p}+\hat{8}$, $\hat{p}_3=\hat{p}+\widehat{14}$, $\hat{p}_4=\hat{p}+\widehat{18}$, $\hat{p}_5=\hat{p}+\widehat{20}$, $\hat{p}_6=\hat{p}+\widehat{24}$, $\hat{p}_7=\hat{p}+\widehat{30}$, $\hat{p}_8=\hat{p}+\widehat{32}$, $\hat{p}_9=\hat{p}+\widehat{38}$, $\hat{p}_{10}=\hat{p}+\widehat{42}$, $\hat{p}_{11}=\hat{p}+\widehat{44}$, $\hat{p}_{12}=\hat{p}+\widehat{48}$, $\hat{p}_{13}=\hat{p}+\widehat{50}$.

$$J_2(\omega)=300\prod_{29\le p\le p_i}(p-14)\neq 0.$$

$$\pi_{14}(N,2)=\frac{J_2(\omega)\omega^{13}}{\phi^{14}(\omega)}\frac{N}{\log^{14}N}(1+O(1))$$

$$= \frac{5}{608256}(\frac{676039}{110592})^{13}\prod_{29\le p\le p_i}\frac{p^{13}(p-14)}{(p-1)^{14}}\frac{N}{\log^{14}N}(1+O(1)).$$

We have $\pi_{14}(N,2)=1,2$ if $N=10^{17},10^{18}$.

Note that Theorems 3.1.8. to 3.1.19. are Forbes isoprime theorems. Since $J_2(\omega)\to\infty$ as $\omega\to\infty$, there exist infinitely many k-tuples of isoprimes, where $k=3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14$. Forbes proved that there exists one k-tuple [5]. They are the simplest theorems. One can understand and prove them.

Theorem 3.1.20. $\hat{p}_1=\hat{p}+\hat{2}$, $\hat{p}_2=\hat{p}+\hat{6}$, $\hat{p}_3=\hat{p}+\hat{8}$, $\hat{p}_4=\hat{p}+\widehat{12}$, $\hat{p}_5=\hat{p}+\widehat{18}$, $\hat{p}_6=\hat{p}+\widehat{20}$, $\hat{p}_7=\hat{p}+\widehat{26}$, $\hat{p}_8=\hat{p}+\widehat{30}$, $\hat{p}_9=\hat{p}+\widehat{32}$, $\hat{p}_{10}=\hat{p}+\widehat{36}$, $\hat{p}_{11}=\hat{p}+\widehat{42}$, $\hat{p}_{12}=\hat{p}+\widehat{48}$, $\hat{p}_{13}=\hat{p}+\widehat{50}$, $\hat{p}_{14}=\hat{p}+\widehat{56}$.

$$J_2(\omega)=96\prod_{29\le p\le p_i}(p-15)\neq 0.$$

$$\pi_{15}(N,2) = \frac{J_2(\omega)\omega^{14}}{\phi^{15}(\omega)} \frac{N}{\log^{15} N}(1+O(1)).$$

Theorem 3.1.21. $\hat{p}_1 = \hat{p} + \hat{2},\ \hat{p}_2 = \hat{p} + \hat{6},\ \hat{p}_3 = \hat{p} + \hat{8},\ \hat{p}_4 = \hat{p} + \widehat{12},\ \hat{p}_5 = \hat{p} + \widehat{14}.$

$$J_2(5) = 0.$$

They cannot all be prime, for at least one of the five is divisible by 5.

Theorem 3.1.22. $\hat{p}_1 = \hat{p} + \hat{4},\ \hat{p}_2 = \hat{p} + \hat{6},\ \hat{p}_3 = \hat{p} + \widehat{10},\ \hat{p}_4 = \hat{p} + \widehat{12},\ \hat{p}_5 = \hat{p} + \widehat{16},\ \hat{p}_6 = \hat{p} + \widehat{22}.$

$$J_2(7) = 0.$$

They cannot all be prime, for at least one of the six is divisible by 7.

Theorem 3.1.23. $\hat{p}_1 = \hat{p}^{\hat{2}} + \hat{p} + \hat{I}$ and $\hat{p}_2 = \hat{p}^{\hat{2}} + \hat{3}\hat{\times}\hat{p} + \hat{3}$.

$$J_2(\omega) = \prod_{3 \le p \le p_i} (p - 3 - \chi(p)) \ne 0,$$

where

$$\chi(p) = \begin{cases} -1 & \text{if } \hat{p} = \hat{3} \\ 2 & \text{if } \hat{p} \hat{\equiv} \hat{I} (\text{mod } \hat{3}) \\ -2 & \text{if } \hat{p} \hat{\equiv} -\hat{I} (\text{mod } \hat{3}). \end{cases}$$

Theorem 3.1.24. $\hat{p}_1 = \hat{p}^{\hat{2}} + \hat{p} + \hat{I}$, $\hat{p}_2 = \hat{p}^{\hat{2}} + \hat{3}\hat{\times}\hat{p} + \hat{3}$ and $\hat{p}_3 = \hat{p}^{\hat{2}} + \hat{5}\hat{\times}\hat{p} + \hat{7}$.

$$J_2(3) = 0.$$

Theorem 3.1.25. $\hat{p}_1 = \hat{p}^{\hat{2}} + \hat{p} + \widehat{41}$, $\hat{p}_2 = \hat{p}^{\hat{2}} + \hat{3}\hat{\times}\hat{p} + \widehat{43}$.

$$J_2(\omega) = \prod_{3 \le p \le p_i} (p - 3 - \chi(p)) \ne 0,$$

where

$$\chi(p) = \begin{cases} 0 & \text{if } \hat{p} = \widehat{41} \text{ and } \hat{p} = \widehat{163}, \\ 1 & \text{if } \hat{p} = \widehat{43}, \\ 2 & \text{if } (-\frac{\widehat{163}}{\hat{p}}) = 1, \\ -2 & \text{if } (-\frac{\widehat{163}}{\hat{p}}) = -1. \end{cases}$$

Theorem 3.1.26. $\hat{p}_1 = \hat{p}^{\hat{2}} + \hat{p} + \widehat{41}$, $\hat{p}_2 = \hat{p}^{\hat{2}} + \hat{3}\hat{\times}\hat{p} + \widehat{43}$, $\hat{p}_3 = \hat{p}^{\hat{2}} + \hat{5}\hat{\times}\hat{p} + \widehat{47}$.

$$J_2(\omega) = \prod_{3 \le p \le p_i} (p - 4 - \chi(p)) \neq 0,$$

where

$$\chi(p) = \begin{cases} 0 & \text{if } \hat{p} = \widehat{41} \text{ and } \hat{p} = \widehat{163}, \\ 1 & \text{if } \hat{p} = \widehat{43}, \\ 2 & \text{if } \hat{p} = \widehat{47}, \\ 3 & \text{if } (-\frac{\widehat{163}}{\hat{p}}) = 1, \\ -3 & \text{if } (-\frac{\widehat{163}}{\hat{p}}) = -1. \end{cases}$$

Theorem 3.2. If there exist infinitely many isoprimes $\hat{p}_j$ (for $j = 1, \cdots, n-1$) such that isopolynomials $\hat{f}_i(\hat{p}_j)$ (for $i = 1, \cdots, k-1$) are all isoprimes, then $\hat{f}_i(\hat{p}_j)$ must satisfy two necessary and sufficient conditions:

(I) Let $\hat{f}_i(\hat{p}_j)$ be $k-1$ distinct irreducible isopolynomials with isointegral coefficients and $n-1$ variables: $\hat{p}_j$.

(II) There exists an arithmetic function $J_n(\omega)$ that is to separate the number of subequation k-tuples from (3.2). It is also the number of solutions of

$$\left(\prod_{i=1}^{k-1} \hat{f}_i(\hat{p}_{\alpha_j}),\ \hat{\omega} \right) = \hat{I}, \tag{3.17}$$

where $1 \le \alpha_j \le \phi(\omega),\ j = 1, \cdots, n-1$.

Since $J_n(\omega) \le \phi^{n-1}(\omega)$, $J_n(\omega)$ can be expressed as the form

$$J_n(\omega) = \prod_{3 \le p \le p_i} ((p-1)^{n-1} - H(p)), \tag{3.18}$$

where $H(p)$ is the number of solutions of isocongruence

$$\prod_{i=1}^{k-1} \hat{f}_i(\hat{q}_j) \hat{\equiv} 0 \ (\bmod \hat{p}), \tag{3.19}$$

$\hat{q}_j = \hat{1}, \hat{2}, \cdots, \hat{p} - \hat{1};\ j = 1, \cdots, n-1$.

If $H(p) = (p-1)^{n-1}$ for some isoprime, then $J_n(p) = 0$, there exist finitely many k-tuples of isoprimes; If $J_n(\omega) \neq 0$, then there exist infinitely many k-tuples of isoprimes. Since $(p-1)^{n-1} = \frac{(p-1)^n - (-1)^n}{p} + \frac{(p-1)^{n-1} - (-1)^{n-1}}{p}$, $J_n(\omega)$ can also be expressed as the form

$$J_n(\omega) = \prod_{3 \le p \le P_i} \left(\frac{(p-1)^n - (-1)^n}{p} - \chi(p) \right),\ \chi(p) = 0, \pm 1, \pm 2, \cdots. \tag{3.20}$$

In the same way as in Theorem 3.1., we can derive the asymptotic formula

$$\pi_k(N,n) =\mid \{\hat{p}_j : \hat{p}_j \leq \hat{N},\ \hat{f}_i(\hat{p}_j) = \text{isoprime}\} \mid$$

$$= \prod_{i=1}^{k-1} (\deg \hat{f}_i)^{-1} \times \frac{J_n(\omega)\omega^{k-1}}{(n-1)!\phi^{n+k-2}(\omega)} \frac{N^{n-1}}{(\log N)^{n+k-2}} (1+O(1)). \tag{3.21}$$

The prove of the theorems is transformed into studying the arithmetic functions $J_n(\omega)$.

By using the $J_n(\omega)$ we prove the following theorems:

Theorem 3.2.1. $\hat{p}_n = \hat{N} - \sum_{i=1}^{n-1} \hat{p}_i$.

$$J_n(\omega) = \prod_{3 \leq p \leq p_i} \left(\frac{(p-1)^n - (-1)^n}{p} \right) \prod_{p|N} \left(1 + \frac{(-1)^n p}{(p-1)^n - (-1)^n} \right) \neq 0.$$

If $n = 2$, see Theorem 3.1.2. It is isogoldbach theorem.

Theorem 3.2.2. $\hat{p}_n = \sum_{i=1}^{n-1} \hat{p}_i + \hat{b}$.

$$J_n(\omega) = \prod_{3 \leq p \leq p_i} \left(\frac{(p-1)^n - (-1)^n}{p} \right) \prod_{p|b} \left(1 + \frac{(-1)^n p}{(p-1)^n - (-1)^n} \right) \neq 0.$$

If $n = 2$, see Theorem 3.1.3. It is twin isoprime theorem.

Theorem 3.2.3. $\hat{p}_n = \hat{p}_1 \hat{\times} \cdots \hat{\times} \hat{p}_{n-1} + \hat{b}$.

$$J_n(\omega) = \phi^{n-2}(\omega) \prod_{3 \leq p \leq p_i} (p-2) \prod_{p|b} \frac{p-1}{p-2} \neq 0.$$

If $n = 2$, it is twin isoprime theorem.

Theorem 3.2.4. $\hat{p}_n = \hat{N} - \hat{p}_1 \hat{\times} \cdots \hat{\times} \hat{p}_{n-1}$.

$$J_n(\omega) = \phi^{n-2}(\omega) \prod_{3 \leq p \leq p_i} (p-2) \prod_{p|N} \frac{p-1}{p-2} \neq 0.$$

If $n = 2$, it is isogoldbach theorem. If $n = 3$, it is three isoprime theorem called (1+2). If $n = 4$, it is called (1+3). If $n = 5$, (1+4) [5].

Theorem 3.2.5. $\hat{p}_n = \hat{p}_1 \hat{\times} \cdots \hat{\times} \hat{p}_{n-r} + \sum_{i=n+1-r}^{n-1} \hat{p}_i + \hat{b}$.

$$J_n(\omega) = \prod_{3 \leq p \leq p_i} \left(\frac{(p-1)^n - (-1)^n}{p} - \chi(p) \right) \neq 0,$$

where

$$\chi(p) = (-1)^r \frac{(p-1)^{n-r} - (-1)^{n-r}}{p} \quad \text{if } \hat{p} \hat{|} \hat{b};$$

$$\chi(p) = (-1)^{r+1} \frac{(p-1)^{n-r-1} - (-1)^{n-r-1}}{p} \quad \text{if } \hat{p} \hat{\not|} \hat{b}.$$

Let $n - r = 1$, see Theorem 3.2.2. Let $r = 1$, see Theorem 3.2.3. Let $n = 2$, and $r = 1$, it is twin isoprime theorem.

Theorem 3.2.6. $\hat{p}_n = \hat{N} - \hat{p}_1 \hat{\times} \cdots \hat{\times} \hat{p}_{n-r} - \sum_{i=n+1-r}^{n-1} \hat{p}_i$.

$$J_n(\omega) = \prod_{3 \le p \le p_i} \left(\frac{(p-1)^n - (-1)^n}{p} - \chi(p) \right) \neq 0,$$

where

$$\chi(p) = (-1)^r \frac{(p-1)^{n-r} - (-1)^{n-r}}{p} \quad \text{if } \hat{p} \hat{|} \hat{N};$$

$$\chi(p) = (-1)^{r+1} \frac{(p-1)^{n-r-1} - (-1)^{n-r-1}}{p} \quad \text{if } \hat{p} \hat{\not|} \hat{N}.$$

Let $n - r = 1$, see Theorem 3.2.1. Let $r = 1$, see Theorem 3.2.4. Let $n = 2$, and $r = 1$, it is isogoldbach theorem. Let $n = 3$ and $r = 1$, it is three isoprime theorem.

Theorem 3.2.7. $\hat{p}_n = \sum_{i=1}^{n-r} \hat{p}_i + \hat{p}_{n+1-r} \hat{\times} \cdots \hat{\times} \hat{p}_{n-1} + \hat{b}$.

$$J_n(\omega) = \prod_{3 \le p \le p_i} \left(\frac{(p-1)^n - (-1)^n}{p} - \chi(p) \right) \neq 0$$

where

$$\chi(p) = (-1)^{n-r+1} \frac{(p-1)^{r-1} - (-1)^{r-1}}{p} \quad \text{if } \hat{p} \hat{|} \hat{b},$$

$$\chi(p) = (-1)^{n-r+2} \frac{(p-1)^{r-2} - (-1)^{r-2}}{p} \quad \text{if } \hat{p} \hat{\not|} \hat{b}, \ r \ge 2.$$

Theorem 3.2.8. $\hat{p}_n = \hat{N} - \sum_{i=1}^{n-r} \hat{p}_i - \hat{p}_{n+1-r} \hat{\times} \cdots \hat{\times} \hat{p}_{n-1}$.

$$J_n(\omega) = \prod_{3 \le p \le p_i} \left(\frac{(p-1)^n - (-1)^n}{p} \chi(p) \right) \neq 0,$$

where

$$\chi(p) = (-1)^{n-r+1} \frac{(p-1)^{r-1} - (-1)^{r-1}}{p} \quad \text{if } \hat{p} \hat{|} \hat{N};$$

$$\chi(p)=(-1)^{n-r+2}\frac{(p-1)^{r-2}-(-1)^{r-2}}{p} \quad \text{if } \hat{p} \hat{\not|} \hat{N},\ r\geq 2.$$

For $r=2$, see Theorem 3.2.1. For $n=r$, see Theorem 3.2.4. For $n=r=2$, it is isogoldbach theorem. For $n=r=3$, it is (1+2). Note that Theorem 3.2.1. to 3.2.8. are n isoprime theorems.

Theorem 3.2.9. $\hat{p}_4=(\hat{p}_1+\hat{p}_2+\hat{p}_3-\hat{I})^{\hat{2}}+\hat{I}$ and $\hat{p}_5=(\hat{p}_1+\hat{p}_2+\hat{p}_3+\hat{I})^{\hat{2}}+\hat{I}$.

$$J_4(\omega)=\prod_{3\leq p\leq p_i}\left(\frac{(p-1)^4-1}{p}-\chi(p)\right)\neq 0,$$

where

$$\chi(p)=(p^2-3p+3)(1+2(-1)^{\frac{p-1}{2}}).$$

Theorem 3.2.10. $\hat{p}_4=(\hat{p}_1+\hat{p}_2+\hat{p}_3-\hat{3})^{\hat{2}}+\hat{3}$ and $\hat{p}_5=(\hat{p}_1+\hat{p}_2+\hat{p}_3+\hat{3})^{\hat{2}}+\hat{3}$.

$$J_4(\omega)=\prod_{3\leq p\leq p_i}\left(\frac{(p-1)^4-1}{p}-\chi(p)\right)\neq 0,$$

where

$$\chi(3)=-1,\ \chi(p)=(p^2-3p+3)\left(1+2(\frac{-\hat{3}}{\hat{p}})\right).$$

Theorem 3.2.11. $\hat{p}_3=\hat{p}_1+\hat{p}_2+\hat{I},\ \hat{p}_4=(\hat{p}_1+\hat{p}_2)^{\hat{2}}+\hat{I}$.

$$J_3(\omega)=\prod_{3\leq p\leq p_i}(p^2-3p+3-\chi(p))\neq 0,$$

where

$$\chi(p)=(p-2)(1+(-1)^{\frac{p-1}{2}}).$$

Theorem 3.2.12. $\hat{p}_3=(\hat{p}_1+\hat{p}_2)^{\hat{2}}+\hat{b},\ \hat{2}\ \hat{\not|}\,\hat{b},\ \hat{b}\neq-\hat{a}^{\hat{2}}$.

$$J_3(\omega)=\prod_{3\leq p\leq p_i}(p^2-3p+3-\chi(p))\neq 0,$$

where $\chi(p)=1$ if $\hat{p}\ \hat{|}\ \hat{b}$; $\chi(p)=(p-2)(\frac{-\hat{b}}{\hat{p}})$ if $\hat{p}\ \hat{\not|}\,\hat{b}$.

Theorem 3.2.13. $\hat{p}_3=\hat{N}-(\hat{p}_1+\hat{p}_2)^{\hat{2}},\ \hat{2}\ \hat{\not|}\,\hat{N},\ \hat{N}\neq\hat{a}^{\hat{2}}$.

$$J_3(\omega)=\prod_{3\leq p\leq p_i}(p^2-3p+3-\chi(p))\neq 0,$$

where $\chi(p) = 1$ if $\hat{p} \hat{|} \hat{N}$; $\chi(p) = (p-2)(\frac{\hat{N}}{\hat{p}})$ otherwise.

Theorem 3.2.14. $\hat{p}_4 = (\hat{p}_1 + \hat{p}_2 + \hat{p}_3)^{\hat{2}} + \hat{b},\ \hat{2} \hat{|} \hat{b},\ \hat{b} \neq -\hat{a}^{\hat{2}}$.

$$J_4(\omega) = \prod_{3 \le p \le p_i} \left(\frac{(p-1)^4 - 1}{p} - \chi(p) \right) \neq 0,$$

where $\chi(p) = -1$ if $\hat{p} \hat{|} \hat{b}$; $\chi(p) = (p^2 - 3p + 3)(-\frac{\hat{b}}{\hat{p}})$ if $\hat{p}\ \hat{\not|}\ \hat{b}$.

Theorem 3.2.15. $\hat{p}_4 = \hat{N} - (\hat{p}_1 + \hat{p}_2 + \hat{p}_3)^{\hat{2}},\ \hat{2} \hat{|} \hat{N},\ \hat{N} \neq \hat{a}^{\hat{2}}$.

$$J_4(\omega) = \prod_{3 \le p \le p_i} \left(\frac{(p-1)^4 - 1}{p} - \chi(p) \right) \neq 0,$$

where $\chi(p) = -1$ if $\hat{p} \hat{|} \hat{N}$; $\chi(p) = (p^2 - 3p + 3)(-\frac{\hat{N}}{\hat{p}})$ if $\hat{p}\ \hat{\not|}\ \hat{N}$.

Theorem 3.2.16. $\hat{p}_4 = \hat{p}_1 + \hat{p_2}^{\hat{3}} + \hat{p_3}^{\hat{3}}$.

$$J_4(\omega) = \prod_{3 \le p \le p_i} \left(\frac{(p-1)^4 - 1}{p} - \chi(p) \right) \neq 0.$$

where $\chi(p) = 2p - 1$ if $\hat{p} \hat{\equiv} \hat{I} (\text{mod } \hat{3})$; $\chi(p) = 1$ otherwise.

Theorem 3.2.17. $\hat{p}_4 = \hat{p}_1 + \hat{p}_2 - \hat{p}_3$

$$J_4(\omega) = \prod_{3 \le p \le p_i} \left(\frac{(p-1)^4 - 1}{p} + 1 \right) \neq 0.$$

Theorem 3.2.18. $\hat{p}_6 = \hat{p}_1 + \hat{p}_2 + \hat{p}_3 - \hat{p}_4 - \hat{p}_5$

$$J_6(\omega) = \prod_{3 \le p \le p_i} \left(\frac{(p-1)^6 - 1}{p} + 1 \right) \neq 0.$$

Theorem 3.2.19. $\hat{p}_3 = \hat{p}_1 + \hat{p_2}^{\hat{2}} + \hat{b}$

$$J_3(\omega) = \prod_{3 \le p \le p_i} (p^2 - 3p + 3 + \chi(p)) \neq 0,$$

where $\chi(p) = -1$ if $\hat{p} \hat{|} \hat{b}$; $\chi(p) = (-\frac{\hat{b}}{\hat{p}})$ otherwise.

Theorem 3.2.20. $\hat{p}_3 = \hat{N} - \hat{p}_1 - \hat{p}_2^{\hat{2}}$

$$J_3(\omega) = \prod_{3\le p\le p_i} (p^2 - 3p + 3 + \chi(p)) \ne 0,$$

where $\chi(p) = -1$ if $\hat{p} \hat{|} \hat{b}$; $\chi(p) = (\frac{\hat{N}}{\hat{p}})$ otherwise.

Theorem 3.2.21. $\hat{p}_3 = \hat{p}_1 + \hat{p}_2^{\hat{3}} + \hat{b}$

$$J_3(\omega) = \prod_{3\le p\le p_i} (p^2 - 3p + 3 + \chi(p)) \ne 0,$$

where $\chi(p) = 1$ if $\hat{p} \hat{|} \hat{b}$; $\chi(p) = 2$ if iso $b^{\frac{p-1}{3}} \hat{\equiv} \hat{I}$ (mod $\hat{p}$);
$\chi(p) = -1$ if iso $b^{\frac{p-1}{3}} \hat{\not\equiv} \hat{I}$(mod $\hat{p}$), $\chi(p) = 0$ otherwise.

Theorem 3.2.22. $\hat{p}_3 = \hat{N} - \hat{p}_1 - \hat{p}_2^{\hat{3}}$

$$J_3(\omega) = \prod_{3\le p\le p_i} (p^2 - 3p + 3 + \chi(p)) \ne 0,$$

where $\chi(p) = -1$ if $\hat{p}\hat{|}\hat{N}$; $\chi(p) = 2$ if iso$(N)^{\frac{p-1}{3}} \hat{\equiv} \hat{I}$ (mod $\hat{p}$);
$\chi(p) = -1$ if iso$(N)^{\frac{p-1}{3}} \hat{\not\equiv} \hat{I}$ (mod $\hat{p}$), $\chi(p) = 0$ otherwise.

Theorem 3.2.23. $\hat{p}_3 = \hat{p}_1 + \hat{p}_2^{\hat{4}} + \hat{b}$

$$J_3(\omega) = \prod_{3\le p\le p_i} (p^2 - 3p + 3 + \chi(p)) \ne 0,$$

where $\chi(p) = -1$ if $\hat{p} \hat{|} \hat{b}$; $\chi(p) = 3$ if iso$(-b)^{\frac{p-1}{4}} \hat{\equiv} \hat{I}$(mod $\hat{p}$);
$\chi(p) = -1$ if iso$(-b)^{\frac{p-1}{4}} \hat{\not\equiv} \hat{I}$(mod $\hat{p}$); $\chi(p) = (-\frac{\hat{b}}{\hat{p}})$ otherwise.

Theorem 3.2.24. $\hat{p}_3 = \hat{N} - \hat{p}_1 - \hat{p}_2^{\hat{4}}$

$$J_3(\omega) = \prod_{3\le p\le p_i} (p^2 - 3p + 3 + \chi(p)) \ne 0,$$

where $\chi(p) = -1$ if $\hat{p} \hat{|} \hat{N}$; $\chi(p) = 3$ if iso$(N)^{\frac{p-1}{4}} \hat{\equiv} \hat{I}$(mod $\hat{p}$);
$\chi(p) = -1$ if $iso(N)^{\frac{p-1}{4}} \hat{\not\equiv} \hat{I}$(mod $\hat{p}$); $\chi(p) = \left(\frac{\hat{N}}{\hat{p}}\right)$ otherwise.

Theorem 3.2.25. $\hat{p}_4 = \hat{p}_1 + \hat{p}_2 + \hat{p}_3 + \hat{2},\ \hat{p}_5 = \hat{p}_1 + \hat{p}_2 + \hat{p}_3 + 4$

$$J_4(\omega) = \prod_{3 \le p \le p_i} \left(\frac{(p-1)^4 - 1}{p} - \frac{(p-1)^3 + 1}{p} \right) \neq 0.$$

Theorem 3.3. The arithmetic (sifting) function $J_n(\omega)$ has the following properties:

(1) $J_n(\omega) = \prod_{3 \le p \le p_i} (\frac{(p-1)^n - (-1)^n}{p} - \chi(p))$

(2) $J_n(2^m) = \phi^{n-1}(2^m) = 2^{(n-1)(m-1)}$

(3) $J_n(1) = J_1(\omega) = J_n(2) = 1$

(4) $J_n(ab) = J_n(a)J_n(b),\ (a,b) = 1$

(5) $a \mid b \longrightarrow J_n(a) \mid J_n(b), n > 1$

(6) $J_n(\omega^m) = \omega^{(n-1)(m-1)} J_n(\omega)$

(7) $J_n(\omega) = \sum_{\alpha=1}^{\phi(\omega)} J_{n-1}(\omega, p_\alpha)$

(8) $J_n(ab) = \frac{d^{n-1} J_n(a) J_n(b)}{J_n(d)}, (a,b) = d.$

(9) $\frac{(p-1)^n - (-1)^n}{p} - \frac{(p-1)^{n-2} - (-1)^{n-2}}{p} = (p-1)^{n-2}(p-2)$

(10) $J_n(\omega, k-2) \ge J_n(\omega, k-1)$

(11) $\frac{J_n(\omega^m)(\omega^m)^{k-1}}{\phi^{n+k-2}(\omega^m)} = \frac{J_n(\omega)\omega^{k-1}}{\phi^{n+k-2}(\omega)}$

(12) $(p-1)^{n-1} = \frac{(p-1)^n - (-1)^n}{p} + \frac{(p-1)^{n-1} - (-1)^{n-1}}{p}$
$= J_n(p) + H(p).$

We have

$$J_n(p) = \frac{(p-1)^n - (-1)^n}{p} + \frac{(p-1)^{n-1} - (-1)^{n-1}}{p} - H(p)$$
$$= \frac{(p-1)^n - (-1)^n}{p} - \chi(p).$$

where $\chi(p) = H(p) - \frac{(p-1)^{n-1}-(-1)^{n-1}}{p}$; $\chi(p)$ depends on the coefficients and degrees of $\hat{f}_i$.

For example. $\hat{p}_1 = \hat{p} + \hat{2},\ \chi(p) = 0;\ \hat{p}_3 = \hat{p}_1 + \hat{p}_2 + \hat{I},\ \chi(p) = 0;\ \hat{p}_4 = \hat{p}_1 + \hat{p}_2 + \hat{p}_3 + \hat{2},\ \chi(p) = 0.$

4. Derivations of the best asymptotic formulas

4.1. $\hat{p}_2 = \hat{N} - \hat{p}_1$.

From (3.2) we have a subequation

$$\hat{E}_{p_2}(K_2) = \hat{N} - \hat{E}_{p_1}(K_1). \tag{4.1}$$

We define the sequence

$$K_1 = 0, 1, \ldots, m. \tag{4.2}$$

From (4.2) we take the average value formula

$$T_1 \sim \frac{[\pi_1(\omega m)]^2}{m}, \tag{4.3}$$

where T_1 denotes the number of solutions in (4.1), $\hat{E}_{p_1}(K_1)$ and $\hat{E}_{p_2}(K_2)$ are two isoprimes, $\pi_1(\omega m)$ the number of primes less than m in $\hat{E}_{p_1}(K_1)$. From (2.5) we have

$$\pi_1(\omega m) \sim \frac{\pi(\omega m)}{\phi(\omega)}, \tag{4.4}$$

where $\pi(N)$ denotes the number of primes less than N. Let $N = \omega m$ and $\pi(N) \sim N/\log N$. From (4.3) and (4.4) we have

$$T_1 = \frac{\omega N}{\phi^2(\omega)\log^2 N}(1 + O(1)). \tag{4.5}$$

From (4.5) we have the asymptotic formula

$$\pi_2(N, 2) = \sum_{\hat{p}_2 = \hat{N} - \hat{p}_1} 1 = J_2(\omega) T_1$$

$$= 2 \prod_{2 < p \le p_i} \left(1 - \frac{1}{(p-1)^2}\right) \prod_{p | N, 2 < p \le p_i} \frac{p-1}{p-2} \frac{N}{\log^2 N}(1 + O(1)), \tag{4.6}$$

where $J_2(\omega)$ denotes the number of the subequations.

$$J_2(\omega) = \sum_{\alpha=1}^{\phi(\omega)} \left[\frac{1}{(\omega, N - p_\alpha)}\right]$$

$$= \prod_{2<p\le p_i} \left(\frac{(p-1)^2-(-1)^2}{p} \right) \prod_{p|N, 2<p\le p_i} \left(1 + \frac{(-1)^2 p}{(p-1)^2-(-1)^2} \right). \quad (4.7)$$

4.2. $\hat{p}_3 = \hat{N} - \hat{p}_1 - \hat{p}_2$.
From (3.2) we have

$$\hat{E}_{p_3}(K) = \hat{N} - \hat{E}_{p_1}(K_1) - \hat{E}_{p_1}(K_2), \; p_1 \neq p_2. \quad (4.8)$$

From (4.2) we take the average value formula

$$T_1 \sim \frac{[\pi_1(\omega m)]^3}{m}, \quad (4.9)$$

where T_1 denotes the number of solutions in (4.8), $\hat{E}_{p_1}(K_1)$, $\hat{E}_{p_2}(K_2)$, and $\hat{E}_{p_3}(K)$ are three isoprimes. In the same manner as in §4.1, from (4.9) we have the asymptotic formula

$$T_1 = \frac{\omega}{\phi^3(\omega)} \frac{N^2}{\log^3 N}(1+O(1)). \quad (4.10)$$

In (4.8) there are two subequations: $\hat{E}_{p_3}(K_3) = \hat{N} - \hat{E}_{p_1}(K_1) - \hat{E}_{p_2}(K_2)$, and $\hat{E}_{p_3}(K_4) = \hat{N} - \hat{E}_{p_2}(K_1) - \hat{E}_{p_1}(K_2)$. From (4.10) we have the asymptotic formula of every subequation

$$T_2 = \frac{1}{2} \frac{\omega}{\phi^3(\omega)} \frac{N^2}{\log^3 N}(1+O(1)). \quad (4.11)$$

From (4.11) we have

$$\pi_2(N,3) = \sum_{\hat{p}_3 = \hat{N} - \hat{p}_1 - \hat{p}_2} 1 = J_3(\omega) T_2 = \frac{J_3(\omega)\omega}{2\phi^3(\omega)} \frac{N^2}{\log^3 N}(1+O(1)). \quad (4.12)$$

where $J_3(\omega)$ denotes the number of the subequations

$$J_3(\omega) = \sum_{\alpha=1}^{\phi(\omega)} J_2(\omega, N - p_\alpha)$$

$$= \prod_{2<p\le p_i} \left(\frac{(p-1)^3-(-1)^3}{p} \right) \prod_{p|N, 2<p\le p_i} \left(1 + \frac{(-1)^3 p}{(p-1)^3-(-1)^3} \right). \quad (4.13)$$

Substituting (4.13) into (4.12) we have

$$\pi_2(N,3) = \prod_{2<p\le p_i} \left(1 + \frac{1}{(p-1)^3} \right) \prod_{p|N, 2<p\le p_i} \left(1 - \frac{1}{p^2-3p+3} \right) \frac{N^2}{\log^3 N}(1+O(1)). \quad (4.14)$$

In 1937, I. M. Vinogradov [6] obtained (4.14). Eq. (4.14) is the best asymptotic formula.

4.3. $\hat{p}_4 = \hat{N} - \hat{p}_1 - \hat{p}_2 - \hat{p}_3$.

From (3.2) we have

$$\hat{E}_{p_4}(K) = \hat{N} - \hat{E}_{p_1}(K_1) - \hat{E}_{p_2}(K_2) - \hat{E}_{p_3}(K_3), \tag{4.15}$$

where p_1, p_2 and p_3 are three unequalable numbers. In the same manner as in §4.1, we have the asymptotic formula of (4.15)

$$T_1 = \frac{\omega}{\phi^4(\omega)} \frac{N^3}{\log^4 N}(1 + O(1)), \tag{4.16}$$

where T_1 denotes the number of solutions of (4.15); $\hat{E}_{p_1}(K_1)$, $\hat{E}_{p_2}(K_2)$, $\hat{E}_{p_3}(K_3)$ and $\hat{E}_{p_4}(K)$ are isoprimes. In (4.15) there are six subequations:

$$\hat{E}_{p_4}(K_4) = \hat{N} - \hat{E}_{p_1}(K_1) - \hat{E}_{p_2}(K_2) - \hat{E}_{p_3}(K_3),$$
$$\hat{E}_{p_4}(K_5) = \hat{N} - \hat{E}_{p_1}(K_1) - \hat{E}_{p_3}(K_2) - \hat{E}_{p_2}(K_3),$$
$$\hat{E}_{p_4}(K_6) = \hat{N} - \hat{E}_{p_2}(K_1) - \hat{E}_{p_1}(K_2) - \hat{E}_{p_3}(K_3),$$
$$\hat{E}_{p_4}(K_7) = \hat{N} - \hat{E}_{p_2}(K_1) - \hat{E}_{p_3}(K_2) - \hat{E}_{p_1}(K_3),$$
$$\hat{E}_{p_4}(K_8) = \hat{N} - \hat{E}_{p_3}(K_1) - \hat{E}_{p_1}(K_2) - \hat{E}_{p_2}(K_3),$$
$$\hat{E}_{p_4}(K_9) = \hat{N} - \hat{E}_{p_3}(K_1) - \hat{E}_{p_2}(K_2) - \hat{E}_{p_1}(K_3).$$

From (4.16) we have the asymptotic formula of every subequation

$$T_2 = \frac{\omega}{6\phi^4(\omega)} \frac{N^3}{\log^4 N}(1 + O(1)). \tag{4.17}$$

From (4.17) we have

$$\pi_2(N, 4) = \sum_{\hat{p}_4 = \hat{N} - \hat{p}_1 - \hat{p}_2 - \hat{p}_3} 1 = J_4(\omega) T_2 = J_4(\omega) T_2 = \frac{J_4(\omega)\omega}{6\phi^4(\omega)} \frac{N^3}{\log^4 N}(1 + O(1)), \tag{4.18}$$

where $J_4(\omega)$ denotes the number of the subequations. From (4.7) and (4.13) we have

$$J_4(\omega) = \sum_{\alpha=1}^{\phi(\omega)} J_3(\omega, N - p_\alpha)$$

$$= \prod_{2<p\le p_i} \left(\frac{(p-1)^4 - (-1)^4}{p} \right) \prod_{p|N, 2<p\le p_i} \left(1 + \frac{(-1)^4 p}{(p-1)^4 - (-1)^4} \right). \tag{4.19}$$

4.4. $\hat{p}_n = \hat{N} - \sum_{i=1}^{n-1} \hat{p}_i$

From (3.2) we have

$$\hat{E}_{p_n}(K) = \hat{N} - \sum_{\alpha=1}^{n-1} \hat{E}_{p_\alpha}(K_\alpha), \tag{4.20}$$

where p_α are $(n-1)$ unequalable numbers. In the same manner as in §4.1 we have the asymptotic formula of (4.20)

$$T_1 = \frac{\omega}{\phi(\omega)} \frac{N^{n-1}}{\log^N}(1 + O(1)), \tag{4.21}$$

where T_1 denotes the number of solutions of (4.20); $\hat{E}_{p_n}(K)$ and $\hat{E}_{p_\alpha}(K_\alpha)$ are n isoprimes. In (4.20) there are $(n-1)!$ subequations. From (4.21) we have the asymptotic formula of every subequation

$$T_2 = \frac{\omega}{(n-1)!\phi^n(\omega)} \frac{N^{n-1}}{\log^N}(1 + O(1)). \tag{4.22}$$

From (4.22) we have

$$\pi_2(N, n) = \sum_{\hat{p}_n = \hat{N} - \sum_{i=1}^{n-1} \hat{P}_i} 1 = J_n(\omega) T_2 = \frac{J_n(\omega)\omega}{(n-1)!\phi^n(\omega)} \frac{N^{n-1}}{\log^n N}(1 + O(1)), \tag{4.23}$$

where $J_n(\omega)$ denotes the number of the subequations. From (4.7), (4.13) and (4.19) we have

$$J_n(\omega) = \sum_{\alpha=1}^{\phi(\omega)} J_{n-1}(\omega, N - p_\alpha)$$

$$= \prod_{2<p\le p_i} \left(\frac{(p-1)^n - (-1)^n}{p}\right) \prod_{p|N, 2<p\le p_i} \left(1 + \frac{(-1)^n p}{(p-1)^n - (-1)^n}\right). \tag{4.24}$$

In the same way it is possible to derive all asymptotic formulas in the additive isoprime number theory. They have the same form but differ in $J_n(\omega)$.

5. Determinations of the number of the subequations

Let $\omega = 6$. From (3.2) we have

$$\hat{E}_5(k) = (6K + 5)\hat{I}, \ \hat{E}_7(K) = (6K + 7)\hat{I}, \tag{5.1}$$

where $K = 0, 1, 2, \ldots$

5.1. $\hat{N} = \hat{p}_1 + \hat{p}_2$.

Suppose that

$$\hat{N} = (6m + h)\hat{I}, \tag{5.2}$$

where $h = 10, 12, 14;\ m = 0, 1, 2, \ldots$

If $\hat{3}\hat{|}\hat{N}$, from (4.7) we have $J_2(6) = 2$.

From (5.1) and (5.2) we have

$$\hat{N} = (6m + 12)\hat{I} = \hat{E}_5(K_1) + \hat{E}_7(K_2) \text{ (2 subequations).} \tag{5.3}$$

If $\hat{3}\ \hat{\not|}\hat{N}$, from (4.7) we have $J_2(6) = 1$. From (5.1) and (5.2) we have

$$\hat{N} = (6m + 10)\hat{I} = \hat{E}_5(K_1) + \hat{E}_5(K_2) \text{ (1 subequations),} \tag{5.4}$$

$$\hat{N} = (6m + 14)\hat{I} = \hat{E}_7(K_1) + \hat{E}_7(K_2) \text{ (1 subequations).} \tag{5.5}$$

5.2. $\hat{N} = \hat{p}_1 + \hat{p}_2 + \hat{p}_3$.

Suppose that

$$\hat{N} = (6m + h)\hat{I}, \tag{5.6}$$

where $h = 17, 19, 21;\ m = 0, 1, 2, \ldots$

If $\hat{3}\hat{|}\hat{N}$, from (4.13) we have $J_3(6) = 2$.

From (5.1) and (5.6) we have

$$\hat{N} = (6m + 21)\hat{I} = \hat{E}_7(K_1) + \hat{E}_7(K_2) + \hat{E}_7(K_3) \text{ (1 subequations),}$$

$$\hat{N} = [6(m - 1) + 21)]\hat{I} = \hat{E}_5(K_1) + \hat{E}_5(K_2) + \hat{E}_5(K_3) \text{ (1 subequations).} \quad (5.7),$$

If $\hat{3}\ \hat{\not|}\hat{N}$, from (4.13) we have $J_3(6) = 3$. From (5.1) and (5.6) we have

$$\hat{N} = (6m + 17)\hat{I} = \hat{E}_5(K_1) + \hat{E}_5(K_2) + \hat{E}_7(K_3) \text{ (3 subequations);} \tag{5.8}$$

$$\hat{N} = (6m + 19)\hat{I} = \hat{E}_5(K_1) + \hat{E}_7(K_2) + \hat{E}_7(K_3) \text{ (3 subequations).} \tag{5.9}$$

5.3. $\hat{N} = \hat{p}_1 + \hat{p}_2 + \hat{p}_3 + \hat{p}_4$.

Suppose that

$$\hat{N} = (6m + h)\hat{I} \tag{5.10}$$

where $h = 24, 26, 28;\ m = 0, 1, 2, \ldots$

If $\hat{3}\hat{|}\hat{N}$, from (4.19) we have $J_4(6) = 6$.

From (5.1) and (5.10) we have

$$\hat{N} = (6m + 24)\hat{I} = \hat{E}_5(K_1) + \hat{E}_5(K_2) + \hat{E}_7(K_3) + \hat{E}_7(K_4) \text{(6 subequations).} \tag{5.11}$$

If $\hat{3}\ \hat{\not|}\hat{N}$, from (4.19) we have $J_4(6) = 5$. From (5.1) and (5.10) we have

$$\hat{N} = (6m + 26)\hat{I} = \hat{E}_5(K_1) + \hat{E}_7(K_2) + \hat{E}_7(K_3) + \hat{E}_7(K_4) \text{(4 subequations),}$$

$$\hat{N} = [6(m-1)+26]\hat{I} = \hat{E}_5(K_1)+\hat{E}_5(K_2)+\hat{E}_5(K_3)+\hat{E}_5(K_4)(1 \text{ subequations}); \quad (5.12)$$

$$\hat{N} = (6m+28)\hat{I} = \hat{E}_7(K_1)+\hat{E}_7(K_2)+\hat{E}_7(K_3)+\hat{E}_7(K_4)(1 \text{ subequations}),$$

$$\hat{N} = [6(m-1)+28]\hat{I} = \hat{E}_5(K_1)+\hat{E}_5(K_2)+\hat{E}_5(K_3)+\hat{E}_7(K_4)(4 \text{ subequations}). \quad (5.13)$$

5.4. $\hat{N} = \hat{p}_1 + \hat{p}_2 + \hat{p}_3 + \hat{p}_4 + \hat{p}_5$.

Suppose that

$$\hat{N} = (6m+h)\hat{I}, \tag{5.14}$$

where $h = 31, 33, 35;\ m = 0, 1, \ldots$

If $\hat{3}\hat{|}\hat{N}$, from (4.24) we have $J_5(6) = 10$.

From (5.1) and (5.14) we have

$$\hat{N} = (6m+33)\hat{I} = \hat{E}_5(K_1)+\hat{E}_7(K_2)+\hat{E}_7(K_3)+\hat{E}_7(K_4)+\hat{E}_7(K_5)$$

(5 subequations),

$$\hat{N} = [6(m-1)+33]\hat{I} = \hat{E}_5(K_1)+\hat{E}_5(K_2)+\hat{E}_5(K_3)+\hat{E}_5(K_4)+\hat{E}_7(K_5)$$

(5 subequations). (5.15)

If $\hat{3}\ \hat{\not|}\hat{N}$, from (4.24) we have $J_5(6) = 11$. From (5.1) and (5.14) we have

$$\hat{N} = (6m+31)\hat{I} = \hat{E}_5(K_1)+\hat{E}_5(K_2)+\hat{E}_7(K_3)+\hat{E}_7(K_4)+\hat{E}_7(K_5)$$

(10 subequations),

$$\hat{N} = [6(m-1)+31]\hat{I} = \hat{E}_5(K_1)+\hat{E}_5(K_2)+\hat{E}_5(K_3)+\hat{E}_5(K_4)+\hat{E}_5(K_5)$$

(1 subequations); (5.16)

$$\hat{N} = (6m+35)\hat{I} = \hat{E}_7(K_1)+\hat{E}_7(K_2)+\hat{E}_7(K_3)+\hat{E}_7(K_4)+\hat{E}_7(K_5)$$

(1 subequations),

$$\hat{N} = [6(m-1)+35]\hat{I} = \hat{E}_5(K_1)+\hat{E}_5(K_2)+\hat{E}_5(K_3)+\hat{E}_7(K_4)+\hat{E}_7(K_5)$$

(10 subequations). (5.17)

5.5. Let $\omega = 30$. From (3.2) we have

$$\hat{E}_{p_\alpha}(K) = (30K + p_\alpha)\hat{I}, \tag{5.18}$$

where $K = 0, 1, 2, \ldots;\ p_\alpha = 7, 11, 13, 17, 19, 23, 29, 31$.

5.5.1. $\hat{N} = \hat{p}_1 + \hat{p}_2$.

Suppose that

$$\hat{N} = (30m + h)\hat{I}, \tag{5.19}$$

where $m = 0, 1, 2, \ldots; h = 18, 20, 22, 24, 26, 28, 30, 32, 34, 36, 38, 40, 42, 44, 46$. From (5.18) and (5.19) we have

$$\hat{N} = (30m + h)\hat{I} = \hat{E}_{p_1}(K_1) + \hat{E}_{p_2}(K_2) = [30(K_1 + K_2) + p_1 + p_2]\hat{I}. \tag{5.20}$$

where $\hat{E}_{p_1}(K)$ and $\hat{E}_{p_2}(K)$ are two isoprimes. From (5.20) we have

$$m = K_1 + K_2,\ \ h = p_1 + p_2 \tag{5.21}$$

$m = K_1 + K_2$ is called Yu's mathematical problems, that is integers m greater than 1 is the sum of K_1 and K_2. To prove $m = K_1 + K_2$ is transformed into studying $\hat{N} = \hat{E}_{p_1}(K_1) + \hat{E}_{p_2}(K_2)$.

If $\hat{3}\hat{|}\hat{N}$, from (4.7) we have $J_2(30) = 6$. From (5.18) and (5.20) we have six subequations: $6 = 2 + 2 + 2$.

$$\hat{N} = (30m + 18)\hat{I} = \hat{E}_7(K_1) + \hat{E}_{11}(K_2)$$

$$= \hat{E}_{17}(K_1) + \hat{E}_{31}(K_2) = \hat{E}_{19}(K_1) + \hat{E}_{29}(K_2), \tag{5.22}$$

$$\hat{N} = (30m + 24)\hat{I} = \hat{E}_7(K_1) + \hat{E}_{17}(K_2)$$

$$= \hat{E}_{11}(K_1) + \hat{E}_{13}(K_2) = \hat{E}_{23}(K_1) + \hat{E}_{31}(K_2), \tag{5.23}$$

$$\hat{N} = (30m + 36)\hat{I} = \hat{E}_7(K_1) + \hat{E}_{29}(K_2)$$

$$= \hat{E}_{13}(K_1) + \hat{E}_{23}(K_2) = \hat{E}_{17}(K_1) + \hat{E}_{19}(K_2), \tag{5.24}$$

$$\hat{N} = (30m + 42)\hat{I} = \hat{E}_{11}(K_1) + \hat{E}_{31}(K_2)$$

$$= \hat{E}_{13}(K_1) + \hat{E}_{29}(K_2) = \hat{E}_{19}(K_1) + \hat{E}_{23}(K_2). \tag{5.25}$$

If $\hat{5}\hat{|}\hat{N}$, from (4.7) we have $J_2(30) = 4$. From (5.18) and (5.20) we have four subequations: $4 = 2 + 2$,

$$\hat{N} = (30m + 20)\hat{I} = \hat{E}_7(K_1) + \hat{E}_{13}(K_2) = \hat{E}_{19}(K_1) + \hat{E}_{31}(K_2), \tag{5.26}$$

$$\hat{N} = (30m + 40)\hat{I} = \hat{E}_{11}(K_1) + \hat{E}_{29}(K_2) = \hat{E}_{17}(K_1) + \hat{E}_{23}(K_2). \tag{5.27}$$

If $\hat{3},\ \hat{5}\ \hat{\not|}\hat{N}$, from (4.7) we have $J_2(30) = 3$. From (5.18) and (5.20) we have three subequations: $3 = 2 + 1$,

$$\hat{N} = (30m + 22)\hat{I} = \hat{E}_{23}(K_1) + \hat{E}_{29}(K_2) = \hat{E}_{11}(K_1) + \hat{E}_{11}(K_2), \tag{5.28}$$

$$\hat{N} = (30m + 26)\hat{I} = \hat{E}_7(K_1) + \hat{E}_{19}(K_2) = \hat{E}_{13}(K_1) + \hat{E}_{13}(K_2), \tag{5.29}$$

$$\hat{N} = (30m + 28)\hat{I} = \hat{E}_{11}(K_1) + \hat{E}_{17}(K_2) = \hat{E}_{29}(K_1) + \hat{E}_{29}(K_2), \tag{5.30}$$

$$\hat{N} = (30m + 32)\hat{I} = \hat{E}_{13}(K_1) + \hat{E}_{19}(K_2) = \hat{E}_{31}(K_1) + \hat{E}_{31}(K_2), \quad (5.31)$$

$$\hat{N} = (30m + 34)\hat{I} = \hat{E}_{11}(K_1) + \hat{E}_{23}(K_2) = \hat{E}_{17}(K_1) + \hat{E}_{17}(K_2), \quad (5.32)$$

$$\hat{N} = (30m + 38)\hat{I} = \hat{E}_{7}(K_1) + \hat{E}_{31}(K_2) = \hat{E}_{19}(K_1) + \hat{E}_{19}(K_2), \quad (5.33)$$

$$\hat{N} = (30m + 44)\hat{I} = \hat{E}_{13}(K_1) + \hat{E}_{31}(K_2) = \hat{E}_{7}(K_1) + \hat{E}_{7}(K_2), \quad (5.34)$$

$$\hat{N} = (30m + 46)\hat{I} = \hat{E}_{17}(K_1) + \hat{E}_{29}(K_2) = \hat{E}_{23}(K_1) + \hat{E}_{23}(K_2). \quad (5.35)$$

If $\hat{3}$, $\hat{5}\hat{|}\hat{N}$, from (4.7) we have $J_2(30) = 8$. From (5.18) and (5.20) we have eight subequations: $8 = 2 + 2 + 2 + 2$.

$$\hat{N} = (30m + 30)\hat{I} = \hat{E}_{7}(K_1) + \hat{E}_{23}(K_2) = \hat{E}_{11}(K_1) + \hat{E}_{19}(K_2)$$
$$= \hat{E}_{13}(K_1) + \hat{E}_{17}(K_2) = \hat{E}_{29}(K_1) + \hat{E}_{31}K_2. \quad (5.36)$$

For every subequation we have the arithmetic function

$$J_2(\omega > 30) = \prod_{7 \le p \le p_i} (p - 2) \prod_{p|N, 7 \le p \le p_i} \frac{p-1}{p-2} \ne 0, \ p_1 \ne p_2. \quad (5.37)$$

Since $J_2(\omega > 30) \ne 0$, we prove Yu's mathematical problems. From (3.16) we have the best asymptotic formula of solutions for Yu's mathematical problems [3]

$$\pi_2(N, 2) = \sum_{m = K_1 + K_2} 1 = \sum_{\hat{N} = \hat{E}_{p_1}(K_1) + \hat{E}_{p_2}(K_2)} 1$$
$$= \frac{15}{32} \prod_{7 \le p \le p_i} (1 - \frac{1}{(p-1)^2}) \prod_{p|N} \frac{p-1}{p-2} \frac{N}{\log^2 N} (1 + O(1)). \quad (5.38)$$

5.5.2. $\hat{N} = \hat{p}_1 + \hat{p}_2 + \hat{p}_3$.

Suppose that

$$\hat{N} = (30m + h)\hat{I}, \quad (5.39)$$

where $m = 0, 1, 2, \ldots h = 25, 27, 29, 31, 33, 35, 37, 39, 41, 43, 45, 47, 49, 51, 53$. If $\hat{3}\hat{|}\hat{N}$, from (4.13) we have $J_3(30) = 26$. From (5.18) and (5.39) we have twenty-six subequations: $26 = 3 + 3 + 3 + 3 + 6 + 6 + 1 + 1$,

$$\begin{aligned}
\hat{N} &= (30m + 27)\hat{I} \\
&= \hat{E}_{7}(K_1) + \hat{E}_{7}(K_2) + \hat{E}_{13}(K_3) = \hat{E}_{13}(K_1) + \hat{E}_{13}(K_2) + \hat{E}_{31}(K_3) \\
&= \hat{E}_{17}(K_1) + \hat{E}_{17}(K_2) + \hat{E}_{23}(K_3) = \hat{E}_{23}(K_1) + \hat{E}_{23}(K_2) + \hat{E}_{11}(K_3) \\
&= \hat{E}_{7}(K_1) + \hat{E}_{19}(K_2) + \hat{E}_{31}(K_3) = \hat{E}_{11}(K_1) + \hat{E}_{17}(K_2) + \hat{E}_{29}(K_3) \\
&= \hat{E}_{19}(K_1) + \hat{E}_{19}(K_2) + \hat{E}_{19}(K_3) = \hat{E}_{29}(K_1) + \hat{E}_{29}(K_2) + \hat{E}_{29}(K_3), \quad (5.40)
\end{aligned}$$

$$\begin{aligned}\hat{N} &= (30m+33)\hat{I}\\ &= \hat{E}_7(K_1)+\hat{E}_7(K_2)+\hat{E}_{19}(K_3)=\hat{E}_{13}(K_1)+\hat{E}_{13}(K_2)+\hat{E}_7(K_3)\\ &= \hat{E}_{17}(K_1)+\hat{E}_{17}(K_2)+\hat{E}_{29}(K_3)=\hat{E}_{23}(K_1)+\hat{E}_{23}(K_2)+\hat{E}_{17}(K_3)\\ &= \hat{E}_{11}(K_1)+\hat{E}_{23}(K_2)+\hat{E}_{29}(K_3)=\hat{E}_{13}(K_1)+\hat{E}_{19}(K_2)+\hat{E}_{31}(K_3)\\ &= \hat{E}_{11}(K_1)+\hat{E}_{11}(K_2)+\hat{E}_{11}(K_3)=\hat{E}_{31}(K_1)+\hat{E}_{31}(K_2)+\hat{E}_{31}(K_3),\end{aligned} \tag{5.41}$$

$$\begin{aligned}\hat{N} &= (30m+39)\hat{I}\\ &= \hat{E}_{11}(K_1)+\hat{E}_{11}(K_2)+\hat{E}_{17}(K_3)=\hat{E}_{19}(K_1)+\hat{E}_{19}(K_2)+\hat{E}_{31}(K_3)\\ &= \hat{E}_{29}(K_1)+\hat{E}_{29}(K_2)+\hat{E}_{11}(K_3)=\hat{E}_{31}(K_1)+\hat{E}_{31}(K_2)+\hat{E}_7(K_3)\\ &= \hat{E}_7(K_1)+\hat{E}_{13}(K_2)+\hat{E}_{19}(K_3)=\hat{E}_{17}(K_1)+\hat{E}_{23}(K_2)+\hat{E}_{29}(K_3)\\ &= \hat{E}_{13}(K_1)+\hat{E}_{13}(K_2)+\hat{E}_{13}(K_3)=\hat{E}_{23}(K_1)+\hat{E}_{23}(K_2)+\hat{E}_{23}(K_3),\end{aligned} \tag{5.42}$$

$$\begin{aligned}\hat{N} &= (30m+51)\hat{I}\\ &= \hat{E}_{11}(K_1)+\hat{E}_{11}(K_2)+\hat{E}_{29}(K_3)=\hat{E}_{19}(K_1)+\hat{E}_{19}(K_2)+\hat{E}_{13}(K_3)\\ &= \hat{E}_{29}(K_1)+\hat{E}_{29}(K_2)+\hat{E}_{23}(K_3)=\hat{E}_{31}(K_1)+\hat{E}_{31}(K_2)+\hat{E}_{19}(K_3)\\ &= \hat{E}_7(K_1)+\hat{E}_{13}(K_2)+\hat{E}_{31}(K_3)=\hat{E}_{11}(K_1)+\hat{E}_{17}(K_2)+\hat{E}_{23}(K_3)\\ &= \hat{E}_7(K_1)+\hat{E}_7(K_2)+\hat{E}_7(K_3)=\hat{E}_{17}(K_1)+\hat{E}_{17}(K_2)+\hat{E}_{17}(K_3).\end{aligned} \tag{5.43}$$

If $\hat{5}\hat{|}\hat{N}$, from (4.13) we have $J_3(30)=36$. From (5.18) and (5.28) we have thirty-six subequations: $36=3+3+3+3+6+6+6+6$,

$$\begin{aligned}\hat{N} &= (30m+25)\hat{I}\\ &= \hat{E}_7(K_1)+\hat{E}_7(K_2)+\hat{E}_{11}(K_3)=\hat{E}_{13}(K_1)+\hat{E}_{13}(K_2)+\hat{E}_{29}(K_3)\\ &= \hat{E}_{19}(K_1)+\hat{E}_{19}(K_2)+\hat{E}_{17}(K_3)=\hat{E}_{31}(K_1)+\hat{E}_{31}(K_2)+\hat{E}_{23}(K_3)\\ &= \hat{E}_7(K_1)+\hat{E}_{17}(K_2)+\hat{E}_{31}(K_3)=\hat{E}_7(K_1)+\hat{E}_{19}(K_2)+\hat{E}_{29}(K_3)\\ &= \hat{E}_{11}(K_1)+\hat{E}_{13}(K_2)+\hat{E}_{31}(K_3)=\hat{E}_{13}(K_1)+\hat{E}_{19}(K_2)+\hat{E}_{23}(K_3),\end{aligned} \tag{5.44}$$

$$\begin{aligned}\hat{N} &= (30m+35)\hat{I}\\ &= \hat{E}_{11}(K_1)+\hat{E}_{11}(K_2)+\hat{E}_{13}(K_3)=\hat{E}_{17}(K_1)+\hat{E}_{17}(K_2)+\hat{E}_{31}(K_3)\\ &= \hat{E}_{23}(K_1)+\hat{E}_{23}(K_2)+\hat{E}_{19}(K_3)=\hat{E}_{29}(K_1)+\hat{E}_{29}(K_2)+\hat{E}_7(K_3)\\ &= \hat{E}_7(K_1)+\hat{E}_{11}(K_2)+\hat{E}_{17}(K_3)=\hat{E}_{11}(K_1)+\hat{E}_{23}(K_2)+\hat{E}_{31}(K_3)\\ &= \hat{E}_{11}(K_1)+\hat{E}_{23}(K_2)+\hat{E}_{29}(K_3)=\hat{E}_{13}(K_1)+\hat{E}_{19}(K_2)+\hat{E}_{29}(K_3).\end{aligned} \tag{5.45}$$

If $\hat{3}$, $\hat{5}\ \hat{\not|}\ \hat{N}$, from (4.13) we have $J_3(30)=39$. From (5.18) and (5.28) we have thirty-nine subequations: $39=3+3+3+6+6+6+6+6$,

$$\begin{aligned}\hat{N} &= (30m+29)\hat{I}\\ &= \hat{E}_{11}(K_1)+\hat{E}_{11}(K_2)+\hat{E}_7(K_3)=\hat{E}_{23}(K_1)+\hat{E}_{23}(K_2)+\hat{E}_{13}(K_3)\\ &= \hat{E}_{29}(K_1)+\hat{E}_{29}(K_2)+\hat{E}_{31}(K_3)=\hat{E}_7(K_1)+\hat{E}_{23}(K_2)+\hat{E}_{29}(K_3)\\ &= \hat{E}_{11}(K_1)+\hat{E}_{17}(K_2)+\hat{E}_{31}(K_3)=\hat{E}_{11}(K_1)+\hat{E}_{19}(K_2)+\hat{E}_{29}(K_3)\\ &= \hat{E}_{13}(K_1)+\hat{E}_{17}(K_2)+\hat{E}_{29}(K_3)=\hat{E}_{17}(K_1)+\hat{E}_{19}(K_2)+\hat{E}_{23}(K_3),\end{aligned} \tag{5.46}$$

$$\begin{aligned}
\hat{N} &= (30m+31)\hat{I} \\
&= \hat{E}_7(K_1)+\hat{E}_7(K_2)+\hat{E}_{17}(K_3)=\hat{E}_{19}(K_1)+\hat{E}_{19}(K_2)+\hat{E}_{23}(K_3) \\
&= \hat{E}_{31}(K_1)+\hat{E}_{31}(K_2)+\hat{E}_{29}(K_3)=\hat{E}_7(K_1)+\hat{E}_{11}(K_2)+\hat{E}_{13}(K_3) \\
&= \hat{E}_7(K_1)+\hat{E}_{23}(K_2)+\hat{E}_{31}(K_3)=\hat{E}_{11}(K_1)+\hat{E}_{19}(K_2)+\hat{E}_{31}(K_3) \\
&= \hat{E}_{13}(K_1)+\hat{E}_{17}(K_2)+\hat{E}_{31}(K_3)=\hat{E}_{13}(K_1)+\hat{E}_{19}(K_2)+\hat{E}_{29}(K_3), \quad (5.47)
\end{aligned}$$

$$\begin{aligned}
\hat{N} &= (30m+37)\hat{I} \\
&= \hat{E}_7(K_1)+\hat{E}_7(K_2)+\hat{E}_{23}(K_3)=\hat{E}_{13}(K_1)+\hat{E}_{13}(K_2)+\hat{E}_{11}(K_3) \\
&= \hat{E}_{19}(K_1)+\hat{E}_{19}(K_2)+\hat{E}_{29}(K_3)=\hat{E}_7(K_1)+\hat{E}_{11}(K_2)+\hat{E}_{19}(K_3) \\
&= \hat{E}_7(K_1)+\hat{E}_{13}(K_2)+\hat{E}_{17}(K_3)=\hat{E}_7(K_1)+\hat{E}_{29}(K_2)+\hat{E}_{31}(K_3) \\
&= \hat{E}_{13}(K_1)+\hat{E}_{23}(K_2)+\hat{E}_{31}(K_3)=\hat{E}_{17}(K_1)+\hat{E}_{19}(K_2)+\hat{E}_{31}(K_3), \quad (5.48)
\end{aligned}$$

$$\begin{aligned}
\hat{N} &= (30m+41)\hat{I} \\
&= \hat{E}_{11}(K_1)+\hat{E}_{11}(K_2)+\hat{E}_{19}(K_3)=\hat{E}_{17}(K_1)+\hat{E}_{17}(K_2)+\hat{E}_7(K_3) \\
&= \hat{E}_{29}(K_1)+\hat{E}_{29}(K_2)+\hat{E}_{13}(K_3)=\hat{E}_7(K_1)+\hat{E}_{11}(K_2)+\hat{E}_{23}(K_3) \\
&= \hat{E}_{11}(K_1)+\hat{E}_{13}(K_2)+\hat{E}_{17}(K_3)=\hat{E}_{11}(K_1)+\hat{E}_{29}(K_2)+\hat{E}_{31}(K_3) \\
&= \hat{E}_{17}(K_1)+\hat{E}_{23}(K_2)+\hat{E}_{31}(K_3)=\hat{E}_{19}(K_1)+\hat{E}_{23}(K_2)+\hat{E}_{29}(K_3), \quad (5.49)
\end{aligned}$$

$$\begin{aligned}
\hat{N} &= (30m+43)\hat{I} \\
&= \hat{E}_7(K_1)+\hat{E}_7(K_2)+\hat{E}_{29}(K_3)=\hat{E}_{13}(K_1)+\hat{E}_{13}(K_2)+\hat{E}_{17}(K_3) \\
&= \hat{E}_{31}(K_1)+\hat{E}_{31}(K_2)+\hat{E}_{11}(K_3)=\hat{E}_7(K_1)+\hat{E}_{13}(K_2)+\hat{E}_{23}(K_3) \\
&= \hat{E}_7(K_1)+\hat{E}_{17}(K_2)+\hat{E}_{19}(K_3)=\hat{E}_{11}(K_1)+\hat{E}_{13}(K_2)+\hat{E}_{19}(K_3) \\
&= \hat{E}_{13}(K_1)+\hat{E}_{29}(K_2)+\hat{E}_{31}(K_3)=\hat{E}_{19}(K_1)+\hat{E}_{23}(K_2)+\hat{E}_{31}(K_3), \quad (5.50)
\end{aligned}$$

$$\begin{aligned}
\hat{N} &= (30m+47)\hat{I} \\
&= \hat{E}_{17}(K_1)+\hat{E}_{17}(K_2)+\hat{E}_{13}(K_3)=\hat{E}_{23}(K_1)+\hat{E}_{23}(K_2)+\hat{E}_{31}(K_3) \\
&= \hat{E}_{29}(K_1)+\hat{E}_{29}(K_2)+\hat{E}_{19}(K_3)=\hat{E}_7(K_1)+\hat{E}_{11}(K_2)+\hat{E}_{29}(K_3) \\
&= \hat{E}_7(K_1)+\hat{E}_{17}(K_2)+\hat{E}_{23}(K_3)=\hat{E}_{11}(K_1)+\hat{E}_{13}(K_2)+\hat{E}_{23}(K_3) \\
&= \hat{E}_{11}(K_1)+\hat{E}_{17}(K_2)+\hat{E}_{19}(K_3)=\hat{E}_{17}(K_1)+\hat{E}_{29}(K_2)+\hat{E}_{31}(K_3), \quad (5.51)
\end{aligned}$$

$$\begin{aligned}
\hat{N} &= (30m+49)\hat{I} \\
&= \hat{E}_{13}(K_1)+\hat{E}_{13}(K_2)+\hat{E}_{23}(K_3)=\hat{E}_{19}(K_1)+\hat{E}_{19}(K_2)+\hat{E}_{11}(K_3) \\
&= \hat{E}_{31}(K_1)+\hat{E}_{31}(K_2)+\hat{E}_{17}(K_3)=\hat{E}_7(K_1)+\hat{E}_{11}(K_2)+\hat{E}_{31}(K_3) \\
&= \hat{E}_7(K_1)+\hat{E}_{13}(K_2)+\hat{E}_{29}(K_3)=\hat{E}_7(K_1)+\hat{E}_{19}(K_2)+\hat{E}_{23}(K_3) \\
&= \hat{E}_{13}(K_1)+\hat{E}_{17}(K_2)+\hat{E}_{19}(K_3)=\hat{E}_{19}(K_1)+\hat{E}_{29}(K_2)+\hat{E}_{31}(K_3), \quad (5.52)
\end{aligned}$$

$$\begin{aligned}
\hat{N} &= (30m+53)\hat{I} \\
&= \hat{E}_{11}(K_1)+\hat{E}_{11}(K_2)+\hat{E}_{31}(K_3)=\hat{E}_{17}(K_1)+\hat{E}_{17}(K_2)+\hat{E}_{19}(K_3) \\
&= \hat{E}_{23}(K_1)+\hat{E}_{23}(K_2)+\hat{E}_7(K_3)=\hat{E}_7(K_1)+\hat{E}_{17}(K_2)+\hat{E}_{29}(K_3) \\
&= \hat{E}_{11}(K_1)+\hat{E}_{13}(K_2)+\hat{E}_{29}(K_3)=\hat{E}_{11}(K_1)+\hat{E}_{19}(K_2)+\hat{E}_{23}(K_3) \\
&= \hat{E}_{13}(K_1)+\hat{E}_{17}(K_2)+\hat{E}_{23}(K_3)=\hat{E}_{23}(K_1)+\hat{E}_{29}(K_2)+\hat{E}_{31}(K_3). \quad (5.53)
\end{aligned}$$

If $\hat{3},\ \hat{5}|\hat{N}$ from (4.13) we have $J_3(30) = 24$. From (5.18) and (5.39) we have twenty-four subequations: $24 = 3+3+3+3+3+3+3+3$

$$\begin{aligned}\hat{N} &= (30m+45)\hat{I}\\ &= \hat{E_7}(K_1)+\hat{E_7}(K_2)+\hat{E_{31}}(K_3) = \hat{E_{11}}(K_1)+\hat{E_{11}}(K_2)+\hat{E_{23}}(K_3)\\ &= \hat{E_{13}}(K_1)+\hat{E_{13}}(K_2)+\hat{E_{19}}(K_3) = \hat{E_{17}}(K_1)+\hat{E_{17}}(K_2)+\hat{E_{11}}(K_3)\\ &= \hat{E_{19}}(K_1)+\hat{E_{19}}(K_2)+\hat{E_7}(K_3) = \hat{E_{23}}(K_1)+\hat{E_{23}}(K_2)+\hat{E_{29}}(K_3)\\ &= \hat{E_{29}}(K_1)+\hat{E_{29}}(K_2)+\hat{E_{17}}(K_3) = \hat{E_{31}}(K_1)+\hat{E_{31}}(K_2)+\hat{E_{13}}(K_3). \quad (5.54)\end{aligned}$$

The number of the subequations is very exactly determined by $J_n(\omega)$. $J_n(\omega)\to\infty$ as $\omega\to\infty$ is proof of the existence of infinitely many isoprimes similar to Euler's proof. It is a generalization of Euler's proof of the existence of infinitely many primes.

5.5.3. $p_2 = p_1 + 2$.

We have

$$J_2(\omega) = \prod_{3\le p\le p_i}(p-2) \neq 0. \quad (5.55)$$

From (5.55) we have $J_2(30) = 3$. From (5.18) we have three subequations

$$E_{13}(K) = E_{11}(K)+2, E_{19}(K) = E_{17}(K)+2, E_{31} = E_{29}(K)+2. \quad (5.56)$$

Every subequation has infinitely many prime solutions. Since $J_2(\omega)\to\infty$ as $\omega\to\infty$, there exist infinitely many subequations. Each has infinitely many prime solutions.

5.5.4. $p_2 = (p_1+1)^2 + 1$.

We have

$$J_2(\omega) = \prod_{3\le p\le p_i}(p-2-\chi(p)) \neq 0, \quad (5.57)$$

where $\chi(p) = (-1)^{\frac{p-1}{2}}$.

From (5.57) we have $J_2(30) = 4$. From (5.18) we have four subequations

$$E_{17}(K_2) = (E_{13}(K_1)+1)^2+1, \quad E_7(K_2) = (E_{19}(K_1)+1)^2+1,$$

$$E_7(K_2) = (E_{23}(K_1)+1)^2+1, \quad E_{31}(K_2) = (E_{29}(K_1)+1)^2+1. \quad (5.58)$$

Every subequation has infinitely many prime solutions. Since $J_2(\omega)\to\infty$ as $\omega\to\infty$, there exist infinitely many subequations. Each has infinitely many prime solutions.

5.5.5. $p_2 = (p_1+3)^2 + 1$.

We have

$$J_2(\omega) = \prod_{3\le p\le p_i}(p-2-\chi(p)) \neq 0,$$

where $\chi(5) = 0, \chi(p) = (-1)^{\frac{p-1}{2}}$ otherwise.

From (5.58) we have $J_2(30) = 6$. From (5.18) we have six subequations

$$E_{11}(K_2) = (E_7(K_1) + 3)^2 + 1, \quad E_{17}(K_2) = (E_{11}(K_1) + 3)^2 + 1,$$
$$E_{17}(K_2) = (E_{13}(K_1) + 3)^2 + 1, \quad E_{11}(K_2) = (E_{17}(K_1) + 3)^2 + 1,$$
$$E_{17}(K_2) = (E_{23}(K_1) + 3)^2 + 1, \quad E_{17}(K_2) = (E_{31}(K_1) + 3)^2 + 1. \tag{5.59}$$

Every subequation has infinitely many prime solutions. Since $J_2(\omega) \to \infty$ as $\omega \to \infty$, there exist infinitely many subequations. Each has infinitely many prime solutions.

5.5.6. $p_2 = (p_1 + 5)^2 + 1$.

We have
$$J_2(\omega) = \prod_{3 \le p \le p_i} (p - 2 - \chi(p)) \neq 0, \tag{5.60}$$
where $\chi(13) = 0, \chi(p) = (-1)^{\frac{p-1}{2}}$ otherwise.

From (5.60) we have $J_2(30) = 4$. From (5.18) we have four subequations

$$E_{17}(K_2) = (E_{11}(K_1) + 5)^2 + 1, E_7(K_2) = (E_{19}(K_1) + 5)^2 + 1,$$
$$E_{17}(K_2) = (E_{29}(K_1) + 5)^2 + 1, E_7(K_2) = (E_{31}(K_1) + 5)^2 + 1. \tag{5.61}$$

Every subequation has infinitely many prime solutions. Since $J_2(\omega) \to \infty$ as $\omega \to \infty$, there exist infinitely many subequations. Each has infinitely many prime solutions.

5.5.7. $p_2 = (p_1 + 7)^2 + 1$.

We have
$$J_2(\omega) = \prod_{3 \le p \le p_i} (p - 2 - \chi(p)) \neq 0, \tag{5.62}$$
where $\chi(5) = 0, \chi(p) = (-1)^{\frac{p-1}{2}}$ otherwise.

From (5.62) we have $J_2(30) = 6$. From (5.18) we have six subequations

$$E_{17}(K_2) = (E_7(K_1) + 7)^2 + 1, \quad E_{11}(K_2) = (E_{13}(K_1) + 7)^2 + 1,$$
$$E_7(K_2) = (E_{17}(K_1) + 7)^2 + 1, \quad E_{17}(K_2) = (E_{19}(K_1) + 7)^2 + 1,$$
$$E_{31}(K_2) = (E_{23}(K_1) + 7)^2 + 1, \quad E_7(K_2) = (E_{29}(K_1) + 7)^2 + 1. \tag{5.63}$$

Every subequation has infinitely many prime solutions. Since $J_2(\omega) \to \infty$ as $\omega \to \infty$, there exist infinitely many subequations. Each has infinitely many prime solutions.

5.5.8. $p_2 = (p_1 + 9)^2 + 1$.

We have
$$J_2(\omega) = \prod_{3 \le p \le p_i} (p - 2 - \chi(p)) \neq 0, \tag{5.64}$$

where $\chi(41)=0, \chi(p)=(-1)^{\frac{p-1}{2}}$ otherwise.

From (5.64) we have $J_2(30)=4$. From (5.18) we have four subequations

$$E_{17}(K_2)=(E_7(K_1)+9)^2+1, E_{11}(K_2)=(E_{11}(K_1)+9)^2+1,$$

$$E_{17}(K_2)=(E_{17}(K_1)+9)^2+1, E_{11}(K_2)=(E_{31}(K_1)+9)^2+1. \tag{5.65}$$

5.5.9. $p_2=(p_1+11)^2+1$.

We have

$$J_2(\omega)=\prod_{3\le p\le p_i}(p-2-\chi(p))\ne 0, \tag{5.66}$$

where $\chi(61)=0, \chi(p)=(-1)^{\frac{p-1}{2}}$ otherwise.

From (5.66) we have $J_2(30)=4$. From (5.18) we have four subequations

$$E_7(K_2)=(E_{13}(K_1)+11)^2+1, E_{31}(K_2)=(E_{19}(K_1)+11)^2+1,$$

$$E_{17}(K_2)=(E_{23}(K_1)+11)^2+1, E_{11}(K_2)=(E_{29}(K_1)+11)^2+1. \tag{5.67}$$

Since $J_2(\omega)\to\infty$ as $\omega\to\infty$, there exist infinitely many subequations.

5.5.10. $p_2=(p_1+2)^2+2$.

We have

$$J_2(\omega)=\prod_{3\le p\le p_i}(p-2-\chi(p))\ne 0, \tag{5.68}$$

where $\chi(3)=0, \chi(p)=(\frac{-2}{p})$ otherwise.

From (5.68) we have $J_2(30)=4$. From (5.18) we have four subequations

$$E_{23}(K_2)=(E_7(K_1)+2)^2+2, E_{17}(K_2)=(E_{13}(K_1)+2)^2+2,$$

$$E_{23}(K_2)=(E_{19}(K_1)+2)^2+2, E_{11}(K_2)=(E_{31}(K_1)+2)^2+2. \tag{5.69}$$

Every subequation has infinitely many prime solutions. Since $J_2(\omega)\to\infty$ as $\omega\to\infty$, there exist infinitely many subequations. Each has infinitely many prime solutions.

5.5.11. $p_2=(p_1+4)^2+2$.

We have

$$J_2(\omega)=\prod_{3\le p\le p_i}(p-2-\chi(p))\ne 0, \tag{5.70}$$

where $\chi(3)=0, \chi(p)=(\frac{-2}{p})$ otherwise.

From (5.70) we have $J_2(30)=4$. From (5.18) we have four subequations

$$E_{17}(K_2)=(E_{11}(K_1)+4)^2+2, E_{23}(K_2)=(E_{17}(K_1)+4)^2+2,$$

$$E_{11}(K_2) = (E_{23}(K_1) + 4)^2 + 2, E_{11}(K_2) = (E_{29}(K_1) + 4)^2 + 2. \tag{5.71}$$

5.5.12. $p_2 = (p_1 + 6)^2 + 2.$

We have

$$J_2(\omega) = \prod_{3 \le p \le p_i} (p - 2 - \chi(p)) \ne 0, \tag{5.72}$$

where $\chi(3) = 0, \chi(p) = \frac{-2}{3})$. There do not exist prime solutions except $p_2 = (3 + 6)^2 + 2 = 83$.

5.5.13. $p_2 = (p_1 + a)^2 + 2$. where $6|a$

We have $J_2(3) = 0$. There do not exist prime solutions except $p_2 = (3+a)^2+2 =$ prime.

5.5.14. $p_2 = (p_1 + a)^2 + 2$. where $6 \not| a$

We have $J_2(\omega) \ne 0$. There exist infinitely many prime solutions.

5.5.15. $p_2 = (p_1 + 1)^2 + 3.$

We have

$$J_2(\omega) = \prod_{3 \le p \le p_i} (p - 2 - \chi(p)) \ne 0, \tag{5.73}$$

where $\chi(3) = 0, \chi(p) = (\frac{-3}{p})$ otherwise.

From (5.73) we have $J_2(30) = 4$. From (5.18) we have four subequations

$$E_7(K_2) = (E_7(K_1) + 1)^2 + 3, E_{19}(K_2) = (E_{13}(K_1) + 1)^2 + 3,$$

$$E_{13}(K_2) = (E_{19}(K_1) + 1)^2 + 3, E_7(K_2) = (E_{31}(K_1) + 1)^2 + 3. \tag{5.74}$$

Every subequation has infinitely many prime solutions. Since $J_2(\omega) \to \infty$ as $\omega \to \infty$, there exist infinitely many subequations and each has infinitely many prime solutions.

5.5.16. $p_2 = (p_1 + 3)^2 + 3.$

We have

$$J_2(\omega) = \prod_{3 \le p \le p_i} (p - 2 - \chi(p)) \ne 0, \tag{5.75}$$

where $\chi(3) = -1, \chi(p) = (\frac{-3}{p})$ otherwise.

From (5.75) we have $J_2(30) = 8$. From (5.18) we have eight subequations

$$E_{11}(K_2) = (E_7(K_1) + 3)^2 + 3, E_{17}(K_2) = (E_{11}(K_1) + 3)^2 + 3,$$

$$E_{17}(K_2) = (E_{13}(K_1) + 3)^2 + 3, E_{13}(K_2) = (E_{17}(K_1) + 3)^2 + 3,$$

$$E_7(K_2) = (E_{19}(K_1) + 3)^2 + 3, E_{19}(K_2) = (E_{23}(K_1) + 3)^2 + 3,$$
$$E_7(K_2) = (E_{29}(K_1) + 3)^2 + 3, E_{17}(K_2) = (E_{31}(K_1) + 3)^2 + 3. \tag{5.76}$$

5.5.17. $p_2 = (p_1 + 5)^2 + 3$.

We have
$$J_2(\omega) = \prod_{3 \le p \le p_i} (p - 2 - \chi(p)) \ne 0, \tag{5.77}$$
where $\chi(3) = \chi(7) = 0, \chi(p) = (\frac{-3}{p})$ otherwise.

From (5.77) we have $J_2(30) = 4$. From (5.18) we have four subequations
$$E_{19}(K_2) = (E_{11}(K_1) + 5)^2 + 3, E_7(K_2) = (E_{17}(K_1) + 5)^2 + 3,$$
$$E_7(K_2) = (E_{23}(K_1) + 5)^2 + 3, E_{19}(K_2) = (E_{29}(K_1) + 5)^2 + 3. \tag{5.78}$$

5.5.18. $p_2 = (p_1 + 2)^2 + 4$.

We have
$$J_2(\omega) = \prod_{3 \le p \le p_i} (p - 2 - (-1)^{\frac{p-1}{2}}) \ne 0, \tag{5.79}$$
From (5.79) we have $J_2(30) = 4$. From (5.18) we have four subequations
$$E_{23}(K_2) = (E_{11}(K_1) + 2)^2 + 4, E_{19}(K_2) = (E_{13}(K_1) + 2)^2 + 4,$$
$$E_{29}(K_2) = (E_{23}(K_1) + 2)^2 + 4, E_{13}(K_2) = (E_{31}(K_1) + 2)^2 + 4. \tag{5.80}$$
Every subequation has infinitely many prime solutions. Since $J_2(\omega) \to \infty$ as $\omega \to \infty$, there exist infinitely many subequations. Each has infinitely many prime solutions.

5.5.19. $p_2 = (p_1 + 5)^2 + 5$.

We have
$$J_2(\omega) = \prod_{3 \le p \le p_i} (p - 2 - \chi(p)) \ne 0, \tag{5.81}$$
where $\chi(3) = 0, \chi(5) = -1, \chi(p) = (\frac{-5}{p})$ otherwise.

From (5.81) we have $J_2(30) = 4$. From (5.18) we have four subequations
$$E_{29}(K_2) = (E_7(K_1) + 5)^2 + 5, E_{29}(K_2) = (E_{13}(K_1) + 5)^2 + 5,$$
$$E_{11}(K_2) = (E_{19}(K_1) + 5)^2 + 5, E_{11}(K_2) = (E_{31}(K_1) + 5)^2 + 5. \tag{5.82}$$
Every subequation has infinitely many prime solutions. Since $J_2(\omega) \to \infty$ as $\omega \to \infty$, there exist infinitely many subequations. Each has infinitely many prime solutions.

5.5.20. $p_2 = p_1^3 + 2$

We have

$$J_2(\omega) = \prod_{3 \le p \le p_i} (p - 2 - \chi(p)) \neq 0, \tag{5.83}$$

where $\chi(p) = 2$ if $2^{\frac{p-1}{3}} \equiv 1 \pmod{p}$, $\chi(p) = -1$ if $2^{\frac{p-1}{3}} \not\equiv = 1 (\mathrm{mod}\ p)$, $\chi(p) = 0$. otherwise.

From (5.83) we have $J_2(30) = 3$. From (5.18) we have three subequations

$$E_{13}(K_2) = (E_{11}(K_1))^3 + 2, E_{19}(K_2) = (E_{23}(K_1))^3 + 2,$$

$$E_{31}(K_2) = (E_{29}(K_1))^3 + 2. \tag{5.84}$$

Every subequation has infinitely many prime solutions. Since $J_2(\omega) \to \infty$ as $\omega \to \infty$, there exist infinitely many subequations. Each has infinitely many prime solutions.

5.5.21. $p_2 = (p_1 + 1)^4 + 1$

We have

$$J_2(\omega) = \prod_{3 \le p \le p_i} (p - 2 - \chi(p)) \neq 0, \tag{5.85}$$

where $\chi(p) = 3$ if $p \equiv 1 \pmod{8}$, $\chi(p) = -1$ if $p \not\equiv 1 (\mathrm{mod}\ 8)$. From (5.85) we have $J_2(30) = 8$. From (5.18) we have eight subequations

$$E_{17}(K_2) = (E_7(K_1) + 1)^4 + 1, E_7(K_2) = (E_{11}(K_1) + 1)^4 + 1,$$

$$E_{17}(K_2) = (E_{13}(K_1) + 1)^4 + 1, E_7(K_2) = (E_{17}(K_1) + 1)^4 + 1,$$

$$E_{11}(K_2) = (E_{19}(K_1) + 1)^4 + 1, E_7(K_2) = (E_{23}(K_1) + 1)^4 + 1,$$

$$E_{31}(K_2) = (E_{29}(K_1) + 1)^4 + 1, E_{17}(K_2) = (E_{31}(K_1) + 1)^4 + 1. \tag{5.86}$$

Every subequation has infinitely many prime solutions. Since $J_2(\omega) \to \infty$ as $\omega \to \infty$, there exist infinitely many subequations. Each has infinitely many prime solutions.

5.5.22. 3-tuple: $p_2 = p_1 + 2$ and $p_3 = p_1 + 6$

We have

$$J_2(\omega) = \prod_{5 \le p \le p_i} (p - 3) \neq 0, \tag{5.87}$$

From (5.87) we have $J_2(30) = 2$. From (5.18) we have two 2-tuples of subequations

$$E_{13}(K) = E_{11}(K) + 2, \quad E_{17}(K) = (E_{11}(K) + 6;$$

$$E_{19}(K) = E_{17}(K) + 2, \quad E_{23}(K) = E_{17}(K) + 6. \tag{5.88}$$

Every 2-tuple of subequations has infinitely many 3-tuples of primes. Since $J_2(\omega) \to \infty$ as $\omega \to \infty$, there exist infinitely many 2-tuples of subequations. Each has infinitely many 3-tuples of primes.

5.5.23. 3-tuple: $p_2 = p_1 + 6$ and $p_3 = p_1 + 12$

We have

$$J_2(\omega) = 2 \prod_{5 \le p \le p_i} (p-3) \ne 0, \tag{5.89}$$

From (5.89) we have $J_2(30) = 4$. From (5.18) we have four 2-tuples of subequations

$$E_{13}(K) = E_7(K) + 6, \quad E_{19}(K) = E_7(K) + 12;$$

$$E_{17}(K) = E_{11}(K) + 6, \quad E_{23}(K) = E_{11}(K) + 12;$$

$$E_{23}(K) = E_{17}(K) + 6, \quad E_{29}(K) = E_{17}(K) + 12;$$

$$E_7(K) = E_{31}(K_1) + 6, \quad E_{13}(K_1) = E_{31}(K) + 12. \tag{5.90}$$

Every 2-tuple of subequations has infinitely many 3-tuples of primes. Since $J_2(\omega) \to \infty$ as $\omega \to \infty$, there exist infinitely many 2-tuples of subequations. Each has infinitely many prime solutions.

5.5.24. 3-tuple: $p_2 = p_1 + 4$ and $p_3 = p_1^2 + 4$

We have

$$J_2(\omega) = 2 \prod_{7 \le p \le p_i} (p - 3 - (-1)^{(p-1)/2}) \ne 0, \tag{5.91}$$

From (5.91) we have $J_2(30) = 2$. From (5.18) we have two 2-tuples of subequations

$$E_{11}(K) = E_7(K) + 4, \quad E_{23}(K_1) = (E_7(K))^2 + 4,$$

$$E_{17}(K) = E_{13}(K) + 4, \quad E_{23}(K_1) = (E_{13}(K))^2 + 4. \tag{5.92}$$

Every 2-tuple of subequations has infinitely many 3-tuples of primes. Since $J_2(\omega) \to \infty$ as $\omega \to \infty$, there exist infinitely many 2-tuples of subequations. Each has infinitely many 3-tuples of primes.

5.5.25. 4-tuple: $p_2 = p_1 + 4$ and $p_3 = p_1 + 16$ and $p_4 = p_1 + 36$

We have

$$J_2(\omega) = 2 \prod_{7 \le p \le p_i} (p-4) \ne 0, \tag{5.93}$$

From (5.93) we have $J_2(30) = 2$. From (5.18) we have two 3-tuples of subequations

$$E_{11}(K) = E_7(K) + 4, \quad E_{23}(K) = E_7(K) + 16, \quad E_{13}(K) = E_7(K_1) + 36;$$

$$E_{17}(K) = E_{13}(K) + 4, \quad E_{29}(K) = E_{13}(K) + 16, \quad E_{19}(K) = E_{13}(K_1) + 36. \tag{5.94}$$

Every 3-tuple of subequations has infinitely many 4-tuples of primes. Since $J_2(\omega) \to \infty$ as $\omega \to \infty$, there exist infinitely many 3-tuples of subequations. Each has infinitely many 4-tuples of primes.

5.6. Let $\omega = 210$. From (3.1) we have

$$E_{p\alpha}(K) = 210 + p_\alpha, \tag{5.95}$$

where $K = 0, 1, 2, ..., \phi(210) = 48$, that is the number of p_α, p_α = 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61, 67, 71,73, 79, 83, 89, 97, 101, 103, 107, 109, 113, 121, 127, 131, 137, 139, 143,149, 151, 157, 163, 167, 169, 173, 179, 181, 187, 191, 193, 197, 199, 209,211.

Since $\phi(\omega) \to \infty$ as $\omega \to \infty$, there exist many arithmetic progressions and each has infinitely many primes. We have

$$\lim_{\omega\to\infty} \frac{\omega}{\phi(\omega)} = \sum_{n=1}^{\infty} \frac{1}{n} = \infty,$$

and $(\omega, \omega+1) = (\omega, p_\alpha) = 1$. It is an Euclid-Euler proof of the existence of an infinity of primes. $J_2(\omega)$ is a generalization of Euler function $\phi(\omega)$.

5.6.1. 4-tuple: $p_2 = p_1 + 2, p_3 = p_1 + 6$ and $p_4 = p_1 + 8$.

We have

$$J_2(\omega) = \prod_{5 \le p \le p_i} (p - 4) \ne 0. \tag{5.96}$$

From (5.96) we have $J_2(210) = 3$. From (5.95) we have three 3-tuples of subequations

$$E_{13}(K) = E_{11}(K) + 2, \quad E_{17}(K) = E_{11}(K) + 6, \quad E_{19}(K) = E_{11}(K) + 8;$$

$$E_{103}(K) = E_{101}(K) + 2, \quad E_{107}(K) = E_{101}(K) + 6, \quad E_{109}(K) = E_{101}(K) + 8;$$

$$E_{193}(K) = E_{191}(K)+2, \quad E_{197}(K) = E_{191}(K)+6, \quad E_{199}(K) = E_{191}(K)+8. \tag{5.97}$$

Every 3-tuple of subequations has infinitely many 4-tuples of primes. Since $J_2(\omega) \to \infty$ as $\omega \to \infty$, there exist infinitely many 3-tuples of subequations. Each has infinitely many 4-tuples of primes.

5.6.2. 5-tuple: $p_2 = p_1 + 4, p_3 = p_1 + 6, p_4 = p_1 + 36, p_5 = p_1 + 64$.

We have

$$J_2(\omega) = 6 \prod_{11 \le p \le p_i} (p - 5) \ne 0. \tag{5.98}$$

From (5.98) we have $J_2(30) = 6$. From (5.95) we have six 4-tuples of subequations

$$E_{41}(K) = E_{37}(K) + 4, \quad E_{53}(K) = E_{37}(K) + 16,$$

$$E_{73}(K)=E_{37}(K)+36,\quad E_{101}(K)=E_{37}(K)+64;$$
$$E_{47}(K)=E_{43}(K)+4,\quad E_{59}(K)=E_{43}(K)+16,$$
$$E_{79}(K)=E_{43}(K)+36,\quad E_{107}(K)=E_{43}(K)+64;$$
$$E_{71}(K)=E_{67}(K)+4,\quad E_{83}(K)=E_{67}(K)+16,$$
$$E_{103}(K)=E_{67}(K)+36,\quad E_{131}(K)=E_67(K)+64;$$
$$E_{131}(K)=E_{127}(K)+4,\quad E_{143}(K)=E_{127}(K)+16,$$
$$E_{163}(K)=E_{127}(K)+36,\quad E_{191}(K)=E_{127}(K)+64;$$
$$E_{167}(K)=E_{163}(K)+4,\quad E_{179}(K)=E_{163}(K)+16,$$
$$E_{199}(K)=E_{163}(K)+36,\quad E_{17}(K_1)=E_{163}(K)+64;$$
$$E_{197}(K)=E_{193}(K)+4,\quad E_{209}(K)=E_{193}(K)+16,$$
$$E_{19}(K_1)=E_{193}(K)+36,\quad E_{47}(K_1)=E_{193}(K)+64. \tag{5.99}$$

Every 4-tuple of subequations has infinitely many 5-tuples of primes. Since $J_2(\omega)\to\infty$ as $\omega\to\infty$, there exist infinitely many 4-tuples of subequations. Each has infinitely many 5-tuples of primes.

5.6.3. 5-tuple: $p_2=p_1+2, p_3=p_1+6, p_4=p_1+8$ and $p_5=p_1+12$.

We have

$$J_2(\omega)=\prod_{7\le p\le p_i}(p-5)\neq 0. \tag{5.100}$$

From (5.100) we have $J_2(210)=2$. From (5.95) we have two 4-tuples of subequations

$$E_{13}(K)=E_{11}(K)+2,\quad E_{17}(K)=E_{11}(K)+6,$$
$$E_{19}(K)=E_{11}(K)+8,\quad E_{23}(K)=E_{11}(K)+12;$$
$$E_{103}(K)=E_{101}(K)+2,\quad E_{107}(K)=E_{101}(K)+6,$$
$$E_{109}(K)=E_{101}(K)+8,\quad E_{113}(K)=E_{101}(K)+12. \tag{5.101}$$

Every 4-tuple of subequations has infinitely many 5-tuples of primes. Since $J_2(\omega)\to\infty$ as $\omega\to\infty$, there exist infinitely many 4-tuples of subequations. Each has infinitely many 5-tuples of primes.

5.6.4. 6-tuple: $p_2=p_1+4, p_3=p_1+6, p_4=p_1+10, p_5=p_1+12, p_6=p_1+16$.

We have

$$J_2(\omega)=\prod_{7\le p\le p_i}(p-6)\neq 0. \tag{5.102}$$

From (5.102) we have $J_2(210) = 1$. From (5.95) we have one 5-tuple of subequations

$$E_{101}(K) = E_{97}(K) + 4, \quad E_{103}(K) = E_{97}(K) + 6, \quad E_{107}(K) = E_{97}(K) + 10,$$
$$E_{109}(K) = E_{97}(K) + 12, \quad E_{113}(K) = E_{97}(K) + 16. \tag{5.103}$$

(5.103) has infinitely many 6-tuples of primes. Since $J_2(\omega) \to \infty$ as $\omega \to \infty$, there exist infinitely many 5-tuples of subequations. Each has infinitely many 6-tuples of primes.

5.6.5. 7-tuple: $p_2 = p_1 + 4, p_3 = p_1 + 6, p_4 = p_1 + 10, p_5 = p_1 + 16, p_6 = p_1 + 18$ and $p_7 = p_1 + 24$.

We have

$$J_2(\omega) = 2 \prod_{11 \le p \le p_i} (p - 7) \ne 0. \tag{5.104}$$

From (5.104) we have $J_2(210) = 2$. From (5.95) we have two 6-tuples of subequations

$$E_{17}(K) = E_{13}(K) + 4, \quad E_{19}(K) = E_{13}(K) + 6, \quad E_{23}(K) = E_{13}(K) + 10,$$
$$E_{29}(K) = E_{13}(K) + 16, \quad E_{31}(K) = E_{13}(K) + 18, \quad E_{37}(K) = E_{13}(K) + 24;$$
$$E_{167}(K) = E_{163}(K) + 4, \quad E_{169}(K) = E_{163}(K) + 6, \quad E_{173}(K) = E_{163}(K) + 10,$$
$$E_{179}(K) = E_{163}(K) + 16, \quad E_{181}(K) = E_{163}(K) + 18, \quad E_{187}(K) = E_{163}(K) + 24. \tag{5.105}$$

Every 6-tuple of subequations has infinitely many 7-tuples of primes. Since $J_2(\omega) \to \infty$ as $\omega \to \infty$, there exist infinitely many 6-tuples of subequations. Each has infinitely many 7-tuples of primes.

5.6.6. 7-tuple: $p_2 = p_1 + 6, p_3 = p_1 + 8, p_4 = p_1 + 14, p_5 = p_1 + 18, p_6 = p_1 + 20$ and $p_7 = p_1 + 24$.

We have

$$J_2(\omega) = 2 \prod_{11 \le p \le p_i} (p - 7) \ne 0. \tag{5.106}$$

From (5.106) we have $J_2(210) = 2$. From (5.95) we have two 6-tuples of subequations

$$E_{29}(K) = E_{23}(K) + 6, \quad E_{31}(K) = E_{23}(K) + 8, \quad E_{37}(K) = E_{23}(K) + 14,$$
$$E_{41}(K) = E_{23}(K) + 18, \quad E_{43}(K) = E_{23}(K) + 20, \quad E_{47}(K) = E_{23}(K) + 24;$$
$$E_{179}(K) = E_{173}(K) + 6, \quad E_{181}(K) = E_{173}(K) + 8, \quad E_{187}(K) = E_{173}(K) + 14,$$
$$E_{191}(K) = E_{173}(K) + 18, \quad E_{193}(K) = E_{173}(K) + 20, \quad E_{197}(K) = E_{173}(K) + 24. \tag{5.107}$$

Every 6-tuple of subequations has infinitely many 7-tuples of primes. Since $J_2(\omega) \to \infty$ as $\omega \to \infty$, there exist infinitely many 6-tuples of subequations. Each has infinitely many 7-tuples of primes.

5.6.7. 7-tuple: $p_2 = p_1 + 42, p_3 = p_1 + 48, p_4 = p_1 + 50, p_5 = p_1 + 56, p_6 = p_1 + 60$ and $p_7 = p_1 + 62$.

We have

$$J_2(\omega) = 3 \prod_{11 \le p \le p_i} (p - 7 - \chi(p)) \neq 0. \tag{5.108}$$

where $X(31) = -1, \chi(p) = 0$ otherwise.

From (5.108) we have $J_2(210) = 3$. From (5.95) we have three 6-tuples of subequations

$$E_{53}(K) = E_{11}(K) + 42, \quad E_{59}(K) = E_{11}(K) + 48, \quad E_{61}(K) = E_{11}(K) + 50,$$

$$E_{67}(K) = E_{11}(K) + 56, \quad E_{71}(K) = E_{11}(K) + 60, \quad E_{73}(K) = E_{11}(K) + 62;$$

$$E_{173}(K) = E_{131}(K) + 42, \quad E_{179}(K) = E_{131}(K) + 48, \quad E_{181}(K) = E_{131}(K) + 50,$$

$$E_{187}(K) = E_{131}(K) + 56, \quad E_{191}(K) = E_{131}(K) + 60, \quad E_{193}(K) = E_{131}(K) + 62;$$

$$E_{23}(K_1) = E_{191}(K) + 42, \quad E_{29}(K_1) = E_{191}(K) + 48, \quad E_{31}(K_1) = E_{191}(K) + 50,$$

$$E_{37}(K_1) = E_{191}(K) + 56, \quad E_{41}(K_1) = E_{191}(K) + 60, \quad E_{43}(K_1) = E_{191}(K) + 62. \tag{5.109}$$

Every 6-tuple of subequations has infinitely many 7-tuples of primes. Since $J_2(\omega) \to \infty$ as $\omega \to \infty$, there exist infinitely many 6-tuples of subequations. Each has infinitely many 7-tuples of primes.

5.7. We define

$$E_{p_\alpha}(K) = 36K + p_\alpha \tag{5.110}$$

where $K = 0, 1, 2, ..., p_\alpha = 5, 7, 11, 13, 17, 19, 23, 25, 29, 31, 35, 37$.

5.7.1 $p_2 = p_1 + 2$

We have $J_2(36) = 6$. From (5.110) we have six 6-tuples of subequations of twin primes

$$E_7(K) = E_5(K) + 2, \quad E_{13}(K) = E_{11}(K) + 2, \quad E_{19}(K) = E_{17}(K) + 2,$$

$$E_{25}(K) = E_{23}(K) + 2, \quad E_{31}(K) = E_{29}(K) + 2, \quad E_{37}(K) = E_{35}(K) + 2. \tag{5.111}$$

5.8. We define

$$E_{p_\alpha}(K) = 4K + p_\alpha \tag{5.112}$$

where $K = 0, 1, 2, ..., p_\alpha = 3, 5$.

5.8.1. $p_2 = p_1 + 2$

We have $J_2(4) = \phi(4) = 2$. From (5.112) we have two subequations of twin primes

$$E_5(K) = E_3(K) + 2, E_3(K_1) = E_5(K) + 2. \tag{5.113}$$

5.9. We define

$$E_{p_\alpha}(K) = 5K + p_\alpha \tag{5.114}$$

where $K = 0, 1, 2, ..., p_\alpha = 2, 3, 4, 6$.

5.9.1. $p_2 = p_1 + 2$

We have $J_2(5) = 3$. From (5.114) we have three subequations of twin primes

$$E_4(K) = E_2(K) + 2, E_6(K) = E_4(K) + 2, E_3(K) = E_6(K_1) + 2. \tag{5.115}$$

5.10. We define

$$E_{p_\alpha}(K) = 8K + p_\alpha \tag{5.116}$$

where $K = 0, 1, 2, ..., p_\alpha = 3, 5, 7, 9$.

5.10.1. $p_2 = p_1 + 2$

We have $J_2(8) = \phi(8) = 4$. From (5.116) we have four subequations of twin primes

$$E_5(K) = E_3(K)+2, E_7(K) = E_5(K)+2, E_9(K) = E_7(K)+2, E_3(K_1) = E_9(K)+2. \tag{5.117}$$

5.11. We define

$$E_{p_\alpha}(K) = 16K + p_\alpha \tag{5.118}$$

where $K = 0, 1, 2, ..., p_\alpha = 3, 5, 7, 9, 11, 13, 15, 17$.

5.11.1. $p_2 = p_1 + 2$

We have $J_2(16) = \phi(16) = 8$. From (5.118) we have eight subequations of twin primes

$$E_5(K) = E_3(K) + 2, \quad E_7(K) = E_5(K) + 2, \quad E_9(K) = E_7(K) + 2,$$
$$E_{11}(K) = E_9(K) + 2, \quad E_{13}(K) = E_{11}(K) + 2, \quad E_{15}(K) = E_{13}(K) + 2,$$
$$E_{17}(K) = E_{15}(K) + 2, \quad E_3(K_1) = E_{17}(K) + 2. \tag{5.119}$$

5.12. We define

$$E_{p_\alpha}(K) = 2^n K + p_\alpha \tag{5.120}$$

where $K = 0, 1, 2, ..., \phi(2^n) = 2^{n-1}, (2^n, p_\alpha) = 1, p_\alpha = 3, 5, 7, ..., 2^n + 1$.

5.12.1. $p_2 = p_1 + 2$

We have $J_2(2^n) = \phi(2^n) = 2^{n-1}$. From (5.120) we have 2^{n-1} subequations of twin primes.

$$E_5(K) = E_3(K) + 2, \quad E_7(K) = E_5(K) + 2, \quad .... \tag{5.121}$$

Since $J_2(2^n) \to \infty$ as $n \to \infty$, there exist infinitely many subequations of twin primes and each has infinitely many twin primes.

We have the asymptotic formula

$$\pi_2(N, 2) \approx 2\frac{N}{\log^2 N} \tag{5.122}$$

5.12.2. $p_2 = p_1 + 2, p_3 = p_1 + 6$.

We have $J_2(2^n) = \phi(2^n) = 2^{n-1}$. From (5.120) we have 2^{n-1} 2-tuples of subequations

$$E_5(K) = E_3(K) + 2, E_9(K) = E_3(K) + 6; \cdots. \tag{5.123}$$

Since $J_2(2^n) \to \infty$ as $n \to \infty$, there exist infinitely many 2-tuples of subequations and each has infinitely many 3-tuples of primes.

We have the asymptotic formula

$$\pi_3(N, 2) \approx \frac{4N}{\log^3 N} \tag{5.124}$$

5.12.3. $p_2 = p_1 + 2, p_3 = p_1 + 6, p_4 = p_1 + 8$.

We have $J_2(2^n) = \phi(2^n) = 2^{n-1}$. From (5.120) we have 2^{n-1} 3-tuples of subequations

$$E_5(K) = E_3(K) + 2, E_9(K) = E_3(K) + 6, E_{11}(K) = E_3(K) + 8; \cdots. \tag{5.125}$$

Since $J_2(2^n) \to \infty$ as $n \to \infty$, there exist infinitely many 3-tuples of subequations and each has infinitely many 4-tuples of primes.

We have the asymptotic formula

$$\pi_4(N, 2) \approx \frac{8N}{\log^4 N}. \tag{5.126}$$

6. Santilli's Isoprime m-Chains

While studying the Cunningham chains we discover the Santilli's isoprime m-chains that show novel properties for sufficiently large isoprimes and prove them using the arithmetic function $J_2(\omega)$ below.

Theorem 6.1. An increasing sequence of isoprimes $\hat{p}_1 < \hat{p}_2 < \cdots < p_k$ is called a Santilli's isoprime m-chain of the first kind of length k if

$$\hat{p}_{j+1} = \hat{m}\hat{\times}\hat{p}_j + \hat{m} - \hat{I} = (m^j(p_1+1)-1)\hat{I} \tag{6.1}$$

for $j = 1, \cdots, k-1$, where $m > 1$ is any positive integer.

There exist infinitely many isoprimes $\hat{p}_1$ such that $\hat{p}_2, \cdots, \hat{p}_k$ are all isoprimes for any length k.

Let $\hat{I} = 1$. From (6.1) we have

$$p_{j+1} = mp_j + m - 1 = m^j(p_1+1) - 1 \tag{6.2}$$

for $j = 1, \cdots, k-1$.

Proof: We have the arithmetic function

$$J_2(\omega) = \left|\left\{p_\alpha : 1 \le \alpha \le \phi(\omega), \left(\prod_{j=1}^{k-1}(m^j(p_\alpha+1)-1), \omega\right) = 1\right\}\right| = \prod_{3\le p\le p_i}(p-\chi(p)) \ne 0 \tag{6.3}$$

We now calculate $\chi(p)$. The smallest positive integer S such that

$$m^S \equiv 1(\mathrm{mod}\ p), (m, p) = 1 \tag{6.4}$$

is called the exponent of modulo p, and is denoted by writing

$$S = \exp_p(m). \tag{6.5}$$

$\chi(p) = k$ if $k < S$; $\chi(p) = S$ if $k \ge S$; $\chi(p) = 1$ if $p|m(m-1)$.

From the Euler-Fermat theorem we have

$$S \le \phi(p) = p - 1 \tag{6.6}$$

Then we have

$$\chi(p) \le p - 1 \text{ and } J_2(p) \ge 1. \tag{6.7}$$

We prove that

$$J_2(\omega) \ne 0. \tag{6.8}$$

Since $J_2(\omega) \to \infty$ as $\omega \to \infty$, there exist infinitely many isoprimes $\hat{p}_1$ such that $\hat{p}_2, \cdots, \hat{p}_k$ are all isoprimes for any length k. It is a generalization of Euler's proof of the existence of infinitely many primes. We have the best asymptotic formula of the number of k-tuples of isoprimes

$$\pi_k(N, 2) \approx \frac{J_2(\omega)\omega^{k-1}}{\phi^k(\omega)}\frac{N}{\log^k N}. \tag{6.9}$$

Theorem 6.2 An increasing sequence of isoprimes $\hat{p}_1 < \hat{p}_2 < \cdots < p_\alpha$ is called a Santilli's isoprime m-chain of the second kind of length k if

$$\hat{p}_{j+1} = \hat{m}\hat{\times}\hat{p}_j - \hat{m} + \hat{I} = (m^j(p_1 - 1) + 1)\hat{I} \tag{6.10}$$

for $j = 1, \cdots, k-1$.

There exist infinitely many isoprimes $\hat{p}_1$ such that $\hat{p}_2, \cdots, \hat{p}_k$ are all isoprimes.

Let $\hat{I} = 1$. From (6.10) we have

$$p_{j+1} = mp_j - m + 1 = m^j(p_1 - 1) + 1 \tag{6.11}$$

for $j = 1, \cdots, k-1$.

Proof: Theorems 6.1 and 6.2 have the same arithmetic function

$$J_2(\omega) = |\{p_\alpha : 1 \le \alpha \le \phi(\omega), (\prod_{j=1}^{k-1}(m^j(p_\alpha - 1) + 1), \omega) = 1\}|$$

$$= |\{p_\alpha : 1 \le \alpha \le \phi(\omega), (\prod_{j=1}^{k-1}(m^j(p_\alpha+1)-1), \omega) = 1\}| = \prod_{3 \le p \le p_i}(p-\chi(p)) \neq 0. \tag{6.12}$$

Since $J_2(\omega) \to \infty$ as $\omega \to \infty$, there exist infinitely many isoprimes $\hat{p}_1$ such that $\hat{p}_2, \cdots, \hat{p}_k$ are all isoprimes for any length k.

Theorem 6.3 We deal with the Santilli's isoprime 2-chain of the first kind of any length k. From (6.1) we have

$$\hat{p}_{j+1} = \hat{2}\hat{\times}\hat{p}_j + \hat{I} = (2^j(p_1 + 1) - 1)\hat{I} \tag{6.13}$$

for $j = 1, \cdots, k-1$.

Let $\hat{I} = 1$. From (6.13) we have

$$p_{j+1} = 2p_j + 1 = 2^j(p_1 + 1) - 1 \tag{6.14}$$

for $j = 1, \cdots, k-1$.

From (6.3) we have the arithmetic function

$$J_2(\omega) = \prod_{3 \le p \le p_i}(p - \chi(p)) \neq 0 \tag{6.15}$$

We now calculate $\chi(p)$. We define the smallest positive integer S such that

$$2^S \equiv 1(\text{mod } p). \tag{6.16}$$

We have

Table 6.1

p	3	5	7	11	13	17	19	23	29	31	37	41	43	47	53	59	61	67	71
s	2	4	3	10	12	8	18	11	28	5	36	20	14	23	52	58	60	66	35

p	73	79	83	89	97	101	103	107	109	113	127	131	139	149	151
s	9	39	82	11	48	100	51	106	36	28	7	130	138	148	15

$\chi(p) = k$ if $k < S$; $\chi(p) = S$ if $k \geq S$. From the Euler-Fermat theorem we have

$$S \leq \phi(p) = p - 1. \tag{6.17}$$

From (6.15) we have

$$J_2(p) \geq 1. \tag{6.18}$$

Then we have $J_2(\omega) \neq 0$. There exist infinitely many k-tuples of isoprimes for any length k.

From (6.9) we have

$$\pi_k(N, 2) \approx \frac{J_2(\omega)\omega^{k-1}}{\phi^k(\omega)} \frac{N}{\log^k N}. \tag{6.19}$$

Note: $p_{j+1} = 2p_j + 1$ is the DNA Mathematics.

To understand theorem 6.3 we yield the more detailed proofs below.

(1) Let $k = 2$. From (6.15) we have

$$J_2(\omega) = \prod_{3 \leq p \leq p_i} (p - 2) \neq 0.$$

Since $J_2(\omega) \neq 0$, there exist infinitely many isoprimes $\hat{p}_1$ such that $\hat{p}_2$ is also an isoprime.

From (6.19) we have

$$\pi_2(N, 2) \approx 2 \prod_{3 \leq p \leq p_i} \left(1 - \frac{1}{(p-1)^2}\right) \frac{N}{\log^2 N}.$$

(2) Let $k = 3$. We have

$$J_2(\omega) = \prod_{5 \leq p \leq p_i} (p - 3) \neq 0.$$

Since $J_2(\omega) \neq 0$, there exist infinitely many 3-tuples of isoprimes.

We have

$$\pi_3(N, 2) \approx \frac{1}{4}\left(\frac{15}{4}\right)^2 \prod_{7 \leq p \leq p_i} \frac{p^2(p-3)}{(p-1)^3} \frac{N}{\log^3 N}.$$

(3) Let $k = 4$. We have

$$J_2(\omega) = 4 \prod_{11 \leq p \leq p_i} (p - 4) \neq 0.$$

Since $J_2(\omega) \neq 0$, there exist infinitely many 4-tuples of isoprimes.
We have

$$\pi_4(N,2) \approx \frac{1}{12}\left(\frac{35}{8}\right)^3 \prod_{11 \leq p \leq p_i} \frac{p^3(p-4)}{(p-1)^4} \frac{N}{\log^4 N}.$$

(4) Let $k = 5$. We have

$$J_2(\omega) = 4 \prod_{11 \leq p \leq p_i} (p-5) \neq 0.$$

Since $J_2(\omega) \neq 0$, there exist infinitely many 5-tuples of isoprimes.
We have

$$\pi_5(N,2) \approx \frac{1}{12}\left(\frac{35}{8}\right)^4 \prod_{11 \leq p \leq p_i} \frac{p^4(p-5)}{(p-1)^5} \frac{N}{\log^5 N}.$$

(5) Let $k = 6$. We have

$$J_2(\omega) = 4 \prod_{11 \leq p \leq p_i} (p-\chi(p)) \neq 0.$$

where $\chi(31) = 5, \chi(p) = 6$ otherwise.

Since $J_2(\omega) \neq 0$, there exist infinitely many 6-tuples of isoprimes.
We have

$$\pi_6(N,2) \approx \frac{1}{12}\left(\frac{35}{8}\right)^5 \prod_{11 \leq p \leq p_i} \frac{p^5(p-\chi(p))}{(p-1)^6} \frac{N}{\log^6 N}.$$

(6) Let $k = 7$. We have

$$J_2(\omega) = 4 \prod_{11 \leq p \leq p_i} (p-\chi(p)) \neq 0.$$

where $\chi(31) = 5, \chi(p) = 7$ otherwise.

Since $J_2(\omega) \neq 0$, there exist infinitely many 7-tuples of isoprimes.
We have

$$\pi_7(N,2) \approx \frac{1}{12}\left(\frac{35}{8}\right)^6 \prod_{11 \leq p \leq p_i} \frac{p^6(p-\chi(p))}{(p-1)^7} \frac{N}{\log^7 N}.$$

(7) Let $k = 8$. We have

$$J_2(\omega) = 4 \prod_{11 \leq p \leq p_i} (p-\chi(p)) \neq 0.$$

where $\chi(31) = 5, \chi(127) = 7, \chi(p) = 8$ otherwise.

Since $J_2(\omega) \neq 0$, there exist infinitely many 8-tuples of isoprimes.

We have

$$\pi_8(N,2) \approx \frac{1}{12}\left(\frac{35}{8}\right)^7 \prod_{11\le p\le p_i} \frac{p^7(p-\chi(p))}{(p-1)^8}\frac{N}{\log^8 N}.$$

(8) Let $k=9$. We have

$$J_2(\omega) = 4\prod_{11\le p\le p_i}(p-\chi(p)) \ne 0.$$

where $\chi(p) = 5,7,8$ If $p = 31,127,17$; $\chi(p) = 9$ otherwise.

Since $J_2(\omega) \ne 0$, there exist infinitely many 9-tuples of isoprimes.

We have

$$\pi_9(N,2) \approx \frac{1}{12}\left(\frac{35}{8}\right)^8 \prod_{11\le p\le p_i} \frac{p^8(p-\chi(p))}{(p-1)^9}\frac{N}{\log^9 N}.$$

(9) Let $k=10$. We have

$$J_2(\omega) = 4\prod_{11\le p\le p_i}(p-\chi(p)) \ne 0.$$

where $\chi(p) = 5,7,8,9$. If $p = 31,127,17,73$; $\chi(p) = 10$ otherwise.

Since $J_2(\omega) \ne 0$, there exist infinitely many 10-tuples of isoprimes.

We have

$$\pi_{10}(N,2) \approx \frac{1}{12}\left(\frac{35}{8}\right)^9 \prod_{11\le p\le p_i} \frac{p^9(p-\chi(p))}{(p-1)^{10}}\frac{N}{\log^{10} N}.$$

(10) Let $k=11$. We have

$$J_2(\omega) = 4\prod_{13\le p\le p_i}(p-\chi(p)) \ne 0.$$

where $\chi(p) = 5,7,8,9$. If $p = 31,127,17,73$; $\chi(p) = 11$ otherwise.

Since $J_2(\omega) \ne 0$, there exist infinitely many 11-tuples of isoprimes.

We have

$$\pi_{11}(N,2) \approx \frac{1}{120}\left(\frac{77}{16}\right)^{10} \prod_{13\le p\le p_i} \frac{p^{10}(p-\chi(p))}{(p-1)^{11}}\frac{N}{\log^{11} N}.$$

(11) Let $k=12$. We have

$$J_2(\omega) = 4\prod_{13\le p\le p_i}(p-\chi(p)) \ne 0,$$

where $\chi(p) = 5,7,8,9,11,11$. If $p = 31,127,17,73,23,89$; $\chi(p) = 12$ otherwise.

We have

$$\pi_{12}(N,2) \approx \frac{1}{120}\left(\frac{77}{16}\right)^{11} \prod_{13\le p\le p_i} \frac{p^{11}(p-\chi(p))}{(p-1)^{12}} \frac{N}{\log^{12} N}.$$

(12) Let $k=13$. We have

$$J_2(\omega) = 4 \prod_{17\le p\le p_i} (p-\chi(p)) \neq 0.$$

where $\chi(p) = 5,7,8,9,11,11$. If $p = 31,127,17,73,23,89$; $\chi(p)=13$ otherwise.

We have

$$\pi_{13}(N,2) \approx \frac{1}{1440}\left(\frac{1001}{192}\right)^{12} \prod_{17\le p\le p_i} \frac{p^{12}(p-\chi(p))}{(p-1)^{13}} \frac{N}{\log^{13} N}.$$

(13) Let $k=14$. We have

$$J_2(\omega) = 4 \prod_{17\le p\le p_i} (p-\chi(p)) \neq 0.$$

where $\chi(p) = 5,7,8,9,11,11,13$. If $p = 31,127,17,73,23,89,8191$; $\chi(p)=14$ otherwise.

We have

$$\pi_{14}(N,2) \approx \frac{1}{1440}\left(\frac{1001}{192}\right)^{13} \prod_{17\le p\le p_i} \frac{p^{13}(p-\chi(p))}{(p-1)^{14}} \frac{N}{\log^{14} N}.$$

(14) Let $k=15$. We have

$$J_2(\omega) = 4 \prod_{17\le p\le p_i} (p-\chi(p)) \neq 0.$$

where $\chi(p) = 5,7,8,9,11,11,13,14$. If $p = 31,127,17,73,23,89,8191,43$; $\chi(p)=15$ otherwise.

We have

$$\pi_{15}(N,2) \approx \frac{1}{1440}\left(\frac{1001}{192}\right)^{14} \prod_{17\le p\le p_i} \frac{p^{14}(p-\chi(p))}{(p-1)^{15}} \frac{N}{\log^{15} N}.$$

(15) Let $k=20$. Its smallest solution is $\pi_{20}(N,2)\approx 1$ if $N=10^{26}$.
(16) Let $k=30$. Its smallest solution is $\pi_{30}(N,2)\approx 4$ if $N=10^{44}$.
(17) Let $k=40$. Its smallest solution is $\pi_{40}(N,2)\approx 3$ if $N=10^{63}$.
(18) Let $k=50$. Its smallest solution is $\pi_{50}(N,2)\approx 2$ if $N=10^{83}$.
(19) Let $k=100$. Its smallest solution is $\pi_{100}(N,2)\approx 1$ if $N=10^{195}$.

(20) Let $k = 150$. Its smallest solution is $\pi_{150}(N, 2) \approx 2$ if $N = 10^{317}$.

Since $J_2(\omega) \neq 0$, there exist infinitely many 150-tuples of isoprimes.

Theorem 6.4 We deal with the Santilli's isoprime 3-chain of the second kind of any length k. From (6.10) we have

$$\hat{p}_{j+1} = \hat{3}\hat{\times}\hat{p}_j - \hat{2} \tag{6.20}$$

for $j = 1, \cdots, k-1$.

Let $\hat{I} = 1$. From (6.20) we have

$$p_{j+1} = 3p_j - 2. \tag{6.21}$$

From (6.12) we have

$$J_2(\omega) = \prod_{3 \le p \le p_i} (p - \chi(p)) \neq 0. \tag{6.22}$$

Now we calculate $\chi(p)$. We define the smallest positive integer S such that

$$3^S \equiv 1(\text{mod } p). \tag{6.23}$$

We obtain

Table 6.2

p	5	7	11	13	17	19	23	29	31	37	41	43	47	53	59	61	67	71
s	4	6	5	3	16	18	11	28	30	18	8	42	23	52	29	10	22	35

p	73	79	83	89	97	101	103	107	109	113	127	131	137	139	149	151
s	12	78	41	88	48	100	34	53	54	112	126	65	136	138	148	150

$\chi(p) = k$ if $k < S; \chi(p) = S$ if $k \geq S, \chi(3) = 1$. From the Euler-Fermat theorem we have

$$S \leq \phi(p) = p - 1. \tag{6.24}$$

From (6.22) we have

$$J_2(p) \geq 1. \tag{6.25}$$

Then we have $J_2(\omega) \neq 0$. There exist infinitely many k-tuples of isoprimes for any length k.

We have

$$\pi_k(N, 2) \approx \frac{J_2(\omega)\omega^{k-1}}{\phi^k(\omega)} \frac{N}{\log^k N} \tag{6.26}$$

To further understand Theorem 6.4 we yield the more detailed proofs below. (1) Let $k = 2$. We have

$$J_2(\omega) = 2 \prod_{5 \le p \le p_i} (p - 2) \neq 0.$$

Since $J_2(\omega) \neq 0$, there exist infinitely many isoprimes $\hat{p}_1$ such that $\hat{p}_2$ is also an isoprime.

We have

$$\pi_2(N,2) \approx 3 \prod_{5\leq p\leq p_i} \frac{p(p-2)}{(p-1)^2}) \frac{N}{\log^2 N}.$$

(2) Let $k=3$. We have

$$J_2(\omega) = 2 \prod_{5\leq p\leq p_i} (p-3) \neq 0.$$

Since $J_2(\omega) \neq 0$, there exist infinitely many 3-tuples of isoprimes.

We have

$$\pi_3(N,2) \approx 9 \prod_{5\leq p\leq p_i} \frac{p^2(p-3)}{(p-1)^3}) \frac{N}{\log^3 N}.$$

(3) Let $k=4$. We have

$$J_2(\omega) = 2 \prod_{5\leq p\leq p_i} (p-\chi(p)) \neq 0,$$

where $\chi(13)=2, \chi(p)=4$ otherwise.

Since $J_2(\omega) \neq 0$, there exist infinitely many 4-tuples of isoprimes.

We have

$$\pi_4(N,2) \approx 27 \prod_{5\leq p\leq p_i} \frac{p^3(p-\chi(p))}{(p-1)^4} \frac{N}{\log^4 N}.$$

(4) Let $k=5$. We have

$$J_2(\omega) = 2 \prod_{7\leq p\leq p_i} (p-\chi(p)) \neq 0,$$

where $\chi(13)=2$, $\chi(p)=5$ otherwise.

We have

$$\pi_5(N,2) \approx \frac{1}{4}(\frac{15}{4})^4 \prod_{7\leq p\leq p_i} \frac{p^4(p-\chi(p))}{(p-1)^5} \frac{N}{\log^5 N}.$$

(5) Let $k=6$. We have

$$J_2(\omega) = 2 \prod_{7\leq p\leq p_i} (p-\chi(p)) \neq 0.$$

where $\chi(p)=3,5$ if $p=13,11$; $\chi(p)=6$ otherwise.

Since $J_2(\omega) \neq 0$, there exist infinitely many 6-tuples of isoprimes.

We have

$$\pi_6(N,2) \approx \frac{1}{4}(\frac{15}{4})^5 \prod_{7\leq p\leq p_i} \frac{p^5(p-\chi(p))}{(p-1)^6} \frac{N}{\log^6 N}.$$

(6) Let $k=7$. We have

$$J_2(\omega)=2\prod_{11\le p\le p_i}(p-\chi(p))\neq 0.$$

where $\chi(p)=3,5$, if $p=13,11$; $\chi(p)=7$ otherwise.

We have

$$\pi_7(N,2)\approx\frac{1}{24}(\frac{35}{8})^6\prod_{11\le p\le p_i}\frac{p^6(p-\chi(p))}{(p-1)^7}\frac{N}{\log^7 N}.$$

(7) Let $k=8$. We have

$$J_2(\omega)=2\prod_{11\le p\le p_i}(p-\chi(p))\neq 0.$$

where $\chi(p)=3,5,7$ if $p=13,11,1093$; $\chi(p)=8$ otherwise.

We have

$$\pi_8(N,2)\approx\frac{1}{24}(\frac{35}{8})^7\prod_{11\le p\le p_i}\frac{p^7(p-\chi(p))}{(p-1)^8}\frac{N}{\log^8 N}.$$

(8) Let $k=9$. We have

$$J_2(\omega)=2\prod_{11\le p\le p_i}(p-\chi(p))\neq 0.$$

where $\chi(p)=3,5,7,8$. If $p=13,11,1093,41$; $\chi(p)=9$ otherwise.

We have

$$\pi_9(N,2)\approx\frac{1}{24}(\frac{35}{8})^8\prod_{11\le p\le p_i}\frac{p^8(p-\chi(p))}{(p-1)^9}\frac{N}{\log^9 N}.$$

(9) Let $k=10$. We have

$$J_2(\omega)=2\prod_{11\le p\le p_i}(p-\chi(p))\neq 0.$$

where $\chi(p)=3,5,7,8,9$. If $p=13,11,1093,41,757$; $\chi(p)=10$ otherwise.

We have

$$\pi_{10}(N,2)\approx\frac{1}{24}(\frac{35}{8})^9\prod_{11\le p\le p_i}\frac{p^9(p-\chi(p))}{(p-1)^{10}}\frac{N}{\log^{10} N}.$$

(10) Let $k=11$. We have

$$J_2(\omega)=120\prod_{17\le p\le p_i}(p-\chi(p))\neq 0.$$

where $\chi(p)=7,8,9,10$. If $p=1093,41,757,61$; $\chi(p)=11$ otherwise.

We have

$$\pi_{11}(N,2) \approx \frac{1}{48}(\frac{1001}{192})^{10} \prod_{17 \le p \le p_i} \frac{p^{10}(p-\chi(p))}{(p-1)^{11}} \frac{N}{\log^{11} N}.$$

(11) Let $k = 20$. Its smallest solution is $\pi_{20}(N,2) \approx 1$ if $N = 10^{25}$.
(12) Let $k = 30$. Its smallest solution is $\pi_{30}(N,2) \approx 1$ if $N = 10^{44}$.
(13) Let $k = 50$. Its smallest solution is $\pi_{50}(N,2) \approx 1$ if $N = 10^{82}$.
(14) Let $k = 100$. Its smallest solution is $\pi_{100}(N,2) \approx 1$ if $N = 10^{194}$.
(15) Let $k = 150$. Its smallest solution is $\pi_{150}(N,2) \approx 1$ if $N = 10^{321}$.
Since $J_2(\omega) \neq 0$, there exist infinitely many 150-tuples of isoprimes, which show the novel properties for sufficiently large isoprimes.

Theorem 6.5. We deal with the Santilli's isoprime 4-chain of the first kind of any length k. From (6.1) we have

$$\hat{p}_{j+1} = \hat{4}\hat{\times}\hat{p}_j + 3 \tag{6.27}$$

for $j = 1, \cdots, k-1$.
Let $\hat{I} = 1$. From (6.27) we have

$$p_{j+1} = 4p_j + 3 \tag{6.28}$$

for $j = 1, \cdots, k-1$.
From (6.3) we have

$$J_2(\omega) = \prod_{3 \le p \le p_i} (p - \chi(p)) \neq 0 \tag{6.29}$$

Now we calculate $\chi(p)$. We define the smallest positive integer S such that

$$4^S \equiv 1(\text{modp}). \tag{6.30}$$

We obtain

Table 6.3

p	5	7	11	13	17	19	23	29	31	37	41	43	47	53	59	61	67	71
s	2	3	5	6	4	9	11	14	5	18	10	7	23	26	29	30	33	35

p	73	79	83	89	97	101	103	107	109	113	127	131	137	139	149
s	9	39	41	11	24	50	51	53	18	14	7	65	34	69	74

$\chi(p) = k$ if $k < S$; $\chi(p) = S$ if $k \geq S, \chi(3) = 1$. From the Euler-Fermat theorem we have

$$S \le \frac{\phi(p)}{2} = \frac{p-1}{2}. \tag{6.31}$$

From (6.29) we have

$$J_2(p) \geq \frac{p+1}{2}. \tag{6.32}$$

Then we have $J_2(\omega) \neq 0$. There exist infinitely many k-tuples of isoprimes for any length k.

We have

$$\pi_k(N,2) \approx \frac{J_2(\omega)\omega^{k-1}}{\phi^k(\omega)} \frac{N}{\log^k N} \tag{6.33}$$

(1) Let $k = 2$. We have

$$J_2(\omega) = 2 \prod_{5 \le p \le p_i} (p-2) \neq 0.$$

Since $J_2(\omega) \neq 0$, there exist infinitely many isoprimes $\hat{p}_1$ such that $\hat{p}_2$ is also an isoprime.

We have

$$\pi_2(N,2) \approx 3 \prod_{5 \le p \le p_i} \frac{p(p-2)}{(p-1)^2} \frac{N}{\log^2 N}.$$

(2) Let $k = 3$. We have

$$J_2(\omega) = 6 \prod_{7 \le p \le p_i} (p-3) \neq 0.$$

Since $J_2(\omega) \neq 0$, there exist infinitely many 3-tuples of isoprimes.

We have

$$\pi_3(N,2) \approx \frac{3}{4}\left(\frac{15}{4}\right)^2 \prod_{7 \le p \le p_i} \frac{p^2(p-3)}{(p-1)^3} \frac{N}{\log^3 N}.$$

(3) Let $k = 4$. We have

$$J_2(\omega) = 6 \prod_{7 \le p \le p_i} (p-\chi(p)) \neq 0,$$

where $\chi(7) = 3, \chi(p) = 4$ otherwise.

Since $J_2(\omega) \neq 0$, there exist infinitely many 4-tuples of isoprimes.

We have

$$\pi_4(N,2) \approx \frac{3}{4}\left(\frac{15}{4}\right)^3 \prod_{7 \le p \le p_i} \frac{p^3(p-\chi(p))}{(p-1)^4} \frac{N}{\log^4 N}.$$

(4) Let $k = 5$. We have

$$J_2(\omega) = 6 \prod_{7 \le p \le p_i} (p-\chi(p)) \neq 0,$$

where $\chi(p) = 3, 4$ if $p = 7, 17$; $\chi(p) = 5$ otherwise.

Since $J_2(\omega) \neq 0$, there exist infinitely many 5-tuples of isoprimes.

We have

$$\pi_5(N,2) \approx \frac{3}{4}\left(\frac{15}{4}\right)^4 \prod_{7\le p\le p_i} \frac{p^4(p-\chi(p))}{(p-1)^5}\frac{N}{\log^5 N}.$$

(5) Let $k=6$. We have

$$J_2(\omega) = 24 \prod_{11\le p\le p_i} (p-\chi(p)) \neq 0.$$

where $\chi(p) = 4,5,5$ if $p = 17,11,31$; $\chi(p) = 6$ otherwise.

Since $J_2(\omega) \neq 0$, there exist infinitely many 6-tuples of isoprimes.

We have

$$\pi_6(N,2) \approx \frac{1}{2}\left(\frac{35}{8}\right)^5 \prod_{11\le p\le p_i} \frac{p^5(p-\chi(p))}{(p-1)^6}\frac{N}{\log^6 N}.$$

(6) Let $k=7$. We have

$$J_2(\omega) = 24 \prod_{11\le p\le p_i} (p-\chi(p)) \neq 0.$$

where $\chi(p) = 4,5,5,6$ if $p = 17,11,31,13$; $\chi(p) = 7$ otherwise.

We have

$$\pi_7(N,2) \approx \frac{1}{2}\left(\frac{35}{8}\right)^6 \prod_{11\le p\le p_i} \frac{p^6(p-\chi(p))}{(p-1)^7}\frac{N}{\log^7 N}.$$

(7) Let $k=8$. We have

$$J_2(\omega) = 24 \prod_{11\le p\le p_i} (p-\chi(p)) \neq 0.$$

where $\chi(p) = 4,5,5,6,7,7$ if $p = 17,11,31,13,43,127$; $\chi(p) = 8$ otherwise.

We have

$$\pi_8(N,2) \approx \frac{1}{2}\left(\frac{35}{8}\right)^7 \prod_{11\le p\le p_i} \frac{p^7(p-\chi(p))}{(p-1)^8}\frac{N}{\log^8 N}.$$

(8) Let $k=9$. We have

$$J_2(\omega) = 24 \prod_{11\le p\le p_i} (p-\chi(p)) \neq 0.$$

where $\chi(p) = 4,5,5,6,7,7,8$. If $p = 17,11,31,13,43,127,257$; $\chi(p) = 9$ otherwise.

We have

$$\pi_9(N,2) \approx \frac{1}{2}\left(\frac{35}{8}\right)^8 \prod_{11\le p\le p_i} \frac{p^8(p-\chi(p))}{(p-1)^9}\frac{N}{\log^9 N}.$$

(9) Let $k = 10$. We have

$$J_2(\omega) = 24 \prod_{11 \le p \le p_i} (p - \chi(p)) \ne 0.$$

where $\chi(p) = 4, 5, 5, 6, 7, 7, 8, 9, 9$. If $p = 17, 11, 31, 13, 43, 127, 257, 19, 73$; $\chi(p) = 10$ otherwise.

We have

$$\pi_{10}(N, 2) \approx \frac{1}{2}\left(\frac{35}{8}\right)^9 \prod_{11 \le p \le p_i} \frac{p^9(p - \chi(p))}{(p-1)^{10}} \frac{N}{\log^{10} N}.$$

(10) Let $k = 20$. Its smallest solution is $\pi_{20}(N, 2) \approx 4$ if $N = 10^{20}$.
(11) Let $k = 30$. Its smallest solution is $\pi_{30}(N, 2) \approx 1$ if $N = 10^{35}$.
(12) Let $k = 40$. Its smallest solution is $\pi_{40}(N, 2) \approx 1$ if $N = 10^{52}$.
(13) Let $k = 50$. Its smallest solution is $\pi_{50}(N, 2) \approx 1$ if $N = 10^{70}$.
(14) Let $k = 100$. Its smallest solution is $\pi_{100}(N, 2) \approx 2$ if $N = 10^{172}$.
(15) Let $k = 150$. Its smallest solution is $\pi_{150}(N, 2) \approx 3$ if $N = 10^{287}$.

Since $J_2(\omega) \ne 0$, there exist infinitely many 150-tuples of isoprimes.
Note. The number of solutions of 4-chains is more than 2-chains.

Theorem 6.6 We deal with the Santilli's isoprime 5-chain of the first kind of length k. From (6.1) we have

$$\hat{p}_{j+1} = \hat{5}\hat{\times}\hat{p}_j + \hat{4} \tag{6.34}$$

for $j = 1, \cdots, k-1$.

Let $\hat{I} = 1$. From (6.34) we have

$$p_{j+1} = 5p_j + 4 \tag{6.35}$$

for $j = 1, \cdots, k-1$.

From (6.3) we have

$$J_2(\omega) = \prod_{3 \le p \le p_i} (p - \chi(p)) \ne 0 \tag{6.36}$$

Now we calculate $\chi(p)$. We define the smallest positive integer S such that

$$5^S \equiv 1 (\text{mod p}). \tag{6.37}$$

We obtain

Table 6.4

p	3	7	11	13	17	19	23	29	31	37	41	43	47	53	59	61	67	71
s	2	6	5	4	16	9	22	14	3	36	20	42	46	52	29	30	22	5

p	73	79	83	89	97	101	103	107	109	113	127	131	137	139	149
s	72	39	82	44	96	25	102	106	54	112	42	65	136	69	37

$\chi(p) = k$ if $k < S; \chi(p) = S$ if $k \geq S, \chi(5) = 1$. From the Euler-Fermat theorem we have

$$S \leq \phi(p) = p - 1. \tag{6.38}$$

From (6.38) we have

$$J_2(p) \geq 1. \tag{6.39}$$

Then we have $J_2(\omega) \neq 0$. There exist infinitely many k-tuples of isoprimes for any length k.

We have

$$\pi_k(N, 2) \approx \frac{J_2(\omega)\omega^{k-1}}{\phi^k(\omega)} \frac{N}{\log^k N} \tag{6.40}$$

(1) Let $k = 2$. We have

$$J_2(\omega) = 4 \prod_{7 \leq p \leq p_i} (p - 2) \neq 0.$$

Since $J_2(\omega) \neq 0$, there exist infinitely many isoprimes $\hat{p}_1$ such that $\hat{p}_2$ is also an isoprime.

We have

$$\pi_2(N, 2) \approx \frac{15}{8} \prod_{7 \leq p \leq p_i} \frac{p(p-2)}{(p-1)^2} \frac{N}{\log^2 N}.$$

(2) Let $k = 3$. We have

$$J_2(\omega) = 4 \prod_{7 \leq p \leq p_i} (p - 3) \neq 0.$$

Since $J_2(\omega) \neq 0$, there exist infinitely many 3-tuples of isoprimes.

We have

$$\pi_3(N, 2) \approx \frac{1}{2}(\frac{15}{4})^2 \prod_{7 \leq p \leq p_i} \frac{p^2(p-3)}{(p-1)^3} \frac{N}{\log^3 N}.$$

(3) Let $k = 4$. We have

$$J_2(\omega) = 4 \prod_{7 \leq p \leq p_i} (p - \chi(p)) \neq 0,$$

where $\chi(31) = 3, \chi(p) = 4$ otherwise.

Since $J_2(\omega) \neq 0$, there exist infinitely many 4-tuples of isoprimes.

We have

$$\pi_4(N,2) \approx \frac{1}{2}\left(\frac{15}{4}\right)^3 \prod_{7\le p\le p_i} \frac{p^3(p-\chi(p))}{(p-1)^4}\frac{N}{\log^4 N}.$$

(4) Let $k=5$. We have

$$J_2(\omega) = 4\prod_{7\le p\le p_i}(p-\chi(p)) \neq 0,$$

where $\chi(p)=3,4$ if $p=31,13;\chi(p)=5$ otherwise.

Since $J_2(\omega) \neq 0$, there exist infinitely many 5-tuples of isoprimes.

We have

$$\pi_5(N,2) \approx \frac{1}{2}\left(\frac{15}{4}\right)^4 \prod_{7\le p\le p_i} \frac{p^4(p-\chi(p))}{(p-1)^5}\frac{N}{\log^5 N}.$$

(5) Let $k=6$. We have

$$J_2(\omega) = 4\prod_{7\le p\le p_i}(p-\chi(p)) \neq 0.$$

where $\chi(p)=3,4,5,5$ if $p=31,13,11,71$; $\chi(p)=6$ otherwise.

We have

$$\pi_6(N,2) \approx \frac{1}{2}\left(\frac{15}{4}\right)^5 \prod_{7\le p\le p_i} \frac{p^5(p-\chi(p))}{(p-1)^6}\frac{N}{\log^6 N}.$$

(6) Let $k=7$. We have

$$J_2(\omega) = 4\prod_{11\le p\le p_i}(p-\chi(p)) \neq 0.$$

where $\chi(p)=3,4,5,5$ if $p=31,13,11,71$; $\chi(p)=7$ otherwise.

We have

$$\pi_7(N,2) \approx \frac{1}{12}\left(\frac{35}{8}\right)^6 \prod_{11\le p\le p_i} \frac{p^6(p-\chi(p))}{(p-1)^7}\frac{N}{\log^7 N}.$$

(7) Let $k=8$. We have

$$J_2(\omega) = 4\prod_{11\le p\le p_i}(p-\chi(p)) \neq 0.$$

where $\chi(p)=3,4,5,5,7$ if $p=31,13,11,71,19531$; $\chi(p)=8$ otherwise.

We have

$$\pi_8(N,2) \approx \frac{1}{2}\left(\frac{35}{8}\right)^7 \prod_{11\le p\le p_i} \frac{p^7(p-\chi(p))}{(p-1)^8}\frac{N}{\log^8 N}.$$

(8) Let $k = 20$. Its smallest solution is $\pi_{20}(N, 2) \approx 1$ if $N = 10^{24}$.
(9) Let $k = 30$. Its smallest solution is $\pi_{30}(N, 2) \approx 3$ if $N = 10^{42}$.
(10) Let $k = 50$. Its smallest solution is $\pi_{50}(N, 2) \approx 3$ if $N = 10^{83}$.
(11) Let $k = 100$. Its smallest solution is $\pi_{100}(N, 2) \approx 1$ if $N = 10^{194}$.
(12) Let $k = 150$. Its smallest solution is $\pi_{150}(N, 2) \approx 1$ if $N = 10^{321}$.

Theorem 6.7 We deal with the Santilli's isoprime 6-chain of the first kind of length k. From (6.1) we have

$$\hat{p}_{j+1} = \hat{6}\hat{\times}\hat{p}_j + \hat{5} \tag{6.41}$$

for $j = 1, \cdots, k-1$.

Let $\hat{I} = 1$. From (6.41) we have

$$p_{j+1} = 6p_j + 5 \tag{6.42}$$

for $j = 1, \cdots, k-1$.

From (6.3) we have

$$J_2(\omega) = \prod_{3 \le p \le p_i} (p - \chi(p)) \neq 0 \tag{6.43}$$

Now we calculate $\chi(p)$. We define the smallest positive integer S such that

$$6^S \equiv 1(\text{modp}). \tag{6.44}$$

We obtain

Table 6.5

p	7	11	13	17	19	23	29	31	37	41	43	47	53	59	61	67	71	73
s	2	10	12	16	9	11	14	6	4	40	3	23	26	58	60	33	35	24

p	79	83	89	97	101	103	107	109	113	127	131	137	139	149	151
s	78	82	88	12	10	102	106	108	112	126	130	136	23	37	150

$\chi(p) = k$ if $k < S; \chi(p) = S$ if $k \geq S, \chi(3) = \chi(5) = 1$. From the Euler-Fermat theorem we have

$$S \leq \phi(p) = p - 1 \tag{6.45}$$

From (6.45) we have

$$J_2(p) \geq 1. \tag{6.46}$$

Then we have $J_2(\omega) \neq 0$. There exist infinitely many k-tuples of isoprimes for any length k.

We have

$$\pi_k(N, 2) \approx \frac{J_2(\omega)\omega^{k-1}}{\phi^k(\omega)} \frac{N}{\log^k N} \tag{6.47}$$

(1) Let $k = 2$. We have

$$J_2(\omega) = 8 \prod_{7 \le p \le p_i} (p-2) \neq 0.$$

Since $J_2(\omega) \neq 0$, there exist infinitely many isoprimes $\hat{p}_1$ such that $\hat{p}_2$ is also an isoprime.

We have

$$\pi_2(N,2) \approx \frac{15}{4} \prod_{7 \le p \le p_i} \frac{p(p-2)}{(p-1)^2} \frac{N}{\log^2 N}.$$

(2) Let $k = 3$. We have

$$J_2(\omega) = 8 \prod_{7 \le p \le p_i} (p - \chi(p)) \neq 0,$$

where $\chi(7) = 2, \chi(p) = 3$ otherwise.

We have

$$\pi_3(N,2) \approx \left(\frac{15}{4}\right)^2 \prod_{7 \le p \le p_i} \frac{p^2(p-\chi(p))}{(p-1)^3} \frac{N}{\log^3 N}.$$

(3) Let $k = 4$. We have

$$J_2(\omega) = 8 \prod_{7 \le p \le p_i} (p - \chi(p)) \neq 0,$$

where $\chi(7) = 2, 3$, if $p = 7, 43$; $\chi(p) = 4$ otherwise.

We have

$$\pi_4(N,2) \approx \left(\frac{15}{4}\right)^3 \prod_{7 \le p \le p_i} \frac{p^3(p-\chi(p))}{(p-1)^4} \frac{N}{\log^4 N}.$$

(4) Let $k = 5$. We have

$$J_2(\omega) = 8 \prod_{7 \le p \le p_i} (p - \chi(p)) \neq 0,$$

where $\chi(p) = 2, 3, 4$ if $p = 7, 42, 37; \chi(p) = 5$ otherwise.

We have

$$\pi_5(N,2) \approx \left(\frac{15}{4}\right)^4 \prod_{7 \le p \le p_i} \frac{p^4(p-\chi(p))}{(p-1)^5} \frac{N}{\log^5 N}.$$

(5) Let $k = 6$. We have

$$J_2(\omega) = 8 \prod_{7 \le p \le p_i} (p - \chi(p)) \neq 0,$$

where $\chi(p) = 2, 3, 4, 5$ if $p = 7, 43, 37, 311$; $\chi(p) = 6$ otherwise.

We have

$$\pi_6(N,2) \approx \left(\frac{15}{4}\right)^5 \prod_{7\le p\le p_i} \frac{p^5(p-\chi(p))}{(p-1)^6} \frac{N}{\log^6 N}.$$

(6) Let $k=7$. We have

$$J_2(\omega) = 8 \prod_{7\le p\le p_i} (p-\chi(p)) \neq 0,$$

where $\chi(p)=2,3,4,5,6$ if $p=7,43,37,311,31$; $\chi(p)=7$ otherwise.

We have

$$\pi_7(N,2) \approx \left(\frac{15}{4}\right)^6 \prod_{7\le p\le p_i} \frac{p^6(p-\chi(p))}{(p-1)^7} \frac{N}{\log^7 N}.$$

(7) Let $k=8$. We have

$$J_2(\omega) = 8 \prod_{7\le p\le p_i} (p-\chi(p)) \neq 0,$$

where $\chi(p)=2,3,4,5,6,7$ if $p=7,43,37,311,31,55987$; $\chi(p)=8$ otherwise.

We have

$$\pi_8(N,2) \approx \left(\frac{15}{4}\right)^7 \prod_{7\le p\le p_i} \frac{p^7(p-\chi(p))}{(p-1)^8} \frac{N}{\log^8 N}.$$

(8) Let $k=20$. Its smallest solution is $\pi_{20}(N,2)\approx 1$ if $N=10^{23}$.
(9) Let $k=30$. Its smallest solution is $\pi_{30}(N,2)\approx 2$ if $N=10^{40}$.
(10) Let $k=50$. Its smallest solution is $\pi_{50}(N,2)\approx 2$ if $N=10^{78}$.
(11) Let $k=100$. Its smallest solution is $\pi_{100}(N,2)\approx 1$ if $N=10^{194}$.
(12) Let $k=150$. Its smallest solution is $\pi_{150}(N,2)\approx 1$ if $N=10^{321}$.

Theorem 6.8 We deal with the Santilli's isoprime 16-chain of the first kind of any length k. From (6.1) we have

$$\hat{p}_{j+1} = \hat{1}6 \hat{\times} \hat{p}_j + \hat{1}5 \tag{6.48}$$

for $j=1,\cdots,k-1$.

Let $\hat{I}=1$. From (6.48) we have

$$p_{j+1} = 16p_j + 15 \tag{6.49}$$

for $j=1,\cdots,k-1$.

From (6.3) we have

$$J_2(\omega) = \prod_{3\le p\le p_i} (p-\chi(p)) \neq 0 \tag{6.50}$$

Now we calculate $\chi(p)$. We define the smallest positive integer S such that

$$16^S \equiv 1 (\text{mod } p). \tag{6.51}$$

We obtain

Table 6.6

p	7	11	13	17	19	23	29	31	37	41	43	47	53	59	61	67	71	73	79
s	3	5	3	2	9	11	7	5	9	5	7	23	13	29	15	33	35	9	39

p	83	89	97	101	103	107	109	113	127	131	137	139	149	151
s	41	11	12	25	51	53	9	7	7	65	17	69	37	15

$\chi(p) = k$ if $k < S; \chi(p) = S$ if $k \geq S, \chi(3) = \chi(5) = 1$. From the Euler-Fermat theorem we have

$$S \leq \frac{\phi(p)}{2} = \frac{p-1}{2} \text{ and } S \leq \frac{\phi(p)}{4} = \frac{p-1}{4}, \text{ if } 4|(p-1). \tag{6.52}$$

From (6.50) we have

$$J_2(p) \geq \frac{p+1}{2} \text{ and } J_2(p) \geq \frac{3p+1}{4}, \text{ if } 4|(p-1). \tag{6.53}$$

We have $J_2(\omega) \neq 0$. There exist infinitely many k-tuples of isoprimes for any length k.

We have

$$\pi_k(N, 2) \approx \frac{J_2(\omega)\omega^{k-1}}{\phi^k(\omega)} \frac{N}{\log^k N} \tag{6.54}$$

To further understand theorem 6.8 we give the detailed proofs below.

(1) Let $k = 2$. We have

$$J_2(\omega) = 8 \prod_{7 \leq p \leq p_i} (p-2) \neq 0.$$

We have

$$\pi_2(N, 2) \approx \frac{15}{4} \prod_{7 \leq p \leq p_i} \frac{p(p-2)}{(p-1)^2} \frac{N}{\log^2 N}.$$

(2) Let $k = 3$. We have

$$J_2(\omega) = 8 \prod_{7 \leq p \leq p_i} (p - \chi(p)) \neq 0,$$

where $\chi(17) = 2$, $\chi(p) = 3$ otherwise.

We have

$$\pi_3(N,2) \approx \left(\frac{15}{4}\right)^2 \prod_{7\le p\le p_i} \frac{p^2(p-\chi(p))}{(p-1)^3} \frac{N}{\log^3 N}.$$

(3) Let $k=4$. We have

$$J_2(\omega) = 8 \prod_{7\le p\le p_i} (p-\chi(p)) \neq 0,$$

where $\chi(p)=2,3,3$ if $p=17,7,13$; $\chi(p)=4$ otherwise.

We have

$$\pi_4(N,2) \approx \left(\frac{15}{4}\right)^3 \prod_{7\le p\le p_i} \frac{p^3(p-\chi(p))}{(p-1)^4} \frac{N}{\log^4 N}.$$

(4) Let $k=5$. We have

$$J_2(\omega) = 8 \prod_{7\le p\le p_i} (p-\chi(p)) \neq 0,$$

where $\chi(p)=2,3,3$ if $p=17,7,13; \chi(p)=5$ otherwise.

We have

$$\pi_5(N,2) \approx \left(\frac{15}{4}\right)^4 \prod_{7\le p\le p_i} \frac{p^4(p-\chi(p))}{(p-1)^5} \frac{N}{\log^5 N}.$$

(5) Let $k=6$. We have

$$J_2(\omega) = 8 \prod_{7\le p\le p_i} (p-\chi(p)) \neq 0,$$

where $\chi(p)=2,3,3,5,5,5$ if $p=17,7,13,11,31,41$; $\chi(p)=6$ otherwise.

We have

$$\pi_6(N,2) \approx \left(\frac{15}{4}\right)^5 \prod_{7\le p\le p_i} \frac{p^5(p-\chi(p))}{(p-1)^6} \frac{N}{\log^6 N}.$$

(6) Let $k=20$. Its smallest solution is $\pi_{20}(N,2)\approx 2$ if $N=10^{18}$.
(7) Let $k=30$. Its smallest solution is $\pi_{30}(N,2)\approx 3$ if $N=10^{34}$.
(8) Let $k=40$. Its smallest solution is $\pi_{40}(N,2)\approx 2$ if $N=10^{50}$.
(9) Let $k=50$. Its smallest solution is $\pi_{50}(N,2)\approx 3$ if $N=10^{68}$.
(10) Let $k=100$. Its smallest solution is $\pi_{100}(N,2)\approx 7$ if $N=10^{170}$.
(11) Let $k=150$. Its smallest solution is $\pi_{150}(N,2)\approx 1$ if $N=10^{284}$.
Since $J_2(\omega)\neq 0$, there exist infinitely many 150-tuples of isoprimes, which show the novel properties for sufficiently large isoprimes

Note. The number of solutions of 16-chains is more than 4-chains.

7. Santilli's Chains of Isoprimes in Arithmetic Progression

Theorem 7.1 Let

$$\omega_g = \prod_{2 \le p \le p_g} p. \tag{7.1}$$

We have the Santilli's chains of isoprimes in arithmetic progression

$$\hat{p}_j = \hat{p} + \hat{\omega}_g \hat{\times} \hat{j} \tag{7.2}$$

for $j = 1, \cdots, k-1$.

If $\hat{I} = 1$, from (7.2) we have

$$p_j = p + \omega_g j \tag{7.3}$$

for $j = 1, \cdots, k-1$.

We have the arithmetic function

$$J_2(\omega) = \prod_{3 \le p \le p_g} (p-1) \prod_{p_{g+1} \le p \le p_i} (p-k). \tag{7.4}$$

$J_2(p_{g+1}) = 0$ if $p_{g+1} = k$; $J_2(\omega) \neq 0$ if $p_{g+1} > k$, there exist infinitely many isoprimes $\hat{p}$ such that $\hat{p}_j$ are all isoprimes for any length k.

We have the best asymptotic formula of the number of k-tuples of isoprimes

$$\pi_k(N,2) = |\{p : p \le N, p + \omega_g j = \text{ prime for } j = 1, \cdots, k-1\}| \approx$$

$$\prod_{2 \le p \le p_g} \left(\frac{p}{p-1}\right)^{k-1} \prod_{p_{g+1} \le p \le p_i} \frac{p^{k-1}(p-k)}{(p-1)^k} \frac{N}{\log^k N}. \tag{7.5}$$

Let

$$\omega_g = p_{g+2} p_{g+3} \prod_{2 \le p \le p_g} (p)^m. \tag{7.6}$$

We have the Santilli's chains of isoprimes in arithmetic progression

$$\hat{p}_j = \hat{p} + \hat{\omega}_g \hat{\times} \hat{j} \tag{7.7}$$

for $j = 1, \cdots, k-1$.

We have the arithmetic function

$$J_2(\omega) = \frac{p_{g+2}-1}{p_{g+2}-k} \frac{p_{g+3}-1}{p_{g+3}-k} \prod_{3 \le p \le P_g} (p-1) \prod_{p_{g+1} \le p \le p_i} (p-k) \neq 0 \tag{7.8}$$

We have the best asymptotic formula of the number of k-tuples of isoprimes

$$\pi_k(N,2) \approx \frac{p_{g+2}-1}{p_{g+2}-k} \frac{p_{g+3}-1}{p_{g+3}-k} \prod_{2 \le p \le p_g} (\frac{p}{p-1})^{k-1} \prod_{p_{g+1} \le p \le p_i} \frac{p^{k-1}(p-k)}{(p-1)^k} \frac{N}{\log^k N}. \tag{7.9}$$

To understand thr theorem 7.1 we have the detailed proofs below.
(1) Let $p_g = 3$ and $\omega_g = 6$. We have

$$p_j = p + 6j$$

for $j = 1, \cdots, k-1$.
We have

$$J_2(\omega) = 2 \prod_{5 \le p \le p_i} (p - k).$$

$J_2(5) = 0$ if $k = 5$; $J_2(\omega) \neq 0$ if $k \le 4$, there exist infinitely many k-tuples of primes. Let $k = 4$. We have

$$p_1 = p + 6, p_2 = p + 12, p_3 = p + 18.$$

We have

$$\pi_4(N, 2) \approx 27 \prod_{5 \le p \le p_i} \frac{p^3(p-4)}{(p-1)^4} \frac{N}{\log^4 N}.$$

(2) Let $p_g = 5$ and $\omega_g = 30$. We have

$$p_j = p + 30j$$

for $j = 1, \cdots, k-1$.
We have

$$J_2(\omega) = 8 \prod_{7 \le p \le p_i} (p - k).$$

$J_2(7) = 0$ if $k = 7$; $J_2(\omega) \neq 0$ if $k \le 6$, there exist infinitely many k-tuples of primes. Let $k = 6$. We have

$$p_j = p + 30j$$

for $j = 1, \cdots, 5$.
We have

$$\pi_6(N, 2) \approx (\frac{15}{4})^5 \prod_{7 \le p \le p_i} \frac{p^5(p-6)}{(p-1)^6} \frac{N}{\log^6 N}.$$

(3) Let $p_g = 7$ and $\omega_g = 210$. We have

$$p_j = p + 210j$$

for $j = 1, \cdots, k-1$. We have

$$J_2(\omega) = 48 \prod_{11 \le p \le p_i} (p - k).$$

$J_2(11) = 0$ if $k = 11$; $J_2(\omega) \neq 0$ if $k \le 10$, there exist infinitely many k-tuples of primes. Let $k = 10$. We have

$$p_j = p + 210j$$

for $j = 1, \cdots, 9$.

We have

$$\pi_{10}(N,2) \approx (\frac{35}{8})^9 \prod_{11 \le p \le p_i} \frac{p^9(p-10)}{(p-1)^{10}} \frac{N}{\log^{10} N}.$$

(4) Let $p_g = 11$ and $\omega_g = 2310$. We have

$$p_j = p + 2310j$$

for $j = 1, \cdots, k-1$.

We have

$$J_2(\omega) = 480 \prod_{13 \le p \le p_i} (p-k).$$

$J_2(13) = 0$ if $k = 13; J_2(\omega) \neq 0$ if $k \le 12$, there exist infinitely many k-tuples of primes. Let $k = 12$. We have

$$p_j = p + 2310j$$

for $j = 1, \cdots, 11$.

We have

$$\pi_{12}(N,2) \approx (\frac{77}{16})^{11} \prod_{13 \le p \le p_i} \frac{p^{11}(p-12)}{(p-1)^{12}} \frac{N}{\log^{12} N}.$$

(5) Let $p_g = 23$ and $\omega_g = \prod_{2 \le p \le 23} p$. We have

$$p_j = p + \omega_g j$$

for $j = 1, \cdots, k-1$.

We have

$$J_2(\omega) = \prod_{3 \le p \le 23} (p-1) \prod_{29 \le p \le p_i} (p-k).$$

$J_2(29) = 0$ if $k = 29; J_2(\omega) \neq 0$ if $k \le 28$, there exist infinitely many k-tuples of primes. Let $k = 28$. We have

$$p_j = p + \omega_g j$$

for $j = 1, \cdots, 27$.

We have

$$\pi_{28}(N,2) \approx \prod_{2 \le p \le 23} \left(\frac{p}{p-1}\right)^{27} \prod_{29 \le p \le p_i} \frac{p^{27}(p-28)}{(p-1)^{28}} \frac{N}{\log^{28} N}.$$

Note.Theorem 7.1 is the simplest one in our theory. The number of k-tuples of primes is very accurately calculated by (7.5) and (7.9).

(6) Let

$$p_j = p^2 + 6j, j = 1, 2, 3.$$

We have

$$J_2(\omega) = 2 \prod_{5 \le p \le p_i} \left(p - 4 - (\frac{-6}{p}) - (\frac{-3}{p}) - (\frac{-2}{p}) \right).$$

$$\pi_4(N, 2) \approx \frac{J_2(\omega)\omega^3}{8\phi^4(\omega)} \frac{N}{\log^4 N}.$$

(7) Let

$$p_j = p^3 + 6j, j = 1, 2, 3, 4.$$

We have

$$J_2(5) = 0.$$

(8) Let

$$p_j = p^4 + 6j, j = 1, 2, 3, 4.$$

We have

$$J_2(5) = J_2(7) = 0.$$

(9) Let

$$p_j = p^5 + 6j, j = 1, 2, 3, 4.$$

We have

$$J_2(5) = 0.$$

(10) Let

$$p_j = p^6 + 6j, j = 1, 2, 3, 4.$$

We have

$$J_2(7) = 0.$$

(11) Let

$$p_j = p^7 + 6j, j = 1, 2, 3, 4.$$

We have

$$J_2(5) = 0.$$

(12) Let

$$p_j = p^8 + 6j, j = 1, 2, 3, 4.$$

We have

$$J_2(5) = J_2(7) = 0.$$

(13) Let

$$p_j = p^9 + 6j, j = 1, 2, 3, 4.$$

We have

$$J_2(5) = 0.$$

(14) Let

$$p_j = p^{10} + 6j, j = 1, 2, 3, 4.$$

We have

$$J_2(5) = J_2(7) = 0.$$

(15) Let

$$p_j = p^{11} + 6j, j = 1, 2, 3, 4.$$

We have

$$J_2(5) = 0.$$

(16) Let

$$p_j = p^{12} + 6j, j = 1, 2, 3, 4.$$

We have

$$J_2(7) = 0.$$

(17) Let

$$p_j = p^2 + 30j, j = 1, 2, 3, 4, 5.$$

We have

$$J_2(\omega) = 8 \prod_{7 \le p \le p_i} \left(p - 6 - \sum_{j=1}^{5} (\frac{-30j}{p}) \right) \neq 0.$$

$$\pi_6(N, 2) \approx \frac{J_2(\omega)\omega^5}{32\phi^6(\omega} \frac{N}{\log^6 N}.$$

(18) Let

$$p_j = p^2 + 30j, j = 1, \cdots, 6.$$

We have

$$J_2(7) = 0.$$

(19) Let

$$p_j = p^3 + 30j, j = 1, 2, 3, 4.$$

We have

$$J_2(7) = 0.$$

(20) Let

$$p_j = p^4 + 30j, j = 1, \cdots, 6.$$

We have

$$J_2(7) = 0.$$

(21) Let

$$p_j = p^5 + 30j, j = 1, \cdots, 6.$$

We have

$$J_2(7) = 0.$$

(22) Let

$$p_j = p^6 + 30j, j = 1, 2, 3.$$

We have

$$J_2(7) = 0.$$

(23) Let

$$p_j = p^7 + 30j, j = 1, \cdots, 6.$$

We have

$$J_2(7) = 0.$$

(24) Let

$$p_j = p^8 + 30j, j = 1, \cdots, 6.$$

We have

$$J_2(7) = 0.$$

(25) Let

$$p_j = p^9 + 30j, j = 1, 2, 3, 4.$$

.

We have

$$J_2(7) = 0.$$

(26) Let

$$p_j = p^{10} + 30j, j = 1, 2, 3, 4.$$

We have

$$J_2(11) = 0.$$

(27) Let

$$p_j = p^2 + 210j, j = 1, \cdots, 9.$$

We have

$$J_2(\omega) = 48 \prod_{11 \leq p \leq p_i} \left(p - 10 - \sum_{j=1}^{9} \left(\frac{-210j}{p}\right) \right) \neq 0.$$

$$\pi_{10}(N, 2) \approx \frac{J_2(\omega)\omega^9}{512\phi^{10}(\omega)} \frac{N}{\log^{10} N}.$$

(28) Let

$$p_j = p^2 + 210j, j = 1, \cdots, 10.$$

We have

$$J_2(11) = 0.$$

(29) Let

$$p_j = p^3 + 210j, j = 1, \cdots, 9.$$

We have

$$J_2(13) = 0.$$

(30) Let

$$p_j = p^4 + 210j, j = 1, \cdots, 6.$$

We have

$$J_2(13) = 0.$$

(31) Let

$$p_j = p^5 + 210j, j = 1, \cdots, 10.$$

We have

$$J_2(11) = J_2(13) = 0.$$

(32) Let

$$p_j = p^6 + 210j, j = 1, \cdots, 7.$$

We have

$$J_2(13) = 0.$$

(33) Let

$$p_j = p^7 + 210j, j = 1, \cdots, 10.$$

We have

$$J_2(11) = J_2(13) = 0.$$

(34) Let

$$p_j = p^8 + 210j, j = 1, \cdots, 6.$$

We have

$$J_2(13) = 0.$$

(35) Let

$$p_j = p^9 + 210j, j = 1, \cdots, 9.$$

We have

$$J_2(13) = 0.$$

(36) Let

$$p_j = p^{10} + 210j, j = 1, \cdots, 10.$$

We have

$$J_2(11) = J_2(13) = 0.$$

Note. If $J_2(\omega) = 0$, then there are no prime solutions.

8. Chains of Prime-Producing Quadratics

Theorem 8.1. Let

$$p_j = 2^j(p^2+1)-1 \tag{8.1}$$

for $j = 1, \cdots, k-1$.

Since $p_1 = 2p^2+1$ has no prime solutions except $p = 3$ and $p_1 = 19$, (8.1) has no prime solutions.

Theorem 8.2 Let

$$p_j = 2^j(p^2-1)+1 \tag{8.2}$$

for $j = 1, \cdots, k-1$.

We have

$$J_2(\omega) \neq 0. \tag{8.3}$$

Since $J_2(\omega) \neq 0$, there exist infinitely many prime solutions for any k. We have

$$\pi_k(N,2) \approx \frac{J_2(\omega)\omega^{k-1}}{2^{k-1}\phi^k(\omega)}\frac{N}{\log^k N}. \tag{8.4}$$

(1) Let $k = 3$. We have $p_1 = 2p^2 - 1$ and $p_2 = 4p^2 - 3$.

We have

$$J_2(\omega) = \prod_{5\le p\le p_i}\left(p-3-(\frac{2}{p})-(\frac{3}{p})\right).$$

We have

$$\pi_3(N,2) \approx \frac{J_2(\omega)\omega^2}{4\phi^3(\omega)}\frac{N}{\log^3 N}.$$

(2) Let $k = 4$. We have $p_1 = 2p^2-1, p_2 = 4p^2-3, p_3 = 8p^2-7$.

We have

$$J_2(\omega) = 4\prod_{11\le p\le p_i}\left(p-4-(\frac{2}{p})-(\frac{3}{p})-(\frac{14}{p})\right) \neq 0.$$

We have

$$\pi_4(N,2) \approx \frac{J_2(\omega)\omega^3}{8\phi^4(\omega)}\frac{N}{\log^4 N}.$$

(3) Let $k = 5$. We have $p_1 = 2p^2-1, p_2 = 4p^2-3, p_3 = 8p^2-7, p_4 = 16p^2-15$.

We have

$$J_2(\omega) = 4\prod_{11\le p\le p_i}\left(p-5-(\frac{2}{p})-(\frac{3}{p})-(\frac{14}{p})-(\frac{15}{p})\right) \neq 0.$$

We have

$$\pi_5(N,2) \approx \frac{J_2(\omega)\omega^4}{16\phi^5(\omega)}\frac{N}{\log^5 N}.$$

Theorem 8.3 Let

$$p_j = 3^j(p^2+1)-1 \tag{8.5}$$

for $j = 1, \cdots, k-1$.

For $k < 6$ (8.5) there exist infinitely many prime solutions. For $k = 6$ we have $J_2(7) = 0$, therefore (8.5) has no prime solutions.

(1) Let $k = 5$. We have $p_1 = 3p^2+2, p_2 = 9p^2+8, p_3 = 27p^2+26, p_4 = 81p^2+80$.

We have

$$J_2(\omega) = 4\prod_{7\le p\le p_i}\left(p-5-(\frac{-6}{p})-(\frac{-2}{p})-(\frac{-78}{p})-(\frac{-5}{p})\right) \neq 0.$$

Since $J_2(\omega) \neq 0$, there exist infinitely many prime solutions.

We have

$$\pi_5(N,2) \approx \frac{J_2(\omega)\omega^4}{16\phi^5(\omega)}\frac{N}{\log^5 N}.$$

Theorem 8.4. Let

$$p_j = 3^j(p^2-1)+1 \tag{8.6}$$

for $j = 1, \cdots, k-1$.

We have

$$J_2(\omega) \neq 0. \tag{8.7}$$

Since $J_2(\omega) \neq 0$, (8.6) has infinitely many prime solutions for any length k.

We have

$$\pi_k(N,2) \approx \frac{J_2(\omega)\omega^{k-1}}{2^{k-1}\phi^k(\omega)}\frac{N}{\log^k N}. \tag{8.8}$$

Theorem 8.5. Let

$$p_j = 4^j(p^2+1)-1 \tag{8.9}$$

for $j = 1, \cdots, k-1$.

We have

$$J_2(\omega) \neq 0. \tag{8.10}$$

Since $J_2(\omega) \neq 0$, there exist infinitely many prime solutions for any length k.

We have

$$\pi_k(N,2) \approx \frac{J_2(\omega)\omega^{k-1}}{2^{k-1}\phi^k(\omega)}\frac{N}{\log^k N}.$$

(1) Let $k = 3$. We have $p_1 = 4p^2+3, p_2 = 16p^2+15$.

We have

$$J_2(\omega) = 8\prod_{7\le p\le p_i}\left(p-3-(\frac{-3}{p})-(\frac{-15}{p})\right) \neq 0.$$

We have

$$\pi_3(N,2) \approx \frac{J_2(\omega)\omega^2}{4\phi^3(\omega)} \frac{N}{\log^3 N}.$$

Theorem 8.6. Let

$$p_j = 4^j(p^2 - 1) + 1 \tag{8.11}$$

for $j = 1, \cdots, k-1$.

We have

$$J_2(\omega) \neq 0. \tag{8.12}$$

Since $J_2(\omega) \neq 0$, there exist infinitely many prime solutions for any length k. We have

$$\pi_k(N,2) \approx \frac{J_2(\omega)\omega^{k-1}}{2^{k-1}\phi^k(\omega)} \frac{N}{\log^k N}. \tag{8.13}$$

Theorem 8.7. Let

$$p_j = 5^j(p^2 + 1) - 1 \tag{8.14}$$

for $j = 1, \cdots, k-1$.

Since $p_1 = 5p^2 + 4$ has no prime solutions, (8.14) has no prime solutions.

Theorem 8.8. Let

$$p_j = 5^j(p^2 - 1) + 1 \tag{8.15}$$

for $j = 1, \cdots, k-1$.

We have

$$J_2(\omega) \neq 0. \tag{8.16}$$

Since $J_2(\omega) \neq 0$, (8.6) has infinitely many prime solutions for any length k.

We have

$$\pi_k(N,2) \approx \frac{J_2(\omega)\omega^{k-1}}{2^{k-1}\phi^k(\omega)} \frac{N}{\log^k N}. \tag{8.17}$$

Theorem 8.9. Let

$$p_j = 6^j(p^2 + 1) - 1 \tag{8.18}$$

for $j = 1, \cdots, k-1$.

For $k < 10$ (8.18) has infinitely many prime solutions. For $k = 10$ we have $J_2(11) = 0$, therefore (8.18) has no prime solutions.

Theorem 8.10. Let

$$p_j = 6^j(p^2 - 1) + 1 \tag{8.19}$$

for $j = 1, \cdots, k-1$.

We have

$$J_2(\omega) \neq 0. \tag{8.20}$$

We have

$$\pi_k(N,2) \approx \frac{J_2(\omega)\omega^{k-1}}{2^{k-1}\phi^k(\omega)}\frac{N}{\log^k N}. \tag{8.21}$$

Theorem 8.11. Let

$$p_j = m^j(p^2 - 1) + 1 \tag{8.22}$$

for $j = 1, \cdots, k-1$.

We have

$$J_2(\omega) \neq 0. \tag{8.23}$$

Since $J_2(\omega) \neq 0$, there exist infinitely many prime solutions for any k and m.

We have

$$\pi_k(N,2) \approx \frac{J_2(\omega)\omega^{k-1}}{2^{k-1}\phi^k(\omega)}\frac{N}{\log^k N}. \tag{8.24}$$

Theorem 8.12. Let

$$p_j = m^j(p^3 + 1) - 1 \tag{8.25}$$

for $j = 1, \cdots, k-1$.

We have

$$J_2(\omega) \neq 0. \tag{8.26}$$

There exist infinitely many prime solutions for any k and m.

We have

$$\pi_k(N,2) \approx \frac{J_2(\omega)\omega^{k-1}}{3^{k-1}\phi^k(\omega)}\frac{N}{\log^k N}. \tag{8.27}$$

Theorem 8.13. Let

$$p_j = m^j(p^3 - 1) + 1. \tag{8.28}$$

for $j = 1, \cdots, k-1$.

We have

$$J_2(\omega) \neq 0. \tag{8.29}$$

There exist infinitely many prime solutions for any k and m.

We have

$$\pi_k(N,2) \approx \frac{J_2(\omega)\omega^{k-1}}{3^{k-1}\phi^k(\omega)}\frac{N}{\log^k N}. \tag{8.30}$$

Theorem 8.14. Let

$$p_j = m^j(p^4 - 1) + 1. \tag{8.31}$$

for $j = 1, \cdots, k-1$.

We have $J_2(\omega)\neq 0$, there exist infinitely many prime solutions for any k and m. We have

$$\pi_k(N,2)\approx\frac{J_2(\omega)\omega^{k-1}}{4^{k-1}\phi^k(\omega)}\frac{N}{\log^k N}. \tag{8.32}$$

Theorem 8.15. Let

$$p_j=2j(p+1)+1 \tag{8.33}$$

for $j=1,\cdots,k-1$.

We have

$$J_2(\omega)=\prod_{3\le p\le p_i}(p-\chi(p))\neq 0, \tag{8.34}$$

where $\chi(p)=k$ or $k-1$ if $k<p$; $\chi(p)=p-1$ if $k\ge p$.

Since $J_2(\omega)\neq 0$, there exist infinitely many k-tuples of primes for any k.

We have

$$\pi_k(N,2)\approx\frac{J_2(\omega)\omega^{k-1}}{\phi^k(\omega)}\frac{N}{\log^k N}. \tag{8.35}$$

(1) Let $k=3$. We have

$$p_1=2p+3, p_2=4p+5.$$

We have

$$J_2(\omega)=3\prod_{7\le p\le p_i}(p-3)\neq 0.$$

We have

$$\pi_3(N,2)\approx\frac{3}{8}(\frac{15}{4})^2\prod_{7\le p\le p_i}\frac{p^2(p-3)}{(p-1)^3}\frac{N}{\log^3 N}.$$

(2) Let $k=4$. We have

$$p_1=2p+3, p_2=4p+5, p_3=6p+7.$$

We have

$$J_2(\omega)=8\prod_{11\le p\le p_i}(p-4)\neq 0.$$

We have

$$\pi_4(N,2)\approx\frac{1}{6}(\frac{35}{8})^2\prod_{11\le p\le p_i}\frac{p^3(p-4)}{(p-1)^4}\frac{N}{\log^4 N}.$$

(3) Let $k=5$. We have

$$p_j=2j(p+1)+1$$

for $j=1,2,3,4$.

We have

$$J_2(\omega) = 3 \prod_{11 \le p \le p_i} (p-5) \neq 0.$$

We have

$$\pi_5(N,2) \approx \frac{1}{16}(\frac{35}{8})^4 \prod_{11 \le p \le p_i} \frac{p^4(p-5)}{(p-1)^5} \frac{N}{\log^5 N}.$$

(4) Let $k = 13$. We have

$$p_j = 2j(p+1) + 1$$

for $j = 1, \cdots, 12$.

We have

$$J_2(\omega) = \prod_{17 \le p \le p_i} (p-13) \neq 0.$$

We have

$$\pi_{13}(N,2) \approx \frac{1}{5760}(\frac{1001}{192})^{12} \prod_{17 \le p \le p_i} \frac{p^{12}(p-13)}{(p-1)^{13}} \frac{N}{\log^{13} N}.$$

(5) Let $k = 23$. We have

$$p_j = 2j(p+1) + 1$$

for $j = 1, \cdots, 22$.

We have

$$J_2(\omega) = \prod_{29 \le p \le p_i} (p-23) \neq 0.$$

We have

$$\pi_{23}(N,2) \approx \frac{J_2(\omega)\omega^{22}}{\phi^{23}(\omega)} \frac{N}{\log^{23} N}.$$

Theorem 8.16. Let

$$p_1 = 6p + 1, p_2 = 9p + 2.$$

We have

$$J_2(\omega) = 2 \prod_{5 \le p \le p_i} (p-3) \neq 0,$$

Since $J_2(\omega) \neq 0$, there exist infinitely many 3-tuples of primes.

We have

$$\pi_3(N,2) \approx \frac{J_2(\omega)\omega^2}{\phi^3(\omega)} \frac{N}{\log^3 N}.$$

Theorem 8.17. Let

$$p_1 = 10p + 1, p_2 = 15p + 2, p_3 = 20p + 3, p_4 = 25p + 4.$$

We have

$$J_2(\omega) = 4 \prod_{7 \le p \le p_i} (p-5) \neq 0$$

Since $J_2(\omega) \neq 0$, there exist infinitely many 5-tuples of primes.

We have

$$\pi_5(N,2) \approx \frac{J_2(\omega)\omega^4}{\phi^5(\omega)} \frac{N}{\log^5 N}.$$

Theorem 8.18. Let

$$p_j = 7(j+1)p + j.$$

for $j = 1, \cdots, 6$.

We have

$$J_2(\omega) = 6 \prod_{11 \le p \le p_i} (p-7) \neq 0,$$

Since $J_2(\omega) \neq 0$, there exist infinitely many 7-tuples of primes.

We have

$$\pi_7(N,2) \approx \frac{J_2(\omega)\omega^6}{\phi^7(\omega)} \frac{N}{\log^7 N}.$$

Theorem 8.19. Let

$$p_j = 11(j+1)p + j.$$

for $j = 1, \cdots, 10$.

We have

$$J_2(\omega) = 10 \prod_{13 \le p \le p_i} (p-11) \neq 0,$$

Since $J_2(\omega) \neq 0$, there exist infinitely many 11-tuples of primes.

We have

$$\pi_{11}(N,2) \approx \frac{J_2(\omega)\omega^{10}}{\phi^{11}(\omega)} \frac{N}{\log^{11} N}.$$

Theorem 8.20. Let

$$p_j = p_0(j+1)p + j.$$

for $j = 1, \cdots, p_0 - 1$, where p_0 is an odd prime.

We have

$$J_2(\omega) = (p_0 - 1) \prod_{p_0 < p \le p_i} (p - p_0) \neq 0,$$

Since $J_2(\omega) \neq 0$, there exist infinitely many p_0-tuples of primes.

We have

$$\pi_{P_0}(N,2) \approx \frac{J_2(\omega)\omega^{p_0-1}}{\phi^{p_0}(\omega)} \frac{N}{\log^{p_0} N}.$$

Theorem 8.21. Let

$$p_1 = 3p+2, p_2 = 2p+3, p_3 = 5p+2, p_4 = 2p+5.$$

We have

$$J_2(\omega) = 42 \prod_{13 \le p \le p_i} (p-5) \neq 0,$$

Since $J_2(\omega) \neq 0$, there exist infinitely many 5-tuples of primes.

We have

$$\pi_5(N,2) \approx \frac{J_2(\omega)\omega^4}{\phi^5(\omega)} \frac{N}{\log^5 N}.$$

Theorem 8.22. Let

$$p_1 = 3p+2, p_2 = 2p+3, p_3 = 7p+2, p_4 = 2p+7.$$

We have

$$J_2(\omega) = 8 \prod_{11 \le p \le p_i} (p-5) \neq 0,$$

We have

$$\pi_5(N,2) \approx \frac{J_2(\omega)\omega^4}{\phi^5(\omega)} \frac{N}{\log^5 N}.$$

Theorem 8.23. Let

$$p_1 = 3p+2, p_2 = 2p+3, p_3 = 9p+2, p_4 = 2p+9.$$

We have

$$J_2(\omega) = 42 \prod_{13 \le p \le p_i} (p-5) \neq 0,$$

We have

$$\pi_5(N,2) \approx \frac{J_2(\omega)\omega^4}{\phi^5(\omega)} \frac{N}{\log^5 N}.$$

It is the best asymptotic formula.

Theorem 8.24. Let

$$p_1 = 3p+4, p_2 = 4p+3.$$

We have

$$J_2(\omega) = 20 \prod_{11 \le p \le p_i} (p-3) \neq 0,$$

Since $J_2(\omega) \neq 0$, there exist infinitely many 3-tuples of primes.

We have

$$\pi_3(N,2) \approx \frac{J_2(\omega)\omega^2}{\phi^3(\omega)}\frac{N}{\log^3 N}.$$

Let

$$p_1 = 3p^2+4, p_2 = 4p^2+3.$$

We have

$$J_2(\omega) = 32 \prod_{11\le p\le p_i}(p-3-2(\frac{-3}{p})) \neq 0,$$

Since $J_2(\omega) \neq 0$, there exist infinitely many 3-tuples of primes.
We have

$$\pi_3(N,2) \approx \frac{J_2(\omega)\omega^2}{4\phi^3(\omega)}\frac{N}{\log^3 N}.$$

Theorem 8.25. Let

$$p_1 = 5p+6, p_2 = 6p+5.$$

We have

$$J_2(\omega) = 288 \prod_{13\le p\le p_i}(p-3) \neq 0,$$

We have

$$\pi_3(N,2) \approx \frac{J_2(\omega)\omega^2}{\phi^3(\omega)}\frac{N}{\log^3 N}.$$

Let

$$p_1 = 5p^2+6, p_2 = 6p^2+5.$$

We have

$$J_2(\omega) = 384 \prod_{13\le p\le p_i}(p-3-2(\frac{-30}{p})) \neq 0,$$

We have

$$\pi_3(N,2) \approx \frac{J_2(\omega)\omega^2}{4\phi^3(\omega)}\frac{N}{\log^3 N}.$$

Theorem 8.26. Let

$$p_1 = 6p+7, p_2 = 7p+6.$$

We have

$$J_2(\omega) = 2112 \prod_{17\le p\le p_i}(p-3) \neq 0,$$

We have

$$\pi_3(N,2) \approx \frac{J_2(\omega)\omega^2}{\phi^3(\omega)}\frac{N}{\log^3 N}.$$

Let

$$p_1 = 6p^2 + 7, p_2 = 7p^2 + 6.$$

We have

$$J_2(\omega) = 4800 \prod_{17 \le p \le p_i} (p - 3 - 2(\frac{-42}{p})) \ne 0,$$

We have

$$\pi_3(N, 2) \approx \frac{J_2(\omega)\omega^2}{4\phi^3(\omega)} \frac{N}{\log^3 N}.$$

Theorem 8.27. Let

$$p_1 = 8p + 9, p_2 = 9p + 8.$$

We have

$$J_2(\omega) = 19200 \prod_{19 \le p \le p_i} (p - 3) \ne 0,$$

We have

$$\pi_3(N, 2) \approx \frac{J_2(\omega)\omega^2}{\phi^3(\omega)} \frac{N}{\log^3 N}.$$

Let

$$p_1 = 8p^2 + 9, p_2 = 9p^2 + 8.$$

We have

$$J_2(\omega) = 48384 \prod_{19 \le p \le p_i} (p - 3 - 2(\frac{-2}{p})) \ne 0,$$

Since $J_2(\omega) \ne 0$, there exist infinitely many 3-tuples of primes.
We have

$$\pi_3(N, 2) \approx \frac{J_2(\omega)\omega^2}{4\phi^3(\omega)} \frac{N}{\log^3 N}.$$

Theorem 8.28. Let

$$p_1 = 9p + 10, p_2 = 10p + 9.$$

We have

$$J_2(\omega) = 609280 \prod_{23 \le p \le p_i} (p - 3) \ne 0,$$

We have

$$\pi_3(N, 2) \approx \frac{J_2(\omega)\omega^2}{\phi^3(\omega)} \frac{N}{\log^3 N}.$$

Let

$$p_1 = 9p^2 + 10, p_2 = 10p^2 + 9.$$

We have

$$J_2(\omega) = 208896 \prod_{23\le p\le p_i}(p-3-2(\frac{-10}{p}))\neq 0,$$

Since $J_2(\omega)\neq 0$, there exist infinitely many 3-tuples of primes.

We have

$$\pi_3(N,2)\approx \frac{J_2(\omega)\omega^2}{4\phi^3(\omega)}\frac{N}{\log^3 N}.$$

9. An Application of Santilli's Isonumber Theory

In this section we prove that the limit for the periodic table of the stable elements is Uranium with an atomic number of 92.

In studying the stability of the many-body problems we have two theorems:

(I) The stable isoprime theorem. An isoprime number is irreducible over $\hat{F}(\hat{a},+,\hat{\times})$. It seems therefore natural to associate it with the most stable subsystem. We show that $\hat{p}=\hat{I},\hat{3},\hat{5},\hat{7},\hat{11},\hat{23},\hat{47}$ are the most stable isoprimes.

(II) The stable isoeven theorem. The most stable configuration of two isoprimes is then the most stable symmetric system in nature. We show that $\hat{2}\hat{\times}\hat{p}=\hat{2},\hat{6},\hat{10},\hat{14},\hat{22},\hat{46},\hat{94}$ are the most stable isoeven numbers.

We speculate that in an atom there are three configurations: the outermost electron configuration, the middlemost neutron one and the innermost proton one. By using the stable isoprime theorem and the stable isoeven theorem we study the innermost proton configuration.

The total number of electrons in all shells must equal the number of protons in the nucleus. Protons arrange themselves in shells in a nucleus because they take up configurations which are analogous to those of electrons in atoms, with preferred stable shells of protons.

The total quantum isonumber $\hat{n}$ and orbital quantum isonumber $\hat{p}$ determine the proton configurations of many-proton atoms[7]:

$$\begin{array}{rcccccccc} \text{Proton shells } \hat{n} & = & \hat{I} & \hat{2} & \hat{3} & \hat{4} & \hat{5} & \hat{6} & \ldots \\ & & K & L & M & N & O & P & \ldots \\ \text{Proton subshells } \hat{2}\hat{\times}\hat{p} & = & \hat{2} & \hat{6} & \widehat{10} & \widehat{14} & \widehat{18} & \widehat{22} & \ldots \\ & & s & p & d & f & g & h & \ldots \end{array}$$

An atomic subshell that contains its full quota of protons is said to be closed. A closed s subshell holds two protons, a closed p subshell six protons, a closed d subshell ten protons, a closed f subshell fourteen protons, a closed g subshell eighteen protons, a closed h subshell twenty-two protons, and so on. The Pauli principle permits $\hat{2}\hat{\times}\hat{p}$ particles per subshell. It has been proved that $\hat{2}\hat{\times}\hat{p} = \hat{2}, \hat{6}, \widehat{10}, \widehat{14}, \widehat{22}$, and $\widehat{46}$ are stable and $\hat{2}\hat{\times}\hat{p} = \widehat{18}$ is unstable [8]. s, p, d and f subshells are stable and g subshell is unstable which are called the Pauli-Santilli principle.

Table 9.1 shows the proton configurations of the elements. We shall omit hat $\wedge$ for short. From 1 to 92 of the atomic numbers every subshell is stable. It has been proved that the heaviest element that occurs naturally is Uranium with an atomic number of 92. Beginning at the atomic number of 93 there is an unstable subshell ($5g$) in it. These elements get more and more unstable. Element 114 is the most unstable. The island of stability around atomic number 114 does not exist.

In the middlemost neutron configuration there are no shell one. Magic numbers and isotopes are closely related to the neutron numbers. The electron configuration and proton one have the same subshell one. Since $5g$ is unstable, in $6s$, $6p$, $6d$, $6f$ and $7s$ subshells there are no electrons.

Remark: Pythagorean claims that *everything is number.* We claim that *everything is stable number*, that is, it obeys the stable isoprime theorem and the stable isoeven theorem. Above two theorems are foundations of isobiology's periodic table and structural genomics in the human genome project. Homo-sapiens is so advanced because we have 46 chromosomes (23 pairs) in a cell. There are the most stable sequences: 3-bp, 5-bp, and 7-bp in tRNA. The three nucleotides are able to form only stable sequences: $4^3 = 64$. The five nucleotides are able to form the most stable sequences: $4^5 = 1024$. The above studies can be further extended to whole biological field. From above two theorems we suggest new evolution theory in the biology. The evolution of the living organism starts with mutant of the prime number. The living organisms mutants from a prime number system to another new one which may be produced a new species to raise up seed and its structure tended to stabilize in given environment. From the above two theorems we find the analogies between the Chinese poem and the English poem, such as the iambic pentameter in English poem and five or seven characters in Chinese poem. Although the languages are different, the human brains are the same. The brain structures and the structures of the nervous system can be studied. For example, there are Tyr-Gly-Gly-Phe-Met, Met-enkephalin and Tyr-Gly-Gly-Phe-Leu, Leu-enkephalin in brain [9–10].

Table 9.1. Proton Configurations of the Elements

		K	L		M			N				O					P	
		1*s*	2*s*	2*p*	3*s*	3*p*	3*d*	4*s*	4*p*	4*d*	4*f*	5*s*	5*p*	5*d*	5*f*	5*g*	6*s*	6*p*
1	H	1																
2	He	2																
3	Li	2	1															
4	Be	2	2															
5	B	2	2	1														
6	C	2	2	2														
7	N	2	2	3														
8	O	2	2	4														
9	P	2	2	5														
10	Ne	2	2	6														
11	Na	2	2	6	1													
12	Mg	2	2	6	2													
13	Al	2	2	6	2	1												
14	Si	2	2	6	2	2												
15	P	2	2	6	2	3												
16	S	2	2	6	2	4												
17	Cl	2	2	6	2	5												
18	Ar	2	2	6	2	6												
19	K	2	2	6	2	6	1											
20	Ca	2	2	6	2	6	2											
21	Sc	2	2	6	2	6	3											
22	Ti	2	2	6	2	6	4											
23	V	2	2	6	2	6	5											
24	Cr	2	2	6	2	6	6											
25	Mn	2	2	6	2	6	7											
26	Fe	2	2	6	2	6	8											
27	Co	2	2	6	2	6	9											
28	Ni	2	2	6	2	6	10											
29	Cu	2	2	6	2	6	10	1										
30	Zn	2	2	6	2	6	10	2										
31	Ga	2	2	6	2	6	10	2	1									
32	Ge	2	2	6	2	6	10	2	2									
33	As	2	2	6	2	6	10	2	3									
34	Se	2	2	6	2	6	10	2	4									
35	Br	2	2	6	2	6	10	2	5									
36	Kr	2	2	6	2	6	10	2	6									
37	Rb	2	2	6	2	6	10	2	6	1								
38	Sr	2	2	6	2	6	10	2	6	2								
39	Y	2	2	6	2	6	10	2	6	3								
40	Zr	2	2	6	2	6	10	2	6	4								
41	Nb	2	2	6	2	6	10	2	6	5								
42	Mo	2	2	6	2	6	10	2	6	6								
43	Tc	2	2	6	2	6	10	2	6	7								
44	Ru	2	2	6	2	6	10	2	6	8								
45	Rh	2	2	6	2	6	10	2	6	9								
46	Pd	2	2	6	2	6	10	2	6	10								
47	Ag	2	2	6	2	6	10	2	6	10	1							
48	Cd	2	2	6	2	6	10	2	6	10	2							
49	In	2	2	6	2	6	10	2	6	10	3							
50	Sn	2	2	6	2	6	10	2	6	10	4							
51	Sb	2	2	6	2	6	10	2	6	10	5							
52	Te	2	2	6	2	6	10	2	6	10	6							
53	I	2	2	6	2	6	10	2	6	10	7							
54	Xe	2	2	6	2	6	10	2	6	10	8							
55	Cs	2	2	6	2	6	10	2	6	10	9							
56	Ba	2	2	6	2	6	10	2	6	10	10							
57	La	2	2	6	2	6	10	2	6	10	11							
58	Ce	2	2	6	2	6	10	2	6	10	12							
59	Pr	2	2	6	2	6	10	2	6	10	13							

Table 9.1. Continued

						K	L		M			N				O		
		$1s$	$2s$	$2p$	$3s$	$3p$	$3d$	$4s$	$4p$	$4d$	$4f$	$5s$	$5p$	$5d$	$5f$	$5g$	$6s$	$6p$
60	Nd	2	2	6	2	6	10	2	6	10	14							
61	Pm	2	2	6	2	6	10	2	6	10	14	1						
62	Sm	2	2	6	2	6	10	2	6	10	14	2						
63	Eu	2	2	6	2	6	10	2	6	10	14	2	1					
64	Gd	2	2	6	2	6	10	2	6	10	14	2	2					
65	Tb	2	2	6	2	6	10	2	6	10	14	2	3					
66	Dy	2	2	6	2	6	10	2	6	10	14	2	4					
67	Ho	2	2	6	2	6	10	2	6	10	14	2	5					
68	Er	2	2	6	2	6	10	2	6	10	14	2	6					
69	Tm	2	2	6	2	6	10	2	6	10	14	2	6	1				
70	Yb	2	2	6	2	6	10	2	6	10	14	2	6	2				
71	Lu	2	2	6	2	6	10	2	6	10	14	2	6	3				
72	Hf	2	2	6	2	6	10	2	6	10	14	2	6	4				
73	Ta	2	2	6	2	6	10	2	6	10	14	2	6	5				
74	W	2	2	6	2	6	10	2	6	10	14	2	6	6				
75	Re	2	2	6	2	6	10	2	6	10	14	2	6	7				
76	Os	2	2	6	2	6	10	2	6	10	14	2	6	8				
77	Ir	2	2	6	2	6	10	2	6	10	14	2	6	9				
78	Pt	2	2	6	2	6	10	2	6	10	14	2	6	10				
79	Au	2	2	6	2	6	10	2	6	10	14	2	6	10	1			
80	Hg	2	2	6	2	6	10	2	6	10	14	2	6	10	2			
81	Tl	2	2	6	2	6	10	2	6	10	14	2	6	10	3			
82	Pb	2	2	6	2	6	10	2	6	10	14	2	6	10	4			
83	Bi	2	2	6	2	6	10	2	6	10	14	2	6	10	5			
84	Po	2	2	6	2	6	10	2	6	10	14	2	6	10	6			
85	At	2	2	6	2	6	10	2	6	10	14	2	6	10	7			
86	Rn	2	2	6	2	6	10	2	6	10	14	2	6	10	8			
87	Fr	2	2	6	2	6	10	2	6	10	14	2	6	10	9			
88	Ra	2	2	6	2	6	10	2	6	10	14	2	6	10	10			
89	Ac	2	2	6	2	6	10	2	6	10	14	2	6	10	11			
90	Th	2	2	6	2	6	10	2	6	10	14	2	6	10	12			
91	Pa	2	2	6	2	6	10	2	6	10	14	2	6	10	13			
92	U	2	2	6	2	6	10	2	6	10	14	2	6	10	14			
93	Np	2	2	6	2	6	10	2	6	10	14	2	6	10	14	1		
94	Pu	2	2	6	2	6	10	2	6	10	14	2	6	10	14	2		
95	Am	2	2	6	2	6	10	2	6	10	14	2	6	10	14	3		
96	Cm	2	2	6	2	6	10	2	6	10	14	2	6	10	14	4		
97	Bk	2	2	6	2	6	10	2	6	10	14	2	6	10	14	5		
98	Cf	2	2	6	2	6	10	2	6	10	14	2	6	10	14	6		
99	Es	2	2	6	2	6	10	2	6	10	14	2	6	10	14	7		
100	Fm	2	2	6	2	6	10	2	6	10	14	2	6	10	14	8		
101	Md	2	2	6	2	6	10	2	6	10	14	2	6	10	14	9		
102	No	2	2	6	2	6	10	2	6	10	14	2	6	10	14	10		
103	Lr	2	2	6	2	6	10	2	6	10	14	2	6	10	14	11		
104	Rf	2	2	6	2	6	10	2	6	10	14	2	6	10	14	12		
105	Ha	2	2	6	2	6	10	2	6	10	14	2	6	10	14	13		
106	Sg	2	2	6	2	6	10	2	6	10	14	2	6	10	14	14		
107	Ns	2	2	6	2	6	10	2	6	10	14	2	6	10	14	15		
108	Hs	2	2	6	2	6	10	2	6	10	14	2	6	10	14	16		
109	Mt	2	2	6	2	6	10	2	6	10	14	2	6	10	14	17		
110		2	2	6	2	6	10	2	6	10	14	2	6	10	14	18		
111		2	2	6	2	6	10	2	6	10	14	2	6	10	14	18	1	
112		2	2	6	2	6	10	2	6	10	14	2	6	10	14	18	2	
113		2	2	6	2	6	10	2	6	10	14	2	6	10	14	18	2	1
114		2	2	6	2	6	10	2	6	10	14	2	6	10	14	18	2	2
115		2	2	6	2	6	10	2	6	10	14	2	6	10	14	18	2	3
116		2	2	6	2	6	10	2	6	10	14	2	6	10	14	18	2	4
117		2	2	6	2	6	10	2	6	10	14	2	6	10	14	18	2	5
118		2	2	6	2	6	10	2	6	10	14	2	6	10	14	18	2	6

Note. 5g is an unstable subshell.

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Chapter 2

SANTILLI'S ISONUMBER THEORY PART II: ISONUMBER THEORY OF THE SECOND KIND

The organic unity of mathematics is inherent in the nature of this science, for mathematics is the foundation of all exact knowledge of natural phenomena.

David Hilbert

1. Introduction

In the seminal works[1,2] Santilli has introduced a generalization of real, complex and quaternionic numbers $a = n, c, q$ based on the lifting of the unit 1 of conventional numbers into an invertible and well behaved quantity with arbitrary functional dependences on local variables

$$1 \to \hat{I}(t, x, \dot{x}, \cdots) = 1/\hat{T} \neq 1 \tag{1.1}$$

while jointly lifting the product $ab = a \times b$ of conventional numbers into the form

$$ab \to a\hat{\times}b = a\hat{T}b \tag{1.2}$$

under which $\hat{I} = 1/\hat{T}$ is the correct left and right new unit

$$\hat{I}\hat{\times}a = \hat{T}^{-1}\hat{T}a = a\hat{\times}\hat{I} = a\hat{T}\hat{T}^{-1} = a \tag{1.3}$$

for all possible $a = n, c, q$.

Since the new multiplication $a\hat{\times}b$ is associative, Santilli [1,2] has then proved that the new numbers verify all axioms of a field. The above liftings were then called *isotopic* in the Greek sense of being axiom-preserving. The prefix *iso* is then used whenever the original axioms are preserved.

Let $F(a,+,\times)$ be a conventional field with numbers $a = n, c, q$ equipped with the conventional sum $a+b \in F$, product $ab = a\times b \in F$, additive unit $0 \in F$ and their multiplicative unit $1 \in F$.

Definition 1.1. Santilli's isofields of the first kind $\hat{F} = \hat{F}(\hat{a},+,\hat{\times})$ are the rings with elements

$$\hat{a} = a\hat{I} \tag{1.4}$$

called *isonumbers*, where $a = n, c, q \in F, \hat{I} = 1/\hat{T}$ is a well behaved, invertible and Hermitean quantity outside the original field $\hat{I} = 1/\hat{T} \notin F$ and $a\hat{I}$ is the multiplication in F equipped with the isosum

$$\hat{a} + \hat{b} = (a+b)\hat{I} \tag{1.5}$$

with conventional additive unit $0 = 0\hat{I} = 0,\ \hat{a}+0 = 0+\hat{a} = \hat{a}, \quad \forall \hat{a} \in \hat{F}$ and the isoproduct

$$\hat{a}\hat{\times}\hat{b} = \hat{a}\hat{T}\hat{b} = a\hat{I}\hat{T}b\hat{I} = (ab)\hat{I} \tag{1.6}$$

under which $\hat{I} = 1/\hat{T}$ is the correct left and right new unit ($\hat{I}\hat{\times}\hat{a} = \hat{a}\hat{\times}\hat{I} = \hat{a}, \forall \hat{a} \in \hat{F}$) called isounit.

Lemma 1.1. The isofields $\hat{F}(\hat{a},+,\hat{\times})$ of Def.1.1 verify all axioms of a field. The lifting $F \to \hat{F}$ is then an isotopy. All operations depending on the product must then be lifted in $\hat{F}$ for consistency.

The Santilli's commutative isogroup of the first kind

$$\hat{a}^{\hat{I}} = a\hat{I}, \quad \hat{a}^{-\hat{I}} = a^{-1}\hat{I}, \quad \hat{a}^{\hat{I}}\hat{\times}\hat{a}^{-\hat{I}} = \hat{a}^{\hat{0}} = \hat{I} = \hat{T}^{-1} \neq 1.$$

$$\hat{a}^{\hat{b}} = a^{b}\hat{I}, \quad \hat{a}^{-\hat{b}} = a^{-b}\hat{I}, \quad \hat{a}^{\hat{b}}\hat{\times}\hat{a}^{-\hat{b}} = \hat{a}^{\hat{0}} = \hat{I} = \hat{T}^{-1} \neq 1.$$

$$\hat{a}^{\widehat{c/b}} = a^{\frac{c}{b}}\hat{I}, \quad \hat{a}^{-\widehat{c/b}} = a^{-\frac{c}{b}}\hat{I}, \quad \hat{a}^{\widehat{c/b}}\hat{\times}\hat{a}^{-\widehat{c/b}} = \hat{a}^{\hat{0}} = \hat{I} = \hat{T}^{-1} \neq 1.$$

$$\hat{a}\hat{\times}\hat{I} = \hat{a}\hat{T}\hat{I} = \hat{a}. \quad \hat{a}\hat{\times}\hat{b} = a\hat{I}\hat{T}b\hat{I} = ab\hat{I} = \widehat{ab}.$$

$$\hat{a}\hat{\times}\hat{b}^{-\hat{2}} = a\hat{I}\hat{T}b^{-2}\hat{I} = a/b^{2}\hat{I}.$$

where $\hat{I}$ is called an isounit, $\hat{T}$ is called an isoinverse of $\hat{I}$; $\hat{a}^{-\hat{b}}$ is called an isoinverse of $\hat{a}^{\hat{b}}$; $(\hat{a}^{\hat{b}}, \hat{\times})$ is called the Santilli's commutative isogroup of the first kind.

Lemma 1.2. Sentilli's isofields of the second kind $\hat{F} = \hat{F}(a, +, \times)$ (that is, $a \in F$ is not lifted to $\hat{a} = a\hat{I}$) also verify all the axioms of a field, if and only if the isounit is an element of the original field.

$$\hat{I} = 1/\hat{T} \in F \tag{1.7}$$

The isoproduct is defined by

$$a\hat{\times}b = a\hat{T}b \in \hat{F}. \tag{1.8}$$

We then have the isoquotient, isopower, isosquare root, etc.,

$$a\hat{\div}b = (a/b)\hat{I},\ a^{\hat{n}} = a\hat{\times}\cdots\hat{\times}a = a^n(\hat{T})^{n-1},\ a^{\hat{1}/2} = a^{1/2}(\hat{I})^{1/2}. \tag{1.9}$$

Definition 1.2. Isodual isomultiplication is defined by

$$\bar{\times} = -\hat{T}\times . \tag{1.10}$$

We then have isodual isoproduct

$$a\bar{\times}b = -a\hat{T}b. \tag{1.11}$$

We then have the following isomultiplicative operations of second kind:

$$a^{\hat{I}} = a, \quad a^{-\hat{I}} = a^{-1}\hat{I}^2, \quad a^{\hat{I}}\hat{\times}a^{-\hat{I}} = a^{\hat{0}} = \hat{I} = \hat{T}^{-1} \neq 1.$$

$$a^{\hat{2}} = a^2\hat{T}, \quad a^{-\hat{2}} = a^{-2}\hat{I}^3, \quad a^{\hat{2}}\hat{\times}a^{-\hat{2}} = a^{\hat{0}} = \hat{I} = \hat{T}^{-1} \neq 1.$$

$$a^{\hat{3}} = a^3\hat{T}^2, \quad a^{-\hat{3}} = a^{-3}\hat{I}^4, \quad a^{\hat{3}}\hat{\times}a^{-\hat{3}} = a^{\hat{0}} = \hat{I} = \hat{T}^{-1} \neq 1.$$

$$\cdots\cdots$$

$$a^{\hat{n}} = a^n\hat{T}^{n-1}, \quad a^{-\hat{n}} = a^{-n}\hat{I}^{n+1}, \quad a^{\hat{n}}\hat{\times}a^{-\hat{n}} = a^{\hat{0}} = \hat{I} = \hat{T}^{-1} \neq 1.$$

$$a^{\widehat{1/2}} = a^{\frac{1}{2}}(\hat{I})^{\frac{1}{2}}, \quad a^{-\widehat{1/2}} = a^{-\frac{1}{2}}(\hat{I})^{\frac{3}{2}}, \quad a^{\widehat{1/2}}\hat{\times}a^{-\widehat{1/2}} = a^{\hat{0}} = \hat{I} = \hat{T}^{-1} \neq 1.$$

$$a^{\widehat{1/3}} = a^{\frac{1}{3}}(\hat{I})^{\frac{2}{3}}, \quad a^{-\widehat{1/3}} = a^{-\frac{1}{3}}(\hat{I})^{\frac{4}{3}}, \quad a^{\widehat{1/3}}\hat{\times}a^{-\widehat{1/3}} = a^{\hat{0}} = \hat{I} = \hat{T}^{-1} \neq 1.$$

$$a^{\widehat{1/n}} = a^{\frac{1}{n}}(\hat{I})^{1-\frac{1}{n}}, \quad a^{-\widehat{1/n}} = a^{-\frac{1}{n}}(\hat{I})^{1+\frac{1}{n}}, \quad a^{\widehat{1/n}}\hat{\times}a^{-\widehat{1/n}} = a^{\hat{0}} = \hat{I} = \hat{T}^{-1} \neq 1.$$

$$a^{\widehat{c/b}} = a^{\frac{c}{b}}(\hat{I})^{1-\frac{c}{b}}, \quad a^{-\widehat{c/b}} = a^{-\frac{c}{b}}(\hat{I})^{1+\frac{c}{b}}, \quad a^{\widehat{c/b}}\hat{\times}a^{-\widehat{c/b}} = a^{\hat{0}} = \hat{I} = \hat{T}^{-1} \neq 1.$$

$$a^{\hat{b}} = a^b(\hat{I})^{1-b} = a^b\hat{T}^{b-1}, \quad a^{-\hat{b}} = a^{-b}(\hat{I})^{1+b}, \quad a^{\hat{b}}\hat{\times}a^{-\hat{b}} = a^{\hat{0}} = \hat{I} = \hat{T}^{-1} \neq 1.$$

$$a^{\hat{I}}\hat{\times}b^{\hat{I}} = a\hat{T}b, \quad a^{\hat{I}}\hat{\times}b^{-\hat{I}} = ab^{-1}\hat{I}.$$

where $\hat{I}$ is called an isounit, $\hat{T}$ an isoinverse of $\hat{I}$, $a^{-\hat{b}}$ an isoinverse of $a^{\hat{b}}$, and $(a^{\hat{b}}, \hat{\times})$ the Santilli's commutative isogroup of the second kind. The following examples are devoted to an exposition of the simplest properties of isomultiplications.

Example 1:

$$a^{\widehat{3/2}} = a^{\hat{I}} \hat{\times} a^{\widehat{1/2}} = a^{\frac{3}{2}} (\hat{T})^{\frac{1}{2}}, \quad a^{\widehat{3/2}} = a^{\widehat{1/2}} \hat{\times} a^{\widehat{1/2}} \hat{\times} a^{\widehat{1/2}} = a^{\frac{3}{2}} (\hat{T})^{\frac{1}{2}},$$

$$a^{\widehat{3/2}} = a^{\hat{2}} \hat{\times} a^{-\widehat{1/2}} = a^{\frac{3}{2}} (\hat{T})^{\frac{1}{2}}.$$

Example 2:

$$a^{-\widehat{3/2}} = a^{-\hat{I}} \hat{\times} a^{-\widehat{1/2}} = a^{-\widehat{1/2}} \hat{\times} a^{-\widehat{1/2}} \hat{\times} a^{-\widehat{1/2}} = a^{-\hat{2}} \hat{\times} a^{\widehat{1/2}} = a^{-\frac{3}{2}} (\hat{I})^{\frac{5}{2}}.$$

Example 3:

$$a^{\widehat{5/6}} = a^{\widehat{1/2}} \hat{\times} a^{\widehat{1/3}} = a^{\widehat{1/6}} \hat{\times} a^{\widehat{1/6}} \hat{\times} a^{\widehat{1/6}} \hat{\times} a^{\widehat{1/6}} \hat{\times} a^{\widehat{1/6}} = a^{\hat{I}} \hat{\times} a^{-\widehat{1/6}} = a^{\frac{5}{6}} (\hat{I})^{\frac{1}{6}}.$$

Example 4:

$$a^{-\widehat{5/6}} = a^{-\widehat{1/2}} \hat{\times} a^{-\widehat{1/3}} = a^{-\widehat{1/6}} \hat{\times} a^{-\widehat{1/6}} \hat{\times} a^{-\widehat{1/6}} \hat{\times} a^{-\widehat{1/6}} \hat{\times} a^{-\widehat{1/6}} = a^{-\hat{I}} \hat{\times} a^{\widehat{1/6}} = a^{-\frac{5}{6}} (\hat{I})^{\frac{11}{6}}.$$

Example 5:

$$a^{\hat{2}} = a^{\widehat{1/2}} \hat{\times} a^{\widehat{1/2}} \hat{\times} a^{\widehat{1/2}} \hat{\times} a^{\widehat{1/2}} = a^{\hat{3}} \hat{\times} a^{-\hat{I}} = a^2 \hat{T}.$$

Example 6:

$$a^{\widehat{3/4}} = a^{\widehat{1/4}} \hat{\times} a^{\widehat{1/4}} \hat{\times} a^{\widehat{1/4}} = a^{\hat{I}} \hat{\times} a^{-\widehat{1/4}} = a^{\widehat{1/2}} \hat{\times} a^{\widehat{1/4}} = a^{\frac{3}{4}} (\hat{I})^{\frac{1}{4}}.$$

Example 7:

$$a^{\widehat{13/6}} = a^{\widehat{2/3}} \hat{\times} a^{\widehat{3/2}} = a^{\hat{2}} \hat{\times} a^{\widehat{1/6}} = a^{\frac{13}{6}} (\hat{T})^{\frac{7}{6}}.$$

Example 8:

$$a^{-\widehat{1/6}} = a^{-\widehat{1/2}} \hat{\times} a^{\widehat{1/3}} = a^{\hat{I}} \hat{\times} a^{-\widehat{7/6}} = a^{-\frac{1}{6}} (\hat{I})^{\frac{7}{6}}.$$

Example 9:

$$a^{\widehat{1/2}} \hat{\times} b^{\widehat{1/3}} \hat{\times} c^{\widehat{1/5}} = a^{\frac{1}{2}} b^{\frac{1}{3}} c^{\frac{1}{5}} (\hat{T})^{\frac{1}{30}}.$$

Let $a = b = c$, we have $a^{\widehat{31/30}} = a^{\frac{31}{30}} (\hat{T})^{\frac{1}{30}}$.

Example 10:

$$a^{\widehat{1/2}} \hat{\times} b^{\widehat{1/3}} \hat{\times} c^{\widehat{1/5}} \hat{\times} d^{\widehat{1/6}} = a^{\frac{1}{2}} b^{\frac{1}{3}} c^{\frac{1}{5}} d^{\frac{1}{6}} (\hat{T})^{\frac{1}{5}}.$$

Let $a = b = c = d$, we have $a^{\widehat{6/5}} = a^{\frac{6}{5}} (\hat{T})^{\frac{1}{5}}$

Example 11:

$$a^{\hat{I}} = a^{\widehat{1/3}} \hat{\times} a^{\widehat{2/3}} = a\hat{I}\hat{T} = a.$$

Example 12:

$$a^{-\hat{I}} = a^{-\widehat{1/3}}\hat{\times}a^{-\widehat{2/3}} = a^{-1}\hat{I}^3\hat{T} = a^{-1}\hat{I}^2.$$

Example 13:

$$a^{\hat{2}}\hat{\times}b^{-\hat{2}} = (\frac{a}{b})^2\hat{I}. \quad a^{\hat{2}}\hat{\times}b^{-\hat{3}} = a^2b^{-3}\hat{I}^2.$$

Example 14:

$$a\hat{\times}y^{\hat{2}} + b\hat{\times}y + c = a(y\hat{T})^2 + b(y\hat{T}) + c.$$

Example 15:

$$a_3\hat{\times}y^{\hat{3}} + a_2\hat{\times}y^{\hat{2}} + a_1\hat{\times}y + a_0 = a_3(y\hat{T})^3 + a_2(y\hat{T})^2 + a_1(y\hat{T}) + a_0.$$

In this chapter we study Santilli's isonumber theory of the second kind based on isofields $\hat{F} = \hat{F}(a, +, \hat{\times})$ [3].

2. Foundations of Santilli's isonumber theory of the second kind

By lifting $F(a, +, \times) \to \hat{F}(a, +, \hat{\times})$ we study Santilli's isonumber theory of the second kind.

We can partition the positive integers into four classes:

1. The unit: 1,
2. The isounit: $\hat{I}$ or $\hat{T}$,
3. The prime numbers: $2, 3, 5, \ldots,$
4. The composite numbers: $4, 6, 8, \ldots.$

The Santilli's isonumber theory of the second kind is primarily concerned with isodivisibility properties of integers.

Definition 2.1. Definition of isodivisibility. We say that a nonzero integer a isodivides an integer b, if there exists an integer c such that $a\hat{\times}c = a\hat{T}c = b,\ c = b\hat{I}/a$. If a isodivides b, we write $a\hat{|}b = a|\hat{I}b$. Then we have $a|b$ or $a|\hat{I}$. If a does not isodivide b, we write $a \hat{\not|} b = \ a \not| b\hat{I}$. Then we have $a \not| b$ and $a \not| \hat{I}$.

The following theorem gives the key properties of isodivisibility.

Theorem 2.1.

(1) If a is a nonzero integer, then $a\hat{|}a = a|a\hat{I}$.
(2) If a is an integer, then $1\hat{|}a = 1|a\hat{I}$.
(3) If $a\hat{|}b = a|b\hat{I}$ and $b\hat{|}c = b|c\hat{I}$, then $a\hat{|}c = a|c\hat{I}$
(4) If $a\hat{|}b = a|b\hat{I}$ and c is a nonzero integer, then $ac\hat{|}bc = ac|bc\hat{I}$ and $a\hat{|}bc = a|bc\hat{I}$.

(5) If $a\hat{|}b = a|b\hat{I}$ and $a\hat{|}c = a|c\hat{I}$, then for all integer m and n we have $a\hat{|}(mb+nc) = a|\hat{I}(mb+nc)$.
(6) If $a\hat{|}b = a|b\hat{I}$ and $b\hat{|}a = b|a\hat{I}$, then $a = \pm b$ and $\hat{I} = 1$.
(7) If $a\hat{|}b = a|b\hat{I}$ and a and b are positive integers, then $a > b,\ a < \hat{I}$ and $a|\hat{I}$ or $a > \hat{I}, a < b$ and $a|b$.

Definition 2.2. If d divides two integer a and b, then d is called a common divisor of a and b. The number d is called the greatest common divisor (gcd) of a and b and is denoted by (a,b). If $(a,b)=1$, then a and b are said to be relatively prime.

Theorem 2.2. The prime number theorem for isoarithmetic progressions

$$E_a(K) = \omega\hat{\times}K + a = \omega\hat{T}k + a, \tag{2.1}$$

where $k = 0, 1, 2, \ldots; (\omega\hat{T}, a) = 1$. We have

$$\pi_a(N) = \frac{1}{\phi(\omega\hat{T})}\frac{N}{\log N}(1+O(1)), \tag{2.2}$$

where $\pi_a(N)$ denotes the number of prime in $E_a(K) \leq N$ and $\phi(\omega\hat{T})$ Euler's ϕ-function.

Santilli's isoadditive prime problems:

$$\hat{p}_2 = 2\hat{\times}\hat{p}_1 + 1 = 2\hat{T}p_1 + 1, \quad p_3 = 4\hat{\times}p_1 + 1 = 4\hat{T}p_1 + 1. \tag{2.3}$$

Let $\hat{T} = 1$, we have

$$p_2 = 2p_1 + 1, \quad p_3 = 4p + 1. \tag{2.4}$$

They cannot all be prime, for at least one of the three is divisible by 3.
There exist no 3-tuples of primes except $p_1 = 3, \quad p_2 = 7, \quad p_3 = 13$. Let $\hat{T} = 2$, and $\hat{I} = \frac{1}{2}$ we have

$$p_2 = 4p_1 + 1, \quad p_3 = 8p_1 + 1. \tag{2.5}$$

There exist no 3-tuples of primes.

Let $\hat{T} = 3$ and $\hat{I} = \frac{1}{3}$, we have

$$p_2 = 6p_1 + 1, p_3 = 12p_1 + 1. \tag{2.6}$$

There exist infinitely many 3-tuples of primes: 5, 31, 61; 13, 79, 157; 23, 139, 277; 61, 367, 733; ...

$$p_4 = (p_1 + p_2 + p_3 + 1)^{\hat{2}} + 1 = \hat{T}(p_1 + p_2 + p_3 + 1)^2 + 1. \tag{2.7}$$

Let $\hat{T} = \{\hat{T}_1, \ldots, \hat{T}_n\}$ and $\hat{I} = \{\hat{I}_1, \ldots, \hat{I}_n\}$. In (2.7) there are n additive prime equations. Every equation has an isounit.

Fermat-Santilli equations:

$$x^{\hat{n}} + y^{\hat{n}} = 1. \tag{2.8}$$

From(2.8) we have

$$x^n + y^n = (\hat{I})^{n-1}, \quad (x, y) = 1. \tag{2.9}$$

For $n > 3$, (2.9) has no rational solutions.

Pell-Santilli equations:

$$x^{\hat{2}} - p\hat{\times}y^{\hat{2}} = \pm 1. \tag{2.10}$$

By Santilli's isonumber theory we can extend the additive prime equations and Diophantine equations.

Definition 2.3. Given integers a, b, m with $m > 0$. We say that a is isocongruent to b modulo m and we write

$$a \,\hat{\equiv} b \text{ (mod } m). \tag{2.11}$$

If m isodivides the difference $a - b$, the number m is called the modulus of isocongruence. The isocongruence (2.11) is equivalent to the isodivisibility relation

$$m \,\hat{|}\, (a - b) = m \;\; | \;\; \hat{I}(a - b). \tag{2.12}$$

If $m \;\hat{\not|}(a - b)$ we write

$$a \overset{\wedge}{\not\equiv} b \text{ (mod } m), \tag{2.13}$$

and we say that a and b are nonisocongruent (mod m).

Theorem 2.3. The isocongruence is an equivalence relation:

(1) $a\hat{\equiv}a$ (mod m) (reflexivity)
(2) $a\hat{\equiv}b$ (mod m) implies $b\hat{\equiv}a$ (mod m) (symmetry)
(3) $a\hat{\equiv}b$ (mod) and$b\hat{\equiv}c$ (mod) implies $a\hat{\equiv}c$(mod m) (transitivity).

Definition 2.4. The quadratic isocongruence

$$x^{\hat{2}}\hat{\equiv}n \text{ (mod } p) \tag{2.14}$$

where p is an odd prime. Let $(\hat{I}, p) = 1$ so we can cancel $\hat{I}$. Eq. (2.14) can be written as

$$x^2 \equiv n\hat{I} \text{ (mod } p). \tag{2.15}$$

If congruence (2.15) has a solution and we say that n is a quadratic residue mod p and we write $(\frac{n\hat{I}}{p}) = 1$, where $(\frac{n\hat{I}}{p})$ is Legendre symbol. If (2.15) has no solution we say that n is a quadratic nonresidue mod p and we write $(\frac{n\hat{I}}{p}) = -1$.

Theorem 2.4.

$$\sum_{i=1}^{n} x_i \mathrel{\overset{\wedge}{\equiv}} a \pmod p \tag{2.16}$$

When $(\hat{I}, p) = 1$, we can cancel $\hat{I}$. (2.16) can be written as

$$\sum_{i=1}^{n} x_i \equiv a \pmod p \tag{2.17}$$

where p is an odd prime.

(2.17) has exactly $J_n(p) + (-1)^n$ solutions, where $J_n(p) = \frac{(p-1)^n - (-1)^n}{p}$, if $p|a$ and (2.17) $J_n(p)$ solutions if $p \not| a$.

Theorem 2.5.

$$x_1^{\hat{2}} + x_2 + \ldots + x_n \hat{\equiv} a \pmod p. \tag{2.18}$$

When $(\hat{I}, p) = 1$, we can cancel $\hat{I}$, (2.18) can be written as

$$x_1^2 + \hat{I}(x_2 + \ldots + x_n) \equiv \hat{I}a \pmod p. \tag{2.19}$$

(2.19) has exactly $J_n(p) - (-1)^n$ solutions if $(\frac{a\hat{I}}{p}) = 1$ and (2.19) $J_n(p) + (-1)^n$ solutions if $(\frac{a\hat{I}}{p}) = -1$ and $p|a$.

3. Santilli's Isoadditive Prime Theory

Definition 3.1. We define the arithmetic progressions [3]

$$E_{p_\alpha}(K) = \omega K + p_\alpha, \tag{3.1}$$

where $K = 0, 1, 2, \ldots;$

$$\omega = \prod_{2 \le p \le p_i} p; \;\; (\omega, p_\alpha) = 1;$$

$$p_i < p_\alpha = p_1, p_2, \ldots, p_{\phi(\omega)} = \omega + 1; \;\; \phi(\omega) = \sum_{\substack{(p_\alpha,\, \omega) = 1 \\ 1 \le \alpha \le \phi(\omega)}} 1 = \prod_{3 \le p \le p_i} (p-1)$$

$\phi(\omega)$ is Euler's ϕ-function.

For every $Ep_\alpha(K)$, there exist infinitely many primes.We have

$$\pi_{p_\alpha}(N) = \frac{1}{\phi(\omega)} \frac{N}{\log N}(1 + O(1)), \tag{3.2}$$

where $\pi_{p_\alpha}(N)$ denotes the number of primes $p \le N$ in $E_{p_\alpha}(K)$. Since $\pi_{p_\alpha}(N)$ is independent of p_α, the primes seem to be equally distributed among the $\phi(\omega)$ reduced residue classes mod ω, and (3.2) is a precise statement of this fact.

We deal with the prime twins: $p_2 = p_1 + 2$. It can be written as the form of the arithmetic progressions

$$E_{p_\alpha+2}(K) = E_{p_\alpha}(K) + 2. \tag{3.3}$$

We define the arithmetic function of the prime twins

$$J_2(\omega) = \sum_{\substack{(p_\alpha+2,\omega)=1 \\ 1\le\alpha\le\phi(\omega)}} 1 = \prod_{3\le p\le p_i} (p-2). \tag{3.4}$$

Since $J_2(\omega) < \phi(\omega)$, it is a generalization of Euler's ϕ function $\phi(\omega)$. Since $(p_\alpha + 2, \omega) = 1$, (3.3) has the infinitude of the prime twins.

Let $p_i = 3$. From (3.1) we have

$$E_5(K) = 6K + 5, \quad E_7(K) = 6K + 7, \tag{3.5}$$

where $K = 0, 1, 2, \ldots$

From (3.4) we have $J_2(6) = 1$. From (3.5) we have one subequation of the prime twins

$$E_7(K) = E_5(K) + 2. \tag{3.6}$$

Since $(7, 6) = 1$, (3.6) has the infinitude of the prime twins.

Let $p_i = 5$. From (3.1) we have

$$E_{p_\alpha}(K) = 30K + p_\alpha, \tag{3.7}$$

where $K = 0, 1, \ldots;\ p_\alpha = 7, 11, 13, 17, 19, 23, 29, 31$.

From (3.4) we have $J_2(30) = 3$. From (3.7) we have three subequations of the prime twins

$$E_{13}(K) = E_{11}(K) + 2, E_{19}(K) = E_{17}(K) + 2, E_{31}(K) = E_{29}(K) + 2. \tag{3.8}$$

Since $(p_\alpha + 2, 30) = 1$, every subequation has the infinitude of the prime twins. The prime twins seem to be equally distributed among the $J_2(30)$ reduced residue classes mod 30. It is a generalization of Direchlet's theorem. For $J_2(\omega) \to \infty$ as $\omega \to \infty$, there exist infinitely many subequations of the prime twins, every subequation has the infinitude of the prime twins. By using this method and Santilli's isonumber theory we found the new branch of number theory: Santilli's isoadditive prime theory.

By lifting $F(a, +, \times) \to \hat{F}(a, +, \hat{\times})$ from (3.1) we have isoarithmetic progressions

$$E_{p_\alpha}(K) = \omega \hat{\times} K + p_\alpha = \omega \hat{T} K + p_\alpha. \tag{3.9}$$

Let $\hat{T} = \omega^{m-1}$. From (3.9) we have

$$E_{p_\alpha}(K) = \omega^m K + p_\alpha, \tag{3.10}$$

where

$$p_i < p_\alpha = p_1, \ldots, p_{\phi(\omega^m)} = \omega^m + 1;\ \phi(\omega^m) = \sum_{(p_\alpha . \omega^m)=1} 1 = \omega^{m-1}\phi(\omega).$$

Theorem 3.1. If there exist infinitely many primes p_j (for $j = 1, \ldots, n-1$) such that the absolute values of polynomials $f_i(p_j)$ (for $i = 1, \ldots, k-1$) are all prime, then $f_i(p_j)$ must satisfy two necessary and sufficient conditions:

(I) Let $f_i(p_j)$ be $k-1$ distinct polynomials with integral coefficients irreducible over the integers.

(II) There exists an arithmetic function $J_n(\omega^m)$, that is to separate the number of k-tuples of subequations from (3.10). It is also the number of solutions of

$$\left(\prod_{i=1}^{k-1} f_i(p_{\alpha_j}), \omega^m\right) = 1, \tag{3.11}$$

where $1 \leq \alpha_j \leq \phi(\omega^m),\ j = 1,\ \ldots,\ n-1$.

Since $J_n(\omega^m) \leq \phi^{n-1}(\omega^m),\ J_n(\omega^m)$. can be expressed as the form

$$\begin{aligned} J_n(\omega^m) &= \sum_{\alpha_{n-1}=1}^{\phi(\omega^m)} \cdots \sum_{\alpha_1=1}^{\phi(\omega^m)} \left[\frac{1}{(\prod_{i=1}^{k-1} f_i(p_{\alpha_j}), \omega^m)}\right] \\ &= \omega^{(n-1)(m-1)} \prod_{3\leq p\leq p_i} ((p-1)^{n-1} - H(p)), \end{aligned} \tag{3.12}$$

where $H(p)$ is the number of solutions of congruence

$$\prod_{i=1}^{k-1} f_i(q_j) \equiv 0 \pmod{p}, \tag{3.13}$$

$q_j = 1, 2, \ldots, p-1; j = 1, \ldots, n-1$.

Since $(p-1)^{n-1} = \frac{(p-1)^n - (-1)^n}{p} + \frac{(p-1)^{n-1} - (-1)^{n-1}}{p}$, $J_n(\omega^m)$ can also be expressed as the form

$$J_n(\omega^m) = \omega^{(n-1)(m-1)} \prod_{3\leq p\leq p_i} (\frac{(p-1)^n - (-1)^n}{p} - \chi(p)), \tag{3.14}$$

where $\chi(p) = 0, \pm 1, \ldots$

In the same way as in Chapter 1, we can derive the best asymptotic formula

$$\pi_k(N, n) = |\{p_j : p_j \leq N,\ f_i(p_j) = \text{prime}\}|$$

$$= \prod_{i=1}^{k-1} (\deg f_i)^{-1} \hat{\times} \frac{J_n(\omega^m)(\omega^m)^{k-1}}{(n-1)!\phi^{n+k-2}(\omega^m)} \frac{N^{n-1}}{(\log N)^{n+k-2}} (1 + O(1)). \tag{3.15}$$

We have

$$\frac{J_n(\omega^m)(\omega^m)^{k-1}}{\phi^{n+k-2}(\omega^m)} = \frac{J_n(\omega)\omega^{k-1}}{\phi^{n+k-2}(\omega)}. \tag{3.16}$$

Substituting (3.16) into (3.15) we have

$$\pi_k(N,n) = \prod_{i=1}^{k-1}(\deg f_i)^{-1}\hat{\times}\frac{J_n(\omega)\omega^{k-1}}{(n-1)!\phi^{n+k-2}(\omega)}\frac{N^{n-1}}{(\log N)^{n+k-2}}(1+O(1)). \tag{3.17}$$

By the degree is to be understood the highest degree of any individual term, where the degree of a term such as $x_1\ x_2^3\ x_3^4$ is taken to be $1+3+4=8$. We prove that $\pi_k(N,n)$ is independent of m. (3.17) can be written as the form

$$\pi_k(N,n) = J_n(\omega)t_1, \tag{3.18}$$

where

$$t_1 = \prod_{i=1}^{k-1}(\deg f_i)^{-1}\hat{\times}\frac{\omega^{k-1}N^{n-1}}{(n-1)!(\phi(\omega)\log N)^{n+k-2}}(1+O(1)). \tag{3.19}$$

t_1 denoting the number of k-tuples of primes in one k-tuple of subequations. t_1 can be applied to any k-tuple of subequations and is called the common factor in Santilli's additive prime theory. $t_1=0$ if $N<\omega$, $t_1\neq 0$ if $N>\omega$ similar to (3.1). $t_1\neq 0$ implies that there exist infinitely many prime solutions. If $J_n(\omega)=0$ then $\pi_k(N,n)=0$,there exist finitely many k-tuples of primes. If $J_n(\omega)\to\infty$ as $\omega\to\infty$,then there exist infinitely many k-tuples of primes. It is a generalization of Euler proof of the existence of infinitely many primes.

Let $n=2$ and $k=1$. From (3.19) we have

$$t_1 = \frac{N}{\phi(\omega)\log N}(1+O(1)). \tag{3.20}$$

It is the prime number theorem of the arithmetic progressions.

Since $k=1$, we have $J_2(\omega)=\phi(\omega)$. Substituting (3.20) into (3.18) we have

$$\pi_1(N,2) = \frac{N}{\log N}(1+O(1)). \tag{3.21}$$

It is the prime number theorem.

(3.17) is a unified asymptotic formula in the Santilli's isoadditive prime theory. To prove it is transformed into studying the arithmetic functions $J_n(\omega)$. By using the $J_n(\omega)$ we prove the following Santilli's isoadditive prime theorems:

Theorem 3.1.1. $p_2 = a\hat{\times}p_1 + b = a\hat{T}p_1 + b$, where $(a\hat{T}, b) = 1$ and $2|a\hat{T}b$.

We have

$$J_2(\omega) = \sum_{\substack{(a\hat{T}p_\alpha + b,\, \omega) = 1 \\ 1 \le \alpha \le \phi(\omega)}} 1 = \prod_{3\le p\le p_i} (p-2) \prod_{p|a\hat{T}b} \frac{p-1}{p-2} \neq 0,$$

$$\pi_2(N,2) = 2 \prod_{3\le p\le p_i} (1 - \frac{1}{(p-1)^2}) \prod_{p|a\hat{T}b} \frac{p-1}{p-2} \frac{N}{\log^2 N}(1+O(1)).$$

Since $J_2(\omega) \neq= 0$, there exist infinitely many primes p_1 such that p_2 is also a prime. Let $a = 1$, it is the prime twins theorem.

Theorem 3.1.2.

$$p_2 = a\bar{\times}p_1 + N = -a\hat{T}p_1 + N,$$

Let $a = \hat{T} = 1$. We have the Goldbach's theorem: $p_2 = N - p_1$.[4]
We have

$$J_2(\omega) = \sum_{\substack{(N - p_\alpha,\, \omega) = 1 \\ 1 \le \alpha \le \phi(\omega)}} 1 = \prod_{3\le p\le p_i} (p-2) \prod_{p|N} \frac{p-1}{p-2} \neq 0,$$

$$\pi(N,2) = 2 \prod_{3\le p\le p_i} (1 - \frac{1}{(p-1)^2}) \prod_{p|N} \frac{p-1}{p-2} \frac{N}{\log^2 N}(1+O(1)).$$

Since $J_2(\omega) \neq 0$, every even number greater than 4 is the sum of two primes. It is the simplest theorem in Santilli's isoadditive prime theory.

Theorem 3.1.3. $p_2 = p_1^{\hat{2}} + p_1 + 1 = \hat{T}p_1^2 + p_1 + 1$, where $\hat{T}$ is an odd.
Since $J_2(\omega) \neq 0$, there exist infinitely many primes p_1 such that p_2 is also a prime.
Let $\hat{T} = 1$. we have

$$J_2(\omega) = \sum_{\substack{(p_\alpha^2 + p_\alpha + 1,\, \omega) = 1 \\ 1 \le \alpha \le \phi(\omega)}} 1 = \prod_{3\le p\le p_i} (p - 2 - \chi(p)) \neq 0,$$

where $\chi(3) = 0; \chi(p) = 1$ if $p \equiv 1 (\mathrm{mod}\ 3)$, $\chi(p) = -1$ if $p \equiv -1\ (\mathrm{mod}\ \ 3)$,

$$\pi_2(N,2) = \frac{J_2(\omega)\omega}{2\phi^2(\omega)} \frac{N}{\log^2 N}(1+O(1)).$$

Theorem 3.1.4. $p_2 = (p_1+1)^{\hat{2}} + 1 = \hat{T}(p_1+1)^2 + 1$
Since $J_2(\omega) \neq 0$, there exist infinitely many primes p_1 such that p_2 is also a prime.
Let $\hat{T} = 1$, we have

$$J_2(\omega) = \sum_{\substack{((p_\alpha + 1)^2 + 1,\, \omega) = 1 \\ 1 \le \alpha \le \phi(\omega)}} 1 = \prod_{3\le p\le p_i} \left(p - 2 - (-1)^{\frac{p-1}{2}}\right) \neq 0,$$

$$\pi_2(N,2)=\frac{J_2(\omega)\omega}{2\phi^2(\omega)}\frac{N}{\log^2 N}(1+O(1)).$$

Theorem 3.1.5. $p_2=p_1^{\hat{3}}+2=\hat{T}^2p_1^3+2$, where $\hat{T}$ is an odd.
Since $J_2(\omega)\neq 0$, there exist infinitely many primes p_1 such that p_2 is also a prime.
Let $\hat{T}=1$. We have

$$J_2(\omega)=\sum_{\substack{(p_\alpha^3+2,\omega)=1\\ 1\le\alpha\le\phi(\omega)}}1=\prod_{3\le p\le p_i}(p-2-\chi(p))\neq 0,$$

where $\chi(p)=2$ if $2^{\frac{p-1}{3}}\equiv 1(\text{mod } p)$; $\chi(p)=-1$ if $2^{\frac{p-1}{3}}\not\equiv 1(\text{mod } p)$; $\chi(p)=0$ otherwise.

$$\pi_2(N,2)=\frac{J_2(\omega)\omega}{3\phi^2(\omega)}\frac{N}{\log^2 N}(1+O(1)).$$

Theorem 3.1.6. $p_2=(p_1+1)^{\hat{4}}+1$.
If $J_2(\omega)\neq 0$, then there exist infinitely many primes p_1 such that p_2 is also *a* prime.
Let $\hat{T}=1$. We have

$$J_2(\omega)=\sum_{\substack{((p_\alpha+1)^4+1,\omega)=1\\ 1\le\alpha\le\phi(\omega)}}1=\prod_{3\le p\le p_i}(p-2-\chi(p))\neq 0,$$

where $\chi(p)=3$ if $p\equiv 1(\text{mod } 8);\chi(p)=-1$ if $p\not\equiv 1(\text{mod } 8)$,

$$\pi_2(N,2)=\frac{J_2(\omega)\omega}{4\phi^2(\omega)}\frac{N}{\log^2 N}(1+O(1)).$$

Theorem 3.1.7. $p_2=p_1^{\hat{5}}+2$, where $\hat{T}$ is an odd.
Since $J_2(\omega)\neq 0$, there exist infinitely many primes p_1 such that p_2 is also a prime.
Let $\hat{T}=1$. We have

$$J_2(\omega)=\sum_{\substack{(p_\alpha^5+2,\omega)=1\\ 1\le\alpha\le\phi(\omega)}}1=\prod_{3\le p\le p_i}(p-2-\chi(p))\neq 0,$$

where $\chi(p)=4$ if $2^{\frac{p-1}{5}}\equiv 1(\text{mod } p)$; $\chi(p)=-1$ if $2^{\frac{p-1}{5}}\not\equiv 1(\text{mod } p)$; $\chi(p)=0$ otherwise.

$$\pi_2(N,2)=\frac{J_2(\omega)\omega}{5\phi^2(\omega)}\frac{N}{\log^2 N}(1+O(1)).$$

Theorem 3.1.8. $p_2 = (p_1 + 4)^{\hat{6}} + 4$ where $\hat{T}$ is an odd.
Since $J_2(\omega) \neq 0$, there exist infinitely many primes p_1 such that p_2 is also a prime.
Let $\hat{T} = 1$. We have

$$J_2(\omega) = \sum_{\substack{((p_\alpha + 4)^6 + 4,\, \omega) = 1 \\ 1 \leq \alpha \leq \phi(\omega)}} 1 = \prod_{3 \leq p \leq p_i} (p - 2 - \chi(p)) \neq 0,$$

where $\chi(p) = 5$ if $(4)^{\frac{p-1}{6}} \equiv (-1)^{\frac{p-1}{6}} (\text{mod } p)$; $\chi(p) = -1$ if $(4)^{\frac{p-1}{6}} \not\equiv (-1)^{\frac{p-1}{6}} (\text{mod } p)$; $\chi(p) = (-1)^{\frac{p-1}{2}}$ otherwise.

$$\pi_2(N, 2) = \frac{J_2(\omega)\omega}{6\phi^2(\omega)} \frac{N}{\log^2 N}(1 + O(1)).$$

Theorem 3.1.9. $p_2 = p_1^{\hat{6}} + p_1^{\hat{3}} + 1$.
Since $J_2(\omega) \neq 0$, there exist infinitely many primes p_1 such that p_2 is also a prime.
Let $\hat{T} = 1$. We have

$$J_2(\omega) = \sum_{\substack{(p_\alpha^6 + p_\alpha^3 + 1,\, \omega) = 1 \\ 1 \leq \alpha \leq \phi(\omega)}} 1 = \prod_{3 \leq p \leq p_i} (p - 2 - \chi(p)) \neq 0,$$

where $\chi(3) = 0$; $\chi(p) = 5$ if $p \equiv 1 (\text{mod } 18)$; $\chi(p) = -1$ otherwise.

$$\pi_2(N, 2) = \frac{J_2(\omega)\omega}{6\phi^2(\omega)} \frac{N}{\log^2 N}(1 + O(1)).$$

Theorem 3.1.10. $p_2 = p_1^{\hat{7}} + 2$, where $\hat{T}$ is an odd.
Since $J_2(\omega) \neq 0$, there exist infinitely many primes p_1 such that p_2 is also a prime.
Let $\hat{T} = 1$. We have

$$J_2(\omega) = \sum_{\substack{(p_\alpha^7 + 2,\, \omega) = 1 \\ 1 \leq \alpha \leq \phi(\omega)}} 1 = \prod_{3 \leq p \leq p_i} (p - 2 - \chi(p)) \neq 0,$$

where $\chi(p) = 6$ if $2^{\frac{p-1}{7}} \equiv 1 (\text{mod } p)$; $\chi(p) = -1$ if $2^{\frac{p-1}{7}} \not\equiv 1 (\text{mod } p)$; $\chi(p) = 0$ otherwise.

$$\pi_2(N, 2) = \frac{J_2(\omega)\omega}{7\phi^2(\omega)} \frac{N}{\log^2 N}(1 + O(1)).$$

Theorem 3.1.11. $p_2 = (p_1 + 1)^{\hat{8}} + 1$.
Since $J_2(\omega) \neq 0$, there exist infinitely many primes p_1 such that p_2 is also a prime.

Let $\hat{T}=1$. We have

$$J_2(\omega)=\sum_{\substack{((p_\alpha+1)^8+1,\omega)=1\\ 1\le\alpha\le\phi(\omega)}} 1=\prod_{3\le p\le p_i}(p-2-\chi(p))\neq 0,$$

where $\chi(p)=7$ if $p\equiv 1(\mathrm{mod}\ 16), \chi(p)=-1$ if $p\not\equiv 1(\mathrm{mod}\ 16)$,

$$\pi_2(N,2)=\frac{J_2(\omega)\omega}{8\phi^2(\omega)}\frac{N}{\log^2 N}(1+O(1)).$$

Theorem 3.1.12. $p_2=(p_1+1)^{\widehat{16}}+1$.
Since $J_2(\omega)\neq 0$, there exist infinitely many primes p_1 such that p_2 is also a prime.
Let $\hat{T}=1$. We have

$$J_2(\omega)=\sum_{\substack{((P_\alpha+1)^{16}+1,\omega)=1\\ 1\le\alpha\le\phi(\omega)}} 1=\prod_{3\le p\le p_i}(p-2-\chi(p))\neq 0$$

where $\chi(p)=15$ if $p\equiv 1(\mathrm{mod}\ 32);\ \chi(p)=-1$ if $p\not\equiv 1(\mathrm{mod}\ 32)$,

$$\pi_2(N,2)=\frac{J_2(\omega)\omega}{16\phi^2(\omega)}\ \frac{N}{\log^2 N}(1+0(1)).$$

Theorem 3.1.13. $p_3=p_1^{\hat{4}}+p_1^{\hat{3}}+p_1^{\hat{2}}+p_1+1$, where $\hat{T}$ is an odd.
Since $J_2(\omega)\neq 0$, there exist infinitely many primes p_1 such that p_2 is also a prime.
Let $\hat{T}=1$, we have

$$J_2(\omega)=\sum_{\substack{(p_\alpha^4+p_\alpha^3+p_\alpha^2+p_\alpha+1,\omega)=1\\ 1\le\alpha\le\phi(\omega)}} 1=\prod_{3\le p\le p_i}(p-2-\chi(p))\neq 0.$$

where

$$\chi(5)=0;\chi(p)=3 \text{ if } p\equiv 1(\mathrm{mod}\ 5);\chi(p)=-1 \text{ if } p\not\equiv 1(\mathrm{mod}\ 5),$$

$$\pi_2(N,2)=\frac{J_2(\omega)\omega}{4\phi^2(\omega)}\ \frac{N}{\log^2 N}(1+O(1)).$$

Theorem 3.1.14. $p_2=2\hat{\times}p_1+1,\ p_3=4\hat{\times}p_1+1$.

$$J_2(\omega)\neq 0 \text{ if } 3|\hat{T};\ J_2(\omega)=0 \text{ if } 3\not|\hat{T}.$$

Let $\hat{T}=3$, we have

$$J_2(\omega)=\sum_{\alpha=1}^{\phi(\omega)}\left[\frac{2}{(6p_\alpha+1,\omega)+(12p_\alpha+1,\omega)}\right]=2\prod_{5\le p\le p_i}(p-3)\neq 0$$

Here [ ] denotes the greatest integer,

$$\pi_3(N,2)=\frac{J_2(\omega)\omega^2}{\phi^3(\omega)}\frac{N}{\log^3 N}(1+O(1)).$$

Theorem 3.1.15. $p_2=3\hat{\times}p_1+2,\ p_3=2\hat{\times}p_1+3$, where $\hat{T}$ is a prime greater than 3.

Since $J_2(\omega)\neq 0$, there are infinitely many 3-tuples of primes.

Let $\hat{T}=1$, we have

$$J_2(\omega)=\sum_{\alpha=1}^{\phi(\omega)}\left[\frac{2}{(3p_\alpha+2,\omega)+(2p_\alpha+3,\omega)}\right]=6\prod_{7\le p\le p_i}(p-3)\neq 0.$$

$$\pi_3(N,2)=\frac{J_2(\omega)\omega^2}{\phi^3(\omega)}\frac{N}{\log^3 N}(1+O(1))).$$

Theorem 3.1.16. $p_2=30\hat{\times}p_1+1,\ p_3=60\hat{\times}p_1-1$.

Since $J_2(\omega)\neq 0$, there are infinitely many 3-tuples of primes.

Let $\hat{T}=1$, we have

$$J_2(\omega)=\sum_{\alpha=1}^{\phi(\omega)}\left[\frac{2}{(30p_\alpha+1,\omega)+(60p_\alpha-1,\omega)}\right]=8\prod_{7\le p\le p_i}(p-3)\neq 0,$$

$$\pi_3(N,2)=\frac{J_2(\omega)\omega^2}{\phi^3(\omega)}\frac{N}{\log^3 N}(1+O(1)).$$

Theorem 3.1.17. $p_2=p_1+4,\ p_3=p_1^{\hat{2}}+4$, where T is an odd and $3\not|(\hat{T}+4)$.

Since $J_2(\omega)\neq 0$ there are infinitely many 3-tuples of primes.

Let $\hat{T}=1$, we have

$$J_2(\omega)=\sum_{\alpha=1}^{\phi(\omega)}\left[\frac{2}{(p_\alpha+4,\omega)+(p_\alpha^2+4,\omega)}\right]=2\prod_{7\le p\le p_i}(p-3-(-1)^{\frac{p-1}{2}})\neq 0.$$

$$\pi_3(N,2)=\frac{J_2(\omega)\omega^2}{2\phi^3(\omega)}\frac{N}{\log^3 N}(1+O(1)).$$

Theorem 3.1.18. $p_2 = p_1 + 2,\ p_3 = p_1^{\hat{2}} + 30$, where $\hat{T}$ is a prime greater than 5. Since $J_2(\omega) \neq 0$, there are infinitely many 3-tuples of primes.
Let $\hat{T} = 1$, we have

$$J_2(\omega) = \sum_{\alpha=1}^{\phi(\omega)} \left[\frac{2}{(p_\alpha + 2, \omega) + (p_\alpha^2 + 30, \omega)} \right] = 3 \prod_{7 \leq p \leq p_i} \left(p - 3 - \left(-\frac{30}{p} \right) \right) \neq 0,$$

$$\pi_3(N, 2) = \frac{J_2(\omega)\omega^2}{2\phi^3(\omega)} \frac{N}{\log^3 N}(1 + O(1)).$$

Theorem 3.1.19. $p_2 = p_1^{\hat{2}} + 1,\ p_3 = p_1^{\hat{2}} + 3$

$$J_2(\omega) \neq 0 \text{ if } 3|(\hat{T} - 4);\ J_2(\omega) = 0 \text{ otherwise}$$

Let $\hat{T} = 4$, we have

$$J_2(\omega) = \sum_{\alpha=1}^{\phi(\omega)} \left[\frac{2}{(4p_\alpha^2 + 1, \omega) + (4p_\alpha^2 + 3, \omega)} \right]$$

$$= 2 \prod_{5 \leq p \leq p_i} \left(p - 3 - (-1)^{\frac{p-1}{2}} - \left(\frac{-3}{p} \right) \right) \neq 0,$$

$$\pi_3(N, 2) = \frac{J_2(\omega)\omega^2}{4\phi^3(\omega)} \frac{N}{\log^3 N}(1 + O(1)).$$

Theorem 3.1.20. $p_2 = 2\hat{\times}p_1 + 1,\ p_3 = 6\hat{\times}p_1 + 1,\ p_4 = 8\hat{\times}p_1 + 1$.
Since $J_2(\omega) \neq 0$, there are infinitely many 4-tuples of primes.
Let $\hat{T} = 1$, we have

$$J_2(\omega) = \sum_{\alpha=1}^{\phi(\omega)} \left[\frac{3}{(2p_\alpha + 1, \omega) + (6p_\alpha + 1, \omega) + (8p_\alpha + 1, \omega)} \right]$$

$$= \prod_{5 \leq p \leq p_i} (p - 4) \neq 0.$$

$$\pi_4(N, 2) = \frac{J_2(\omega)\omega^3}{\phi^4(\omega)} \frac{N}{\log^4 N}(1 + O(1)).$$

Theorem 3.1.21 $p_2 = 2\hat{\times}p_1 + 1,\ p_3 = 3\hat{\times}p_1 + 2,\ p_4 = 4\hat{\times}p_1 + 3$, where $\hat{T}$ is a prime greater than 3.

Since $J_2(\omega) \neq 0$, there are infinitely many 4-tuples of primes.
Let $\hat{T} = 1$, we have

$$J_2(\omega) = \sum_{\alpha=1}^{\phi(\omega)} \left[\frac{3}{(2p_\alpha + 1, \omega) + (3p_\alpha + 2, \omega) + (4p_\alpha + 3, \omega)}\right]$$

$$= \prod_{5 \leq p \leq p_i} (p - 4) \neq 0.$$

$$\pi_4(N, 2) = \frac{J_2(\omega)\omega^3}{\phi^4(\omega)} \frac{N}{\log^4 N}(1 + O(1)).$$

Theorem 3.1.22. $p_2 = 30\hat{\times}p_1 + 1,\ p_3 = 60\hat{\times}p_1 + 1,\ p_4 = 90\hat{\times}p_1 + 1.$
Since $J_2(\omega) \neq 0$, there are infinitely many 4-tuples of primes.
Let $\hat{T} = 1$, we have

$$J_2(\omega) = \sum_{\alpha=1}^{\phi(\omega)} \left[\frac{3}{\sum_{i=1}^{3}(30ip_\alpha + 1, \omega)}\right] = 8 \prod_{7 \leq p \leq p_i} (p - 4) \neq 0.$$

$$\pi_4(N, 2) = \frac{J_2(\omega)\omega^3}{\phi^4(\omega)} \frac{N}{\log^4 N}(1 + O(1)).$$

Theorem 3.1.23. $p_2 = p_1^{\hat{2}} + 30,\ p_3 = p_1^{\hat{2}} + 60,\ p_4 = p_1^{\hat{2}} + 90,$
where $\hat{T}$, is a prime greater than 5.
Let $\hat{T} = 1$, we have

$$J_2(\omega) = \sum_{\alpha=1}^{\phi(\omega)} \left[\frac{3}{\sum_{i=1}^{3}(p_\alpha^2 + 30i, \omega)}\right]$$

$$= 8 \prod_{7 \leq p \leq p_i} \left(p - 4 - \left(\frac{-30}{p}\right) - \left(\frac{-15}{p}\right) - \left(\frac{-10}{p}\right)\right) \neq 0.$$

$$\pi_4(N, 2) = \frac{J_2(\omega)\omega^3}{8\phi^4(\omega)} \frac{N}{\log^4 N}(1 + O(1)).$$

Theorem 3.1.24. $p_2 = p_1^{\hat{3}} + 30,\ p_3 = p_1^{\hat{3}} + 60,\ p_4 = p_1^{\hat{3}} + 90$, where $\hat{T}$ is a prime greater than 5.
Since $J_2(\omega) \neq 0$, there are infinitely many 4-tuples of primes.

Let $\hat{T} = 1$, we have

$$J_2(\omega) = \sum_{\alpha=1}^{\phi(\omega)} \left[\frac{3}{\sum_{i=1}^{3}(p_\alpha^3 + 30i, \omega)} \right] = 8 \prod_{7 \le p \le p_i} (p - 4 - \chi_1(p) - \chi_2(p) - \chi_3(p)) \neq 0.$$

where $\chi_i(p) = 2$ if $B_i^{\frac{p-1}{3}} \equiv 1 (\bmod\ p)$; $\chi_i(p) = -1$ if $B_i^{\frac{p-1}{3}} \not\equiv 1 (\bmod\ p)$, $B_1 = 30$, $B_2 = 60$, $B_3 = 90$; $\chi(p) = 0$ otherwise.

$$\pi_4(N, 2) = \frac{J_2(\omega)\omega^3}{27\phi^4(\omega)} \frac{N}{\log^4 N}(1 + O(1)).$$

Theorem 3.1.25. $p_2 = 2\hat{\times}p_1 + 1$, $p_3 = 6\hat{\times}p_1 + 1$, $p_4 = 8\hat{\times}p_1 + 1$, $p_5 = 12\hat{\times}p_1 + 1$.
Since $J_2(\omega) \neq 0$, there are infinitely many 5-tuples of primes.
Let $\hat{T} = 1$, we have

$$J_2(\omega) = \sum_{\alpha=1}^{\phi(\omega)} \left[\frac{1}{(2p_\alpha + 1, \omega)\ (6p_\alpha + 1, \omega)\ (8p_\alpha + 1, \omega)\ (12p_\alpha + 1, \omega)} \right]$$

$$= \prod_{7 \le p \le p_i} (p - 5) \neq 0.$$

$$\pi_5(N, 2) = \frac{J_2(\omega)\omega^4}{\phi^5(\omega)} \frac{N}{\log^5 N}(1 + O(1)).$$

Theorem 3.1.26. $p_2 = 30\hat{\times}p_1 + 1$, $p_3 = 60\hat{\times}p_1 + 1$, $p_4 = 90\hat{\times}p_1 + 1$, $p_5 = 120\hat{\times}p_1 + 1$.
Since $J_2(\omega) \neq 0$, there are infinitely many 5-tuples of primes.
Let $\hat{T} = 1$, we have

$$J_2(\omega) = \sum_{\alpha=1}^{\phi(\omega)} \left[\frac{4}{\sum_{i=1}^{4}(30ip_\alpha + 1, \omega)} \right] = 8 \prod_{7 \le p \le p_i} (p - 5) \neq 0.$$

$$\pi_5(N, 2) = \frac{J_2(\omega)\omega^4}{\phi^5(\omega)} \frac{N}{\log^5 N}(1 + O(1)).$$

Theorem 3.1.27. $p_2 = p_1^{\hat{2}} + 30$, $p_3 = p_1^{\hat{2}} + 60$, $p_4 = p_1^{\hat{2}} + 90$, $p_5 = p_1^{\hat{2}} + 120$, where $\hat{T}$ is a prime greater than 5.
If $J_2(\omega) \neq 0$, then there are infinitely many 5-tuples of primes.

Let $\hat{T}=1$, we have

$$J_2(\omega)=\sum_{\alpha=1}^{\phi(\omega)}\left[\frac{4}{\sum_{i=1}^{4}(p_\alpha^2+30i,\omega)}\right]$$

$$=8\prod_{7\le p\le p_i}(p-5-2(\frac{-30}{p})-(\frac{-15}{p})-(\frac{-10}{p}))\neq 0.$$

$$\pi_5(N,2)=\frac{J_2(\omega)\omega^4}{16\phi^5(\omega)}\frac{N}{\log^5 N}(1+O(1)).$$

Theorem 3.1.28. $p_2=p_1^{\hat{3}}+30,\ p_3=p_1^{\hat{3}}+60,\ p_4=p_1^{\hat{3}}+90,\ p_5=p_1^{\hat{3}}+120$, where $\hat{T}$ is a prime greater than 5.

If $J_2(\omega)\neq 0$, then there are infinitely many 5-tuples of primes.If $J_2(\omega)=0$, then there are no 5-tuples of primes.

Let $\hat{T}=1$, we have $J_2(7)=0$. then there are no 5-tuples of primes.

Theorem 3.1.29. $p_2=4\hat{\times}p_1+1,\ p_3=6\hat{\times}p_1+1,\ p_4=10\hat{\times}p_1+1,\ p_5=12\hat{\times}p_1+1,\ p_6=16\hat{\times}p_1+1$.

Since $J_2(\omega)\neq 0$, there are infinitely many 6-tuples of primes.

Let $\hat{T}=1$, we have

$$J_2(\omega)=$$

$$\sum_{\alpha=1}^{\phi(\omega)}\left[\frac{1}{(4p_\alpha+1,\omega)(6p_\alpha+1,\omega)(10p_\alpha+1,\omega)(12p_\alpha+1,\omega)(16p_\alpha+1,\omega)}\right]$$

$$=\prod_{7\le p\le p_i}(p-6)\neq 0.$$

$$\pi_6(N,2)=\frac{J_2(\omega)\omega^5}{\phi^6(\omega)}\frac{N}{\log^6 N}(1+O(1)).$$

Theorem 3.1.30. $p_2=30\hat{\times}p_1+1,\ p_3=60\hat{\times}p_1+1,\ p_4=90\hat{\times}p_1+1,\ p_5=120\hat{\times}p_1+1,\ p_6=150\hat{\times}p_1+1$.

Since $J_2(\omega)\neq 0$, there are infinitely many 6-tuples of primes.

Let $\hat{T}=1$, we have

$$J_2(\omega)=\sum_{\alpha=1}^{\phi(\omega)}\left[\frac{5}{\sum_{i=1}^{5}(30ip_\alpha+1,\omega)}\right]=8\prod_{7\le p\le p_i}(p-6)\neq 0.$$

$$\pi_6(N,2)=\frac{J_2(\omega)\omega^5}{\phi^6(\omega)}\frac{N}{\log^6 N}(1+O(1)).$$

Theorem 3.1.31. $p_2=p_1^{\hat{2}}+30,\ p_3=p_1^{\hat{2}}+60,\ p_4=p_1^{\hat{2}}+90,\ p_5=p_1^{\hat{2}}+120.\ p_6=p_1^{\hat{2}}+150$, where $\hat{T}$ is a prime greater than 5.

If $J_2(\omega)\neq 0$, then there are infinitely many 6-tuples of primes.

Let $\hat{T}=1$, we have

$$J_2(\omega)=\sum_{\alpha=1}^{\phi(\omega)}\left[\frac{5}{\sum_{i=1}^{5}(p_\alpha^2+30i,\omega)}\right]$$

$$=8\prod_{7\le p\le p_i}\left(p-6-2\left(\frac{-30}{p}\right)-\left(\frac{-6}{p}\right)-\left(\frac{-10}{p}\right)-\left(\frac{-15}{p}\right)\right)\neq 0.$$

$$\pi_6(N,2)=\frac{J_2(\omega)\omega^5}{32\phi^6(\omega)}\frac{N}{\log^6 N}(1+O(1)).$$

Theorem 3.1.32. $p_2=p_1^{\hat{4}}+30,\ p_3=p_1^{\hat{4}}+60,\ p_4=p_1^{\hat{4}}+90,\ p_5=p_1^{\hat{4}}+120,\ p_6=p_1^{\hat{4}}+150$, where $\hat{T}$ is a prime greater than 5.

If $J_2(\omega)\neq 0$, then there are infinitely many 6-tuples of primes.

Let $\hat{T}=1$, we have

$$J_2(\omega)=\sum_{\alpha=1}^{\phi(\omega)}\left[\frac{5}{\sum_{i=1}^{5}(p_\alpha^4+30i,\omega)}\right]=8\prod_{7\le p\le p_i}\left(p-6-\sum_{i-1}^{5}\chi_i(p)\right)\neq 0.$$

where $\chi_i(p)=3$ if $B_i^{\frac{p-1}{4}}\equiv(-1)^{\frac{p-1}{4}}(\text{mod}p), \chi(p)=-1$ if $B_i^{\frac{p-1}{4}}\not\equiv(-1)^{\frac{p-1}{4}}(\text{mod}p)$, $B_1=30,\ B_2=60,\ B_3=90,\ B_4=120,\ B_5=150;\ \chi_i(p)=(-\frac{B_i}{p})$ otherwise,

$$\pi_6(N,2)=\frac{J_2(\omega)\omega^5}{1024\phi^6(\omega)}\frac{N}{\log^6 N}(1+O(1)).$$

Theorem 3.1.33. $p_3=p_1\hat{\times}p_2+b$, where $(\hat{T},b)=1$ and $2|\hat{T}b$.

We have

$$J_3(\omega)=\sum_{\alpha_2=1}^{\phi(\omega)}\sum_{\alpha_1=1}^{\phi(\omega)}\left[\frac{1}{(\hat{T}p_{\alpha_1}p_{\alpha_2}+b,\omega)}\right]=\phi(\omega)\prod_{3\le p\le p_i}(p-2)\prod_{p|\hat{T}b}\frac{p-1}{p-2}\neq 0,$$

$$\pi_2(N,3)=\frac{1}{2}\prod_{3\le p\le p_i}(1-\frac{1}{(p-1)^2}\prod_{p|\hat{T}b}\frac{p-1}{p-2}\frac{N^2}{\log^3 N}(1+O(1)).$$

This is three primes theorem called (1+2).

Theorem 3.1.34. $p_3 = N - p_1 \hat{\times} p_2$, $J_3(\omega) = 0$ if $(N, \hat{T}) > 1$; $J_3(\omega) \neq 0$ if $(N, \hat{T}) = 1$.

Let $\hat{T} = 1$, we have

$$J_3(\omega) = \sum_{\alpha_2=1}^{\phi(\omega)} \sum_{\alpha_1=1}^{\phi(\omega)} \left[\frac{1}{(N - p_{\alpha_1} p_{\alpha_2}, \omega)} \right] = \phi(\omega) \prod_{3 \leq p \leq p_i} (p-2) \prod_{p|N} \frac{p-1}{p-2} \neq 0,$$

$$\pi_2(N,3) = |\{p_1, p_2 : p_1, p_2 \leq N, p_3 = |N - p_1 p_2|\}|$$
$$= \frac{1}{2} \prod_{3 \leq p \leq p_i} \left(1 - \frac{1}{(p-1)^2}\right) \prod_{p|N} \frac{p-1}{p-2} \frac{N^2}{\log^3 N}(1 + O(1)).$$

This is three primes theorem called (1+2).

Theorem 3.1.35. $p_3 = (p_1 + p_2)^{\hat{2}} + b$, where b is an odd and $3 \not| (\hat{T} + b)$.

We have

$$J_3(\omega) = \sum_{\alpha_2=1}^{\phi(\omega)} \sum_{\alpha_1=1}^{\phi(\omega)} \left[\frac{1}{(\hat{T}(p_{\alpha_1} + p_{\alpha_2})^2 + b, \omega)} \right] = \prod_{3 \leq p \leq p_i} (p^2 - 3p + 3 - \chi(p)) \neq 0,$$

where $\chi(p) = (p-2)$ if $p|\hat{T}b$; $\chi(p) = (p-2)(\frac{-b\hat{T}}{p})$ otherwise,

$$\pi_2(N,3) = \frac{J_3(\omega)\omega}{4\phi^3(\omega)} \frac{N^2}{\log^3 N}(1 + O(1)).$$

Theorem 3.1.36. $p_3 = (p_1 + p_2)^{\hat{3}} + 3$, where $3 \not| \hat{T}$

Since $J_3(\omega) \neq 0$, there exist infinitely many primes p_1 and p_2 such that p_3 is also a prime.

Let $\hat{T} = 1$, we have

$$J_3(\omega) = \sum_{\alpha_2=1}^{\phi(\omega)} \sum_{\alpha_1=1}^{\phi(\omega)} \left[\frac{1}{((p_{\alpha_1} + p_{\alpha_2})^3 + 3, \omega)} \right] = \prod_{3 \leq p \leq p_i} (p^2 - 3p + 3 - \chi(p)) \neq 0,$$

where $\chi(3) = 1, \chi(p) = 2(p-2)$ if $3^{\frac{p-1}{3}} \equiv 1 (\text{mod } p)$; $\chi(p) = -(p-2)$ if $3^{\frac{p-1}{3}} \not\equiv 1 (\text{mod } p)$; $\chi(p) = 0$ otherwise,

$$\pi_2(N,3) = \frac{J_3(\omega)\omega}{6\phi^3(\omega)} \frac{N^2}{\log^3 N}(1 + O(1)).$$

Theorem 3.1.37. $p_3 = (p_1 + p_2)^{\hat{4}} + 1$.

If $J_3(\omega) = 0$, then there are finitely many prime solutions. If $J_3(\omega) \neq 0$, then there are infinitely many prime solutions.

Let $\hat{T} = 1$, we have

$$J_3(\omega) = \sum_{\alpha_2=1}^{\phi(\omega)} \sum_{\alpha_1=1}^{\phi(\omega)} \left[\frac{1}{((p_{\alpha_1} + p_{\alpha_2})^4 + 1, \omega)} \right] = \prod_{3 \le p \le p_i} (p^2 - 3p + 3 - \chi(p)) \neq 0,$$

where $\chi(p) = 3(p-2)$ if $p \equiv 1 (\mathrm{mod}\ 8)$; $\chi(p) = -(p-2)$ if $p \not\equiv 1\ (\mathrm{mod}\ 8)$.

$$\pi_2(N, 3) = \frac{J_3(\omega)\omega}{8\phi^3(\omega)} \frac{N^2}{\log^3 N}(1 + O(1)).$$

Theorem 3.1.38. $p_3 = p_1 \hat{\times} p_2^{\hat{2}} + b$, where $(\hat{T}, b) = 1$ and $2|\hat{T}b$.

Since $J_3(\omega) \neq 0$ there exist infinitely many prime solutions.

Let $\hat{T} = 1$, we have

$$J_3(\omega) = \sum_{\alpha_2=1}^{\phi(\omega)} \sum_{\alpha_1=1}^{\phi(\omega)} \left[\frac{1}{(p_{\alpha_1} p_{\alpha_2}^2 + b, \omega)} \right] = \phi(\omega) \prod_{3 \le p \le p_i} (p-2) \prod_{p|b} \frac{p-1}{p-2} \neq 0,$$

$$\pi_2(N, 3) = \frac{J_3(\omega)\omega}{6\phi^3(\omega)} \frac{N^2}{\log^3 N}(1 + O(1)).$$

Theorem 3.1.39. $p_3 = p_1 \hat{\times} p_2^{\hat{3}} + b$, where $(\hat{T}, b) = 1$ and $2|\hat{T}b$.

Since $J_3(\omega) \neq 0$, there are infinitely many prime solutions.

Let $\hat{T} = 1$, we have

$$J_3(\omega) = \sum_{\alpha_2=1}^{\phi(\omega)} \sum_{\alpha_1=1}^{\phi(\omega)} \left[\frac{1}{(p_{\alpha_1} p_{\alpha_2}^3 + b, \omega)} \right] = \phi(\omega) \prod_{3 \le p \le p_i} (p-2) \prod_{p|b} \frac{p-1}{p-2} \neq 0,$$

$$\pi_2(N, 3) = \frac{J_3(\omega)\omega}{8\phi^3(\omega)} \frac{N^2}{\log^3 N}(1 + O(1)).$$

Theorem 3.1.40. $p_3 = p_1 \hat{\times} p_2^{\hat{4}} + b$, where $(\hat{T}, b) = 1$ and $2|\hat{T}b$.

Since $J_3(\omega) \neq 0$, there are infinitely many prime solutions.

Let $\hat{T} = 1$, we have

$$J_3(\omega) = \sum_{\alpha_2=1}^{\phi(\omega)} \sum_{\alpha_1=1}^{\phi(\omega)} \left[\frac{1}{(p_{\alpha_1} p_{\alpha_2}^4 + b, \omega)} \right] = \phi(\omega) \prod_{3 \le p \le p_i} (p-2) \prod_{p|b} \frac{p-1}{p-2} \neq 0,$$

$$\pi_2(N,3) = \frac{J_3(\omega)\omega}{10\phi^3(\omega)} \frac{N^2}{\log^3 N}(1+O(1)).$$

Theorem 3.1.41. $p_4 = p_1 \hat{\times} p_2 \hat{\times} p_3 + b$, where $(\hat{T}, b) = 1$ and $2|\hat{T}b$.
Since $J_4(\omega) \neq 0$, there are infinitely many prime solutions.
Let $\hat{T} = 1$, we have

$$J_4(\omega) = \sum_{\alpha_3=1}^{\phi(\omega)} \sum_{\alpha_2=1}^{\phi(\omega)} \sum_{\alpha_1=1}^{\phi(\omega)} \left[\frac{1}{(p_{\alpha_1} p_{\alpha_2} p_{\alpha_3} + b, \omega)}\right]$$

$$= \phi^2(\omega) \prod_{3 \leq p \leq p_i} (p-2) \prod_{p|b} \frac{p-1}{p-2} \neq 0,$$

$$\pi_2(N,4) = \frac{1}{9} \prod_{3 \leq p \leq p_i} \left(1 - \frac{1}{(p-1)^2}\right) \prod_{p|b} \frac{p-1}{p-2} \frac{N^3}{\log^4 N}(1+O(1)).$$

It is the four primes theorem called (1+3).

Theorem 3.1.42. $p_4 = N - p_1 \hat{\times} p_2 \hat{\times} p_3$.

$$J_4(\omega) = 0 \text{ if } (N, \hat{T}) > 1; \;\; J_4(\omega) \neq 0 \text{if} (N, \hat{T}) = 1.$$

Let $\hat{T} = 1$, we have

$$J_4(\omega) = \sum_{\alpha_3=1}^{\phi(\omega)} \sum_{\alpha_2=1}^{\phi(\omega)} \sum_{\alpha_1=1}^{\phi(\omega)} \left[\frac{1}{(N - p_{\alpha_1} p_{\alpha_2} p_{\alpha_3}, \omega)}\right]$$

$$= \phi^2(\omega) \prod_{3 \leq p \leq p_i} (p-2) \prod_{p|b} \frac{p-1}{p-2} \neq 0,$$

$$\pi_2(N,4) = |\{p_1, p_2, p_3 : p_1, p_2, p_3 \leq N, p_4 = |N - p_1 p_2 p_3|\}|$$

$$= \frac{1}{9} \prod_{3 \leq p \leq p_i} (1 - \frac{1}{(p-1)^2}) \prod_{p|N} \frac{p-1}{p-2} \frac{N^3}{\log^4 N}(1+O(1)).$$

It is the four primes theorem called (1+3).

Theorem 3.1.43. $p_4 = p_1 \hat{\times} p_2 + p_3 + b$, where $\hat{T}$ and b are both odds or both evens.

Since $J_4(\omega) \neq 0$, there are infinitely many prime solutions.
Let $\hat{T} = 1$, we have

$$J_4(\omega) = \sum_{\alpha_3=1}^{\phi(\omega)} \sum_{\alpha_2=1}^{\phi(\omega)} \sum_{\alpha_1=1}^{\phi(\omega)} \left[\frac{1}{(p_{\alpha_1} p_{\alpha_2} + p_{\alpha_3} + b, \omega)}\right]$$

$$= \prod_{3 \le p \le p_i} \left(\frac{(p-1)^4 - 1}{p} - \chi(p) \right) \neq 0,$$

where $\chi(p) = p - 2$ if $p|b$; $\chi(p) = -1$ if $p \not| b$.

$$\pi_2(N, 4) = \frac{J_4(\omega)\omega}{18\phi^4(\omega)} \frac{N^3}{\log^4 N}(1 + O(1)).$$

Theorem 3.1.44. $p_4 = p_1 \hat{\times} p_2^{\hat{2}} + p_3 + b$, where $\hat{T}$ and b are both odds or both evens.

Since $J_4(\omega) \neq 0$, there are infinitely many prime solutions.

Let $\hat{T} = 1$, we have

$$J_4(\omega) = \sum_{\alpha_3=1}^{\phi(\omega)} \sum_{\alpha_2=1}^{\phi(\omega)} \sum_{\alpha_1=1}^{\phi(\omega)} \left[\frac{1}{(p_{\alpha_1} p_{\alpha_2}^2 + p_{\alpha_3} + b, \omega)} \right]$$

$$= \prod_{3 \le p \le p_i} \left(\frac{(p-1)^4 - 1}{p} - \chi(p) \right) \neq 0,$$

where $\chi(p) = p - 2$ if $p|b$; $\chi(p) = -1$ if $p \not| b$.

$$\pi_2(N, 4) = \frac{J_4(\omega)\omega}{12\phi^4(\omega)} \frac{N^3}{\log^4 N}(1 + O(1)).$$

Theorem 3.1.45. $p_4 = (p_1 + p_2 + p_3 - 1)^{\hat{2}} + 1,\ p_5 = (p_1 + p_2 + p_3 + 1)^{\hat{2}} + 1.$

Let $\hat{T} = 1$, we have

$$J_4(\omega) = \sum_{\alpha_3=1}^{\phi(\omega)} \sum_{\alpha_2=1}^{\phi(\omega)} \sum_{\alpha_1=1}^{\phi(\omega)} \left[\frac{1}{((p_{\alpha_1}+p_{\alpha_2}+p_{\alpha_3}-1)^2+1,\omega)((p_{\alpha_1}+p_{\alpha_2}+p_{\alpha_3}+1)^2+1,\omega)} \right]$$

$$= \prod_{3 \le p \le p_i} \left(\frac{(p-1)^4 - 1}{p} - \chi(p) \right) \neq 0,$$

where $\chi(p) = (1 + 2(-1)^{\frac{p-1}{2}})(p^2 - 3p + 3)$,

$$\pi_3(N, 4) = \frac{J_4(\omega)\omega^2}{24\phi^5(\omega)} \frac{N^3}{\log^5 N}(1 + O(1)).$$

Let $\hat{T} = 4$, we have

$$J_4(\omega) = \sum_{\alpha_3=1}^{\phi(\omega)} \sum_{\alpha_2=1}^{\phi(\omega)} \sum_{\alpha_1=1}^{\phi(\omega)} \left[\frac{1}{(4(p_{\alpha_1}+p_{\alpha_2}+p_{\alpha_3}-1)^2+1,\omega)(4(p_{\alpha_1}+p_{\alpha_2}+p_{\alpha_3}+1)^2+1,\omega)} \right]$$

$$= \prod_{3\le p\le p_i} (\frac{(p-1)^4-1}{p} - \chi(p)) \neq 0,$$

where $\chi(5)=25,\ \chi(p)=(1+2(-1)^{\frac{p-1}{2}})(p^2-3p+3)$,

$$\pi_3(N,4) = \frac{J_4(\omega)\omega^2}{24\phi^5(\omega)} \frac{N^3}{\log^5 N}(1+O(1)).$$

Since $J_4(\omega)\neq 0$, there exist infinitely many prime solutions.

Theorem 3.1.46. $p_5 = p_1\hat{\times}p_2\hat{\times}p_3\hat{\times}p_4 + b$, where $(\hat{T},b)=1$ and $2|\hat{T}b$.
Since $J_5(\omega)\neq 0$, there are infinitely many prime solutions.
Let $\hat{T}=1$, we have

$$J_5(\omega) = \sum_{\alpha_4=1}^{\phi(\omega)}\sum_{\alpha_3=1}^{\phi(\omega)}\sum_{\alpha_2=1}^{\phi(\omega)}\sum_{\alpha_1=1}^{\phi(\omega)} \left[\frac{1}{(p_{\alpha_1}p_{\alpha_2}p_{\alpha_3}p_{\alpha_4}+b,\omega)}\right]$$

$$= \phi^3(\omega)\prod_{3\le p\le p_i}(p-2)\prod_{p|b}\frac{p-1}{p-2} \neq 0,$$

$$\pi_2(N,5) = \frac{J_5(\omega)\omega}{96\phi^5(\omega)} \frac{N^4}{\log^5 N}(1+O(1)).$$

It is the five primes theorem called (1+4).

Theorem 3.1.47. $p_5 = N - p_1\hat{\times}p_2\hat{\times}p_3\hat{\times}p_4$
$J_5(\omega)=0$ if $(\hat{T},N)>1$; $J_5(\omega)\neq 0$ if $(\hat{T},N)=1$.
Let $\hat{T}=1$, we have

$$J_5(\omega) = \sum_{\alpha_4=1}^{\phi(\omega)}\sum_{\alpha_3=1}^{\phi(\omega)}\sum_{\alpha_2=1}^{\phi(\omega)}\sum_{\alpha_1=1}^{\phi(\omega)} \left[\frac{1}{(N-p_{\alpha_1}p_{\alpha_2}p_{\alpha_3}p_{\alpha_4},\omega)}\right]$$

$$= \phi^3(\omega)\prod_{3\le p\le p_i}(p-2)\prod_{p|N}\frac{p-1}{p-2} \neq 0,$$

$$\pi_2(N,5) = |\{p_1,p_2,p_3,p_4 : p_1,p_2,p_3,p_4 \le N, p_5 = |N-p_1p_2p_3p_4|\}|$$

$$= \frac{1}{48}\prod_{3\le p\le p_i}\left(1-\frac{1}{(p-1)^2}\right)\prod_{p|N}\frac{p-1}{p-2}\frac{N^4}{\log^5 N}(1+O(1)).$$

It is the five primes theorem called (1+4).

Theorem 3.1.48. $p_5 = p_1 + p_2 + p_3^{\hat{2}}\hat{\times}p_4^{\hat{3}}$, where $\hat{T}$ is an odd.
Since $J_5(\omega)\neq 0$ there are infinitely many prime solutions.

Let $\hat{T} = 1$, we have

$$J_5(\omega) = \sum_{\alpha_4=1}^{\phi(\omega)} \sum_{\alpha_3=1}^{\phi(\omega)} \sum_{\alpha_2=1}^{\phi(\omega)} \sum_{\alpha_1=1}^{\phi(\omega)} \left[\frac{1}{(p_{\alpha_1} + p_{\alpha_2} + p_{\alpha_3}^2 p_{\alpha_4}^3, \omega)}\right]$$

$$= \prod_{3 \le p \le p_i} \left(\frac{(p-1)^5 + 1}{p} + p - 2\right) \neq 0.$$

$$\pi_2(N, 5) = \frac{J_5(\omega)\omega}{120\phi^5(\omega)} \frac{N^4}{\log^5 N}(1 + O(1)).$$

Theorem 3.1.49. $p_5 = p_1 + p_2 + p_3^{\hat{2}} \hat{\times} p_4^{\hat{3}} + b$, where b is an even and $\hat{T}$ is an odd. Let $\hat{T} = 1$, we have

$$J_5(\omega) = \sum_{\alpha_4=1}^{\phi(\omega)} \sum_{\alpha_3=1}^{\phi(\omega)} \sum_{\alpha_2=1}^{\phi(\omega)} \sum_{\alpha_1=1}^{\phi(\omega)} \left[\frac{1}{(p_{\alpha_1} + p_{\alpha_2} + p_{\alpha_3}^2 p_{\alpha_4}^3 + b, \omega)}\right]$$

$$= \prod_{3 \le p \le p_i} \left(\frac{(p-1)^5 + 1}{p} - \chi(p)\right) \neq 0.$$

where $\chi(p) = 1$ if $p \nmid b$; $\chi(p) = -(p-2)$ if $p|b$.

$$\pi_2(N, 5) = \frac{J_5(\omega)\omega}{120\phi^5(\omega)} \frac{N^4}{\log^5 N}(1 + O(1)).$$

Theorem 3.1.50. $p_5 = p_1 + p_2 \hat{\times} (p_3 + p_4) + 2$.
Since $J_5(\omega) \neq 0$, there are infinitely many prime solutions.
Let $\hat{T} = 1$, we have

$$J_5(\omega) = \sum_{\alpha_4=1}^{\phi(\omega)} \sum_{\alpha_3=1}^{\phi(\omega)} \sum_{\alpha_2=1}^{\phi(\omega)} \sum_{\alpha_1=1}^{\phi(\omega)} \left[\frac{1}{(p_{\alpha_1} + p_{\alpha_2}(p_{\alpha_3} + p_{\alpha_4}) + 2, \omega)}\right]$$

$$= \prod_{3 \le p \le p_i} \left(\frac{(p-1)^5 + 1}{p} - 1\right) \neq 0.$$

$$\pi_2(N, 5) = \frac{J_5(\omega)\omega}{48\phi^5(\omega)} \frac{N^4}{\log^5 N}(1 + O(1)).$$

Theorem 3.1.51. $p_n = p_1 \hat{\times} p_2 \hat{\times} \cdots \hat{\times} p_{n-1} + b$, where $(b, \hat{T}) = 1$ and $2|b\hat{T}$.

We have

$$J_n(\omega) = \sum_{\alpha_{n-1}=1}^{\phi(\omega)} \cdots \sum_{\alpha_1=1}^{\phi(\omega)} \left[\frac{1}{(\hat{T}^{n-2} p_{\alpha_1} \cdots p_{\alpha_{n-1}} + b, \omega)} \right]$$

$$= \phi^{n-2}(\omega) \prod_{3 \le p \le p_i} (p-2) \prod_{p|\hat{T}b} \frac{p-1}{p-2} \neq 0,$$

$$\pi_2(N,n) = \frac{2}{(n-1)(n-1)!} \prod_{3 \le p \le p_i} (1 - \frac{1}{(p-1)^2}) \prod_{p|\hat{T}b} \frac{p-1}{p-2} \frac{N^{n-1}}{\log^n N} (1 + O(1)).$$

Let $n = 2$, it is the prime twins theorem.

Theorem 3.1.52. $p_n = N - p_1 \hat{\times} p_2 \hat{\times} \cdots \hat{\times} p_{n-1}$.
$J_n(\omega) = 0$ if $(N, \hat{T}) > 1$; $J_n(\omega) \neq 0$ if $(N, \hat{T}) = 1$.
Let $\hat{T} = 1$, we have

$$J_n(\omega) = \sum_{\alpha_{n-1}=1}^{\phi(\omega)} \cdots \sum_{\alpha_1=1}^{\phi(\omega)} \left[\frac{1}{(N - p_{\alpha_1} \cdots p_{\alpha_{n-1}}, \omega)} \right]$$

$$= \phi^{n-2}(\omega) \prod_{3 \le p \le p_i} (p-2) \prod_{p|N} \frac{p-1}{p-2} \neq 0,$$

$$\pi_2(N,n) = |\,\{p_1, \cdots, p_{n-1} : p_1, \cdots, p_{n-1} \le N, p_n = |N - p_1 \cdots p_{n-1}|\}\,|$$

$$= \frac{2}{(n-1)(n-1)!} \prod_{3 \le p \le p_i} (1 - \frac{1}{(p-1)^2}) \prod_{p|N} \frac{p-1}{p-2} \frac{N^{n-1}}{\log^n N} (1 + O(1)),$$

It is the Rényi's theorem. Let $n = 2$, it is the Goldbach's theorem, see theorem 3.1.2.

Note. All sieve methods obtain only the upper estimates, but the lower estimates are more difficult. Unfortunately, it turns out there is no method which will give such a formula for general sifting function. $J_n(\omega)$ is a precise shifting function. The Santilli's isoadditive prime theory will take the the place of all sieve methods.

Theorem 3.1.53. $p_3 = (p_1 + 1) \hat{\hat{\times}} (p_2 + 1) + 1$, where p_1, p_2 and p_3 are the odd primes.
Since $J_3(\omega) \neq 0$, there are infinitely many prime solutions.
For $\hat{T} = 1$, we have

$$J_3(\omega) = \prod_{3 \le p \le p_i} (p^2 - 3p + 4) \neq 0.$$

$$\pi_2(N,3) = \frac{J_3(\omega)\omega}{4\phi^3(\omega)} \frac{N^2}{\log^3 N}(1+O(1)).$$

$(1+O(1))$ means that the error $O(1)$ is negligible in comparison to 1.

Theorem 3.1.54.

$$p_4 = (p_1+1)\hat{\times}(p_2+1)\hat{\times}(p_3+1)+1.$$

Since $J_4(\omega) \neq 0$, there are infinitely many prime solutions.

For $\hat{T}=1$, we have

$$J_4(\omega) = \prod_{3 \le p \le p_i} \Big[(p-1)^3 - H(p)\Big] \neq 0,$$

where $H(p) = (p-3)(p-2)+1$.

$$\pi_2(N,4) = \frac{J_4(\omega)\omega}{18\phi^4(\omega)} \frac{N^3}{\log^4 N}(1+O(1)).$$

Theorem 3.1.55. $p_5 = (p_1+1)\hat{\times}(p_2+1)\hat{\times}(p_3+1)\hat{\times}(p_4+1)+1$
Since $J_5(\omega) \neq 0$, there are infinitely many prime solutions.

For $\hat{T}=1$, we have

$$J_5(\omega) = \prod_{3 \le p \le p_i} \Big[(p_1)^4 - H(p)\Big] \neq 0.$$

where $H(p) = (p-3)\left[(p-2)^2+1\right]$

$$\pi_2(N,5) = \frac{J_5(\omega)\omega}{96\phi^5(\omega)} \frac{N^4}{\log^4 N}(1+O(1)).$$

Theorem 3.1.56. $p_6 = (p_1+1)\hat{\times}\cdots\hat{\times}(p_5+1)+1$
Since $J_6(\omega) \neq 0$, there are infinitely many prime solutions.

For $\hat{T}=1$, we have

$$J_6(\omega) = \prod_{3 \le p \le p_i} \Big[(p-1)^5 - H(p)\Big] \neq 0,$$

where $H(p) = (p-3)\left[(p-2)^3+(p-2)\right]+1$

$$\pi_2(N,6) = \frac{J_6(\omega)\omega}{5\times 5!\phi^6(\omega)} \frac{N^5}{\log^6 N}(1+O(1)).$$

Theorem 3.1.57. $p_7 = (p_1+1)\hat{\times}\cdots\hat{\times}(p_6+1)+1$
Since $J_7(\omega) \neq 0$, there are infinitely many prime solutions.

For $\hat{T} = 1$, we have

$$J_7(\omega) = \prod_{3 \le p \le p_i} \Big[(p-1)^6 - H(p)\Big] \ne 0,$$

where $H(p) = (p-3)\left[(p-2)^4 + (p-2) + 1\right]$

$$\pi_2(N,7) = \frac{J_7(\omega)\omega}{6 \times 6!\phi^7(\omega)} \frac{N^6}{\log^7 N}(1 + O(1)).$$

Theorem 3.1.58. $p_8 = (p_1 + 1)\hat{\times} \cdots \hat{\times}(p_7 + 1) + 1$
Since $J_8(\omega) \ne 0$, there are infinitely many prime solutions.
For $\hat{T} = 1$, we have

$$J_8(\omega) = \prod_{3 \le p \le p_i} \Big[(p-1)^7 - H(p)\Big] \ne 0,$$

where $H(p) = (p-3)\left[(p-2)^5 + (p-2)^3 + (p-2)\right] + 1$

$$\pi_2(N,8) = \frac{J_8(\omega)\omega}{7 \times 7!\phi^8(\omega)} \frac{N^7}{\log^8 N}(1 + O(1)).$$

Theorem 3.1.59. $p_9 = (p_1 + 1)\hat{\times} \cdots \hat{\times}(p_8 + 1) + 1$
Since $J_9(\omega) \ne 0$, there are infinitely many prime solutions.
For $\hat{T} = 1$, we have

$$J_9(\omega) = \prod_{3 \le p \le p_i} \Big[(p-1)^8 - H(p)\Big] \ne 0,$$

where $H(p) = (p-3)\left[(p-2)^6 + (p-2)^4 + (p-2) + 1\right]$

$$\pi_2(N,9) = \frac{J_9(\omega)\omega}{8 \times 8!\phi^9(\omega)} \frac{N^8}{\log^9 N}(1 + O(1)).$$

Theorem 3.1.60. $p_{2n+1} = (p_1 + 1)\hat{\times} \ldots \hat{\times}(p_{2n} + 1) + 1$
Since $J_{2n+1}(\omega) \ne 0$, there are infinitely many prime solutions.
For $\hat{T} = 1$, we have

$$J_{2n+1}(\omega) = \prod_{3 \le p \le p_i} \Big[(p-1)^{2n} - H(p)\Big] \ne 0,$$

where $H(p) = (p-3)\left[(p-2)^{2n-2} + (p-2)^{2n-4} + \ldots + 1\right]$

$$\pi_2(N,2n+1) = \frac{J_{2n+1}(\omega)\omega}{(2n) \times (2n)!\phi^{2n+1}(\omega)} \frac{N^{2n}}{\log^{2n+1} N}(1 + O(1)).$$

Theorem 3.1.61. $p_{2n+2} = (p_1+1)\hat{\times}\ldots\hat{\times}(p_{2n+1}+1)+1$
Since $J_{2n+2}(\omega) \neq 0$, there are infinitely many prime solutions.
For $\hat{T}=1$, we have

$$J_{2n+2}(\omega) = \prod_{3\le p\le p_i}\left[(p-1)^{2n+1} - H(p)\right] \neq 0,$$

where $H(p) = (p-2)\left[(p-2)^{2n-1} + (p-2)^{2n-3} + \ldots + (p-2)\right] + 1$

$$\pi_2(N, 2n+2) = \frac{J_{2n+2}(\omega)\omega}{(2n+1)\times(2n+1)!\phi^{2n+2}(\omega)}\frac{N^{2n+1}}{\log^{2n+2} N}(1+O(1)).$$

Theorem 3.1.62. $p_3 = (p_1+1)\hat{\times}(p_2+1)+3$
Since $J_3(\omega) \neq 0$, there are infinitely many prime solutions.
For $\hat{T}=1$, we have

$$J_3(\omega) = \prod_{3\le p\le p_i}\left[(p-1)^2 - H(p)\right] \neq 0,$$

where $H(3)=3,\ H(p)=p-3$ otherwise.

$$\pi_2(N,3) = \frac{J_3(\omega)\omega}{4\phi^3(\omega)}\frac{N^2}{\log^3 N}(1+O(1)).$$

Theorem 3.1.63. $p_3 = (p_1+1)\hat{\times}(p_2+1)+5$
Since $J_3(\omega) \neq 0$, there are infinitely many prime solutions.
For $\hat{T}=1$, we have

$$J_3(\omega) = \prod_{3\le p\le p_i}\left[(p-1)^2 - H(p)\right] \neq 0,$$

where $H(3)=1,\ H(5)=7,\ H(p)=p-3$ otherwise.

$$\pi_2(N,3) = \frac{J_3(\omega)\omega}{4\phi^3(\omega)}\frac{N^2}{\log^3 N}(1+O(1)).$$

Theorem 3.1.64. $p_3 = (p_1+1)\hat{\times}(p_2+1)+7$
Since $J_3(\omega) \neq 0$, there are infinitely many prime solutions.
For $\hat{T}=1$, we have

$$J_3(\omega) = \prod_{3\le p\le p_i}\left[(p-1)^2 - H(p)\right] \neq 0,$$

where $H(7)=11,\ H(p)=p-3$ otherwise

$$\pi_2(N,3) = \frac{J_2(\omega)\omega}{4\phi^3(\omega)}\frac{N^2}{\log^3 N}(1+O(1)).$$

Theorem 3.1.65. $p_3 = (p_1 + 1)\hat{\times}(p_2 + 1) + 9$
Since $J_3(\omega) \neq 0$, there are infinitely many prime solutions.
For $\hat{T} = 1$, we have

$$J_3(\omega) = \prod_{3 \leq p \leq p_i} \Big[(p-1)^2 - H(p)\Big] \neq 0,$$

where $H(3) = 3,\ H(5) = 3,\ H(p) = p - 3$ otherwise

$$\pi_2(N, 3) = \frac{J_3(\omega)\omega}{4\phi^3(\omega)} \frac{N^2}{\log^3 N}(1 + O(1)).$$

Theorem 3.1.66. $p_3 = (p_1 + 1)\hat{\times}(p_2 + 1) + b$, where $2 \not| b$
Since $J_3(\omega) \neq 0$, there are infinitely many prime solutions.
For $\hat{T} = 1$, we have

$$J_2(\omega) = \prod_{3 \leq p \leq p_i} \Big[(p-1)^2 - H(p)\Big] \neq 0,$$

where $H(p) = p - 2$, if $p \mid (b + 1)$; $H(p) = 2p - 3$, if $p \mid b$; $H(p) = p - 3$ otherwise.

$$\pi_2(N, 3) = \frac{J_2(\omega)\omega}{4\phi^3(\omega)} \frac{N^2}{\log^3 N}(1 + O(1)).$$

Theorem 3.1.67. $p_4 = (p_1 + 1)\hat{\times}(p_2 + 1)\hat{\times}(p_3 + 1) + b$, where $2 \not| b$
Since $J_4(\omega) \neq 0$, there are infinitely many prime solutions.
For $\hat{T} = 1$, we have

$$J_4(\omega) = \prod_{3 \leq p \leq p_i} \Big[(p-1)^3 - H(p)\Big] \neq 0,$$

where $H(p) = 3p^2 - 9p + 7$, if $p \mid b$; $H(p) = (p - 3)(p - 2)$, if $p \mid (b + 1)$; $H(p) = (p - 3)(p - 2) + 1$ otherwise

$$\pi_2(N, 4) = \frac{J_4(\omega)\omega}{18\phi^4(\omega)} \frac{N^3}{\log^4 N}(1 + O(1)).$$

Theorem 3.1.68. $p_5 = (p_1 + 1)\hat{\times}(p_2 + 1)\hat{\times}(p_3 + 1)\hat{\times}(p_4 + 1) + b,\ 2 \not| b$
Since $J_5(\omega) \neq 0$, there are infinitely many prime solutions.
For $\hat{T} = 1$, we have

$$J_5(\omega) = \prod_{3 \leq p \leq p_i} \Big[(p-1)^4 - H(p)\Big] \neq 0,$$

where $H(p) = (p-1)^4 - (p-2)^4$, if $p \mid b$; $H(p) = (p-2)\left[(p-3)(p-2)+1\right]$, if $p \mid (b+1)$; $H(p) = (p-3)\left[(p-2)^2+1\right]$ otherwise

$$\pi_2(N,5) = \frac{J_5(\omega)\omega}{96\phi^5(\omega)} \frac{N^4}{\log^5 N}(1+O(1)).$$

Theorem 3.1.69. $p_3 = (p_1+1)\hat{\times}(p_2+1) - 1$
Since $J_3(\omega) \neq 0$, there are infinitely many prime solutions.

For $\hat{T} = 1$, we have

$$J_3(\omega) = \prod_{3 \le p \le p_i} (p^2 - 3p + 3) \neq 0,$$

$$\pi_2(N,3) = \frac{J_3(\omega)\omega}{4\phi^3(\omega)} \frac{N^2}{\log^3 N}(1+O(1)).$$

Theorem 3.1.70. $p_4 = (p_1+1)\hat{\times}(p_2+1)\hat{\times}(p_3+1) - 1$
Since $J_4(\omega) \neq 0$, there are infinitely many prime solutions.

For $\hat{T} = 1$, we have

$$J_4(\omega) = \prod_{3 \le p \le p_i} \left[(p-1)^3 - H(p)\right] \neq 0,$$

where $H(p) = (p-2)^2 - (p-2)$

$$\pi_2(N,4) = \frac{J_4(\omega)\omega}{18\phi^4(\omega)} \frac{N^3}{\log^4 N}(1+O(1)).$$

Theorem 3.1.71. $p_5 = (p_1+1)\hat{\times}(p_2+1)\hat{\times}(p_3+1)\hat{\times}(p_4+1) - 1$
Since $J_4(\omega) \neq 0$, there are infinitely many prime solutions.

For $\hat{T} = 1$, we have

$$J_5(\omega) = \prod_{3 \le p \le p_i} \left[(p-1)^4 - H(p)\right] \neq 0,$$

where $H(p) = (p-2)\left[(p-2)^2 - (p-2) + 1\right]$

$$\pi_2(N,5) = \frac{J_5(\omega)\omega}{96\phi^5(\omega)} \frac{N^4}{\log^5 N}(1+O(1)).$$

Theorem 3.1.72. $p_{n+1} = (p_1+1)\hat{\times}\ldots\hat{\times}(p_n+1) - 1$
Since $J_{n+1}(\omega) \neq 0$, there are infinitely many prime solutions.

For $\hat{T} = 1$, we have

$$J_{n+1}(\omega) = \prod_{3 \le p \le p_i} \left[(p-1)^n - H(p)\right] \neq 0,$$

where $H(p) = \frac{(p-2)\left[(p-2)^{n-1}+(-1)^n\right]}{p-1}$.

$$\pi_2(N,n) = \frac{J_{n+1}(\omega)\omega}{n\times n!\phi^{n+1}(\omega)}\frac{N^n}{\log^{n+1}N}(1+O(1)).$$

Theorem 3.1.73. $p_3 = (p_1-1)\hat{\times}(p_2-1)+1$
Since $J_3(\omega)\neq 0$, there are infinitely many prime solutions.

For $\hat{T}=1$, we have

$$J_3(\omega) = \prod_{3\leq p\leq p_i}(p^2-3p+4)\neq 0,$$

$$\pi_2(N,3) = \frac{J_3(\omega)\omega}{4\phi^3(\omega)}\frac{N^2}{\log^3 N}(1+O(1)).$$

Theorem 3.1.74. $p_4 = (p_1-1)\hat{\times}(p_2-1)\hat{\times}(p_3-1)+1$
For any integer $\hat{T}$, $J_4(\omega)\neq 0$ there are infinitely many prime solutions.
For $\hat{T}=1$, we have

$$J_4(\omega) = \prod_{3\leq p\leq p_i}\Big[(p-1)^3 - H(p)\Big]\neq 0,$$

where $H(p) = (p-2)^2-(p-2)$

$$\pi_2(N,4) = \frac{J_4(\omega)\omega}{18\phi^4(\omega)}\frac{N^3}{\log^4 N}(1+O(1)).$$

Theorem 3.1.75. $p_5 = (p_1-1)\hat{\times}\ldots\hat{\times}(p_4-1)+1$
For $\hat{T}=1$, we have

$$J_5(\omega) = \prod_{3\leq p\leq p_i}\Big[(p-1)^4 - H(p)\Big]\neq 0,$$

where $H(p) = (p-2)^3-(p-2)^2+(p-3)$

$$\pi_2(N,5) = \frac{J_5(\omega)\omega}{96\phi^5(\omega)}\frac{N^4}{\log^5 N}(1+O(1)).$$

Theorem 3.1.76. $p_{n+1} = (p_1-1)\hat{\times}\ldots\hat{\times}(p_n-1)+1,\ n\geq 4$
For $\hat{T}=1$, we have

$$J_{n+1}(\omega) = \prod_{3\leq p\leq p_i}[(p-1)^n - H(p)]\neq 0,$$

where $H(p) = \frac{\left[(p-2)^4-1\right](p-2)^{n-4}}{p-1}$

$$\pi_2(N, n+1) = \frac{J_{n+1}(\omega)\omega}{n \times n!\phi^{n+1}(\omega)} \frac{N^n}{\log^{n+1} N}(1 + O(1)).$$

Theorem 3.1.77. $p_3 = (p_1 - 1)\hat{\times}(p_2 - 1) - 1$
For $\hat{T} = 1$, we have

$$J_3(\omega) = \prod_{3 \le p \le p_i} (p^2 - 3p + 3) \ne 0,$$

$$\pi_2(N, 3) = \frac{J_3(\omega)\omega}{4\phi^3(\omega)} \frac{N^2}{\log^3 N}(1 + O(1)).$$

Theorem 3.1.78. $p_4 = (p_1 - 1)\hat{\times}(p_2 - 1)\hat{\times}(p_3 - 1) - 1$
For $\hat{T} = 1$, we have

$$J_4(\omega) = \prod_{3 \le p \le p_i} \left[(p-1)^3 - H(p)\right] \ne 0,$$

where $H(p) = (p - 3)(p - 2) + 1$

$$\pi_2(N, 4) = \frac{J_4(\omega)\omega}{18\phi^5(\omega)} \frac{N^4}{\log^5 N}(1 + O(1)).$$

Theorem 3.1.79. $p_{n+1} = (p_1 - 1)\hat{\times} \dots \hat{\times}(p_n - 1) - 1,\ n \ge 3$
For $\hat{T} = 1$, we have

$$J_{n+1}(\omega) = \prod_{3 \le p \le p_i} [(p-1)^n - H(p)] \ne 0,$$

where $H(p) = (p - 2)^{n-3}\left[(p - 3)(p - 2) + 1\right]$

$$\pi_2(N, n+1) = \frac{J_{n+1}(\omega)\omega}{n \times n!\phi^{n+1}(\omega)} \frac{N^n}{\log^{n+1} N}(1 + O(1)).$$

Theorem 3.1.80. $p_3 = (p_1 + 1)\hat{\times}(p_2 - 1) + 1$
For $\hat{T} = 1$, we have

$$J_3(\omega) = \prod_{3 \le p \le p_i} (p^2 - 3p + 3) \ne 0,$$

$$\pi_2(N, 3) = \frac{J_3(\omega)\omega}{4\phi^3(\omega)} \frac{N^2}{\log^3 N}(1 + O(1)).$$

Theorem 3.1.81. $p_4 = (p_1 + 1)\hat{\times}(p_2 - 1)\hat{\times}(p_3 + 1) + 1$
For $\hat{T} = 1$, we have

$$J_4(\omega) = \prod_{3 \le p \le p_i} \left[(p-1)^3 - H(p)\right] \ne 0,$$

where $H(p) = (p-3)(p-2)$

$$\pi_2(N,4) = \frac{J_4(\omega)\omega}{18\phi^4(\omega)} \frac{N^3}{\log^4 N}(1+O(1)).$$

Theorem 3.1.82. $p_5 = (p_1+1)\hat{\times}(p_2-1)\hat{\times}(p_3+1)\hat{\times}(p_4-1)+1$
For $\hat{T}=1$, we have

$$J_5(\omega) = \prod_{3\le p\le p_i} \Big[(p-1)^4 - H(p)\Big] \ne 0,$$

where $H(p) = (p-2)^3 - (p-2)^2 + (p-3)$

$$\pi_2(N,5) = \frac{J_5(\omega)\omega}{96\phi^5(\omega)} \frac{N^4}{\log^5 N}(1+O(1)).$$

Theorem 3.1.83. $p_6 = (p_1+1)\hat{\times}(p_2-1)\hat{\times}(p_3+1)\hat{\times}(p_4-1)\hat{\times}(p_5+1)+1$
For $\hat{T}=1$, we have

$$J_6(\omega) = \prod_{3\le p\le p_i} \Big[(p-1)^5 - H(p)\Big] \ne 0,$$

where $H(p) = (p-2)^4 - (p-2)^3 + (p-2)^2 - (p-2) + 1$

$$\pi_2(N,6) = \frac{J_6(\omega)\omega}{600\phi^6(\omega)} \frac{N^5}{\log^6 N}(1+O(1)).$$

Theorem 3.1.84. $p_3 = (p_1+1)\hat{\times}(p_2-1)-1$
For $\hat{T}=1$, we have

$$J_3(\omega) = \prod_{3\le p\le p_i} [(p-1)^2 - H(p)] \ne 0,$$

where $H(3)=0;\ H(p) = p-3$ otherwise

$$\pi_2(N,3) = \frac{J_3(\omega)\omega}{4\phi^3(\omega)} \frac{N^2}{\log^3 N}(1+O(1)).$$

Theorem 3.1.85. $p_4 = (p_1+1)\hat{\times}(p_2-1)\hat{\times}(p_3+1)-1$
For $\hat{T}=1$, we have

$$J_4(\omega) = \prod_{3\le p\le p_i} \Big[(p-1)^3 - H(p)\Big] \ne 0,$$

where $H(p) = (p-2)^2 - (p-2) + 1, H(3) = 1$

$$\pi_2(N,4) = \frac{J_4(\omega)\omega}{18\phi^4(\omega)} \frac{N^3}{\log^4 N}(1+O(1)).$$

Theorem 3.1.86. $p_5 = (p_1+1)\hat{\times}(p_2-1)\hat{\times}(p_3+1)\hat{\times}(p_4-1)-1$
For $\hat{T}=1$, we have

$$J_5(\omega) = \prod_{3\le p\le p_i}\Big[(p-1)^4 - H(p)\Big] \neq 0,$$

where $H(p) = (p-2)^3-(p-2)^2+(p-2), H(3)=1$.

$$\pi_2(N,5) = \frac{J_5(\omega)\omega}{96\phi^5(\omega)}\frac{N^4}{\log^5 N}(1+O(1)).$$

Theorem 3.1.87. $p_6 = (p_1+1)\hat{\times}(p_2-1)\hat{\times}(p_3+1)\hat{\times}(p_4-1)\hat{\times}(p_5+1)-1$
For $\hat{T}=1$, we have

$$J_6(\omega) = \prod_{3\le p\le p_i}\Big[(p-1)^5 - H(p)\Big] \neq 0,$$

where $H(p) = (p-2)^4-(p-2)^3+(p-2)^2-(p-2), H(3)=0$.

$$\pi_2(N,6) = \frac{J_6(\omega)\omega}{600\phi^6(\omega)}\frac{N^5}{\log^6 N}(1+O(1)).$$

Theorem 3.1.88. $p_3 = (p_1+b)\hat{\times}(p_2+b)+b$
For $\hat{T}=1$, we have

$$J_3(\omega) = \prod_{3\le p\le p_i}\Big[(p-1)^2 - H(p)\Big] \neq 0,$$

where $H(p)=0$, if $p \mid b$; $H(p)=p-2$ if $p \mid (b+1)$; $H(p)=p-3$ otherwise.

$$\pi_2(N,3) = \frac{J_3(\omega)\omega}{4\phi^3(\omega)}\frac{N^2}{\log^3 N}(1+O(1)).$$

Theorem 3.1.89. $p_3 = (p_1+1)\hat{\times}(p_2+2)+1$
For $\hat{T}=1$, we have

$$J_3(\omega) = \prod_{3\le p\le p_i}[(p-1)^2 - H(p)] \neq 0,$$

where $H(3)=1, H(p)=p-3$ otherwise.

$$\pi_2(N,3) = \frac{J_3(\omega)\omega}{4\phi^3(\omega)}\frac{N^2}{\log^3 N}(1+O(1)).$$

Theorem 3.1.90. $p_4 = (p_1+1)\hat{\times}(p_2+2)\hat{\times}(p_3+3)+1$
For $\hat{T}=1$, we have

$$J_4(\omega) = \prod_{3\le p\le p_i}\Big[(p-1)^3 - H(p)\Big] \neq 0,$$

where $H(p) = (p-3)(p-2)+1$

$$\pi_2(N,4) = \frac{J_4(\omega)\omega}{18\phi^4(\omega)}\frac{N^3}{\log^4 N}(1+O(1)).$$

Theorem 3.1.91. $p_2 = p_1 + 2^{\hat{2}} = p_1 + 4\hat{T}$
(1) $\hat{T} = 2^j,\ j = 0, 1, \cdots,$ we have

$$J_2(\omega) = \prod_{3\le p\le p_i}(p-2) \ne 0.$$

(2) $\hat{T} = 3.6.9$, we have

$$J_2(\omega) = 2\prod_{5\le p\le p_i}(p-2) \ne 0.$$

(3) $\hat{T} = 5, 10$, we have

$$J_2(\omega) = 4\prod_{7\le p\le p_i}(p-2) \ne 0.$$

(4) $\hat{T} = 7, 14$, we have

$$J_2(\omega) = 18\prod_{11\le p\le p_i}(p-2) \ne 0.$$

(5) $\hat{T} = 11, 22$, we have

$$J_2(\omega) = 150\prod_{13\le p\le p_i}(p-2) \ne 0.$$

Since $J_2(\omega) \to \infty$ as $\omega \to \infty$, there exist infinitely many prime solutions. We have the best asymptotic formula

$$\pi_2(N,2) = \frac{J_2(\omega)\omega}{\phi^2(\omega)}\frac{N}{\log^2 N}(1+O(1)).$$

Theorem 3.1.92. $p_2 = p_1 + 2^{\hat{2}},\ p_3 = p_1 + 4^{\hat{2}}$

(1) $\hat{T} = 2^j, j = 0, 1, 2,$ we have

$$J_2(\omega) = \prod_{5\le p\le p_i}(p-3) \ne 0.$$

(2) $\hat{T} = 3, 6, 9$, we have

$$J_2(\omega) = 2\prod_{5\le p\le p_i}(p-3) \ne 0.$$

(3) $\hat{T}=5,10$, we have

$$J_2(\omega)=4\prod_{11\le p\le p_i}(p-3)\neq 0.$$

(4) $\hat{T}=7,14$, we have

$$J_2(\omega)=12\prod_{11\le p\le p_i}(p-3)\neq 0.$$

Since $J_2(\omega)\to\infty$ as $\omega\to\infty$, there exist infinitely many 3-tuples of primes. We have

$$\pi_3(N,2)=\frac{J_2(\omega)\omega^2}{\phi^3(\omega)}\frac{N}{\log^3 N}(1+O(1)).$$

Theorem 3.1.93. $p_2=p_1+2^{\hat{2}},\ p_3=p_1+4^{\hat{2}},\ p_4=p_1+6^{\hat{2}}$

(1) $\hat{T}=2^j, j=0,1,2,\cdots$, we have

$$J_2(\omega)=2\prod_{7\le p\le p_i}(p-4)\neq 0.$$

(2) $\hat{T}=3,6,9$, we have

$$J_2(\omega)=4\prod_{7\le p\le p_i}(p-4)\neq 0.$$

(3) $\hat{T}=5,10$, we have

$$J_2(\omega)=4\prod_{7\le p\le p_i}(p-4)\neq 0.$$

Since $J_2(\omega)\to\infty$ as $\omega\to\infty$, there exist infinitely many 4-tuples of primes. We have

$$\pi_4(N,2)=\frac{J_2(\omega)\omega^3}{\phi^4(\omega)}\frac{N}{\log^4 N}(1+O(1)).$$

Theorem 3.1.94. $p_2=p_1+2^{\hat{2}},\ p_3=p_1+4^{\hat{2}},\ p_4=p_1+6^{\hat{2}},\ p_5=p_1+8^{\hat{2}}$

(1) $\hat{T}=2^j, j=0,1,2,\cdots$, we have

$$J_2(\omega)=6\prod_{11\le p\le p_i}(p-5)\neq 0.$$

(2) $\hat{T}=3,6,9$, we have

$$J_2(\omega)=12\prod_{11\le p\le p_i}(p-5)\neq 0.$$

(3) $\hat{T} = 5, 10$, we have

$$J_2(\omega) = 12 \prod_{11 \le p \le p_i} (p-5) \neq 0.$$

Since $J_2(\omega) \to \infty$ as $\omega \to \infty$, there exist infinitely many 5-tuples of primes. We have

$$\pi_5(N, 2) = \frac{J_2(\omega)\omega^4}{\phi^5(\omega)} \frac{N}{\log^5 N}(1 + O(1)).$$

Theorem 3.1.95. $p_2 = p_1 + 2^{\hat{2}},\ p_3 = p_1 + 4^{\hat{2}},\ p_4 = p_1 + 6^{\hat{2}},\ p_5 = p_1 + 8^{\hat{2}},$ $p_6 = p_1 + 10^{\hat{2}}$

(1) $\hat{T} = 2^j, j = 0, 1, 2, \cdots$, we have

$$J_2(\omega) = 6 \prod_{11 \le p \le p_i} (p-6) \neq 0.$$

(2) $\hat{T} = 3, 6, 9$, we have

$$J_2(\omega) = 12 \prod_{11 \le p \le p_i} (p-6) \neq 0.$$

(3) $\hat{T} = 5, 10$, we have

$$J_2(\omega) = 12 \prod_{11 \le p \le p_i} (p-6) \neq 0.$$

Since $J_2(\omega) \to \infty$ as $\omega \to \infty$, there exist infinitely many 6-tuples of primes. We have

$$\pi_6(N, 2) = \frac{J_2(\omega)\omega^5}{\phi^6(\omega)} \frac{N}{\log^6 N}(1 + O(1)).$$

Theorem 3.1.96. $p_{j+1} = p_1 + (2j)^{\hat{2}},\ j = 1, 2, \cdots, n.$

(1) $\hat{T} = 2^j, j = 0, 1, 2, \cdots$, we have

$$J_2(\omega) = \prod_{3 \le p \le p_e} \frac{p-1}{2} \prod_{p_m \le p \le p_i} (p-n-1) \neq 0.$$

where $p_e \le 2n-1, p_m > 2n-1$.

(2) $\hat{T} = 3, 6, 9$, we have

$$J_2(\omega) = 2 \prod_{5 \le p \le p_e} \frac{p-1}{2} \prod_{p_m \le p \le p_i} (p-n-1) \neq 0.$$

where $p_e \le 2n-1, p_m > 2n-1$.

Since $J_2(\omega) \to \infty$ as $\omega \to \infty$, there exist infinitely many $(n+1)$-tuples of primes. We have the best asymptotic formula of the number for $(n+1)$-tuples of primes.

$$\pi_{n+1}(N,2) = \frac{J_2(\omega)\omega^n}{\phi^{n+1}(\omega)} \frac{N}{\log^{n+1} N}(1+O(1)).$$

Theorem 3.1.97. $p_{j+1} = p_1 + (2j)^{\hat{4}}, j = 1, \cdots, 6$
For $\hat{T} = 1$, we have

$$J_2(\omega) = 4860 \prod_{19 \le p \le p_i} (p - 7 - \chi(p)) \neq 0.$$

Where $\chi(29) = \chi(37) = -1,\ \chi(61) = -2,\ \chi(p) = 0$ otherwise

$$\pi_7(N,2) = \frac{J_2(\omega)\omega^6}{\phi^7(\omega)} \frac{N}{\log^7 N}(1+O(1)).$$

Theorem 3.1.98. $p_{j+1} = p_1 + (2j)^{\hat{6}}, j = 1, \cdots, 6$
For $\hat{T} = 1$, we have

$$J_2(\omega) = 90000 \prod_{23 \le p \le p_i} (p - 7 - \chi(p)) \neq 0.$$

Where $\chi(31) = -2,\ \chi(37) = \chi(43) = \chi(61) = -1,\ \chi(p) = 0$ otherwise.

$$\pi_7(N,2) = \frac{J_2(\omega)\omega^6}{\phi^7(\omega)} \frac{N}{\log^7 N}(1+O(1)).$$

Theorem 3.1.99. $p_{j+1} = p_1 + (2j)^{\hat{8}}, j = 1, \cdots, 6$
For $\hat{T} = 1$, we have

$$J_2(\omega) = 5670 \prod_{19 \le p \le p_i} (p - 7 - \chi(p)) \neq 0.$$

Where $\chi(29) = \chi(37) = \chi(41) = -1,\ \chi(p) = 0$, otherwise

$$\pi_7(N,2) = \frac{J_2(\omega)\omega^6}{\phi^7(\omega)} \frac{N}{\log^7 N}(1+O(1)).$$

Theorem 3.1.100. $p_{j+1} = p_1 + (2j)^{\hat{10}}, j = 1, \cdots, 6$
For $\hat{T} = 1$, we have

$$J_2(\omega) = 54 \prod_{13 \le p \le p_i} (p - 7 - \chi(p)) \neq 0.$$

Where $\chi(31) = -3,\ \chi(41) = -1, \chi(p) = 0$ otherwise

$$\pi_7(N,2) = \frac{J_2(\omega)\omega^6}{\phi^7(\omega)}\frac{N}{\log^7 N}(1+O(1)).$$

Theorem 3.1.101. $p_{j+1} = p_1 + (2j)^{\hat{12}}, j = 1, \cdots, 6$
For $\hat{T} = 1$, we have

$$J_2(\omega) = 148500 \prod_{23 \le p \le p_i}(p - 7 - \chi(p)) \neq 0.$$

Where $\chi(37) = -3, \chi(29) = \chi(31) = \chi(43) = -1,\ \chi(p) = 0$, otherwise.

$$\pi_7(N,2) = \frac{J_2(\omega)\omega^6}{\phi^7(\omega)}\frac{N}{\log^7 N}(1+O(1)).$$

Theorem 3.1.102. $p_{j+1} = p_1 + (2j)^{\hat{14}}, j = 1, \cdots, 6$
For $\hat{T} = 1$, we have

$$J_2(\omega) = 30 \prod_{13 \le p \le p_i}(p - 7 - \chi(p)) \neq 0.$$

Where $\chi(29) = \chi(43) = -4,\ \chi(p) = 0$ otherwise.

$$\pi_7(N,2) = \frac{J_2(\omega)\omega^6}{\phi^7(\omega)}\frac{N}{\log^7 N}(1+O(1)).$$

Theorem 3.1.103. $p_{j+1} = p_1 + (2j)^{\hat{16}}, j = 1, \cdots, 6$
For $\hat{T} = 1$, we have

$$J_2(\omega) = 6075 \prod_{19 \le p \le p_i}(p - 7 - \chi(p)) \neq 0.$$

Where $\chi(29) = \chi(37) = -1,\ \chi(41) = -3,\ \chi(p) = 0$, otherwise.

$$\pi_7(N,2) = \frac{J_2(\omega)\omega^6}{\phi^7(\omega)}\frac{N}{\log^7 N}(1+O(1)).$$

Theorem 3.1.104. $p_{j+1} = p_1 + (2j)^{\hat{18}}, j = 1, \cdots, 6$
For $\hat{T} = 1$, we have

$$J_2(\omega) = 500 \prod_{17 \le p \le p_i}(p - 7 - \chi(p)) \neq 0.$$

Where $\chi(19) = -5,\ \chi(31) = -1,\ \chi(37) = -4,\ \chi(p) = 0$ otherwise.

$$\pi_7(N,2) = \frac{J_2(\omega)\omega^6}{\phi^7(\omega)}\frac{N}{\log^7 N}(1+O(1)).$$

Theorem 3.1.105. $p_{j+1} = p_1 + (2j)^{\hat{20}}, j = 1, \cdots, 6$
For $\hat{T} = 1$, we have

$$J_2(\omega) = 8748 \prod_{19 \leq p \leq p_i} (p - 7 - \chi(p)) \neq 0.$$

Where $\chi(29) = -1,\ \chi(31) = \chi(37) = -2,\ \chi(41) = -3,\ \chi(p) = 0$, otherwise.

$$\pi_7(N, 2) = \frac{J_2(\omega)\omega^6}{\phi^7(\omega)} \frac{N}{\log^7 N}(1 + O(1)).$$

Theorem 3.1.106. $p_2 = (p_1 - 1)^{2^n} + 1$
Since $J_2(\omega) \neq 0$, there are infinitely may prime solutions.
We have

$$J_2(\omega) = \prod_{3 \leq p \leq p_i} (p - 2 - \chi(p)) \neq 0.$$

where

$$\chi(p) = \begin{cases} 2^n - 1 & \text{if } p \equiv\ 1 (\text{mod } 2^{n+1}) \\ -1 & \text{if } p \not\equiv\ 1 (\text{mod } 2^{n+1}) \end{cases}$$

$$\pi_2(N, 2) = \mid \{p_1 : p_1 \leq N, (p_1 - 1)^{2^n} = p_2\} \mid = \frac{1}{2^n} \frac{J_2(\omega)\omega}{\phi^2(\omega)} \frac{N}{\log^2 N}(1 + O(1)).$$

$$= \frac{1}{2^{n-1}} \prod_{3 \leq p \leq p_i} \frac{p(p - 2 - \chi(p))}{(p - 1)^2} \frac{N}{\log^2 N}(1 + O(1)).$$

Assume that $N \leq 31$ and $n \geq 5$, we have $\pi_2(31, 2) = 0$. Using the theorem 3.1.106, we may prove that Fermat numbers:$F_0 = 3,\ F_1 = 5,\ F_2 = 17,\ F_3 = 257$, and $F_4 = 65537$ are all primes. Beyond F_5 it has no further Fermat primes.

Theorem 3.1.107. $p_3 = p_1 + 2\hat{\times}p_2 + b,\ 2 \mid b$
Since $J_3(\omega) \neq 0$, there are infinitely many prime solutions.
For $\hat{T} = 1$, we have

$$J_3(\omega) = \prod_{3 \leq p \leq p_i} (p^2 - 3p + 3 - \chi(p)) \neq 0.$$

where $\chi(p) = 1$ if $p \mid b;\ \chi(p) = 0$ otherwise..
We have

$$\pi_2(N, 3) = \frac{J_3(\omega)\omega}{2\phi^3(\omega)} \frac{N^2}{\log^3 N}(1 + O(1)).$$

Theorem 3.1.108. $p_3 = N - p_1 - 2\hat{\times}p_2,\ 2 \mid N$
Since $J_3(\omega) \neq 0$ there are infinitely many prime solutions as $N \to \infty$.

For $\hat{T}=1$ we have

$$J_3(\omega)=\prod_{3\le p\le p_i}(p^2-3p+3-\chi(p))\neq 0,$$

where $\chi(p)=1$ if $p\mid N$; $\chi(p)=0$ otherwise..
We have

$$\pi_2(N,3)=\frac{J_3(\omega)\omega}{2\phi^3(\omega)}\frac{N^2}{\log^3 N}(1+O(1)).$$

Theorem 3.1.109. $p_3=p_1+3\hat{\times}p_2+b,\ 2\not| b$
Since $J_3(\omega)\neq 0$ there are infinitely many prime solutions.
For $\hat{T}=1$ we have

$$J_3(\omega)=\prod_{3\le p\le p_i}(p^2-3p+3-\chi(p))\neq 0,$$

where $\chi(3)=1$ if $3\not| b$; $\chi(3)=-1$, if $3\mid b$; $\chi(p)=1$ if $p\mid b$; $\chi(p)=0$ if $p\not| b$
We have

$$\pi_2(N,3)=\frac{J_3(\omega)\omega}{2\phi^3(\omega)}\frac{N^2}{\log^3 N}(1+O(1)).$$

Theorem 3.1.110. $p_3=N-p_1-3\hat{\times}p_2,\ 2\not| N$
Since $J_3(\omega)\neq 0$ there are infinitely many prime solutions as $N\to\infty$.
For $\hat{T}=1$ we have

$$J_3(\omega)=\prod_{3\le p\le p_i}(p^2-3p+3-\chi(p))\neq 0,$$

where $\chi(3)=1$ if $3\not| N$; $\chi(3)=-1$, if $3\mid N$; $\chi(p)=1$ if $p\mid N$; $\chi(p)=0$ if $p\not| N$
We have

$$\pi_2(N,3)=\frac{J_3(\omega)\omega}{2\phi^3(\omega)}\frac{N^2}{\log^3 N}(1+O(1)).$$

Theorem 3.1.111. $p_3=2\hat{\times}p_1+3\hat{\times}p_2+b,\ 2\mid b$
For $\hat{T}=1$ we have

$$J_3(\omega)=\prod_{3\le p\le p_i}(p^2-3p+3-\chi(p))\neq 0,$$

where $\chi(3)=-1$ if $3\mid b$; $\chi(3)=1$ if $3\not| b$; $\chi(p)=1$, if $p\mid b$; $\chi(p)=0$ if $p\not| b$,
We have

$$\pi_2(N,3)=\frac{J_3(\omega)\omega}{2\phi^3(\omega)}\frac{N^2}{\log^3 N}(1+O(1)).$$

Theorem 3.1.112. $p_3 = N - 2\hat{\times}p_1 + 3\hat{\times}p_2,\ 2 \mid N$
For $\hat{T} = 1$ we have

$$J_3(\omega) = \prod_{3\le p\le p_i} (p^2 - 3p + 3 - \chi(p)) \neq 0,$$

where $\chi(3) = -1$ if $3 \mid N$; $\chi(3) = 1$ if $3 \not| N$; $\chi(p) = 1$ if $p \mid N$; $\chi(p) = 0$ if $p \not| N$
We have

$$\pi_2(N, 3) = \frac{J_3(\omega)\omega}{2\phi^3(\omega)} \frac{N^2}{\log^3 N}(1 + O(1)).$$

Theorem 3.1.113. $p_3 = p_1 + 5\hat{\times}p_2 + b,\ 2 \not| b$
For $\hat{T} = 1$ we have

$$J_3(\omega) = \prod_{3\le p\le p_i} (p^2 - 3p + 3 - \chi(p)) \neq 0,$$

where $\chi(5) = -3$ if $5 \mid b$; $\chi(5) = 1$ if $5 \not| b$; $\chi(p) = 1$ if $p \mid b$; $\chi(p) = 0$ if $p \not| N$
We have

$$\pi_2(N, 3) = \frac{J_3(\omega)\omega}{2\phi^3(\omega)} \frac{N^2}{\log^3 N}(1 + O(1)).$$

Theorem 3.1.114. $p_3 = N - p_1 - 5\hat{\times}p_2,\ 2 \not| N$
For $\hat{T} = 1$ we have

$$J_3(\omega) = \prod_{3\le p\le p_i} (p^2 - 3p + 3 - \chi(p)) \neq 0,$$

where $\chi(5) = -3$ if $5 \mid N$; $\chi(5) = 1$ if $5 \not| N$; $\chi(p) = 1$ if $p \mid N$; $\chi(p) = 0$ if $p \not| N$
We have

$$\pi_2(N, 3) = \frac{J_3(\omega)\omega}{2\phi^3(\omega)} \frac{N^2}{\log^3 N}(1 + O(1)).$$

Theorem 3.1.115. $p_3 = p_1 + 7\hat{\times}p_2 + b,\ 2 \not| b$
For $\hat{T} = 1$ we have

$$J_3(\omega) = \prod_{3\le p\le p_i} (p^2 - 3p + 3 - \chi(p)) \neq 0,$$

where $\chi(7) = 1$ if $7 \not| b$; $\chi(7) = -5$ if $7 \mid b$; $\chi(p) = 1$ if $p \mid b$; $\chi(p) = 0$ if $p \not| b$
We have

$$\pi_2(N, 3) = \frac{J_3(\omega)\omega}{2\phi^3(\omega)} \frac{N^2}{\log^3 N}(1 + O(1)).$$

Theorem 3.1.116. $p_3 = N - p_1 - 7\hat{\times}p_2,\ 2 \not| N$
For $\hat{T} = 1$ we have

$$J_3(\omega) = \prod_{3\le p\le p_i} (p^2 - 3p + 3 - \chi(p)) \neq 0,$$

where $\chi(7)=1$ if $7\not|N$; $\chi(7)=-5$ if $7\mid N$; $\chi(p)=1$ if $p\mid N$; $\chi(p)=0$ if $p\not|N$
We have

$$\pi_2(N,3)=\frac{J_3(\omega)\omega}{2\phi^3(\omega)}\frac{N^2}{\log^3 N}(1+O(1)).$$

Theorem 3.1.117. $p_3=p_1+15\hat{\times}p_2+b,\ 2\not|b$
For $\hat{T}=1$ we have

$$J_3(\omega)=\prod_{3\le p\le p_i}(p^2-3p+3-\chi(p))\neq 0,$$

where $\chi(3)=1$ if $3\not|b$; $\chi(3)=-1$ if $3\mid b$; $\chi(5)=1$ if $5\not|b$; $\chi(5)=-3$ if $5\mid b$; $\chi(p)=1$ if $p\mid b$; $\chi(p)=0$ if $p\not|b$
We have

$$\pi_2(N,3)=\frac{J_3(\omega)\omega}{2\phi^3(\omega)}\frac{N^2}{\log^3 N}(1+O(1)).$$

Theorem 3.1.118. $p_3=N-p_1-15\hat{\times}p_2,\ 2\not|N$
For $\hat{T}=1$ we have

$$J_3(\omega)=\prod_{3\le p\le p_i}(p^2-3p+3-\chi(p))\neq 0,$$

where $\chi(3)=1$ if $3\not|N$; $\chi(3)=-1$ if $3\mid N$; $\chi(5)=1$ if $5\not|N$; $\chi(5)=-3$ if $5\mid N$; $\chi(p)=1$ if $p\mid N$; $\chi(p)=0$ if $p\not|N$
We have

$$\pi_2(N,3)=\frac{J_3(\omega)\omega}{2\phi^3(\omega)}\frac{N^2}{\log^3 N}(1+O(1)).$$

Theorem 3.1.119. $p_4=p_1+3\hat{\times}p_2+5\hat{\times}p_3+b,\ 2\mid b$
For $\hat{T}=1$ we have

$$J_4(\omega)=\prod_{3\le p\le p_i}(\frac{(p-1)^4-1}{p}-\chi(p))\neq 0,$$

where $\chi(3)=1$ if $3\mid b$; $\chi(3)=-1$ if $3\not|b$; $\chi(5)=3$ if $5\mid b$; $\chi(5)=-1$ if $5\not|b$; $\chi(p)=-1$ if $p\mid b$; $\chi(p)=0$ if $p\not|b$
We have

$$\pi_2(N,4)=\frac{J_4(\omega)\omega}{6\phi^4(\omega)}\frac{N^3}{\log^4 N}(1+O(1)).$$

Theorem 3.1.120. $p_4=N-p_1-3\hat{\times}p_2-5\hat{\times}p_3,\ 2\mid N$
For $\hat{T}=1$ we have

$$J_4(\omega)=\prod_{3\le p\le p_i}(\frac{(p-1)^4-1}{p}-\chi(p))\neq 0,$$

where $\chi(3)=1$ if $3\mid N$; $\chi(3)=-1$ if $3\not|N$; $\chi(5)=3$ if $5\mid N$; $\chi(5)=-1$ if $5\not|N$; $\chi(p)=-1$ if $p\mid N$; $\chi(p)=0$ if $p\not|N$
We have

$$\pi_2(N,4)=\frac{J_4(\omega)\omega}{6\phi^4(\omega)}\frac{N^3}{\log^4 N}(1+O(1)).$$

Theorem 3.1.121. $p_4=p_1+5\hat{\times}p_2+5\hat{\times}p_3+b,\ 2\mid b$
Since $J_4(\omega)\neq 0$ there are infinitely many prime solutions.

For $\hat{T}=1$ we have

$$J_4(\omega)=\prod_{3\le p\le p_i}(\frac{(p-1)^4-1}{p}-\chi(p))\neq 0,$$

where $\chi(5)=-13$ if $5\mid b$; $\chi(5)=3$ if $5\not|b$; $\chi(p)=-1$ if $p\mid b$; $\chi(p)=0$ if $p\not|b$.
We have

$$\pi_2(N,4)=\frac{J_4(\omega)\omega}{6\phi^4(\omega)}\frac{N^3}{\log^4 N}(1+O(1)).$$

Theorem 3.1.122. $p_4=N-p_1-5\hat{\times}p_2-5\hat{\times}p_3,\ 2\mid N$
Since $J_4(\omega)\neq 0$ there are infinitely many prime solutions, as $N\to\infty$.

For $\hat{T}=1$ we have

$$J_4(\omega)=\prod_{3\le p\le p_i}(\frac{(p-1)^4-1}{p}-\chi(p))\neq 0,$$

where $\chi(5)=-13$ if $5\mid N$; $\chi(5)=3$ if $5\not|N$; $\chi(p)=-1$ if $p\mid N$; $\chi(p)=0$ if $p\not|N$;
We have

$$\pi_2(N,4)=\frac{J_4(\omega)\omega}{6\phi^4(\omega)}\frac{N^3}{\log^4 N}(1+O(1)).$$

Theorem 3.1.123. $p_n=2\hat{\times}p_1\hat{\times}\cdots\hat{\times}p_{n-1}\pm 1.$

For $\hat{T}=1$, we have

$$J_n(p)=\mid\{q_i\cdot q_i=1,\cdots,p-1,i=1,\cdots,n-1,(2q_1\cdots q_{n-1}\pm 1,p)=1\}\mid,$$

$$J_n(\omega)=\phi^{(n-2)}(\omega)\prod_{3\le p\le p_i}(p-2)\neq 0,$$

Since $J_n(\omega)\to\infty$ as $\omega\to\infty$, there must exist infinitely many prime solutions.

We have the best asymptotic formula

$$\pi_2(N,n)=\frac{J_n(\omega)\omega}{(n-1)(n-1)!\phi^n(\omega)}\frac{N^{n-1}}{\log^n N}(1+O(1)).$$

Remark. There exist infinitely many primes p for which $\omega\pm 1$ is prime. There exist infinitely many primes p for which $\omega\pm p_{i+1}$ is prime.

Theorem 3.1.124. $p_n = (2\hat{\times} p_1 \hat{\times} \cdots \hat{\times} p_{n-1})^{\hat{2}} + 1$.

For $\hat{T} = 1$, we have

$$J_n(p) = \mid \{q_i \cdot q_i = 1, \cdots, p-1, i = 1, \cdots, n-1, ((2q_1 \cdots q_{n-1})^2 + 1, p) = 1\} \mid,$$

$$J_n(\omega) = \prod_{3 \le p \le p_i} [(p-1)^{n-1} - H(p)] \neq 0,$$

where $H(p) = 2(p-1)^{n-2}$ if $(\frac{-1}{p}) = 1$; $H(p) = 0$ if $(\frac{-1}{p}) = -1$.

We have

$$\pi_2(N, n) = \frac{J_n(\omega)\omega}{2(n-1)(n-1)!\phi^n(\omega)} \frac{N^{n-1}}{\log^n N}(1 + O(1)).$$

Remark. There exist infinitely many primes p for which $\omega^2 + 1$ is prime. There exist infinitely many primes p for which $\omega^2 \pm p_{i+1}$ is prime.

Theorem 3.1.125. $p_n = (2\hat{\times} p_1 \hat{\times} \cdots \hat{\times} p_{n-1})^{\hat{3}} + b$, $b = 3, 5, 7, 11, 13$.

For $\hat{T} = 1$, we have

$$J_n(\omega) = \prod_{3 \le p \le p_i} [(p-1)^{n-1} - H(p)] \neq 0,$$

where $H(p) = 3(p-1)^{n-2}$ if $b^{\frac{p-1}{3}} \equiv 1 (\text{mod } p)$; $H(p) = 0$ if $b^{\frac{p-1}{3}} \not\equiv 1 (\text{mod } p)$, and $p \mid b$; $H(p) = (p-1)^{n-2}$ otherwise.

We have

$$\pi_2(N, n) = \frac{J_n(\omega)\omega}{3(n-1)(n-1)!\phi^n(\omega)} \frac{N^{n-1}}{\log^n N}(1 + O(1)).$$

Theorem 3.1.126. $p_n = (2\hat{\times} p_1 \hat{\times} \cdots \hat{\times} p_{n-1})^{\hat{4}} + 1$.

For $\hat{T} = 1$, we have

$$J_n(p) = \mid \{q_i \cdot q_i = 1, \cdots, p-1, i = 1, \cdots, n-1, ((2q_1 \cdots q_{n-1})^4 + 1, p) = 1\} \mid,$$

$$J_n(\omega) = \prod_{3 \le p \le p_i} [(p-1)^{n-1} - H(p)] \neq 0,$$

where $H(p) = 4(p-1)^{n-2}$ if $p \equiv 1 (\text{mod } 8)$; $H(p) = 0$ if $p \not\equiv 1 (\text{mod } 8)$.

We have

$$\pi_2(N, n) = \frac{J_n(\omega)\omega}{4(n-1)(n-1)!\phi^n(\omega)} \frac{N^{n-1}}{\log^n N}(1 + O(1)).$$

Theorem 3.1.127. $p_n = (2\hat{\times} p_1 \hat{\times} \cdots \hat{\times} p_{n-1})^{\hat{4}} + b$, $b = 3, 5, 7, 9, 11, 13$.

For $\hat{T} = 1$, we have

$$J_n(\omega) = \prod_{3 \le p \le p_i} [(p-1)^{n-1} - H(p)] \neq 0,$$

where $H(p) = 4(p-1)^{n-2}$ if $b^{\frac{p-1}{4}} \equiv (-1)^{\frac{p-1}{4}} \equiv (\bmod\ p)$; $H(p) = 0$ if $b^{\frac{p-1}{4}} \not\equiv (-1)^{\frac{p-1}{4}} (\bmod\ p)$; $H(p) = 2(p-1)^{n-2}$ if $(\frac{-b}{p}) = 1$ and $4 \not| (p-1)$; $H(p) = 0$ if $(\frac{-b}{p}) = -1$, $4 \not| (p-1)$ and $p \mid b$.

We have

$$\pi_2(N,n) = \frac{J_n(\omega)\omega}{4(n-1)(n-1)!\phi^n(\omega)} \frac{N^{n-1}}{\log^n N}(1+O(1)).$$

Theorem 3.1.128. $p_n = (2\hat{\times}p_1\hat{\times}\cdots\hat{\times}p_{n-1})^{\hat{5}} + b,\ b = 3,5,7,9,11,13.$

For $\hat{T} = 1$, we have

$$J_n(\omega) = \prod_{3\le p\le p_i}[(p-1)^{n-1} - H(p)] \ne 0,$$

where $H(p) = 5(p-1)^{n-2}$ if $b^{\frac{p-1}{5}} \equiv 1(\bmod\ p)$; $H(p) = 0$ if $b^{\frac{p-1}{5}} \not\equiv 1(\bmod\ p)$ and $p \mid b$; $H(p) = (p-1)^{n-2}$ otherwise.

We have

$$\pi_2(N,n) = \frac{J_n(\omega)\omega}{5(n-1)(n-1)!\phi^n(\omega)} \frac{N^{n-1}}{\log^n N}(1+O(1)).$$

Theorem 3.1.129. $p_n = (2\hat{\times}p_1\hat{\times}\cdots\hat{\times}p_{n-1})^{\hat{8}} + 1.$

For $\hat{T} = 1$, we have

$$J_n(\omega) = \prod_{3\le p\le p_i}[(p-1)^{n-1} - H(p)] \ne 0,$$

where $H(p) = 8(p-1)^{n-2}$ if $p \equiv 1(\bmod\ 16)$; $H(p) = 0$ if $p \not\equiv 1(\bmod\ 16)$.

We have the best asymptotic formula

$$\pi_2(N,n) = \frac{J_n(\omega)\omega}{8(n-1)(n-1)!\phi^n(\omega)} \frac{N^{n-1}}{\log^n N}(1+O(1)).$$

Theorem 3.1.130. $p_n = (2\hat{\times}p_1\hat{\times}\cdots\hat{\times}p_{n-1})^{\hat{9}} + b,\ b = 3.5.7.9.11.13.$

For $\hat{T} = 1$, we have

$$J_n(\omega) = \prod_{3\le p\le p_i}[(p-1)^{n-1} - H(p)] \ne 0,$$

where $H(p) = 9(p-1)^{n-2}$ if $b^{\frac{p-1}{9}} \equiv 1(\bmod\ p)$; $H(p) = 0$ if $b^{\frac{p-1}{g}} \not\equiv 1(\bmod\ p)$ and $p \mid b$; $H(p) = (p-1)^{n-2}$ otherwise.

We have

$$\pi_2(N,n) = \frac{J_n(\omega)\omega}{9(n-1)(n-1)!\phi^n(\omega)} \frac{N^{n-1}}{\log^n N}(1+O(1)).$$

Remark. Taking $n = 2$, we have $p_2 = (2p_1)^9 + b$. Since $J_2(\omega) \neq 0$, there must exist infinitely many primes p_1 such that p_2 is also a prime.

Theorem 3.1.131. $p_n = (p_1 \cdots p_{n-1})^2 - 2$.
We have

$$J_n(\omega) = \prod_{3 \leq p \leq p_i} [(p-1)^{n-1} - H(p)] \neq 0,$$

where $H(p) = 2(p-1)^{n-2}$ if $(\frac{2}{p}) = 1$; $H(p) = 0$ if $(\frac{2}{p}) = -1$.
We have

$$\pi_2(N, 2) = \frac{J_n(\omega)\omega}{2(n-1)(n-1)!\phi^n(\omega)} \frac{N^{n-1}}{\log^n N}(1 + O(1)).$$

Theorem 3.1.132. $p_n = (p_1 \cdots p_{n-1})^3 - 2$.
We have

$$J_n(\omega) = \prod_{3 \leq p \leq p_i} [(p-1)^{n-1} - H(p)] \neq 0,$$

where $H(p) = 3(p-1)^{n-2}$ if $2^{\frac{p-1}{3}} \equiv 1 (\bmod p)$; $H(p) = 0$ if $2^{\frac{p-1}{3}} \not\equiv 1 (\bmod p), H(p) = (p-1)^{n-2}$ otherwise.
We have

$$\pi_2(N, n) = \frac{J_n(\omega)\omega}{3(n-1)(n-1)!\phi^n(\omega)} \frac{N^{n-1}}{\log^n N}(1 + O(1)).$$

Theorem 3.1.133. $p_n = (p_1 \cdots p_{n-1})^4 - 2$.
We have

$$J_n(\omega) = \prod_{3 \leq p \leq p_i} [(p-1)^{n-1} - H(p)] \neq 0,$$

where $H(p) = 4(p-1)^{n-2}$ if $2^{\frac{p-1}{4}} \equiv 1 (\bmod p)$; $H(p) = 0$ if $2^{\frac{p-1}{4}} \not\equiv 1 (\bmod p)$; $H(p) = 2(p-1)^{n-2}$ if $(\frac{2}{p}) = 1$ and $4 \not| (p-1)$; $H(p) = 0$ if $(\frac{2}{p}) = -1$ and $4 \not| (p-1)$.
We have

$$\pi_2(N, n) = \frac{J_n(\omega)\omega}{4(n-1)(n-1)!\phi^n(\omega)} \frac{N^{n-1}}{\log^n N}(1 + O(1)).$$

Remark. Taking $n = 2$, we have $p_2 = p_1^4 - 2$. Since $J_2(\omega) \neq 0$, there must exist infinitely many primes p_1 such that p_2 is also a prime.

Theorem 3.1.134. $p_n = (p_1 \cdots p_{n-1})^5 - 2$.
We have

$$J_n(\omega) = \prod_{3 \leq p \leq p_i} [(p-1)^{n-1} - H(p)] \neq 0,$$

where $H(p) = 5(p-1)^{n-2}$ if $2^{\frac{p-1}{5}} \equiv 1 (\bmod p)$; $H(p) = 0$ if $2^{\frac{p-1}{5}} \not\equiv 1 (\bmod p); H(p) = (p-1)^{n-2}$ otherwise.

We have

$$\pi_2(N,n)=\frac{J_n(\omega)\omega}{5(n-1)(n-1)!\phi^n(\omega)}\frac{N^{n-1}}{\log^n N}(1+O(1)).$$

Remark. Taking $n=2$, we have $p_2=p_1^5-2$. Since $J_2(\omega)\neq 0$,there must exist infinitely many primes p_1 such that p_2 is also a prime.

Theorem 3.1.135. $p_n=\frac{(p_1\cdots p_{n-1})^{p_0}-1}{p_1\cdots p_{n-1}-1}$, where p_0 is an odd prime.

We have

$$J_n(\omega)=\prod_{3\leq p\leq p_i}[(p-1)^{n-1}-H(p)]\neq 0,$$

where $H(p)=(p_0-1)(p-1)^{n-2}$ if $p\equiv 1(\bmod p_0)$; $H(p)=0$ if $p\not\equiv 1(\bmod p_0)$; $H(p_0)=(p_0-1)^{n-2}$.

We have

$$\pi_2(N,2)=\frac{J_n(\omega)\omega}{(n-1)(p_0-1)(n-1)!\phi^n(\omega)}\frac{N^{n-1}}{\log^n N}(1+O(1)).$$

Remark. Taking $n=2$, we have $p_2=\frac{p_1^{p_0}-1}{p_1-1}$. Since $J_2(\omega)\neq 0$,there must exist infinitely many primes p_1 such that p_2 is also a prime.

Theorem 3.1.136. $2p_n=(p_1\cdots p_{n-1})^{2^m}+1$,

We have

$$J_n(\omega)=\prod_{3\leq p\leq p_i}[(p-1)^{n-1}-H(p)]\neq 0,$$

where $H(p)=2^m(p-1)^{n-2}$ if $p\equiv 1(\bmod 2^{m+1})$; $H(p)=0$ if $p\not\equiv 1(\bmod 2^{m+1})$.

We have

$$\pi_2(N,n)=\frac{J_n(\omega)\omega}{2^m(n-1)(n-1)!\phi^n(\omega)}\frac{N^{n-1}}{\log^n N}(1+O(1)).$$

Remark. Taking $n=2$, we have $2p_2=p_1^{2^m}+1$. Since $J_2(\omega)\neq 0$, there must exist infinitely many primes p_1 such that p_2 is also a prime.

Theorem 3.1.137. $4p_n=3(p_1\cdots p_{n-1})^2+1$,

We have

$$J_n(\omega)=\prod_{3\leq p\leq p_i}[(p-1)^{n-1}-H(p)]\neq 0,$$

where $H(p)=2(p-1)^{n-2}$ if $(\frac{-3}{p})=1$; $H(p)=0$ if $(\frac{-3}{p})=-1$; $H(3)=0$.

We have

$$\pi_2(N,n)=\frac{J_n(\omega)\omega}{2(n-1)(n-1)!\phi^n(\omega)}\frac{N^{n-1}}{\log^n N}(1+O(1)).$$

Remark. We have $4p_2 = 3p_1^2 + 1, 4p_3 = 3p_2^2 + 1, 4p_4 = 3p_3^2 + 1$. Since $J_2(\omega) \neq 0$ there must exist infinitely many primes p_1 such that p_2, p_3 and p_4 are primes.

Theorem 3.1.138. $3p_n = (p_1 \cdots p_{n-1})^2 + 2$.

We have

$$J_n(\omega) = \prod_{3 \le p \le p_i} [(p-1)^{n-1} - H(p)] \neq 0,$$

where $H(p) = 2(p-1)^{n-2}$ if $(\frac{-2}{p}) = 1$; $H(p) = 0$ if $(\frac{-2}{p}) = -1$; $H(3) = 0$.

We have

$$\pi_2(N, n) = \frac{J_n(\omega)\omega}{2(n-1)(n-1)!\phi^n(\omega)} \frac{N^{n-1}}{\log^n N}(1 + O(1)).$$

Remark. We have $3p_2 = p_1^2 + 2, 3p_3 = {p_2}^2 + 2, 3p_4 = {p_3}^2 + 2$. Since $J_2(\omega) \neq 0$ there must exist infinitely many primes p_1 such that p_2, p_3 and p_4 are primes.

Theorem 3.1.139. $4p_n = (p_1 \cdots p_{n-1})^4 + 3$.

We have

$$J_n(\omega) = \prod_{3 \le p \le p_i} [(p-1)^{n-1} - H(p)] \neq 0,$$

where $H(p) = 4(p-1)^{n-2}$ if $3^{\frac{p-1}{4}} \equiv (-1)^{\frac{p-1}{4}} (\bmod p)$; $H(p) = 0$ if $3^{\frac{p-1}{4}} \not\equiv (-1)^{\frac{p-1}{4}} (\bmod p)$; $H(p) = 2(p-1)^{n-2}$ if $(\frac{-3}{p}) = 1$ and $4 \not| (p-1)$; $H(p) = 0$ if $(\frac{-3}{p}) = -1$ and $4 \not| (p-1)$; $H(3) = 0$.

We have the best asymptotic formula

$$\pi_2(N, n) = \frac{J_n(\omega)\omega}{4(n-1)(n-1)!\phi^n(\omega)} \frac{N^{n-1}}{\log^n N}(1 + O(1)).$$

Remark. taking $n = 2$, we have $4p_2 = p_1^4 + 3$. Since $J_2(\omega) \neq 0$, there must exist infinitely many p_1 such that p_2 is also a prime.

Theorem 3.1.140. $8p_n = (p_1 \cdots p_{n-1})^4 + 7$.

We have

$$J_n(\omega) = \prod_{3 \le p \le p_i} [(p-1)^{n-1} - H(p)] \neq 0,$$

where $H(p) = 4(p-1)^{n-2}$ if $7^{\frac{p-1}{4}} \equiv (-1)^{\frac{p-1}{4}} (\bmod p)$; $H(p) = 0$ if $7^{\frac{p-1}{4}} \not\equiv (-1)^{\frac{p-1}{4}} (\bmod p)$; $H(p) = 2(p-1)^{n-2}$ if $(\frac{-7}{p}) = 1$ and $4 \not| (p-1)$; $H(p) = 0$ if $(\frac{-7}{p}) = -1$ and $4 \not| (p-1)$; $H(7) = 0$.

We have

$$\pi_2(N, n) = \frac{J_n(\omega)\omega}{4(n-1)(n-1)!\phi^n(\omega)} \frac{N^{n-1}}{\log^n N}(1 + O(1)).$$

Remark. Taking $n = 2$, we have $8p_2 = p_1^4 + 7$. Since $J_2(\omega) \neq 0$, there must exist infinitely many primes p_1 such that p_2 is also a prime.

Theorem 3.1.141. $5p_n = (2p_1 \cdots p_{n-1})^4 + 9$.
We have
$$J_n(\omega) = \prod_{3 \le p \le p_i} [(p-1)^{n-1} - H(p)] \ne 0,$$
where $H(p) = 4(p-1)^{n-2}$ if $9^{\frac{p-1}{4}} \equiv (-1)^{\frac{p-1}{4}} (\bmod p)$; $H(p) = 0$ if $9^{\frac{p-1}{4}} \not\equiv (-1)^{\frac{p-1}{4}} (\bmod p)$; $H(p) = 2(p-1)^{n-2}$ if $(\frac{-1}{p}) = 1$ and $4 \not| (p-1)$; $H(p) = 0$ if $(\frac{-1}{p}) = -1$ and $4 \not| (p-1)$; $H(3) = 0$.
We have
$$\pi_2(N, n) = \frac{J_n(\omega)\omega}{4(n-1)(n-1)!\phi^n(\omega)} \frac{N^{n-1}}{\log^n N}(1 + O(1)).$$
Remark. Taking $n = 2$, we have $5p_2 = (2p_1)^4 + 9$. Since $J_2(\omega) \ne 0$ there must exist infinitely many primes p_1 such that p_2 is also a prime.

Theorem 3.1.142. $10p_n = (p_1 \cdots p_{n-1})^4 + 9$.
We have
$$J_n(\omega) = \prod_{3 \le p \le p_i} [(p-1)^{n-1} - H(p)] \ne 0,$$
where $H(p) = 4(p-1)^{n-2}$ if $9^{\frac{p-1}{4}} \equiv (-1)^{\frac{p-1}{4}} (\bmod p)$; $H(p) = 0$ if $9^{\frac{p-1}{4}} \not\equiv (-1)^{\frac{p-1}{4}} (\bmod p)$; $H(p) = 2(p-1)^{n-2}$ if $(\frac{-1}{p}) = 1$ and $4 \not| (p-1)$; $H(p) = 0$ if $(\frac{-1}{p}) = -1$ and $4 \not| (p-1)$; $H(3) = 0$.
We have
$$\pi_2(N, n) = \frac{J_n(\omega)\omega}{4(n-1)(n-1)!\phi^n(\omega)} \frac{N^{n-1}}{\log^n N}(1 + O(1)).$$
Remark. Taking $n = 2$, we have $10p_2 = (p_1)^4 + 9$. Since $J_2(\omega) \ne 0$ there must exist infinitely many primes p_1 such that p_2 is also a prime.

Theorem 3.1.143. $4p_n = (p_1 \cdots p_{n-1})^{2p_0} + 3$, p_0 being a given odd prime.
We have
$$J_n(\omega) = \prod_{3 \le p \le p_i} [(p-1)^{n-1} - H(p)] \ne 0,$$
where $H(p) = 2p_0(p-1)^{n-2}$ if $3^{\frac{p-1}{2p_0}} \equiv (-1)^{\frac{p-1}{2p_0}}$ (mod p); $H(p) = 0$ if $3^{\frac{p-1}{2p_0}} \not\equiv (-1)^{\frac{p-1}{2p_0}}$ (mod p); $H(p) = 2(p-1)^{n-2}$ if $(\frac{-3}{p}) = 1$ and $2p_0 \not| (p-1)$; $H(p) = 0$ if $(\frac{-3}{p}) = -1$ and $2p_0 \not| (p-1)$; $H(3) = 0$.
We have
$$\pi_2(N, n) = \frac{J_n(\omega)\omega}{2p_0(n-1)(n-1)!\phi^n(\omega)} \frac{N^{n-1}}{\log^n N}(1 + O(1)).$$
Remark. Taking $n = 2$, we have $4p_2 = (p_1)^{2p_0} + 3$. Since $J_2(\omega) \ne 0$ there must exist infinitely many primes p_1 such that p_2 is a prime.

Theorem 3.1.144. $p_2 = p_1^4 - p_1^2 + 1$.

We have

$$J_2(\omega) = \prod_{3 \le p \le p_i} (p - 2 - \chi(p)) \neq 0,$$

where $\chi(p) = 3$ if $p \equiv 1 (\text{mod } 12)$; $\chi(p) = -1$ if $p \not\equiv 1 (\text{mod } 12)$.

We have

$$\pi_2(N, 2) = \frac{J_2(\omega)\omega}{4\phi^2(\omega)} \frac{N}{\log^2 N}(1 + O(1)).$$

Theorem 3.1.145. $p_3 = 2p_1^2 - p_2^2$.

We have

$$J_3(\omega) = \prod_{3 \le p \le p_i} [(p - 1)^2 - H(p)] \neq 0,$$

where $H(p) = 2(p - 1)$ if $(\frac{2}{p}) = 1$; $H(p) = 0$ if $(\frac{2}{p}) = -1$.

We have the best asymptotic formula

$$\pi_2(N, 3) = \frac{J_3(\omega)\omega}{4\phi^3(\omega)} \frac{N^2}{\log^3 N}(1 + O(1)).$$

Theorem 3.1.146. $p_3 = 4p_1^2 + p_2^2$.

We have

$$J_3(\omega) = \prod_{3 \le p \le p_i} [(p - 1)^2 - H(p)] \neq 0,$$

where $H(p) = 2(p - 1)$ if $(\frac{-1}{p}) = 1$; $H(p) = 0$ if $(\frac{-1}{p}) = -1$.

We have the best asymptotic formula

$$\pi_2(N, 3) = \frac{J_3(\omega)\omega}{4\phi^3(\omega)} \frac{N^2}{\log^3 N}(1 + O(1)).$$

Theorem 3.1.147. $2p_3 = p_1^2 + p_2^2$.

We have

$$J_3(\omega) = \prod_{3 \le p \le p_i} [(p - 1)^2 - H(p)] \neq 0,$$

where $H(p) = 2(p - 1)$ if $(\frac{-1}{p}) = 1$; $H(p) = 0$ if $(\frac{-1}{p}) = -1$.

We have

$$\pi_2(N, 3) = \frac{J_3(\omega)\omega}{4\phi^3(\omega)} \frac{N^2}{\log^3 N}(1 + O(1)).$$

Theorem 3.1.148. $p_3 = 2 \hat{\times} (p_1 + p_2) + 3$.

For $\hat{T} = 1$, we have

$$J_3(\omega) = \prod_{3 \leq p \leq p_i} (p^2 - 3p + 3 - \chi(p)) \neq 0,$$

where $\chi(3) = 1, \ \chi(p) = 0$ otherwise.

We have the best asymptotic formula

$$\pi_2(N, 3) = \frac{J_3(\omega)\omega}{2\phi^3(\omega)} \frac{N^2}{\log^3 N}(1 + O(1)).$$

Theorem 3.1.149. $p_2 = a\hat{\times}p_1^{\hat{2}} + b\hat{\times}p_1 + c, \ (a, b, c) = 1, \ 2 \not| (a + b + c), \ b^2 - 4ac$ not a perfect square.

For $\hat{T} = 1$, we have

$$J_2(\omega) = \prod_{3 \leq p \leq p_i} (p - 2 - \chi(p)) \neq 0,$$

where $\chi(p) = (\frac{b^2 - 4ac}{p}), \ \chi(p) = 0$ if $p \mid (b^2 - 4ac)$.

We have the best asymptotic formula

$$\pi_2(N, 2) = \frac{J_2(\omega)\omega}{2\phi^2(\omega)} \frac{N}{\log^2 N}(1 + O(1)).$$

Theorem 3.1.150. $p_4 = p_1^3 + p_2^3 + p_3^3$,

We have

$$J_4(\omega) = \prod_{3 \leq p \leq p_i} (\frac{(p-1)^4 - 1}{p} - \chi(p)) \neq 0,$$

where $\chi(p) = -\frac{(p-1)^3 + 1}{p}$ if $p \not\equiv 1 (\text{mod } 3); \ \chi(p) = -1$ if $p \not\equiv 1 (\text{mod } 3)$.

We have the best asymptotic formula

$$\pi_2(N, 4) = \frac{J_4(\omega)\omega}{18\phi^4(\omega)} \frac{N^3}{\log^4 N}(1 + O(1)).$$

Theorem 3.1.151. $p_4 = p_1^3 + p_2^3 + p_3$,

We have

$$J_4(\omega) = \prod_{3 \leq p \leq p_i} (\frac{(p-1)^4 - 1}{p} + \chi(p)) \neq 0,$$

where $\chi(p) = 2p - 1$ if $p \equiv 1 (\text{mod } 3); \ \chi(p) = 1$ otherwise.

We have the best asymptotic formula

$$\pi_2(N,4)=\frac{J_4(\omega)\omega}{18\phi^4(\omega)}\frac{N^3}{\log^4 N}(1+O(1)).$$

Theorem 3.1.152. $p_2=\frac{p_1(p_1+1)}{2}+1$
We have

$$J_2(\omega)=\prod_{3\le p\le p_i}(p-2-\chi(p))\neq 0,$$

where $\chi(p)=(\frac{-7}{p})$, $\chi(7)=0$.
We have

$$\pi_2(N,2)=\frac{J_2(\omega)\omega}{2\phi^2(\omega)}\frac{N}{\log^2 N}(1+O(1)).$$

Theorem 3.1.153. $p_2=\frac{p_1(p_1+1)}{2}-2$
We have

$$J_2(\omega)=\prod_{3\le p\le p_i}(p-2-\chi(p))\neq 0,$$

where $\chi(p)=(\frac{17}{p})$, $\chi(17)=0$.
We have

$$\pi_2(N,2)=\frac{J_2(\omega)\omega}{2\phi^2(\omega)}\frac{N}{\log^2 N}(1+O(1)).$$

Theorem 3.1.154.[5] $p_2=10p_1+1,\ p_3=15p_1+2,\ p_4=6p_1+1$
We have

$$J_2(\omega)=3\prod_{7\le p\le p_i}(p-4)\neq 0,$$

Since $J_2(\omega)\neq 0$, then there must exist infinitely many integers p_1 such that $p_2,\ p_3,\ p_4$ are primes.
We have

$$\pi_4(N,2)=\frac{J_2(\omega)\omega^3}{\phi^4(\omega)}\frac{N}{\log^4 N}(1+O(1)).$$

$\pi_4(N,2)$ denotes the number of triples of consecutive integers.

From above we have $3p_2+1=30p_1+4=2p_3,\ 3p_2+2=30p_1+5=5p_4$. So $3p_2,\ 3p_2+1, 3p_2+2$ are product of two distinct primes. We prove that there must exist infinitely many triples of consecutive integers, each being the product of two distinct primes. For examples $213=3\times 71,\ 214=2\times 107,\ 215=5\times 43; 393=3\times 131,\ 394=2\times 197,\ 395=5\times 79$.

Theorem 3.1.155. $p_2=70p_1+1,\ p_3=105p_1+2,\ p_4=42p_1+1$

We have

$$J_2(\omega) = 18 \prod_{11 \le p \le p_i} (p-4) \ne 0,$$

We have

$$\pi_4(N,2) = \frac{J_2(\omega)\omega^3}{\phi^4(\omega)} \frac{N}{\log^4 N}(1+O(1)).$$

From above we have $3p_2 + 1 = 210p_1 + 4 = 2p_3,\ 3p_2 + 2 = 210p_1 + 5 = 5p_4$. So $3p_2,\ 3p_2+1, 3p_2+2$ are product of two distinct primes. We prove that there must exist infinitely many triples of consecutive integers, each being the product of two distinct primes. For examples $44313 = 3 \times 14771,\ 44314 = 2 \times 22157,\ 44315 = 5 \times 8863$. $\pi_4(N,2)$ denotes the number of triples of consecutive integers, $p_1 \le N$.

Theorem 3.1.156. $p_2 = 2380p_1 + 1,\ p_3 = 2310p_1 + 1,\ p_4 = 2244p_1 + 1$

We have

$$J_2(\omega) = \prod_{3 \le p \le p_i} (p - 4 - \chi(p)) \ne 0,$$

where $\chi(p) = -2$ if $p = 3, 5, 7, 11, 17;\ \chi(p) = 0$ otherwise.

Since $J_2(\omega) \ne 0$, there must exist infinitely many integers p_1 such that $p_2,\ p_3,\ p_4$ are primes.

We have

$$\pi_4(N,2) = \frac{J_2(\omega)\omega^3}{\phi^4(\omega)} \frac{N}{\log^4 N}(1+O(1)).$$

From above we have $33p_2 + 1 = 34p_3,\ 33p_2 + 2 = 35p_4$. So $33p_2,\ 33p_2 + 1,\ 33p_2 + 2$ are product of three distinct primes. We have prove that there must exist infinitely many triples of consecutive integers, each being the product of three distinct primes. For example, $1727913 = 3 \times 11 \times 52361,\ 1727914 = 2 \times 17 \times 50821,\ 1727915 = 5 \times 7 \times 49369$. $\pi_4(N,2)$ denotes the number of triples of consecutive integers, $p_1 \le N$.

Theorem 3.1.157. Taking $m_1 = 1727913 = 3 \times 11 \times 52361,\ m_2 = m_1 + 1 = 1727914 = 2 \times 17 \times 50821,\ m_3 = m_1 + 2 = 1727915 = 5 \times 7 \times 49369$. m is least common multiple of integers $m_1,\ m_2,\ m_3$, that is $m = \mathrm{lcm}(m_1, m_2, m_3) = 2 \times 3 \times 5 \times 7 \times 11 \times 17 \times 49369 \times 50821 \times 52361 = m_1 m_2 m_3$.

We define the Diophantine equations

$$p_2 = \frac{2m}{m_1} p_1 + 1,\ p_3 = \frac{2m}{m_2} p_1 + 1,\ p_4 = \frac{2m}{m_3} p_1 + 1$$

We have the arithmetic function

$$J_2(\omega) = \prod_{3 \le p \le p_i} (p - 4 - \chi(p)) \ne 0,$$

where $\chi(p) = -2$ if $p|m,\ \chi(p) = 0$ otherwise.

Since $J_2(\omega) \neq 0$, there must exist infinitely many integers p_1 such that $p_2,\ p_3,\ p_4$ are primes.

We have

$$\pi_4(N,2) = \frac{J_2(\omega)\omega^3}{\phi^4(\omega)} \frac{N}{\log^4 N}(1+O(1)).$$

From above we have $m_1p_2+1 = 2mp_1+m_1+1 = m_2p_3,\ m_1p_2+2 = 2mp_1+m_1+2 = m_3p_4$ So $m_1p_2,\ m_1p_2+1,\ m_1p_2+2$ are product of four distinct primes. We have prove that there must exist infinitely many triples of consecutive integers, each being the product of four distinct primes. $\pi_4(N,2)$ denotes the number of triples of consecutive integers, $p_1 \leq N$.

Theorem 3.1.158. Suppose that $m_1,\ m_2 = m_1+1,\ m_3 = m_1+2$ are three consecutive integers, each being the product of $k-1$ distinct primes. m is least common multiple of the integers $m_1,\ m_2,\ m_3$, that is $m = m_1m_2m_3$

We define the Diophantine equations

$$p_2 = \frac{2m}{m_1}p_1+1,\ p_3 = \frac{2m}{m_2}p_1+1,\ p_4 = \frac{2m}{m_3}p_1+1$$

We have

$$J_2(\omega) = \prod_{3\leq p\leq p_i}(p-4-\chi(p)) \neq 0,$$

where $\chi(p) = -2$ if $p \mid m,\ \chi(p) = 0$ otherwise.

Since $J_2(\omega) \neq 0$, there must exist infinitely many integers p_1 such that $p_2,\ p_3,\ p_4$ are primes.

We have

$$\pi_4(N,2) = \frac{J_2(\omega)\omega^3}{\phi^4(\omega)} \frac{N}{\log^4 N}(1+O(1)).$$

From above we have $m_1p_2+1 = 2mp_1+m_1+1 = m_2p_3,\ m_1p_2+2 = 2mp_1+m_1+2 = m_3p_4$. So $m_1p_2,\ m_1p_2+1, m_1p_2+2$ are product of k distinct primes. We prove that there exist infinitely many triples of consecutive integers, each being the product of k distinct primes. $\pi_4(N,2)$ denotes the number of triples of consecutive integers, p_1 such that $p_1 \leq N$.

Now we study the Carmichael numbers. By definition[5], the Carmichael numbers are the composites n such that $a^{n-1} \equiv 1 (\text{mod } n)$ for every integer $a,\ 1 < a < n$, such that a is relatively prime to n. In 1939, Chernick gave the following method to obtain Carmichael numbers. Let $m \geq 1$ and $M_3(m) = (6m+1)(12m+1)(18m+1)$. If m is such that all three factors above are primes, then $M_3(m)$ is Carmichael number. This yields Carmichael numbers with three prime factors. Hitherto it is not even known if there exist infinitely many Carmichael numbers, which are product of three primes.

If $k \geq 4, m \geq 1$, let

$$M_k(m) = (6m+1)(12m+1)\prod_{i=1}^{k-2}(9 \times 2^i m + 1)$$

If m is such that all k factors are primes, then $M_k(m)$ is a Carmichael number with k prime factors.

Theorem 3.1.159. $p_1 = 6m+1,\ p_2 = 12m+1,\ p_3 = 18m+1.$

We have

$$J_2(\omega) = 2 \prod_{5 \leq p \leq p_i} (p-4) \neq 0.$$

Since $J_2(\omega) \neq 0$,there must exist infinitely many integers m such that $p_1,\ p_2,\ p_3$ are primes.

We have

$$\pi_4(N,2) = \frac{J_2(\omega)\omega^3}{\phi^4(\omega)} \frac{N}{\log^4 N}(1+O(1)).$$

$\pi_4(N,2)$ denotes the number of Carmichael numbers m such that $m \leq N$.

We have proved that there exist infinitely many Carmichael numbers, which are product of exactly three primes.

Theorem 3.1.160. $p_1 = 6m+1,\ p_2 = 12m+1,\ p_3 = 18m+1\ p_4 = 36m+1.$

We have

$$J_2(\omega) = 2 \prod_{7 \leq p \leq p_i} (p-5) \neq 0.$$

Since $J_2(\omega) \neq 0$, there exist infinitely many integers m such that $p_1,\ p_2,\ p_3,\ p_4$ are primes.

We have

$$\pi_5(N,2) = \frac{J_2(\omega)\omega^4}{\phi^5(\omega)} \frac{N}{\log^5 N}(1+O(1)).$$

$\pi_5(N,2)$ denotes the number of Carmichael numbers m such that $m \leq N$.

We have proved that there exist infinitely many Carmichael numbers, which are product of exactly three primes.

Theorem 3.1.161. $p_1 = 6m+1,\ p_2 = 12m+1,\ p_3 = 18m+1\ p_4 = 36m+1,\ p_5 = 72m+1.$

We have

$$J_2(\omega) = 12 \prod_{13 \leq p \leq p_i} (p-6) \neq 0.$$

Since $J_2(\omega) \neq 0$, there must exist infinitely many integers m such that $p_1,\ p_2,\ p_3,\ p_4,\ p_5$ are primes.

We have

$$\pi_6(N,2)=\frac{J_2(\omega)\omega^5}{\phi^6(\omega)}\frac{N}{\log^6 N}(1+O(1)).$$

$\pi_6(N,2)$ denotes the number of Carmichael numbers m such that $m\le N$.

We have proved that there exist infinitely many Carmichael numbers, which are product of exactly five primes.

Theorem 3.1.162. $p_1+\cdots+p_m=p_{m+1}+\cdots+p_{2m},\ p_i\le N,\ i=1,\ \cdots,\ 2m-1, m>1$.

We have

$$J_{2m}(\omega)=\prod_{3\le p\le p_i}\left(\frac{(p-1)^{2m}-1}{p}+1\right)\neq 0.$$

We have

$$\pi_2(N,2m)=\frac{J_{2m}(\omega)\omega}{(2m-1)!\phi^{2m}(\omega)}\frac{N^{2m-1}}{\log^{2m} N}(1+O(1))$$

$$=\frac{2}{(2m-1)!}\prod_{3\le p\le p_i}\left(1+\frac{1}{(p-1)^{2m-1}}\right)\frac{N^{2m-1}}{\log^{2m} N}(1+O(1)).$$

Theorem 3.1.163. Every positive integer m may be written in finitely many ways in the form[5]:

$$m=\frac{p_2\pm 1}{p_1\pm 1},$$

where $p_1,\ p_2$ are odd primes.

From above we have the Diophantine equation

$$p_2=mp_1\pm(m-1).$$

Both have the same arithmetic function

$$J_2(\omega)=\prod_{3\le p\le p_i}(p-2)\prod_{p|m(m-1)}\frac{p-1}{p-2}\neq 0.$$

Since $J_2(\omega)\neq 0$, there must exist infinitely many primes p_1 such that p_2 is a prime for every positive integer m.

We have

$$\pi_2(N,2)=\frac{J_2(\omega)\omega}{\phi^2(\omega)}\frac{N}{\log^2 N}(1+O(1)).$$

Theorem 3.1.164. Every positive integer m may be written in finitely many ways in the form:

$$m=\frac{p_2-1}{p_1+1},$$

$p_1,\ p_2$ being odd primes.

From above we have the Diophantine equation

$$p_2 = mp_1 + m + 1.$$

We have

$$J_2(\omega) = \prod_{3\le p\le p_i}(p-2)\prod_{p|m(m+1)}\frac{p-1}{p-2} \neq 0.$$

Since $J_2(\omega) \neq 0$, there must exist infinitely many primes p_1 such that p_2 is a prime for every positive integer m.

We have

$$\pi_2(N,2) = \frac{J_2(\omega)\omega}{\phi^2(\omega)}\frac{N}{\log^2 N}(1+O(1)).$$

Remark. Taking $m = 1$, we have $p_2 = p_1 + 2$. It is the twin primes theorem.

Theorem 3.1.165. Every positive integer m may be written in infinitely many ways in the form:

$$m = \frac{p_2+1}{p_1^2-1},$$

where $p_1,\ p_2$ are odd primes.

From above we have the Diophantine equation

$$p_2 = mp_1^2 - m - 1.$$

We have

$$J_2(\omega) = \prod_{3\le p\le p_i}(p-2-\chi(p)) \neq 0,$$

where $\chi(p) = (\frac{m(m+1)}{p})$; $\chi(p) = -1$ if $p \mid m(m+1)$.

Since $J_2(\omega) \neq 0$, there must exist infinitely many primes p_1 such that p_2 is a prime for every positive integer m.

We have

$$\pi_2(N,2) = \frac{J_2(\omega)\omega}{2\phi^2(\omega)}\frac{N}{\log^2 N}(1+O(1)).$$

Theorem 3.1.166. Every positive integer $m > 1$ may be written in infinitely many ways in the form:

$$m = \frac{p_2-1}{p_1^2-1},$$

$p_1,\ p_2$ being odd primes.

From above we have the Diophantine equation

$$p_2 = mp_1^2 - m + 1.$$

We have

$$J_2(\omega) = \prod_{3 \le p \le p_i} (p - 2 - \chi(p)) \neq 0,$$

where $\chi(p) = (\frac{m(m-1)}{p})$; $\chi(p) = -1$ if $p \mid m(m-1)$.

We have

$$\pi_2(N, 2) = \frac{J_2(\omega)\omega}{2\phi^2(\omega)} \frac{N}{\log^2 N}(1 + O(1)).$$

Since $J_2(\omega) \neq 0$, there must exist infinitely many primes p_1 such that p_2 is a prime for every positive integer $m > 1$.

Theorem 3.1.167. Every positive integer m may be written in infinitely many ways in the form:

$$m = \frac{p_2 \pm 1}{p_1^3 \mp 1},$$

p_1, p_2 being odd primes.

We define The Diophantine equation

$$p_2 = mp_1^3 \mp (m + 1).$$

We have

$$J_2(\omega) = \prod_{3 \le p \le p_i} (p - 1 - \chi(p)) \neq 0,$$

where $\chi(p) = 3$ if $[m^2(m+1)]^{\frac{p-1}{3}} \equiv 1 \pmod p$; $\chi(p) = 0$ if $[m^2(m+1)]^{\frac{p-1}{3}} \not\equiv 1 \pmod p$; $\chi(p) = 0$ if $p \mid m(m+1)$; $\chi(p) = 1$ otherwise.

Since $J_2(\omega) \neq 0$ there exist infinitely many primes p_1 such that p_2 is a prime for every positive integer m.

We have

$$\pi_2(N, 2) = \frac{J_2(\omega)\omega}{3\phi^2(\omega)} \frac{N}{log^2 N}(1 + O(1)).$$

Remark. Let $m = f^3$, we have $\chi(p) = 3$ if $(m+1)^{\frac{p-1}{3}} \equiv 1 \pmod p$.

Theorem 3.1.168. Every positive integer $m > 1$ may be written in infinitely many ways in the form:

$$m = \frac{p_2 - 1}{p_1^4 - 1},$$

where p_1, p_2 are odd primes.

We define the Diophantine equation

$$p_2 = mp_1^4 - m + 1.$$

We have

$$J_2(\omega) = \prod_{3\le p\le p_i} (p - 1 - \chi(p)) \ne 0,$$

where $\chi(p) = b = 2, 4$ if $[m^3(m-1)]^{\frac{p-1}{b}} \equiv 1 \pmod{p}$; $\chi(p) = 0$ if $[m^3(m-1)]^{\frac{p-1}{b}} \not\equiv 1 \pmod{p}$; $\chi(p) = 0$ if $p \mid m(m-1)$;

Since $J_4(\omega) \ne 0$ there exist infinitely many primes p_1 such that p_2 is a prime for every positive integer $m > 1$.

We have

$$\pi_2(N, 2) = \frac{J_2(\omega)\omega}{4\phi^2(\omega)} \frac{N}{\log^2 N}(1 + O(1)).$$

Remark. Let $m = f^b$, we have $\chi(p) = b = 2, 4$, if $(m-1)^{\frac{p-1}{b}} \equiv 1 \pmod{p}$.

Theorem 3.1.169. Every positive integer m may be written in infinitely many ways in the form:

$$m = \frac{p_2 + 1}{p_1^4 - 1},$$

$p_1,\ p_2$ being odd primes.

We define the Diophantine equation

$$p_2 = mp_1^4 - m - 1.$$

We have

$$J_2(\omega) = \prod_{3\le p\le p_i} (p - 1 - \chi(p)) \ne 0,$$

where $\chi(p) = b = 2, 4$ if $[m^3(m+1)]^{\frac{p-1}{b}} \equiv 1 \pmod{p}$; $\chi(p) = 0$ if $[m^3(m+1)]^{\frac{p-1}{b}} \not\equiv 1 \pmod{p}$; $\chi(p) = 0$ if $p \mid m(m+1)$;

Since $J_2(\omega) \ne 0$ there exist infinitely many primes p_1 such that p_2 is a prime for every positive integer m.

We have

$$\pi_2(N, 2) = \frac{J_2(\omega)\omega}{4\phi^2(\omega)} \frac{N}{\log^2 N}(1 + O(1)).$$

Remark. Let $m = f^b$, we have $\chi(p) = b = 2, 4$, if $(m+1)^{\frac{p-1}{b}} \equiv 1 \pmod{p}$.

Theorem 3.1.170. Every positive integer m may be written in infinitely many ways in the form:

$$m = \frac{p_2 \pm 1}{p_1^5 \mp 1},$$

$p_1,\ p_2$ being odd primes.

We define the Diophantine equation

$$p_2 = mp_1^5 \mp (m + 1).$$

We have

$$J_2(\omega) = \prod_{3 \le p \le p_i} (p - 1 - \chi(p)) \ne 0,$$

where $\chi(p) = 5$ if $[m^4(m+1)]^{\frac{p-1}{5}} \equiv 1 \pmod p$; $\chi(p) = 0$ if $[m^4(m+1)]^{\frac{p-1}{5}} \not\equiv 1 \pmod p$; $\chi(p) = 0$ if $p \mid m(m+1)$, $\chi(p) = 1$ otherwise.

Since $J_2(\omega) \ne 0$ there must exist infinitely many primes p_1 such that p_2 is a prime for every positive integer m.

We have

$$\pi_2(N, 2) = \frac{J_2(\omega)\omega}{5\phi^2(\omega)} \frac{N}{\log^2 N}(1 + O(1)).$$

Remark. Let $m = f^5$, we have $\chi(p) = 5$, if $(m+1)^{\frac{p-1}{5}} \equiv 1 \pmod p$.

Theorem 3.1.171. Every positive integer m may be written in infinitely many ways in the form:

$$m = \frac{p_2 + 1}{p_1^6 - 1},$$

where p_1, p_2 are odd primes.

We define the Diophantine equation

$$p_2 = mp_1^6 - m - 1.$$

We have

$$J_2(\omega) = \prod_{3 \le p \le p_i} (p - 1 - \chi(p)) \ne 0,$$

where $\chi(p) = b = 2, 6$ if $[m^5(m+1)]^{\frac{p-1}{b}} \equiv 1 \pmod p$; $\chi(p) = 0$ if $[m^5(m+1)]^{\frac{p-1}{b}} \not\equiv 1 \pmod p$; $\chi(p) = 0$ if $p \mid m(m+1)$.

Since $J_2(\omega) \ne 0$ there are infinitely many primes p_1 such that p_2 is a prime for every positive integer m.

We have

$$\pi_2(N, 2) = \frac{J_2(\omega)\omega}{6\phi^2(\omega)} \frac{N}{\log^6 N}(1 + O(1)).$$

Remark. Let $m = f^b$, we have $\chi(p) = b = 2, 6$, if $(m+1)^{\frac{p-1}{b}} \equiv 1 \pmod p$.

Theorem 3.1.172. Every integer m may be written in infinitely many ways in the form:

$$m = \frac{p_2 + 1}{p_1^8 - 1},$$

p_1, p_2 being odd primes.

We define the Diophantine equation

$$p_2 = mp_1^8 - m - 1.$$

We have

$$J_2(\omega) = \prod_{3\le p\le p_i} (p-1-\chi(p)) \neq 0,$$

where $\chi(p) = b = 2, 4, 8$ if $[m^7(m+1)]^{\frac{p-1}{b}} \equiv 1 \pmod p$; $\chi(p) = 0$ if $[m^7(m+1)]^{\frac{p-1}{b}} \not\equiv 1 \pmod p$; $\chi(p) = 0$ if $p \mid m(m+1)$.

Since $J_2(\omega) \neq 0$ there are infinitely many primes p_1 such that p_2 is a prime for every integer m.

We have

$$\pi_2(N, 2) = \frac{J_2(\omega)\omega}{8\phi^2(\omega)} \frac{N}{\log^2 N}(1 + O(1)).$$

Remark. Let $m = f^b$, we have $\chi(p) = b = 2, 4, 8$, if $(m+1)^{\frac{p-1}{b}} \equiv 1 \pmod p$.

Theorem 3.1.173. Every integer m may be written in finitely many ways in the form:

$$m = \frac{p_2 \pm 1}{p_1^9 \mp 1},$$

$p_1,\ p_2$ being odd primes.

We define the Diophantine equation

$$p_2 = mp_1^9 \mp (m+1).$$

We have

$$J_2(\omega) = \prod_{3\le p\le p_i} (p-1-\chi(p)) \neq 0,$$

where $\chi(p) = b = 3, 9$ if $[m^8(m+1)]^{\frac{p-1}{b}} \equiv 1 \pmod p$; $\chi(p) = 0$ if $[m^8(m+1)]^{\frac{p-1}{b}} \not\equiv 1 \pmod p$; $\chi(p) = 0$ if $p \mid m(m+1)$; $\chi(p) = 1$ otherwise.

Since $J_2(\omega) \neq 0$ there must exist infinitely many primes p_1 such that p_2 is a prime for every integer m.

We have

$$\pi_2(N, 2) = \frac{J_2(\omega)\omega}{9\phi^2(\omega)} \frac{N}{\log^2 N}(1 + O(1)).$$

Remark. Let $m = f^b$, we have $\chi(p) = b = 3, 9$, if $(m+1)^{\frac{p-1}{b}} \equiv 1 \pmod p$.

Theorem 3.1.174. Every integer m may be written in infinitely many ways in the form:

$$m = \frac{p_2 + 1}{p_1^{12} - 1},$$

$p_1,\ p_2$ being odd primes.

We define the Diophantine equation

$$p_2 = mp_1^{12} - m - 1.$$

We have the arithmetic function

$$J_2(\omega) = \prod_{3 \le p \le p_i} (p - 1 - \chi(p)) \neq 0,$$

where $\chi(p) = b = 2, 4, 6, 12$ if $[m^{11}(m+1)]^{\frac{p-1}{b}} \equiv 1 \pmod{p}$; $\chi(p) = 0$ if $[m^{11}(m+1)]^{\frac{p-1}{b}} \not\equiv 1 \pmod{p}$; $\chi(p) = 0$ if $p \mid m(m+1)$.

Since $J_2(\omega) \neq 0$ there exist infinitely many primes p_1 such that p_2 is a prime for every integer m.

We have

$$\pi_2(N, 2) = \frac{J_2(\omega)\omega}{12\phi^2(\omega)} \frac{N}{\log^2 N}(1 + O(1)).$$

Remark. Let $m = f^b$, we have $\chi(p) = b = 2, 4, 6, 12$, if $(m+1)^{\frac{p-1}{b}} \equiv 1 \pmod{p}$.

Theorem 3.1.175. Every integer m may be written in infinitely many ways in the form:

$$m = \frac{p_2 \mp 1}{p_1^{15} \pm 1},$$

p_1, p_2 being odd primes.

We define the Diophantine equation

$$p_2 = mp_1^{15} \pm (m + 1).$$

We have the arithmetic function

$$J_2(\omega) = \prod_{3 \le p \le p_i} (p - 1 - \chi(p)) \neq 0,$$

where $\chi(p) = b = 3, 5, 15$ if $[m^{14}(m+1)]^{\frac{p-1}{b}} \equiv 1 \pmod{p}$; $\chi(p) = 0$ if $[m^{14}(m+1)]^{\frac{p-1}{b}} \not\equiv 1 \pmod{p}$; $\chi(p) = 0$ if $p \mid m(m+1)$; $\chi(p) = 1$ otherwise.

Since $J_2(\omega) \neq 0$ there are infinitely many primes p_1 such that p_2 is a prime for every integer m.

We have

$$\pi_2(N, 2) = \frac{J_2(\omega)\omega}{15\phi^2(\omega)} \frac{N}{\log^2 N}(1 + O(1)).$$

Remark. Let $m = f^b$, we have $\chi(p) = b = 3, 5, 15$, if $(m+1)^{\frac{p-1}{b}} \equiv 1 \pmod{p}$.

Theorem 3.1.176. Every integer m can be written in infinitely many ways in the form:

$$m = \frac{p_2 + 1}{p_1^{16} - 1},$$

p_1, p_2 being odd primes.

We define the Diophantine equation

$$p_2 = mp_1^{16} - m - 1.$$

We have the arithmetic function

$$J_2(\omega) = \prod_{3 \le p \le p_i} (p - 1 - \chi(p)) \neq 0,$$

where $\chi(p) = b = 2, 4, 8, 16$ if $[m^{15}(m+1)]^{\frac{p-1}{b}} \equiv 1 \pmod{p}$; $\chi(p) = 0$ if $[m^{15}(m+1)]^{\frac{p-1}{b}} \not\equiv 1 \pmod{p}$; $\chi(p) = 0$ if $p \mid m(m+1)$.

Since $J_2(\omega) \neq 0$, there are infinitely many primes p_1 such that p_2 is a prime for every integer m.

We have

$$\pi_2(N, 2) = \frac{J_2(\omega)\omega}{16\phi^2(\omega)} \frac{N}{\log^2 N}(1 + O(1)).$$

Remark. Let $m = f^b$, we have $\chi(p) = b = 2, 4, 8, 16$, if $(m+1)^{\frac{p-1}{b}} \equiv 1 \pmod{p}$.

Theorem 3.1.177. Every integer m can be written in infinitely many ways in the form:

$$m = \frac{p_2 + 1}{p_1^{18} - 1},$$

$p_1,\ p_2$ being odd primes.

We define the Diophantine equation

$$p_2 = mp_1^{18} - m - 1.$$

We have the arithmetic function

$$J_2(\omega) = \prod_{3 \le p \le p_i} (p - 1 - \chi(p)) \neq 0,$$

where $\chi(p) = b = 2, 4, 8, 18$ if $[m^{17}(m+1)]^{\frac{p-1}{b}} \equiv 1 \pmod{p}$; $\chi(p) = 0$ if $[m^{17}(m+1)]^{\frac{p-1}{b}} \not\equiv 1 \pmod{p}$; $\chi(p) = 0$ if $p \mid m(m+1)$.

Since $J_2(\omega) \neq 0$ there are infinitely many primes p_1 such that p_2 is a prime for every integer m.

We have

$$\pi_2(N, 2) = \frac{J_2(\omega)\omega}{18\phi^2(\omega)} \frac{N}{\log^2 N}(1 + O(1)).$$

Remark. Let $m = f^b$, we have $\chi(p) = b = 2, 6, 18$, if $(m+1)^{\frac{p-1}{b}} \equiv 1 \pmod{p}$.

Theorem 3.1.178. Every integer m can be written in infinitely many ways in the form:

$$m = \frac{p_2 + 1}{p_1^{24} - 1},$$

We define the Diophantine equation

$$p_2 = mp_1^{24} - m - 1.$$

We have the arithmetic function

$$J_2(\omega) = \prod_{3 \le p \le p_i} (p - 1 - \chi(p)) \neq 0,$$

where $\chi(p) = b = 2, 4, 6, 8, 12, 24$ if $[m^{23}(m+1)]^{\frac{p-1}{b}} \equiv 1 \pmod{p}$; $\chi(p) = 0$ if $[m^{23}(m+1)]^{\frac{p-1}{b}} \not\equiv 1 \pmod{p}$; $\chi(p) = 0$ if $p \mid m(m+1)$.

Since $J_2(\omega) \neq 0$ there must exist infinitely many primes p_1 such that p_2 is a prime for every integer m.

We have

$$\pi_2(N, 2) = \frac{J_2(\omega)\omega}{24\phi^2(\omega)} \frac{N}{\log^2 N}(1 + O(1)).$$

Remark. Let $m = f^b$, we have $\chi(p) = b = 2, 4, 6, 8, 12, 24$, if $(m+1)^{\frac{p-1}{b}} \equiv 1 \pmod{p}$.

Theorem 3.1.179. Every integer m can be written in infinitely many ways in the form:

$$m = \frac{p_2 \mp 1}{p_1^d \pm 1},$$

d being a given odd prime.

We define the Diophantine equation

$$p_2 = mp_1^d \pm (m+1).$$

We have the arithmetic function

$$J_2(\omega) = \prod_{3 \le p \le p_i} (p - 1 - \chi(p)) \neq 0,$$

where $\chi(p) = d$ if $[m^{d-1}(m+1)]^{\frac{p-1}{d}} \equiv 1 \pmod{p}$; $\chi(p) = 0$ if $[m^{d-1}(m+1)]^{\frac{p-1}{b}} \not\equiv 1 \pmod{p}$; $\chi(p) = 0$ if $p \mid m(m+1)$; $\chi(p) = 1$ otherwise.

Since $J_2(\omega) \neq 0$ there are infinitely many primes p_1 such that p_2 is a prime for every integer m.

We have

$$\pi_2(N, 2) = \frac{J_2(\omega)\omega}{d\phi^2(\omega)} \frac{N}{\log^2 N}(1 + O(1)).$$

Remark. Taking $m = f^d$, we have $\chi(p) = d$, if $(m+1)^{\frac{p-1}{d}} \equiv 1 \pmod{p}$.

Theorem 3.1.180. Every integer m can be written in infinitely many ways in the form:

$$m = \frac{p_2 \mp 1}{p_1^{d_1 d_2} \pm 1}.$$

where d_1 and d_2 are the given odd primes.

We have the Diophantine equation

$$p_2 = mp_1^{d_1 d_2} \pm (m+1).$$

We have the arithmetic function

$$J_2(\omega) = \prod_{3 \le p \le p_i} (p - 1 - \chi(p)) \neq 0,$$

where $\chi(p) = b = d_1, d_2, d_1 d_2$ if $[m^{d_1 d_2 - 1}(m+1)]^{\frac{p-1}{b}} \equiv 1 \pmod{p}$; $\chi(p) = 0$ if $[m^{d_1 d_2 - 1}(m+1)]^{\frac{p-1}{b}} \not\equiv 1 \pmod{p}$; $\chi(p) = 0$ if $p \mid m(m+1)$; $\chi(p) = 1$ otherwise.

Since $J_2(\omega) \neq 0$, there are infinitely many primes p_1 such that p_2 is a prime for every integer m.

We have

$$\pi_2(N, 2) = \frac{J_2(\omega)\omega}{d_1 d_2 \phi^2(\omega)} \frac{N}{\log^2 N}(1 + O(1)).$$

Remark. Taking $m = f^b$ we have $\chi(p) = b = d_1, d_2, d_1 d_2$, if $(m+1)^{\frac{p-1}{b}} \equiv 1 \pmod{p}$.

Theorem 3.1.181. Every integer m can be written in infinitely many ways in the form:

$$m = \frac{p_2 + 1}{p_1^{2d} - 1}.$$

where d is a given odd prime.

We have the Diophantine equation

$$p_2 = mp_1^{2d} - m - 1.$$

We have the arithmetic function

$$J_2(\omega) = \prod_{3 \le p \le p_i} (p - 1 - \chi(p)) \neq 0,$$

where $\chi(p) = b = 2, 2d$ if $[m^{2d-1}(m+1)]^{\frac{p-1}{b}} \equiv 1 \pmod{p}$; $\chi(p) = 0$ if $[m^{2d-1}(m+1)]^{\frac{p-1}{b}} \not\equiv 1 \pmod{p}$; $\chi(p) = 0$ if $p \mid m(m+1)$.

Since $J_2(\omega) \neq 0$, there must exist infinitely many primes p_1 such that p_2 is a prime for every integer m.

We have the best asymptotic formula

$$\pi_2(N, 2) = \frac{J_2(\omega)\omega}{2d\phi^2(\omega)} \frac{N}{\log^2 N}(1 + O(1)).$$

An equation is correct but it does not convey the more precise information.

$O(1)$ tells us that the asymptotic value of $\pi_2(N,2)$ is $\frac{J_2(\omega)\omega}{2d\phi^2(\omega)}\frac{N}{\log^2 N}$.

Remark. Taking $m=f^b$, we have $\chi(p)=b=2,2d$, if $(m+1)^{\frac{p-1}{b}}\equiv 1 \pmod p$.

Theorem 3.1.182. The prime 3-tuples.

(1) $p+a,\ a=0,2,4.$

We have the arithmetic function

$$J_2(3)=0.$$

There are no prime 3-tuples if $p\neq 3$

(2) $p_1+ap_2\cdots p_{n-1}$.

We have

$$J_n(3)=0.$$

There are no prime 3-tuples if $p_1,\cdots,p_{n-1}\neq 3$

(3) $p+3a$.

We have

$$J_2(\omega)=2\prod_{5\le p\le p_i}(p-3)\neq 0.$$

Since $J_2(\omega)\neq 0$, there exist the infinitely many prime 3-tuples. We have the best asymptotic formula of the number of the prime 3-tuples

$$\pi_3(N,3)\sim\frac{J_2(\omega)\omega^2}{\phi^3(\omega)}\frac{N}{\log^3 N}.$$

(4) $p_1+3ap_2\cdots p_{n-1}$.

We have

$$J_n(\omega)=\phi^{n-2}(\omega)J_2(\omega)\neq 0.$$

Since $J_n(\omega)\neq 0$, there exist the infinitely many prime 3-tuples. We have the best asymptotic formula of the number of the prime 3-tuples

$$\pi_3(N,n)\sim\frac{J_n(\omega)\omega^2}{(n-1)!\phi^{n+1}(\omega)}\frac{N^{n-1}}{\log^{n+1} N}.$$

Theorem 3.1.183. The prime 5-tuples.

(1) $p+a,\ a=0,2,6,8,14.$

We have the arithmetic function

$$J_2(5)=0.$$

There are no prime 5-tuples if $p\neq 5$

(2) $p_1+ap_2\cdots p_{n-1}$.

We have

$$J_n(5) = 0.$$

There are no prime 5-tuples if $p_1, \cdots, p_{n-1} \neq 5$

(3) $p + 5a$.

We have

$$J_2(\omega) = 12 \prod_{11 \leq p \leq p_i} (p - 5) \neq 0.$$

Since $J_2(\omega) \neq 0$, there exist the infinitely many prime 5-tuples. We have the best asymptotic formula of the number of the prime 5-tuples

$$\pi_5(N, 3) \sim \frac{J_2(\omega)\omega^4}{\phi^5(\omega)} \frac{N}{\log^5 N}.$$

(4) $p_1 + 5ap_2 \cdots p_{n-1}$.

We have

$$J_n(\omega) = \phi^{n-2}(\omega) J_2(\omega) \neq 0.$$

Since $J_n(\omega) \neq 0$, there exist the infinitely many prime 5-tuples. We have the best asymptotic formula of the number of the prime 5-tuples

$$\pi_5(N, n) \sim \frac{J_n(\omega)\omega^4}{(n-1)!\phi^{n+3}(\omega)} \frac{N^{n-1}}{\log^{n+3} N}.$$

Theorem 3.1.184. The prime 7-tuples.

(1) $p + a,\ a = 0, 4, 6, 10, 12, 16, 22$.

We have the arithmetic function

$$J_2(7) = 0.$$

There are no prime 7-tuples if $p \neq 7$

(2) $p_1 + ap_2 \cdots p_{n-1}$.

We have

$$J_n(7) = 0.$$

There are no prime 7-tuples if $p_1, \cdots, p_{n-1} \neq 7$

(3) $p + 7a$.

We have

$$J_2(\omega) = 30 \prod_{13 \leq p \leq p_i} (p - 7) \neq 0.$$

Since $J_2(\omega) \neq 0$, there exist the infinitely many prime 7-tuples. We have the best asymptotic formula of the number of the prime 7-tuples

$$\pi_7(N, 2) \sim \frac{J_2(\omega)\omega^6}{\phi^7(\omega)} \frac{N}{\log^7 N}.$$

(4) $p_1 + 7ap_2 \cdots p_{n-1}$.
We have

$$J_n(\omega) = \phi^{n-2}(\omega) J_2(\omega) \neq 0.$$

Since $J_n(\omega) \neq 0$, there exist the infinitely many prime 7-tuples. We have the best asymptotic formula of the number of the prime 7-tuples

$$\pi_7(N, n) \sim \frac{J_n(\omega)\omega^6}{(n-1)!\phi^{n+5}(\omega)} \frac{N^{n-1}}{\log^{n+5} N}.$$

Theorem 3.1.185. The prime 11-tuples.
(1) $p + a,\ a = 0, 2, 6, 8, 12, 18, 20, 26, 32, 36, 60$.
We have the arithmetic function

$$J_2(11) = 0.$$

There are no prime 11-tuples if $p \neq 11$
(2) $p_1 + ap_2 \cdots p_{n-1}$.
We have

$$J_2(11) = 0.$$

There are no prime 11-tuples if $p_1, \cdots, p_{n-1} \neq 11$
(3) $p + 11a$.
We have

$$J_2(\omega) = 400 \prod_{19 \leq p \leq p_i} (p - 11) \neq 0.$$

Since $J_2(\omega) \neq 0$, there exist the infinitely many prime 11-tuples. We have the best asymptotic formula of the number of the prime 11-tuples

$$\pi_{11}(N, 2) \sim \frac{J_2(\omega)\omega^{10}}{\phi^{11}(\omega)} \frac{N}{\log^{11} N}.$$

(4) $p_1 + 11ap_2 \cdots p_{n-1}$.
We have

$$J_n(\omega) = \phi^{n-2}(\omega) J_2(\omega) \neq 0.$$

Since $J_n(\omega) \neq 0$, there exist the infinitely many prime 11-tuples. We have the best asymptotic formula of the number of the prime 11-tuples

$$\pi_{11}(N, n) \sim \frac{J_n(\omega)\omega^{10}}{(n-1)!\phi^{n+9}(\omega)} \frac{N^{n-1}}{\log^{n+9} N}.$$

Theorem 3.1.186. The prime 13-tuples.
(1) $p + a,\ a = 0, 4, 6, 10, 16, 18, 24, 28, 34, 40, 46, 48, 90$.

We have the arithmetic function

$$J_2(13) = 0.$$

There are no prime 13-tuples if $p \neq 13$

(2) $p_1 + ap_2 \cdots p_{n-1}$.

We have

$$J_n(13) = 0.$$

There are no prime 13-tuples if $p_1, \cdots, p_{n-1} \neq 13$

(3) $p + 13a$.

We have

$$J_2(\omega) = 216 \prod_{19 \leq p \leq p_i} (p - 13) \neq 0.$$

Since $J_2(\omega) \neq 0$, there exist the infinitely many prime 13-tuples. We have the best asymptotic formula of the number of the prime 13-tuples

$$\pi_{13}(N, 2) \sim \frac{J_2(\omega)\omega^{12}}{\phi^{13}(\omega)} \frac{N}{\log^{13} N}.$$

(4) $p_1 + 13ap_2 \cdots p_{n-1}$.

We have

$$J_n(\omega) = \phi^{n-2}(\omega) J_2(\omega) \neq 0.$$

Since $J_n(\omega) \neq 0$, there exist the infinitely many prime 13-tuples. We have the best asymptotic formula of the number of the prime 13-tuples

$$\pi_1 3(N, n) \sim \frac{J_n(\omega)\omega^{12}}{(n-1)!\phi^{n+11}(\omega)} \frac{N^{n-1}}{\log^{n+11} N}.$$

Theorem 3.1.187. The prime k-tuples.

(1) $p + a$, $a = 0, 2, 6, 12, 14, 20, 24, 26, 30, 42, 44, 50, 56, 62, 66, 72, 86$.

We have the arithmetic function

$$J_2(17) = 0.$$

There are no prime 17-tuples if $p \neq 17$

(2) $p + a$, $a = 0, 4, 10, 12, 18, 22, 24, 28, 34, 40, 52, 54, 64, 70, 82, 84, 112, 144, 172$.

We have

$$J_2(19) = 0$$

There are no prime 19-tuples if $p \neq 19$.

(3) $p + a$, $a = 0, 6, 8, 14, 18, 20, 24, 30, 36, 38, 44, 48, 50, 56, 74, 78, 80, 86, 104, 108, 114, 134, 210$.

We have

$$J_2(23) = 0$$

There are no prime 23-tuples if $p \neq 23$.

(4) $p + a,\ a = 0, 2, 8, 12, 14, 18, 24, 30, , 32, 38, 42, 44, 50, 54, 68, 74, 78, 80, 84, 98, 110, 120, 122, 144, 150, 152, 162, 164, 288.$

We have

$$J_2(29) = 0.$$

There are no prime 29-tuples if $p \neq 29$.

Theorem 3.1.188. $p_j = m^j p \pm n^j,\ (m, n) = 1, 2|mn$, for $j = 1, \cdots, k-1$.

We have the arithmetic function

$$J_2(\omega) = \prod_{3 \leq p \leq p_i} (p - \chi(p)).$$

We now calculate $\chi(p)$.The smallest positive integer S satisfies the congruence

$$n^S \equiv m^S (\text{mod}\ \ p).$$

$\chi(p) = k$ if $k < S; \chi(p) = S + 1$ if $k \geq S; \chi(p) = 1$if $p|mn; J_2(p) = 0$if $S = p - 1$.

We have the best asymptotic formula of the number of the prime 13-tuples

$$\pi_k(N, 2) \sim \frac{J_2(\omega)\omega^{k-1}}{\phi^k(\omega)} \frac{N}{\log^k N}.$$

Theorem 3.1.189. $p_j = m^{2j} p \pm n^{2j},\ (m, n) = 1, 2|mn$, for $j = 1, \cdots, k-1$.

We have the arithmetic function

$$J_2(\omega) = \prod_{3 \leq p \leq p_i} (p - \chi(p)) \neq 0.$$

We now calculate $\chi(p)$.The smallest positive integer S satisfies the congruence

$$n^{2S} \equiv m^{2S} (\text{mod}\ \ p).$$

$\chi(p) = k$ if $k < S; \chi(p) = S + 1$ if $k \geq S; \chi(p) = 1$ if $p|mn; J_2(p) = 0$ if $S = p - 1$.

Since $S \leq \frac{p-1}{2}$, we have $J_2(p) \geq \frac{p-1}{2}$. There exist infinitely many k-tuples of primes for any k.

We have the best asymptotic formula

$$\pi_k(N, n) \sim \frac{J_n(\omega)\omega^{k-1}}{\phi^k(\omega)} \frac{N}{\log^k N}.$$

Theorem 3.1.190. $p_j = 16^j(p^2 - 1) + 1$ for $j = 1, \cdots, k-1$.

We define the smallest positive integer S such that

$$16^S \equiv 1(\bmod \ p).$$

We have the arithmetic function

$$J_2(\omega) = 8 \prod_{7 \le p \le p_i} (p - n - \sum_{j=1}^{n-1} (\frac{16^j - 1}{p}) \ne 0.$$

$n = k$ if $k < S; n = S$ if $k \ge S$.

We have the best asymptotic formula

$$\pi_k(N, 2) \sim \frac{J_2(\omega)\omega^{k-1}}{2^{k-1}\phi^k(\omega)} \frac{N}{\log^k N}.$$

Theorem 3.1.191. $p_3 = p_1 + p_2 + 1, p_4 = p_1 + p_2 + 3.$
We have the arithmetic function

$$J_3(\omega) = \prod_{5 \le p \le p_i} (p^2 - 4p + 5) \ne 0.$$

We have the best asymptotic formula

$$\pi_3(N, 2) \sim \frac{J_2(\omega)\omega^2}{\phi^4(\omega)} \frac{N^2}{\log^4 N}.$$

Theorem 3.1.192. $p_3 = p_1 + p_2 + 1, p_4 = p_1 + p_2 + 3, p_5 = p_1 + p_2 + 7.$
We have the arithmetic function

$$J_2(\omega) = 140 \prod_{11 \le p \le p_i} (p^2 - 5p + 7) \ne 0.$$

We have the best asymptotic formula

$$\pi_4(N, 3) \sim \frac{J_3(\omega)\omega^3}{\phi^5(\omega)} \frac{N^2}{\log^5 N}.$$

Theorem 3.1.193. $p_3 = p_1 + p_2 + 1, p_4 = p_1 + p_2 + 3, p_5 = p_1 + p_2 + 7, p_6 = p_1 + p_2 + 9.$
We have the arithmetic function

$$J_2(\omega) = 60 \prod_{11 \le p \le p_i} (p^2 - 6p + 9) \ne 0.$$

We have the best asymptotic formula

$$\pi_5(N,2) \sim \frac{J_3(\omega)\omega^4}{\phi^6(\omega)} \frac{N^2}{\log^6 N}.$$

Theorem 3.1.194. $p_3 = p_1 + p_2 + 1, p_4 = N - p_1 - p_2, 2 \not| N$.
We have the arithmetic function

$$J_2(\omega) = \prod_{3 \le p \le p_i} (p^2 - 3p + 3 - \chi(p)) \ne 0,$$

where $\chi(p) = 0$ if $p|(N+1)$; $\chi(p) = p - 1$ if $p|N$; $\chi(p) = p - 2$ if$p \not| N$.
We have the best asymptotic formula

$$\pi_3(N,3) \sim \frac{J_3(\omega)\omega^2}{\phi^4(\omega)} \frac{N^2}{\log^4 N}.$$

4. A disproof of the Riemann's hypothesis

Let $s = \sigma + ti$, where σ and t are real, $i = \sqrt{-1}$. We have the Riemann's zeta function

$$\zeta(s) = \prod_p \frac{1}{1 - \frac{1}{p^s}}, \tag{4.1}$$

where p ranges over all primes.

In 1859 Riemann [7] stated that nontrivial zeros of $\zeta(s)$ all lie on the line $\sigma = 1/2$ called the Riemann's hypothesis. In 1990 Hilbert listed the problem of proving or disproving the Riemann's hypothesis as one of the most important problems confronting twentieth century mathematicians. To this day it remains unsolved. The arithmetic (sifting) function $J_n(\omega)$ is able to take the place of the Riemann's hypothesis and the generalized Riemann's hypothesis.

Theorem 4.1.

$$\zeta(s) \ne 0, 0 \le Re(s). \tag{4.2}$$

Proof. From (4.1) we have

$$\frac{1}{\zeta(s)} = \prod_p (1 - \frac{1}{p^s}) = Re^{\theta i}, \tag{4.3}$$

where

$$R = \prod_p R_p, \quad R_p = \sqrt{1 - \frac{2\cos(t \log p)}{p^\sigma} + \frac{1}{p^{2\sigma}}}, \tag{4.4}$$

$$\theta = \sum_p \theta_p, \quad \theta_p = \tan^{-1} \frac{\sin(t \log p)}{p^\sigma - \cos(t \log p)}. \tag{4.5}$$

$\zeta(s) = 0$ if and only if Re $\zeta(s) = 0$ and Im $\zeta(s) = 0$, that is $R = \infty$. From (4.4) we have that if $\cos(t\log p) \leq 0$ then $R_p > 1$ and if $\cos(t\log p) > 0$ then $R_p < 1$.

If

$$\cos(t \log p_1) > 0, \ \cdots, \ \cos(t \log p_2) > 0, \tag{4.6}$$

we have $p_2 < e^{\frac{\pi}{t}} p_1$ and $t \log p_2 < t \log p_1 + \pi$.

If

$$\cos(t\log p_1) < 0, \ \cdots, \ \cos(t\log p_2) < 0, \tag{4.7}$$

we have $p_2 < e^{\frac{\pi}{t}} p_1$ and $t\log p_2 < t\log p_1 + \pi$.

$\cos(t\log p)$ is independent of the real part σ, but may well depend on prime p and imaginary part t. We write $m_+(t)$ for the number of primes p satisfying $\cos(t\log p) > 0$, $m_-(t)$ for the number of primes p satisfying $\cos(t\log p) \leq 0$.

For $\cos(t\log p) > 0$ we have

$$1 > R_p(1 + ti) > R_p(0.5 + ti). \tag{4.8}$$

If $m_+(t_1)$ is much greater than $m_-(t_1)$ such that $R(0.5 + t_1 i)$=min. From (4.4), (4.5) and (4.8) we have for given t_1

$$\min R(\sigma_1 + t_1 i) > \min R(1 + t_1 i) > \min R(0.5 + t_1 i) > \min R(\sigma_2 + t_1 i) \to 0, \tag{4.9}$$

$$\theta(\sigma_1 + t_1 i) = \theta(1 + t_1 i) = \theta(0.5 + t_1 i) = \theta(\sigma_2 + t_1 i) = \text{const}, \tag{4.10}$$

where $\sigma_1 > 1$ and $0 < \sigma_2 < 0.5$.

Since $|\zeta(s)| = \frac{1}{R}$ from (4.9) we have

$$\max|\zeta(\sigma_1 + t_1 i)| < \max|\zeta(1 + t_1 i)| < \max|\zeta(0.5 + t_1 i)| < \max|\zeta(\sigma_2 + t_1 i)| \to \infty. \tag{4.11}$$

For $\cos(t\log p) < 0$ we have

$$1 < R_p(0.5 + ti) < R_p(0.4 + ti) < R_p(0.3 + ti). \tag{4.12}$$

If $m_-(t_1)$ is much greater than $m_+(t_1)$ such that $R(0.5 + t_1 i) = \max$.

From (4.4), (4.5) and (4.12) we have for given t_1

$$\max R(\sigma_1 + t_1 i) < \max R(0.5 + t_1 i) < \max R(0.4 + t_1 i)$$
$$< \max R(0.3 + t_1 i) < \max R(\sigma_2 + t_1 i) \neq \infty, \tag{4.13}$$

$$\theta(\sigma_1 + t_1 i) = \theta(0.5 + t_1 i) = \theta(0.4 + t_1 i) = \theta(0.3 + t_1 i) = \theta(\sigma_2 + t_1 i) = \text{const}, \tag{4.14}$$

where $\sigma_1 > 0.5$ and $0 < \sigma_2 < 0.3$.

Since $|\zeta(s)| = \frac{1}{R}$ from (4.13) we have

$$\min|\zeta(\sigma_1 + t_1 i)| > \min|\zeta(0.5 + t_1 i)| > \min|\zeta(0.4 + t_1 i)|$$
$$> \min|\zeta(0.3 + t_1 i)| > \min|\zeta(\sigma_2 + t_1 i)| \neq 0. \tag{4.15}$$

$\zeta(s)$ satisfies the functional equation

$$\pi^{(-s/2)}\Gamma(s/2)\zeta(s) = \pi^{(-(1-s)/2)}\Gamma((1-s)/2)\zeta(1-s). \tag{4.16}$$

From (4.16) we have

$$\zeta(ti) \neq 0. \tag{4.17}$$

We disprove again the Riemann's hypothesis. We define the Riemann's zeta function

$$\zeta(s) = \prod_p (1 - p^{-s})^{-1} = \sum_{n=1}^{\infty} \frac{1}{n^s}. \tag{4.18}$$

where $s = \sigma + ti, i = \sqrt{-1}$, σ and t are real, p ranges over all primes. We have the exponential formula of the Riemann's zeta function

$$\zeta(s) = Re^{i\theta}, \tag{4.19}$$

where

$$R = \prod_p R_p, \quad R_p = \frac{p^\sigma}{\sqrt{p^{2\sigma} + 1 - 2p^\sigma \cos(t \log p)}}. \tag{4.20}$$

$$\theta = \sum_p \theta_p, \quad \theta_p = \tan^{-1} \frac{\sin(t \log p)}{\cos(t \log p) - p^\sigma}. \tag{4.21}$$

We define the beta function

$$\beta(s) = \prod_p (1 + p^{-s})^{-1} = \sum_{n=1}^{\infty} \frac{\lambda(n)}{n^s}, \tag{4.22}$$

where $\lambda(1) = 1, \lambda(n) = (-1)^{a_1 + \cdots + a_k}$ if $n = p_1^{a_1} \cdots p_k^{a_k}$.

We have the exponential formula of the beta function

$$\beta(s) = \bar{R}e^{i\bar{\theta}}. \tag{4.23}$$

where

$$\bar{R} = \prod_p \bar{R}_p, \quad \bar{R}_p = \frac{p^\sigma}{\sqrt{p^{2\sigma} + 1 + 2p^\sigma \cos(t \log p)}}. \tag{4.24}$$

$$\bar{\theta} = \sum_p \bar{\theta}_p, \quad \bar{\theta}_p = \tan^{-1} \frac{\sin(t \log p)}{\cos(t \log p) + p^\sigma}. \tag{4.25}$$

From (4.18) and (4.22) we have

$$\zeta(2s) = \zeta(s)\beta(s). \tag{4.26}$$

$\zeta(s)$ and $\beta(s)$ have the same property, see (4.20) and (4.24).

Theorem 4.2 In 1896 J. Hadamard and C.J.de la Vallee Poussion proved independently $\zeta(1+ti) \neq 0$. From (4.26) we have

$$|\zeta(1+2ti)| = |\zeta(\frac{1}{2}+ti)||\beta(\frac{1}{2}+ti)| \neq 0. \tag{4.27}$$

From (4.27) we have

$$|\zeta(\frac{1}{2}+ti)| \neq 0 \quad \text{and} \quad |\beta(\frac{1}{2}+ti)| \neq 0. \tag{4.28}$$

We have [8]

$$|\zeta(\frac{1}{2}+ti)| \neq \infty. \tag{4.29}$$

Therefore

$$|\beta(\frac{1}{2}+ti)| \neq 0. \tag{4.30}$$

$\zeta(s)$ and $\beta(s)$ are the dual functions. We have

$$|\beta(\frac{1}{2}+ti)| \neq \infty. \tag{4.31}$$

Therefore

$$|\zeta(\frac{1}{2}+ti)| \neq 0. \tag{4.32}$$

Theorem 4.3 For $\sigma > 1$ we have

$$\log\zeta(s) = \sum_p \sum_{m=1}^{\infty} m^{-1}p^{-m\sigma}\exp(-itm\log p). \tag{4.33}$$

If $\zeta(s)$ had a zero at $\frac{1}{2}+ti$, then $\log|\zeta(\sigma+ti)|$ would tend to $-\infty$ as σ tends to $\frac{1}{2}$ from the right.

From (4.33) we have

$$\log|\zeta(s)| = \sum_p \sum_{m=1}^{\infty} m^{-1}p^{-m\sigma}\cos(tm\log p) \tag{4.34}$$

with t replaced by $0, t, 2t, \cdots, Ht$. It gives

$$\sum_{j=0}^{H-1} \binom{2H}{j} \log|\zeta(\sigma+(H-j)ti)| + \frac{1}{2}\binom{2H}{H}\log\zeta(\sigma) = \sum_p \sum_{m=1}^{\infty} m^{-1}p^{-m\sigma}A \geq 0, \tag{4.35}$$

where

$$A=\sum_{j=0}^{H-1}\binom{2H}{j}\cos\Big((H-j)tm\log p\Big)+\frac{1}{2}\binom{2H}{H}=2^{H-1}\Big[1+\cos(tm\log p)\Big]^{H}\geq 0. \tag{4.36}$$

H is an even number.

From (4.35) we have

$$\Big(\zeta(\sigma)\Big)^{\frac{1}{2}\binom{2H}{H}}\prod_{j=0}^{H-1}\Big|\zeta(\sigma+(H-j)ti)\Big|^{\binom{2H}{j}}\geq 1. \tag{4.37}$$

Since $|\zeta(\frac{1}{2}+eti)|\neq\infty$[8], where $e=1,2,\cdots,H$, from (4.37) we have $|\zeta(\frac{1}{2}+eti)|\neq 0$ for sufficiently large even number H.

Min $|\zeta(\frac{1}{2}+ti)\approx 0$ but $\neq 0$. The computation of all nontrivial zeros of $\zeta(\frac{1}{2}+ti)$ is an error, which satisfies the Riemann's error hypothesis[9].

From (4.36) we have

$$\cos 2\theta+4\cos\theta+3=2(1+\cos\theta)^2.$$

$$\cos 4\theta+8\cos 3\theta+28\cos 2\theta+56\cos\theta+35=8(1+\cos\theta)^4.$$

$$\cos 6\theta+12\cos 5\theta+66\cos 4\theta+220\cos 3\theta+495\cos 2\theta+792\cos\theta+462=32(1+\cos\theta)^6.$$

In the same way we may prove $|\zeta(\sigma+ti)|\neq 0$, where $\sigma\geq 0$. Since the Riemann hypothesis is false, all theorems and all conjectures related to the Riemann hypothesis are also false.

The Riemann hypothesis is a simple simple mathematical problem, which has tantalized many famous mathematicians for 150 years.

5. Santilli's Theory for a Table of Primes

Fundamental Theorem in a Table of Prime. If there exist infinitely many primes p such that each of $p+a_j$, where $j=1,2,\cdots,k-1$, is simultaneously a prime, then $p+a_j$ must satisfy two necessary and sufficient conditions:

(I) $p+a_j,\ a_j$ is an even number.

(II) There exists an arithmetic function $J_2(\omega)\neq 0$ which denotes the number of k-tuples of subequations. It is also the number of k-tuples of solutions for

$$(p_\alpha,\omega)=1,\Big(\prod_{j=1}^{k-1}(p_\alpha+a_j),\omega\Big)=1, 1\leq\alpha\leq\phi(\omega) \tag{5.1}$$

From (5.1) defining $J_2(\omega)$ can be written in the form

$$J_2(\omega)=\sum_{\alpha=1}^{\phi(\omega)}\left[\frac{k-1}{\sum_{j=1}^{k-1}(p_\alpha+a_j,\ \omega)}\right]=\prod_{3\leq p\leq p_i}(p-1-\chi(p)), \tag{5.2}$$

where $\chi(p)$ is the number of solutions of congruence

$$\prod_{j=1}^{k-1}(q+a_j)\equiv 0(\text{mod}\ \ p), q=1,\cdots,p-1. \tag{5.3}$$

If $J_2(\omega)\neq 0$, there exist infinitely many k-tuples of primes. If $J_2(\omega)=0$, there exist finitely many k-tuples of primes. The arithmetic function $J_2(\omega)$ is a generalization of Euler function $\phi(\omega)$. We shall obtain exact asymptotic formula by introducing the $J_2(\omega)$.

$$\pi_k(N)\sim\frac{J_2(\omega)\omega^{k-1}}{\phi^k(\omega)}Li_k(N) \tag{5.4}$$

where

$$Li_k(N)=\int_2^N\frac{dx}{(\log x)^k}$$

We not only prove the existence of infinitely many k-tuples of primes, but we give that number of k-tuples of primes is very accurately calculated by (5.4).

Theorem 5.1. k-tuple: $p, p+a_j$; where $a_j=p_j-p_0,\ j=1,2,\cdots,k-1, p_0,p_1,\cdots,p_{k-1}$ are the consecutive primes in a table of primes.

If $J_2(p_0)=0$, there exists one k-tuple of primes. If $J_2(\omega)\neq 0$, there exist infinitely many k-tuples of primes, where k is a finite number.

In order to be understandable, the author needs to have supplied many details below.

Theorem 5.1.1. 2-tuples:
(1) $p,\ p+a_1,\ a_1=p_1-p_0,\ p_0=3,\ p_1=5.$
We have

$$J_2(\omega)=\prod_{3\le p\le p_i}(p-2)\neq 0.$$

It is the first theorem in a table of primes. So far, no one has found a proof.

Since $J_2(\omega)\to\infty$ as $\omega\to\infty$, there exist infinitely many twin primes. It is the simplest theorem. No one could ever find the "largest" twin prime.

(2) $p,\ p+a_1,\ a_1=p_1-p_0,\ p_0=7,\ p_1=11.$ We have

$$J_2(\omega)=\prod_{3\le p\le p_i}(p-2)\neq 0.$$

(3) $p,\ p+a_1,\ a_1=p_1-p_0,\ p_0=23,\ p_1=29.$ We have

$$J_2(\omega)=2\prod_{5\le p\le p_i}(p-2)\neq 0.$$

Since $J_2(\omega)\to\infty$ as $\omega\to\infty$, there exist infinitely many prime solutions.

We have the best asymptotic formula

$$\pi_2(N) \sim \frac{J_2(\omega)\omega}{\phi^2(\omega)} Li_2(N)$$

the Theorem 5.1.1 is the subtable of prime numbers.

Theorem 5.1.2. 3-tuples:
(1) $p,\ p+a_j,\ a_j = p_j - p_0,\ p_0 = 3,\ p_1 = 5,\ p_2 = 7$. We have $J_2(3) = 0$. There does not exist any 3-tuple of primes except 3.5.7.
(2) $p,\ p+a_j,\ a_j = p_j - p_0,\ p_0 = 5,\ p_1 = 7,\ p_2 = 11$. We have

$$J_2(\omega) = \prod_{5 \le p \le p_i} (p-3) \neq 0.$$

(3) $p,\ p+a_j,\ a_j = p_j - p_0,\ p_0 = 7,\ p_1 = 11,\ p_2 = 13$. We have

$$J_2(\omega) = \prod_{5 \le p \le p_i} (p-3) \neq 0.$$

(4) $p,\ p+a_j,\ a_j = p_j - p_0,\ p_0 = 19,\ p_1 = 23,\ p_2 = 29$. We have

$$J_2(\omega) = 3 \prod_{7 \le p \le p_i} (p-3) \neq 0.$$

Since $J_2(\omega) \to \infty$ as $\omega \to \infty$, there exist infinitely many 3-tuples of primes. We have

$$\pi_3(N) \sim \frac{J_2(\omega)\omega^2}{\phi^3(\omega)} Li_3(N)$$

The Theorem 5.1.2 is the subtable of prime numbers.

Theorem 5.1.3. 4-tuples:
(1) $p,\ p+a_j,\ a_j = p_j - p_0,\ p_0 = 5,\ p_1 = 7,\ p_2 = 11,\ p_3 = 13$. We have

$$J_2(\omega) = \prod_{7 \le p \le p_i} (p-4) \neq 0.$$

(2) $p,\ p+a_j,\ a_j = p_j - p_0,\ p_0 = 7,\ p_1 = 11,\ p_2 = 13.\ p_3 = 17$. We have

$$J_2(\omega) = 2 \prod_{7 \le p \le p_i} (p-4) \neq 0.$$

(3) $p,\ p+a_j,\ a_j = p_j - p_0,\ p_0 = 11,\ p_1 = 13,\ p_2 = 17,\ p_3 = 19$. We have

$$J_2(\omega) = \prod_{7 \le p \le p_i} (p-4) \neq 0.$$

(4) $p,\ p+a_j,\ a_j=p_j-p_0,\ p_0=13,\ p_1=17,\ p_2=19,\ p_3=23$. We have

$$J_2(\omega)=2\prod_{7\le p\le p_i}(p-4)\neq 0.$$

(5) $p,\ p+a_j,\ a_j=p_j-p_0,\ p_0=23,\ p_1=29,\ p_2=31,\ p_3=37$. We have

$$J_2(\omega)=4\prod_{11\le p\le p_i}(p-4)\neq 0.$$

(6) $p,\ p+a_j,\ a_j=p_j-p_0,\ p_0=31,\ p_1=37,\ p_2=41,\ p_3=43$. We have

$$J_2(\omega)=2\prod_{7\le p\le p_i}(p-4)\neq 0.$$

Since $J_2(\omega)\to\infty$ as $\omega\to\infty$, there exist infinitely many 4-tuples of primes. We have

$$\pi_4(N)\sim\frac{J_2(\omega)\omega^3}{\phi^4(\omega)}Li_4(N)$$

The Theorem 5.1.3 is the subtable of prime numbers.

Theorem 5.1.4. 5-tuples:
(1) $p,\ p+a_j,\ a_j=p_j-p_0,\ p_0=5,\ p_1=7,\ p_2=11,\ p_3=13,\ p_4=17$. We have

$$J_2(\omega)=\prod_{7\le p\le p_i}(p-5)\neq 0.$$

(2) $p,\ p+a_j,\ a_j=p_j-p_0,\ p_0=7,\ p_j=11,13,17,19$. We have

$$J_2(\omega)=\prod_{7\le p\le p_i}(p-5)\neq 0.$$

(3) $p,\ p+a_j,\ a_j=p_j-p_0,\ p_0=13,\ p_j=17,19,23,29$. We have

$$J_2(\omega)=2\prod_{7\le p\le p_i}(p-5)\neq 0.$$

(4) $p,\ p+a_j,\ a_j=p_j-p_0,\ p_0=17,\ p_j=19,23,29,31$. We have

$$J_2(\omega)=3\prod_{11\le p\le p_i}(p-5)\neq 0.$$

(5) $p,\ p+a_j,\ a_j=p_j-p_0,\ p_0=31,\ p_j=37,41,43,47$. We have

$$J_2(\omega)=4\prod_{11\le p\le p_i}(p-5)\neq 0.$$

Since $J_2(\omega) \to \infty$ as $\omega \to \infty$, there exist infinitely many 5-tuples of primes. We have

$$\pi_5(N) \sim \frac{J_2(\omega)\omega^4}{\phi^5(\omega)} Li_5(N)$$

The Theorem 5.1.4 is the subtable of prime numbers.

Theorem 5.1.5. 6-tuples:

(1) $p,\ p+a_j,\ a_j = p_j - p_0,\ p_0 = 5,\ p_j = 7, 11, 13, 17, 19$. We have $J_2(3) \neq 0$ and $J_2(5) = 0$. There does not exist any 6-tuple of primes except one 6-tuple of primes: 5,7,11,13,17,19.

(2) $p,\ p+a_j,\ a_j = p_j - p_0,\ p_0 = 7,\ p_j = 11, \cdots, 23$. We have

$$J_2(\omega) = \prod_{7 \le p \le p_i} (p-6) \neq 0.$$

(3) $p,\ p+a_j,\ a_j = p_j - p_0,\ p_0 = 11,\ p_j = 13, \cdots, 29$. We have

$$J_2(\omega) = \prod_{7 \le p \le p_i} (p-6) \neq 0.$$

(4) $p,\ p+a_j,\ a_j = p_j - p_0,\ p_0 = 13,\ p_j = 17, \cdots, 31$. We have

$$J_2(\omega) = 2 \prod_{11 \le p \le p_i} (p-6) \neq 0.$$

(5) $p,\ p+a_j,\ a_j = p_j - p_0,\ p_0 = 17,\ p_j = 19, \cdots, 37$. We have

$$J_2(\omega) = 3 \prod_{11 \le p \le p_i} (p-6) \neq 0.$$

(6) $p,\ p+a_j,\ a_j = p_j - p_0,\ p_0 = 19,\ p_j = 23, \cdots, 41$. We have

$$J_2(\omega) = 12 \prod_{13 \le p \le p_i} (p-6) \neq 0.$$

Since $J_2(\omega) \to \infty$ as $\omega \to \infty$, there exist infinitely many 6-tuples of primes. We have

$$\pi_6(N) \sim \frac{J_2(\omega)\omega^5}{\phi^6(\omega)} Li_6(N)$$

The Theorem 5.1.5 is the subtable of prime numbers.

Theorem 5.1.6. 7-tuples:

(1) $p,\ p+a_j,\ a_j = p_j - p_0,\ p_0 = 7,\ p_j = 11, \cdots, 29$. We have $J_2(3) \neq 0$, $J_2(5) \neq 0$ and $J_2(7) = 0$. It has no 7-tuples of primes except one 7-tuples of primes: 7,11,13,17,19,23,29.

(2) $p,\ p+a_j,\ a_j=p_j-p_0,\ p_0=11,\ p_j=13,\cdots,31$. We have

$$J_2(\omega)=2\prod_{11\le p\le p_i}(p-7)\ne 0.$$

(3) $p,\ p+a_j,\ a_j=p_j-p_0,\ p_0=13,\ p_j=17,\cdots,37$. We have

$$J_2(\omega)=2\prod_{11\le p\le p_i}(p-7)\ne 0.$$

(4) $p,\ p+a_j,\ a_j=p_j-p_0,\ p_0=17,\ p_j=19,\cdots,41$. We have

$$J_2(\omega)=10\prod_{13\le p\le p_i}(p-7)\ne 0.$$

(5) $p,\ p+a_j,\ a_j=p_j-p_0,\ p_0=19,\ p_j=23,\cdots,43$. We have

$$J_2(\omega)=10\prod_{13\le p\le p_i}(p-7)\ne 0.$$

(6) $p,\ p+a_j,\ a_j=p_j-p_0,\ p_0=23,\ p_j=29,\cdots,47$. We have

$$J_2(\omega)=2\prod_{11\le p\le p_i}(p-7)\ne 0.$$

(7) $p,\ p+a_j,\ a_j=p_j-p_0,\ p_0=29,\ p_j=31,\cdots,53$. We have

$$J_2(\omega)=5\prod_{13\le p\le p_i}(p-7)\ne 0.$$

(8) $p,\ p+a_j,\ a_j=p_j-p_0,\ p_0=31,\ p_j=37,\cdots,59$. We have

$$J_2(\omega)=6\prod_{13\le p\le p_i}(p-7)\ne 0.$$

Since $J_2(\omega)\to\infty$ as $\omega\to\infty$, there exist infinitely many 7-tuples of primes. We have

$$\pi_7(N)\sim\frac{J_2(\omega)\omega^6}{\phi^7(\omega)}Li_7(N)$$

Theorem 5.1.7. 8-tuples:
(1) $p,\ p+a_j,\ a_j=p_j-p_0,\ p_0=11,\ p_j=13,\cdots,37$. We have

$$J_2(\omega)=18\prod_{17\le p\le p_i}(p-8)\ne 0.$$

(2) $p,\ p+a_j,\ a_j=p_j-p_0,\ p_0=13,\ p_j=17,\cdots,41$. We have

$$J_2(\omega)=8\prod_{13\le p\le p_i}(p-8)\neq 0.$$

(3) $p,\ p+a_j,\ a_j=p_j-p_0,\ p_0=17,\ p_j=19,\cdots,43$. We have

$$J_2(\omega)=48\prod_{17\le p\le p_i}(p-8)\neq 0.$$

(4) $p,\ p+a_j,\ a_j=p_j-p_0,\ p_0=19,\ p_j=23,\cdots,47$. We have

$$J_2(\omega)=8\prod_{13\le p\le p_i}(p-8)\neq 0.$$

(5) $p,\ p+a_j,\ a_j=p_j-p_0,\ p_0=23,\ p_j=29,\cdots,53$. We have

$$J_2(\omega)=4\prod_{13\le p\le p_i}(p-8)\neq 0.$$

(6) $p,\ p+a_j,\ a_j=p_j-p_0,\ p_0=29,\ p_j=31,\cdots,59$. We have

$$J_2(\omega)=5\prod_{13\le p\le p_i}(p-8)\neq 0.$$

Since $J_2(\omega)\to\infty$ as $\omega\to\infty$, there exist infinitely many 8-tuples of primes. We have

$$\pi_8(N)\sim\frac{J_2(\omega)\omega^7}{\phi^8(\omega)}Li_8(N)$$

Theorem 5.1.8. 9-tuples:
(1) $p,\ p+a_j,\ a_j=p_j-p_0,\ p_0=11,\ p_j=13,\cdots,41$. We have

$$J_2(\omega)=15\prod_{17\le p\le p_i}(p-9)\neq 0.$$

(2) $p,\ p+a_j,\ a_j=p_j-p_0,\ p_0=13,\ p_j=17,\cdots,43$. We have

$$J_2(\omega)=30\prod_{17\le p\le p_i}(p-9)\neq 0.$$

(3) $p,\ p+a_j,\ a_j=p_j-p_0,\ p_0=17,\ p_j=19,\cdots,47$. We have

$$J_2(\omega)=30\prod_{17\le p\le p_i}(p-9)\neq 0.$$

(4) $p,\ p+a_j,\ a_j=p_j-p_0,\ p_0=19,\ p_j=23,\cdots,53$. We have

$$J_2(\omega)=144\prod_{19\le p\le p_i}(p-9)\ne 0.$$

(5) $p,\ p+a_j,\ a_j=p_j-p_0,\ p_0=23,\ p_j=29,\cdots,59$. We have

$$J_2(\omega)=4\prod_{13\le p\le p_i}(p-9)\ne 0.$$

(6) $p,\ p+a_j,\ a_j=p_j-p_0,\ p_0=29,\ p_j=31,\cdots,61$. We have

$$J_2(\omega)=4\prod_{13\le p\le p_i}(p-9)\ne 0.$$

(7) $p,\ p+a_j,\ a_j=p_j-p_0,\ p_0=31,\ p_j=37,\cdots,67$. We have

$$J_2(\omega)=20\prod_{17\le p\le p_i}(p-9)\ne 0.$$

Since $J_2(\omega)\to\infty$ as $\omega\to\infty$, there exist infinitely many 9-tuples of primes. We have

$$\pi_9(N)\sim\frac{J_2(\omega)\omega^8}{\phi^9(\omega)}Li_9(N)$$

Theorem 5.1.9. 10-tuples:

(1) $p,\ p+a_j,\ a_j=p_j-p_0,\ p_0=11,\ p_j=13,\cdots,43$. We have

$$J_2(\omega)=10\prod_{17\le p\le p_i}(p-10)\ne 0.$$

(2) $p,\ p+a_j,\ a_j=p_j-p_0,\ p_0=13,\ p_j=17,\cdots,47$. We have

$$J_2(\omega)=128\prod_{19\le p\le p_i}(p-10)\ne 0.$$

(3) $p,\ p+a_j,\ a_j=p_j-p_0,\ p_0=17,\cdots,53$. We have

$$J_2(\omega)=96\prod_{19\le p\le p_i}(p-10)\ne 0.$$

(4) $p,\ p+a_j,\ a_j=p_j-p_0,\ p_0=19,\ p_j=23,\cdots,59$. We have

$$J_2(\omega)=96\prod_{19\le p\le p_i}(p-10)\ne 0.$$

(5) $p,\ p+a_j,\ a_j=p_j-p_0,\ p_0=23,\ p_j=29,\cdots,61$. We have

$$J_2(\omega)=630\prod_{23\le p\le p_i}(p-10)\neq 0.$$

(6) $p,\ p+a_j,\ a_j=p_j-p_0,\ p_0=29,\ p_j=31,\cdots,67$. We have

$$J_2(\omega)=840\prod_{23\le p\le p_i}(p-10)\neq 0.$$

Since $J_2(\omega)\to\infty$ as $\omega\to\infty$, there exist infinitely many 10-tuples of primes. We have

$$\pi_{10}(N)\sim\frac{J_2(\omega)\omega^9}{\phi^{10}(\omega)}Li_{10}(N)$$

Theorem 5.1.10. 11-tuples:
(1) $p,\ p+a_j,\ a_j=p_j-p_0,\ p_0=11,\ p_j=13,\cdots,47$. We have

$$J_2(\omega)=28\prod_{19\le p\le p_i}(p-11)\neq 0.$$

(2) $p,\ p+a_j,\ a_j=p_j-p_0,\ p_0=13,\ p_j=17,\cdots,53$. We have

$$J_2(\omega)=48\prod_{19\le p\le p_i}(p-11)\neq 0.$$

(3) $p,\ p+a_j,\ a_j=p_j-p_0,\ p_0=17,\ p_j=19,\cdots,59$. We have

$$J_2(\omega)=63\prod_{19\le p\le p_i}(p-11)\neq 0.$$

(4) $p,\ p+a_j,\ a_j=p_j-p_0,\ p_0=19,\ p_j=23,\cdots,61$. We have

$$J_2(\omega)=378\prod_{23\le p\le p_i}(p-11)\neq 0.$$

(5) $p,\ p+a_j,\ a_j=p_j-p_0,\ p_0=23,\ p_j=29,\cdots,67$. We have

$$J_2(\omega)=540\prod_{23\le p\le p_i}(p-11)\neq 0.$$

(6) $p,\ p+a_j,\ a_j=p_j-p_0,\ p_0=29,\ p_j=31,\cdots,71$. We have

$$J_2(\omega)=378\prod_{23\le p\le p_i}(p-11)\neq 0.$$

(7) $p,\ p+a_j,\ a_j=p_j-p_0,\ p_0=31,\ p_j=37,\cdots,73$. We have

$$J_2(\omega)=378\prod_{23\le p\le p_i}(p-11)\neq 0.$$

Since $J_2(\omega)\to\infty$ as $\omega\to\infty$, there exist infinitely many 11-tuples of primes. We have

$$\pi_{11}(N)\sim\frac{J_2(\omega)\omega^{10}}{\phi^{11}(\omega)}Li_{11}(N)$$

Theorem 5.1.11. 12-tuples:

(1) $p,\ p+a_j,\ a_j=p_j-p_0,\ p_0=11,\ p_j=13,\cdots,53$. We have

$$J_2(\omega)=21\prod_{19\le p\le p_i}(p-12)\neq 0.$$

(2) $p,\ p+a_j,\ a_j=p_j-p_0,\ p_0=13,\ p_j=17,\cdots,59$. We have

$$J_2(\omega)=2352\prod_{29\le p\le p_i}(p-12)\neq 0.$$

(3) $p,\ p+a_j,\ a_j=p_j-p_0,\ p_0=17,\ p_j=19,\cdots,61$. We have

$$J_2(\omega)=288\prod_{23\le p\le p_i}(p-12)\neq 0.$$

(4) $p,\ p+a_j,\ a_j=p_j-p_0,\ p_0=19,\ p_j=23,\cdots,67$. We have

$$J_2(\omega)=324\prod_{23\le p\le p_i}(p-12)\neq 0.$$

(5) $p,\ p+a_j,\ a_j=p_j-p_0,\ p_0=23,\ p_j=29,\cdots,71$. We have

$$J_2(\omega)=216\prod_{23\le p\le p_i}(p-12)$$

(6) $p,\ p+a_j,\ a_j=p_j-p_0,\ p_0=29,\ p_j=31,\cdots,73$. We have

$$J_2(\omega)=288\prod_{23\le p\le p_i}(p-12)\neq 0.$$

(7) $p,\ p+a_j,\ a_j=p_j-p_0,\ p_0=31,\ p_j=37,\cdots,79$. We have

$$J_2(\omega)=192\prod_{23\le p\le p_i}(p-12)\neq 0.$$

Since $J_2(\omega) \to \infty$ as $\omega \to \infty$, there exist infinitely many 12-tuples of primes. We have

$$\pi_{12}(N) \sim \frac{J_2(\omega)\omega^{11}}{\phi^{12}(\omega)} Li_{12}(N)$$

Theorem 5.1.12. 13-tuples:

(1) $p,\ p + a_j,\ a_j = p_j - p_0,\ p_0 = 11,\ p_j = 13, \cdots, 59$. We have

$$J_2(\omega) = 792 \prod_{29 \le p \le p_i} (p - 13) \neq 0.$$

(2) $p,\ p + a_j,\ a_j = p_j - p_0,\ p_0 = 13,\ p_j = 17, \cdots, 61$. We have

$$J_2(\omega) = 924 \prod_{29 \le p \le p_i} (p - 13) \neq 0.$$

(3) $p,\ p + a_j,\ a_j = p_j - p_0,\ p_0 = 17,\ p_j = 19, \cdots, 67$. We have

$$J_2(\omega) = 240 \prod_{23 \le p \le p_i} (p - 13) \neq 0.$$

(4) $p,\ p + a_j,\ a_j = p_j - p_0,\ p_0 = 19,\ p_j = 23, \cdots, 71$. We have

$$J_2(\omega) = 192 \prod_{23 \le p \le p_i} (p - 13) \neq 0.$$

(5) $p,\ p + a_j,\ a_j = p_j - p_0,\ p_0 = 23,\ p_j = 29, \cdots, 73$. We have

$$J_2(\omega) = 160 \prod_{23 \le p \le p_i} (p - 13) \neq 0.$$

(6) $p,\ p + a_j,\ a_j = p_j - p_0,\ p_0 = 29,\ p_j = 31, \cdots, 79$. We have

$$J_2(\omega) = 120 \prod_{23 \le p \le p_i} (p - 13) \neq 0.$$

(7) $p,\ p + a_j,\ a_j = p_j - p_0,\ p_0 = 31,\ p_j = 37, \cdots, 83$. We have

$$J_2(\omega) = 1540 \prod_{29 \le p \le p_i} (p - 13) \neq 0.$$

Since $J_2(\omega) \to \infty$ as $\omega \to \infty$, there exist infinitely many 13-tuples of primes. We have

$$\pi_{13}(N) \sim \frac{J_2(\omega)\omega^{12}}{\phi^{13}(\omega)} Li_{13}(N)$$

Theorem 5.1.13. 14-tuples:

(1) $p,\ p+a_j,\ a_j=p_j-p_0,\ p_0=11,\ p_j=13,\cdots,61$. We have

$$J_2(\omega)=300\prod_{29\le p\le p_i}(p-14)\ne 0.$$

(2) $p,\ p+a_j,\ a_j=p_j-p_0,\ p_0=13,\ p_j=17,\cdots,67$. We have

$$J_2(\omega)=700\prod_{29\le p\le p_i}(p-14)\ne 0.$$

(3) $p,\ p+a_j,\ a_j=p_j-p_0,\ p_0=17,\ p_j=19,\cdots,71$. We have

$$J_2(\omega)=140\prod_{23\le p\le p_i}(p-14)\ne 0.$$

(4) $p,\ p+a_j,\ a_j=p_j-p_0,\ p_0=19,\ p_j=23,\cdots,73$. We have

$$J_2(\omega)=140\prod_{23\le p\le p_i}(p-14)\ne 0.$$

(5) $p,\ p+a_j,\ a_j=p_j-p_0,\ p_0=23,\ p_j=29,\cdots,79$. We have

$$J_2(\omega)=64\prod_{23\le p\le p_i}(p-14)\ne 0.$$

(6) $p,\ p+a_j,\ a_j=p_j-p_0,\ p_0=29,\ p_j=31,\cdots,83$. We have

$$J_2(\omega)=840\prod_{29\le p\le p_i}(p-14)\ne 0.$$

(7) $p,\ p+a_j,\ a_j=p_j-p_0,\ p_0=31,\ p_j=37,\cdots,89$. We have

$$J_2(\omega)=16896\prod_{31\le p\le p_i}(p-14)\ne 0.$$

Since $J_2(\omega)\to\infty$ as $\omega\to\infty$, there exist infinitely many 14-tuples of primes. We have

$$\pi_{14}(N)\sim\frac{J_2(\omega)\omega^{13}}{\phi^{14}(\omega)}Li_{14}(N)$$

Theorem 5.1.14. 15-tuples:
(1) $p,\ p+a_j,\ a_j=p_j-p_0,\ p_0=11,\ p_j=13,\cdots,67$. We have

$$J_2(\omega)=216\prod_{29\le p\le p_i}(p-15)\ne 0.$$

(2) $p,\ p+a_j,\ a_j=p_j-p_0,\ p_0=13,\ p_j=17,\cdots,71$. We have

$$J_2(\omega)=4050\prod_{31\le p\le p_i}(p-15)\ne 0.$$

(3) $p,\ p+a_j,\ a_j=p_j-p_0,\ p_0=17,\ p_j=19,\cdots,73$. We have

$$J_2(\omega)=96\prod_{23\le p\le p_i}(p-15)\ne 0.$$

(4) $p,\ p+a_j,\ a_j=p_j-p_0,\ p_0=19,\ p_j=23,\cdots,79$. We have

$$J_2(\omega)=56\prod_{23\le p\le p_i}(p-15)\ne 0.$$

(5) $p,\ p+a_j,\ a_j=p_j-p_0,\ p_0=23,\ p_j=29,\cdots,83$. We have

$$J_2(\omega)=378\prod_{29\le p\le p_i}(p-15)\ne 0.$$

(6) $p,\ p+a_j,\ a_j=p_j-p_0,\ p_0=29,\ p_j=31,\cdots,89$. We have

$$J_2(\omega)=8100\prod_{31\le p\le p_i}(p-15)\ne 0.$$

(7) $p,\ p+a_j,\ a_j=p_j-p_0,\ p_0=31,\ p_j=37,\cdots,97$. We have

$$J_2(\omega)=10800\prod_{31\le p\le p_i}(p-15)\ne 0.$$

Since $J_2(\omega)\to\infty$ as $\omega\to\infty$, there exist infinitely many 15-tuples of primes. We have

$$\pi_{15}(N)\sim\frac{J_2(\omega)\omega^{14}}{\phi^{15}(\omega)}Li_{15}(N)$$

Theorem 5.1.15. 16-tuples:

(1) $p,\ p+a_j,\ a_j=p_j-p_0,\ p_0=11,\ p_j=13,\cdots,71$. We have $J_2(p)\ne 0,\ p=3,5,7$ and $J_2(11)=0$, there is no 16-tuples of primes except one 16-tuples of primes: $11,\cdots,71$.

(2) $p,\ p+a_j,\ a_j=p_j-p_0,\ p_0=13,\ p_j=17,\cdots,73$. We have

$$J_2(\omega)=2240\prod_{31\le p\le p_i}(p-16)\ne 0.$$

(3) $p,\ p+a_j,\ a_j=p_j-p_0,\ p_0=17,\ p_j=19,\cdots,79$. We have

$$J_2(\omega)=52416\prod_{37\le p\le p_i}(p-16)\ne 0.$$

(4) $p,\ p+a_j,\ a_j=p_j-p_0,\ p_0=19,\ p_j=23,\cdots,83$. We have

$$J_2(\omega)=288\prod_{29\le p\le p_i}(p-16)\neq 0.$$

(5) $p,\ p+a_j,\ a_j=p_j-p_0,\ p_0=23,\ p_j=29,\cdots,89$. We have

$$J_2(\omega)=3024\prod_{31\le p\le p_i}(p-16)\neq 0.$$

(6) $p,\ p+a_j,\ a_j=p_j-p_0,\ p_0=29,\ p_j=31,\cdots,97$. We have

$$J_2(\omega)=6804\prod_{31\le p\le p_i}(p-16)\neq 0.$$

(7) $p,\ p+a_j,\ a_j=p_j-p_0,\ p_0=31,\ p_j=37,\cdots,101$. We have

$$J_2(\omega)=6075\prod_{31\le p\le p_i}(p-16)\neq 0.$$

Since $J_2(\omega)\to\infty$ as $\omega\to\infty$, there exist infinitely many 16-tuples of primes. We have

$$\pi_{16}(N)\sim\frac{J_2(\omega)\omega^{15}}{\phi^{16}(\omega)}Li_{16}(N)$$

Theorem 5.1.16. 17-tuples:
(1) $p,\ p+a_j,\ a_j=p_j-p_0,\ p_0=13,\ p_j=17,\cdots,79$. We have

$$J_2(\omega)=20475\prod_{37\le p\le p_i}(p-17)\neq 0.$$

(2) $p,\ p+a_j,\ a_j=p_j-p_0,\ p_0=17,\ p_j=19,\cdots,83$. We have

$$J_2(\omega)=25200\prod_{37\le p\le p_i}(p-17)\neq 0.$$

(3) $p,\ p+a_j,\ a_j=p_j-p_0,\ p_0=19,\ p_j=23,\cdots,89$. We have

$$J_2(\omega)=27040\prod_{37\le p\le p_i}(p-17)\neq 0.$$

(4) $p,\ p+a_j,\ a_j=p_j-p_0,\ p_0=23,\ p_j=29,\cdots,97$. We have

$$J_2(\omega)=34944\prod_{37\le p\le p_i}(p-17)\neq 0.$$

(5) $p,\ p+a_j,\ a_j=p_j-p_0,\ p_0=29,\ p_j=31,\cdots,101$. We have

$$J_2(\omega)=47040\prod_{37\le p\le p_i}(p-17)\neq 0.$$

(6) $p,\ p+a_j,\ a_j=p_j-p_0,\ p_0=31,\ p_j=37,\cdots,103$. We have

$$J_2(\omega)=26880\prod_{37\le p\le p_i}(p-17)\neq 0.$$

Since $J_2(\omega)\to\infty$ as $\omega\to\infty$, there exist infinitely many 17-tuples of primes. We have

$$\pi_{17}(N)\sim\frac{J_2(\omega)\omega^{16}}{\phi^{17}(\omega)}Li_{17}(N)$$

Theorem 5.1.17. 18-tuples:

(1) $p,\ p+a_j,\ a_j=p_j-p_0,\ p_0=13,\ p_j=17,\cdots,83$. We have

$$J_2(\omega)=9408\prod_{37\le p\le p_i}(p-18)\neq 0.$$

(2) $p,\ p+a_j,\ a_j=p_j-p_0,\ p_0=17,\ p_j=19,\cdots,89$. We have

$$J_2(\omega)=28224\prod_{37\le p\le p_i}(p-18)\neq 0.$$

(3) $p,\ p+a_j,\ a_j=p_j-p_0,\ p_0=19,\ p_j=23,\cdots,97$. We have

$$J_2(\omega)=436800\prod_{41\le p\le p_i}(p-18)\neq 0.$$

(4) $p,\ p+a_j,\ a_j=p_j-p_0,\ p_0=23,\ p_j=29,\cdots,101$. We have

$$J_2(\omega)=1820\prod_{31\le p\le p_i}(p-18)\neq 0.$$

(5) $p,\ p+a_j,\ a_j=p_j-p_0,\ p_0=29,\ p_j=31,\cdots,103$. We have

$$J_2(\omega)=203840\prod_{41\le p\le p_i}(p-18)\neq 0.$$

(6) $p,\ p+a_j,\ a_j=p_j-p_0,\ p_0=31,\ p_j=37,\cdots,107$. We have

$$J_2(\omega)=11648\prod_{37\le p\le p_i}(p-18)\neq 0.$$

Since $J_2(\omega) \to \infty$ as $\omega \to \infty$, there exist infinitely many 18-tuples of primes.

$$\pi_{18}(N) \sim \frac{J_2(\omega)\omega^{17}}{\phi^{18}(\omega)} Li_{18}(N)$$

Theorem 5.1.18. 22-tuple: $p,\ p+a_j,\ a_j = p_j - p_0,\ p_0 = 13,\ p_j = 17,\ \cdots,\ 103$. We have $J_2(p) \neq 0,\ p = 3, 5, 7, 11$ and $J_2(13) = J_2(17) = 0$, there is no 22-tuples of primes except one 22-tuple of primes: $13,\ \cdots,\ 103$.

Theorem 5.1.19. 21-tuple: $p,\ p+a_j,\ a_j = p_j - p_0,\ p_0 = 17,\ p_j = 19,\ \cdots,\ 103$. We have $J_2(p) \neq 0,\ p = 3, 5, 7, 11, 13$ and $J_2(17) = 0$, there is no 21-tuples of primes except one 21-tuples of prime: $17,\ \cdots,\ 103$.

Theorem 5.1.20. 19-tuple: $p,\ p+a_j,\ a_j = p_j - p_0,\ p_0 = 19,\ p_j = 23,\ \cdots,\ 101$. Since $J_2(p) \geq 1$, where $p = 3,\ \cdots,\ 19$, we have $J_2(\omega) \neq 0$. There exist infinitely many 19-tuples of primes. We have

$$\pi_{19}(N) \sim \frac{J_2(\omega)\omega^{18}}{\phi^{19}(\omega)} Li_{19}(N).$$

Theorem 5.1.21. 36-tuple: $p,\ p+a_j,\ a_j = p_j - p_0,\ p_0 = 19,\ p_j = 23,\ \cdots,\ 191$. We have $J_2(p) \neq 0,\ p = 3, 5, 7, 11, 13, 17$ and $J_2(19) = 0$, there is no 36-tuples of primes except one: $19,\ \cdots,\ 191$.

Theorem 5.1.22. 23-tuple: $p,\ p+a_j,\ a_j = p_j - p_0,\ p_0 = 23,\ p_j = 29,\ \cdots,\ 127$. Since $J_2(p) \geq 1$, where $p = 23,\ \cdots,\ 127$, we have $J_2(\omega) \neq 0$ There exist infinitely many 23-tuples of primes. We have

$$\pi_{23}(N) \sim \frac{J_2(\omega)\omega^{22}}{\phi^{23}(\omega)} Li_{23}(N).$$

Theorem 5.1.23. 43-tuple: $p,\ p+a_j,\ a_j = p_j - p_0,\ p_0 = 23,\ p_j = 29,\ \cdots,\ 233$. We have $J_2(p) \neq 0,\ p = 3,\ \cdots,\ 19$ and $J_2(23) = 0$, there are no 43-tuples of primes except one 43-tuples of primes: $23,\ \cdots,\ 233$.

Theorem 5.1.24. 29-tuple: $p,\ p+a_j,\ a_j = p_j - p_0,\ p_0 = 29,\ p_j = 31,\ \cdots,\ 163$. Since $J_2(p) \geq 1$, where $p = 3,\ \cdots,\ 29$, we have $J_2(\omega) \neq 0$ There exist infinitely many 29-tuples of primes. We have

$$\pi_{29}(N) \sim \frac{J_2(\omega)\omega^{28}}{\phi^{29}(\omega)} Li_{29}(N).$$

Theorem 5.1.25. 57-tuple: $p,\ p+a_j,\ a_j = p_j - p_0,\ p_0 = 29,\ p_j = 31,\ \cdots,\ 317$. We have $J_2(p) \neq 0,\ p = 3,\ \cdots,\ 23$ and $J_2(29) = 0$, there are no 57-tuples of primes except one 57-tuples of primes: $29,\ \cdots,\ 317$.

Theorem 5.1.26. 31-tuple: $p,\ p+a_j,\ a_j=p_j-p_0,\ p_0=31,\ p_j=37,\ \cdots,\ 179$. Since $J_2(p)\geq 1$, where $p=3,\ \cdots,\ 31$, we have $J_2(\omega)\neq 0$ There exist infinitely many 31-tuples of primes. We have

$$\pi_{31}(N)\sim\frac{J_2(\omega)\omega^{30}}{\phi^{31}(\omega)}Li_{31}(N).$$

Theorem 5.1.27. 75-tuple: $p,\ p+a_j,\ a_j=p_j-p_0,\ p_0=31,\ p_j=37,\ \cdots,\ 439$. We have $J_2(p)\neq 0,\ p=3,\ \cdots,\ 31$ and $J_2(37)=0$, there are no 75-tuples of primes except one 75-tuples of primes: $31,\ \cdots,\ 439$.

Theorem 5.1.28. 37-tuple: $p,\ p+a_j,\ a_j=p_j-p_0,\ p_0=37,\ p_j=41,\ \cdots,\ 223$. Since $J_2(p)\geq 1$, where $p=3,\ \cdots,\ 37$, we have $J_2(\omega)\neq 0$ There exist infinitely many 37-tuples of primes. We have

$$\pi_{37}(N)\sim\frac{J_2(\omega)\omega^{36}}{\phi^{37}(\omega)}Li_{37}(N).$$

Theorem 5.1.29. 74-tuple: $p,\ p+a_j,\ a_j=p_j-p_0,\ p_0=37,\ p_j=41,\ \cdots,\ 439$. We have $J_2(37)=0$, there are no 74-tuples of primes except one 74-tuples of primes: $37,\ \cdots,\ 439$.

Theorem 5.1.30. 79-tuple: $p,\ p+a_j,\ a_j=p_j-p_0,\ p_0=41,\ p_j=43,\ \cdots,\ 467$. We have $J_2(43)=0$, there are no 79-tuples of primes except one: $41,\ \cdots,\ 467$.

Theorem 5.1.31. 78-tuple: $p,\ p+a_j,\ a_j=p_j-p_0,\ p_0=43,\ p_j=47,\ \cdots,\ 467$. We have $J_2(43)=0$, there are no 78-tuples of primes except one: $43,\ \cdots,\ 467$.

Theorem 5.1.32. 106-tuple: $p,\ p+a_j,\ a_j=p_j-p_0,\ p_0=47,\ p_j=53,\ \cdots,\ 659$. We have $J_2(53)=0$, there are no 106-tuples of primes except one: $47,\ \cdots,\ 659$.

Theorem 5.1.33. 105-tuple: $p,\ p+a_j,\ a_j=p_j-p_0,\ p_0=53,\ p_j=59,\ \cdots,\ 659$. We have $J_2(53)=0$, there are no 105-tuples of primes except one: $53,\ \cdots,\ 659$.

Theorem 5.1.34. 111-tuple: $p,\ p+a_j,\ a_j=p_j-p_0,\ p_0=59,\ p_j=61,\ \cdots,\ 709$. We have $J_2(59)=0$, there are no 111-tuples of primes except one: $59,\ \cdots,\ 709$.

Theorem 5.1.35. 133-tuple: $p,\ p+a_j,\ a_j=p_j-p_0,\ p_0=61,\ p_j=67,\ \cdots,\ 863$. We have $J_2(71)=0$, there are no 153-tuples of primes except one: $61,\ \cdots,\ 863$.

Theorem 5.1.36. 152-tuple: $p,\ p+a_j,\ a_j=p_j-p_0,\ p_0=67,\ p_j=71,\ \cdots,\ 1013$. We have $J_2(67)=0$, there are no 152-tuples of primes except one: $67,\ \cdots,\ 1013$.

Theorem 5.1.37. 197-tuple: $p,\ p+a_j,\ a_j=p_j-p_0,\ p_0=71,\ p_j=73,\ \cdots,\ 1321$. We have $J_2(71)=0$, there are no 197-tuples of primes except one: $71,\ \cdots,\ 1321$.

Theorem 5.1.38. 195-tuple: $p,\ p{+}a_j,\ a_j = p_j{-}p_0,\ p_0 = 73,\ p_j = 79,\ \cdots,\ 1319.$ We have $J_2(79) = 0$, there are no 195-tuples of primes except one: $73,\ \cdots,\ 1319.$

Theorem 5.1.39. 194-tuple: $p,\ p{+}a_j,\ a_j = p_j{-}p_0,\ p_0 = 79,\ p_j = 83,\ \cdots,\ 1319.$ We have $J_2(79) = 0$, there are no 194-tuples of primes except one: $79,\ \cdots,\ 1319.$

Theorem 5.1.40. 239-tuple: $p,\ p{+}a_j,\ a_j = p_j{-}p_0,\ p_0 = 83,\ p_j = 89,\ \cdots,\ 1663.$ We have $J_2(83) = 0$, there are no 239-tuples of primes except one: $89,\ \cdots,\ 1663.$

Theorem 5.1.41. 216-tuple: $p,\ p{+}a_j,\ a_j = p_j{-}p_0,\ p_0 = 89,\ p_j = 97,\ \cdots,\ 1499.$ We have $J_2(97) = 0$, there are no 216-tuples of primes except one: $89,\ \cdots,\ 1499.$

Theorem 5.1.42. 215-tuple: $p,\ p{+}a_j,\ a_j = p_j{-}p_0,\ p_0 = 97,\ p_j = 103,\ \cdots,\ 1499.$ We have $J_2(97) = 0$, there are no 215-tuples of primes except one: $97,\ \cdots,\ 1499.$

Theorem 5.1.43. 273-tuple: $p,\ p + a_j,\ a_j = p_j - p_0,\ p_0 = 101,\ p_j = 103,\ \cdots,\ 1973.$ We have $J_2(103) = 0$, there are no 273-tuples of primes except one: $101,\ \cdots,\ 1973.$

Theorem 5.1.44. 272-tuple: $p,\ p + a_j,\ a_j = p_j - p_0,\ p_0 = 103,\ p_j = 107,\ \cdots,\ 1973.$ We have $J_2(103) = 0$, there are no 272-tuples of primes except one: $103,\ \cdots,\ 1973.$

Theorem 5.1.45. 10^4-tuple: $p,\ p + a_j,\ j = 1,\ \cdots,\ 9999,\ a_j = p_j - p_0,\ p_0 = 9973,\ p_j = 10007,\ \cdots.$

Since $J_2(p) \geq 1$, where $p = 3,\ \cdots,\ 9973$, we have $J_2(\omega) \neq 0$. There exist infinitely many 10^4-tuples of primes. We have

$$\pi_{10^4}(N) \sim \frac{J_2(\omega)\omega^{10^4-1}}{\phi^{10^4}(\omega)} Li_{10^4}(N).$$

Theorem 5.1.46. 10^5-tuple: $p,\ p + a_j,\ j = 1,\ \cdots,\ 99999,\ a_j = p_j - p_0,\ p_0 = 99971,\ p_j = 99989,\ \cdots.$

Since $J_2(p) \geq 1$, where $p = 3,\ \cdots,\ 99989$, we have $J_2(\omega) \neq 0$. There exist infinitely many 10^5-tuples of primes.

The best asymptotic formula for the number of 10^5-tuples of primes is

$$\pi_{10^5}(N) \sim \frac{J_2(\omega)\omega^{10^5-1}}{\phi^{10^5}(\omega)} Li_{10^5}(N).$$

Just as theorem 5.1.1(twin prime) is the simplest theorem, so theorem 5.1.46 is the simplest theorem. It is simple to prove and hard to calculate.

If $\pi_{10^5}(N) > 0$, then $d_N > 10^5$, where d_N is a long gap between primes. one should restudy the theory of prime numbers, because the now theory is finite and surmisable. There exist many great prime clusters in a table of primes.

Theorem 5.2. k-tuple: $p,\ p+b_j$, where $b_j = p_j + p_0$. $j = 1, 2, \cdots, k-1$, $p_0, p_1, \cdots, p_{k-1}$ are the consecutive primes in a table of primes.

If $J_2(\omega) = 0$, there exist finitely many k-tuples of primes. If $J_2(\omega) \neq 0$, there exist infinitely many k-tuples of primes,where k is a finite integer.

In order to be understandable,we need to have supplied many details below.

Theorem 5.2.1. 2-tuples:

(1) $p,\ p+b_1,\ b_1 = p_1 + p_0,\ p_0 = 3,\ p_1 = 5$. We have

$$J_2(\omega) = \prod_{3 \leq p \leq p_i} (p-2) \neq 0.$$

(2) $p,\ p+b_1,\ b_1 = p_1 + p_0,\ p_0 = 11,\ p_1 = 13$. We have

$$J_2(\omega) = 2 \prod_{5 \leq p \leq p_i} (p-2) \neq 0.$$

(3) $p,\ p+b_1,\ b_1 = p_1 + p_0,\ p_0 = 19,\ p_1 = 23$. We have

$$J_2(\omega) = 36 \prod_{11 \leq p \leq p_i} (p-2) \neq 0.$$

Since $J_2(\omega) \to \infty$ as $\omega \to \infty$, there exist infinitely many primes solations.We have

$$\pi_2(N) \sim \frac{J_2(\omega)\omega}{\phi^2(\omega)} Li_2(N).$$

Theorem 5.2.2. 3-tuples:

(1) $p,\ p+b_j,\ b_j = p_j + p_0,\ p_0 = 3,\ p_j = 5, 7$. We have $J_2(3) = 0$ There does not exist any 3-tuples of primes except one:$3, 11, 13$.

(2) $p,\ p+b_j,\ b_j = p_j + p_0,\ p_0 = 5,\ p_j = 7, 11$. We have

$$J_2(\omega) = \prod_{5 \leq p \leq p_i} (p-3) \neq 0.$$

(3) $p,\ p+b_j,\ b_j = p_j + p_0,\ p_0 = 11,\ p_j = 11, 13$. We have

$$J_2(\omega) = 3 \prod_{7 \leq p \leq p_i} (p-3) \neq 0.$$

(4) $p,\ p+b_j,\ b_j = p_j + p_0,\ p_0 = 11,\ p_j = 13, 17$. We have

$$J_2(\omega) = 10 \prod_{11 \leq p \leq p_i} (p-3) \neq 0.$$

Since $J_2(\omega) \neq 0$, there exist infinitely many 3-tuples of primes. We have

$$\pi_3(N) \sim \frac{J_2(\omega)\omega^2}{\phi^3(\omega)} Li_3(N)$$

Theorem 5.2.3. 4-tuples:
(1) $p,\ p+b_j,\ b_j = p_j + p_0,\ p_0 = 5,\ p_j = 7, 11, 13$. We have

$$J_2(\omega) = \prod_{5 \leq p \leq p_i} (p-4) \neq 0.$$

(2) $p,\ p+b_j,\ b_j = p_j + p_0,\ p_0 = 7,\ p_j = 11, 13, 17$. We have

$$J_2(\omega) = 2 \prod_{7 \leq p \leq p_i} (p-4) \neq 0.$$

(3) $p,\ p+b_j,\ b_j = p_j + p_0,\ p_0 = 11,\ p_j = 13, 17, 19$. We have

$$J_2(\omega) = 8 \prod_{11 \leq p \leq p_i} (p-4) \neq 0.$$

(4) $p,\ p+b_j,\ b_j = p_j + p_0,\ p_0 = 13,\ p_j = 17, 19, 23$. We have

$$J_2(\omega) = 2 \prod_{7 \leq p \leq p_i} (p-4) \neq 0.$$

Since $J_2(\omega) \to \infty$ as $\omega \to \infty$, there exist infinitely many 4-tuples of primes. We have

$$\pi_4(N) \sim \frac{J_2(\omega)\omega^3}{\phi^4(\omega)} Li_4(N).$$

Theorem 5.2.4. 5-tuples:
(1) $p,\ p+b_j,\ b_j = p_j + p_0,\ p_0 = 5,\ p_j = 7, 11, 13, 17$. We have

$$J_2(\omega) = 14 \prod_{13 \leq p \leq p_i} (p-5) \neq 0.$$

(2) $p,\ p+b_j,\ b_j = p_j + p_0,\ p_0 = 7,\ p_j = 11, 13, 17, 19$. We have

$$J_2(\omega) = 108 \prod_{17 \leq p \leq p_i} (p-5) \neq 0.$$

(3) $p,\ p+b_j,\ b_j = p_j + p_0,\ p_0 = 11,\ p_j = 13, 17, 19, 23$. We have

$$J_2(\omega) = 6 \prod_{11 \leq p \leq p_i} (p-5-\chi(p)) \neq 0.$$

where $\chi(17) = -1,\ \chi(p) = 0$ otherwise.

(4) $p,\ p + b_j,\ b_j = p_j + p_0,\ p_0 = 13,\ p_j = 17, 19, 23, 29$. We have

$$J_2(\omega) = 6 \prod_{11 \le p \le p_i} (p - 5) \neq 0.$$

Since $J_2(\omega) \to \infty$ as $\omega \to \infty$, there exist infinitely many 5-tuples of primes. We have

$$\pi_5(N) \sim \frac{J_2(\omega)\omega^4}{\phi^5(\omega)} Li_5(N).$$

Theorem 5.2.5. 6-tuples:

(1) $p,\ p + b_j,\ b_j = p_j + p_0,\ p_0 = 5,\ p_j = 7, 11, 13, 17, 19$. We have $J_2(5) = 0$, there does not exist any 6-tuples of primes.

(2) $p,\ p + b_j,\ b_j = p_j + p_0,\ p_0 = 7,\ p_j = 11, 13, 17, 19, 23$. We have

$$J_2(\omega) = 40 \prod_{17 \le p \le p_i} (p - 6) \neq 0.$$

(3) $p,\ p + b_j,\ b_j = p_j + p_0,\ p_0 = 11,\ p_j = 13, 17, 19, 23, 29$. We have

$$J_2(\omega) = 4 \prod_{11 \le p \le p_i} (p - 6 - \chi(p)) \neq 0.$$

where $\chi(17) = -1,\ \chi(p) = 0$ otherwise.

(4) $p,\ p + b_j,\ b_j = p_j + p_0,\ p_0 = 13,\ p_j = 17, 19, 23, 29, 31$. We have

$$J_2(\omega) = 18 \prod_{13 \le p \le p_i} (p - 6) \neq 0.$$

Since $J_2(\omega) \neq 0$, there exist infinitely many 6-tuples of primes. We have

$$\pi_6(N) \sim \frac{J_2(\omega)\omega^5}{\phi^6(\omega)} Li_6(N).$$

Theorem 5.2.6. 7-tuples:

(1) $p,\ p + b_j,\ b_j = p_j + p_0,\ p_0 = 7,\ p_j = 11, 13, 17, 19, 23, 29$. We have $J_2(7) = 0$, there does not exist any 7-tuple of primes.

(2) $p,\ p + b_j,\ b_j = p_j + p_0,\ p_0 = 11,\ p_j = 13, \cdots, 31$. We have

$$J_2(\omega) = 528 \prod_{19 \le p \le p_i} (p - 7) \neq 0.$$

(3) $p,\ p + b_j,\ b_j = p_j + p_0,\ p_0 = 13,\ p_j = 17, \cdots, 37$. We have

$$J_2(\omega) = 15 \prod_{13 \le p \le p_i} (p - 7) \neq 0.$$

(4) $p,\ p+b_j,\ b_j=p_j+p_0,\ p_0=17,\ p_j=19,\cdots,41$. We have

$$J_2(\omega)=5\prod_{13\le p\le p_i}(p-7-\chi(p))\ne 0,$$

where $\chi(23)=\chi(29)=-1,\ \chi(p)=0$ otherwise.

Since $J_2(\omega)\ne 0$, there exist infinitely many 7-tuples of primes. We have

$$\pi_7(N)\sim\frac{J_2(\omega)\omega^6}{\phi^7(\omega)}Li_7(N).$$

Theorem 5.2.7. 8-tuples:

(1) $p,\ p+b_j,\ b_j=p_j+p_0,\ p_0=11,\ p_j=13,\cdots,37$. We have

$$J_2(\omega)=300\prod_{19\le p\le p_i}(p-8)\ne 0.$$

(2) $p,\ p+b_j,\ b_j=p_j+p_0,\ p_0=13,\ p_j=17,\cdots,41$. We have

$$J_2(\omega)=10\prod_{13\le p\le p_i}(p-8)\ne 0.$$

(3) $p,\ p+b_j,\ b_j=p_j+p_0,\ p_0=17,\ p_j=19,\cdots,43$. We have

$$J_2(\omega)=4\prod_{13\le p\le p_i}(p-8-\chi(p))\ne 0.$$

where $\chi(23)=\chi(29)=-1,\ \chi(p)=0$ otherwise.

(4) $p,\ p+b_j,\ b_j=p_j+p_0,\ p_0=19,\ p_j=23,\cdots,47$. We have

$$J_2(\omega)=8\prod_{13\le p\le p_i}(p-8)\ne 0.$$

Since $J_2(\omega)\to\infty$ as $\omega\to\infty$, there exist infinitely many 8-tuples of primes. We have

$$\pi_8(N)\sim\frac{J_2(\omega)\omega^7}{\phi^8(\omega)}Li_8(N)$$

Theorem 5.2.8. 9-tuples:

(1) $p,\ p+b_j,\ b_j=p_j+p_0,\ p_0=11,\ p_j=13,\cdots,41$. We have

$$J_2(\omega)=270\prod_{19\le p\le p_i}(p-9)\ne 0.$$

(2) $p,\ p+b_j,\ b_j=p_j+p_0,\ p_0=13,\ p_j=17,\cdots,43$. We have

$$J_2(\omega)=40\prod_{17\le p\le p_i}(p-9)\neq 0.$$

(3) $p,\ p+b_j,\ b_j=p_j+p_0,\ p_0=17,\ p_j=19,\cdots,47$. We have

$$J_2(\omega)=3\prod_{13\le p\le p_i}(p-9-\chi(p))\neq 0.$$

where $\chi(23)=\chi(29)=-1,\ \chi(p)=0$ otherwise.

(4) $p,\ p+b_j,\ b_j=p_j+p_0,\ p_0=19,\ p_j=23,\cdots,53$. We have

$$J_2(\omega)=4\prod_{13\le p\le p_i}(p-9)\neq 0.$$

Since $J_2(\omega)\to\infty$ as $\omega\to\infty$, there exist infinitely many 9-tuples of primes. We have

$$\pi_9(N)\sim\frac{J_2(\omega)\omega^8}{\phi^9(\omega)}Li_9(N)$$

Theorem 5.2.9. 10-tuples:

(1) $p,\ p+b_j,\ b_j=p_j+p_0,\ p_0=11,\ p_j=13,\cdots,43$. We have

$$J_2(\omega)=160\prod_{19\le p\le p_i}(p-10)\neq 0.$$

(2) $p,\ p+b_j,\ b_j=p_j+p_0,\ p_0=13,\ p_j=17,\cdots,47$. We have

$$J_2(\omega)=24\prod_{17\le p\le p_i}(p-10)\neq 0.$$

(3) $p,\ p+b_j,\ b_j=p_j+p_0,\ p_0=17,\ p_j=19,\cdots,53$. We have

$$J_2(\omega)=181440\prod_{31\le p\le p_i}(p-10)\neq 0.$$

(4) $p,\ p+b_j,\ b_j=p_j+p_0,\ p_0=19,\ p_j=23,\cdots,59$. We have

$$J_2(\omega)=16\prod_{17\le p\le p_i}(p-10)\neq 0.$$

Since $J_2(\omega)\to\infty$ as $\omega\to\infty$, there exist infinitely many 10-tuples of primes.We have

$$\pi_{10}(N)\sim\frac{J_2(\omega)\omega^9}{\phi^{10}(\omega)}Li_{10}(N)$$

Theorem 5.2.10. 11-tuples:

(1) $p,\ p+b_j,\ b_j=p_j+p_0,\ p_0=11,\ p_j=13,\cdots,47$. We have

$$J_2(\omega)=116736\prod_{31\le p\le p_i}(p-11)\neq 0.$$

(2) $p,\ p+b_j,\ b_j=p_j+p_0,\ p_0=13,\ p_j=17,\cdots,53$. We have

$$J_2(\omega)=63\prod_{19\le p\le p_i}(p-11)\neq 0.$$

(3) $p,\ p+b_j,\ b_j=p_j+p_0,\ p_0=17,\ p_j=19,\cdots,59$. We have

$$J_2(\omega)=93366\prod_{31\le p\le p_i}(p-11)\neq 0.$$

(4) $p,\ p+b_j,\ b_j=p_j+p_0,\ p_0=19,\ p_j=23,\cdots,61$. We have

$$J_2(\omega)=486\prod_{23\le p\le p_i}(p-11)\neq 0.$$

Since $J_2(\omega)\to\infty$ as $\omega\to\infty$, there exist infinitely many 11-tuples of primes. We have

$$\pi_{11}(N)\sim\frac{J_2(\omega)\omega^{10}}{\phi^{11}(\omega)}Li_{11}(N)$$

Theorem 5.2.11. 12-tuples:

(1) $p,\ p+b_j,\ b_j=p_j+p_0,\ p_0=11,\ p_j=13,\cdots,53$. We have

$$J_2(\omega)=33264\prod_{31\le p\le p_i}(p-12)\neq 0.$$

(2) $p,\ p+b_j,\ b_j=p_j+p_0,\ p_0=13,\ p_j=17,\cdots,59$. We have

$$J_2(\omega)=36\prod_{19\le p\le p_i}(p-12)\neq 0.$$

(3)$p,\ p+b_j,\ b_j=p_j+p_0,\ p_0=17,\ p_j=19,\cdots,61$. We have

$$J_2(\omega)=46656\prod_{31\le p\le p_i}(p-12)\neq 0.$$

(4) $p,\ p+b_j,\ b_j=p_j+p_0,\ p_0=19,\ p_j=23,\cdots,67$. We have

$$J_2(\omega)=405\prod_{23\le p\le p_i}(p-12)\neq 0.$$

Since $J_2(\omega) \neq 0$, there exist infinitely many 12−tuples of primes. We have

$$\pi_{12}(N) \sim \frac{J_2(\omega)\omega^{11}}{\phi^{12}(\omega)} Li_{12}(N)$$

Theorem 5.2.12. 13-tuples:

(1) $p,\ p+b_j,\ b_j = p_j + p_0,\ p_0 = 11,\ p_j = 13, \cdots, 59$. We have

$$J_2(\omega) = 15708 \prod_{31 \leq p \leq p_i} (p-13) \neq 0.$$

(2) $p,\ p+b_j,\ b_j = p_j + p_0,\ p_0 = 13,\ p_j = 17, \cdots, 61$. We have

$$J_2(\omega) = 105 \prod_{23 \leq p \leq p_i} (p-13-\chi(p)) \neq 0.$$

where $\chi(37) = -1,\ \chi(p) = 0$ otherwise.

(3) $p,\ p+b_j,\ b_j = p_j + p_0,\ p_0 = 17,\ p_j = 19, \cdots, 67$. We have

$$J_2(\omega) = 33660 \prod_{31 \leq p \leq p_i} (p-13) \neq 0.$$

(4) $p,\ p+b_j,\ b_j = p_j + p_0,\ p_0 = 19,\ p_j = 23, \cdots, 71$. We have

$$J_2(\omega) = 160 \prod_{23 \leq p \leq p_i} (p-13) \neq 0.$$

Since $J_2(\omega) \neq 0$, there exist infinitely many 13-tuples of primes. We have

$$\pi_{13}(N) \sim \frac{J_2(\omega)\omega^{12}}{\phi^{13}(\omega)} Li_{13}(N).$$

Theorem 5.2.13. 14-tuples:

(1) $p,\ p+b_j,\ b_j = p_j + p_0,\ p_0 = 11,\ p_j = 13, \cdots, 61$. We have

$$J_2(\omega) = 5760 \prod_{31 \leq p \leq p_i} (p-14) \neq 0.$$

(2) $p,\ p+b_j,\ b_j = p_j + p_0,\ p_0 = 13,\ p_j = 17, \cdots, 67$. We have

$$J_2(\omega) = 84 \prod_{23 \leq p \leq p_i} (p-14-\chi(p)) \neq 0.$$

where $\chi(37) = -1,\ \chi(p) = 0$ otherwise.

(3) $p,\ p+b_j,\ b_j=p_j+p_0,\ p_0=17,\ p_j=19,\cdots,71$. We have

$$J_2(\omega)=25600\prod_{31\le p\le p_i}(p-14)\neq 0.$$

(4) $p,\ p+b_j,\ b_j=p_j+p_0,\ p_0=19,\ p_j=23,\cdots,73$. We have

$$J_2(\omega)=1120\prod_{29\le p\le p_i}(p-14)\neq 0.$$

Since $J_2(\omega)\neq 0$, there exist infinitely many 14-tuples of primes. We have

$$\pi_{14}(N)\sim\frac{J_2(\omega)\omega^{13}}{\phi^{14}(\omega)}Li_{14}(N).$$

Theorem 5.2.14. 15-tuples:

(1) $p,\ p+b_j,\ b_j=p_j+p_0,\ p_0=11,\ p_j=13,\cdots,67$. We have

$$J_2(\omega)=4050\prod_{31\le p\le p_i}(p-15)\neq 0.$$

(2) $p,\ p+b_j,\ b_j=p_j+p_0,\ p_0=13,\ p_j=17,\cdots,71$. We have

$$J_2(\omega)=48\prod_{23\le p\le p_i}(p-15-\chi(p))\neq 0.$$

where $\chi(37)=-1,\ \chi(p)=0$ otherwise.

(3) $p,\ p+b_j,\ b_j=p_j+p_0,\ p_0=17,\ p_j=23,\cdots,73$. We have

$$J_2(\omega)=15120\prod_{31\le p\le p_i}(p-15)\neq 0.$$

(4) $p,\ p+b_j,\ b_j=p_j+p_0,\ p_0=19,\ p_j=23,\cdots,79$. We have

$$J_2(\omega)=378\prod_{29\le p\le p_i}(p-15)\neq 0.$$

Since $J_2(\omega)\neq 0$, there exist infinitely many 15-tuples of primes. We have

$$\pi_{15}(N)\sim\frac{J_2(\omega)\omega^{14}}{\phi^{15}(\omega)}Li_{15}(N).$$

Theorem 5.2.15. 16-tuples:

(1) $p,\ p+b_j,\ b_j=p_j+p_0,\ p_0=11,\ p_j=13,\cdots,71$. We have $J_2(11)=0$, there does not exist any 16-tuple of primes.

(2) $p,\ p+b_j,\ b_j=p_j+p_0,\ p_0=13,\ p_j=17,\cdots,73$. We have

$$J_2(\omega)=30\prod_{23\le p\le p_i}(p-16-\chi(p))\ne 0.$$

where $\chi(37)=\chi(43)=-1,\ \chi(p)=0$ otherwise.
(3) $p,\ p+b_j,\ b_j=p_j+p_0,\ p_0=17,\ p_j=23,\cdots,79$. We have

$$J_2(\omega)=4704\prod_{31\le p\le p_i}(p-16)\ne 0.$$

(4) $p,\ p+b_j,\ b_j=p_j+p_0,\ p_0=19,\ p_j=23,\cdots,83$. We have

$$J_2(\omega)=324\prod_{29\le p\le p_i}(p-16)\ne 0.$$

Since $J_2(\omega)\ne 0$, there exist infinitely many 16-tuples of primes. We have

$$\pi_{16}(N)\sim\frac{J_2(\omega)\omega^{15}}{\phi^{16}(\omega)}Li_{16}(N).$$

Theorem 5.2.16. 17-tuples:

(1) $p,\ p+b_j,\ b_j=p_j+p_0,\ p_0=13,\ p_j=17,\cdots,79$. We have

$$J_2(\omega)=70\prod_{29\le p\le p_i}(p-17-\chi(p))\ne 0.$$

where $\chi(31)=\chi(37)=\chi(43)=-1,\ \chi(p)=0$ otherwise.
(2) $p,\ p+b_j,\ b_j=p_j+p_0,\ p_0=17,\ p_j=19,\cdots,83$. We have

$$J_2(\omega)=2496\prod_{31\le p\le p_i}(p-17)\ne 0.$$

(3) $p,\ p+b_j,\ b_j=p_j+p_0,\ p_0=19,\ p_j=23,\cdots,89$. We have

$$J_2(\omega)=2340\prod_{31\le p\le p_i}(p-17)\ne 0.$$

(4) $p,\ p+b_j,\ b_j=p_j+p_0,\ p_0=23,\ p_j=29,\cdots,97$. We have

$$J_2(\omega)=5616\prod_{31\le p\le p_i}(p-17)\ne 0.$$

Since $J_2(\omega)\ne 0$, there exist infinitely many 17-tuples of primes. We have

$$\pi_{17}(N)\sim\frac{J_2(\omega)\omega^{16}}{\phi^{17}(\omega)}Li_{17}(N).$$

Theorem 5.2.17. 18-tuples:

(1) $p,\ p+b_j,\ b_j=p_j+p_0,\ p_0=13,\ p_j=17,\cdots,83$. We have

$$J_2(\omega)=28\prod_{29\le p\le p_i}(p-18-\chi(p))\neq 0.$$

where $\chi(31)=\chi(37)=\chi(43)=-1,\ \chi(p)=0$ otherwise.

(2) $p,\ p+b_j,\ b_j=p_j+p_0,\ p_0=17,\ p_j=19,\cdots,89$. We have

$$J_2(\omega)=1040\prod_{31\le p\le p_i}(p-18)\neq 0.$$

(3) $p,\ p+b_j,\ b_j=p_j+p_0,\ p_0=19,\ p_j=23,\cdots,97$. We have

$$J_2(\omega)=2080\prod_{31\le p\le p_i}(p-18-\chi(p))\neq 0.$$

where $\chi(37)=-1,\ \chi(p)=0$ otherwise.

(4) $p,\ p+b_j,\ b_j=p_j+p_0,\ p_0=23,\ p_j=29,\cdots,101$. We have

$$J_2(\omega)=2730\prod_{31\le p\le p_i}(p-18)\neq 0.$$

Since $J_2(\omega)\neq 0$, there exist infinitely many 18-tuples of primes. We have

$$\pi_{18}(N)\sim\frac{J_2(\omega)\omega^{17}}{\phi^{18}(\omega)}Li_{18}(N).$$

Theorem 5.2.18. 22-tuple: $p,\ p+b_j,\ b_j=p_j+p_0,\ p_0=13,\ p_j=17,\cdots,103$. We have $J_2(13)=J_2(17)=0$, there does not exist any 22-tuple of primes.

Theorem 5.2.19. 21-tuple: $p,\ p+b_j,\ b_j=p_j+p_0,\ p_0=17,\ p_j=19,\cdots,103$. We have $J_2(17)=0$, there does not exist any 21-tuple of primes.

Theorem 5.2.20. 36-tuple: $p,\ p+b_j,\ b_j=p_j+p_0,\ p_0=19,\ p_j=23,\cdots,191$. We have $J_2(19)=0$, there does not exist any 36-tuple of primes.

Theorem 5.2.21. 43-tuple: $p,\ p+b_j,\ b_j=p_j+p_0,\ p_0=23,\ p_j=29,\cdots,233$. We have $J_2(23)=0$, there does not exist any 43-tuple of primes.

Theorem 5.2.22. 57-tuple: $p,\ p+b_j,\ b_j=p_j+p_0,\ p_0=29,\ p_j=31,\cdots,317$. We have $J_2(29)=0$.

Theorem 5.2.23. 75-tuple: $p,\ p+b_j,\ b_j=p_j+p_0,\ p_0=31,\ p_j=37,\cdots,439$. We have $J_2(37)=0$.

Theorem 5.2.24. 74-tuple: $p,\ p+b_j,\ b_j=p_j+p_0,\ p_0=37,\ p_j=41,\cdots,439$. We have $J_2(37)=0$.

Theorem 5.2.25. 79-tuple: $p,\ p+b_j,\ b_j=p_j+p_0,\ p_0=41,\ p_j=43,\cdots,467$. We have $J_2(43)=0$.

Theorem 5.2.26. 78-tuple: $p,\ p+b_j,\ b_j=p_j+p_0,\ p_0=43,\ p_j=47,\cdots,467$. We have $J_2(43)=0$.

Theorem 5.2.27. 106-tuple: $p,\ p+b_j,\ b_j=p_j+p_0,\ p_0=47,\ p_j=53,\cdots,659$. We have $J_2(53)=0$.

Theorem 5.2.28. 105-tuple: $p,\ p+b_j,\ b_j=p_j+p_0,\ p_0=53,\ p_j=59,\cdots,659$. We have $J_2(53)=0$.

Theorem 5.2.29. 111-tuple: $p,\ p+b_j,\ b_j=p_j+p_0,\ p_0=59,\ p_j=61,\cdots,709$. We have $J_2(59)=0$.

Theorem 5.2.30. 133-tuple: $p,\ p+b_j,\ b_j=p_j+p_0,\ p_0=61,\ p_j=67,\cdots,863$. We have $J_2(71)=0$.

Theorem 5.2.31. 152-tuple: $p,\ p+b_j,\ b_j=p_j+p_0,\ p_0=67,\ p_j=71,\cdots,1013$. We have $J_2(67)=0$.

Theorem 5.2.32. 197-tuple: $p,\ p+b_j,\ b_j=p_j+p_0,\ p_0=71,\ p_j=73,\cdots,1321$. We have $J_2(71)=0$.

Theorem 5.2.33. 195-tuple: $p,\ p+b_j,\ b_j=p_j+p_0,\ p_0=73,\ p_j=79,\cdots,1319$. We have $J_2(79)=0$.

Theorem 5.2.34. 194-tuple: $p,\ p+b_j,\ b_j=p_j+p_0,\ p_0=79,\ p_j=83,\cdots,1319$. We have $J_2(79)=0$.

Theorem 5.2.35. 239-tuple: $p,\ p+b_j,\ b_j=p_j+p_0,\ p_0=83,\ p_j=89,\cdots,1663$. We have $J_2(83)=0$.

Theorem 5.2.36. 216-tuple: $p,\ p+b_j,\ b_j=p_j+p_0,\ p_0=97,\ p_j=101,\cdots,1499$. We have $J_2(97)=0$.

Theorem 5.2.37. 215-tuple: $p,\ p+b_j,\ b_j=p_j+p_0,\ p_0=97,\ p_j=101,\cdots,1499$. We have $J_2(97)=0$.

Theorem 5.2.38. 273-tuple: $p,\ p+b_j,\ b_j=p_j+p_0,\ p_0=101,\ p_j=103,\cdots,1973$. We have $J_2(103)=0$.

Theorem 5.2.39. 272-tuple: $p,\ p+b_j,\ b_j=p_j+p_0,\ p_0=103,\ p_j=107,\cdots,1973$. We have $J_2(103)=0$.

Theorem 5.2.40. 10^4-tuple: $p,\ p+b_j, j=1,\cdots,9999,\ b_j=p_j+p_0,\ p_0=9973,\ p_j=10007,\ \cdots$.

Since $J_2(p) \geq 1$, where $p = 3, \cdots, 9973$, we have $J_2(\omega) \neq 0$. There exist infinitely many 10^4-tuples of primes. We have

$$\pi_{10^4}(N) \sim \frac{J_2(\omega)\omega^{10^4-1}}{\phi^{10^4}(\omega)} Li_{10^4}(N).$$

Theorem 5.2.41. 10^5-tuple: $p,\ p+b_j,\ b_j = p_j + p_0, j = 1, \cdots, 99999,\ \ p_0 = 99971,\ p_j = 99989,\ \cdots$.

Since $J_2(p) \geq 1$, where $p = 3,\ \cdots,\ 99989$, we have $J_2(\omega) \neq 0$. There exist infinitely many 10^5-tuples of primes. The best asymptotic formula for the number of 10^5-tuples of primes is

$$\pi_{10^5}(N) \sim \frac{J_2(\omega)\omega^{10^5-1}}{\phi^{10^5}(\omega)} Li_{10^5}(N).$$

Theorem 5.3. k-tuple: $p,\ a_jp+1,\ a_j = p_j - p_0,\ j = 1, \cdots, k-1, p_0, p_1, \cdots, p_{k-1}$ are the consecutive primes in a table of primes. If $J_2(\omega) = 0$, there does not exist any k-tuple of primes. If $J_2(\omega) \neq 0$, there exist infinitely many k-tuples of primes, where k is a finite number.

Theorem 5.3.1. 3-tuple: $p,\ a_jp+1,\ a_j = p_j - p_0,\ p_0 = 3,\ p_j = 5, 7$.

We have $J_2(3) = 0$, there does not exist any 3-tuple of primes except one: 3.17.13.

Theorem 5.3.2. 6-tuple: $p,\ a_jp+1,\ a_j = p_j - p_0,\ p_0 = 5,\ p_j = 7, 11, 13, 17, 19$. We have $J_2(5) = 0$, there does not exist any 6-tuple of primes.

Theorem 5.3.3. 7-tuple: $p,\ a_jp+1,\ a_j = p_j - p_0,\ p_0 = 5,\ p_j = 7,\ p_j = 11, \cdots, 29$. We have $J_2(7) = 0$, there does not exist any 7-tuple of primes.

Theorem 5.4. k-tuple: $p,\ b_jp+1,\ b_j = p_j+p_0,\ j = 1,\ \cdots,\ k-1,\ p_0,\ p_1,\ p_2,\ \cdots,\ p_{k-1}$ are the consecutive primes in a table of primes. If $J_2(\omega) = 0$, there does not exist any k-tuple of primes. If $J_2(\omega) \neq 0$, there exist infinitely many k-tuples of primes, where k is a finite number.

Theorem 5.4.1. 3-tuple: $p,\ b_jp+1,\ b_j = p_j + p_0,\ p_0 = 3,\ p_j = 5, 7$. We have $J_2(3) = 0$, there does not exist any 3-tuple of primes.

Theorem 5.4.2. 6-tuple: $p,\ b_jp+1,\ b_j = p_j + p_0,\ p_0 = 5,\ p_j = 7, 11, 13, 17, 19$. We have $J_2(5) = 0$, there does not exist any 6-tuple of primes.

Theorem 5.4.3. 7-tuple: $p,\ b_jp+1,\ b_j = p_j+p_0,\ p_0 = 7,\ p_j = 11, 13, 17, 19, 23, 29$. We have $J_2(7) = 0$, there does not exist any 7-tuple of primes.

Theorem 5.5. 6-tuple: $p,\ p+a_j,\ a_j=p_j-p_0,\ p_0=3,\ p_j=5,11,17,23,29.$ We have $J_2(5)=0$.

Theorem 5.6. 9-tuple: $p,\ p+a_j,\ a_j=p_j-p_0,\ p_0=3,\ p_j=7,\ 13,\ 19,\ 31,\ 37,\ 43,\ 61,\ 67$. We have $J_2(7)=0$.

Theorem 5.7. 14-tuple: $p,\ p+a_j,\ a_j=p_j-p_0,\ p_0=3,\ p_j=11,\ 17,\ 23,\ 29,\ 41,\ 53,\ 59,\ 71,\ 83,\ 101,\ 107,\ 113,\ 131$. We have $J_2(11)=0$.

Theorem 5.8. 9-tuple: $p,\ p+a_j,\ a_j=p_j-p_0,\ p_0=5,\ p_j=11,\ 13,\ 17,\ 23,\ 31,\ 37,\ 41,\ 43$. We have $J_2(7)=0$.

Theorem 5.9. 15-tuple: $p,\ p+a_j,\ a_j=p_j-p_0,\ p_0=5,\ p_j=11,\cdots,73$ without $19,29,59$. We have $J_2(11)=0$.

The DNA in all normal human cells is in 23 pairs of pieces, neatly packaged into 46 chromosomes. The 23 is the prime principle, and the 46 is the symmetric principle.

Theorem 5.10.1. 3-tuples: $p,p+a_j^m,a_j=p_j\pm p_0,p_0=3,p_j=5,7.$

(1) Let m be an even number. We have $J_2(\omega)\neq 0$,

$$\pi_3(N)\sim\frac{J_2(\omega)\omega^2}{\phi^3(\omega)}Li_3(N).$$

(2) Let m be an odd number. We have $J_2(3)=0$.

Theorem 5.10.2. 6-tuples: $p,p+a_j^m,a_j=p_j\pm p_0,p_0=5,p_j=7,11,13,17,19.$

(1) Let m be an even number. We have $J_(\omega)\neq 0$,

$$\pi_6(N)\sim\frac{J_2(\omega)\omega^5}{\phi^6(\omega)}Li_6(N).$$

(2) Let m be an odd number. We have $J_2(5)=0$.

Theorem 5.10.3. 7-tuples: $p,p+a_j^m,a_j=p_j\pm p_0,p_0=7,p_j=11,\cdots,29.$

(1) Let m be an even number. We have $J_2(\omega)\neq 0$,

$$\pi_7(N)\sim\frac{J_2(\omega)\omega^6}{\phi^7(\omega)}Li_7(N).$$

(2) Let m be an odd number, $3|m$. We have $J_2(\omega)\neq 0$,

$$\pi_7(N)\sim\frac{J_2(\omega)\omega^6}{\phi^7(\omega)}Li_7(N).$$

(3) Let m be an odd number, $3 \not| m$. We have $J_2(7) = 0$. There does not exist any 7-tuple of primes.

Theorem 5.10.4. 16-tuples: $p, p + a_j^m, a_j = p_j \pm p_0, p_0 = 11, p_j = 13, ..., 71$.

(1) Let m be an even number. We have $J_2(\omega) \neq 0$,

$$\pi_{16}(N) \sim \frac{J_2(\omega)\omega^{15}}{\phi^{16}(\omega)} Li_{16}(N).$$

(2) Let m be an odd number, $5|m$. We have $J_2(\omega) \neq 0$,

$$\pi_{16}(N) \sim \frac{J_2(\omega)\omega^{15}}{\phi^{16}(\omega)} Li_{16}(N).$$

(3) Let m be an odd number, $5 \not| m$. We have $J_2(11) = 0$. There does not exist any 16-tuple of primes.

Theorem 5.10.5. 22-tuples: $p, p + a_j^m, a_j = p_j \pm p_0, p_0 = 13, p_j = 17, ..., 103$.

(1) Let m be an even number. We have $J_2(\omega) \neq 0$,

$$\pi_{22}(N) \sim \frac{J_2(\omega)\omega^{21}}{\phi^{22}(\omega)} Li_{22}(N).$$

(2) Let m be an odd number. We have $J_2(17) = 0$. There does not exist any 22-tuple of primes.

Theorem 5.10.6. 21-tuples: $p, p + a_j^m, a_j = p_j \pm p_0, p_0 = 17, p_j = 19, ..., 103$.

(1) Let m be an even number. We have $J_2(\omega) \neq 0$,

$$\pi_{21}(N) \sim \frac{J_2(\omega)\omega^{20}}{\phi^{21}(\omega)} Li_{21}(N).$$

(2) Let m be an odd number. We have $J_2(17) = 0$. There does not exist any 21-tuple of primes.

Theorem 5.10.7. 36-tuples: $p, p + a_j^m, a_j = p_j \pm p_0, p_0 = 19, p_j = 23, ..., 191$.

(1) Let m be an even number. We have $J_2(\omega) \neq 0$,

$$\pi_{36}(N) \sim \frac{J_2(\omega)\omega^{35}}{\phi^{36}(\omega)} Li_{36}(N).$$

(2) Let m be an odd number, $3|m$. We have $J_2(\omega) \neq 0$,

$$\pi_{36}(N) \sim \frac{J_2(\omega)\omega^{35}}{\phi^{36}(\omega)} Li_{36}(N).$$

(3) Let m be an odd number, $3 \nmid m$. We have $J_2(19) = 0$. There does not exist any 36-tuple of primes.

Theorem 5.10.8. 43-tuples: $p, p + a_j^m, a_j = p_j \pm p_0, p_0 = 23, p_j = 29, ..., 233$.

(1) Let m be an even number. We have $J_2(\omega) \neq 0$,

$$\pi_{43}(N) \sim \frac{J_2(\omega)\omega^{42}}{\phi^{43}(\omega)} Li_{43}(N).$$

(2) Let m be an odd number, $11|m$. We have $J_2(\omega) \neq 0$,

$$\pi_{43}(N) \sim \frac{J_2(\omega)\omega^{42}}{\phi^{43}(\omega)} Li_{43}(N).$$

(3) Let m be an odd number, $11 \nmid m$. We have $J_2(23) = 0$. There does not exist any 43-tuple of primes.

Theorem 5.10.9. 57-tuples: $p, p + a_j^m, a_j = p_j \pm p_0, p_0 = 29, p_j = 31, ..., 317$.

(1) Let m be an even number. We have $J_2(\omega) \neq 0$,

$$\pi_{57}(N) \sim \frac{J_2(\omega)\omega^{56}}{\phi^{57}(\omega)} Li_{57}(N).$$

(2) Let m be an odd number, $7|m$. We have $J_2(\omega) \neq 0$,

$$\pi_{57}(N) \sim \frac{J_2(\omega)\omega^{56}}{\phi^{57}(\omega)} Li_{57}(N).$$

(3) Let m be an odd number, $7 \nmid m$. We have $J_2(29) = 0$. There does not exist any 57-tuple of primes.

Theorem 5.10.10. 75-tuples: $p, p + a_j^m, a_j = p_j \pm p_0, p_0 = 31, p_j = 37, ..., 439$.

(1) Let m be an even number. We have $J_2(\omega) \neq 0$,

$$\pi_{75}(N) \sim \frac{J_2(\omega)\omega^{74}}{\phi^{75}(\omega)} Li_{75}(N).$$

(2) Let m be an old number, $3|m$ or $5|m$. We have $J_2(\omega) \neq 0$,

$$\pi_{75}(N) \sim \frac{J_2(\omega)\omega^{74}}{\phi^{75}(\omega)} Li_{75}(N).$$

(3) Let m be an odd number, $3, 5 \nmid m$. We have $J_2(37) = 0$. There does not exist any 75-tuple of primes.

Theorem 5.10.11. 74-tuples: $p, p + a_j^m, a_j = p_j \pm p_0, p_0 = 37, p_j = 41, ..., 439$.

(1) Let m be an even number. We have $J_2(\omega) \neq 0$,

$$\pi_{74}(N) \sim \frac{J_2(\omega)\omega^{73}}{\phi^{74}(\omega)} Li_{74}(N).$$

(2) Let m be an odd number, $3|m$. We have $J_2(\omega) \neq 0$,

$$\pi_{74}(N) \sim \frac{J_2(\omega)\omega^{73}}{\phi^{74}(\omega)} Li_{74}(N).$$

(3) Let m be an odd number, $3 \not| m$. We have $J_2(37) = 0$. There does not exist any 74-tuple of primes.

Theorem 5.10.12. 79-tuples: $p, p + a_j^m, a_j = p_j \pm p_0, p_0 = 41, p_j = 43, ..., 467$.

(1) Let m be an even number. We have $J_2(\omega) \neq 0$,

$$\pi_{79}(N) \sim \frac{J_2(\omega)\omega^{78}}{\phi^{79}(\omega)} Li_{79}(N).$$

(2) Let m be an old number, $5|m$. We have $J_2(\omega) \neq 0$,

$$\pi_{79}(N) \sim \frac{J_2(\omega)\omega^{78}}{\phi^{79}(\omega)} Li_{79}(N).$$

(3) Let m be an odd number $5 \not| m$. We have $J_2(43) = 0$. There does not exist any 79-tuple of primes.

Theorem 5.10.13. 78-tuples: $p, p + a_j^m, a_j = p_j \pm p_0, p_0 = 43, p_j = 47, ..., 467$.

(1) Let m be an even number. We have $J_2(\omega) \neq 0$,

$$\pi_{78}(N) \sim \frac{J_2(\omega)\omega^{77}}{\phi^{78}(\omega)} Li_{78}(N).$$

(2) Let m be an odd number, $3|m$ or $7|m$. We have $J_2(\omega) \neq 0$,

$$\pi_{78}(N) \sim \frac{J_2(\omega)\omega^{77}}{\phi^{78}(\omega)} Li_{78}(N).$$

(3) Let m be an odd number, $3, 7 \not| m$. We have $J_2(47) = 0$. There does not exist any 78-tuple of primes.

Theorem 5.10.14. 106-tuples: $p, p + a_j^m, a_j = p_j \pm p_0, p_0 = 47, p_j = 53, ..., 659$.

(1) Let m be an even number. We have $J_2(\omega) \neq 0$,

$$\pi_{106}(N) \sim \frac{J_2(\omega)\omega^{105}}{\phi^{106}(\omega)} Li_{106}(N).$$

(2) Let m be an old number, $23|m$. We have $J_2(\omega) \neq 0$,

$$\pi_{106}(N) \sim \frac{J_2(\omega)\omega^{105}}{\phi^{106}(\omega)} Li_{106}(N).$$

(3) Let m be an odd number, $23 \not| m$. We have $J_2(53) = 0$. There does not exist any 106-tuple of primes.

Theorem 5.10.15. 105-tuples: $p, p + a_j^m, a_j = p_j \pm p_0, p_0 = 53, p_j = 59, ..., 659$.

(1) Let m be an even number. We have $J_2(\omega) \neq 0$,

$$\pi_{105}(N) \sim \frac{J_2(\omega)\omega^{104}}{\phi^{105}(\omega)} Li_{105}(N).$$

(2) Let m be an odd number, $3|m$ or $13|m$. We have $J_2(\omega) \neq 0$,

$$\pi_{105}(N) \sim \frac{J_2(\omega)\omega^{104}}{\phi^{105}(\omega)} Li_{105}(N).$$

(3) Let m be an odd number, $3, 13 \not| m$. We have $J_2(53) = 0$. There does not exist any 105-tuple of primes.

Theorem 5.10.16. 111-tuples: $p, p + a_j^m, a_j = p_j \pm p_0, p_0 = 59, p_j = 61, ..., 709$.

(1) Let m be an even number. We have $J_2(\omega) \neq 0$,

$$\pi_{111}(N) \sim \frac{J_2(\omega)\omega^{110}}{\phi^{111}(\omega)} Li_{111}(N).$$

(2) Let m be an odd number, $29|m$. We have $J_2(\omega) \neq 0$,

$$\pi_{111}(N) \sim \frac{J_2(\omega)\omega^{110}}{\phi^{111}(\omega)} Li_{111}(N).$$

(3) Let m be an odd number, $29 \not| m$. We have $J_2(59) = 0$. There does not exist any 111-tuple of primes.

Theorem 5.10.17. 133-tuples: $p, p + a_j^m, a_j = p_j \pm p_0, p_0 = 61, p_j = 67, ..., 863$.

(1) Let m be an even number. We have $J_2(\omega) \neq 0$,

$$\pi_{133}(N) \sim \frac{J_2(\omega)\omega^{132}}{\phi^{133}(\omega)} Li_{133}(N).$$

(2) Let m be an odd number, $3|m$ or $5|m$. We have $J_2(\omega) \neq 0$,

$$\pi_{133}(N) \sim \frac{J_2(\omega)\omega^{132}}{\phi^{133}(\omega)} Li_{133}(N).$$

(3) Let m be an odd number, $3,5 \not| m$. We have $J_2(71)=0$. There does not exist any 133-tuple of primes.

Theorem 5.10.18. 152-tuples: $p, p+a_j^m, a_j = p_j \pm p_0, p_0 = 67, p_j = 71, ..., 1013$.

(1) Let m be an even number. We have $J_2(\omega) \neq 0$,

$$\pi_{152}(N) \sim \frac{J_2(\omega)\omega^{151}}{\phi^{152}(\omega)} Li_{152}(N).$$

(2) Let m be an odd number, $3|m$ or $11|m$. We have $J_2(\omega) \neq 0$,

$$\pi_{152}(N) \sim \frac{J_2(\omega)\omega^{151}}{\phi^{152}(\omega)} Li_{152}(N).$$

(3) Let m be an odd number, $3,11 \not| m$. We have $J_2(67)=0$. There does not exist any 152-tuple of primes.

Theorem 5.10.19. 197-tuples: $p, p+a_j^m, a_j = p_j \pm p_0, p_0 = 71, p_j = 73, ..., 1321$.

(1) Let m be an even number. We have $J_2(\omega) \neq 0$,

$$\pi_{197}(N) \sim \frac{J_2(\omega)\omega^{196}}{\phi^{197}(\omega)} Li_{197}(N).$$

(2) Let m be an odd number, $5|m$ or $7|m$. We have $J_2(\omega) \neq 0$,

$$\pi_{197}(N) \sim \frac{J_2(\omega)\omega^{196}}{\phi^{197}(\omega)} Li_{197}(N).$$

(3) Let m be an odd number, $5,7 \not| m$. We have $J_2(71)=0$. There does not exist any 197-tuple of primes.

Theorem 5.10.20. 195-tuples: $p, p+a_j^m, a_j = p_j \pm p_0, p_0 = 73, p_j = 79, ..., 1319$.

(1) Let m be an even number. We have $J_2(\omega) \neq 0$,

$$\pi_{195}(N) \sim \frac{J_2(\omega)\omega^{194}}{\phi^{195}(\omega)} Li_{195}(N).$$

(2) Let m be an old number, $3|m$. We have $J_2(\omega) \neq 0$,

$$\pi_{195}(N) \sim \frac{J_2(\omega)\omega^{194}}{\phi^{195}(\omega)} Li_{195}(N).$$

(3) Let m be an odd number, $3 \not| m$. We have $J_2(79)=0$. There does not exist any 195-tuple of primes.

Theorem 5.10.21. 194-tuples: $p, p+a_j^m, a_j = p_j \pm p_0, p_0 = 79, p_j = 83, ..., 1319$.

(1) Let m be an even number. We have $J_2(\omega) \neq 0$,

$$\pi_{194}(N) \sim \frac{J_2(\omega)\omega^{193}}{\phi^{194}(\omega)} Li_{194}(N).$$

(2) Let m be an odd number, $3|m$ or $13|m$. We have $J_2(\omega) \neq 0$,

$$\pi_{194}(N) \sim \frac{J_2(\omega)\omega^{193}}{\phi^{194}(\omega)} Li_{194}(N).$$

(3) Let m be an odd number, $3, 13 \not| m$. We have $J_2(79) = 0$. There does not exist any 194-tuple of primes.

Theorem 5.10.22. 239-tuples: $p, p + a_j^m, a_j = p_j \pm p_0, p_0 = 83, p_j = 89, ..., 1663$.

(1) Let m be an even number. We have $J_2(\omega) \neq 0$,

$$\pi_{239}(N) \sim \frac{J_2(\omega)\omega^{238}}{\phi^{239}(\omega)} Li_{239}(N).$$

(2) Let m be an odd number, $41|m$. We have $J_2(\omega) \neq 0$,

$$\pi_{239}(N) \sim \frac{J_2(\omega)\omega^{238}}{\phi^{239}(\omega)} Li_{239}(N).$$

(3) Let m be an odd number, $41 \not| m$. We have $J_2(83) = 0$. There does not exist any 239-tuple of primes.

Theorem 5.10.23. 216-tuples: $p, p + a_j^m, a_j = p_j \pm p_0, p_0 = 89, p_j = 97, ..., 1499$.

(1) Let m be an even number. We have $J_2(\omega) \neq 0$,

$$\pi_{216}(N) \sim \frac{J_2(\omega)\omega^{215}}{\phi^{216}(\omega)} Li_{216}(N).$$

(2) Let m be an odd number, $11|m$. We have $J_2(\omega) \neq 0$,

$$\pi_{216}(N) \sim \frac{J_2(\omega)\omega^{215}}{\phi^{216}(\omega)} Li_{216}(N).$$

(3) Let m be an odd number, $11 \not| m$. We have $J_2(97) = 0$. There does not exist any 216-tuple of primes.

Theorem 5.10.24. 215-tuples: $p, p + a_j^m, a_j = p_j \pm p_0, p_0 = 97, p_j = 101, ..., 1499$.

(1) Let m be an even number. We have $J_2(\omega) \neq 0$,

$$\pi_{215}(N) \sim \frac{J_2(\omega)\omega^{214}}{\phi^{215}(\omega)} Li_{215}(N).$$

(2) Let m be an odd number, $3|m$. We have $J_2(\omega) \neq 0$,

$$\pi_{215}(N) \sim \frac{J_2(\omega)\omega^{214}}{\phi^{215}(\omega)} Li_{215}(N).$$

(3) Let m be an odd number, $3 \not| m$. We have $J_2(97) = 0$. There does not exist any 215-tuple of primes.

Theorem 5.10.25. 273-tuples: $p, p+a_j^m, a_j = p_j \pm p_0, p_0 = 101, p_j = 103, ..., 1973$.

(1) Let m be an even number. We have $J_2(\omega) \neq 0$,

$$\pi_{273}(N) \sim \frac{J_2(\omega)\omega^{272}}{\phi^{273}(\omega)} Li_{273}(N).$$

(2) Let m be an odd number, $5|m$. We have $J_2(\omega) \neq 0$,

$$\pi_{273}(N) \sim \frac{J_2(\omega)\omega^{272}}{\phi^{273}(\omega)} Li_{273}(N).$$

(3) Let m be an odd number, $5 \not| m$. We have $J_2(103) = 0$. There does not exist any 273-tuple of primes.

Theorem 5.10.26. 272-tuples: $p, p+a_j^m, a_j = p_j \pm p_0, p_0 = 103, p_j = 107, ..., 1973$.

(1) Let m be an even number. We have $J_2(\omega) \neq 0$,

$$\pi_{272}(N) \sim \frac{J_2(\omega)\omega^{271}}{\phi^{272}(\omega)} Li_{272}(N).$$

(2) Let m be an odd number, $3|m$ or $17|m$. We have $J_2(\omega) \neq 0$,

$$\pi_{272}(N) \sim \frac{J_2(\omega)\omega^{271}}{\phi^{272}(\omega)} Li_{272}(N).$$

(3) Let m be an odd number, $3 \not| m$ or $17 \not| m$. We have $J_2(103) = 0$. There does not exist any 272-tuple of primes.

Theorem 5.10.27. k-tuples: $p, p + a_j^m, j = 1, ..., k-1; a_j = p_j \pm p_0, p_j = p_1, p_2, ..., p_{k-1}$. Suppose that $m = 1$, we have $J_2(\omega) = 0$.

(1) Let m be an even number. We have $J_2(\omega) \neq 0$,

$$\pi_k(N) \sim \frac{J_2(\omega)\omega^{k-1}}{\phi^k(\omega)} Li_k(N).$$

(2) Let m be an odd number, $m|(p_0 - 1)$. If $J_2(p_0) \neq 0, J_2(p_1) = 0$ We have $J_2(\omega) \neq 0$,

$$\pi_k(N) \sim \frac{J_2(\omega)\omega^{k-1}}{\phi^k(\omega)} Li_k(N).$$

(3) Let m be an odd number, $m > 1, m \not| (p_0 - 1)$. We have $J_2(\omega) = 0$. There does not exist any k-tuple of primes.

Theorem 5.10.28. 16-tuples: $p, p + a_j^3, a_j = p_j \pm p_0, p_0 = 7, p_j = 11, ..., 67$. We have $J_2(\omega) \neq 0$,

$$\pi_{16}(N) \sim \frac{J_2(\omega)\omega^{15}}{\phi^{16}(\omega)} Li_{16}(N).$$

Since $J_2(\omega) \to \infty$ as $\omega \to \infty$, there exist infinitely many 16-tuples of primes.

Theorem 5.10.29. 17-tuples: $p, p + a_j^3, a_j = p_j \pm p_0, p_0 = 7, p_j = 11, ..., 71$. We have $J_2(11) = 0$. There does not exist any 17-tuple of primes.

Theorem 5.10.30. 21-tuples: $p, p + a_j^5, a_j = p_j \pm p_0, p_0 = 11, p_j = 13, ..., 101$. We have $J_2(\omega) \neq 0$,

$$\pi_{21}(N) \sim \frac{J_2(\omega)\omega^{20}}{\phi^{21}(\omega)} Li_{21}(N).$$

Since $J_2(\omega) \to \infty$ as $\omega \to \infty$, there exist infinitely many 21-tuples of primes.

Theorem 5.10.31. 22-tuples: $p, p + a_j^5, a_j = p_j \pm p_0, p_0 = 11, p_j = 13, ..., 103$. We have $J_2(13) = 0$. There does not exist any 22-tuple of primes.

One of the oldest problems in the theory of numbers, and indeed in the whole of mathematics, is the so-called prime twins. It asserts that "there exist infinitely many primes p such that $p + 2$ is a prime". It is the first theorem in a table of primes. Now we prove k-tuples of prime twins.

Theorem 5.11.1. 3-tuples of prime twins:

(1) $p, p + a_j, a_j = p_j - p_0, p_0 = 11, p_j = 13; 17, 19; 29, 31$. We have

$$J_2(\omega) = 2 \prod_{11 \le p \le p_i} (p - 6) \neq 0.$$

(2) $p, p + a_j, a_j = p_j - p_0, p_0 = 17, p_j = 19; 29, 31; 41, 43$. We have

$$J_2(\omega) = 144 \prod_{17 \le p \le p_i} (p - 6) \neq 0.$$

(3) $p, p + a_j, a_j = p_j - p_0, p_0 = 29, p_j = 31; 41, 43; 59, 61$. We have

$$J_2(\omega) = 6 \prod_{11 \le p \le p_i} (p - 6) \neq 0.$$

We have the best asymptotic formula of the number of 3-tuples of prime twins

$$\pi_6(N) \sim \frac{J_2(\omega)\omega^5}{\phi^6(\omega)} Li_6(N).$$

Since $J_2(\omega) \to \infty$ as $\omega \to \infty$, there exist infinitely many 3-tuples of prime twins.

Theorem 5.11.2. 4-tuples of prime twins:

(1) $p, p+a_j, a_j = p_j - p_0, p_0 = 11, p_j = 13; 17, 19; 29, 31; 41, 43$. We have

$$J_2(\omega) = 48 \prod_{17 \le p \le p_i} (p-8) \ne 0.$$

(2) $p, p+a_j, a_j = p_j - p_0, p_0 = 17, p_j = 19; 29, 31; 41, 43; 59, 61$. We have

$$J_2(\omega) = 90 \prod_{17 \le p \le p_i} (p-8) \ne 0.$$

(3) $p, p+a_j, a_j = p_j - p_0, p_0 = 29, p_j = 31; 41, 43; 59, 61; 71, 73$. We have

$$J_2(\omega) = 24 \prod_{13 \le p \le p_i} (p-8) \ne 0.$$

Since $J_2(\omega) \to \infty$ as $\omega \to \infty$, there exist infinitely many 4-tuples of prime twins. We have the best asymptotic formula of the number of 4-tuples of prime twins.

$$\pi_8(N) \sim \frac{J_2(\omega)\omega^7}{\phi^8(\omega)} Li_8(N).$$

Theorem 5.11.3. 5-tuples of prime twins:

(1) $p, p+a_j, a_j = p_j - p_0, p_0 = 11, p_j = 13; 17, 19; 29, 31; 41, 43; 59, 61$. We have

$$J_2(\omega) = 21168 \prod_{29 \le p \le p_i} (p-10) \ne 0.$$

(2) $p, p+a_j, a_j = p_j - p_0, p_0 = 17, p_j = 19; 29, 31; 41, 43; 59, 61; 71, 73$. We have

$$J_2(\omega) = 60 \prod_{17 \le p \le p_i} (p-10) \ne 0.$$

(3) $p, p+a_j, a_j = p_j - p_0, p_0 = 29, p_j = 31; 41, 43; 59, 61; 71, 73; 101, 103$. We have

$$J_2(\omega) = 19459440 \prod_{37 \le p \le p_i} (p-10) \ne 0.$$

Since $J_2(\omega) \to \infty$ as $\omega \to \infty$, there exist infinitely many 5-tuples of prime twins. We have the best asymptotic formula of the number of 5-tuples of prime twins.

$$\pi_{10}(N) \sim \frac{J_2(\omega)\omega^9}{\phi^{10}(\omega)} Li_{10}(N).$$

Theorem 5.11.4. 6-tuples of prime twins:

(1) $p, p+a_j, a_j = p_j - p_0, p_0 = 11, p_j = 13; 17, 19; 29, 31; 41, 43; 59, 61; 71, 73$. We have

$$J_2(\omega) = 90720 \prod_{31 \le p \le p_i} (p-12) \ne 0.$$

(2) $p, p+a_j, a_j = p_j - p_0, p_0 = 17, p_j = 19; 29, 31; 41, 43; 59, 61; 71, 73; 101, 103$. We have

$$J_2(\omega) = 3742200 \prod_{37 \le p \le p_i} (p-12) \ne 0.$$

(3) $p, p+a_j, a_j = p_j - p_0, p_0 = 29, p_j = 31; 41, 43; 59, 61; 71, 73; 101, 103; 107, 109$. We have

$$J_2(\omega) = 2449440 \prod_{37 \le p \le p_i} (p-12) \ne 0.$$

Since $J_2(\omega) \to \infty$ as $\omega \to \infty$, there exist infinitely many 6-tuples of prime twins. We have

$$\pi_{12}(N) \sim \frac{J_2(\omega)\omega^{11}}{\phi^{12}(\omega)} Li_{12}(N).$$

Theorem 5.11.5. 7-tuples of prime twins:

(1) $p, p+a_j, a_j = p_j - p_0, p_0 = 11, p_j = 13; 17, 19; 29, 31; 41, 43; 59, 61; 71, 73; 101, 103$. We have

$$J_2(\omega) = 10200 \prod_{31 \le p \le p_i} (p-14) \ne 0.$$

(2) $p, p+a_j, a_j = p_j - p_0, p_0 = 17, p_j = 19; 29, 31; 41, 43; 59, 61; 71, 73; 101, 103; 107, 109$. We have

$$J_2(\omega) = 1555200 \prod_{37 \le p \le p_i} (p-14) \ne 0.$$

(3) $p, p+a_j, a_j = p_j - p_0, p_0 = 29, p_j = 31; 41, 43; 59, 61; 71, 73; 101, 103; 107, 109; 137, 139$. We have

$$J_2(\omega) = 1399680 \prod_{37 \le p \le p_i} (p-14) \ne 0.$$

We have

$$\pi_{14}(N) \sim \frac{J_2(\omega)\omega^{13}}{\phi^{14}(\omega)} Li_{14}(N).$$

Theorem 5.11.6. 8-tuples of prime twins:

(1) $p, p+a_j, a_j = p_j - p_0, p_0 = 11, p_j = 13; 17, 19; 29, 31; 41, 43; 59, 61; 71, 73; 101, 103; 107, 109$. We have

$$J_2(\omega) = 2700 \prod_{31 \le p \le p_i} (p-16) \ne 0.$$

(2) $p, p+a_j, a_j = p_j - p_0, p_0 = 17, p_j = 19; 29, 31; 41, 43; 59, 61; 71, 73; 101,$ $103; 107, 109; 137, 139$. We have

$$J_2(\omega) = 967680 \prod_{37 \le p \le p_i} (p - 16) \ne 0.$$

(3) $p, p+a_j, a_j = p_j - p_0, p_0 = 29, p_j = 31; 41, 43; 59, 61; 71, 73; 101, 103; 107,$ $109; 137, 139; 149, 151$. We have

$$J_2(\omega) = 1036800 \prod_{37 \le p \le p_i} (p - 16) \ne 0.$$

We have

$$\pi_1 6(N) \sim \frac{J_2(\omega)\omega^{15}}{\phi^{16}(\omega)} Li_{16}(N).$$

Theorem 5.11.7. 13-tuples of prime twins: $p, p+a_j, a_j = p_j - p_0, p_0 = 11, p_j = 13; ...; 197, 199$. We have $J_2(11) = 0$. There does not exist any 13-tuple of prime twins except one 13-tuple of prime twins: 11,13; ... ; 197,199.

Theorem 5.11.8. 11-tuples of prime twins: $p, p+a_j, a_j = p_j - p_0, p_0 = 17, p_j = 19; ...; 191, 193$. We have $J_2(17) = 0$. There does not exist any 11-tuple of prime twins except one 11-tuple of prime twins: 17,19; ... ; 191,193.

Theorem 5.11.9. 27-tuples of prime twins: $p, p+a_j, a_j = p_j - p_0, p_0 = 29, p_j = 31; ...; 821, 823$. We have $J_2(29) = 0$. There does not exist any 27-tuple of prime twins except one 27-tuple of prime twins: 29,31; ... ; 821,823.

Remark. The Santilli's theory of a table of primes is the queen of number theory. Everything is stable number. The table of primes hints at the inner laws of nature, for example, the law of the genebank of biology. We conjectured that there would be internal relations between the sunspot active cycle of 11 years and the Homo-sapiens on earth.

6. Forbes Theorem

Forbes defines a prime k-tuplet as a sequence of consecutive primes $p_1, \cdots, p_k$, such that $p_k - p_1 = s(k)$ and $p_i - p_1 = b_i; i = 2, \cdots, k$. The difference between the first and the last is as small as possible [10]. Forbes conjectures that the prime k-tuplets occur infinitely often for each k and each admissible set called the Forbes Theorem.

Theorem 6.1. 3-tuplets: $s(3) = 6$.
(1) $p + b : b = 0, 2, 6$.
(2) $p + b : b = 0, 4, 6$.

They have the same arithmetic function

$$J_2(\omega) = \prod_{5\leq p\leq p_i}(p-3) \neq 0.$$

Since $J_2(\omega) \neq 0$, there exist infinitely many primes p such that $p+b$ is also a prime.

We have the best asymptotic formula

$$\pi_3(N,2) \approx \frac{J_2(\omega)\omega^2}{\phi^3(\omega)} Li_3(N).$$

Theorem 6.2. 4-tuplets:$s(4) = 8$.
(1) $p+b : b = 0, 2, 6, 8.$
We have the arithmetic function

$$J_2(\omega) = \prod_{5\leq p\leq p_i}(p-4) \neq 0.$$

Since $J_2(\omega) \neq 0$, there exist infinitely many primes p such that $p+b$ is also a prime.

We have the best asymptotic formula

$$\pi_4(N,2) \approx \frac{J_2(\omega)\omega^3}{\phi^4(\omega)} Li_4(N).$$

Theorem 6.3. 5-tuplets: $s(5) = 12$.
(1) $p+b : b = 0, 2, 6, 8, 12.$
(2) $p+b : b = 0, 4, 6, 10, 12.$
They have the same arithmetic function

$$J_2(\omega) = \prod_{7\leq p\leq p_i}(p-5) \neq 0.$$

Since $J_2(\omega) \neq 0$, there exist infinitely many primes p such that $p+b$ is also a prime.

We have the best asymptotic formula

$$\pi_5(N,2) \approx \frac{J_2(\omega)\omega^4}{\phi^5(\omega)} Li_5(N).$$

Theorem 6.4. 6-tuplets: $s(6) = 16$.
(1) $p+b : b = 0, 4, 6, 10, 12, 16.$
We have the arithmetic function

$$J_2(\omega) = \prod_{7\leq p\leq p_i}(p-6) \neq 0.$$

Since $J_2(\omega) \neq 0$, there exist infinitely many primes p such that $p+b$ is also a prime.

We have the best asymptotic formula

$$\pi_6(N,2) \approx \frac{J_2(\omega)\omega^5}{\phi^6(\omega)} Li_6(N).$$

Theorem 6.5. 7-tuplets: $s(7) = 20$.
(1) $p+b : b = 0, 2, 6, 8, 12, 18, 20.$
(2) $p+b : b = 0, 2, 8, 12, 14, 18, 20.$
They have the same arithmetic function

$$J_2(\omega) = \prod_{11 \le p \le p_i} (p-7) \neq 0.$$

Since $J_2(\omega) \neq 0$, there exist infinitely many primes p such that $p+b$ is also a prime.

We have the best asymptotic formula

$$\pi_7(N,2) \approx \frac{J_2(\omega)\omega^6}{\phi^7(\omega)} Li_7(N).$$

Theorem 6.6. 8-tuplets: $s(8) = 26$.
(1) $p+b : b = 0, 2, 6, 8, 12, 18, 20, 26.$
(2) $p+b : b = 0, 6, 8, 14, 18, 20, 24, 26.$
They have the arithmetic function

$$J_2(\omega) = 18 \prod_{17 \le p \le p_i} (p-8) \neq 0.$$

(3) $p+b : b = 0, 2, 6, 12, 14, 20, 24, 26.$
We have the arithmetic function

$$J_2(\omega) = 48 \prod_{17 \le p \le p_i} (p-8) \neq 0.$$

Since $J_2(\omega) \neq 0$, there exist infinitely many primes p such that $p+b$ is also a prime.

We have the best asymptotic formula

$$\pi_8(N,2) \approx \frac{J_2(\omega)\omega^7}{\phi^8(\omega)} Li_8(N).$$

Theorem 6.7. 9-tuplets: $s(9) = 30$.
(1) $p+b : b = 0, 4, 6, 10, 16, 18, 24, 28, 30.$
(2) $p+b : b = 0, 2, 6, 12, 14, 20, 24, 26, 30.$

They have the same arithmetic function

$$J_2(\omega) = 30 \prod_{17 \le p \le p_i} (p-9) \neq 0.$$

(3) $p+b : b = 0, 4, 10, 12, 18, 22, 24, 28, 30.$
(4) $p+b : b = 0, 2, 6, 8, 12, 18, 20, 26, 30.$
They have the same arithmetic function

$$J_2(\omega) = 15 \prod_{17 \le p \le p_i} (p-9) \neq 0.$$

Since $J_2(\omega) \neq 0$, there exist infinitely many primes p such that $p+b$ is also a prime.
We have the best asymptotic formula

$$\pi_9(N, 2) \approx \frac{J_2(\omega)\omega^8}{\phi^9(\omega)} Li_9(N).$$

Theorem 6.8. 10-tuplets: $s(10) = 32$.
(1) $p+b : b = 0, 2, 6, 8, 12, 18, 20, 26, 30, 32.$
(2) $p+b : b = 0, 2, 6, 12, 14, 20, 24, 26, 30, 32.$
They have the same arithmetic function

$$J_2(\omega) = 10 \prod_{17 \le p \le p_i} (p-10) \neq 0.$$

Since $J_2(\omega) \neq 0$, there exist infinitely many primes p such that $p+b$ is also a prime.
We have the best asymptotic formula

$$\pi_{10}(N, 2) \approx \frac{J_2(\omega)\omega^9}{\phi^{10}(\omega)} Li_{10}(N).$$

Theorem 6.9 11-tuplets: $s(11) = 36$.
(1) $p+b : b = 0, 4, 6, 10, 16, 18, 24, 28, 30, 34, 36.$
(2) $p+b : b = 0, 2, 6, 8, 12, 18, 20, 26, 30, 32, 36.$
They have the same arithmetic function

$$J_2(\omega) = 28 \prod_{19 \le p \le p_i} (p-11) \neq 0.$$

Since $J_2(\omega) \neq 0$, there exist infinitely many primes p such that $p+b$ is also a prime.
We have the best asymptotic formula

$$\pi_{11}(N, 2) \approx \frac{J_2(\omega)\omega^{10}}{\phi^{11}(\omega)} Li_{11}(N).$$

Theorem 6.10. 12-tuplets: $s(12) = 42$.
(1) $p + b : b = 0, 2, 6, 8, 12, 18, 20, 26, 30, 32, 36, 42$.
(2) $p + b : b = 0, 6, 10, 12, 16, 22, 24, 30, 34, 36, 40, 42$.
They have the same arithmetic function

$$J_2(\omega) = 21 \prod_{19 \le p \le p_i} (p - 12) \ne 0.$$

Since $J_2(\omega) \ne 0$, there exist infinitely many 12-tuplets of primes. We have the best asymptotic formula

$$\pi_{12}(N, 2) \approx \frac{J_2(\omega)\omega^{11}}{\phi^{12}(\omega)} Li_{12}(N).$$

Theorem 6.11. 13-tuplets: $s(13) = 48$
(1) $p + b : b = 0, 6, 12, 16, 18, 22, 28, 30, 36, 40, 42, 46, 48$.
(2) $p + b : b = 0, 2, 6, 8, 12, 18, 20, 26, 30, 32, 36, 42, 48$.
They have the same arithmetic function

$$J_2(\omega) = 792 \prod_{29 \le p \le p_i} (p - 13) \ne 0.$$

(3) $p + b : b = 0, 4, 6, 10, 16, 18, 24, 28, 30, 34, 36, 46, 48$.
(4) $p + b : b = 0, 2, 12, 14, 18, 20, 24, 30, 32, 38, 42, 44, 48$.
They have the same arithmetic function

$$J_2(\omega) = 770 \prod_{29 \le p \le p_i} (p - 13) \ne 0.$$

(5) $p + b : b = 0, 2, 8, 14, 18, 20, 24, 30, 32, 38, 42, 44, 48$.
(6) $p + b : b = 0, 4, 6, 10, 16, 18, 24, 28, 30, 34, 40, 46, 48$.
They have the same arithmetic function

$$J_2(\omega) = 924 \prod_{29 \le p \le p_i} (p - 13) \ne 0.$$

Since $J_2(\omega) \ne 0$, there exist infinitely many 13-tuplets of primes. We have the best asymptotic formula

$$\pi_{13}(N, 2) \approx \frac{J_2(\omega)\omega^{12}}{\phi^{13}(\omega)} Li_{13}(N).$$

Theorem 6.12. 14-tuplets: $s(14) = 50$
(1) $p + b : b = 0, 2, 6, 8, 12, 18, 20, 26, 30, 32, 36, 42, 48, 50$.
(2) $p + b : b = 0, 2, 8, 14, 18, 20, 24, 30, 32, 38, 42, 44, 48, 50$.

They have the same arithmetic function

$$J_2(\omega) = 300 \prod_{29 \le p \le p_i} (p - 14) \ne 0.$$

Since $J_2(\omega) \ne 0$, there exist infinitely many 14-tuplets of primes. We have the best asymptotic formula

$$\pi_{14}(N, 2) \approx \frac{J_2(\omega)\omega^{13}}{\phi^{14}(\omega)} Li_{14}(N).$$

Theorem 6.13. 15-tuplets: $s(15) = 56$.
(1) $p + b : b = 0, 2, 6, 8, 12, 18, 20, 26, 30, 32, 36, 42, 48, 50, 56.$
(2) $p + b : b = 0, 6, 8, 14, 20, 24, 26, 30, 36, 38, 44, 48, 50, 54, 56.$
They have the same arithmetic function

$$J_2(\omega) = 216 \prod_{29 \le p \le p_i} (p - 15) \ne 0.$$

(3) $p + b : b = 0, 2, 6, 12, 14, 20, 24, 26, 30, 36, 42, 44, 50, 54, 56.$
(4) $p + b : b = 0, 2, 6, 12, 14, 20, 26, 30, 32, 36, 42, 44, 50, 54, 56.$
They have the same arithmetic function

$$J_2(\omega) = 96 \prod_{29 \le p \le p_i} (p - 15) \ne 0.$$

Since $J_2(\omega) \ne 0$, there are infinitely many 15-tuplets of primes. We have the best asymptotic formula

$$\pi_{15}(N, 2) \approx \frac{J_2(\omega)\omega^{14}}{\phi^{15}(\omega)} Li_{15}(N).$$

Theorem 6.14. 16-tuplets: $s(16) = 60$.
(1) $p + b : b = 0, 2, 6, 12, 14, 20, 26, 30, 32, 36, 42, 44, 50, 54, 56, 60.$
(2) $p + b : b = 0, 4, 6, 10, 16, 18, 24, 28, 30, 34, 40, 46, 48, 54, 58, 60.$
Both have the same arithmetic function

$$J_2(\omega) = 2240 \prod_{31 \le p \le p_i} (p - 16) \ne 0.$$

Since $J_2(\omega) \ne 0$, there are infinitely many 16-tuplets of primes. We have the best asymptotic formula

$$\pi_{16}(N, 2) \approx \frac{J_2(\omega)\omega^{15}}{\phi^{16}(\omega)} Li_{16}(N).$$

Theorem 6.15. 17-tuplets: $s(17) = 66$.

(1) $p+b:b=0,4,10,12,16,22,24,30,36,40,42,46,52,54,60,64,66.$
(2) $p+b:b=0,2,6,12,14,20,24,26,30,36,42,44,50,54,56,62,66.$
Both have the same arithmetic function

$$J_2(\omega)=25200\prod_{37\le p\le p_i}(p-17)\neq 0.$$

(3) $p+b:b=0,4,6,10,16,18,24,28,30,34,40,46,48,54,58,60,66.$
(4) $p+b:b=0,6,8,12,18,20,26,32,36,38,42,48,50,56,60,62,66.$
Both have the same arithmetic function

$$J_2(\omega)=20475\prod_{37\le p\le p_i}(p-17)\neq 0.$$

Since $J_2(\omega)\neq 0$, there are infinitely many 17-tuplets of primes. We have the best asymptotic formula

$$\pi_{17}(N,2)\approx\frac{J_2(\omega)\omega^{16}}{\phi^{17}(\omega)}Li_{17}(N).$$

Theorem 6.16. 18-tuplets: $s(18)=70$.
(1) $p+b:b=0,4,10,12,16,22,24,30,36,40,42,46,52,54,60,64,66,70.$
(2) $p+b:b=0,4,6,10,16,18,24,28,30,34,40,46,48,54,58,60,66,70.$
Both have the same arithmetic function

$$J_2(\omega)=9408\prod_{37\le p\le p_i}(p-18)\neq 0.$$

Since $J_2(\omega)\neq 0$, there are infinitely many 18-tuplets of primes. We have the best asymptotic formula

$$\pi_{18}(N,2)\approx\frac{J_2(\omega)\omega^{17}}{\phi^{18}(\omega)}Li_{18}(N).$$

Theorem 6.17. 19-tuplets: $s(19)=76$.
(1) $p+b:b=0,6,10,16,18,22,28,30,36,42,46,48,52,58,60,66,70,72,76.$
(2) $p+b:b=0,4.6,10,16,18,24,28,30,34,40,46,48,54,58,60,66,70,76.$
Both have the same arithmetic function

$$J_2(\omega)=4368\prod_{37\le p\le p_i}(p-19)\neq 0.$$

(3) $p+b:b=0,4,6,10,16,22,24,30,34,36,42,46,52,60,64,66,70,72,76.$
(4) $p+b:b=0,4,6,10,12,16,24,30,34,40,42,46,52,54,60,66,70,72,76.$
Both have the same arithmetic function

$$J_2(\omega)=14784\prod_{37\le p\le p_i}(p-19)\neq 0.$$

Since $J_2(\omega) \neq 0$, there exist infinitely many 19-tuplets of primes. We have the best asymptotic formula

$$\pi_{19}(N,2) \approx \frac{J_2(\omega)\omega^{18}}{\phi^{19}(\omega)} Li_{19}(N).$$

Theorem 6.18. 20-tuplets: $s(20) = 80$.
(1) $p + b : b = 0, 2, 6, 8, 12, 20, 26, 30, 36, 38, 42, 48, 50, 56, 62, 66, 68, 72, 78, 80.$
(2) $p + b : b = 0, 2, 8, 12, 14, 18, 24, 30, 32, 38, 42, 44, 50, 54, 60, 68, 72, 74, 78, 80.$
Both have the same arithmetic function

$$J_2(\omega) = 123552 \prod_{41 \leq p \leq p_i} (p - 20) \neq 0.$$

Since $J_2(\omega) \neq 0$, there are infinitely many 20-tuplets of primes. We have the best asymptotic formula

$$\pi_{20}(N,2) \approx \frac{J_2(\omega)\omega^{19}}{\phi^{20}(\omega)} Li_{20}(N).$$

Theorem 6.19. 21-tuplets: $s(21) = 84$.
(1) $p + b_i : b_i = p_i - p_1, p_1 = 29, \cdots, p_{21} = 113$
We have the arithmetic function

$$J_2(\omega) = \prod_{17 \leq p \leq p_i} (p - \chi(p)) \neq 0.$$

where $\chi(p) = 15, 15, 17, 19, 19, 20, 20$, if $p = 17, 19, 23, 29, 31, 37, 41$;
$\chi(p) = 21$ otherwise.
Since $J_2(\omega) \neq 0$, there are infinitely many 21-tuplets of primes. We have the best asymptotic formula

$$\pi_{21}(N,2) \approx \frac{J_2(\omega)\omega^{20}}{\phi^{21}(\omega)} Li_{21}(N).$$

Theorem 6.20. 22-tuplets: $s(22) = 90$.
(1) $p + b_i : b_i = p_i - p_1, p_1 = 23, \cdots, p_{22} = 113$
We have the arithmetic function

$$J_2(\omega) = \prod_{17 \leq p \leq p_i} (p - \chi(p)) \neq 0.$$

where $\chi(p) = 15, 15, 17, 20, 20, 21, 21, 21$, if $p = 17, 19, 23, 29, 31, 37, 41, 43$;
$\chi(p) = 22$ otherwise.
(2) $p + b_i : b_i = p_i - p_1, p_i = 19, \cdots, p_{22} = 109$

We have the arithmetic function

$$J_2(\omega) = \prod_{19 \le p \le p_i} (p - \chi(p)) \neq 0.$$

where $\chi(p) = 16, 19, 20, 20, 20, 21$, if $p = 19, 23, 29, 31, 37, 41$;
$\chi(p) = 22$ otherwise.

Since $J_2(\omega) \neq 0$, there are infinitely many 22-tuplets of primes. We have the best asymptotic formula

$$\pi_{22}(N, 2) \approx \frac{J_2(\omega)\omega^{21}}{\phi^{22}(\omega)} Li_{22}(N).$$

Theorem 6.21. 23-tuplets: $s(23) = 94$.
(1) $p + b_i : b_i = p_i - p_1, p_1 = 19, \cdots, p_{23} = 113$
We have the arithmetic function

$$J_2(\omega) = \prod_{19 \le p \le p_i} (p - \chi(p)) \neq 0.$$

where $\chi(p) = 16, 19, 21, 21, 21, 22, 22, 22$, if $p = 19, 23, 29, 31, 37, 41, 43, 47$;
$\chi(p) = 23$ otherwise.

Since $J_2(\omega) \neq 0$, there are infinitely many 23-tuplets of primes. We have the best asymptotic formula

$$\pi_{23}(N, 2) \approx \frac{J_2(\omega)\omega^{22}}{\phi^{23}(\omega)} Li_{23}(N).$$

Theorem 6.22. 24-tuplets: $s(24) = 108$.
(1) $p + b_i : b_i = p_i - p_1, p_1 = 19, \cdots, p_{24} = 127$
We have the arithmetic function

$$J_2(\omega) = \prod_{19 \le p \le p_i} (p - \chi(p)) \neq 0.$$

where $\chi(p) = 16, 20, 22, 22, 20, 22, 22, 23$, if $p = 19, 23, 29, 31, 37, 41, 43, 47$;
$\chi(p) = 24$ otherwise.
(2) $p + b_i : b_i = p_i - p_1, p_1 = 23, \cdots, p_{24} = 131$
We have the arithmetic function

$$J_2(\omega) = \prod_{13 \le p \le p_i} (p - \chi(p)) \neq 0.$$

where $\chi(p) = 11, 16, 16, 19, 21, 22, 21, 24, 22, 25$, if $p = 13, 17, 19, 23, 29, 31, 37, 41, 43, 47$;
$\chi(p) = 24$ otherwise.
(3) $p + b_i : b_i = p_i - p_1, p_1 = 29, \cdots, p_{24} = 137$

We have the arithmetic function

$$J_2(\omega) = \prod_{17 \le p \le p_i} (p - \chi(p)) \neq 0.$$

where $\chi(p) = 15, 15, 20, 21, 22, 22, 23, 23, 22, 23,$ if $p = 17, 19, 23, 29, 31, 37, 41, 43, 47, 53;$
$\chi(p) = 24$ otherwise.
(4) $p + b_i : b_i = p_i - p_1, p_1 = 31, \cdots, p_{24} = 139$
We have the arithmetic function

$$J_2(\omega) = \prod_{17 \le p \le p_i} (p - \chi(p)) \neq 0.$$

where $\chi(p) = 15, 16, 19, 20, 22, 23, 23, 22, 22, 23,$ if $p = 17, 19, 23, 29, 31, 37, 41, 43, 47, 53;$
$\chi(p) = 24$ otherwise.

Since $J_2(\omega) \neq 0$, there are infinitely many 24-tuplets of primes. We have the best asymptotic formula

$$\pi_{24}(N, 2) \approx \frac{J_2(\omega)\omega^{23}}{\phi^{24}(\omega)} Li_{24}(N).$$

Note: $s(k)$ is an important parameter in a table of prime numbers. It can apply to the study of the stable structure of DNA and the stable genomic sequences.

Theorem 6.23. Euler showed that there must exist infinitely many primes because a certain expression formed with all the primes is infinitive. We have

$$\prod_{i=1}^{n} \left(1 - \frac{1}{p_i}\right)^{-1} = \frac{\omega}{\phi(\omega)} = \sum_{n=1}^{\infty} \frac{1}{n}.$$

The series $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent; being a series of positive terms, the order of summation is irrelevant, so the right-hand side is infinite, while the left-hand side is clearly finite. This is absurd. It is equivalent to $\phi(\omega) \to \infty$ as $\omega \to \infty$. Along this line one suggests the Riemann's hypothesis and other conjectures, but they are useless in the study of the distribution of prime numbers.

By Euler's idea we prove that there must exist infinitely many twin primes: $p_2 = p_1 + 2$. We have

$$\prod_{i=1}^{n} \left(1 - \frac{\chi(p_i)}{p_i}\right)^{-1} = \frac{\omega}{J_2(\o mega)} = \sum_{n=1}^{\infty} \frac{\chi(n)}{n} = 1 + \frac{1}{2} + \frac{2}{3} + \frac{1}{4} + \frac{2}{5} + \frac{2}{6} + \frac{2}{7} + \cdots,$$

where $\chi(2) = 1, \chi(p) = 2$ if $p_i > 2$.

The series $\sum_{n=1}^{\infty} (\frac{\chi(n)}{n})$ is divergent; being a series of positive terms, the order of summation is irrelevant, so the right-hand side is infinite, while the left-hand side is clearly finite. This is absurd. It is equivalent to $J_2(\omega) \to \infty$ as $\omega \to \infty$.

Now we prove that there must exist infinitely many 3-tuplets of primes: $p_2 = p_1 + 2, p_3 = p_1 + 6$. We have

$$\prod_{i=1}^{n}\left(1-\frac{\chi(p_i)}{p_i}\right)^{-1} = \frac{\omega}{J_2(\o mega)} = \sum_{n=1}^{\infty}\frac{\chi(n)}{n} = 1+\frac{1}{2}+\frac{2}{3}+\frac{1}{4}+\frac{3}{5}+\frac{2}{6}+\frac{3}{7}+\cdots,$$

where $\chi(2)=1, \chi(3)=2, \chi(p_i)=3$ if $p_i>3$.

The series $\sum_{n=1}^{\infty}(\frac{\chi(n)}{n})$ is divergent; being a series of positive terms, the order of summation is irrelevant, so the right-hand side is infinite, while the left-hand side is clearly finite. This is absurd. It is equivalent to $J_2(\omega)\to\infty$ as $\omega\to\infty$.

Theorem 6.24 We have the following equations

$$\log\prod_{k<p\le N}\left(1-\frac{k}{p}\right) \approx -\sum_{k<p\le N}\frac{k}{p}. \tag{6.1}$$

$$-\sum_{k<p\le N}\frac{1}{p} \approx -\log\log N. \tag{6.2}$$

From (6.1) and (6.2) we have

$$\prod_{k<p\le N}\left(1-\frac{k}{p}\right) \approx \frac{c_k}{\log^k N}. \tag{6.3}$$

From (6.3) we have

$$\frac{\phi(\omega)}{\omega} = \prod_{2\le p\le N}\left(1-\frac{1}{p}\right) \approx \frac{c_1}{\log N}. \tag{6.4}$$

From (6.4) we have the prime number theorem

$$\pi(N) \approx \frac{N}{\log N}. \tag{6.5}$$

$\pi(N)$ denotes the number of primes $\le N$.

We have the arithmetic function for prime k-tuplets

$$J_2(\omega) = B\prod_{k<p\le N}(p-k). \tag{6.6}$$

From (6.3) and (6.6) we have

$$\frac{J_2(\omega)}{\omega} \approx B_1\prod_{k<p\le N}\left(1-\frac{k}{p}\right) \approx \frac{B_2}{\log^k N}. \tag{6.7}$$

From (6.7) we have the prime k-tuplets theorem

$$\pi_k(N) \approx A_k \frac{N}{\log^k N}. \tag{6.8}$$

$\pi_k(N)$ denotes the number of the prime k-tuplets $\leq N$.

If $A_k = 0$, then there exist finite prime k-tuplets. If $A_k \neq 0$, then there must exist infinitely many prime k-tuplets.

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Chapter 3

FERMAT'S LAST THEOREM AND ITS APPLICATIONS

> A theory is the more impressive, the simpler are its premises, the more distinct are the things it connects, and the broader is its range of applicability.
>
> *Albert Einstein*

1. Introduction

As reviewed in preceding chapters, in the seminal works [1, 2] Santilli has introduced a generalization of real, complex and quaternionic numbers $a = n, c, q$, based on the lifting of the unit 1 of conventional numbers into an invertible and well behaved quantity with arbitrary functional dependence on local variables

$$1 \to \hat{I}(t, x, \dot{x}, \cdots), \tag{1.1}$$

while jointly lifting the product $ab = a \times b$ of conventional numbers into the form

$$ab \to a\hat{\times}b = a\hat{T}b, \tag{1.2}$$

under which $\hat{I} = (\hat{T})^{-1}$ is the correct left and right new unit

$$\hat{I}\hat{\times}a = (\hat{T})^{-1}\hat{T}a = a\hat{\times}\hat{I} = a\hat{T}\hat{I} = a, \tag{1.3}$$

for all possible $a = n, c, q$.

Since the new multiplication $a\hat{\times}b$ is associative, Santilli has then proved that the new numbers verify all axioms of a field. The above liftings were then called *isotopic* in the Greek sense of being axiom-preserving. Again, the prefix *iso* will be used whenever the original axioms are preserved.

Let $F = F(a, +, \times)$ be a conventional field with numbers $a = n, c, q$ equipped with the conventional sum $(a + b) \in F$, product $ab = a \times b \in F$, additive unit $0 \in F$ and their multiplicative unit $1 \in F$.

Definition 1.1. Santilli's isofields of the first kind $\hat{F} = \hat{F}(\hat{a}, +, \hat{\times})$ are rings with elements

$$\hat{a} = a\hat{I}, \tag{1.4}$$

called *isonumbers*, where $a = n, c, q \in F$, $\hat{I} = 1/\hat{T}$ is a well behaved, invertible and Hermitean quantity outside the original field, $\hat{I} = 1/\hat{T} \notin F$ and $a\hat{I}$ is the multiplication in F equipped with the isosum

$$\hat{a} + \hat{b} = (a + b)\hat{I}, \tag{1.5}$$

with conventional additive unit $0 = 0 \cdot \hat{I} = 0$, $\hat{a} + 0 = 0 + \hat{a} = \hat{a}$, $\forall \hat{a} \in \hat{F}$, and the isoproduct

$$\hat{a}\hat{\times}\hat{b} = \hat{a}\hat{T}\hat{b} = (ab)\hat{I}, \tag{1.6}$$

under which $\hat{I} = 1/\hat{T}$ is the correct left and right new unit ($\hat{I}\hat{\times}\hat{a} = \hat{a}\hat{\times}\hat{I} = \hat{a}, \ \forall \hat{a} \in \hat{F}$) called isounit.

Lemma 1.1. Isofields $\hat{F}(\hat{a}, +, \hat{\times})$ of Definition 1.1 verify all axioms of a field. The lifting $F \to \hat{F}$ is then an isotopy. All operations depending on the product must then be lifted in $\hat{F}$ for consistency.

Lemma 1.2. Santilli's isofields of the second kind $\hat{F} = \hat{F}(a, +, \hat{\times})$ (that is, when $a \in F$ is not lifted to $\hat{a} = a\hat{I}$) also verify all axioms of a field, if and only if the isounit is an element of the original field

$$\hat{I} = \frac{1}{\hat{T}} \in F. \tag{1.7}$$

In the latter case the isoproduct is defined by

$$a\hat{\times}b = a\hat{T}b \in \hat{F}. \tag{1.8}$$

In this Chapter by using the Santilli's isonumber theory, we obtain the Fermat-Santilli isotheorem. By using Euler's and Fermat's methods we prove it.

We establish the Fermat's mathematics. From the Fermat's mathematics, we suggest the chaotic mathematics.

2. Class I Fermat-Santilli Isotheorems

Let $b = n$, where n is an odd. From (8.5) we have the complex hyperbolic functions S_i of order n with $(n-1)$ variables [3]–[8],

$$S_i = \frac{1}{n}\left[e^A + 2\sum_{j=1}^{(n-1)/2}(-1)^{(i-1)j}e^{B_j}\cos\left(\theta_j + (-1)^j\frac{(i-1)j\pi}{n}\right)\right], \tag{2.1}$$

where $i = 1, \cdots, n$;

$$A = \sum_{\alpha=1}^{n-1} t_\alpha, \quad B_j = \sum_{\alpha=1}^{n-1} t_\alpha(-1)^{\alpha j}\cos\frac{\alpha j\pi}{n},$$

$$\theta_j = (-1)^{j+1}\sum_{\alpha=1}^{n-1} t_\alpha(-1)^{\alpha j}\sin\frac{\alpha j\pi}{n}, \quad A + 2\sum_{j=1}^{(n-1)/2} B_j = 0. \tag{2.2}$$

(2.1) may be written in the matrix form

$$\begin{bmatrix} S_1 \\ S_2 \\ S_3 \\ \vdots \\ S_n \end{bmatrix} = \frac{1}{n}\begin{bmatrix} 1 & 1 & 0 & \cdots & 0 \\ 1 & -\cos\frac{\pi}{n} & -\sin\frac{\pi}{n} & \cdots & -\sin\frac{(n-1)\pi}{2n} \\ 1 & \cos\frac{2\pi}{n} & \sin\frac{2\pi}{n} & \cdots & -\sin\frac{(n-1)\pi}{n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & \cos\frac{(n-1)\pi}{n} & \sin\frac{(n-1)\pi}{n} & \cdots & -\sin\frac{(n-1)^2\pi}{2n} \end{bmatrix}$$

$$\cdot\begin{bmatrix} e^A \\ 2e^{B_1}\cos\theta_1 \\ 2e^{B_1}\sin\theta_1 \\ \vdots \\ 2e^{B_{\frac{n-1}{2}}}\sin\theta_{\frac{n-1}{2}} \end{bmatrix}, \tag{2.3}$$

where $(n-1)/2$ is an even. From (2.3) we have its inverse transformation

$$\begin{bmatrix} e^A \\ e^{B_1}\cos\theta_1 \\ e^{B_1}\sin\theta_1 \\ \vdots \\ e^{(B_{\frac{n-1}{2}})}\sin(\theta_{\frac{n-1}{2}}) \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & -\cos\frac{\pi}{n} & \cos\frac{2\pi}{n} & \cdots & \cos\frac{(n-1)\pi}{n} \\ 0 & -\sin\frac{\pi}{n} & \sin\frac{2\pi}{n} & \cdots & \sin\frac{(n-1)\pi}{n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & -\sin\frac{(n-1)\pi}{2n} & -\sin\frac{(n-1)\pi}{n} & \cdots & -\sin\frac{(n-1)^2\pi}{2n} \end{bmatrix} \begin{bmatrix} S_1 \\ S_2 \\ S_3 \\ \vdots \\ S_n \end{bmatrix}. \tag{2.4}$$

From (2.4) we have

$$e^A = \sum_{i=1}^{n} S_i, \quad e^{Bj}\cos\theta_j = S_1 + \sum_{i=1}^{n-1} S_{1+i}(-1)^{ij}\cos\frac{ij\pi}{n},$$

$$e^{Bj}\sin\theta_j = (-1)^{j+1}\sum_{i=1}^{n-1} S_{1+i}(-1)^{ij}\sin\frac{ij\pi}{n}. \tag{2.5}$$

(2.2) and (2.5) have the same form.

Assume in (2.5), $S_1 \neq 0, S_2 \neq 0, S_i = 0$, where $i = 3, \cdots, n$. $S_i = 0$ are $(n-2)$ indeterminate equations with $(n-1)$ variables [9,10]. From (2.5) we have

$$e^A = S_1 + S_2, \quad e^{2Bj} = S_1^2 + S_2^2 + 2S_1S_2(-1)^j\cos\frac{j\pi}{n}. \tag{2.6}$$

Theorem 2.1. Let $n = \prod p_i$, where p_i ranges over all odd primes. From (2.6) we have

$$\exp\left(A + 2\sum_{j=1}^{\frac{n-1}{2}} B_j\right) = S_1^n + S_2^n, \tag{2.7}$$

$$\exp\left(A + 2\sum_{j=1}^{\frac{p_i-1}{2}} B_{\frac{n}{p_i}j}\right) = S_1^{p_i} + S_2^{p_i}. \tag{2.8}$$

From (2.2) we have

$$\exp\left(A + 2\sum_{j=1}^{\frac{n-1}{2}} B_j\right) = 1, \tag{2.9}$$

$$\exp\left(A + 2\sum_{j=1}^{\frac{p_i-1}{2}} B_{\frac{n}{p_i}j}\right) = \left[\exp\left(\sum_{\alpha=1}^{\frac{n}{p_i}-1} t_{p_i\alpha}\right)\right]^{p_i}. \tag{2.10}$$

From (2.7), (2.8), (2.9) and (2.10) we have the Fermat's equations

$$S_1^n + S_2^n = 1, \tag{2.11}$$

$$S_1^{p_i} + S_2^{p_i} = \left[\exp\left(\sum_{\alpha=1}^{\frac{n}{p_i}-1} t_{p_i\alpha}\right)\right]^{p_i}. \tag{2.12}$$

Note. If $S_i \neq 0, i = 1, 2, \cdots, n$, then (2.11) and (2.12) have infinitely many rational solutions.

Euler [11] proved (2.11), therefore (2.12) has no rational solutions for any prime $p_i > 3$. Using this method we proved Fermat's last theorem in 1991 [3]–[8] and the Fermat's last theorem was proved second in 1994 by A. Wiles [12].

Theorem 2.2. By lifting $F(a, +, \times) \to \hat{F}(\hat{a}, +, \hat{\times})$ we have the Fermat-Santilli equations of the first kind from (2.11) and (2.12)

$$\hat{S_1}^{\hat{n}} + \hat{S_2}^{\hat{n}} = \hat{I}, \tag{2.13}$$

$$\hat{S_1}^{\hat{p_i}} + \hat{S_2}^{\hat{p_i}} = \left[\hat{\exp}\left(\sum_{\alpha=1}^{\frac{n}{p_i}-1} \hat{t}_{p_i\alpha}\right)\right]^{\hat{p_i}}. \tag{2.14}$$

Euler [11] proved (2.13), therefore (2.14) has not isorational solutions for any isoprime $\hat{p_i} > \hat{3}$.

Theorem 2.3. By lifting $F(a, +, \times) \to \hat{F}(a, +, \hat{\times})$ we have the Fermat-Santilli equations of the second kind from (2.11) and (2.12)

$$S_1^{\hat{n}} + S_2^{\hat{n}} = 1, \tag{2.15}$$

$$S_1^{\hat{p_i}} + S_2^{\hat{p_i}} = \left[\exp\left(\sum_{\alpha=1}^{\frac{n}{p_i}-1} t_{p_i\alpha}\right)\right]^{\hat{p_i}}. \tag{2.16}$$

Euler [11] proved (2.15) for $\hat{p_i} = \hat{3}$, therefore (2.16) has no rational solutions for any odd isonumber $\hat{n} > \hat{3}$.

From (2.2), (2.5) and (8.6) we have the cyclic determinant

$$\exp(A + 2\sum_{j=1}^{(n-1)/2} B_j) = \begin{vmatrix} S_1 & S_n & \dots & S_2 \\ S_2 & S_1 & \dots & S_3 \\ S_3 & S_2 & \dots & S_4 \\ \dots & \dots & \dots & \dots \\ S_n & S_{n-1} & \dots & S_1 \end{vmatrix} = 1. \tag{2.17}$$

Theorem 2.4. Let $n = p$, where p is an odd prime. Suppose all $S_i = 0$ except $S_a \neq 0$ and $S_b \neq 0, a \neq b$. $S_i = 0$ are $(p - 2)$ Diophantine equations with $(p - 1)$ variables [9, 10].

From (2.17) we have $p(p-1)/2$ Fermat's equations:

$$S_a^p + S_b^p = 1. \tag{2.18}$$

It is sufficient to prove the theorem 2.4, but the proof has great difficulty [9, 10]. In the following theorems we consider n is the composite number.

Theorem 2.5. Let $n = 3p$, where p is an odd prime. Suppose all $S_i = 0$ except $S_a \neq 0$ and $S_b \neq 0$, $a \neq b$. $S_i = 0$ are $(3p-2)$ Diophantine equations with $(3p-1)$ variables [9]. From (2.17) we have $3p(3p-1)/2$ Fermat's equations. Every Fermat's equation has no rational solutions.

Proof. If $(a-b, 3p) = p$, from (2.17) we have $3p$ Fermat's equations:

$$(S_a^3 + S_b^3)^p = 1. \tag{2.19}$$

From (2.19) we have

$$S_a^3 + S_b^3 = 1. \tag{2.20}$$

If $(a-b, 3p) = 3$, from (2.17) we have $3p(p-1)/2$ Fermat's equations:

$$(S_a^p + S_b^P)^3 = 1. \tag{2.21}$$

From (2.21) we have

$$S_a^p + S_b^p = 1. \tag{2.22}$$

(2.20) and (2.22) are twin Fermat's equations. Both equations have no rational solutions.

If $(a-b, 3p) = 1$, from (2.17) we have $3p(p-1)$ Fermat's equations:

$$S_a^{3p} + S_b^{3p} = 1. \tag{2.23}$$

From (2.2) and (2.5) we have

$$\exp(A + 2B_p) = S_a^3 + S_b^3 = \left[\exp(\sum_{\alpha=1}^{p-1} t_{3\alpha})\right]^3, \tag{2.24}$$

$$\exp\left(A + 2\sum_{j=1}^{(p-1)/2} B_{3j}\right) = S_a^p + S_b^p = [\exp(t_p + t_{2p})]^p. \tag{2.25}$$

(2.24) and (2.25) are twin Fermat's equations. Both equations have no rational solutions.

Euler [11] proved (2.20), therefore (2.22)–(2.25) have no rational solutions for any prime $p > 3$. Using the theorem 2.5 we proved the theorem 2.4, because (2.18) and (2.22) are the same Fermat's equation.

Theorem 2.6. Let $n = 5p$, where p is an odd prime. Suppose all $S_i = 0$ except $S_a \neq 0$ and $S_b \neq 0$, $a \neq b$. $S_i = 0$ are $(5p-2)$ Diophantine equations with $(5p-1)$ variables. From (2.17) we have $5p(5p-1)/2$ Fermat's equations. Every Fermat's equation has no rational solutions.

Proof. If $(a-b, 5p) = p$, from (2.17) we have $10p$ Fermat's equations:

$$(S_a^5 + S_b^5)^p = 1. \tag{2.26}$$

From (2.26) we have

$$S_a^5 + S_b^5 = 1. \tag{2.27}$$

If $(a-b, 5p) = 5$, from (2.17) we have $5p(p-1)/2$ Fermat's equations:

$$(S_a^p + S_b^p)^5 = 1. \tag{2.28}$$

From (2.28) we have

$$S_a^p + S_b^p = 1. \tag{2.29}$$

(2.27) and (2.29) are twin Fermat's equations. Both equations have no rational solutions.

If $(a-b, 5p) = 1$, from (2.17) we have $10p(p-1)$ Fermat's equations:

$$S_a^{5p} + S_b^{5p} = 1. \tag{2.30}$$

From (2.2) and (2.5) we have

$$\exp(A + 2B_p + 2B_{2p}) = S_a^5 + S_b^5 = [\exp(\sum_{\alpha=1}^{p-1} t_{5\alpha})]^5. \tag{2.31}$$

$$\exp(A + 2\sum_{j=1}^{(p-1)/2} B_{5j}) = S_a^p + S_b^p = [\exp(\sum_{\alpha=1}^{4} t_{p\alpha})]^p. \tag{2.32}$$

(2.31) and (2.32) are twin Fermat's equations. Both equations have no rational solutions.

Dirichlet and Legendre [11] proved (2.27), therefore (2.29)–(2.32) have no rational solutions for any prime $p > 5$. If (2.29) has a rational solution, then it contradicts (2.30). Using the theorem 2.6, we prove the theorem 2.4, because (2.18) and (2.29) are the same Fermat's equation.

Theorem 2.7. Let $n = 7p$, where p is an odd prime. Suppose all $S_i = 0$ except $S_a \neq 0$ and $S_b \neq 0$, $a \neq b$. $S_i = 0$ are $(7p-2)$ Diophantine equations with $(7p-1)$ variables [9-10]. From (2.17) we have $7p(7p-1)/2$ Fermat's equations. Every Fermat's equation has no rational solutions.

Proof. If $(a-b, 7p) = p$, from (2.17) we have $21p$ Fermat's equations:

$$(S_a^7 + S_b^7)^p = 1. \tag{2.33}$$

From (2.33) we have

$$S_a^7 + S_b^7 = 1. \tag{2.34}$$

If $(a-b, 7p) = 7$, from (2.17) we have $7p(p-1)/2$ Fermat's equations:

$$(S_a^p + S_b^p)^7 = 1. \tag{2.35}$$

From (2.35) we have

$$S_a^p + S_b^p = 1. \tag{2.36}$$

(2.34) and (2.36) are twin Fermat's equations. Both equations have no rational solutions.

If $(a-b, 7p) = 1$, from (2.17) we have

$$S_a^{7p} + S_b^{7p} = 1. \tag{2.37}$$

From (2.2) and (2.5) we have

$$\exp(A + 2\sum_{j=1}^{3} B_{pj}) = S_a^7 + S_b^7 = [\exp(\sum_{\alpha=1}^{p-1} t_{7\alpha})]^7, \tag{2.38}$$

$$\exp(A + 2\sum_{j=1}^{(p-1)/2} B_{7j}) = S_a^p + S_b^p = [\exp(\sum_{\alpha=1}^{6} t_{p\alpha})]^p. \tag{2.39}$$

(2.38) and (2.39) are twin Fermat's equations. Both equations have no rational solutions.

Lame[11] proved (2.34), therefore (2.36) − (2.39) have no rational solutions for any prime $p > 7$. If (2.36)has a rational solution, then it contradicts (2.37). Using the theorem 2.7, we prove the theorem 2.4, because (2.18) and (2.36) are the same Fermat's equation.

Theorem 2.8. Let $n = \prod p$, where p ranges over all odd primes. Suppose all $S_i = 0$ except $S_a \neq 0$ and $S_b \neq 0$, $a \neq b$. $S_i = 0$ are $(n-2)$ Diophantine equations with $(n-1)$ variables [9-10]. From (2.17) we have $n(n-1)/2$ Fermat's equations. Every Fermat's equation has no rational solutions.

Proof. If $(a-b, n) = n/3$, from (2.17) we have n Fermat's equations:

$$(S_a^3 + S_b^3)^{n/3} = 1. \tag{2.40}$$

From (2.40) we have

$$S_a^3 + S_b^3 = 1. \tag{2.41}$$

If $(a-b, n) = n/5$, from (2.17) we have $2n$ Fermat's equations:

$$(S_a^5 + S_b^5)^{n/5} = 1. \tag{2.42}$$

From (2.42) we have

$$S_a^5 + S_b^5 = 1. \tag{2.43}$$

If $(a-b, n) = n/7$, from (2.17) we have $3n$ Fermat's equations:

$$(S_a^7 + S_b^7)^{n/7} = 1. \tag{2.44}$$

From (2.44) we have

$$S_a^7 + S_b^7 = 1. \tag{2.45}$$

If $(a-b, n) = n/f$, where f is a factor of n, from (2.17) we have $n\phi(f)/2$, where $\phi(f)$ is Euler function, Fermat's equations:

$$(S_a^f + S_b^f)^{n/f} = 1. \tag{2.46}$$

From (2.46) we have

$$S_a^f + S_b^f = 1. \tag{2.47}$$

If $(a-b, n) = n/p$, from (2.17) we have $n(p-1)/2$ Fermat's equations:

$$(S_a^p + S_b^p)^{n/p} = 1. \tag{2.48}$$

From (2.48) we have

$$S_a^p + S_b^p = 1. \tag{2.49}$$

If $(a-b, n) = 1$, from (2.17) we have $n\phi(n)/2$ Fermat's equations:

$$S_a^n + S_b^n = 1. \tag{2.50}$$

From (2.2) and (2.5) we have

$$\exp(A + 2\sum_{j=1}^{(p-1)/2} B_{[(n/p)j]}) = S_a^p + S_b^p = [\exp(\sum_{\alpha=1}^{n/p-1} t_{p\alpha})]^p. \tag{2.51}$$

Euler proved (2.41), therefore (2.43), (2.45), (2.47), (2.49), (2.50) and (2.51) have no rational solutions, that is all Fermat's equations in (2.17) have no rational solutions. Using the theorem 2.8, we prove the theorem 2.4, because (2.18) and (2.49) are the same Fermat's equation.

By lifting $F(a, +, \times) \to \hat{F}(\hat{a}, +, \hat{\times})$ and $F(a, +, \times) \to \hat{F}(a, +, \hat{\times})$ theorems 2.4 to 2.8 are the Fermat-Santilli isotheorems.

3. Class II Fermat-Santilli Isotheorems

Let $b = 2n$, where n is an odd. From (8.5) we have the complex hyperbolic functions S_i of order $2n$ with $(2n-1)$ variables.

$$S_i = \frac{1}{2n}\left[e^{A_1} + 2\sum_{j=1}^{\frac{n-1}{2}}(-1)^{(i-1)j}e^{B_j}\cos\left(\theta_j + (-1)^j\frac{(i-1)j\pi}{n}\right)\right]$$

$$+\frac{(-1)^{(i-1)}}{2n}\left[e^{A_2} + 2\sum_{j=1}^{\frac{n-1}{2}}(-1)^{(i-1)j}e^{D_j}\cos\left(\phi_j + (-1)^{j+1}\frac{(i-1)j\pi}{n}\right)\right], \tag{3.1}$$

where $i = 1, \cdots, 2n$;

$$A_1 = \sum_{\alpha=1}^{2n-1} t_\alpha, \quad B_j = \sum_{\alpha=1}^{2n-1} t_\alpha(-1)^{\alpha j}\cos\frac{\alpha j\pi}{n}, \quad \theta_j = (-1)^{(j+1)}\sum_{\alpha=1}^{2n-1} t_\alpha(-1)^{\alpha j}\sin\frac{\alpha j\pi}{n},$$

$$A_2 = \sum_{\alpha=1}^{2n-1} t_\alpha(-1)^{\alpha}, \quad D_j = \sum_{\alpha=1}^{2n-1} t_\alpha(-1)^{(j-1)\alpha}\cos\frac{\alpha j\pi}{n},$$

$$\phi_j = (-1)^j\sum_{\alpha=1}^{2n-1} t_\alpha(-1)^{(j-1)\alpha}\sin\frac{\alpha j\pi}{n}, \quad A_1 + A_2 + 2\sum_{j=1}^{\frac{n-1}{2}}(B_j + D_j) = 0. \tag{3.2}$$

In the same way as in (2.3) and (2.4), from (3.1) we have its inverse transformation

$$e^{A_1} = \sum_{i=1}^{2n} S_i, \quad e^{A_2} = \sum_{i=1}^{2n} S_i(-1)^{1+i},$$

$$e^{B_j}\cos\theta_j = S_1 + \sum_{i=1}^{2n-1} S_{1+i}(-1)^{ij}\cos\frac{ij\pi}{n},$$

$$e^{B_j}\sin\theta_j = (-1)^{(j+1)}\sum_{i=1}^{2n-1} S_{1+i}(-1)^{ij}\sin\frac{ij\pi}{n},$$

$$e^{D_j}\cos\phi_j = S_1 + \sum_{i=1}^{2n-1} S_{1+i}(-1)^{(j-1)i}\cos\frac{ij\pi}{n},$$

$$e^{D_j}\sin\phi_j = (-1)^j\sum_{i=1}^{2n-1} S_{1+i}(-1)^{(j-1)i}\sin\frac{ij\pi}{n}. \tag{3.3}$$

(3.2) and (3.3) have the same form.

Assume in (3.3) $S_1 \neq 0, S_2 \neq 0, S_i = 0$, where $i = 3, \cdots, 2n$. $S_i = 0$ are $(2n-2)$ indeterminate equations with $(2n-1)$ variables. From (3.3) we have

$$e^{A_1} = S_1 + S_2, \quad e^{A_2} = S_1 - S_2, \quad e^{2B_j} = S_1^2 + S_2^2 + 2S_1S_2(-1)^j \cos\frac{j\pi}{n},$$

$$e^{2D_j} = S_1^2 + S_2^2 + 2S_1S_2(-1)^{j+1} \cos\frac{j\pi}{n}. \tag{3.4}$$

Theorem 3.1. Let $n = \prod p_i$, where p_i ranges over all odd primes. From (3.4) we have

$$\exp\left[A_1 + A_2 + 2\sum_{j=1}^{\frac{n-1}{2}}(B_j + D_j)\right] = S_1^{2n} - S_2^{2n}, \tag{3.5}$$

$$\exp\left(A_1 + 2\sum_{j=1}^{\frac{p_i-1}{2}} B_{\frac{n}{p_i}j}\right) = S_1^{p_i} + S_2^{p_i}. \tag{3.6}$$

From (3.2) we have

$$\exp\left[A_1 + A_2 + 2\sum_{j=1}^{\frac{n-1}{2}}(B_j + D_j)\right] = 1, \tag{3.7}$$

$$\exp\left(A_1 + 2\sum_{j=1}^{\frac{p_i-1}{2}} B_{\frac{n}{p_i}j}\right) = \left[exp\left(\sum_{\alpha=1}^{\frac{2n}{p_i}-1} t_{p_i\alpha}\right)\right]^{p_i}. \tag{3.8}$$

From (3.5), (3.6), (3.7) and (3.8) we have the Fermat's equations

$$S_1^{2n} - S_2^{2n} = 1, \tag{3.9}$$

$$S_1^{p_i} + S_2^{p_i} = \left[\exp\left(\sum_{\alpha=1}^{\frac{2n}{p_i}-1} t_{p_i\alpha}\right)\right]^{p_i}. \tag{3.10}$$

Euler [11] proved (3.9), therefore (3.10) has no rational solutions for any prime $p_i > 3$.

Theorem 3.2. By lifting $F(a, +, \times) \to \hat{F}(\hat{a}, +, \hat{\times})$ we have the Fermat-Santilli equations of the first kind from (3.9) and (3.10)

$$\hat{S}_1^{\hat{2n}} - \hat{S}_2^{\hat{2n}} = \hat{I}, \tag{3.11}$$

$$\hat{S}_1^{\hat{p}_i} + \hat{S}_2^{\hat{p}_i} = \left[\hat{\exp} \left(\sum_{\alpha=1}^{\frac{2n}{p_i}-1} \hat{t}_{p_i\alpha} \right) \right]^{\hat{p}_i}. \tag{3.12}$$

Euler [11] proved (3.11), therefore (3.12) has no isorational solutions for any isoprime $\hat{p}_i > \hat{3}$.

Theorem 3.3. By lifting $F(a,+,\times) \to \hat{F}(\hat{a},+,\hat{\times})$ we have the Fermat-Santilli equations of the second kind from (3.9) and (3.10)

$$S_1^{\hat{2n}} - S_2^{\hat{2n}} = 1, \tag{3.13}$$

$$S_1^{\hat{p_i}} + S_2^{\hat{p_i}} = \left[\exp \left(\sum_{\alpha=1}^{\frac{2n}{p_i}-1} t_{p_i\alpha} \right) \right]^{\hat{p_i}}. \tag{3.14}$$

Euler [11] proved (3.14) for $\hat{p_i} = \hat{3}$, therefore (3.13) has no rational solutions for any isonumber $\hat{2n}$.

From (3.2), (3.3) and (8.6) we have the cyclic determinant

$$\exp[A_1 + A_2 + 2\sum_{j=1}^{(n-1)/2}(B_j + D_j)] = \begin{vmatrix} S_1 & S_{2n} & \dots & S_2 \\ S_2 & S_1 & \dots & S_3 \\ S_3 & S_2 & \dots & S_4 \\ \dots & \dots & \dots & \dots \\ S_{2n} & S_{2n-1} & \dots & S_1 \end{vmatrix} = 1 \tag{3.15}$$

Theorem 3.4. Let $n = p$, where p is an odd prime. Suppose all $S_i = 0$ except $S_a \neq 0$ and $S_b \neq 0$, $a \neq b$. $S_i = 0$ are $(2p-2)$ Diophantine equations with $(2p-1)$ variables. From (3.15) we have $2p(p-1)$ Fermat's equations. Every real Fermat's equation has no rational solutions.

If $(a-b, 2p) = 2$, from (3.15) we have $p(p-1)$ Fermat's equations:

$$(S_a^p + S_b^p)^2 = \pm 1. \tag{3.16}$$

The $-$ sign can be rejected. From (3.16) we have $p(p-1)/2$ real Fermat's equations:

$$S_a^p + S_b^p = 1. \tag{3.17}$$

If $(a-b, 2p) = 1$, from (3.15) we have $p(p-1)$ Fermat's equations:

$$S_a^{2p} - S_b^{2p} = 1. \tag{3.18}$$

It is sufficient to prove the theorem 3.4, but the proof has great difficulty. In the following theorems we consider n is the composite number.

Theorem 3.5. Let $n = 3p$, where p is an odd prime. Suppose all $S_i = 0$ except $S_a \neq 0$ and $S_b \neq 0$, $a \neq b$. $S_i = 0$ are $(6p-2)$ Diophantine equations with $(6p-1)$ variables. From (3.15) we have $6p(3p-1)$ Fermat's equations. Every real Fermat's equation has no rational solutions.

Proof. If $(a-b, 6p) = 2p$, from (3.15) we have $6p$ Fermat's equations:

$$(S_a^3 + S_b^3)^{2p} = \pm 1. \tag{3.19}$$

The $-$ sign can be rejected. From (3.19) we have $3p$ real Fermat's equations:

$$S_a^3 + S_b^3 = 1. \tag{3.20}$$

If $(a-b, 6p) = 6$, from (3.15) we have $3p(p-1)$ Fermat's equations:

$$(S_a^p + S_b^p)^6 = \pm 1. \tag{3.21}$$

The $-$ sign can be rejected. From (3.21) we have $3p(p-1)/2$ real Fermat's equations:

$$S_a^p + S_b^p = 1. \tag{3.22}$$

(3.20) and (3.22) are twin Fermat's equations. Both equations have no rational solutions.

If $(a-b, 6p) = 3$, from (3.15) we have $3p(p-1)$ Fermat's equations:

$$(S_a^{2p} - S_b^{2p})^3 = \pm 1. \tag{3.23}$$

From (3.23) we have $3p(p-1)/2$ Fermat's equations:

$$S_a^{2p} - S_b^{2p} = 1. \tag{3.24}$$

If $(a-b, 6p) = 2$, from (3.15) we have $6p(p-1)$ Fermat's equations:

$$(S_a^{3p} + S_b^{3p})^2 = \pm 1. \tag{3.25}$$

From (3.25) we have $3p(p-1)$ real Fermat's equations:

$$S_a^{3p} + S_b^{3p} = 1. \tag{3.26}$$

If $(a-b, 6p) = 1$, from (3.15) we have $6p(p-1)$ Fermat's equations:

$$S_a^{6p} - S_b^{6p} = 1. \tag{3.27}$$

From (3.2) and (3.3) we have

$$\exp(A_1 + 2B_p) = S_a^3 + S_b^3 = [\exp(\sum_{\alpha=1}^{2p-1} t_{3\alpha})]^3. \tag{3.28}$$

$$\exp(A_1 + 2\sum_{j=1}^{(p-1)/2} B_{3j}) = S_a^p + S_b^p = [\exp(\sum_{\alpha=1}^{5} t_{p\alpha})]^p. \tag{3.29}$$

Eqs. (3.28) and (3.29) are twin Fermat's equations. Both equations have no rational solutions.

Euler proved (3.20), therefore (3.22), (3.24) and (3.26) − (3.29) have no rational solutions. Using the theorem 3.5 we prove the theorem 3.4, because (3.17) and (3.22) are the same Fermat's equation.

Theorem 3.6. Let $n = 5p$, where p is an odd prime. Suppose all $S_i = 0$ except $S_a \neq 0$ and $S_b \neq 0$, $a \neq b$. $S_i = 0$ are $(10p-2)$ Diophantine equations with $(10p-1)$ variables. From (3.15) we have $10p(5p-1)$ Fermat's equations. Every real Fermat's equation has no rational solutions.

Proof. If $(a-b, 10p) = 2p$, from (3.15) we have $20p$ Fermat's equations:

$$(S_a^5 + S_b^5)^{2p} = \pm 1. \tag{3.30}$$

From (3.30) we have $10p$ real Fermat's equations:

$$S_a^5 + S_b^5 = 1. \tag{3.31}$$

If $(a-b, 10p) = 10$, from (3.15) we have $5p(p-1)$ Fermat's equations:

$$(S_a^p + S_b^p)^{10} = \pm 1. \tag{3.32}$$

From (3.32) we have $5p(p-1)/2$ real Fermat's equations:

$$S_a^p + S_b^p = 1. \tag{3.33}$$

(3.31) and (3.33) are twin Fermat's equations. Both equations have no rational solutions.

If $(a-b, 10p) = 5$, from (3.15) we have $5p(p-1)$ Fermat's equations:

$$(S_a^{2p} - S_b^{2p})^5 = \pm 1. \tag{3.34}$$

From (3.34) we have $5p(p-1)/2$ Fermat's equations:

$$S_a^{2p} - S_b^{2p} = 1. \tag{3.35}$$

If $(a-b, 10p) = 2$, from (3.15) we have $20p(p-1)$ Fermat's equations:

$$(S_a^{5p} + S_b^{5p})^2 = \pm 1. \tag{3.36}$$

From (3.36) we have $10p(p-1)$ real Fermat's equations:

$$S_a^{5p} + S_b^{5p} = 1. \tag{3.37}$$

If $(a-b,10p)=1$, from (3.15) we have $20p(p-1)$ Fermat's equations:

$$S_a^{10p} - S_b^{10p} = 1. \tag{3.38}$$

From (3.2) and (3.3) we have

$$\exp(A_1 + B_p + B_2 p) = S_a^5 + S_b^5 = [\exp(\sum_{\alpha=1}^{2p-1} t_{5\alpha})]^5. \tag{3.39}$$

$$\exp(A_1 + 2\sum_{j=1}^{(p-1)/2} B_{5j}) = S_a^p + S_b^p = [\exp(\sum_{\alpha=1}^{9} t_{p\alpha})]^p. \tag{3.40}$$

(3.39) and (3.40) are twin Fermat's equations. Both equations have no rational solutions.

Dirichlet and Legendre proved (3.31), therefore (3.33), (3.35) and (3.37) – (3.40) have no rational solutions. Using the theorem 3.6, we prove the theorem 3.4, because (3.17) and (3.33) are the same Fermat's equation.

Theorem 3.7. Let $n = 7p$, where p is an odd prime. Suppose all $S_i = 0$ except $S_a \neq 0$ and $S_b \neq 0$, $a \neq b$. $S_i = 0$ are $(14p-2)$ Diophantine equations with $(14p-1)$ variables. From (3.15) we have $14p(7p-1)$ Fermat's equations. Every real Fermat's equation has no rational solutions.

Proof. If $(a-b,14p)=2p$, from (3.15) we have $42p$ Fermat's equations:

$$(S_a^7 + S_b^7)^{2p} = \pm 1. \tag{3.41}$$

From (3.41) we have $21p$ real Fermat's equations:

$$S_a^7 + S_b^7 = 1. \tag{3.42}$$

If $(a-b,14p)=14$, from (3.15) we have $7p(p-1)$ Fermat's equations:

$$(S_a^p + S_b^p)^{14} = \pm 1. \tag{3.43}$$

From (3.43) we have $7p(p-1)/2$ real Fermat's equations:

$$S_a^p + S_b^p = 1. \tag{3.44}$$

(3.42) and (3.44) are twin Fermat's equations. Both equations have no rational solutions.

If $(a-b,14p)=7$, from (3.15) we have $7p(p-1)$ Fermat's equations:

$$(S_a^{2p} - S_b^{2p})^7 = \pm 1. \tag{3.45}$$

From (3.45) we have $7p(p-1)/2$ Fermat's equations:

$$S_a^{2p} - S_b^{2p} = 1. \tag{3.46}$$

If $(a-b, 14p) = 2$, from (3.15) we have $42p(p-1)$ Fermat's equations:

$$(S_a^{7p} + S_b^{7p})^2 = \pm 1. \tag{3.47}$$

From (3.47) we have $21p(p-1)$ real Fermat's equations:

$$S_a^{7p} + S_b^{7p} = 1. \tag{3.48}$$

If $(a-b, 14p) = 1$, from (3.15) we have $42p(p-1)$ Fermat's equations:

$$S_a^{14p} - S_b^{14p} = 1. \tag{3.49}$$

From (3.2) and (3.3) we have

$$\exp(A_1 + 2B_p + 2B_{2p} + 2B_{3p}) = S_a^7 + S_b^7 = \left[\exp\left(\sum_{\alpha=1}^{2p-1} t_{7\alpha}\right)\right]^7. \tag{3.50}$$

$$\exp(A_1 + 2\sum_{j=1}^{(p-1)/2} B_{7j}) = S_a^p + S_b^p = \left[\exp\left(\sum_{\alpha=1}^{13} t_{p\alpha})\right]^p. \tag{3.51}$$

(3.50) and (3.51) are twin Fermat's equations. Both equations have no rational solutions.

Lame proved (3.42), therefore (3.44), (3.46) and (3.48) − (3.51) have no rational solutions. Using the theorem 3.7, we prove the theorem 3.4, because (3.17) and (3.44) are the same Fermat's equation.

Theorem 3.8. Let $n = \prod p$, where p ranges over all odd primes. Suppose all $S_i = 0$ except $S_a \neq 0$ and $S_b \neq 0$, $a \neq b$. $S_i = 0$ are $(2n-2)$ Diophantine equations with $(2n-1)$ variables. From (3.15) we have $2n(n-1)$ Fermat's equations. Every real Fermat's equation has no rational solutions.

Proof. If $(a-b, 2n) = 2n/3$, from (3.15) we have $2n$ Fermat's equations:

$$(S_a^3 + S_b^3)^{2n/3} = \pm 1. \tag{3.52}$$

From (3.52) we have n real Fermat's equations:

$$S_a^3 + S_b^3 = 1. \tag{3.53}$$

If $(a-b, 2n) = 2n/5$, from (3.15) we have $4n$ Fermat's equations:

$$(S_a^5 + S_b^5)^{2n/5} = \pm 1. \tag{3.54}$$

From (3.54) we have $2n$ real Fermat's equations:

$$S_a^5 + S_b^5 = 1. \tag{3.55}$$

If $(a-b, 2n) = 2n/7$, from (3.15) we have $6n$ Fermat's equations:

$$(S_a^7 + S_b^7)^{2n/7} = \pm 1. \tag{3.56}$$

From (3.56) we have $3n$ real Fermat's equations:

$$S_a^7 + S_b^7 = 1. \tag{3.57}$$

If $(a-b, 2n) = 2n/p$, from (3.15) we have $n(p-1)$ Fermat's equations:

$$(S_a^p + S_b^p)^{2n/p} = \pm 1. \tag{3.58}$$

From (3.58) we have $n(p-1)/2$ real Fermat's equations:

$$S_a^p + S_b^p = 1. \tag{3.59}$$

If $(a-b, 2n) = 1$, from (3.15) we have $n\phi(n)$ Fermat's equations:

$$S_a^{2n} - S_b^{2n} = 1 \tag{3.60}$$

From (3.2) and (3.3) we have

$$\exp(A_1 + 2\sum_{j=1}^{(p-1)/2} B_{nj/p}) = S_a^p + S_b^p = [\exp(\sum_{\alpha=1}^{2n/p-1} t_{p\alpha})]^p. \tag{3.61}$$

Euler proved (3.53), therefore (3.55), (3.57) and (3.59) - (3.61) have no rational solutions, that is all real Fermat's equations in (3.15) have no rational solutions.

By lifting $F(a,+,\times) \to \hat{F}(\hat{a},+,\hat{\times})$ and $F(a,+,\times) \to \hat{F}(a,+,\hat{\times})$ the theorem 3.4 to 3.8 are the Fermat-Santilli isotheorems.

4. Class III Fermat-Santilli Isotheorems

Let $b = 4m, m = 1, 2, \cdots$. From (8.5) we have the complex hyperbolic functions S_i of order $4m$ with $(4m-1)$ variables,

$$S_i = \frac{1}{4m}\left[e^{A_1} + 2e^H \cos\left(\beta + \frac{(i-1)\pi}{2}\right) + 2\sum_{j=1}^{m-1} e^{B_j}\cos\left(\theta_j + \frac{(i-1)j\pi}{2m}\right)\right]$$

$$+\frac{(-1)^{(i-1)}}{4m}\left[e^{A_2} + 2\sum_{j=1}^{m-1} e^{D_j}\cos\left(\phi_j - \frac{(i-1)j\pi}{2m}\right)\right], \tag{4.1}$$

where $i = 1, \cdots, 4m$;

$$A_1 = \sum_{\alpha=1}^{4m-1} t_\alpha, \quad A_2 = \sum_{\alpha=1}^{4m-1} t_\alpha(-1)^\alpha,$$

$$H = \sum_{\alpha=1}^{2m-1} t_{2\alpha}(-1)^\alpha, \quad \beta = \sum_{\alpha=1}^{2m} t_{2\alpha-1}(-1)^\alpha,$$

$$B_j = \sum_{\alpha=1}^{4m-1} t_\alpha \cos\frac{\alpha j\pi}{2m}, \quad \theta_j = -\sum_{\alpha=1}^{4m-1} t_\alpha \sin\frac{\alpha j\pi}{2m},$$

$$D_j = \sum_{\alpha=1}^{4m-1} t_\alpha(-1)^\alpha \cos\frac{\alpha j\pi}{2m}, \quad \phi_j = \sum_{\alpha=1}^{4m-1} t_\alpha(-1)^\alpha \sin\frac{\alpha j\pi}{2m},$$

$$A_1 + A_2 + 2H + 2\sum_{j=1}^{m-1}(B_j + D_j) = 0. \tag{4.2}$$

In the same way as in (2.3) and (2.4), from (4.1) we have its inverse transformation

$$e^{A_1} = \sum_{i=1}^{4m} S_i, \quad e^{A_2} = \sum_{i=1}^{4m} S_i(-1)^{1+i},$$

$$e^H \cos\beta = \sum_{i=1}^{2m} S_{2i-1}(-1)^{1+i}, \quad e^H \sin\beta = \sum_{i=1}^{2m} S_{2i}(-1)^i,$$

$$e^{B_j}\cos\theta_j = S_1 + \sum_{i=1}^{4m-1} S_{1+i}\cos\frac{ij\pi}{2m}, \quad e^{B_j}\sin\theta_j = -\sum_{i=1}^{4m-1} S_{1+i}\sin\frac{ij\pi}{2m},$$

$$e^{D_j}\cos\phi_j = S_1 + \sum_{i=1}^{4m-1} S_{1+i}(-1)^i\cos\frac{ij\pi}{2m}, \quad e^{D_j}\sin\phi_j = \sum_{i=1}^{4m-1} S_{1+i}(-1)^i\sin\frac{ij\pi}{2m}. \tag{4.3}$$

(4.2) and (4.3) have the same form.

Assume in (4.3) $S_1 \neq 0, S_2 \neq 0, S_i = 0$, where $i = 3, \cdots, 4m$. $S_i = 0$ are $(4m-2)$ indeterminate equations with $(4m-1)$ variables. From (4.3) we have

$$e^{A_1} = S_1 + S_2, \quad e^{A_2} = S_1 - S_2, \quad e^{2H} = S_1^2 + S_2^2,$$

$$e^{2B_j} = S_1^2 + S_2^2 + 2S_1S_2\cos\frac{j\pi}{2m}, \quad e^{2D_j} = S_1^2 + S_2^2 - 2S_1S_2\cos\frac{j\pi}{2m}. \tag{4.4}$$

Theorem 4.1. Let $m = \prod p_i$, where p_i ranges over all odd primes.
From (4.4) we have

$$\exp[A_1 + A_2 + 2H + 2\sum_{j=1}^{m-1}(B_j + D_j)] = S_1^{4m} - S_2^{4m}, \tag{4.5}$$

$$\exp(A_1 + 2D_{\frac{2m}{p_i}} + 2B_{\frac{4m}{p_i}} + 2D_{\frac{6m}{p_i}} + 2B_{\frac{8m}{p_i}} + \cdots) = S_1^{p_i} + S_2^{p_i}. \tag{4.6}$$

From (4.2) we have

$$\exp\left[A_1 + A_2 + 2H + 2\sum_{j=1}^{m=1}(B_j + D_j)\right] = 1, \tag{4.7}$$

$$\exp\left(A_1 + 2D_{\frac{2m}{p_i}} + 2B_{\frac{4m}{p_i}} + \cdots\right) = \left[\exp\left(\sum_{\alpha=1}^{\frac{4m}{p_i}-1} t_{p_i\alpha}\right)\right]^{p_i}, \tag{4.8}$$

From (4.5)–(4.8) we have the Fermat's equations

$$S_1^{4m} - S_2^{4m} = 1, \tag{4.9}$$

$$S_1^{p_i} + S_2^{p_i} = \left[\exp\left(\sum_{\alpha=1}^{\frac{4m}{p_i}-1} t_{p_i\alpha}\right)\right]^{p_i}. \tag{4.10}$$

Fermat [11] proved (4.9), therefore (4.10) has no rational solutions for any prime $p_i > 3$. It is a Fermat's marvellous proof and can fit in the margin[5].

Theorem 4.2. By lifting $F(a, +, \times) \to \hat{F}(\hat{a}, +, \hat{\times})$ we have the Fermat-Santilli equations of the first kind from (4.9) and (4.10)

$$\hat{S}_1^{\hat{4m}} - \hat{S}_2^{\hat{4m}} = \hat{I}, \tag{4.11}$$

$$\hat{S}_1^{\hat{p}_i} + \hat{S}_2^{\hat{p}_i} = \left[\hat{\exp}\left(\sum_{\alpha=1}^{\frac{4m}{p_i}-1} \hat{t}_{p_i\alpha}\right)\right]^{\hat{p}_i}. \tag{4.12}$$

Fermat [11] proved (4.11), therefore (4.12) has no isorational solutions for any isoprime $\hat{p}_i > \hat{3}$

Theorem 4.3. By lifting $F(a, +, \times) \to \hat{F}(a, +, \hat{\times})$ we have the Fermat-Santilli equations of the second kind from (4.9) and (4.10)

$$S_1^{\hat{4m}} - S_2^{\hat{4m}} = 1, \tag{4.13}$$

$$S_1^{\hat{p}_i} + S_2^{\hat{p}_i} = \left[\exp\left(\sum_{\alpha=1}^{\frac{4m}{p_i}-1} t_{p_i\alpha}\right)\right]^{\hat{p}_i}. \tag{4.14}$$

Euler [11] proved (4.14) for isoprime $\hat{p}_i = \hat{3}$, therefore (4.13) has no rational solutions for any isonumber $\hat{4m}$.

From (4.2), (4.3) and (8.6) we have the circular determinant

$$\exp[A_1 + A_2 + 2H + 2\sum_{j=1}^{m-1}(B_j + D_j)] = \begin{vmatrix} S_1 & S_{4m} & \dots & S_2 \\ S_2 & S_1 & \dots & S_3 \\ S_3 & S_2 & \dots & S_4 \\ \dots & \dots & \dots & \dots \\ S_{4m} & S_{4m-1} & \dots & S_1 \end{vmatrix} = 1 \qquad (4.15)$$

Theorem 4.4. Let $m = 16$. Suppose all $S_i = 0$ except $S_a \neq 0$ and $S_b \neq 0$, $a \neq b$. $S_i = 0$ are 62 Diophantine equations with 63 variables. From (4.15) we have 1984 Fermat's equations. Every real Fermat's equation has no rational solutions.

Proof. If $(a - b, 64) = 16$, from (4.15) we have 64 Fermat's equations:

$$(S_a^4 - S_b^4)^{16} = \pm 1. \qquad (4.16)$$

The $-$ sign can be rejected. From (4.16) we have 32 real Fermat's equations:

$$S_a^4 - S_b^4 = 1. \qquad (4.17)$$

If $(a - b, 64) = 8$, from (4.15) we have 120 Fermat's equations:

$$(S_a^8 - S_b^8)^8 = \pm 1. \qquad (4.18)$$

From (4.18) we have 60 real Fermat's equations:

$$S_a^8 - S_b^8 = 1. \qquad (4.19)$$

If $(a - b, 64) = 4$, from (4.15) we have 240 Fermat's equations:

$$(S_a^{16} - S_b^{16})^4 = \pm 1. \qquad (4.20)$$

From (4.20) we have 120 real Fermat's equations:

$$S_a^{16} - S_b^{16} = 1. \qquad (4.21)$$

If $(a - b, 64) = 2$, from (4.15) we have 480 Fermat's equations:

$$(S_a^{32} - S_b^{32})^2 = \pm 1. \qquad (4.22)$$

From (4.22) we have 240 real Fermat's equations:

$$S_a^{32} - S_b^{32} = 1. \qquad (4.23)$$

If $(a - b, 64) = 1$, from (4.15) we have 960 Fermat's equations:

$$S_a^{64} - S_b^{64} = 1. \qquad (4.24)$$

Fermat proved (4.17), therefore (4.19), (4.21) and (4.23) − (4.24) have no rational solutions.

From the theorem 3.4 we follow that all real Fermat's equations in (4.15) have no rational solutions.

Theorem 4.5. Let $m = p$, where p is an odd prime. Suppose all $S_i = 0$ except $S_a \neq 0$ and $S_b \neq 0$, $a \neq b$. $S_i = 0$ are $(4p-2)$ Diophantine equations with $(4p-1)$ variables. From (4.15) we have $4p(2p-1)$ Fermat's equations. Every real Fermat's equation has no rational solutions.

Proof. If $(a-b, 4p) = p$, from (4.15) we have $4p$ Fermat's equations:

$$(S_a^4 - S_b^4)^p = \pm 1. \tag{4.25}$$

From (4.25) we have

$$S_a^4 - S_b^4 = 1. \tag{4.26}$$

If $(a-b, 4p) = 4$, from (4.15) we have $2p(p-1)$ Fermat's equations:

$$(S_a^p + S_b^p)^4 = \pm 1. \tag{4.27}$$

From (4.27) we have $p(p-1)$ real Fermat's equations:

$$S_a^p + S_b^p = 1. \tag{4.28}$$

If $(a-b, 4p) = 2$, from (4.15) we have $2p(p-1)$ Fermat's equations:

$$(S_a^{2p} - S_b^{2p})^2 = \pm 1. \tag{4.29}$$

From (4.29) we have $p(p-1)$ Fermat's equations:

$$S_a^{2p} - S_b^{2p} = 1. \tag{4.30}$$

If $(a-b, 4p) = 1$, from (4.15) we have $4p(p-1)$ Fermat's equations:

$$S_a^{4p} - S_b^{4p} = 1. \tag{4.31}$$

From (4.2) and (4.3) we have

$$\exp(A_1 + 2D_2 + 2B_4 + \ldots) = S_a^p + S_b^p = [\exp(\sum_{\alpha=1}^{3} t_{p\alpha})]^p. \tag{4.32}$$

Fermat proved (4.26), therefore (4.28), (4.3), (4.31) and (4.32) have no rational solutions.

Theorem 4.6. Let $m = 3p$, where p is an odd prime. Suppose all $S_i = 0$ except $S_a \neq 0$ and $S_b \neq 0$, $a \neq b$. $S_i = 0$ are $(12p-2)$ Diophantine equations with $(12p-1)$

variables. From (4.15) we have $12p(6p-1)$ Fermat's equations. Every real Fermat's equation has no rational solutions.

Proof. If $(a-b, 12p) = 3p$, from (4.15) we have $12p$ Fermat's equations:

$$(S_a^4 - S_b^4)^{3p} = \pm 1. \tag{4.33}$$

From (4.33) we have

$$S_a^4 - S_b^4 = 1. \tag{4.34}$$

If $(a-b, 12p) = 4p$, from (4.15) we have $12p$ Fermat's equations:

$$(S_a^3 + S_b^3)^{4p} = \pm 1. \tag{4.35}$$

From (4.35) we have $6p$ real Fermat's equations:

$$S_a^3 + S_b^3 = 1. \tag{4.36}$$

If $(a-b, 12p) = 12$, from (4.15) we have $6p(p-1)$ Fermat's equations:

$$(S_a^p + S_b^p)^{12} = \pm 1. \tag{4.37}$$

From (4.37) we have $3p(p-1)$ Fermat's equations:

$$S_a^p + S_b^p = 1. \tag{4.38}$$

If $(a-b, 12p) = 6$, from (4.15) we have $6p(p-1)$ Fermat's equations:

$$(S_a^{2p} - S_b^{2p})^6 = \pm 1. \tag{4.39}$$

From (4.39) we have $3p(p-1)$ real Fermat's equations:

$$S_a^{2p} - S_b^{2p} = 1. \tag{4.40}$$

If $(a-b, 12p) = 4$, from (4.15) we have $12p(p-1)$ Fermat's equations:

$$(S_a^{3p} + S_b^{3p})^4 = \pm 1. \tag{4.41}$$

From (4.41) we have $6p(p-1)$ real Fermat's equations:

$$S_a^{3p} + S_b^{3p} = 1. \tag{4.42}$$

If $(a-b, 12p) = 3$, from (4.15) we have $12p(p-1)$ Fermat's equations:

$$(S_a^{4p} - S_b^{4p})^3 = \pm 1. \tag{4.43}$$

From (4.43) we have $6p(p-1)$ real Fermat's equations:

$$S_a^{4p} - S_b^{4p} = 1. \tag{4.44}$$

If $(a-b,12p)=2$, from (4.15) we have $12p(p-1)$ Fermat's equations:

$$(S_a^{6p}-S_b^{6p})^2=\pm 1. \tag{4.45}$$

From (4.45) we have $6p(p-1)$ real Fermat's equations:

$$S_a^{6p}-S_b^{6p}=1. \tag{4.46}$$

If $(a-b,12p)=1$ from (4.15) we have $24p(p-1)$ Fermat's equations:

$$S_a^{12p}-S_b^{12p}=1. \tag{4.47}$$

From (4.2) and (4.3) we have

$$\exp(A_1+2D_6+2B_{12}+\ldots)=S_a^p+S_b^p=[\exp(\sum_{\alpha=1}^{11}t_{p\alpha})]^p \tag{4.48}$$

Fermat proved (4.34), therefore (4.36), (4.38), (4.4) and (4.46) – (4.48) have no rational solutions.

Theorem 4.7. Let $m=\prod p$, where p ranges over all odd primes. Suppose all $S_i=0$ except $S_a\neq 0$ and $S_b\neq 0$, $a\neq b$. $S_i=0$ are $(4m-2)$ Diophantine equations with $(4m-1)$ variables. From (4.15) we have $4m(2m-1)$ Fermat's equations. Every real Fermat's equation has no rational solutions.

Proof. If $(a-b,4m)=m$, from (4.15) we have $4m$ Fermat's equations:

$$(S_a^4-S_b^4)^m=\pm 1. \tag{4.49}$$

From (4.49) we have

$$S_a^4-S_b^4=1. \tag{4.50}$$

If $(a-b,4m)=4m/3$, from (4.15) we have $4m$ Fermat's equations:

$$(S_a^3+S_b^3)^{4m/3}=\pm 1. \tag{4.51}$$

From (4.51) we have $2m$ real Fermat's equations:

$$S_a^3+S_b^3=1. \tag{4.52}$$

If $(a-b,4m)=4m/5$, from (4.15) we have $8m$ Fermat's equations:

$$(S_a^5+S_b^5)^{4m/5}=\pm 1. \tag{4.53}$$

From (4.53) we have $4m$ real Fermat's equations:

$$S_a^5+S_b^5=1. \tag{4.54}$$

If $(a-b, 4m) = 4m/p$, from (4.15) we have $2m(p-1)$ Fermat's equations:

$$(S_a^p + S_b^p)^{4m/p} = \pm 1. \tag{4.55}$$

From (4.55) we have $m(p-1)$ real Fermat's equations:

$$S_a^p + S_b^p = 1. \tag{4.56}$$

If $(a-b, 4m) = 2$, from (4.15) we have $2m\phi(m)$ Fermat's equations:

$$(S_a^{2m} - S_b^{2m})^2 = \pm 1. \tag{4.57}$$

From (4.57) we have $m\phi(m)$ real Fermat's equations:

$$S_a^{2m} - S_b^{2m} = 1. \tag{4.58}$$

If $(a-b, 4m) = 1$, from (4.15) we have $2m\phi(4m)$ real Fermat's equations:

$$S_a^{4m} - S_b^{4m} = 1. \tag{4.59}$$

Fermat proved (4.50), therefore (4.52), (4.54), (4.56), (4.58) and (4.59) have no rational solutions, that is all real Fermat's equations in (4.15) have no rational solutions.

By lifting $F(a, +, \times) \to \hat{F}(\hat{a}, +, \hat{\times})$ and $F(a, +, \times) \to \hat{F}(a, +, \hat{\times})$ the theorems 4.4 to 4.7 are the Fermat-Santilli isotheorems.

5. Class IV Fermat-Santilli Isotheorems

Let $b = n$, where n is an odd. From (8.13) we have the complex trigonometric functions S_i of order n with $(n-1)$ variables,

$$S_i = \frac{(-1)^{i-1}}{n}\left[e^A + 2\sum_{j=1}^{\frac{n-1}{2}}(-1)^{(i-1)j}e^{B_j}\cos\left(\theta_j + (-1)^j\frac{(i-1)j\pi}{n}\right)\right], \tag{5.1}$$

where $i = 1, \cdots, n$;

$$A = \sum_{\alpha=1}^{n-1} t_\alpha(-1)^\alpha, \quad B_j = \sum_{\alpha=1}^{n-1} t_\alpha(-1)^{(j-1)\alpha}\cos\frac{\alpha j\pi}{n},$$

$$\theta_j = (-1)^{j+1}\sum_{\alpha=1}^{n-1} t_\alpha(-1)^{(j-1)\alpha}\sin\frac{\alpha j\pi}{n}, \quad A + 2\sum_{j=1}^{\frac{n-1}{2}} B_j = 0. \tag{5.2}$$

In the same way as in (2.3) and (2.4), we have its inverse transformation from (5.1)

$$e^{A}=\sum_{i=1}^{n}S_j(-1)^{i+1},\quad e^{B_j}\cos\theta_j=S_1+\sum_{i=1}^{n-1}S_{1+i}(-1)^{(j-1)i}\cos\frac{ij\pi}{n},$$

$$e^{B_j}\sin\theta_j=(-1)^{j+1}\sum_{i=1}^{n-1}S_{1+i}(-1)^{(j-1)i}\sin\frac{ij\pi}{n}. \tag{5.3}$$

(5.2) and (5.3) have the same form.

Assume in (5.3) $S_1\neq 0, S_2\neq 0, S_i=0$, where $i=3,\cdots,n$. $S_i=0$ are $(n-2)$ indeterminate equations with $(n-1)$ variables. From (5.3) we have

$$e^{A}=S_1-S_2,\quad e^{2B_j}=S_1^2+S_2^2+2S_1S_2(-1)^{j-1}\cos\frac{j\pi}{n}. \tag{5.4}$$

Theorem 5.1. Let $n=\prod p_i$, where p_i ranges over all odd primes. From (5.4) we have

$$\exp\left(A+2\sum_{j=1}^{\frac{n-1}{2}}B_j\right)=S_1^n-S_2^n, \tag{5.5}$$

$$\exp\left(A+2\sum_{j=1}^{\frac{p_i-1}{2}}B_{\frac{n}{p_i}j}\right)=S_1^{p_i}-S_2^{p_i}. \tag{5.6}$$

From (5.2) we have

$$\exp\left(A+2\sum_{j=1}^{\frac{n-1}{2}}B_j\right)=1, \tag{5.7}$$

$$\exp\left(A+2\sum_{j=1}^{\frac{p_i-1}{2}}B_{\frac{n}{p_i}j}\right)=\left[\exp\left(\sum_{\alpha=1}^{\frac{n}{p_i}-1}t_{p_i\alpha}(-1)^{\alpha}\right)\right]^{p_i}. \tag{5.8}$$

From (5.5)–(5.8) we have the Fermat's equations

$$S_1^n-S_2^n=1, \tag{5.9}$$

$$S_1^{p_i}-S_2^{p_i}=\left[\exp\left(\sum_{\alpha=1}^{\frac{n}{p_i}-1}t_{p_i\ alpha}(-1)^{\alpha}\right)\right]^{p_i}. \tag{5.10}$$

Euler [11] proved (5.9),therefore (5.10) has no rational solutions for any prime $p_i > 3$.

Theorem 5.2. By lifting $F(a,+,\times) \to \hat{F}(\hat{a},+,\hat{\times})$ we have the Fermat-Santilli equations of the first kind from (5.9) and (5.10)

$$\hat{S_1}^{\hat{n}} - \hat{S_2}^{\hat{n}} = \hat{I}, \tag{5.11}$$

$$\hat{S_1}^{\hat{p_i}} - \hat{S_2}^{\hat{p_i}} = \left[\hat{\exp}\left(\sum_{\alpha=1}^{\frac{n}{p_i}-1} \hat{t}_{p_i\alpha}(-1)^\alpha\right)\right]^{\hat{p_i}}. \tag{5.12}$$

Euler [11] proved (5.11), therefore (5.12) has no isorational solutions for any isoprime $\hat{p_i} \geq \hat{3}$.

Theorem 5.3. By lifting $F(a,+,\times) \to \hat{F}(a,+,\hat{\times})$ we have the Fermat-Santilli equations of the second kind from (5.9) and (5.10)

$$S_1^{\hat{n}} - S_2^{\hat{n}} = 1, \tag{5.13}$$

$$S_1^{\hat{p_i}} - S_2^{\hat{p_i}} = \left[\exp\left(\sum_{\alpha=1}^{\frac{n}{p_i}-1} t_{p_i\alpha}(-1)^\alpha\right)\right]^{\hat{p_i}}. \tag{5.14}$$

Euler [11] proved (5.14) for isoprime $\hat{p_i} = \hat{3}$, therefore (5.13) has no rational solutions for any odd isonumber $\hat{n} > \hat{3}$.

From (5.2), (5.3) and (8.14) we have the cyclic determinant

$$\exp(A + 2\sum_{j=1}^{(n-1)/2} B_j) = \begin{vmatrix} S_1 & -S_n & \dots & -S_2 \\ S_2 & S_1 & \dots & -S_3 \\ S_3 & S_2 & \dots & -S_4 \\ \dots & \dots & \dots & \dots \\ S_n & S_{n-1} & \dots & S_1 \end{vmatrix} = 1. \tag{5.15}$$

Theorem 5.4. Let $n = p$, where p is an odd prime. Suppose all $S_i = 0$ except $S_a \neq 0$ and $S_b \neq 0$, $a \neq b$. $S_i = 0$ are $(p-2)$ Diophantine equations with $(p-1)$ variables.

From (5.15) we have $(p^2-1)/8$ Fermat's equations:

$$S_a^p + S_b^p = 1. \tag{5.16}$$

$2(p^2-1)/8$ Fermat's equations:

$$S_a^p - S_b^p = 1. \tag{5.17}$$

$(p-1)(p-3)/8$ Fermat's equations:

$$S_a^p + S_b^p = -1. \tag{5.18}$$

It is sufficient to prove the theorem 5.4, but the proof has great difficulty. In the following theorems we consider n is the composite number.

Theorem 5.5. Let $n = 3p$, where p is an odd prime. Suppose all $S_i = 0$ except $S_a \neq 0$ and $S_b \neq 0$, $a \neq b$. $S_i = 0$ are $(3p-2)$ Diophantine equations with $(3p-1)$ variables. From (5.15) we have $3p(3p-1)/2$ Fermat's equations. Every Fermat's equation has no rational solutions.

Proof. If $(a-b, 3p) = p$, from (5.15) we have $2p$ Fermat's equations:

$$(S_a^3 - S_b^3)^p = 1. \tag{5.19}$$

From (5.19) we have

$$S_a^3 - S_b^3 = 1. \tag{5.20}$$

We have $(p+1)/2$ Fermat's equations:

$$(S_a^3 + S_b^3)^p = 1. \tag{5.21}$$

From (5.21) we have

$$S_a^3 + S_b^3 = 1. \tag{5.22}$$

We have $(p-1)/2$ Fermat's equations:

$$(S_a^3 + S_b^3)^p = -1. \tag{5.23}$$

From (5.23) we have

$$S_a^3 + S_b^3 = -1. \tag{5.24}$$

If $(a-b, 3p) = 3$, from (5.15) we have $3(p^2-1)/4$ Fermat's equations:

$$(S_a^p - S_b^p)^3 = 1. \tag{5.25}$$

From (5.25) we have

$$S_a^p - S_b^p = 1. \tag{5.26}$$

We have $(p-1)(p-3)/8$ Fermat's equations:

$$(S_a^p + S_b^p)^3 = 1. \tag{5.27}$$

From (5.27) we have

$$S_a^p + S_b^p = 1. \tag{5.28}$$

We have $(3p^2 - 8p + 5)/8$ Fermat's equations:

$$(S_a^p + S_b^p)^3 = -1. \tag{5.29}$$

From (5.29) we have

$$S_a^p + S_b^p = -1. \tag{5.30}$$

If $(a-b, 3p) = 1$, from (5.15) we have $(p-1)(3p-1)/2$ Fermat's equations:

$$S_a^{3p} - S_b^{3p} = 1. \tag{5.31}$$

$3(p^2-1)/4$ Fermat's equations:

$$S_a^{3p} + S_b^{3p} = 1. \tag{5.32}$$

$(p-1)(3p-1)/4$ Fermat's equations:

$$S_a^{3p} + S_b^{3p} = -1. \tag{5.33}$$

Euler proved (5.20), (5.22) and (5.24), therefore (5.26) $-$ (5.33) have no rational solutions, that is all Fermat's equations in (5.15) have no rational solutions.

By lifting $F(a,+,\times) \rightarrow \hat{F}(\hat{a},+,\hat{\times})$ and $F(a,+,\times) \rightarrow \hat{F}(a,+,\hat{\times})$ the theorems 5.4 and 5.5 are both Fermat-Santilli isotheorems.

6. Class V Fermat-Santilli Isotheorems

Let $b = 2n$, where n is an odd. From (8.13) we have the complex trigonometric functions S_i of order $2n$ with $(2n-1)$ variables,

$$S_i = \frac{(-1)^{i-1}}{n}\left[e^H \cos\left(\beta + \frac{(i-1)\pi}{2}\right) + \sum_{j=0}^{\frac{n-3}{2}} e^{B_j}\cos\left(\theta_j + \frac{(i-1)(2j+1)\pi}{2n}\right)\right]$$

$$+\frac{1}{n}\sum_{j=0}^{\frac{n-3}{2}} e^{D_j}\cos\left(\phi_j - \frac{(i-1)(2j+1)\pi}{2n}\right), \tag{6.1}$$

where $i = 1, \cdots, 2n$;

$$H = \sum_{\alpha=1}^{n-1} t_{2\alpha}(-1)^\alpha, \quad \beta = \sum_{\alpha=1}^{n} t_{2\alpha-1}(-1)^{1+\alpha},$$

$$B_j = \sum_{\alpha=1}^{2n-1} t_\alpha(-1)^\alpha \cos\frac{(2j+1)\alpha\pi}{2n}, \quad \theta_j = \sum_{\alpha=1}^{2n-1} t_\alpha(-1)^{1+\alpha}\sin\frac{(2j+1)\alpha\pi}{2n},$$

$$D_j = \sum_{\alpha=1}^{2n-1} t_\alpha \cos\frac{(2j+1)\alpha\pi}{2n}, \quad \phi_j = \sum_{\alpha=1}^{2n-1} t_\alpha \sin\frac{(2j+1)\alpha\pi}{2n},$$

$$2H + 2\sum_{j=0}^{\frac{n-3}{2}}(B_j + D_j) = 0. \tag{6.2}$$

In the same way as in (2.3) and (2.4), from (6.1) we have its inverse transformation

$$e^H \cos\beta = \sum_{i=1}^{n} S_{2i-1}(-1)^{1+i}, \quad e^H \sin\beta = \sum_{i=1}^{n} S_{2i}(-1)^{1+i},$$

$$e^{B_j}\cos\theta_j = S_1 + \sum_{i=1}^{2n-1} S_{1+i}(-1)^i \cos\frac{(2j+1)i\pi}{2n},$$

$$e^{B_j}\sin\theta_j = \sum_{i=1}^{2n-1} S_{1+i}(-1)^{1+i}\sin\frac{(2j+1)i\pi}{2n},$$

$$e^{D_j}\cos\phi_j = S_1 + \sum_{i=1}^{2n-1} S_{1+i}\cos\frac{(2j+1)i\pi}{2n},$$

$$e^{D_j}\sin\phi_j = \sum_{i=1}^{2n-1} S_{1+i}\sin\frac{(2j+1)i\pi}{2n}. \tag{6.3}$$

(6.2) and (6.3) have the same form.

Assume in (6.3) $S_1 \neq 0, S_2 \neq 0, S_i = 0$, where $i = 3, \cdots, 2n$. $S_i = 0$ are $(2n-2)$ indeterminate equations with $(2n-1)$ variables. From (6.3) we have

$$e^{2H} = S_1^2 + S_2^2, \quad e^{2B_j} = S_1^2 + S_2^2 - 2S_1S_2\cos\frac{(2j+1)\pi}{2n},$$

$$e^{2D_j} = S_1^2 + S_2^2 + 2S_1S_2\cos\frac{(2j+1)\pi}{2n}. \tag{6.4}$$

Let $n = 1$. We have $H = 0$ and $\beta = t_1$. From (6.4) we have

$$S_1^2 + S_2^2 = 1, \tag{6.5}$$

where $S_1 = \cos t_1$ and $S_2 = \sin t_1$.

(6.5) is Pythagorean theorem. It has infinitely many rational solutions. (6.5) inspired Fermat to write his last theorem. From (6.5) Jiang [3] introduces S_i functions and proves Fermat's last theorem.

Theorem 6.1. Let $n = \prod p_i$, where p_i ranges over all odd primes. From (6.4) we have

$$\exp\left[2H + 2\sum_{j=0}^{\frac{n-3}{2}}(B_j + D_j)\right] = S_1^{2n} + S_2^{2n}, \tag{6.6}$$

$$\exp\left[2H+2\sum_{j=0}^{\frac{p_i-3}{2}}(B_{\frac{n}{p_i}j+\frac{n-p_i}{2p_i}}+D_{\frac{n}{p_i}j+\frac{n-p_i}{2p_i}})\right]=S_1^{2p_i}+S_2^{2p_i}. \tag{6.7}$$

From (6.2) we have

$$\exp\left[2H+2\sum_{j=0}^{\frac{n-3}{2}}(B_j+D_j)\right]=1, \tag{6.8}$$

$$\exp\left[2H+2\sum_{j=0}^{\frac{p_i-3}{2}}\left(B_{\frac{n}{p_i}j+\frac{n-p_i}{2p_i}}+D_{\frac{n}{p_i}j+\frac{n-p_i}{2p_i}}\right)\right]=\left[\exp\left(\sum_{\alpha=1}^{\frac{n}{p_i}-1}t_{2p_i\alpha}(-1)^{\alpha}\right)\right]^{2p_i}. \tag{6.9}$$

From (6.6)–(6.9) we have the Fermat's equations

$$S_1^{2n}+S_2^{2n}=1, \tag{6.10}$$

$$S_1^{2p_i}+S_2^{2p_i}=\left[\exp\left(\sum_{\alpha=1}^{\frac{n}{p_i}-1}t_{2p_i\alpha}(-1)^{\alpha}\right)\right]^{2p_i}. \tag{6.11}$$

Euler [11] proved (6.10), therefore (6.11) has no rational solutions for any prime $p_i>3$.

Theorem 6.2. By lifting $F(a,+,\times)\to\hat{F}(\hat{a},+,\hat{\times})$ we have the Fermat-Santilli equations of the first kind from (6.10) and (6.11)

$$\hat{S_1}^{\hat{2n}}+\hat{S_2}^{\hat{2n}}=\hat{I}, \tag{6.12}$$

$$\hat{S_1}^{2\hat{p}_i}+\hat{S_2}^{2\hat{p}_i}=\left[\hat{\exp}\left(\sum_{\alpha=1}^{\frac{n}{p_i}-1}\hat{t}_{2p_i\alpha}(-1)^{\alpha}\right)\right]^{2\hat{p}_i}. \tag{6.13}$$

Euler [11] proved (6.12), therefore (6.13) has no isorational solutions for any isonumber $2\hat{p}_i$.

Theorem 6.3. By lifting $F(a,+,\times)\to\hat{F}(a,+,\hat{\times})$ we have the Fermat-Santilli equations of the second kind from (6.10) and (6.11)

$$S_1^{\hat{2n}}+S_2^{\hat{2n}}=1, \tag{6.14}$$

$$S_1^{2\hat{p}_i}+S_2^{2\hat{p}_i}=\left[\exp\left(\sum_{\alpha=1}^{\frac{n}{p_i}-1}t_{2p_i\alpha}(-1)^{\alpha}\right)\right]^{2\hat{p}_i}. \tag{6.15}$$

Euler [11] proved (6.15) for isonumber $\hat{2p} = \hat{6}$, therefore (6.14) has no rational solutions for any isonumber $\hat{2n}$.

From (6.2), (6.3) and (8.14) we have the cyclic determinant

$$\exp[2H + 2\sum_{j=0}^{(n-3)/2}(B_j + D_j)] = \begin{vmatrix} S_1 & -S_{2n} & \dots & -S_2 \\ S_2 & S_1 & \dots & -S_3 \\ S_3 & S_2 & \dots & -S_4 \\ \dots & \dots & \dots & \dots \\ S_{2n} & S_{2n-1} & \dots & S_1 \end{vmatrix} = 1 \qquad (6.16)$$

Theorem 6.4. Let $n = p$, where p is an odd prime. Suppose all $S_i = 0$ except $S_a \neq 0$ and $S_b \neq 0$, $a \neq b$. $S_i = 0$ are $(2p - 2)$ Diophantine equations with $(2p - 1)$ variables. From (6.16) we have $2p(p - 1)$ Fermat's equations.

If $(a - b, 2p) = 2$, from (6.16) we have $p(p - 1)$ Fermat's equations:

$$(S_a^p \pm S_b^p)^2 = 1. \qquad (6.17)$$

From (6.17) we have

$$S_a^p \pm S_b^p = 1. \qquad (6.18)$$

If $(a - b, 2p) = 1$, from (6.16) we have $p(p - 1)$ Fermat's equations:

$$S_a^{2p} + S_b^{2p} = 1. \qquad (6.19)$$

It is sufficient to prove the theorem 6.4, but the proof has great difficulty. In the following theorems we consider n is the composite number.

Theorem 6.5. Let $n = 3p$, where p is an odd prime. Suppose all $S_i = 0$ except $S_a \neq 0$ and $S_b \neq 0$, $a \neq b$. $S_i = 0$ are $(6p - 2)$ Diophantine equations with $(6p - 1)$ variables. From (6.16) we have $6p(3p - 1)$ Fermat's equations. Every Fermat's equation has no rational solutions.

Proof. If $(a - b, 6p) = 2p$, from (6.16) we have $6p$ Fermat's equations:

$$(S_a^3 \pm S_b^3)^{2p} = 1. \qquad (6.20)$$

From (6.20) we have

$$S_a^3 \pm S_b^3 = 1. \qquad (6.21)$$

If $(a - b, 6p) = p$, from (6.16) we have $6p$ Fermat's equations:

$$(S_a^6 + S_b^6)^p = 1. \qquad (6.22)$$

From (6.22) we have

$$S_a^6 + S_b^6 = 1. \qquad (6.23)$$

If $(a-b, 6p) = 6$, from (6.16) we have $3p(p-1)$ Fermat's equations:

$$(S_a^p \pm S_b^p)^6 = 1. \tag{6.24}$$

From (6.24) we have

$$S_a^p \pm S_b^p = 1. \tag{6.25}$$

If $(a-b, 6p) = 3$, from (6.16) we have $3p(p-1)$ Fermat's equations:

$$(S_a^{2p} + S_b^{2p})^3 = 1. \tag{6.26}$$

From (6.26) we have

$$S_a^{2p} + S_b^{2p} = 1. \tag{6.27}$$

If $(a-b, 6p) = 2$, from (6.16) we have $6p(p-1)$ Fermat's equations:

$$(S_a^{3p} \pm S_b^{3p})^2 = 1. \tag{6.28}$$

From (6.28) we have

$$S_a^{3p} \pm S_b^{3p} = 1. \tag{6.29}$$

If $(a-b, 6p) = 1$, from (6.16) we have $6p(p-1)$ Fermat's equations:

$$S_a^{6p} + S_b^{6p} = 1. \qquad (6.30$$

Euler proved (6.21), therefore (6.23) – (6.30) have no rational solutions.

Theorem 6.6. Let $n = 5p$, where p is an odd prime. Suppose all $S_i = 0$ except $S_a \neq 0$ and $S_b \neq 0$, $a \neq b$. $S_i = 0$ are $(10p-2)$ Diophantine equations with $(10p-1)$ variables. From (6.16) we have $10p(5p-1)$ Fermat's equations. Every Fermat's equation has no rational solutions.

Proof. If $(a-b, 10p) = 2p$, from (6.16) we have $20p$ Fermat's equations:

$$(S_a^5 \pm S_b^5)^{2p} = 1. \tag{6.31}$$

From (6.31) we have

$$S_a^5 \pm S_b^5 = 1. \tag{6.32}$$

If $(a-b, 10p) = p$, from (6.16) we have $20p$ Fermat's equations:

$$(S_a^{10} + S_b^{10})^p = 1. \tag{6.33}$$

From (6.33) we have

$$S_a^{10} + S_b^{10} = 1. \tag{6.34}$$

If $(a-b, 10p) = 10$, from (6.16) we have $5p(p-1)$ Fermat's equations:

$$(S_a^p \pm S_b^p)^{10} = 1. \tag{6.35}$$

From (6.35) we have

$$S_a^p \pm S_b^p = 1. \tag{6.36}$$

If $(a-b, 10p) = 5$, from (6.16) we have $5p(p-1)$ Fermat's equations:

$$(S_a^{2p} + S_b^{2p})^5 = 1. \tag{6.37}$$

From (6.37) we have

$$S_a^{2p} + S_b^{2p} = 1. \tag{6.38}$$

If $(a-b, 10p) = 2$, from (6.16) we have $20p(p-1)$ Fermat's equations:

$$(S_a^{5p} \pm S_b^{5p})^2 = 1. \tag{6.39}$$

From (6.39) we have

$$S_a^{5p} \pm S_b^{5p} = 1. \tag{6.40}$$

If $(a-b, 10p) = 1$, from (6.16) we have $20p(p-1)$ Fermat's equations:

$$S_a^{10p} + S_b^{10p} = 1. \tag{6.41}$$

Dirichlet and Legendre proved (6.32), therefore (6.34) – (6.41) have no rational solutions.

Theorem 6.7. Let $n = \prod p$, where p ranges over all odd primes. Suppose all $S_i = 0$ except $S_a \neq 0$ and $S_b \neq 0$, $a \neq b$. $S_i = 0$ are $(2n-2)$ Diophantine equations with $(2n-1)$ variables. From (6.16) we have $2n(n-1)$ Fermat's equations. Every Fermat's equation has no rational solutions.

Proof. If $(a-b, 2n) = 2n/3$, from (6.16) we have $2n$ Fermat's equations:

$$(S_a^3 \pm S_b^3)^{2n/3} = 1. \tag{6.42}$$

From (6.42) we have

$$S_a^3 \pm S_b^3 = 1. \tag{6.43}$$

If $(a-b, 2n) = 2n/5$, from (6.16) we have $4n$ Fermat's equations:

$$(S_a^5 \pm S_b^5)^{2n/5} = 1. \tag{6.44}$$

From (6.44) we have

$$S_a^5 \pm S_b^5 = 1. \tag{6.45}$$

If $(a-b, 2n) = 2n/7$, from (6.16) we have $6n$ Fermat's equations:

$$(S_a^7 \pm S_b^7)^{2n/7} = 1. \tag{6.46}$$

From (6.46) we have

$$S_a^7 \pm S_b^7 = 1. \tag{6.47}$$

If $(a-b,2n)=2n/p$, from (6.16) we have $n(p-1)$ Fermat's equations:

$$(S_a^p \pm S_b^p)^{2n/p} = 1. \tag{6.48}$$

From (6.48) we have

$$S_a^p \pm S_b^p = 1. \tag{6.49}$$

If $(a-b,2n)=2$, from (6.16) we have $n\phi(n)$ Fermat's equations:

$$(S_a^n \pm S_b^n)^2 = 1. \tag{6.50}$$

From (6.50) we have

$$S_a^n \pm S_b^n = 1. \tag{6.51}$$

If $(a-b,2n)=1$, from (6.16) we have $n\phi(n)$ Fermat's equations:

$$S_a^{2n} + S_b^{2n} = 1. \tag{6.52}$$

Euler proved (6.43), therefore (6.45) - (6.52) have no rational solutions, that is all Fermat's equations in (6.16) have no rational solutions.

By lifting $F(a,+,\times) \to \hat{F}(\hat{a},+,\hat{\times})$ and $F(a,+,\times) \to \hat{F}(a,+,\hat{\times})$ the theorems 6.4 to 6.7 are Fermat-Santilli isotheorems.

7. Class VI Fermat-Santilli Isotheorems

Let $b=4m, m=1,2,\cdots$. From (8.13) we have the complex trigonometric functions S_i of order $4m$ with $(4m-1)$ variables,

$$S_i = \frac{1}{2m}\left[(-1)^{i-1}\sum_{j=0}^{m-1} e^{B_j}\cos\left(\theta_j + \frac{(i-1)(2j+1)\pi}{4m}\right) + \sum_{j=0}^{m-1} e^{D_j}\right.$$
$$\left.\cos\left(\phi_j - \frac{(i-1)(2j+1)\pi}{4m}\right)\right], \tag{7.1}$$

where $i=1,\cdots,4m$;

$$B_j = \sum_{\alpha=1}^{4m-1} t_\alpha(-1)^\alpha \cos\frac{(2j+1)\alpha\pi}{4m}, \quad \theta_j = \sum_{\alpha=1}^{4m-1} t_\alpha(-1)^{1+\alpha}\sin\frac{(2j+1)\alpha\pi}{4m},$$

$$D_j = \sum_{\alpha=1}^{4m-1} t_\alpha \cos\frac{(2j+1)\alpha\pi}{4m}, \quad \phi_j = \sum_{\alpha=1}^{4m-1} t_\alpha \sin\frac{(2j+1)\alpha\pi}{4m},$$

$$2\sum_{j=0}^{m-1}(B_j + D_j) = 0. \tag{7.2}$$

In the same way as in (2.3) and (2.4), from (7.1) we have its inverse transformation

$$e^{B_j}\cos\theta_j = S_1 + \sum_{i=1}^{4m-1} S_{1+i}(-1)^i \cos\frac{(2j+1)i\pi}{4m},$$

$$e^{B_j}\sin\theta_j = \sum_{i=1}^{4m-1} S_{1+i}(-1)^{1+i} \sin\frac{(2j+1)i\pi}{4m},$$

$$e^{D_j}\cos\phi_j = S_1 + \sum_{i=1}^{4m-1} S_{1+i} \cos\frac{(2j+1)i\pi}{4m},$$

$$e^{D_j}\sin\phi_j = \sum_{i=1}^{4m-1} S_{1+i} \sin\frac{(2j+1)i\pi}{4m}. \tag{7.3}$$

(7.2) and (7.3) have the same form.

Assume in (7.3) $S_1 \neq 0, S_2 \neq 0$, and $S_i = 0$, where $i = 3, \cdots, 4m$. $S_i = 0$ are $(4m-2)$ indeterminate equations with $(4m-1)$ variables. From (7.3) we have

$$e^{2B_j} = S_1^2 + S_2^2 - 2S_1S_2\cos\frac{(2j+1)\pi}{4m},\quad e^{2D_j} = S_1^2 + S_2^2 + 2S_1S_2\cos\frac{(2j+1)\pi}{4m}. \tag{7.4}$$

Theorem 7.1. Let $m = \prod p_i$, where p_i ranges over all odd primes. From (7.4) we have

$$\exp\left[2\sum_{j=0}^{m-1}(B_j + D_j)\right] = S_1^{4m} + S_2^{4m}, \tag{7.5}$$

$$\exp\left[2\sum_{j=0}^{p_i-1}(B_{\frac{m}{p_i}j+\frac{m-p_i}{2p_i}} + D_{\frac{m}{p_i}j+\frac{m-p_i}{2p_i}})\right] = S_1^{4p_i} + S_2^{4p_i}. \tag{7.6}$$

From (7.2) we have

$$\exp\left[2\sum_{j=0}^{m-1}(B_j + D_j)\right] = 1, \tag{7.7}$$

$$\exp\left[2\sum_{j=0}^{p_i-1}\left(B_{\frac{m}{p_i}j+\frac{m-p_i}{2p_i}} + D_{\frac{m}{p_i}j+\frac{m-p_i}{2p_i}}\right)\right] = \left[\exp\left(\sum_{alpha=1}^{\frac{m}{p_i}-1} t_{4p_i\alpha}(-1)^\alpha\right)\right]^{4p_i}. \tag{7.8}$$

From (7.5)–(7.8) we have the Fermat's equations

$$S_1^{4m} + S_2^{4m} = 1, \tag{7.9}$$

$$S_1^{4p_i} + S_2^{4p_i} = \left[\exp\left(\sum_{\alpha=1}^{\frac{m}{p_i}-1} t_{4p_i\alpha}(-1)^\alpha\right)\right]^{4p_i}. \tag{7.10}$$

Fermat [11] proved (7.9), therefore (7.10) has no rational solutions for any prime p_i.

Theorem 7.2. By lifting $F(a, +, \times) \to \hat{F}(\hat{a}, +, \hat{\times})$ we have the Fermat-Santilli equations of the first kind from (7.9) and (7.10)

$$\hat{S}_1^{\hat{4m}} + \hat{S}_2^{\hat{4m}} = \hat{I}, \tag{7.11}$$

$$\hat{S}_1^{\hat{4p_i}} + \hat{S}_2^{\hat{4p_i}} = \left[\hat{\exp}\left(\sum_{\alpha=1}^{\frac{m}{p_i}-1} \hat{t}_{4p_i\alpha}(-1)^\alpha\right)\right]^{\hat{4p_i}}. \tag{7.12}$$

Fermat [11] proved (7.11), therefore (7.12) has no isorational solutions for any isonumber $\hat{4p_i}$.

Theorem 7.3. By lifting $F(a, +, \times) \to \hat{F}(a, +, \hat{\times})$ we have the Fermat-Santilli equations of the second kind from (7.9) and (7.10)

$$S_1^{\hat{4m}} + S_2^{\hat{4m}} = 1, \tag{7.13}$$

$$S_1^{\hat{4p_i}} + S_2^{\hat{4p_i}} = \left[\exp\left(\sum_{\alpha=1}^{\frac{m}{p_i}-1} t_{4p_i\alpha}(-1)^\alpha\right)\right]^{\hat{4p_i}}. \tag{7.14}$$

Fermat [11] proved (7.14), therefore (7.13) has no rational solutions for any isonumber $\hat{4m}$. From (7.2), (7.3) and (8.14) we have the cyclic determinant

$$\exp[2\sum_{j=0}^{m-1}(B_j + D_j)] = \begin{vmatrix} S_1 & -S_{4m} & \dots & -S_2 \\ S_2 & S_1 & \dots & -S_3 \\ S_3 & S_2 & \dots & -S_4 \\ \dots & \dots & \dots & \dots \\ S_{4m} & S_{4m-1} & \dots & S_1 \end{vmatrix} = 1 \tag{7.15}$$

Theorem 7.4. Let $m = 16$. Suppose all $S_i = 0$ except $S_a \neq 0$ and $S_b \neq 0$, $a \neq b$. $S_i = 0$ are 62 Diophantine equations with 63 variables. From (7.15) we have 1984 Fermat's equations. Every Fermat's equation has no rational solutions.

Proof. If $(a - b, 64) = 16$, from (7.15) we have 64 Fermat's equations:

$$(S_a^4 + S_b^4)^{16} = 1. \tag{7.16}$$

From (7.16) we have

$$S_a^4 + S_b^4 = 1. \tag{7.17}$$

If $(a-b, 64) = 8$, from (7.15) we have 128 Fermat's equations:

$$(S_a^8 + S_b^8)^8 = 1. \tag{7.18}$$

From (7.18) we have

$$S_a^8 + S_b^8 = 1. \tag{7.19}$$

If $(a-b, 64) = 4$, from (7.15) we have 256 Fermat's equations:

$$(S_a^{16} + S_b^{16})^4 = 1. \tag{7.20}$$

From (7.20) we have

$$S_a^{16} + S_b^{16} = 1. \tag{7.21}$$

If $(a-b, 64) = 2$, from (7.15) we have 512 Fermat's equations:

$$(S_a^{32} + S_b^{32})^2 = 1. \tag{7.22}$$

From (7.22) we have

$$S_a^{32} + S_b^{32} = 1. \tag{7.23}$$

If $(a-b, 64) = 1$, from (7.15) we have 1024 Fermat's equations:

$$S_a^{64} + S_b^{64} = 1. \tag{7.24}$$

Fermat proved (7.17), therefore (7.19), (7.21), (7.23) and (7.24) have no rational solutions.

From the theorem 7.4, we follow that all Fermat's equations in (7.15) have no rational solutions.

Theorem 7.5. Let $m = p$, where p is an odd prime. Suppose all $S_i = 0$ except $S_a \neq 0$ and $S_b \neq 0$, $a \neq b$. $S_i = 0$ are $(4p-2)$ Diophantine equations with $(4p-1)$ variables. From (7.15) we have $4p(2p-1)$ Fermat's equations. Every Fermat's equation has no rational solutions.

Proof. If $(a-b, 4p) = p$, from (7.15) we have $4p$ Fermat's equations:

$$(S_a^4 + S_b^4)^p = 1. \tag{7.25}$$

From (7.25) we have

$$S_a^4 + S_b^4 = 1. \tag{7.26}$$

If $(a-b, 4p) = 4$, from (7.15) we have $2p(p-1)$ Fermat's equations:

$$(S_a^p \pm S_b^p)^4 = 1. \tag{7.27}$$

From (7.27) we have

$$S_a^p \pm S_b^p = 1. \tag{7.28}$$

If $(a-b, 4p) = 2$, from (7.15) we have $2p(p-1)$ Fermat's equations:

$$(S_a^{2p} + S_b^{2p})^2 = 1. \tag{7.29}$$

From (7.29) we have

$$S_a^{2p} + S_b^{2p} = 1. \tag{7.30}$$

If $(a-b, 4p) = 1$, from (7.15) we have $4p(p-1)$ Fermat's equations:

$$S_a^{4p} + S_b^{4p} = 1. \tag{7.31}$$

Fermat proved (7.26), therefore (7.28), (7.30) and (7.31) have no rational solutions.

Theorem 7.6. Let $m = \prod p$, where p ranges over all odd primes. Suppose all $S_i = 0$ except $S_a \neq 0$ and $S_b \neq 0$, $a \neq b$. $S_i = 0$ are $(4m-2)$ Diophantine equations with $(4m-1)$ variables. From (7.15) we have $4m(2m-1)$ Fermat's equations. Every Fermat's equation has no rational solutions.

Proof. If $(a-b, 4m) = m$, from (7.15) we have $4m$ Fermat's equations:

$$(S_a^4 + S_b^4)^m = 1. \tag{7.32}$$

From (7.32) we have

$$S_a^4 + S_b^4 = 1. \tag{7.33}$$

If $(a-b, 4m) = 4m/3$, from (7.15) we have $4m$ Fermat's equations:

$$(S_a^3 \pm S_b^3)^{4m/3} = 1. \tag{7.34}$$

From (7.34) we have

$$S_a^3 \pm S_b^3 = 1. \tag{7.35}$$

If $(a-b, 4m) = 4m/5$, from (7.15) we have $8m$ Fermat's equations:

$$(S_a^5 \pm S_b^5)^{4m/5} = 1. \tag{7.36}$$

From (7.36) we have

$$S_a^5 \pm S_b^5 = 1. \tag{7.37}$$

If $(a-b, 4m) = 4m/7$, from (7.15) we have $12m$ Fermat's equations:

$$(S_a^7 \pm S_b^7)^{4m/7} = 1. \tag{7.38}$$

From (7.38) we have

$$S_a^7 \pm S_b^7 = 1. \tag{7.39}$$

If $(a-b,4m)=4m/p$, from (7.15) we have $2m(p-1)$ Fermat's equations:

$$(S_a^p \pm S_b^p)^{4m/p} = 1. \tag{7.40}$$

From (7.40) we have

$$S_a^p \pm S_b^p = 1. \tag{7.41}$$

If $(a-b,4m)=1$, from (7.15) we have $4m\phi(m)$ Fermat's equations:

$$S_a^{4m} + S_b^{4m} = 1. \tag{7.42}$$

Fermat proved (7.33), therefore (7.35), (7.37), (7.39), (7.41) and (7.42) have no rational solutions, that is all Fermat's equations in (7.15) have no rational solutions.

By lifting $F(a,+,\times) \to \hat{F}(\hat{a},+,\hat{\times})$ and $F(a,+,\times) \to \hat{F}(a,+,\hat{\times})$ the theorems 7.4 to 7.6 are the Fermat-Santilli isotheorems.

8. Fermat's Mathematics

Definition 8.1. We define the hyperbolic functions of order b: $Jh_{1\cdot b}^{(i)}(t_1)$[3]

$$\exp(t_1 J_b^{b-1}) = \sum_{i=0}^{b-1} Jh_{1\cdot b}^{(i)}(t_1) J_b^i \tag{8.1}$$

where J_b denotes a b-th root of positive unity, $J_b^b = 1$,

$$Jh_{1\cdot b}^{(0)}(t_1) = \sum_{i=0}^{\infty} \frac{t_1^{bi}}{(bi)!} = \frac{1}{b}\Big\{\sum_{j=1}^{b}[\exp(t_1 \cos\frac{2\pi j}{b})]\cos(t_1 \sin\frac{2\pi j}{b})\Big\}, \tag{8.2}$$

$$Jh_{1\cdot b}^{(1)}(t_1) = \frac{d}{dt_1} Jh_{1\cdot b}^{(0)}(t_1) = \frac{1}{b}\Big\{\sum_{j=1}^{b}[\exp(t_1 \cos\frac{2\pi j}{b})]\cos(t_1 \sin\frac{2\pi j}{b} + \frac{2\pi j}{b})\Big\}, \tag{8.3}$$

$$Jh_{1\cdot b}^{(e)}(t_1) = \frac{d^e}{dt_1^e} Jh_{1\cdot b}^{(0)}(t_1) = \frac{1}{b}\Big\{\sum_{j=1}^{b}[\exp(t_1 \cos\frac{2\pi j}{b})]\cos(t_1 \sin\frac{2\pi j}{b} + \frac{2\pi e j}{b})\Big\}, \tag{8.4}$$

$$Jh_{1\cdot b}^{(b)}(t_1) = Jh_{1\cdot b}^{(0)}(t_1)$$

If $b=1$, then $Jh_{1.1}^{(0)}(t_1) = e^{t_1}$.
If $b=2$, then $Jh_{1.2}^{(0)}(t_1) = \mathrm{ch} t_1$.
If $b=n$, where n is an odd, then

$$Jh_{1\cdot n}^{(0)}(t_1) = \frac{1}{n}\Big[e^{t_1} + 2\sum_{j=1}^{\frac{n-1}{2}} \exp(t_1(-1)^j \cos\frac{j\pi}{n})\cos(t_1 \sin\frac{j\pi}{n})\Big].$$

If $b = 2n$, where n is an odd,

$$Jh^{(0)}_{1\cdot 2n}(t_1) = \frac{1}{n}\Big[\mathrm{ch}t_1 + 2\sum_{j=1}^{\frac{n-1}{2}} \mathrm{ch}(t_1 \cos\frac{j\pi}{n})\cos(t_1 \sin\frac{j\pi}{n})\Big].$$

If $b = 4m$, $m = 1, 2, \ldots,$

$$Jh^{(0)}_{1\cdot 4m}(t_1) = \frac{1}{2m}\Big[\cos t_1 + \mathrm{ch}t_1 + 2\sum_{j=1}^{m-1} \cos(t_1 \sin\frac{j\pi}{2m})\mathrm{ch}(t_1 \cos\frac{j\pi}{2m})\Big].$$

Definition 8.2. We define the Euler's formula of the positive hypercomplex numbers similar to the Euler's formula in the complex numbers [3],

$$\exp\left(\sum_{i=1}^{b-1} t_i J_b^i\right) = \sum_{i=1}^{b} S_i J_b^{i-1}, \tag{8.5}$$

where S_i is called the complex hyperbolic functions of order b with $(b-1)$ variables. Substituting (8.1)-(8.4) into (8.5) we can obtain S_i. Let $b = 1$, we have $S_1 = e^{t_1}$. Let $b = 2$, we have $\exp(t_1 J_2) = \mathrm{ch}t_1 + \mathrm{sh}t_1 J_2$. S_i have an identity

$$\begin{vmatrix} S_1 & S_b & \cdots & S_2 \\ S_2 & S_1 & \cdots & S_3 \\ S_3 & S_2 & \cdots & S_4 \\ \vdots & \vdots & & \vdots \\ S_b & S_{b-1} & \cdots & S_1 \end{vmatrix} = \begin{vmatrix} S_1 & (S_1)_1 & \cdots & (S_1)_{b-1} \\ S_2 & (S_2)_1 & \cdots & (S_2)_{b-1} \\ S_3 & (S_3)_1 & \cdots & (S_3)_{b-1} \\ \vdots & \vdots & & \vdots \\ S_b & (S_b)_1 & \cdots & (S_b)_{b-1} \end{vmatrix} = 1, \tag{8.6}$$

where $(S_i)_j = \frac{\partial S_i}{\partial t_j}$. From (8.6) we have the S_i Cauchy-Riemann equations

$$\begin{vmatrix} S_1 & S_b & \cdots & S_2 \\ \| & \| & \cdots & \| \\ (S_2)_1 & (S_1)_1 & \cdots & (S_3)_1 \\ \| & \| & \cdots & \| \\ (S_3)_2 & (S_2)_2 & \cdots & (S_4)_2 \\ \| & \| & \cdots & \| \\ \vdots & \vdots & & \vdots \\ \| & \| & \cdots & \| \\ (S_b)_{b-1} & (S_{b-1})_{b-1} & \cdots & (S_1)_{b-1} \end{vmatrix} \tag{8.7}$$

Let $S_i = \frac{y_i}{R^b}$, where

$$R^b = \begin{vmatrix} x_1 & x_b & \cdots & x_2 \\ x_2 & x_1 & \cdots & x_3 \\ \vdots & \vdots & & \vdots \\ x_b & x_{b-1} & \cdots & x_1 \end{vmatrix}$$

$$y_i = \sum_{\sum_{j=1}^{b} jb_j \equiv i-1 (\mathrm{mod} \quad b)} \frac{b!}{b_1!b_2! \cdots b_b!} x_1^{b_1} x_2^{b_2} \cdots x_b^{b_b}$$

$$\sum_{j=1}^{b} b_j = b$$

If $x_1, \cdots, x_b$ are integers, then S_i are rational numbers. We prove that (8.6) has infinitely many rational solutions. Let $b = 2$. We have

$$R^2 = x_1^2 - x_2^2, \ y_1 = x_1^2 + x_2^2, \ y_2 = 2x_1x_2.$$

Let =3. We have

$$R^3 = x_1^3 + x_2^3 + x_3^3 - 3x_1x_2x_3,$$

$$y_1 = x_1^3 + x_2^3 + x_3^3 + 6x_1x_2x_3,$$

$$y_2 = 3(x_1^2x_2 + x_2^2x_3 + x_3^2x_1),$$

$$y_3 = 3(x_1^2x_3 + x_2^2x_1 + x_3^2x_2).$$

We have the following formulas [3]

$$\sum_{j=0}^{n-1} S_i(j) = \frac{|S_i(n)|}{|S(1)|},$$

where

$$|S(1)| = \begin{vmatrix} S_1(1)-1 & S_b(1) & \cdots & S_2(1) \\ S_2(1) & S_1(1)-1 & \cdots & S_3(1) \\ \vdots & \vdots & & \vdots \\ S_b(1) & S_{b-1}(1) & \cdots & S_1(1)-1 \end{vmatrix},$$

$$S_1(n)-1, S_b(n), \cdots, S_2(n) \to S_1(1)-1, \ S_b(1), \cdots, \ S_2(1).$$

$$|S_1(n)| = |S_1(n)-1| = \begin{vmatrix} S_1(n)-1 & S_b(n) & \cdots & S_2(n) \\ S_2(1) & S_1(1)-1 & \cdots & S_3(1) \\ \vdots & \vdots & & \vdots \\ S_b(1) & S_{b-1}(1) & \cdots & S_1(1)-1 \end{vmatrix},$$

$$S_2(n), S_1(n)-1, \cdots, S_3(n) \to S_1(1)-1, S_b(1), \cdots, S_2(1)$$

$$|S_2(n)| = \begin{vmatrix} S_2(n) & S_1(n)-1 & \cdots & S_3(n) \\ S_2(1) & S_1(1)-1 & \cdots & S_3(1) \\ \vdots & \vdots & & \vdots \\ S_b(1) & S_{b-1}(1) & \cdots & S_1(1)-1 \end{vmatrix},$$

$$\cdots\cdots$$

$$S_b(n), S_{b-1}(n), \cdots, S_1(n)-1 \to S_1(1)-1, S_b(1), \cdots, S_2(1),$$

$$|S_b(n)| = \begin{vmatrix} S_b(n) & S_{b-1}(n) & \cdots & S_1(n)-1 \\ S_2(1) & S_1(1)-1 & \cdots & S_3(1) \\ \vdots & \vdots & & \vdots \\ S_b(1) & S_{b-1}(1) & \cdots & S_1(1)-1 \end{vmatrix}.$$

We have the following formulas [3]

$$\sum_{j=0}^{n-1}(-1)^j S_i(j) = \frac{|(-1)^{n-1}S_i(n)|}{|(S(1))|},$$

where

$$|(S(1))| = \begin{vmatrix} S_1(1)+1 & S_b(1) & \cdots & S_2(1) \\ S_2(1) & S_1(1)+1 & \cdots & S_3(1) \\ \vdots & \vdots & & \vdots \\ S_b(1) & S_{b-1}(1) & \cdots & S_1(1)+1 \end{vmatrix},$$

$$(-1)^{n-1}S_1(n)+1, (-1)^{n-1}S_b(n), \cdots, (-1)^{n-1}S_2(n) \to S_1(1)+1, S_b(1), \cdots, S_2(1).$$

$$|(-1)^{n-1}S_1(n)| = |(-1)^{n-1}S_1(n)+1|$$

$$= \begin{vmatrix} (-1)^{n-1}S_1(n)+1 & (-1)^{n-1}S_b(n) & \cdots & (-1)^{n-1}S_2(n) \\ S_2(1) & S_1(1)+1 & \cdots & S_3(1) \\ \vdots & \vdots & & \vdots \\ S_b(1) & S_{b-1}(1) & \cdots & S_1(1)+1 \end{vmatrix},$$

$$(-1)^{n-1}S_2(n), (-1)^{n-1}S_1(n)+1, \cdots, (-1)^{n-1}S_3(n) \to S_1(1)+1, S_b(1). \cdots, S_2(1),$$

$$|(-1)^{n-1}S_2(n)| = \begin{vmatrix} (-1)^{n-1}S_2(n) & (-1)^{n-1}S_1(n)+1 & \cdots & (-1)^{n-1}S_3(n) \\ S_2(1) & S_1(1)+1 & \cdots & S_3(1) \\ \vdots & \vdots & & \vdots \\ S_b(1) & S_{b-1}(1) & \cdots & S_1(1)+1, \end{vmatrix},$$

$$(-1)^{n-1}S_b(n), (-1)^{n-1}S_{b-1}(n), \cdots, (-1)^{n-1}S_1(n)+1 \to S_1(1)+1, S_b(1), \cdots, S_2(1),$$

$$|(-1)^{n-1}S_b(n)| = \begin{vmatrix} (-1)^{n-1}S_b(n) & (-1)^{n-1}S_{b-1}(n) & \cdots & (-1)^{n-1}S_1(n)+1 \\ S_2(1) & S_1(1)+1 & \cdots & S_3(1) \\ \vdots & \vdots & & \vdots \\ S_b(1) & S_{b-1}(1) & \cdots & S_1(1)+1 \end{vmatrix},$$

$$S_i(1) = S_i(t_1, \cdots, t_{b-1}) \text{ and } S_i(n) = S_i(n) = S_i(nt_1, \cdots, nt_{b-1}).$$

S_i have the recurrent formula similar to De Moivre formula

$$\sum_{i=1}^{b} S_i(mt) J_b^{i-1} = \left(\sum_{i=1}^{b} S_i(t) J_b^{i-1} \right)^m, \tag{8.8}$$

where $m = 0, \pm 1, \pm 2, \cdots,\ \ (t) = t(t_1, t_2, \cdots, t_{b-1}),\ \ (mt) = (mt_1, mt_2, \cdots, mt_{b-1})$.

Example 8.1. Let $b = 3$. From (8.5) we have

$$\exp(t_1 J_3 + t_2 J_3^2) = S_1 + S_2 J_3 + S_3 J_3^2,$$

where

$$S_1 = \frac{1}{3}(e^A + 2e^B \cos\theta)$$

$$S_2 = \frac{1}{3}[e^A - 2e^B \cos(\theta - \frac{\pi}{3})]$$

$$S_3 = \frac{1}{3}[e^A + 2e^B \cos(\theta - \frac{2\pi}{3})].$$

$$A = t_1 + t_2, \quad B = -\frac{t_1 + t_2}{2}, \quad \theta = \frac{\sqrt{3}}{2}(t_2 - t_1)$$

$$t_1 = \frac{1}{2}\left[\ln(S_1 + S_2 + S_3) - \frac{2}{\sqrt{3}} \tan^{-1} \frac{\sqrt{3}(-S_2 + S_3)}{2S_1 - S_2 - S_3} \right],$$

$$t_2 = \frac{1}{2}\left[\ln(S_1+S_2+S_3) + \frac{2}{\sqrt{3}}\tan^{-1}\frac{\sqrt{3}(-S_2+S_3)}{2S_1-S_2-S_3}\right].$$

From (8.6) we have

$$\begin{vmatrix} S_1 & S_3 & S_2 \\ S_2 & S_1 & S_3 \\ S_3 & S_2 & S_1 \end{vmatrix} = \begin{vmatrix} S_1 & (S_1)_1 & (S_1)_2 \\ S_2 & (S_2)_1 & (S_2)_2 \\ S_3 & (S_3)_1 & (S_3)_2 \end{vmatrix} = 1$$

From (8.7) we have

$$\begin{vmatrix} S_1 & S_3 & S_2 \\ \| & \| & \| \\ (S_2)_1 & (S_1)_1 & (S_3)_1 \\ \| & \| & \| \\ (S_3)_2 & (S_2)_2 & (S_1)_2 \end{vmatrix}$$

From (8.8) we have the recurrent formula

$$\sum_{i=1}^{3} S_i(mt_1, mt_2)J_3^{i-1} = (\sum_{i=1}^{3} S_i(t_1,t_2)J_3^{i-1})^m.$$

If $m = 2$, we have

$$S_1(2t_1,2t_2) = S_1^2(t_1,t_2) + 2S_2(t_1,t_2)S_3(t_1,t_2),$$

$$S_2(2t_1,2t_2) = S_3^2(t_1,t_2) + 2S_1(t_1,t_2)S_2(t_1,t_2),$$

$$S_3(2t_1,2t_2) = S_2^2(t_1,t_2) + 2S(t_1,t_2)S_3(t_1,t_2).$$

Definition 8.3. We define the trigonometric functions of order $b : Jh_{2.b}^{(i)}(t_1)$.[3]

$$\exp(t_1 I_n) = Jh_{2.b}^{(0)}(t_1) - \sum_{i=1}^{b-1} Jh_{2.b}^{(b-i)}(t_1)I_b^i, \tag{8.9}$$

where I_b denotes a b-th root of negative unity, $I_b^b = -1$.

$$Jh_{2.b}^{(0)}(t_1) = \sum_{i=0}^{\infty}\frac{t_1^{bi}}{(bi)!}(-1)^i = \frac{1}{b}\Big\{\sum_{j=1}^{b}[\exp(t_1\cos\frac{(2j+1)\pi}{b})]\cos(t_1\sin\frac{(2j+1)\pi}{b})\Big\} \tag{8.10}$$

$$Jh_{2.b}^{(1)}(t_1) = \frac{d}{dt_1}Jh_{2.b}^{(0)}(t_1)$$

$$= \frac{1}{b}\Big\{\sum_{j=1}^{b}[\exp(t_1\cos\frac{(2j+1)\pi}{b})]\cos(t_1\sin\frac{(2j+1)\pi}{b} + \frac{(2j+1)\pi}{b})\Big\}, \tag{8.11}$$

......

$$Jh_{2.b}^{(e)}(t_1) = \frac{d^e}{dt_1^e} Jh_{2.b}^{(0)}(t_1)$$

$$= \frac{1}{b}\left\{\sum_{j=1}^{b}[\exp(t_1 \cos\frac{(2j+1)\pi}{b})]\cos(t_1 \sin\frac{(2j+1)\pi}{b} + \frac{(2j+1)e\pi}{b})\right\}, \tag{8.12}$$

$$Jh_{2.b}^{(b)}(t_1) = -Jh_{2.b}^{(0)}(t_1)$$

If $b = 1$, then $Jh_{2.1}^{(0)}(t_1) = e^{-t_1}$. If $b = 2$, then $Jh_{2.2}^{(0)}(t_1) = \cos t_1$. If $b = n$ is an odd,

$$Jh_{2.n}^{(0)}(t_1) = \frac{1}{n}\left[e^{-t_1} + 2\sum_{j=1}^{\frac{n-1}{2}} \exp(t_1(-1)^{j+1}\cos\frac{j\pi}{n})\cos(t_1 \sin\frac{j\pi}{n})\right].$$

If $b = 2n$, where n is an odd, then

$$Jh_{2.2n}^{(0)}(t_1) = \frac{1}{n}\left[\cos t_1 + 2\sum_{j=0}^{\frac{n-3}{2}} \mathrm{ch}(t_1 \cos\frac{(2j+1)\pi}{2n})\cos(t_1 \sin\frac{(2j+1)\pi}{2n})\right].$$

If $b = 4m, m = 1, 2, ...,$

$$Jh_{2.4m}^{(0)}(t_1) = \frac{1}{m}\left[\sum_{j=0}^{m-1} \mathrm{ch}(t_1 \cos\frac{(2j+1)\pi}{4m})\cos(t_1 \sin\frac{(2j+1)\pi}{4m})\right].$$

Definition 8.4. We define the Euler's formula of the negative hypercomplex numbers similar to the Euler's formula in the complex numbers.[3]

$$\exp\left[\sum_{i=1}^{b-1} t_i I_b^i\right] = \sum_{i=1}^{b} S_i I_b^{i-1}, \tag{8.13}$$

where S_i is called the complex trigonometric functions of order b with $(b-1)$ variables. Substituting (8.1)–(8.4) and (8.9)–(8.12) into (8.13) we can obtain S_i. Let $b = 1$, we have $S_1 = e^{-t_1}$. Let $b = 2$, we have $\exp(t_1 I_2) = \cos t_1 + \sin t_1 I_2$. S_i have an identity

$$\begin{vmatrix} S_1 & -S_b & \cdots & -S_2 \\ S_2 & S_1 & \cdots & -S_3 \\ S_3 & S_2 & \cdots & -S_4 \\ \vdots & \vdots & & \vdots \\ S_b & S_{b-1} & \cdots & S_1 \end{vmatrix} = \begin{vmatrix} S_1 & (S_1)_1 & \cdots & (S_1)_{b-1} \\ S_2 & (S_2)_1 & \cdots & (S_3)_{b-1} \\ \vdots & \vdots & & \vdots \\ S_b & (S_b)_1 & \cdots & (S_b)_{b-1} \end{vmatrix} = 1, \tag{8.14}$$

where $(S_j)_i = \frac{\partial S_j}{\partial t_i}$. From (8.14) we have the S_i Cauchy-Riemann equations

$$\left| \begin{array}{cccc} S_1 & -S_b & \cdots & -S_2 \\ \| & \| & & \| \\ (S_2)_1 & (S_1)_1 & \cdots & -(S_3)_1 \\ \| & \| & & \| \\ (S_3)_2 & (S_2)_2 & \cdots & -(S_4)_2 \\ \| & \| & & \| \\ \vdots & \vdots & & \vdots \\ \| & \| & & \| \\ (S_b)_{b-1} & (S_{b-1})_{b-1} & \cdots & (S_1)_{b-1} \end{array} \right| . \tag{8.15}$$

Let $S_i = \frac{y_i}{R^b}$, where

$$R^b = \left| \begin{array}{cccc} x_1 & -x_b & \cdots & -x_2 \\ x_2 & x_1 & \cdots & -x_3 \\ \vdots & \vdots & & \vdots \\ x_b & x_{b-1} & \cdots & x_1 \end{array} \right|,$$

$$y_i = \sum_{\sum_{j=1}^{b} jb_j = i-1 (\bmod\ b)} \frac{b!}{b_1! b_2! \cdots b_b!} x_1^{b_1} x_2^{b_2} \cdots x_b^{b_b} (-1)^m,$$

$$\sum_{j=1}^{b} b_j = b.$$

$$m = \sum_{j=1}^{b} (j-1) b_j - i + 1.$$

If $x_1, \cdots, x_b$ are integers, then S_i are rational numbers. We prove that (8.14) has infinitely many rational solutions.Let $b = 2$. We have $R^2 = x_1^2 + x_2^2$, $y_1 = x_1^2 - x_2^2$, $y_2 = 2x_1x^2$. Let $b = 3$. We have $R^3 = x_1^3 - x_2^3 + x_3^3 + 3x_1x_2x_3$, $y_1 = x_1^3 - x_2^3 + x_3^3 - 6x_1x_2x_3$,$y_2 = 3(x_1^2x_2 - x_2^2x_3 - x_3^2x_1)$, $y_3 = 3(x_1^2x_3 + x_2^2x_1 - x_3^2x_2)$. We have the following formulas[3]

$$\sum_{j=0}^{n-1} S_i(j) = \frac{|S_i(n)|}{|S(1)|},$$

where

$$|S(1)| = \left| \begin{array}{cccc} S_1(1) - 1 & -S_b(1) & \cdots & -S_2(1) \\ S_2(1) & S_1(1) - 1 & \cdots & -S_3(1) \\ \vdots & \vdots & & \vdots \\ S_b(1) & S_{b-1}(1) & \cdots & S_1(1) - 1 \end{array} \right|,$$

$$S_1(n)-1,-S_b(n),\cdots,-S_2(n)\rightarrow S_1(1)-1,-S_b(1),\cdots,-S_2(1)$$

$$|S_1(n)|=|S_1(n)-1|=\begin{vmatrix} S_1(n)-1 & -S_b(n) & \cdots & -S_2(n)\\ S_2(1) & S_1(1)-1 & \cdots & -S_3(1)\\ \vdots & \vdots & & \vdots\\ S_b(1) & S_{b-1}(1) & \cdots & S_(1)-1 \end{vmatrix},$$

$$S_2(n),S_1(n)-1,\cdots,-S_3(n)\rightarrow S_1(1)-1,-S_b(1),\cdots,-S_2(1),$$

$$|S_2(n)|=\begin{vmatrix} S_2(n) & S_1(n)-1 & \cdots & -S_3(n)\\ S_2(1) & S_1(1)-1 & \cdots & -S_3(1)\\ \vdots & \vdots & & \vdots\\ S_b(1) & S_{b-1}(1) & \cdots & S_1(1)-1 \end{vmatrix},$$

$$\cdots\cdots$$

$$S_b(n),S_{b-1}(n),\cdots,S_1(n)-1\rightarrow S_1(1)-1,-S_b(1),\cdots,-S_2(1),$$

$$|S_b(n)|=\begin{vmatrix} S_b(n) & S_{b-1}(n) & \cdots & S_1(n)-1\\ S_2(1) & S_1(1)-1 & \cdots & -S_3(1)\\ \vdots & \vdots & & \vdots\\ S_b(1) & S_{b-1}(1) & \cdots & S_1(1)-1 \end{vmatrix}.$$

We have the following formulas [3]

$$\sum_{j=0}^{n-1}(-1)^j S_i(j)=\frac{|(-1)^{n-1}S_i(n)|}{|(S(1))|},$$

where

$$|(S(1))|=\begin{vmatrix} S_1(1)+1 & -S_b(1) & \cdots & -S_2(1)\\ S_2(1) & S_1(1)+1 & \cdots & -S_3(1)\\ \vdots & \vdots & & \vdots\\ S_b(1) & S_{b-1}(1) & \cdots & S_1(1)+1 \end{vmatrix},$$

$$(-1)^{n-1}S_1(n)+1,-(-1)^{n-1}S_b(n),\cdots,-(-1)^{n-1}S_2(n)\rightarrow S_1(1)+1,-S_b(1),\cdots,-S_2(1),$$

$$|(-1)^{n-1}S_1(n)| = \begin{vmatrix} (-1)^{n-1}S_1(n)+1 & -(-1)^{n-1}S_b(n) & \cdots & -(-1)^{n-1}S_2(n) \\ S_2(1) & S_1(1)+1 & \cdots & -S_3(1) \\ \vdots & \vdots & & \vdots \\ S_b(1) & S_{b-1}(1) & \cdots & S_1(1)+1 \end{vmatrix},$$

$$(-1)^{n-1}S_2(n), (-1)^{n-1}S_1(n)+1, \cdots, -(-1)^{n-1}S_3(n) \to S_1(1)+1, -S_b(1), \cdots, -S_2(1),$$

$$|(-1)^{n-1}S_2(n)| = \begin{vmatrix} (-1)^{n-1}S_2(n) & (-1)^{n-1}S_1(n)+1 & \cdots & -(-1)^{n-1}S_3(n) \\ S_2(1) & S_1(1)+1 & \cdots & -S_3(1) \\ \vdots & \vdots & & \vdots \\ S_b(1) & S_{b-1}(1) & \cdots & S_1(1)+1 \end{vmatrix},$$

$$\cdots\cdots$$

$$(-1)^{n-1}S_b(n), (-1)^{n-1}S_{b-1}(n), \cdots, (-1)^{n-1}S_1(n)+1 \to S_1(1)+1, -S_b(1), \cdots, -S_2(1),$$

$$|(-1)^{n-1}S_b(n)| = \begin{vmatrix} (-1)^{n-1}S_b(n) & (-1)^{n-1}S_{b-1}(n) & \cdots & (-1)^{n-1}S_1(n)+1 \\ S_2(1) & S_1(1)+1 & \cdots & -S_3(1) \\ \vdots & \vdots & & \vdots \\ S_b(1) & S_{b-1}(1) & \cdots & S_1(1)+1 \end{vmatrix},$$

$$S_i(1) = S_i(t_1, \cdots, t_{b-1}) \text{ and } S_i(n) = S_i(nt_1, \cdots, nt_{b-1})$$

S_i have the recurrent formula similar to De Moivre formula

$$\sum_{i=1}^{b} S_i(mt)I_b^{i-1} = \left(\sum_{i=1}^{b} S_i(t)I_b^{i-1}\right)^m, \tag{8.16}$$

where $m = 0, \pm1, \pm2, \cdots$, $(t) = (t_1, t_2, \cdots, t_{b-1})$, $(mt) = (mt_1, mt_2, \cdots, mt_{b-1})$.

Example 8.2. Let b=3. From (8.13) we have

$$\exp(t_1I_3 + t_2I_3^2) = S_1 + S_2I_3 + S_3I_3^2,$$

where

$$S_1 = \frac{1}{3}[e^A + 2e^B \cos\theta],$$

$$S_2 = \frac{-1}{3}[e^A - 2e^B \cos(\theta - \frac{\pi}{3})],$$
$$S_3 = \frac{1}{3}[e^A + 2e^B \cos(\theta - \frac{2\pi}{3})],$$
$$A = t_2 - t_1,\ B = \frac{t_1 - t_2}{2},\ \theta = \frac{\sqrt{3}}{2}(t_1 + t_2),$$
$$t_1 = \frac{1}{2}\left[\frac{2}{\sqrt{3}} \tan^{-1} \frac{\sqrt{3}(S_2 + S_3)}{2S_1 + S_2 - S_3} - \ln(S_1 - S_2 + S_3)\right],$$
$$t_2 = \frac{1}{2}\left[\frac{2}{\sqrt{3}} \tan^{-1} \frac{\sqrt{3}(S_2 + S_3)}{2S_1 + S_2 - S_3} + \ln(S_1 - S_2 + S_3)\right].$$

From (8.14) we have the identity

$$\begin{vmatrix} S_1 & -S_3 & -S_2 \\ S_2 & S_1 & -S_3 \\ S_3 & S_2 & S_1 \end{vmatrix} = \begin{vmatrix} S_1 & (S_1)_1 & (S_1)_2 \\ S_2 & (S_2)_1 & (S_2)_2 \\ S_3 & (S_3)_1 & (S_3)_2 \end{vmatrix} = 1.$$

From (8.15) we have the Cauchy-Riemann equations

$$\begin{vmatrix} S_1 & -S_3 & -S_2 \\ \| & \| & \| \\ (S_2)_1 & (S_1)_1 & -(S_3)_1 \\ \| & \| & \| \\ (S_3)2 & (S_2)_2 & (S_1)_2 \end{vmatrix}.$$

From (8.16) we have the recurrent formula

$$\sum_{i=1}^{3} S_i(mt_1, mt_2) I_3^{i-1} = \left(\sum_{i=1}^{3} S_i(t_1, t_2) I_3^{i-1}\right)^m.$$

Let $m = 2$, we have

$$S_1(2t_1, 2t_2) = S_1^2(t_1, t_2) - 2S_2(t_1, t_2)S_3(t_1, t_2),$$
$$S_2(2t_1, 2t_2) = -S_3^2(t_1, t_2) + 2S_1(t_1, t_2)S_2(t_1, t_2),$$
$$S_3(2t_1, 2t_2) = S_2^2(t_1, t_2) + 2S_1(t_1, t_2)S_3(t_1, t_2).$$

Definition 8.5. We define the positive hypercomplex numbers [3]

$$x = \begin{bmatrix} x_1 & x_n & \cdots & x_2 \\ x_2 & x_1 & \cdots & x_3 \\ x_3 & x_2 & \cdots & x_4 \\ \cdots & \cdots & \cdots & \cdots \\ x_n & x_{n-1} & \cdots & x_1 \end{bmatrix} = \sum_{i=1}^{n} x_i J_n^{i-1}, \tag{8.17}$$

where

$$J_n = \begin{bmatrix} 0 & 0 & \cdots & 1 \\ 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}, ..., J_n^n = 1,$$

$J_n, J_n^2, ..., J_n^{n-1}$ are called the bases of the positive hypercomplex numbers.

From (8.17) we have

$$\int \frac{dx}{x} = \log x = \int \frac{\sum_{i=1}^n dx_i J_n^{i-1}}{\sum_{i=1}^n x_i J_n^{i-1}} = \log R + \sum_{i=1}^{n-1} t_i J_n^i, \tag{8.18}$$

where R is called the modulus, t_i is called i-th argument,

$$\log R = \int \frac{1}{R^n} \begin{vmatrix} dx_1 & dx_n & \cdots & dx_2 \\ x_2 & x_1 & \cdots & x_3 \\ x_3 & x_2 & \cdots & x_4 \\ \cdots & \cdots & \cdots & \cdots \\ x_n & x_{n-1} & \cdots & x_1 \end{vmatrix}, t_1 = \int \frac{1}{R^n} \begin{vmatrix} dx_2 & dx_1 & \cdots & dx_3 \\ x_2 & x_1 & \cdots & x_3 \\ x_3 & x_2 & \cdots & x_4 \\ \cdots & \cdots & \cdots & \cdots \\ x_n & x_{n-1} & \cdots & x_1 \end{vmatrix},$$

$$t_2 = \int \frac{1}{R^n} \begin{vmatrix} dx_3 & dx_2 & \cdots & dx_4 \\ x_2 & x_1 & \cdots & x_3 \\ x_3 & x_2 & \cdots & x_4 \\ \cdots & \cdots & \cdots & \cdots \\ x_n & x_{n-1} & \cdots & x_1 \end{vmatrix}, \cdots, t_{n-1} = \int \frac{1}{R^n} \begin{vmatrix} dx_n & dx_{x-1} & \cdots & dx_1 \\ x_2 & x_1 & \cdots & x_3 \\ x_3 & x_2 & \cdots & x_4 \\ \cdots & \cdots & \cdots & \cdots \\ x_n & x_{n-1} & \cdots & x_1 \end{vmatrix},$$

$$R^n = \begin{vmatrix} x_1 & x_n & \cdots & x_2 \\ x_2 & x_1 & \cdots & x_3 \\ x_3 & x_2 & \cdots & x_4 \\ \cdots & \cdots & \cdots & \cdots \\ x_n & x_{n-1} & \cdots & x_1 \end{vmatrix}. \tag{8.19}$$

Let $n = 2$. From (8.19) we have

$$t_1 = \int \frac{\begin{vmatrix} dx_2 & dx_1 \\ x_2 & x_1 \end{vmatrix}}{\begin{vmatrix} x_1 & x_2 \\ x_2 & x_1 \end{vmatrix}} = \tanh^{-1} \frac{x_2}{x_1}.$$

Let $n = 3$. From (8.19) we have

$$t_1 = \int \frac{1}{R^3} \begin{vmatrix} dx_2 & dx_1 & dx_3 \\ x_2 & x_1 & x_3 \\ x_3 & x_2 & x_1 \end{vmatrix} = \frac{1}{2}\left(\log \frac{x_1 + x_2 + x_3}{R} - \frac{2}{\sqrt{3}} \tan^{-1} \frac{\sqrt{3}(-x_2 + x_3)}{2x_1 - x_2 - x_3} \right),$$

$$t_2 = \int \frac{1}{R^3} \begin{vmatrix} dx_3 & dx_2 & dx_1 \\ x_2 & x_1 & x_1 \\ x_3 & x_2 & x_1 \end{vmatrix} = \frac{1}{2}\left(\log \frac{x_1 + x_2 + x_3}{R} + \frac{2}{\sqrt{3}} \tan^{-1} \frac{\sqrt{3}(-x_2 + x_3)}{2x_1 - x_2 - x_3} \right),$$

Let $n = 4$. From (8.19) we have

$$t_1 = \int \frac{1}{R^4} \begin{vmatrix} dx_2 & dx_1 & dx_4 & dx_3 \\ x_2 & x_1 & x_4 & x_3 \\ x_3 & x_2 & x_1 & x_4 \\ x_4 & x_3 & x_2 & x_1 \end{vmatrix} = \frac{1}{2}\left(\log \frac{x_1 + x_2 + x_3 + x_4}{x_1 - x_2 + x_3 - x_4} - \tan^{-1} \frac{-x_2 + x_4}{x_1 - x_3} \right),$$

$$t_2 = \int \frac{1}{R^4} \begin{vmatrix} dx_3 & dx_2 & dx_1 & dx_4 \\ x_2 & x_1 & x_4 & x_3 \\ x_3 & x_2 & x_1 & x_4 \\ x_4 & x_3 & x_2 & x_1 \end{vmatrix} = \frac{1}{2} \log \frac{(x_1 + x_3)^2 - (x_2 + x_4)^2}{R_2},$$

$$t_3 = \int \frac{1}{R^4} \begin{vmatrix} dx_4 & dx_3 & dx_2 & dx_1 \\ x_2 & x_1 & x_4 & x_3 \\ x_3 & x_2 & x_1 & x_4 \\ x_4 & x_3 & x_2 & x_1 \end{vmatrix} = \frac{1}{2}\left(\log \frac{x_1 + x_2 + x_3 + x_4}{x_1 - x_2 + x_3 - x_4} + \tan^{-1} \frac{(-x_2 + x_4)}{x_1 - x_3} \right).$$

Let $n = 5$. From (8.19) we have

$$\begin{aligned} \begin{matrix} t_1 \\ t_4 \end{matrix} &= \frac{1}{2}\left(- \frac{B_1 \cos\frac{\pi}{5} + B_2 \cos\frac{2\pi}{5}}{\sqrt{0.325}} \pm \frac{\theta_1 \sin\frac{\pi}{5} - \theta_2 \sin\frac{2\pi}{5}}{1.25} \right), \\ \begin{matrix} t_3 \\ t_2 \end{matrix} &= \frac{1}{2}\left(- \frac{B_1 \cos\frac{2\pi}{5} + B_2 \cos\frac{\pi}{5}}{\sqrt{0.325}} \pm \frac{\theta_1 \sin\frac{2\pi}{5} + \theta_2 \sin\frac{\pi}{5}}{1.25} \right), \end{aligned}$$

where

$$B_1 = \frac{1}{2} \log \frac{[(-x_2 + x_5)\sin\frac{\pi}{5} + (x_3 - x_4)\sin\frac{2\pi}{5}]^2 + [x_1 - (x_2 + x_5)\cos\frac{\pi}{5} + (x_3 + x_4)\cos\frac{2\pi}{5}]^2}{R^2},$$

$$\theta_1 = \tan^{-1} \frac{(-x_2 + x_5)\sin\frac{\pi}{5} + (x_3 - x_4)\sin\frac{2\pi}{5}}{x_1 - (x_2 + x_5)\cos\frac{\pi}{5} + (x_3 + x_4)\cos\frac{2\pi}{5}},$$

$$B_2 = \frac{1}{2} \log \frac{[(-x_2 + x_5)\sin\frac{2\pi}{5} + (-x_3 + x_4)\sin\frac{\pi}{5}]^2 + [x_1 + (x_2 + x_5)\cos\frac{2\pi}{5} - (x_3 + x_4)\cos\frac{\pi}{5}]^2}{R^2},$$

$$\theta_2 = \tan^{-1} \frac{(-x_2 + x_5)\sin\frac{2\pi}{5} + (-x_3 + x_4)\sin\frac{\pi}{5}}{x_1 + (x_2 + x_5)\cos\frac{2\pi}{5} - (x_3 + x_4)\cos\frac{\pi}{5}},$$

Let $n = 6$. From (8.19) we have

$$t_3 = \frac{1}{6}(2B_1 - 2D_1 + A_2 - A_1),$$

$$\begin{matrix} t_2 \\ t_4 \end{matrix} = \frac{1}{2}\Big(- B_1 - D_1 \pm \frac{\theta_1 - \phi_1}{\sqrt{3}}\Big)$$

$$\begin{matrix} t_5 \\ t_1 \end{matrix} = \frac{1}{2}\Big(\frac{-B_1 + D_1 + A_1 - A_2}{3} \pm \frac{\theta_1 + \phi_1}{\sqrt{3}}\Big)$$

where

$$A_1 = \log \frac{\sum_{i=1}^{6} x_i}{R}, \quad A_2 = \log \frac{\sum_{i=1}^{6} x_i(-1)^{i+1}}{R},$$

$$B_1 = \frac{1}{2}\log \frac{[x_1 + x_4 - \frac{1}{2}(x_2 + x_3 + x_5 + x_6)]^2 + \frac{3}{4}(-x_2 + x_3 - x_5 + x_6)^2}{R^2}$$

$$\theta_1 = \tan^{-1} \frac{\sqrt{3}(-x_2 + x_3 - x_5 + x_6)}{2x_1 + 2x_4 - x_2 - x_3 - x_5 - x_6}$$

$$D_1 = \frac{1}{2}\log \frac{[x_1 - x_4 + \frac{1}{2}(x_2 - x_3 - x_5 + x_6)]^2 + \frac{3}{4}(-x_2 - x_3 + x_5 + x_6)^2}{R^2},$$

$$\phi_1 = \tan^{-1} \frac{\sqrt{3}(-x_2 - x_3 + x_5 + x_6)}{2x_1 - 2x_4 + x_2 - x_3 - x_5 + x_6},$$

We can immediately translate many concepts from complex variables into the positive hypercomplex notation. From (8.18) we have the exponential formula of the positive hypercomplex numbers

$$x = R\exp\Big(\sum_{i=1}^{n-1} t_i J_n^i\Big) = R\sum_{i=1}^{n} S_i J_n^{i-1}. \tag{8.20}$$

If $x_i \neq 0$ and $R = 0$, then there are zero divisors in the positive hypercomplex numbers. Therefore the positive hypercomplex numbers H form a ring.

Let $A \subset H$ be an open set and let $f : A \to H$ be a given function

$$f(x) = \begin{bmatrix} y_1 & y_n & \cdots & y_2 \\ y_2 & y_1 & \cdots & y_3 \\ y_3 & y_2 & \cdots & y_4 \\ \cdots & \cdots & \cdots & \cdots \\ y_n & y_{n-1} & \cdots & y_1 \end{bmatrix} = \sum_{i=1}^{n} y_i J_n^{i-1}. \tag{8.21}$$

Theorem 8.1. Let $f : A \subset H \to H$ be a given function, with A an open set. Then $f'(x_0)$ exists if and only if f is differentiable in the sense of real variables and at $(x_{10}, \cdots, x_{n0}) = x_0, y_i$ satisfy

$$\begin{bmatrix} (y_1)_1 & (y_n)_1 & \cdots & (y_2)_1 \\ \| & \| & \cdots & \| \\ (y_2)_2 & (y_1)_2 & \cdots & (y_3)_2 \\ \| & \| & \cdots & \| \\ (y_3)_3 & (y_2)_3 & \cdots & (y_4)_3 \\ \cdots & \cdots & \cdots & \cdots \\ \| & \| & \cdots & \| \\ (y_n)_n & (y_{n-1})_n & \cdots & (y_1)_n \end{bmatrix}, \tag{8.22}$$

where $(y_i)_j = \frac{\partial}{\partial x_j} y_i$, called the Cauchy-Riemann equations.

Thus, if $(y_i)_j$ exist, are continuous on A, and satisfy the Cauchy-Riemann equations, then f is analytic on A. Since $f'(x_0)$ exists and has the same value regardless of how x approaches x_0.

Let $n = 2$. From (8.22) we have the Cauchy-Riemann equations

$$\frac{\partial y_1}{\partial x_1} = \frac{\partial y_2}{\partial x_2}, \frac{\partial y_1}{\partial x_2} = \frac{\partial y_2}{\partial x_1}.$$

Let $n = 3$. From (8.22) we have the Cauchy-Riemann equations

$$\frac{\partial y_1}{\partial x_1} = \frac{\partial y_2}{\partial x_2} = \frac{\partial y_3}{\partial x_3}, \frac{\partial y_3}{\partial x_1} = \frac{\partial y_1}{\partial x_2} = \frac{\partial y_2}{\partial x_3}, \frac{\partial y_2}{\partial x_1} = \frac{\partial y_3}{\partial x_2} = \frac{\partial y_1}{\partial x_3}.$$

Any function satisfying (8.22) is said to be analytic. y_i are n-times continuously differentiable functions and differentiating the Cauchy-Riemann equations (8.22). We obtain the differential equation of hyperbolic type

$$\begin{vmatrix} \frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_n} & \cdots & \frac{\partial}{\partial x_2} \\ \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_1} & \cdots & \frac{\partial}{\partial x_3} \\ \frac{\partial}{\partial x_3} & \frac{\partial}{\partial x_2} & \cdots & \frac{\partial}{\partial x_4} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial}{\partial x_n} & \frac{\partial}{\partial x_{n-1}} & \cdots & \frac{\partial}{\partial x_1} \end{vmatrix} y_1 = 0 \tag{8.23}$$

Any function satisfying (8.23) is said to be harmonic. In a similar fashion $y_2, \cdots, y_n$ are also harmonic.

Let $n = 2$. From (8.23) we have

$$\frac{\partial^2 y_1}{\partial^2 x_1} - \frac{\partial^2 y_1}{\partial^2 x_2} = 0.$$

Let $n = 3$. From (8.23) we have

$$\frac{\partial^3 y_1}{\partial^3 x_1} + \frac{\partial^3 y_1}{\partial^3 x_2} + \frac{\partial^3 y_1}{\partial^3 x_3} - 3\frac{\partial^3 y_1}{\partial x_1 \partial x_2 \partial x_3} = 0.$$

From the above analysis we come to a conclusion that the above equations are isomorphic to the matrix in (8.17).

We discover a relation between the differential equation and the hypercomplex number.

Definition 8.6. We define the negative hypercomplex numbers

$$x = \begin{bmatrix} x_1 & -x_n & -x_{n-1} & \cdots & -x_2 \\ x_2 & x_1 & -x_n & \cdots & -x_3 \\ x_3 & x_2 & x_1 & \cdots & -x_4 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ x_n & x_{n-1} & x_{n-2} & \cdots & x_1 \end{bmatrix} = \sum_{i=1}^{n} x_i I_n^{i-1}, \tag{8.24}$$

where

$$I_n = \begin{bmatrix} 0 & 0 & \cdots & -1 \\ 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}, ..., I_n^n = -1,$$

$I_n, \cdots, I_n^{n-1}$ are called the bases of the negative hypercomplex numbers.

From (8.24) we have

$$\int \frac{dx}{x} = \log x = \int \frac{\sum_{i=1}^{n} dx_i I_n^{i-1}}{\sum_{i=1}^{n} x_i I_n^{i-1}} = \log R + \sum_{i=1}^{n-1} t_i I_n^i, \tag{8.25}$$

where R is called the modulus, t_i is called i-th argument,

$$\log R = \int \frac{1}{R^n} \begin{vmatrix} dx_1 & -dx_n & \cdots & -dx_2 \\ x_2 & x_1 & \cdots & -x_3 \\ x_3 & x_2 & \cdots & -x_4 \\ \cdots & \cdots & \cdots & \cdots \\ x_n & x_{n-1} & \cdots & x_1 \end{vmatrix},$$

$$t_1 = \int \frac{1}{R^n} \begin{vmatrix} dx_2 & dx_1 & \cdots & -dx_3 \\ x_2 & x_1 & \cdots & -x_3 \\ x_3 & x_2 & \cdots & -x_4 \\ \cdots & \cdots & \cdots & \cdots \\ x_n & x_{n-1} & \cdots & x_1 \end{vmatrix},$$

$$t_2 = \int \frac{1}{R^n} \begin{vmatrix} dx_3 & dx_2 & \cdots & -dx_4 \\ x_2 & x_1 & \cdots & -x_3 \\ x_3 & x_2 & \cdots & -x_4 \\ \cdots & \cdots & \cdots & \cdots \\ x_n & x_{n-1} & \cdots & x_1 \end{vmatrix},$$

$$\cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots$$

$$t_{n-1} = \int \frac{1}{R^n} \begin{vmatrix} dx_n & dx_{n-1} & \cdots & dx_1 \\ x_2 & x_1 & \cdots & -x_3 \\ x_3 & x_2 & \cdots & -x_4 \\ \cdots & \cdots & \cdots & \cdots \\ x_n & x_{n-1} & \cdots & x_1 \end{vmatrix},$$

$$R^n = \begin{vmatrix} x_1 & -x_n & \cdots & -x_2 \\ x_2 & x_1 & \cdots & -x_3 \\ x_3 & x_2 & \cdots & -x_4 \\ \cdots & \cdots & \cdots & \cdots \\ x_n & x_{n-1} & \cdots & x_1 \end{vmatrix}. \tag{8.26}$$

Let $n = 2$. From (8.26) we have

$$t_1 = \int \frac{\begin{vmatrix} dx_2 & dx_1 \\ x_2 & x_1 \end{vmatrix}}{\begin{vmatrix} x_1 & -x_2 \\ x_2 & x_1 \end{vmatrix}} = \tan^{-1} \frac{x_2}{x_1}.$$

Let $n = 3$. From (8.26) we have

$$t_1 = \int \frac{1}{R^3} \begin{vmatrix} dx_2 & dx_1 & -dx_3 \\ x_2 & x_1 & -x_3 \\ x_3 & x_2 & x_1 \end{vmatrix} = \frac{1}{2}\left(\frac{2}{\sqrt{3}} \tan^{-1} \frac{\sqrt{3}(x_2 + x_3)}{2x_1 + x_2 - x_3} - \log \frac{x_1 - x_2 + x_3}{R} \right),$$

$$t_2 = \int \frac{1}{R^3} \begin{vmatrix} dx_3 & dx_2 & dx_1 \\ x_2 & x_1 & -x_3 \\ x_3 & x_2 & x_1 \end{vmatrix} = \frac{1}{2}\left(\frac{2}{\sqrt{3}} \tan^{-1} \frac{\sqrt{3}(x_2 + x_3)}{2x_1 + x_2 - x_3} + \log \frac{x_1 - x_2 + x_3}{R} \right).$$

Let $n = 4$. From (8.26) we have

$$t_2 = \frac{1}{2} \tan^{-1} \frac{-x_2^2 + x_4^2 + 2x_1 x_2}{x_1^2 - x_3^2 + 2x_2 x_4},$$

$$\begin{matrix} t_1 \\ t_3 \end{matrix} = \frac{1}{2\sqrt{2}} \left(\tan^{-1} \frac{\sqrt{2}(x_1 x_2 - x_2 x_3 + x_3 x_4 + x_1 x_4)}{x_1^2 - x_2^2 + x_3^2 - x^2 x_4} \right.$$

$$\left. \mp \log \frac{x_1^2 + x_2^2 + x_3^2 + x_4^2 - \sqrt{2}(x_1 x_2 + x_2 x_3 + x_3 x_4 - x_1 x_4)}{R^2} \right).$$

Let $n = 5$ From (8.26) we have

$$\begin{matrix} t_1 \\ t_4 \end{matrix} = \frac{1}{2}\left(\frac{\theta_1 \sin \frac{\pi}{5} + \theta_2 \sin \frac{2\pi}{5}}{1.25} \pm \frac{B_1 \cos \frac{\pi}{5} + B_2 \cos \frac{2\pi}{5}}{\sqrt{0.3125}} \right),$$

$$\begin{matrix} t_2 \\ t_3 \end{matrix} = \frac{1}{2}\Big(\frac{\theta_1 \sin\frac{2\pi}{5} - \theta_2 \sin\frac{\pi}{5}}{1.25} \mp \frac{B_1 \cos\frac{2\pi}{5} + B_2 \cos\frac{\pi}{5}}{\sqrt{0.3125}}\Big),$$

where

$$B_1 = \frac{1}{2}\log\frac{[x_1 + (x_2 - x_5)\cos\frac{\pi}{5} + (x_3 - x_4)\cos\frac{2\pi}{5}]^2 + [(x_2 + x_5)\sin\frac{\pi}{5} + (x_3 + x_4)\sin\frac{2\pi}{5}]^2}{R^2},$$

$$\theta_1 = \tan^{-1}\frac{(x_2 + x_5)\sin\frac{\pi}{5} + (x_3 + x_4)\sin\frac{2\pi}{5}}{x_1 + (x_2 - x_5)\cos\frac{\pi}{5} + (x_3 - x_4)\cos\frac{2\pi}{5}},$$

$$B_2 = \frac{1}{2}\log\frac{[x_1 + (-x_2 + x_5)\cos\frac{2\pi}{5} + (-x_3 + x_4)\cos\frac{\pi}{5}]^2 + [(x_2 + x_5)\sin\frac{2\pi}{5} - (x_3 + x_4)\sin\frac{\pi}{5}]^2}{R^2},$$

$$\theta_2 = \tan^{-1}\frac{(x_2 + x_5)\sin\frac{2\pi}{5} - (x_3 + x_4)\sin\frac{\pi}{5}}{x_1 + (-x_2 + x_5)\cos\frac{2\pi}{5} + (-x_3 + x_4)\cos\frac{\pi}{5}},$$

Let $n = 6$. From (8.26) we have

$$t_3 = \frac{\theta_0 + \phi_0 - \beta}{3}, \quad \begin{matrix} t_1 \\ t_5 \end{matrix} = \frac{1}{2}\Big(\frac{\theta_0 + \phi_0 + 2\beta}{\sqrt{3}} \mp \frac{B_0 - D_0}{\sqrt{3}}\Big), \ \begin{matrix} t_2 \\ t_4 \end{matrix} = \frac{1}{2}\Big(\frac{\phi_0 - \theta_0}{\sqrt{3}} \mp H\Big).$$

where

$$\beta = \tan^{-1}\frac{x_2 - x_4 + x_6}{x_1 - x_3 + x_5}, \quad H = \frac{1}{2}\log\frac{(x_1 - x_3 + x_5)^2 + (x_2 - x_4 + x_6)^2}{R^2},$$

$$B_0 = \frac{1}{2}\log\frac{[x_1 + (-x_2 + x_6)\frac{\sqrt{3}}{2} + \frac{x_3 - x_5}{2}]^2 + [x_4 - (x_3 + x_5)\frac{\sqrt{3}}{2} + \frac{x_2 + x_6}{2}]^2}{R^2},$$

$$\theta_0 = \tan^{-1}\frac{2x_4 + x_2 + x_6 - \sqrt{3}(x_3 + x_5)}{2x_1 + x_3 - x_5 + \sqrt{3}(-x_2 + x_6)},$$

$$D_0 = \frac{1}{2}\log\frac{[x_1 + (x_2 - x_6)\frac{\sqrt{3}}{2} + \frac{x_3 - x_5}{2}]^2 + [x_4 + (x_3 + x_5)\frac{\sqrt{3}}{2} + \frac{x_2 + x_6}{2}]^2}{R^2},$$

$$\phi_0 = \tan^{-1}\frac{2x_4 + x_2 + x_6 + \sqrt{3}(x_3 + x_5)}{2x_1 + x_3 - x_5 + \sqrt{3}(x_2 - x_6)}.$$

We can immediately translate many concepts from complex variables into the negative hypercomplex notation. From (8.25) we have the exponential formula of the negative hypercomplex numbers

$$x = R\exp\Big(\sum_{i=1}^{n-1} t_i I_n^i\Big) = R\sum_{i=1}^{n} S_i I_n^i \tag{8.27}$$

If $x_i \neq 0$ and $R = 0$, then there are zero divisors in the negative hypercomplex numbers. Therefore the negative hypercomplex numbers H form a ring.

Let $A \subset H$ be an open set and let $f : A \to H$ be a given function

$$f(x) = \begin{bmatrix} y_1 & -y_n & \cdots & -y_2 \\ y_2 & y_1 & \cdots & -y_3 \\ y_3 & y_2 & \cdots & -y_4 \\ \cdots & \cdots & \cdots & \cdots \\ y_n & y_{n-1} & \cdots & y_1 \end{bmatrix} = \sum_{i=1}^{n} y_i I_n^{i-1}. \tag{8.28}$$

Theorem 8.2. Let $f : A \subset H \to H$ be a given function, with A an open set. Then $f'(x_0)$ exists if and only if f is differentiable in the sense of real variables and at $(x_{10}, \cdots, x_{n0}) = x_0, y_i$ satisfy

$$\begin{bmatrix} (y_1)_1 & -(y_n)_1 & \cdots & -(y_2)_1 \\ || & || & \cdots & || \\ (y_2)_2 & (y_1)_2 & \cdots & -(y_3)_2 \\ || & || & \cdots & || \\ (y_3)_3 & (y_2)_3 & \cdots & -(y_4)_3 \\ \cdots & \cdots & \cdots & \cdots \\ || & || & \cdots & || \\ (y_n)_n & (y_{n-1})_n & \cdots & (y_1)_n \end{bmatrix}, \tag{8.29}$$

where $(y_i)_j = \frac{\partial}{\partial x_j} y_i$, called the Cauchy Riemann equations.

Thus, if y_i exist, are continuous on A, and satisfy the Cauchy Riemann equations, then f is analytic on A. Since $f'(x_0)$ exists and has the same value regardless of how x approaches x_0.

Let $n = 2$. From (8.29) we have the Cauchy-Riemann equations

$$\frac{\partial y_1}{\partial x_1} = \frac{\partial y_2}{\partial x_2}, -\frac{\partial y_2}{\partial x_1} = \frac{\partial y_1}{\partial x_2}.$$

Let $n = 3$. From (8.29) we have the Cauchy Riemann equations

$$\frac{\partial y_1}{\partial x_1} = \frac{\partial y_2}{\partial x_2} = \frac{\partial y_3}{\partial x_3}, -\frac{\partial y_3}{\partial x_1} = \frac{\partial y_1}{\partial x_2} = \frac{\partial y_2}{\partial x_3}, -\frac{\partial y_2}{\partial x_1} = -\frac{\partial y_3}{\partial x_2} = \frac{\partial y_1}{\partial x_3}.$$

Any function satisfying (8.29) is said to be analytic. y_i are n-times continuously differentiable functions and differentiating the Cauchy-Riemann equations (8.29). We obtain the differential equation of elliptic type

$$\begin{vmatrix} \frac{\partial}{\partial x_1} & -\frac{\partial}{\partial x_n} & \cdots & -\frac{\partial}{\partial x_2} \\ \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_1} & \cdots & -\frac{\partial}{\partial x_3} \\ \frac{\partial}{\partial x_3} & \frac{\partial}{\partial x_2} & \cdots & -\frac{\partial}{\partial x_4} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial}{\partial x_n} & \frac{\partial}{\partial x_{n-1}} & \cdots & \frac{\partial}{\partial x_1} \end{vmatrix} y_1 = 0 \tag{8.30}$$

Any function satisfying (8.30) is said to be harmonic. In a similar fashion $y_2, \cdots, y_n$ are also harmonic.

Let $n = 2$. From (8.30) we have

$$\frac{\partial^2 y_1}{\partial^2 x_1} + \frac{\partial^2 y_1}{\partial^2 x_2} = 0.$$

Let $n = 3$. From (8.30) we have

$$\frac{\partial^3 y_1}{\partial^3 x_1} - \frac{\partial^3 y_1}{\partial^3 x_2} + \frac{\partial^3 y_1}{\partial^3 x_3} + 3\frac{\partial^3 y_1}{\partial x_1 \partial x_2 \partial x_3} = 0.$$

From the above analysis we come to a conclusion that the above equations are isomorphic to the matrix in (8.24).

We discover a relation between the differential equation and the hypercomplex number.

Definition 8.7. We define the negative-positive hypercomplex numbers

$$x = \begin{bmatrix} x_1 & -x_2 & x_3 & -x_4 \\ x_2 & x_1 & x_4 & x_3 \\ x_3 & -x_4 & x_1 & -x_2 \\ x_4 & x_3 & x_2 & x_1 \end{bmatrix} = x_1 + x_2 I_2 + x_3 J_2 + x_4 k =$$

$$= R\exp(t_1 I_2 + t_2 J_2 + t_3 k) = R(S_1 + S_2 I_2 + S_3 J_2 + S_4 k), \tag{8.31}$$

where

$$I_2 = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}, I_2^2 = -1,$$

$$J_2 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, J_2^2 = 1,$$

$$k = I_2 J_2 = \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}, k^2 = -1,$$

$$t_1 = \int \frac{1}{R^4} \begin{vmatrix} dx_2 & dx_1 & dx_4 & dx_3 \\ x_2 & x_1 & x_4 & x_3 \\ x_3 & -x_4 & x_1 & -x_2 \\ x_4 & x_3 & x_2 & x_1 \end{vmatrix} = \frac{1}{2}\tan^{-1}\frac{2x_1x_2 - 2x_3x_4}{x_1^2 + x_4^2 - x_2^2 - x_3^2},$$

$$t_2 = \int \frac{1}{R^4} \begin{vmatrix} dx_3 & -dx_4 & dx_1 & -dx_2 \\ x_2 & x_1 & x_4 & x_3 \\ x_3 & -x_4 & x_1 & -x_2 \\ x_4 & x_3 & x_2 & x_1 \end{vmatrix} = \frac{1}{2} \tanh^{-1} \frac{2x_1x_3 + 2x_2x_4}{x_1^2 + x_2^2 + x_3^2 + x_4^2},$$

$$t_3 = \int \frac{1}{R^4} \begin{vmatrix} dx_4 & dx_3 & dx_2 & dx_1 \\ x_2 & x_1 & x_4 & x_3 \\ x_3 & -x_4 & x_1 & -x_2 \\ x_4 & x_3 & x_2 & x_1 \end{vmatrix} = \frac{1}{2} \tan^{-1} \frac{-2x_2x_3 + 2x_1x_4}{x_1^2 + x_2^2 - x_3^2 - x_4^2},$$

$$S_1 = \frac{1}{2}\Big(e^{t_2}\cos(t_1+t_3) + e^{-t_2}\cos(t_1-t_3)\Big),$$

$$S_2 = \frac{1}{2}\Big(e^{t_2}\sin(t_1+t_3) + e^{-t_2}\sin(t_1-t_3)\Big),$$

$$S_3 = \frac{1}{2}\Big(e^{t_2}\cos(t_1+t_3) - e^{-t_2}\cos(t_1-t_3)\Big),$$

$$S_4 = \frac{1}{2}\Big(e^{t_2}\sin(t_1+t_3) - e^{-t_2}\sin(t_1-t_3)\Big).$$

We have S_i an identity

$$\begin{vmatrix} S_1 & -S_2 & S_3 & -S_4 \\ S_2 & S_1 & S_4 & S_3 \\ S_3 & -S_4 & S_1 & -S_2 \\ S_4 & S_3 & S_2 & S_1 \end{vmatrix} = \begin{vmatrix} S_1 & (S_1)_1 & (S_1)_2 & (S_1)_3 \\ S_2 & (S_2)_1 & (S_2)_2 & (S_2)_3 \\ S_3 & (S_3)_1 & (S_3)_2 & (S_3)_3 \\ S_4 & (S_4)_1 & (S_4)_2 & (S_4)_3 \end{vmatrix} = 1. \qquad (8.32)$$

From (8.31) we have De Moivre formula

$$S_1(m)+S_2(m)I_2+S_3(m)J_2+S_4(m)k = [S_1(1)+S_2(1)I_2+S_3(1)J_2+S_4(1)k]^m. \quad (8.33)$$

where $S_i(1) = S_i(t_1, t_2, t_3)$, $S_i(m) = S_i(mt_1, mt_2, mt_3)$.

Let $n = 2$. From (8.33) we have

$$S_1(2) = S_1^2(1) - S_2^2(1) + S_3^2(1) - S_4^2(1), \quad S_2(2) = 2S_1(1)S_2(1) + 2S_3(1)S_4(1),$$

$$S_3(2) = 2S_1(1)S_3(1) - 2S_2(1)S_4(1), \quad S_4(2) = 2S_1(1)S_4(1) + 2S_2(1)S_3(1).$$

From (8.33) we have

$$\sum_{i=1}^{n-1} S_1(i) = \frac{1}{|\Delta|} \begin{vmatrix} S_1(n)-1 & -S_2(n) & S_3(n) & -S_4(n) \\ S_2(1) & S_1(1)-1 & S_4(1) & S_3(1) \\ S_3(1) & -S_4(1) & S_1(1)-1 & -S_2(1) \\ S_4(1) & S_3(1) & S_2(1) & S_1(1)-1 \end{vmatrix},$$

$$\sum_{i=1}^{n-1} S_2(i) = \frac{1}{|\Delta|} \begin{vmatrix} S_2(n) & S_(n)-1 & S_4(n) & S_3(n) \\ S_2(1) & S_1(1)-1 & S_4(1) & S_3(1) \\ S_3(1) & -S_4(1) & S_1(1)-1 & -S_2(1) \\ S_4(1) & S_3(1) & S_2(1) & S_1(1)-1 \end{vmatrix},$$

$$\sum_{i=1}^{n-1} S_3(i) = \frac{1}{|\Delta|} \begin{vmatrix} S_3(n) & -S_4(n) & S_1(n)-1 & -S_2(n) \\ S_2(1) & S_1(1)-1 & S_4(1) & S_3(1) \\ S_3(1) & -S_4(1) & S_1(1)-1 & -S_2(1) \\ S_4(1) & S_3(1) & S_2(1) & S_1(1)-1 \end{vmatrix},$$

$$\sum_{i=1}^{n-1} S_4(i) = \frac{1}{|\Delta|} \begin{vmatrix} S_4(n) & S_3(n) & S_2(n) & S_1(n)-1 \\ S_2(1) & S_1(1)-1 & S_4(1) & S_3(1) \\ S_3(1) & -S_4(1) & S_1(1)-1 & -S_2(1) \\ S_4(1) & S_3(1) & S_2(1) & S_1(1)-1 \end{vmatrix},$$

where

$$|\Delta| = \begin{vmatrix} S_1(1)-1 & -S_2(1) & S_3(1) & -S_4(-1) \\ S_2(1) & S_1(1)-1 & S_4(1) & S_3(1) \\ S_3(1) & -S_4(1) & S_1(1)-1 & -S_2(1) \\ S_4(1) & S_3(1) & S_2(1) & S_1(1)-1 \end{vmatrix},$$

If $x_i \neq 0$ and $R = 0$, then there are zero divisors in it. Therefore the negative-positive hypercomplex numbers H form a ring.

Let $A \subset H$ be an open set and let $f : A \to H$ be a given function

$$f(x) = \begin{bmatrix} y_1 & -y_2 & y_3 & -y_4 \\ y_2 & y_1 & y_4 & y_3 \\ y_3 & -y_4 & y_1 & -y_2 \\ y_4 & y_3 & y_2 & y_1 \end{bmatrix} = y_1 + y_2 I_2 + y_3 J_2 + y_4 k \tag{8.34}$$

Theorem 8.3. Let $f : A \subset H \to H$ be a given function, with A an open set. Then $f'(x_0)$ exists if and only if f is differentiable in the sense of real variables and at $(x_{10}, x_{20}, x_{30}, x_{40}) = x_0, y_i$ satisfy

$$\frac{\partial y_1}{\partial x_1} = \frac{\partial y_2}{\partial x_2} = \frac{\partial y_3}{\partial x_3} = \frac{\partial y_4}{\partial x_4},$$

$$-\frac{\partial y_2}{\partial x_1} = \frac{\partial y_1}{\partial x_2} = -\frac{\partial y_4}{\partial x_3} = \frac{\partial y_3}{\partial x_4},$$

$$\frac{\partial y_3}{\partial x_1} = \frac{\partial y_4}{\partial x_2} = \frac{\partial y_1}{\partial x_3} = \frac{\partial y_2}{\partial x_4},$$

$$-\frac{\partial y_4}{\partial x_1} = \frac{\partial y_3}{\partial x_2} = -\frac{\partial y_2}{\partial x_3} = \frac{\partial y_1}{\partial x_4}, \tag{8.35}$$

called the Cauchy Riemann equations.

Thus, if $\frac{\partial y_i}{\partial x_j}, i, j = 1, 2, 3, 4$ exist, are continuous on A, and satisfy the Cauchy Riemann equations, then f is analytic on A. Since $f'(x_0)$ exists and has the same value regardless of how x approaches x_0.

y_i are 4-times continuously differentiable functions. From (8.35) we have the partial differential equation of mixed type

$$\begin{vmatrix} \frac{\partial}{\partial x_1} & -\frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_3} & -\frac{\partial}{\partial x_4} \\ \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_4} & \frac{\partial}{\partial x_3} \\ \frac{\partial}{\partial x_3} & -\frac{\partial}{\partial x_4} & \frac{\partial}{\partial x_1} & -\frac{\partial}{\partial x_2} \\ \frac{\partial}{\partial x_4} & \frac{\partial}{\partial x_3} & \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_1} \end{vmatrix} y_i = 0 \tag{8.36}$$

There are hyperbolic type and elliptic type in (8.36). We discover a relation between differential equation and hypercomplex number.

From (8.35) we have

$$\frac{\partial^2 y_1}{\partial^2 x_1} + \frac{\partial^2 y_1}{\partial^2 x_2} = 0,$$

$$\frac{\partial^2 y_1}{\partial^2 x_1} - \frac{\partial^2 y_1}{\partial^2 x_3} = 0,$$

$$\frac{\partial^2 y_1}{\partial^2 x_1} + \frac{\partial^2 y_1}{\partial^2 x_4} = 0,$$

$$\frac{\partial^2 y_2}{\partial^2 x_2} + \frac{\partial^2 y_2}{\partial^2 x_3} = 0,$$

$$\frac{\partial^2 y_3}{\partial^2 x_3} + \frac{\partial^2 y_3}{\partial^2 x_4} = 0,$$

$$\frac{\partial^2 y_2}{\partial^2 x_2} - \frac{\partial^2 y_2}{\partial^2 x_4} = 0.$$

Definition 8.8. We define the positive-positive hypercomplex numbers

$$x = \begin{bmatrix} x_1 & x_2 & x_4 & x_3 & x_6 & x_5 \\ x_2 & x_1 & x_6 & x_5 & x_4 & x_3 \\ x_3 & x_5 & x_1 & x_4 & x_2 & x_6 \\ x_4 & x_6 & x_3 & x_1 & x_5 & x_2 \\ x_5 & x_3 & x_2 & x_6 & x_1 & x_4 \\ x_6 & x_4 & x_5 & x_2 & x_3 & x_1 \end{bmatrix} = x_1 + x_2 J_2 + x_3 J_3 + x_4 J_3^2 + x_5 J_2 J_3 + x_6 J_2 J_3^2, \tag{8.37}$$

where $J_2, J_3, J_3^2, J_2 J_3, J_2 J_3^2$ are called bases, $J_2^2 = 1$ and $J_3^3 = 1$. From (8.37) we have the exponential formula

$$x = R \exp(t_1 J_2 + t_2 J_3 + t_3 J_3^2 + t_4 J_2 J_3 + t_5 J_2 J_3^2)$$

$$= R(S_1 + S_2J_2 + S_3J_3 + S_4J_3^2 + S_5J_2J_3 + S_6J_2J_3^2), \tag{8.38}$$

where

$$S_1 = \frac{1}{6}\Big[e^{A_1} + 2e^{B_1}\cos\theta_1 + e^{A_2} + 2e^{B_2}\cos\theta_2\Big],$$

$$S_2 = \frac{1}{6}\Big[e^{A_1} + 2e^{B_1}\cos\theta_1 - e^{A_2} - 2e^{B_2}\cos\theta_2\Big],$$

$$S_3 = \frac{1}{6}\Big[e^{A_1} - 2e^{B_1}\cos(\theta_1 - \frac{\pi}{3}) + e^{A_2} - 2e^{B_2}\cos(\theta_2 + \frac{\pi}{3})\Big],$$

$$S_4 = \frac{1}{6}\Big[e^{A_1} + 2e^{B_1}\cos(\theta_1 - \frac{2\pi}{3}) + e^{A_2} + 2e^{B_2}\cos(\theta_2 + \frac{2\pi}{3})\Big],$$

$$S_5 = \frac{1}{6}\Big[e^{A_1} - 2e^{B_1}\cos(\theta_1 - \frac{\pi}{3}) - e^{A_2} + 2e^{B_2}\cos(\theta_2 + \frac{\pi}{3})\Big],$$

$$S_6 = \frac{1}{6}\Big[e^{A_1} + 2e^{B_1}\cos(\theta_1 - \frac{2\pi}{3}) - e^{A_2} - 2e^{B_2}\cos(\theta_2 + \frac{2\pi}{3})\Big]. \tag{8.39}$$

with

$$A_1 = \sum_{i=1}^{5} t_i, \quad A_2 = -t_1 + t_2 + t_3 - t_4 - t_5,$$

$$B_1 = t_1 - \frac{t_2 + t_3 + t_4 + t_5}{2}, \quad \theta_1 = \frac{\sqrt{3}}{2}(-t_2 + t_3 - t_4 + t_5),$$

$$B_2 = -t_1 - \frac{t_2 + t_3 - t_4 - t_5}{2}, \quad \theta_2 = \frac{\sqrt{3}}{2}(t_2 - t_3 - t_4 + t_5). \tag{8.40}$$

From (8.39) we have its inverse transformation

$$e^{A_1} = \sum_{i=1}^{6} S_i, \quad e^{A_2} = S_1 - S_2 + S_3 + S_4 - S_5 - S_6,$$

$$e^{B_1}\cos\theta_1 = S_1 + S_2 - \frac{1}{2}(S_3 + S_4 + S_5 + S_6),$$

$$e^{B_1}\sin\theta_1 = \frac{\sqrt{3}}{2}(-S_3 + S_4 - S_5 + S_6),$$

$$e^{B_2}\cos\theta_2 = S_1 - S_2 + \frac{1}{2}(-S_3 - S_4 + S_5 + S_6),$$

$$e^{B_2}\sin\theta_2 = \frac{\sqrt{3}}{2}(S_3 - S_4 - S_5 + S_6). \tag{8.41}$$

(8.40) and (8.41) have the same form, $t_0 = 0 \to S_1, t_1 \to S_2, t_2 \to S_3, t_3 \to S_4, t_4 \to S_5, t_5 \to S_6$. Let $S_i = x_i/R$. From (8.40) and (8.41) we have

$$t_1 = \frac{1}{6}(2B_1 - 2B_2 + A_1 - A_2),$$

$$\begin{matrix} t_2 \\ t_3 \end{matrix} = \frac{1}{2}\left(\frac{A_1 + A_2}{2} \mp \frac{\theta_1 - \theta_2}{\sqrt{3}}\right), \tag{8.42}$$

$$\begin{matrix} t_4 \\ t_5 \end{matrix} = \frac{1}{2}\left(\frac{A_1 - A_2 - B_1 + B_2}{3} \mp \frac{\theta_1 + \theta_2}{\sqrt{3}}\right).$$

where

$$A_1 = \log \frac{x_1 + x_2 + x_3 + x_4 + x_5 + x_6}{R}, \quad A_2 = \log \frac{x_1 - x_2 + x_3 + x_4 - x_5 - x_6}{R},$$

$$B_1 = \frac{1}{2} \log \frac{[x_1 + x_2 - \frac{x_3+x_4+x_5+x_6}{2}]^2 + \frac{3}{4}(-x_3 + x_4 - x_5 + x_6)^2}{R^2},$$

$$\theta_1 = \tan^{-1} \frac{\sqrt{3}(-x_3 + x_4 - x_5 + x_6)}{2x_1 + 2x_2 - x_3 - x_4 - x_5 - x_6},$$

$$B_2 = \frac{1}{2} \log \frac{[x_1 - x_2 + \frac{-x_3-x_4+x_5+x_6}{2}]^2 + \frac{3}{4}(x_3 - x_4 - x_5 + x_6)^2}{R^2},$$

$$\theta_2 = \tan^{-1} \frac{\sqrt{3}(x_3 - x_4 - x_5 + x_6)}{2x_1 - 2x_2 - x_3 - x_4 + x_5 + x_6}.$$

$$R^6 = \det |x|.$$

From (8.39) we have S_i an identity

$$\begin{vmatrix} 1 & 2 & 4 & 3 & 6 & 5 \\ 2 & 1 & 6 & 5 & 4 & 3 \\ 3 & 5 & 1 & 4 & 2 & 6 \\ 4 & 6 & 3 & 1 & 5 & 2 \\ 5 & 3 & 2 & 6 & 1 & 4 \\ 6 & 4 & 5 & 2 & 3 & 1 \end{vmatrix} = \begin{vmatrix} 1 & (1)_1 & (1)_2 & (1)_3 & (1)_4 & (1)_5 \\ 2 & (2)_1 & (2)_2 & (2)_3 & (2)_4 & (2)_5 \\ 3 & (3)_1 & (3)_2 & (3)_3 & (3)_4 & (3)_5 \\ 4 & (4)_1 & (4)_2 & (4)_3 & (4)_4 & (4)_5 \\ 5 & (5)_1 & (5)_2 & (5)_3 & (5)_4 & (5)_5 \\ 6 & (6)_1 & (6)_2 & (6)_3 & (6)_4 & (6)_5 \end{vmatrix} = 1, \tag{8.43}$$

where $j = S_j$; $j = 1, 2, 3, 4, 5, 6$; $(j)_i = \frac{\partial S_j}{\partial t_i}, i = 1, 2, 3, 4, 5.$

Definition 8.9. We define the positive-positive hypercomplex function

$$f(x) = y_1 + y_2 J_2 + y_3 J_3 + y_4 J_3^2 + y_5 J_2 J_3 + y_6 J_2 J_3^2. \tag{8.44}$$

From (8.44) we have the Cauchy-Riemann equations

$$(1)_1 = (2)_2 = (3)_3 = (4)_4 = (5)_5 = (6)_6$$

$$(2)_1 = (1)_2 = (5)_3 = (6)_4 = (3)_5 = (4)_6$$

$$(4)_1 = (6)_2 = (1)_3 = (3)_4 = (2)_5 = (5)_6$$

$$(3)_1 = (5)_2 = (4)_3 = (1)_4 = (6)_5 = (2)_6$$

$$(6)_1 = (4)_2 = (2)_3 = (5)_4 = (1)_5 = (3)_6$$
$$(5)_1 = (3)_2 = (6)_3 = (2)_4 = (4)_5 = (1)_6 \tag{8.45}$$

where $(i)_j = \frac{\partial y_i}{\partial x_j}, i, j = 1, 2, 3, 4, 5, 6.$

If y_i are 6-times differentiable functions. From (8.45) we have the partial differential equation

$$\begin{vmatrix} 1 & 2 & 4 & 3 & 6 & 5 \\ 2 & 1 & 6 & 5 & 4 & 3 \\ 3 & 5 & 1 & 4 & 2 & 6 \\ 4 & 6 & 3 & 1 & 5 & 2 \\ 5 & 3 & 2 & 6 & 1 & 4 \\ 6 & 4 & 5 & 2 & 3 & 1 \end{vmatrix} y_i = 0, \tag{8.46}$$

where $j = \frac{\partial}{\partial x_j}, j = 1, 2, 3, 4, 5, 6.$

Definition 8.10. We define the negative-negative hypercomplex numbers

$$x = \begin{bmatrix} x_1 & -x_2 & -x_4 & -x_3 & x_6 & x_5 \\ x_2 & x_1 & -x_6 & -x_5 & -x_4 & -x_3 \\ x_3 & -x_5 & x_1 & -x_4 & -x_2 & x_6 \\ x_4 & -x_6 & x_3 & x_1 & -x_5 & -x_2 \\ x_5 & x_3 & x_2 & -x_6 & x_1 & -x_4 \\ x_6 & x_4 & x_5 & x_2 & x_3 & x_1 \end{bmatrix} = x_1 + x_2 I_2 + x_3 I_3 + x_4 I_3^2 + x_5 I_2 I_3 + x_6 I_2 I_3^2, \tag{8.47}$$

where $I_2, I_3, I_3^2, I_2 I_3, I_2 I_3^2$ are called bases, $I_2^2 = -1$ and $I_3^3 = -1$. From (8.47) we have the exponential formula

$$\begin{aligned} x &= R \exp(t_1 I_2 + t_2 I_3 + t_3 I_3^2 + t_4 I_2 I_3 + t_5 I_2 I_3^2) \\ &= R(S_1 + S_2 I_2 + S_3 I_3 + S_4 I_3^2 + S_5 I_2 I_3 + S_6 I_2 I_3^2), \end{aligned} \tag{8.48}$$

where

$$S_1 = \frac{1}{3}[e^{B_0} \cos\theta_0 + e^{B_1} \cos\theta_1 + e^{B_2} \cos\theta_2],$$

$$S_2 = \frac{1}{3}[e^{B_0} \sin\theta_0 - e^{B_1} \sin\theta_1 - e^{B_2} \sin\theta_2],$$

$$S_3 = \frac{1}{3}\Big[- e^{B_0} \cos\theta_0 + e^{B_1} \cos(\theta_1 - \frac{\pi}{3}) + e^{B_2} \cos(\theta_2 + \frac{\pi}{3})\Big],$$

$$S_4 = \frac{1}{3}\Big[e^{B_0} \cos\theta_0 + e^{B_1} \cos(\theta_1 - \frac{2\pi}{3}) + e^{B_2} \cos(\theta_2 + \frac{2\pi}{3})\Big],$$

$$S_5 = \frac{1}{3}\Big[- e^{B_0} \sin\theta_0 - e^{B_1} \sin(\theta_1 - \frac{\pi}{3}) - e^{B_2} \sin(\theta_2 + \frac{\pi}{3})\Big],$$

$$S_6 = \frac{1}{3}\Big[e^{B_0}\sin\theta_0 - e^{B_1}\sin(\theta_1 - \frac{2\pi}{3}) - e^{B_2}\sin(\theta_2 + \frac{2\pi}{3})\Big]. \tag{8.49}$$

with

$$B_0 = -t_2 + t_3, \quad \theta_0 = t_1 - t_4 + t_5,$$

$$B_1 = \frac{t_2 - t_3}{2} + \frac{\sqrt{3}}{2}(t_4 + t_5), \quad \theta_1 = -t_1 + \frac{\sqrt{3}}{2}(t_2 + t_3) + \frac{-t_4 + t_5}{2},$$

$$B_2 = \frac{t_2 - t_3}{2} - \frac{\sqrt{3}}{2}(t_4 + t_5), \quad \theta_2 = -t_1 - \frac{\sqrt{3}}{2}(t_2 + t_3) + \frac{-t_4 + t_5}{2}. \tag{8.50}$$

From (8.49) we have its inverse transformation

$$e^{B_0}\cos\theta_0 = S_1 - S_3 + S_4, \quad e^{B_0}\sin\theta_0 = S_2 - S_5 + S_6,$$

$$e^{B_1}\cos\theta_1 = S_1 + \frac{S_3 - S_4}{2} + \frac{\sqrt{3}}{2}(S_5 + S_6),$$

$$e^{B_1}\sin\theta_1 = -S_2 + \frac{\sqrt{3}}{2}(S_3 + S_4) + \frac{-S_5 + S_6}{2},$$

$$e^{B_2}\cos\theta_2 = S_1 + \frac{S_3 - S_4}{2} - \frac{\sqrt{3}}{2}(S_5 + S_6),$$

$$e^{B_2}\sin\theta_2 == -S_2 - \frac{\sqrt{3}}{2}(S_3 + S_4) + \frac{-S_5 + S_6}{2}. \tag{8.51}$$

(8.50) and (8.51) have the same form, $t_0 = 0 \to S_1, t_1 \to S_2, t_2 \to S_3, t_3 \to S_4, t_4 \to S_5, t_5 \to S_6$. Let $S_i = x_i/R$. From (8.50) and (851) we have

$$t_1 = \frac{\theta_0 - \theta_1 - \theta_2}{3},$$

$$\begin{matrix} t_2 \\ t_3 \end{matrix} = \frac{1}{2}\Big(\frac{\theta_1 - \theta_2}{\sqrt{3}} \pm (B_1 + B_2)\Big),$$

$$\begin{matrix} t_4 \\ t_5 \end{matrix} = \frac{1}{2}\Big(\frac{B_1 - B_2}{\sqrt{3}} \mp \frac{2\theta_0 + \theta_1 + \theta_2}{3}\Big). \tag{8.52}$$

where

$$\theta_0 = \tan^{-1}\frac{x_2 - x_5 + x_6}{x_1 - x_3 + x_4},$$

$$B_1 = \frac{1}{2}\log\frac{[x_1 + \frac{x_3 - x_4}{2} + \frac{\sqrt{3}}{2}(x_5 + x_6)]^2 + [-x_2 + \frac{\sqrt{3}}{2}(x_3 + x_4) - \frac{x_5 - x_6}{2}]^2}{R^2},$$

$$\theta_1 = \tan^{-1}\frac{-2x_2 + \sqrt{3}(x_3 + x_4) - x_5 + x_6}{2x_1 + x_3 - x_4 + \sqrt{3}(x_5 + x_6)},$$

$$B_2 = \frac{1}{2}\log\frac{[x_1+\frac{x_3-x_4}{2}-\frac{\sqrt{3}}{2}(x_5+x_6)]^2+[-x_2-\frac{\sqrt{3}}{2}(x_3+x_4)+\frac{-x_5+x_6}{2}]^2}{R^2},$$

$$\theta_2 = \tan^{-1}\frac{-2x_2-\sqrt{3}(x_3+x_4)-x_5+x_6}{2x_1+x_3-x_4-\sqrt{3}(x_5+x_6)},$$

$$R^6 = \det|x|.$$

From (8.49) we have S_i an identity

$$\begin{vmatrix} 1 & -2 & -4 & -3 & 6 & 5 \\ 2 & 1 & -6 & -5 & -4 & -3 \\ 3 & -5 & 1 & -4 & -2 & 6 \\ 4 & -6 & 3 & 1 & -5 & -2 \\ 5 & 3 & 2 & -6 & 1 & -4 \\ 6 & 4 & 5 & 2 & 3 & 1 \end{vmatrix} = \begin{vmatrix} 1 & (1)_1 & (1)_2 & (1)_3 & (1)_4 & (1)_5 \\ 2 & (2)_1 & (2)_2 & (2)_3 & (2)_4 & (2)_5 \\ 3 & (3)_1 & (3)_2 & (3)_3 & (3)_4 & (3)_5 \\ 4 & (4)_1 & (4)_2 & (4)_3 & (4)_4 & (4)_5 \\ 5 & (5)_1 & (5)_2 & (5)_3 & (5)_4 & (5)_5 \\ 6 & (6)_1 & (6)_2 & (6)_3 & (6)_4 & (6)_5 \end{vmatrix} = 1 \qquad (8.53)$$

where $j = S_j; j = 1,2,3,4,5,6; (j)_i = \frac{\partial S_j}{\partial t_i}, i = 1,2,3,4,5.$

Definition 8.11. We define the negative-negative hypercomplex function

$$f(x) = y_1 + y_2 I_2 + y_3 I_3 + y_4 I_3^2 + y_5 I_2 I_3 + y_6 I_2 I_3^2. \qquad (8.54)$$

From (8.54) we have the Cauchy-Riemann equations

$$(1)_1 = (2)_2 = (3)_3 = (4)_4 = (5)_5 = (6)_6$$

$$-(2)_1 = (1)_2 = -(5)_3 = -(6)_4 = (3)_5 = (4)_6$$

$$-(4)_1 = -(6)_2 = (1)_3 = (3)_4 = (2)_5 = (5)_6$$

$$-(3)_1 = -(5)_2 = -(4)_3 = (1)_4 = -(6)_5 = (2)_6$$

$$(6)_1 = -(4)_2 = -(2)_3 = -(5)_4 = (1)_5 = (3)_6$$

$$(5)_1 = -(3)_2 = (6)_3 = -(2)_4 = -(4)_5 = (1)_6, \qquad (8.55)$$

where $(i)_j = \frac{\partial y_i}{\partial x_j}, i,j = 1,2,3,4,5,6.$

If y_i are 6-times differentiable functions, from (8.55) we have the partial differential equation

$$\begin{vmatrix} 1 & -2 & -4 & -3 & 6 & 5 \\ 2 & 1 & -6 & -5 & -4 & -3 \\ 3 & -5 & 1 & -4 & -2 & 6 \\ 4 & -6 & 3 & 1 & -5 & -2 \\ 5 & 3 & 2 & -6 & 1 & -4 \\ 6 & 4 & 5 & 2 & 3 & 1 \end{vmatrix} y_i = 0, \qquad (8.56)$$

where $j = \frac{\partial}{\partial x_j}, j = 1,2,3,4,5,6.$

Definition 8.12. We define the negative-positive hypercomplex numbers

$$x = \begin{bmatrix} x_1 & -x_2 & x_4 & x_3 & -x_6 & -x_5 \\ x_2 & x_1 & x_6 & x_5 & x_4 & x_3 \\ x_3 & -x_5 & x_1 & x_4 & -x_2 & -x_6 \\ x_4 & -x_6 & x_3 & x_1 & -x_5 & -x_2 \\ x_5 & x_3 & x_2 & x_6 & x_1 & x_4 \\ x_6 & x_4 & x_5 & x_2 & x_3 & x_1 \end{bmatrix} = x_1 + x_2 I_2 + x_3 J_3 + x_4 J_3^2 + x_5 I_2 J_3 + x_6 I_2 J_3^2, \tag{8.57}$$

where $I_2, J_3, J_3^2, I_2J_3, I_2J_3^2$ are called bases, $I_2^2 = -1$ and $J_3^3 = 1$.

From (8.57) we have the exponential formula

$$x = R\exp(t_1 I_2 + t_2 J_3 + t_3 J_3^2 + t_4 I_2 J_3 + t_5 I_2 J_3^2) =$$

$$R(S_1 + S_2 I_2 + S_3 J_3 + S_4 J_3^2 + S_5 I_2 J_3 + S_6 I_2 J_3^2), \tag{8.58}$$

where

$$S_1 = \frac{1}{3}\Big[e^{B_1}\cos\theta_1 + e^{B_2}\cos\theta_2 + e^{B_3}\cos\theta_3\Big],$$

$$S_2 = \frac{1}{3}\Big[e^{B_1}\sin\theta_1 - e^{B_2}\sin\theta_2 - e^{B_3}\sin\theta_3\Big],$$

$$S_3 = \frac{1}{3}\Big[e^{B_1}\cos\theta_1 - e^{B_2}\cos(\theta_2 - \frac{\pi}{3}) - e^{B_3}\cos(\theta_3 + \frac{\pi}{3})\Big],$$

$$S_4 = \frac{1}{3}\Big[e^{B_1}\cos\theta_1 + e^{B_2}\cos(\theta_2 - \frac{2\pi}{3}) + e^{B_3}\cos(\theta_3 + \frac{2\pi}{3})\Big],$$

$$S_5 = \frac{1}{3}\Big[e^{B_1}\sin\theta_1 + e^{B_2}\sin(\theta_2 - \frac{\pi}{3}) + e^{B_3}\sin(\theta_3 + \frac{\pi}{3}\Big],$$

$$S_6 = \frac{1}{3}\Big[e^{B_1}\sin\theta_1 - e^{B_2}\sin(\theta_2 - \frac{2\pi}{3}) - e^{B_3}\sin(\theta_3 + \frac{2\pi}{3})\Big]. \tag{8.59}$$

with

$$B_1 = t_2 + t_3, \quad \theta_1 == t_1 + t_4 + t_5,$$

$$B_2 = -\frac{t_2 + t_3}{2} + \frac{\sqrt{3}}{2}(-t_4 + t_5), \quad \theta_2 = -t_1 + \frac{\sqrt{3}}{2}(-t_2 + t_3) + \frac{t_4 + t_5}{2},$$

$$B_3 = -\frac{t_2 + t_3}{2} + \frac{\sqrt{3}}{2}(t_4 - t_5), \quad \theta_3 = -t_1 + \frac{\sqrt{3}}{2}(t_2 - t_3) + \frac{t_4 + t_5}{2}. \tag{8.60}$$

From (8.59) we have its inverse transformation

$$e^{B_1}\cos\theta_1 = S_1 + S_3 + S_4, \quad e^{B_1}\sin\theta_1 = S_2 + S_5 + S_6,$$

$$e^{B_2}\cos\theta_2 = S_1 - \frac{S_3 + S_4}{2} + \frac{\sqrt{3}}{2}(-S_5 + S_6),$$

$$e^{B_2}\sin\theta_2 = -S_2 + \frac{\sqrt{3}}{2}(-S_3+S_4) + \frac{S_5+S_6}{2},$$

$$e^{B_3}\cos\theta_3 = S_1 - \frac{S_3+S_4}{2} + \frac{\sqrt{3}}{2}(S_5-S_6),$$

$$e^{B_3}\sin\theta_3 = -S_2 + \frac{\sqrt{3}}{2}(S_3-S_4) + \frac{S_5+S_6}{2}. \tag{8.61}$$

(8.60) and (8.61) have the same form, $t_0 = 0 \to S_1, t_1 \to S_2, t_2 \to S_3, t_3 \to S_4, t_4 \to S_5, t_5 \to S_6$. Let $S_i = x_i/R$. From (8.60) and (861) we have

$$t_1 = \frac{\theta_1-\theta_2-\theta_3}{3},\ \begin{matrix} t_2 \\ t_3 \end{matrix} = \frac{1}{2}\Big(B_1 \mp (\frac{\theta_2-\theta_3}{\sqrt{3}}\Big),\quad \begin{matrix} t_4 \\ t_5 \end{matrix} = \frac{1}{2}\Big(\frac{2\theta_1+\theta_2+\theta_3}{3} \mp \frac{B_2-B_3}{\sqrt{3}}\Big). \tag{8.62}$$

where

$$B_1 = \frac{1}{2}\log\frac{(x_1+x_3+x_4)^2+(x_2+x_5+x_6)^2}{R^2},$$

$$\theta_1 = \tan^{-1}\frac{x_2+x_5+x_6}{x_1+x_3+x_4},$$

$$B_2 = \frac{1}{2}\log\frac{[x_1-\frac{x_3+x_4}{2}+\frac{\sqrt{3}}{2}(-x_5+x_6)]^2+[-x_2+\frac{\sqrt{3}}{2}(-x_3+x_4)+\frac{x_5+x_6}{2}]^2}{R^2},$$

$$\theta_2 = \tan^{-1}\frac{-2x_2+x_5+x_6+\sqrt{3}(-x_3+x_4)}{2x_1-x_3-x_4+\sqrt{3}(-x_5+x_6)},$$

$$B_3 = \frac{1}{2}\log\frac{[x_1-\frac{x_3+x_4}{2}+\frac{\sqrt{3}}{2}(x_5-x_6)]^2+[-x_2+\frac{\sqrt{3}}{2}(x_3-x_4)+\frac{x_5+x_6}{2}]^2}{R^2},$$

$$\theta_3 = \tan^{-1}\frac{-2x_2+x_5+x_6+\sqrt{3}(x_3-x_4)}{2x_1-x_3-x_4+\sqrt{3}(x_5-x_6)},$$

$$R^6 = \det|x|.$$

From (8.59) we have S_i an identity

$$\begin{vmatrix} 1 & -2 & 4 & 3 & -6 & -5 \\ 2 & 1 & 6 & 5 & 4 & 3 \\ 3 & -5 & 1 & 4 & -2 & -6 \\ 4 & -6 & 3 & 1 & -5 & -2 \\ 5 & 3 & 2 & 6 & 1 & 4 \\ 6 & 4 & 5 & 2 & 3 & 1 \end{vmatrix} = \begin{vmatrix} 1 & (1)_1 & (1)_2 & (1)_3 & (1)_4 & (1)_5 \\ 2 & (2)_1 & (2)_2 & (2)_3 & (2)_4 & (2)_5 \\ 3 & (3)_1 & (3)_2 & (3)_3 & (3)_4 & (3)_5 \\ 4 & (4)_1 & (4)_2 & (4)_3 & (4)_4 & (4)_5 \\ 5 & (5)_1 & (5)_2 & (5)_3 & (5)_4 & (5)_5 \\ 6 & (6)_1 & (6)_2 & (6)_3 & (6)_4 & (6)_5 \end{vmatrix} = 1 \tag{8.63}$$

where $j = S_j, j = 1,2,3,4,5,6; (j)_i = \frac{\partial S_j}{\partial t_i}, i = 1,2,3,4,5.$

Definition 8.13. We define the negative-positive hypercomplex function

$$f(x) = y_1 + y_2 I_2 + y_3 J_3 + y_4 J_3^2 + y_5 I_2 J_3 + y_6 I_2 J_3^2. \tag{8.64}$$

From (8.64) we have the Cauchy-Riemann equations

$$\begin{aligned} (1)_1 &= (2)_2 = (3)_3 = (4)_4 = (5)_5 = (6)_6 \\ -(2)_1 &= (1)_2 = -(5)_3 = -(6)_4 = (3)_5 = (4)_6 \\ (4)_1 &= (6)_2 = (1)_3 = (3)_4 = (2)_5 = (5)_6 \\ (3)_1 &= (5)_2 = (4)_3 = (1)_4 = (6)_5 = (2)_6 \\ -(6)_1 &= (4)_2 = -(2)_3 = -(5)_4 = (1)_5 = (3)_6 \\ -(5)_1 &= (3)_2 = -(6)_3 = -(2)_4 = (4)_5 = (1)_6 \end{aligned} \tag{8.65}$$

where $(i)_j = \frac{\partial y_i}{\partial x_j}, i, j = 1, 2, 3, 4, 5, 6.$

If y_i are 6-times differentiable functions, from (8.65) we have the partial differential equation

$$\begin{vmatrix} 1 & -2 & 4 & 3 & -6 & -5 \\ 2 & 1 & 6 & 5 & 4 & 3 \\ 3 & -5 & 1 & 4 & -2 & -6 \\ 4 & -6 & 3 & 1 & -5 & -2 \\ 5 & 3 & 2 & 6 & 1 & 4 \\ 6 & 4 & 5 & 2 & 3 & 1 \end{vmatrix} y_i = 0, \tag{8.66}$$

where $j = \frac{\partial}{\partial x_j}, j = 1, 2, 3, 4, 5, 6.$

Definition 8.14. We define the positive-negative hypercomplex numbers

$$x = \begin{bmatrix} x_1 & x_2 & -x_4 & -x_3 & -x_6 & -x_5 \\ x_2 & x_1 & -x_6 & -x_5 & -x_4 & -x_3 \\ x_3 & x_5 & x_1 & -x_4 & x_2 & -x_6 \\ x_4 & x_6 & x_3 & x_1 & x_5 & x_2 \\ x_5 & x_3 & x_2 & -x_6 & x_1 & -x_4 \\ x_6 & x_4 & x_5 & x_2 & x_3 & x_1 \end{bmatrix} = x_1 + x_2 J_2 + x_3 I_3 + x_4 I_3^2 + x_5 J_2 I_3 + x_6 J_2 I_3^2, \tag{8.67}$$

where $J_2, I_3, I_3^2, J_2 I_3, J_2 I_3^2$ are called bases, $J_2^2 = 1$ and $I_3^3 = -1$.

From (8.67) we have the exponential formula

$$\begin{aligned} x &= R \exp(t_1 J_2 + t_2 I_3 + t_3 I_3^2 + t_4 J_2 I_3 + t_5 J_2 I_3^2) \\ &= R(S_1 + S_2 J_2 + S_3 I_3 + S_4 I_3^2 + S_5 J_2 I_3 + S_6 J_2 I_3^2), \end{aligned} \tag{8.68}$$

where

$$S_1 = \frac{1}{3}\left[e^{B_1} \cosh\theta_1 + e^{B_2} \cos\theta_2 + e^{B_3} \cos\theta_3\right],$$

$$S_2 = \frac{1}{3}\Big[e^{B_1}\sinh\theta_1 - e^{B_2}\cos\theta_2 - e^{B_3}\cos\theta_3\Big],$$

$$S_3 = \frac{1}{3}\Big[-e^{B_1}\cosh\theta_1 + e^{B_2}\cos(\theta_2+\frac{\pi}{3}) + e^{B_3}\cos(\theta_3-\frac{\pi}{3})\Big],$$

$$S_4 = \frac{1}{3}\Big[e^{B_1}\cosh\theta_1 + e^{B_2}\cos(\theta_2+\frac{2\pi}{3}) + e^{B_3}\cos(\theta_3-\frac{2\pi}{3})\Big],$$

$$S_5 = \frac{1}{3}\Big[-e^{B_1}\sinh\theta_1 - e^{B_2}\cos(\theta_2+\frac{\pi}{3}) + e^{B_3}\cos(\theta_3-\frac{\pi}{3})\Big],$$

$$S_6 = \frac{1}{3}\Big[e^{B_1}\sinh\theta_1 - e^{B_2}\cos(\theta_2+\frac{2\pi}{3}) + e^{B_3}\cos(\theta_3-\frac{2\pi}{3})\Big]. \quad (8.69)$$

with

$$B_1 = -t_2 + t_3, \quad \theta_1 = t_1 - t_4 + t_5,$$

$$B_2 = -t_1 + \frac{t_2 - t_3 - t - 4 + t_5}{2}, \quad \theta_2 = \frac{\sqrt{3}}{2}(-t_2 - t_3 + t_4 + t_5),$$

$$B_3 = t_1 + \frac{t_2 - t_3 + t_4 - t_5}{2}, \quad \theta_3 = \frac{\sqrt{3}}{2}(t_2 + t_3 + t_4 + t_5). \quad (8.70)$$

From (8.69) we have its inverse transformation

$$e^{B_1}\cosh\theta_1 = S_1 - S_3 + S_4, \quad e^{B_1}\sinh\theta_1 = S_2 - S_5 + S_6,$$

$$e^{B_2}\cos\theta_2 = S_1 - S_2 + \frac{S_3 - S_4 - S_5 + S_6}{2}, \quad e^{B_2}\sin\theta_2 = \frac{\sqrt{3}}{2}(-S_3 - S_4 + S_5 + S_6),$$

$$e^{B_3}\cos\theta_3 = S_1 + S_2 + \frac{S_3 - S_4 + S_5 - S_6}{2}, \quad e^{B_3}\sin\theta_3 = \frac{\sqrt{3}}{2}(S_3 + S_4 + S_5 + S_6). \quad (8.71)$$

(8.70) and (8.71) have the same form, $t_0 = 0 \to S_1, t_1 \to S_2, t_2 \to S_3, t_3 \to S_4, t_4 \to S_5, t_5 \to S_6$. Let $S_i = x_i/R$. From (8.70) and (8.71) we have

$$t_1 = \frac{B_3 - B_2 + \theta_1}{3}, \quad \begin{matrix} t_2 \\ t_3 \end{matrix} = \frac{1}{2}\left(\frac{\theta_3 - \theta_2}{\sqrt{3}} \pm (B_2 + B_3)\right), \quad \begin{matrix} t_4 \\ t_5 \end{matrix} = \frac{1}{2}\left(\frac{\theta_2 + \theta_3}{\sqrt{3}} \mp \frac{B_2 - B_3 + 2\theta_1}{3}\right). \quad (8.72)$$

where

$$\theta_1 = \tanh^{-1}\frac{x_2 - x_5 + x_6}{x_1 - x_3 + x_4},$$

$$B_2 = \frac{1}{2}\log\frac{\frac{3}{4}(-x_3 - x_4 + x_5 + x_6)^2 + [x_1 - x_2 + \frac{x_3 - x_4 - x_5 + x_6}{2}]^2}{R^2},$$

$$\theta_2 = \tan^{-1}\frac{\sqrt{3}(-x_3 - x_4 + x_5 + x_6)}{2x_1 - 2x_2 + x_3 - x_4 - x_5 + x_6},$$

$$B_3 = \frac{1}{2}\log\frac{[x_1+x_2+\frac{x_3-x_4+x_5-x_6}{2}]^2+\frac{3}{4}(x_3+x_4+x_5+x_6)^2}{R^2},$$

$$\theta_3 = \tan^{-1}\frac{\sqrt{3}(x_3+x_4+x_5+x_6)}{2x_1+2x_2+x_3-x_4+x_5-x_6},$$

$$R^6 = \det|x|.$$

From (6.69) we have S_i an identity

$$\begin{vmatrix} 1 & 2 & -4 & -3 & -6 & -5 \\ 2 & 1 & -6 & -5 & -4 & -3 \\ 3 & 5 & 1 & -4 & 2 & -6 \\ 4 & 6 & 3 & 1 & 5 & 2 \\ 5 & 3 & 2 & -6 & 1 & -4 \\ 6 & 4 & 5 & 2 & 3 & 1 \end{vmatrix} = \begin{vmatrix} 1 & (1)_1 & (1)_2 & (1)_3 & (1)_4 & (1)_5 \\ 2 & (2)_1 & (2)_2 & (2)_3 & (2)_4 & (2)_5 \\ 3 & (3)_1 & (3)_2 & (3)_3 & (3)_4 & (3)_5 \\ 4 & (4)_1 & (4)_2 & (4)_3 & (4)_4 & (4)_5 \\ 5 & (5)_1 & (5)_2 & (5)_3 & (5)_4 & (5)_5 \\ 6 & (6)_1 & (6)_2 & (6)_3 & (6)_4 & (6)_5 \end{vmatrix} = 1 \qquad (8.73)$$

where $j = S_j$, $j = 1,2,3,4,5,6$; $(j)_i = \frac{\partial S_j}{\partial t_i}, i = 1,2,3,4,5.$

Definition 8.15. We define the positive-negative hypercomplex function

$$f(x) = y_1 + y_2 J_2 + y_3 I_3 + y_4 I_3^2 + y_5 J_2 I_3 + y_6 J_2 I_3^2. \qquad (8.74)$$

From (8.74) we have the Cauchy-Riemann equations

$$(1)_1 = (2)_2 = (3)_3 = (4)_4 = (5)_5 = (6)_6$$

$$(2)_1 = (1)_2 = (5)_3 = (6)_4 = (3)_5 = (4)_6$$

$$-(4)_1 = -(6)_2 = (1)_3 = (3)_4 = (2)_5 = (5)_6$$

$$-(3)_1 = -(5)_2 = -(4)_3 = (1)_4 = -(6)_5 = (2)_6$$

$$-(6)_1 = -(4)_2 = (2)_3 = (5)_4 = (1)_5 = (3)_6$$

$$-(5)_1 = -(3)_2 = -(6)_3 = (2)_4 = -(4)_5 = (1)_6, \qquad (8.75)$$

where $(i)_j = \frac{\partial y_i}{\partial x_j}$, $i,j = 1,2,3,4,5,6.$

If y_i are 6-times differentiable functions, from (8.75) we have the partial differential equation

$$\begin{vmatrix} 1 & 2 & -4 & -3 & -6 & -5 \\ 2 & 1 & -6 & -5 & -4 & -3 \\ 3 & 5 & 1 & -4 & 2 & -6 \\ 4 & 6 & 3 & 1 & 5 & 2 \\ 5 & 3 & 2 & -6 & 1 & -4 \\ 6 & 4 & 5 & 2 & 3 & 1 \end{vmatrix} y_i = 0, \qquad (8.76)$$

where $j = \frac{\partial}{\partial x_j}, j = 1, 2, 3, 4, 5, 6.$

Definition 8.16. We define the generalised positive hypercomplex numbers

$$x = \begin{bmatrix} x_1 & rx_n & rx_{n-1} & \cdots & rx_2 \\ x_2 & x_1 & rx_n & \cdots & rx_3 \\ x_3 & x_2 & x_1 & \cdots & rx_4 \\ \cdots & \cdots & \cdots & \cdots & \\ x_n & x_{n-1} & x_{n-2} & \cdots & x_1 \end{bmatrix} = \sum_{i=1}^{n} x_i J_n^{i-1}, \tag{8.77}$$

where r is a real number,

$$J_n = \begin{bmatrix} 0 & 0 & \cdots & r \\ 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}, ..., J_n^n = r,$$

$J_n, ..., J_n^{n-1}$ are called the bases of the generalised positive hypercomplex numbers.

From (8.77) we have

$$\int \frac{dx}{x} = \log x = \int \frac{\sum_{i=1}^{n} dx_i J_n^{i-1}}{\sum_{i=1}^{n} x_i J_n^{i-1}} = \log R + \sum_{i=1}^{n-1} t_i J_n^i, \tag{8.78}$$

where R is called the modulus, t_i is called i-th argument,

$$\log R = \int \frac{1}{R^n} \begin{vmatrix} dx_1 & rdx_n & \cdots & rdx_2 \\ x_2 & x_1 & \cdots & rx_3 \\ x_3 & x_2 & \cdots & rx_4 \\ \cdots & \cdots & \cdots & \cdots \\ x_n & x_{n-1} & \cdots & x_1 \end{vmatrix}, t_1 = \int \frac{1}{R^n} \begin{vmatrix} dx_2 & dx_1 & \cdots & rdx_3 \\ x_2 & x_1 & \cdots & rx_3 \\ x_3 & x_2 & \cdots & rx_4 \\ \cdots & \cdots & \cdots & \cdots \\ x_n & x_{n-1} & \cdots & x_1 \end{vmatrix},$$

$$t_2 = \int \frac{1}{R^n} \begin{vmatrix} dx_3 & dx_2 & \cdots & rdx_4 \\ x_2 & x_1 & \cdots & rx_3 \\ x_3 & x_2 & \cdots & rx_4 \\ \cdots & \cdots & \cdots & \cdots \\ x_n & x_{n-1} & \cdots & x_1 \end{vmatrix}, \cdots, t_{n-1} = \int \frac{1}{R^n} \begin{vmatrix} dx_n & dx_{n-1} & \cdots & dx_1 \\ x_2 & x_1 & \cdots & rx_3 \\ x_3 & x_2 & \cdots & rx_4 \\ \cdots & \cdots & \cdots & \cdots \\ x_n & x_{n-1} & \cdots & x_1 \end{vmatrix},$$

$$R^n = \det |x|. \tag{8.79}$$

Let $n = 2$. From (8.79) we have

$$t_1 = \int \frac{\begin{vmatrix} dx_2 & dx_1 \\ x_2 & x_1 \end{vmatrix}}{\begin{vmatrix} x_1 & rx_2 \\ x_2 & x_1 \end{vmatrix}} = \frac{1}{\sqrt{r}} \tanh^{-1} \frac{\sqrt{r} x_2}{x_1}.$$

Let $n = 3$. From (8.79) we have

$$t_1 = \int \frac{1}{R^3} \begin{vmatrix} dx_2 & dx_1 & rdx_3 \\ x_2 & x_1 & rx_3 \\ x_3 & x_2 & x_1 \end{vmatrix} =$$

$$= \frac{1}{2r^{\frac{1}{3}}} \left(\log \frac{x_1 + r^{\frac{1}{3}} x_2 + r^{\frac{2}{3}} x_3}{R} - \frac{2}{\sqrt{3}} \tan^{-1} \frac{\sqrt{3}(-r^{\frac{1}{3}} x_2 + r^{\frac{2}{3}} x_3)}{2x_1 - r^{\frac{1}{3}} x_2 - r^{\frac{2}{3}} x_3} \right),$$

$$t_2 = \int \frac{1}{R^3} \begin{vmatrix} dx_3 & dx_2 & dx_1 \\ x_2 & x_1 & rx_3 \\ x_3 & x_2 & x_1 \end{vmatrix} =$$

$$= \frac{1}{2r^{\frac{2}{3}}} \left(\log \frac{x_1 + r^{\frac{1}{3}} x_2 + r^{\frac{2}{3}} x_3}{R} + \frac{2}{\sqrt{3}} \tan^{-1} \frac{\sqrt{3}(-r^{\frac{1}{3}} x_2 + r^{\frac{2}{3}} x_3)}{2x_1 - r^{\frac{1}{3}} x_2 - r^{\frac{2}{3}} x_3} \right),$$

From (8.79) we have the exponential formula

$$x = R \exp\left(\sum_{i=1}^{n-1} t_i J_n^i \right) = R \sum_{i=1}^{n} S_i J_n^{i-1}, \tag{8.80}$$

S_i is called the generalized complex hyperbolic functions.

Let $n = 2$. From (8.80) we have

$$e^{t_1 J_2} = S_1 + S_2 J_2,$$

where $S_1 = \cosh(\sqrt{r} t_1)$, $S_2 = \frac{1}{\sqrt{r}} \sinh(\sqrt{r} t_1)$.

Let $n = 3$. From (8.80) we have

$$S_i = \frac{r^{\frac{1-i}{3}}}{3} \left[e^A + (-1)^{i-1} 2e^B \cos(\theta - \frac{(i-1)\pi}{3}) \right],$$

where

$$i = 1, 2, 3, \quad A = r^{\frac{1}{3}} t_1 + r^{\frac{2}{3}} t_2, \quad B = -\frac{r^{\frac{1}{3}} t_1 + r^{\frac{2}{3}} t_2}{2}, \quad \theta = \frac{\sqrt{3}}{2}(-r^{\frac{1}{3}} t_1 + r^{\frac{2}{3}} t_2).$$

From (8.80) we have S_i an identity

$$\begin{vmatrix} S_1 & rS_n & \cdots & rS_2 \\ S_2 & S_1 & \cdots & rS_3 \\ S_3 & S_2 & \cdots & rS_4 \\ \cdots & \cdots & \cdots & \cdots \\ S_n & S_{n-1} & \cdots & S_1 \end{vmatrix} = \begin{vmatrix} S_1 & (S_1)_1 & \cdots & (S_1)_{n-1} \\ S_2 & (S_2)_1 & \cdots & (S_2)_{n-1} \\ S_3 & (S_3)_1 & \cdots & (S_3)_{n-1} \\ \cdots & \cdots & \cdots & \cdots \\ S_n & (S_n)_1 & \cdots & (S_n)_{n-1} \end{vmatrix} = 1. \tag{8.81}$$

where $(S_i)_j = \frac{\partial S_i}{\partial t_j}, i = 1, \cdots, n; j = 1, \cdots, n-1$.

Definition 8.17. We define the generalized positive hypercomplex functions

$$f(x) = \begin{bmatrix} y_1 & y_n & \cdots & ry_2 \\ y_2 & y_1 & \cdots & ry_3 \\ y_3 & y_2 & \cdots & ry_4 \\ \cdots & \cdots & \cdots & \cdots \\ y_n & y_{n-1} & \cdots & y_1 \end{bmatrix} = \sum_{i=1}^{n} y_i J_n^{i-1}. \tag{8.81}$$

From (8.81) we have the Cauchy-Riemann equations

$$\begin{vmatrix} (y_1)_1 & r(y_n)_1 & \cdots & r(y_2)_1 \\ || & || & \cdots & || \\ (y_2)_2 & (y_1)_2 & \cdots & r(y_3)_2 \\ || & || & \cdots & || \\ (y_3)_3 & (y_2)_3 & \cdots & r(y_4)_3 \\ || & || & \cdots & || \\ \cdots & \cdots & \cdots & \cdots \\ || & || & \cdots & || \\ (y_n)_n & (y_{n-1})_n & \cdots & (y_1)_n \end{vmatrix}, \tag{8.82}$$

where $(y_i)_j = \frac{\partial y_i}{\partial x_j}, \quad i, j = 1, \cdots, n$.

Let $n = 2$. From (8.82) we have

$$\frac{\partial y_1}{\partial x_1} = \frac{\partial y_2}{\partial x_2}$$

$$r\frac{\partial y_2}{\partial x_1} = \frac{\partial y_1}{\partial x_2}$$

Let $n = 3$. From (8.82) we have

$$\frac{\partial y_1}{\partial x_1} = \frac{\partial y_2}{\partial x_2} = \frac{\partial y_3}{\partial x_3},$$

$$r\frac{\partial y_3}{\partial x_1} = \frac{\partial y_1}{\partial x_2} = \frac{\partial y_2}{\partial x_3},$$

$$r\frac{\partial y_2}{\partial x_1} = r\frac{\partial y_3}{\partial x_2} = \frac{\partial y_1}{\partial x_3}.$$

If y_i are n-times differentiable functions, from (8.82) we have the generalized partial differential equation of hyperbolic type

$$\begin{vmatrix} \frac{\partial}{\partial x_1} & \frac{\partial}{r\partial x_n} & \cdots & \frac{\partial}{r\partial x_2} \\ \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_1} & \cdots & \frac{\partial}{r\partial x_3} \\ \frac{\partial}{\partial x_3} & \frac{\partial}{\partial x_2} & \cdots & \frac{\partial}{r\partial x_4} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial}{\partial x_n} & \frac{\partial}{\partial x_{n-1}} & \cdots & \frac{\partial}{\partial x_1} \end{vmatrix} y_i = 0. \tag{8.83}$$

Let $n = 2$. From (8.83) we have

$$\frac{\partial^2 y_1}{\partial^2 x_1} - \frac{\partial^2 y_1}{r\partial^2 x_2} = 0.$$

Let $n = 3$. From (8.83) we have

$$\frac{\partial^3 y_1}{\partial^3 x_1} + \frac{\partial^3 y_1}{r\partial^3 x_2} + \frac{\partial^3 y_1}{r^2\partial^3 x_3} - 3\frac{\partial^3 y_1}{r\partial x_1 \partial x_2 \partial x_3} = 0.$$

Definition 8.18. We define the generalized negative hypercomplex numbers

$$x = \begin{bmatrix} x_1 & -rx_n & -rx_{n-1} & \cdots & -rx_2 \\ x_2 & x_1 & -rx_n & \cdots & -rx_3 \\ x_3 & x_2 & x_1 & \cdots & -rx_4 \\ \cdots & \cdots & \cdots & \cdots & \\ x_n & x_{n-1} & x_{n-2} & \cdots & x_1 \end{bmatrix} = \sum_{i=1}^{n} x_i I_n^{i-1}, \tag{8.84}$$

where r is a real number,

$$I_n = \begin{bmatrix} 0 & 0 & \cdots & -r \\ 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}, ..., I_n^n = -r,$$

$I_n, ..., I_n^{n-1}$ are called the bases of the generalised negative hypercomplex numbers.

From (8.84) we have

$$\int \frac{dx}{x} = \log x = \int \frac{\sum_{i=1}^{n} dx_i I_n^{i-1}}{\sum_{i=1}^{n} x_i I_n^{i-1}} = \log R + \sum_{i=1}^{n-1} t_i I_n^i, \tag{8.85}$$

where R is called the modulus, t_i is called i-th argument,

$$\log R = \int \frac{1}{R^n} \begin{vmatrix} dx_1 & -rdx_n & \cdots & -rdx_2 \\ x_2 & x_1 & \cdots & -rx_3 \\ x_3 & x_2 & \cdots & -rx_4 \\ \cdots & \cdots & \cdots & \cdots \\ x_n & x_{n-1} & \cdots & x_1 \end{vmatrix}, t_1 = \int \frac{1}{R^n} \begin{vmatrix} dx_2 & dx_1 & \cdots & -rdx_3 \\ x_2 & x_1 & \cdots & -rx_3 \\ x_3 & x_2 & \cdots & -rx_4 \\ \cdots & \cdots & \cdots & \cdots \\ x_n & x_{n-1} & \cdots & x_1 \end{vmatrix},$$

$$t_2 = \int \frac{1}{R^n} \begin{vmatrix} dx_3 & dx_2 & \cdots & -rdx_4 \\ x_2 & x_1 & \cdots & -rx_3 \\ x_3 & x_2 & \cdots & -rx_4 \\ \cdots & \cdots & \cdots & \cdots \\ x_n & x_{n-1} & \cdots & x_1 \end{vmatrix}, \cdots, t_{n-1} = \int \frac{1}{R^n} \begin{vmatrix} dx_n & dx_{n-1} & \cdots & dx_1 \\ x_2 & x_1 & \cdots & -rx_3 \\ x_3 & x_2 & \cdots & -x_4 \\ \cdots & \cdots & \cdots & \cdots \\ x_n & x_{n-1} & \cdots & x_1 \end{vmatrix},$$

$$R^n = \det |x|. \tag{8.86}$$

Let $n = 2$. From (8.86) we have

$$t_1 = \int \frac{\begin{vmatrix} dx_2 & dx_1 \\ x_2 & x_1 \end{vmatrix}}{\begin{vmatrix} x_1 & -rx_2 \\ x_2 & x_1 \end{vmatrix}} = \frac{1}{\sqrt{r}} \tan^{-1} \frac{\sqrt{r}x_2}{x_1}.$$

Let $n = 3$. From (8.86) we have

$$t_1 = \int \frac{1}{R^3} \begin{vmatrix} dx_2 & dx_1 & -rdx_3 \\ x_2 & x_1 & -rdx_3 \\ x_3 & x_2 & x_1 \end{vmatrix} =$$

$$= \frac{1}{2r^{\frac{1}{3}}} \left(\frac{2}{\sqrt{3}} \tan^{-1} \frac{\sqrt{3}(r^{\frac{1}{3}}x_2 + r^{\frac{2}{3}}x_3)}{2x_1 + r^{\frac{1}{3}}x_2 - r^{\frac{2}{3}}x_3} - \log \frac{x_1 - r^{\frac{1}{3}}x_2 + r^{\frac{2}{3}}x_3}{R} \right),$$

$$t_2 = \int \frac{1}{R^3} \begin{vmatrix} dx_3 & dx_2 & dx_1 \\ x_2 & x_1 & -rx_3 \\ x_3 & x_2 & x_1 \end{vmatrix} =$$

$$= \frac{1}{2r^{\frac{2}{3}}} \left(\frac{2}{\sqrt{3}} \tan^{-1} \frac{\sqrt{3}(r^{\frac{1}{3}}x_2 + r^{\frac{2}{3}}x_3)}{2x_1 + r^{\frac{1}{3}}x_2 - r^{\frac{2}{3}}x_3} + \log \frac{x_1 - r^{\frac{1}{3}}x_2 + r^{\frac{2}{3}}x_3}{R} \right).$$

From (8.85) we have the exponential formula

$$x = R \exp\left(\sum_{i=1}^{n-1} t_i I_n^i \right) = R \sum_{i=1}^{n} S_i I_n^{i-1}, \tag{8.87}$$

S_i is called the generalised complex trigonometric functions.

Let $n = 2$. From (8.87) we have

$$e^{t_1 I_2} = S_1 + S_2 I_2, \tag{8.88}$$

where $S_1 = \cos(\sqrt{r}t)$, $S_2 = \frac{1}{\sqrt{r}} \sin(\sqrt{r}t_1)$.

(8.88) is the foundations of the generalised complex theory.

Let $n = 3$. From (8.87) we have

$$S_i = \frac{(-1)^{i-1} r^{\frac{1-i}{3}}}{3} \left[e^A + (-1)^{i-1} 2e^B \cos(\theta - \frac{(i-1)\pi}{3}) \right],$$

where

$$i = 1, 2, 3, \quad A = -r^{\frac{1}{3}}t_1 + r^{\frac{2}{3}}t_2, \quad B = \frac{r^{\frac{1}{3}}t_1 - r^{\frac{2}{3}}t_2}{2}, \quad \theta = \frac{\sqrt{3}}{2}(r^{\frac{1}{3}}t_1 + r^{\frac{2}{3}}t_2).$$

From (8.87) we have S_i an identity

$$\begin{vmatrix} S_1 & -rS_n & \cdots & -rS_2 \\ S_2 & S_1 & \cdots & -rS_3 \\ S_3 & S_2 & \cdots & -rS_4 \\ \cdots & \cdots & \cdots & \cdots \\ S_n & S_{n-1} & \cdots & S_1 \end{vmatrix} = \begin{vmatrix} S_1 & (S_1)_1 & \cdots & (S_1)_{n-1} \\ S_2 & (S_2)_1 & \cdots & (S_2)_{n-1} \\ S_3 & (S_3)_1 & \cdots & (S_3)_{n-1} \\ \cdots & \cdots & \cdots & \cdots \\ S_n & (S_n)_1 & \cdots & (S_n)_{n-1} \end{vmatrix} = 1. \tag{8.89}$$

where $(S_i)_j = \frac{\partial S_i}{\partial t_j}, i = 1, \cdots, n; j = 1, \cdots, n-1.$

Definition 8.19. We define the generalised negative hypercomplex functions

$$f(x) = \begin{bmatrix} y_1 & -ry_n & \cdots & -ry_2 \\ y_2 & y_1 & \cdots & -ry_3 \\ y_3 & y_2 & \cdots & -ry_4 \\ \cdots & \cdots & \cdots & \cdots \\ y_n & y_{n-1} & \cdots & y_1 \end{bmatrix} = \sum_{i=1}^{n} y_i I_n^{i-1}. \tag{8.90}$$

From (8.90) we have the Cauchy-Riemann equations

$$\begin{bmatrix} (y_1)_1 & -r(y_n)_1 & \cdots & -r(y_2)_1 \\ || & || & \cdots & || \\ (y_2)_2 & (y_1)_2 & \cdots & -r(y_3)_2 \\ || & || & \cdots & || \\ (y_3)_3 & (y_2)_3 & \cdots & -r(y_4)_3 \\ || & || & \cdots & || \\ \cdots & \cdots & \cdots & \cdots \\ || & || & \cdots & || \\ (y_n)_n & (y_{n-1})_n & \cdots & (y_1)_n \end{bmatrix}, \tag{8.91}$$

where $(y_i)_j = \frac{\partial y_i}{\partial x_j}, \quad i, j = 1, \cdots, n.$

Let $n = 2$. From (8.91) we have

$$\frac{\partial y_1}{\partial x_1} = \frac{\partial y_2}{\partial x_2},$$

$$-r\frac{\partial y_2}{\partial x_1} = \frac{\partial y_1}{\partial x_2},$$

Let $n = 3$. From (8.91) we have

$$\frac{\partial y_1}{\partial x_1} = \frac{\partial y_2}{\partial x_2} = \frac{\partial y_3}{\partial x_3},$$

$$-r\frac{\partial y_3}{\partial x_1}=\frac{\partial y_1}{\partial x_2}=\frac{\partial y_2}{\partial x_3},$$
$$-r\frac{\partial y_2}{\partial x_1}=-r\frac{\partial y_3}{\partial x_2}=\frac{\partial y_1}{\partial x_3}.$$

If y_i are n-times differentiable functions, from (8.91) we have the generalised partial differential equation of elliptic type

$$\begin{vmatrix} \frac{\partial}{\partial x_1} & -\frac{\partial}{r\partial x_n} & \cdots & -\frac{\partial}{r\partial x_2} \\ \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_1} & \cdots & -\frac{\partial}{r\partial x_3} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial}{\partial x_n} & \frac{\partial}{\partial x_{n-1}} & \cdots & \frac{\partial}{\partial x_1} \end{vmatrix} y_i = 0 \tag{8.92}$$

Let $n=2$. From (8.91) we have

$$\frac{\partial^2 y_1}{\partial^2 x_1}+\frac{\partial^2 y_1}{r\partial^2 x_2}=0.$$

Let $n=3$. From (8.91) we have

$$\frac{\partial^3 y_1}{\partial^3 x_1}-\frac{\partial^3 y_1}{r\partial^3 x_2}+\frac{\partial^3 y_1}{r^2\partial^3 x_3}+3\frac{\partial^3 y_1}{r\partial x_1\partial x_2\partial x_3}=0.$$

From (8.77) and ((8.84) we have

$$\begin{bmatrix} x_1 & rx_4 & rx_3 & rx_2 \\ x_2 & x_1 & rx_4 & rx_3 \\ x_3 & x_2 & x_1 & rx_4 \\ x_4 & x_3 & x_2 & x_1 \end{bmatrix} = \begin{bmatrix} x_1 & -(-r)x_4 & -(-r)x_3 & -(-r)x_2 \\ x_2 & x_1 & -(-r)x_4 & -(-r)x_3 \\ x_3 & x_2 & x_1 & -(-r)x_4 \\ x_4 & x_3 & x_2 & x_1 \end{bmatrix} \tag{8.93}$$

From the generalised negative hypercomplex numbers we can obtain all results of the generalised positive hypercomplex numbers and vice versa. They are the dual hypercomplex numbers.

Definition 8.20. We define the generalised positive-negative hypercomplex numbers

$$x=\begin{bmatrix} x_1 & -2x_3 & -2x_2 & -x_4 \\ x_2 & x_1 & -x_4 & -x_3 \\ x_3 & x_4 & x_1 & x_2 \\ x_4 & 2x_2 & -2x_3 & x_1 \end{bmatrix} = x_1+x_2e_1+x_3e_2+x_4e_3, \tag{8.94}$$

where

$$e_1=\begin{bmatrix} 0 & 0 & -2 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 2 & 0 & 0 \end{bmatrix},$$

$$e_2 = \begin{bmatrix} 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 \end{bmatrix},$$

$$e_3 = \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix},$$

where e_1, e_2, e_3 are the mixed bases, $e_1 = I_2 + J_2, e_2 = I_2 - J_2, e_3 = I_2 J_2, I_2^2 = -1, J_2^2 = 1$.

From (8.94) we have the exponential formula

$$x = R\exp(t_1 e_1 + t_2 e_2 + t_3 e_3) = R(S_1 + S_2 e_1 + S_3 e_2 + S_4 e_3), \tag{8.95}$$

where

$$t_1 = \int \frac{1}{R^4} \begin{vmatrix} dx_2 & dx_1 & -dx_4 & -dx_3 \\ x_2 & x_1 & -x_4 & -x_3 \\ x_3 & x_4 & x_1 & x_2 \\ x_4 & 2x_2 & -2x_3 & x_1 \end{vmatrix}$$

$$= \frac{1}{4}\Bigg(\tan^{-1} \frac{2(x_1x_2 + x_1x_3 - x_2x_4 + x_3x_4)}{x_1^2 - 2x_2^2 - 2x_3^2 + x_4^2} + \log \frac{(x_1 + x_2 - x_3)^2 + (x_2 + x_3 + x_4)^2}{R^2},$$

$$t_2 = \int \frac{1}{R^4} \begin{vmatrix} dx_3 & dx_4 & dx_1 & dx_2 \\ x_2 & x_1 & -x_4 & -x_3 \\ x_3 & x_4 & x_1 & x_2 \\ x_4 & 2x_2 & -2x_3 & x_1 \end{vmatrix}$$

$$= \frac{1}{4}\left(\tan^{-1} \frac{2(x_1x_2 + x_1x_3 - x_2x_4 + x_3x_4)}{x_1^2 - 2x_2^2 - 2x_3^2 + x_4^2} - \log \frac{(x_1 + x_2 - x_3)^2 + (x_2 + x_3 + x_4)^2}{R^2} \right),$$

$$t_3 = \int \frac{1}{R^4} \begin{vmatrix} dx_4 & 2dx_2 & -2dx_3 & dx_1 \\ x_2 & x_1 & -x_4 & -x_3 \\ x_3 & x_4 & x_1 & x_2 \\ x_4 & 2x_2 & -2x_3 & x_1 \end{vmatrix}$$

$$= \frac{1}{2} \tan^{-1} \frac{-2x_2^2 + 2x_3^2 + 2x_1x_4}{x_1^2 - x_4^2 + 4x_2x_3}.$$

$$R^4 = \det|x|$$

$$S_1 = \frac{1}{2}(e^{B_1}\cos\theta_1 + e^{B_2}\cos\theta_2),$$

$$S_2 = \frac{1}{4}\Big(e^{B_1}(\cos\theta_1 + \sin\theta_1) + e^{B_2}(-\cos\theta_2 + \sin\theta_2)\Big)$$

$$S_3 = \frac{1}{4}\Big(e^{B_1}(-\cos\theta_1 + \sin\theta_1) + e^{B_2}(\cos\theta_2 + \sin\theta_2)\Big)$$

$$S_4 = \frac{1}{2}(e^{B_1}\sin\theta_1 - e^{B_2}\sin\theta_2), \tag{8.96}$$

with

$$B_1 = t_1 - t_2, \quad \theta_1 = t_1 + t_2 + t_3,$$

$$B_2 = -t_1 + t_2, \quad \theta_2 = t_1 + t_2 - t_3.$$

From (8.96) we have S_i an identity

$$\begin{vmatrix} S_1 & -2S_3 & -2S_2 & -S_4 \\ S_2 & S_1 & -S_4 & -S_3 \\ S_3 & S_4 & S_1 & S_2 \\ S_4 & 2S_2 & -2S_3 & S_1 \end{vmatrix} = \begin{vmatrix} S_1 & (S_1)_1 & (S_1)_2 & (S_1)_3 \\ S_2 & (S_2)_1 & (S_2)_2 & (S_2)_3 \\ S_3 & (S_3)_1 & (S_3)_2 & (S_3)_3 \\ S_4 & (S_4)_1 & (S_4)_2 & (S_4)_3 \end{vmatrix} = 1. \tag{8.97}$$

where $(S_i)_j = \frac{\partial S_i}{\partial t_j}, i = 1,2,3,4; j = 1,2,3.$

Definition 8.21. We define the mixed positive-negative hypercomplex functions

$$f(x) = \begin{bmatrix} y_1 & -2y_3 & -2y_3 & -y_4 \\ y_2 & y_1 & -y_4 & -y_3 \\ y_3 & y_4 & y_1 & y_2 \\ y_4 & 2y_2 & -2y_3 & y_1 \end{bmatrix} = y_1 + y_2e_1 + y_3e_2 + y_4e_3. \tag{8.98}$$

From (8.98) we have the Cauchy-Riemann equations

$$\frac{\partial y_1}{\partial x_1} = \frac{\partial y_2}{\partial x_2} = \frac{\partial y_3}{\partial x_3} = \frac{\partial y_4}{\partial x_4},$$

$$-2\frac{\partial y_3}{\partial x_1} = \frac{\partial y_1}{\partial x_2} = \frac{\partial y_4}{\partial x_3} = \frac{2\partial y_2}{\partial x_4},$$

$$-2\frac{\partial y_2}{\partial x_1} = -\frac{\partial y_4}{\partial x_2} = \frac{\partial y_1}{\partial x_3} = -2\frac{\partial y_3}{\partial x_4},$$

$$-\frac{\partial y_4}{\partial x_1} = -\frac{\partial y_3}{\partial x_2} = \frac{\partial y_2}{\partial x_3} = \frac{\partial y_1}{\partial x_4}, \tag{8.99}$$

If y_i are 4-times continuously differentiable functions, from (8.99) we have the mixed partial differential equation

$$\begin{vmatrix} \frac{\partial}{\partial x_1} & -2\frac{\partial}{\partial x_3} & -2\frac{\partial}{\partial x_2} & -\frac{\partial}{\partial x_4} \\ \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_1} & -\frac{\partial}{\partial x_4} & -\frac{\partial}{\partial x_3} \\ \frac{\partial}{\partial x_3} & \frac{\partial}{\partial x_4} & \frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2} \\ \frac{\partial}{\partial x_4} & 2\frac{\partial}{\partial x_2} & -2\frac{\partial}{\partial x_3} & \frac{\partial}{\partial x_1} \end{vmatrix} y_1 = 0 \tag{8.100}$$

Definition 8.22. We define the generalised negative-positive hypercomplex numbers

$$x = \begin{bmatrix} x_1 & -r_1x_2 & r_2x_3 & -r_1r_2x_4 \\ x_2 & x_1 & r_2x_4 & r_2x_3 \\ x_3 & -r_1x_4 & x_1 & -r_1x_2 \\ x_4 & x_3 & x_2 & x_1 \end{bmatrix} = x_1 + x_2I_2 + x_3J_2 + x_4k, \tag{8.101}$$

where r_1 and r_2 are real numbers

$$I_2 = \begin{bmatrix} 0 & -r_1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -r_1 \\ 0 & 0 & 1 & 0 \end{bmatrix}, I_2^2 = -r_1,$$

$$J_2 = \begin{bmatrix} 0 & 0 & r_2 & 0 \\ 0 & 0 & 0 & r_2 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, J_2^2 = r_2,$$

$$k = I_2J_2 = \begin{bmatrix} 0 & 0 & 0 & -r_1r_2 \\ 0 & 0 & r_2 & 0 \\ 0 & -r_1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}, k^2 = -r_1r_2.$$

From (8.101) we have the exponential formula

$$x = R\exp(t_1I_2 + t_2J_2 + t_3k) = R(S_1 + S_2I_2 + S_3J_2 + S_4k), \tag{8.102}$$

where

$$S_1 = \frac{1}{2}\left(e^{\sqrt{r_2}t_2}\cos(\sqrt{r_1}t_1 + \sqrt{r_1r_2}t_3) + e^{-\sqrt{r_2}t_2}\cos(\sqrt{r_1}t_1 - \sqrt{r_1r_2}t_3)\right),$$

$$S_2 = \frac{1}{2\sqrt{r_1}}\left(e^{\sqrt{r_2}t_2}\sin(\sqrt{r_1}t_1 + \sqrt{r_1r_2}t_3) + e^{-\sqrt{r_2}t_2}\sin(\sqrt{r_1}t_1 - \sqrt{r_1r_2}t_3)\right),$$

$$S_3 = \frac{1}{2\sqrt{r_2}}\left(e^{\sqrt{r_2}t_2}\cos(\sqrt{r_1}t_1 + \sqrt{r_1r_2}t_3) - e^{-\sqrt{r_2}t_2}\cos(\sqrt{r_1}t_1 - \sqrt{r_1r_2}t_3)\right),$$

$$S_4 = \frac{1}{2\sqrt{r_1r_2}}\left(e^{\sqrt{r_2}t_2}\sin(\sqrt{r_1}t_1 + \sqrt{r_1r_2}t_3) - e^{-\sqrt{r_2}t_2}\sin(\sqrt{r_1}t_1 - \sqrt{r_1r_2}t_3)\right). \tag{8.103}$$

with

$$t_1 = \frac{1}{2\sqrt{r_1}}\tan^{-1}\frac{2\sqrt{r_1}(x_1x_2 - x_3x_4r_2)}{x_1^2 - r_1x_2^2 - r_2x_3^2 + r_1r_2x_4^2},$$

$$t_2 = \frac{1}{2\sqrt{r_2}}\tanh^{-1}\frac{2\sqrt{r_2}(x_1x_3 + x_2x_4r_1)}{x_1^2 + r_1x_2^2 + r_2x_3^2 + r_1r_2x_4^2},$$

$$t_3 = \frac{1}{2\sqrt{r_1 r_2}} \tan^{-1} \frac{2\sqrt{r_1 r_2}(-x_2 x_3 + x_1 x_4)}{x_1^2 + r_1 x_2^2 - r_2 x_3^2 - r_1 r_2 x_4^2}.$$

From (8.103) we have S_i an identity

$$\begin{vmatrix} S_1 & -r_1 S_2 & r_2 S_3 & -r_1 r_2 S_4 \\ S_2 & S_1 & r_2 x_4 & r_2 S_3 \\ S_3 & -r_1 S_4 & x_1 & -r_1 S_2 \\ S_4 & S_3 & S_2 & S_1 \end{vmatrix} = \begin{vmatrix} S_1 & (S_1)_1 & (S_1)_2 & (S_1)_3 \\ S_2 & (S_2)_1 & (S_2)_2 & (S_2)_3 \\ S_3 & (S_3)_1 & (S_3)_2 & (S_3)_3 \\ S_4 & (S_4)_1 & (S_4)_2 & (S_4)_3 \end{vmatrix} = 1. \tag{8.104}$$

where $(S_i)_j = \frac{\partial S_i}{\partial t_j}, i = 1, 2, 3, 4, j = 1, 2, 3.$

Definition 8.23. We define the generalised negative positive hypercomplex function

$$f(x) = \begin{bmatrix} y_1 & -r_1 y_2 & r_2 y_3 & -r_1 r_2 y_4 \\ y_2 & y_1 & r_2 y_4 & r_2 y_3 \\ y_3 & -r_1 y_4 & y_1 & -r_1 y_2 \\ y_4 & y_3 & y_2 & y_1 \end{bmatrix} = y_1 + y_2 I_2 + y_3 J_2 + y_4 k. \tag{8.105}$$

From (8.105) we have the Cauchy-Riemann equations

$$\begin{aligned} \frac{\partial y_1}{\partial x_1} &= \frac{\partial y_2}{\partial x_2} = \frac{\partial y_3}{\partial x_3} = \frac{\partial y_4}{\partial x_4}, \\ -r_1 \frac{\partial y_2}{\partial x_1} &= \frac{\partial y_1}{\partial x_2} = -r_1 \frac{\partial y_4}{\partial x_3} = \frac{\partial y_3}{\partial x_4}, \\ r_2 \frac{\partial y_3}{\partial x_1} &= r_2 \frac{\partial y_4}{\partial x_2} = \frac{\partial y_1}{\partial x_3} = \frac{\partial y_2}{\partial x_4}, \\ -r_1 r_2 \frac{\partial y_4}{\partial x_1} &= r_2 \frac{\partial y_3}{\partial x_2} = -r_1 \frac{\partial y_2}{\partial x_3} = \frac{\partial y_1}{\partial x_4}. \end{aligned} \tag{8.106}$$

From (8.106) we have

$$\begin{aligned} \frac{\partial^2 y_1}{\partial^2 x_1} + \frac{\partial^2 y_1}{r_1 \partial^2 x_2} &= 0, \\ \frac{\partial^2 y_1}{\partial^2 x_1} - \frac{\partial^2 y_1}{r_2 \partial^2 x_3} &= 0, \\ \frac{\partial^2 y_1}{\partial^2 x_1} + \frac{\partial^2 y_1}{r_1 r_2 \partial^2 x_4} &= 0. \end{aligned}$$

If y_i are 4-times continuously differentiable functions, from (8.106) we have the generalised partial differential equation of mixed type

$$\begin{vmatrix} \frac{\partial}{\partial x_1} & -\frac{\partial}{r_1 \partial x_2} & \frac{\partial}{r_2 \partial x_3} & -\frac{\partial}{r_1 r_2 \partial x_4} \\ \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_1} & \frac{\partial}{r_2 \partial x_4} & \frac{\partial}{r_2 \partial x_3} \\ \frac{\partial}{\partial x_3} & -\frac{\partial}{r_1 \partial x_4} & \frac{\partial}{\partial x_1} & -\frac{\partial}{r_1 \partial x_2} \\ \frac{\partial}{\partial x_4} & \frac{\partial}{\partial x_3} & \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_1} \end{vmatrix} y_1 = 0 \tag{8.107}$$

9. Chaotic Isomathematics

Over the past decade chaos has become a very lively subject of scientific study. While chaos was known already to Poincaré, the study of chaos found a renaissance after the publication of the Lorenz model of turbulence [15]. There is yet another way to study chaos, one may use difference equations [16, 17].

The prime principle, mostly written after 1979 [19–26], a few ideas dating back to the sixties, all have one central idea: affinity between the prime principle and chaos (nonlinear dynamics). The idea must have been at the back of my mind for many years, but only recently did I fully realize its importance.

It is suggested that the prime principle: a prime number is irreducible in the integers. It seems therefore natural to associate it with most stable system and the symmetrical system: the most stable configuration of two prime numbers is the stable symmetric system [20–24]. The two principles may be applied to the natural and the social sciences [18–24]. The two principles are only to study the stable state of many-body problems. The dynamics of the two principles are to study the nonlinear equations which are classical and quantum chaos.

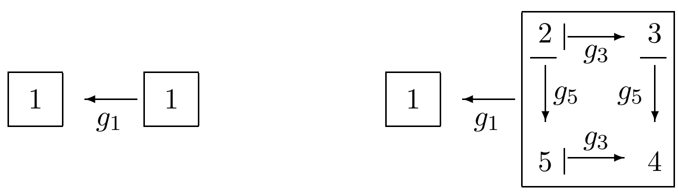
Now we study the dynamics of prime principle. The p-nonlinear equations are derived from the prime principle. It defines the $p-$chaotic functions and studies the local solutions and global stability for the nonlinear equations.

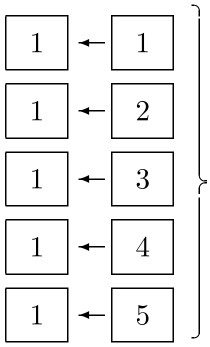
It suggests a new method for studying nonlinear equations that studying the stability and instability of the nonlinear equations is transformed into studying stability and instability for the nonlinear terms of

$$\frac{dA}{dt} = \sum \frac{dN_i}{dt}$$

in the chaotic equations.

Now we study a relationship between the prime principle and the $p-$nonlinear equation. In the previous papers [18, 14] we are able to draw the commutative diagram with a weak map g_1 by using theory of stable group and unstable group for $p = 5$ and 1 to be a stable point.





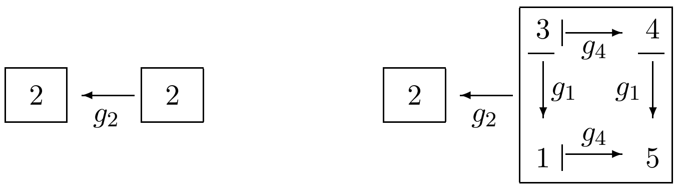
can be represented as

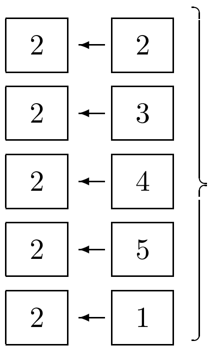
$a_{11}N_1N_1$
$a_{12}N_1N_2$
$a_{13}N_1N_3$
$a_{14}N_1N_4$
$a_{15}N_1N_5$

where $a_{11}, a_{12}, a_{13}, a_{14}$ and a_{15} denote interaction (competition) coefficients. The equation of N_1 growth rate is represented by using a differential equation of first order

$$dN_1/dt = N_1(1 - a_{11}N_1 - a_{12}N_2 - a_{13}N_3 - a_{14}N_4 - a_{15}N_5) \qquad (9.1)$$

If we define 2 to be a stable point, then $(1, 2, 3, 4, 5) \to (2, 3, 4, 5, 1)$. We are able to draw the commutative diagram with a weak map g_2 by using the theory of stable group and unstable group.





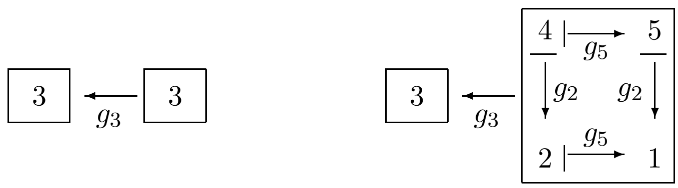
can be represented as

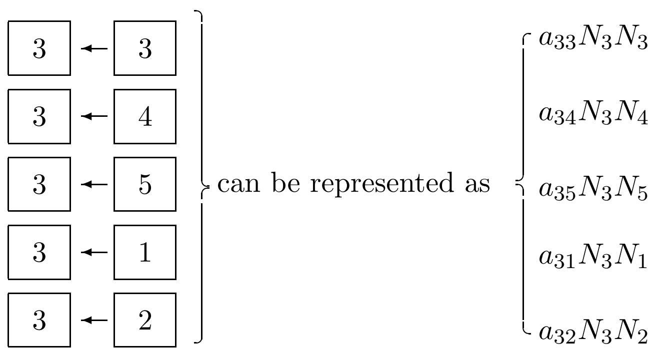
$a_{22}N_2N_2$
$a_{23}N_2N_3$
$a_{24}N_2N_4$
$a_{25}N_2N_5$
$a_{21}N_2N_1$

where $a_{22}, a_{23}, a_{24}, a_{25}$ and a_{21} denote interaction (competition) coefficients. The equation of N_2 growth rate is represented by using a differential equation of first order

$$dN_2/dt = N_2(1 - a_{21}N_1 - a_{22}N_2 - a_{23}N_3 - a_{24}N_4 - a_{25}N_5) \qquad (9.2)$$

If we define 3 to be a stable point, then $(1, 2, 3, 4, 5) \to (3, 4, 5, 1, 2)$. We are able to draw the commutative diagram with a weak map g_3 by using theory of stable group and unstable group.

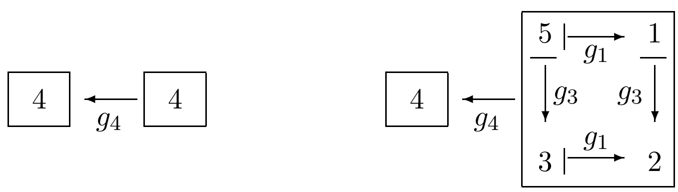


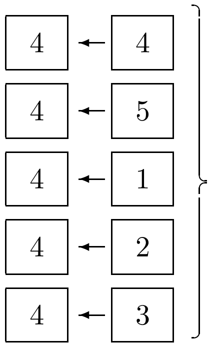


where $a_{33}, a_{34}, a_{35}, a_{31}$ and a_{32} denote interaction (competition) coefficients. The equation of N_3 growth rate is represented by using a differential equation of first order

$$dN_3/dt = N_3(1 - a_{31}N_1 - a_{32}N_2 - a_{33}N_3 - a_{34}N_4 - a_{35}N_5) \tag{9.3}$$

If we define 4 to be a stable point, then $(1, 2, 3, 4, 5) \to (4, 5, 1, 2, 3)$. We are able to draw the commutative diagram with a weak map g_4 by using theory of stable group and unstable group.





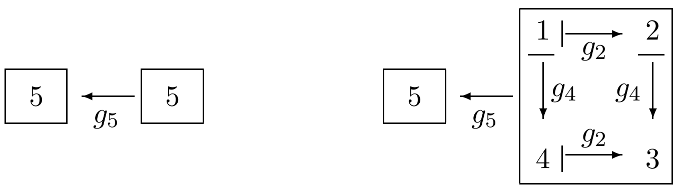
can be represented as

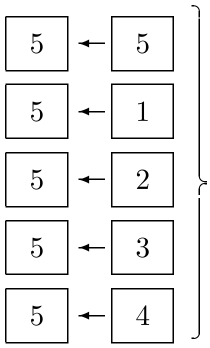
$$\begin{cases} a_{44}N_4N_4 \\ a_{45}N_4N_5 \\ a_{41}N_4N_1 \\ a_{42}N_4N_2 \\ a_{43}N_4N_3 \end{cases}$$

where $a_{44}, a_{45}, a_{41}, a_{42}$ and a_{43} denote interaction (competition) coefficients. The equation of N_4 growth rate is represented by using a differential equation of first order

$$dN_4/dt = N_4(1 - a_{41}N_1 - a_{42}N_2 - a_{43}N_3 - a_{44}N_4 - a_{45}N_5) \tag{9.4}$$

If we define 5 to be a stable point, then $(1, 2, 3, 4, 5) \to (5, 1, 2, 3, 4)$. We are able to draw the commutative diagram with a weak map g_5 by using theory of stable group and unstable group.





can be represented as

$$\begin{cases} a_{55}N_5N_5 \\ a_{51}N_5N_1 \\ a_{52}N_5N_2 \\ a_{53}N_5N_3 \\ a_{54}N_5N_4 \end{cases}$$

where $a_{55}, a_{51}, a_{52}, a_{53}$ and a_{54} denote interaction (competition) coefficients. The equation of N_5 growth rate is represented by using a differential equation of first order

$$dN_5/dt = N_5(1 - a_{51}N_1 - a_{52}N_2 - a_{53}N_3 - a_{54}N_4 - a_{55}N_5) \tag{9.5}$$

From the above analysis we can obtain the $p-$nonlinear equations

$$dN_i/dt = N_i(1 - \sum_{j=1}^{p} a_{ij}N_j), \tag{9.6}$$

where $i = 1, 2, \cdots, p, p =$ prime number

$$(a_{ij}) = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1p} \\ a_{21} & a_{22} & \cdots & a_{2p} \\ \cdots & & & \\ a_{p1} & a_{p2} & \cdots & a_{pp} \end{bmatrix}$$

(a_{ij}) denotes the matrix of interaction (competition) coefficients. The equation (9.6) is defined to be the dynamics of the prime principle which produces the classical and quantum chaos.

9.1. $p-$Nonlinear Dynamics

In order to study solutions of the $p-$nonlinear equation (9.6) first from (2.1) we define the $p-$chaotic function

$$N_i = \frac{1}{p}\Big[A + 2\sum_{j=1}^{(p-1)/2}(-1)^{(i-1)j}B_j\cos(\theta_j + (-1)^j\frac{(i-1)j\pi}{p})\Big], \tag{9.7}$$

where $i = 1, 2, \cdots, p$.

From the equation (9.7) we obtain its inverse transformation

$$A = \sum_{i=1}^{p} N_i,$$

$$B_j\cos\theta_j = N_1 + \sum_{i=1}^{p-1}(-1)^{ij}N_{1+i}\cos\frac{ij\pi}{p}, \tag{9.8}$$

$$B_j\sin\theta_j = (-1)^{j+1}\sum_{i=1}^{p-1}(-1)^{ij}N_{1+i}\sin\frac{ij\pi}{p}.$$

The equations (9.7) and (9.8) are a nonlinear transformation group. From the equation (9.8) we obtain

$$\frac{dA}{dt} = \sum_{i=1}^{p}\frac{dN_i}{dt},$$

$$\frac{dB_j}{dt} = \sum_{i=1}^{p} \frac{dN_i}{dt}(-1)^{(i-1)j} \cos\left(\theta_j + (-1)^j \frac{(i-1)j\pi}{p}\right), \tag{9.9}$$

$$B_j \frac{d\theta_j}{dt} = (-1)^{j+1} \sum_{i=1}^{p} \frac{dN_i}{dt}(-1)^{ij} \sin\left(\theta_j + (-1)^j \frac{(i-1)j\pi}{p}\right),$$

where $j = 1, 2, \cdots, (p-1)/2$.

We study the nonlinear equation (9.6) and other nonlinear equations by using the equations (9.7) and (9.9).

We study the analytic study of the equation (9.6) on the assumption that it is a cyclical matrix. The equation (9.6) may be written into

$$dN_i/dt = N_i\left(1 - \sum_{j=1}^{p} \bar{a}_{ij} N_j\right), \tag{9.10}$$

where

$$(\bar{a}_{ij}) = \begin{bmatrix} 1 & a_1 & \cdots & a_{p-1} \\ a_{p-1} & 1 & \cdots & a_{p-2} \\ \vdots & \vdots & \vdots & \vdots \\ a_1 & a_2 & \cdots & 1 \end{bmatrix}$$

The fixed point solutions of the equation (9.10) may be expressed as points in the $p-$dimensional space: they are the point $(0, 0, \cdots, 0)$; p single-point solutions of the form $(1, 0, \cdots, 0)$; a nontrivial fixed point $(1, 1, \cdots, 1)/\lambda_0$, where

$$\lambda_0 = 1 + \sum_{j=1}^{p-1} a_j.$$

Now we study the local solutions of the equation (9.10) at the fixed point $(1, 1, \ldots, 1)/\lambda_0$. The equation (9.10) may be written into

$$\frac{d \ln N_i}{dt} = 1 - \sum_{j=1}^{p} \bar{a}_{ij} N_j, \tag{9.11}$$

$\frac{d \ln N_i}{dt}$ is the average growth rate of i-th species. Setting $\lambda_0 N_i = e^{y_i}$, the equation (9.11) is transformed into

$$\frac{dy_i}{dt} = 1 - \sum_{j=1}^{p} \bar{a}_{ij} \frac{1}{\lambda_0} e^{y_j}. \tag{9.12}$$

We study the linear solutions of the equation (9.12). Setting $e^{y_j} = 1 + y_j$, the equation (9.12) is transformed into

$$\frac{dy_i}{dt} = -\frac{1}{\lambda_0} \sum_{j=1}^{p} \bar{a}_{ij} y_j. \tag{9.13}$$

The equation (9.13) has the following exact solutions

$$y_i = \frac{1}{p}\left[e^{-t} + 2\sum_{j=1}^{(p-1)/2} (-1)^{(i-1)j} e^{\frac{-\lambda_j}{\lambda_0}t} \cos\left(\frac{\eta_j}{\lambda_0}t + (-1)^j \frac{(i-1)j\pi}{p}\right)\right], \tag{9.14}$$

where $i = 1, 2, \ldots, p, \lambda_0 = 1 + \sum_{j=1}^{p-1} a_j$,

$$\lambda_j = 1 + \sum_{i=1}^{p-1} (-1)^{ij} a_i \cos\frac{ij\pi}{p},$$

$$\eta_j = (-1)^{j+1} \sum_{i=1}^{p-1} (-1)^{ij} a_i \sin\frac{ij\pi}{p}.$$

A necessary and sufficient condition for stability of the equation (9.14) is $\lambda_j > 0$. If $\lambda_j < 0$, then their solutions are unstable.

Now we study the global stability of the equation (9.10) by using the p-chaotic functions. By substituting the equations (9.7) and (9.10) into the $\frac{dA}{dt}$ of the equation (9.9). We obtain the p-chaotic equation

$$\frac{dA}{dt} = A - \frac{1}{p}\left[\lambda_0 A^2 + 2\sum_{j=1}^{(p-1)/2} \lambda_j B_j^2\right] \tag{9.15}$$

The λ_0 and λ_j in both equations (9.14) and (9.15) are the same. We will discuss stability and instability of the nonlinear terms of the equation (9.15)

$$\left(\frac{dA}{dt}\right)_{A^2} = -\frac{1}{p}\lambda_0 A^2. \tag{9.16}$$

For $\lambda_0 > 0$, it is stable,

$$\left(\frac{dA}{dt}\right)_{B_j^2} = -\frac{2}{p}\lambda_j B_j^2. \tag{9.17}$$

If $\lambda_j > 0$, it is stable. If $\lambda_j < 0$, it is unstable. The results obtained from the equations (9.14) and (9.17) are the same. From the above analysis we come to a conclusion that studying stability and instability of the equation (9.10) is transformed into studying stability and instability of the nonlinear terms of

$$\frac{dA}{dt} = \sum \frac{dN_j}{dt}$$

in the equation (9.15). The solutions of the nonlinear equations are simplified by the above method. It provides a useful tool for studying the nonlinear equations.

For $p = 3$ from the equation (9.10) we obtain

$$\frac{dN_i}{dt} = N_i\Big(1 - \sum_{j=1}^{3} \bar{a}_{ij}N_j\Big), \tag{9.18}$$

$$i = 1, 2, 3; \quad (\bar{a}_{ij}) = \begin{bmatrix} 1 & a_1 & a_2 \\ a_2 & 1 & a_1 \\ a_1 & a_2 & 1 \end{bmatrix}$$

where $N_i(t)$ is the number of individuals of the i-th species and a_1 and a_2 denote interaction coefficients. The fixed point solutions may be expressed as points in the 3-dimensional space: they are point $(0, 0, 0)$; 3 single-point solutions of the form $(1, 0, 0)$; a nontrivial fixed point $(1, 1, 1)/\lambda_0$ where $\lambda_0 = 1 + a_1 + a_2$. The equation (9.18) has been studied by Busse in turbulence [27], May and Leonard in the population [28] and Roy and Splimano in biology [29].

We study the local solutions of the equation (9.18) at the fixed point $(1, 1, 1)/\lambda_0$. The equation (9.18) may be written into

$$\begin{aligned} \frac{d\ln N_1}{dt} &= 1 - N_1 - a_1N_2 - a_2N_3, \\ \frac{d\ln N_2}{dt} &= 1 - a_2N_1 - N_2 - a_1N_3, \\ \frac{d\ln N_3}{dt} &= 1 - a_1N_1 - a_2N_2 - N_3. \end{aligned} \tag{9.19}$$

Setting $\lambda_0 N_i = e^{y_i}$, the equation (9.19) is transformed into

$$\begin{aligned} \frac{dy_1}{dt} &= 1 - \frac{1}{\lambda_0}(e^{y_1} + a_1e^{y_2} + a_2e^{y_3}), \\ \frac{dy_2}{dt} &= 1 - \frac{1}{\lambda_0}(a_2e^{y_1} + e^{y_2} + a_1e^{y_3}), \\ \frac{dy_3}{dt} &= 1 - \frac{1}{\lambda_0}(a_1e^{y_1} + a_2e^{y_2} + e^{y_3}). \end{aligned} \tag{9.20}$$

We study the linear solutions of the equation (9.20). Setting $e^{y_i} = 1 + y_i$, the equation (9.20)is transformed into

$$\begin{aligned} \frac{dy_1}{dt} &= -\frac{1}{\lambda_0}(y_1 + a_1y_2 + a_2y_3), \\ \frac{dy_2}{dt} &= -\frac{1}{\lambda_0}(a_2y_1 + y_2 + a_1y_3), \end{aligned} \tag{9.21}$$

$$\frac{dy_3}{dt} = -\frac{1}{\lambda_0}(a_1 y_1 + a_2 y_2 + y_3).$$

The equation (9.21) has the following exact solutions

$$y_1 = \frac{1}{3}\left[e^{-t} + 2e^{\frac{-\lambda_1}{\lambda_0}t}\cos\left(\frac{\eta_1}{\lambda_0}t\right)\right],$$

$$y_1 = \frac{1}{3}\left[e^{-t} - 2e^{\frac{-\lambda_1}{\lambda_0}t}\cos\left(\frac{\eta_1}{\lambda_0}t - \frac{\pi}{3}\right)\right], \tag{9.22}$$

$$y_1 = \frac{1}{3}\left[e^{-t} + 2e^{\frac{-\lambda_1}{\lambda_0}t}\cos\left(\frac{\eta_1}{\lambda_0}t - \frac{2\pi}{3}\right)\right].$$

where $\lambda_0 = 1 + a_1 + a_2, \lambda_1 = 1 - \frac{(a_1+a_2)}{2}, \eta_1 = \frac{\sqrt{3}}{2}(a_1 - a_2)$.

A necessary and sufficient condition for stability of the equation (9.22) is $\lambda_1 > 0$, namely $a_1 + a_2 < 2$. If $\lambda_1 < 0$, namely $a_1 + a_2 > 2$, it is unstable.

We study the global stability of the equation (9.18) by using the 3-chaotic function. From the equation (9.7) we obtain

$$N_1 = \frac{1}{3}[A + 2B\cos\theta]$$

$$N_2 = \frac{1}{3}\left[A - 2B\cos\left(\theta - \frac{\pi}{3}\right)\right], \qquad (9.23)],$$

$$N_3 = \frac{1}{3}\left[A + 2B\cos\left(\theta - \frac{2\pi}{3}\right)\right].$$

From the equation (9.23) we obtain

$$\frac{dA}{dt} = \frac{dN_1}{dt} + \frac{dN_2}{dt} + \frac{dN_3}{dt},$$

$$\frac{dB}{dt} = \cos\theta\frac{dN_1}{dt} - \cos(\theta - \frac{\pi}{3})\frac{dN_2}{dt} + \cos(\theta - \frac{2\pi}{3})\frac{dN_3}{dt}, \tag{9.24}$$

$$B\frac{d\theta}{dt} = -\sin\theta\frac{dN_1}{dt} + \sin(\theta - \frac{\pi}{3})\frac{dN_2}{dt} - \sin(\theta - \frac{2\pi}{3})\frac{dN_3}{dt}.$$

By substituting the equations (9.18) and (9.23) into the equation (9.24) we obtain 3-chaotic equation

$$\frac{dA}{dt} = A - \frac{1}{3}[\lambda_0 A^2 + 2\lambda_1 B^2],$$

$$\frac{dB}{dt} = B - \frac{1}{3}[(\lambda_0 + \lambda_1)AB + B^2(\lambda_1\cos 3\theta + \eta_1\sin 3\theta)], \tag{9.25}$$

$$B\frac{d\theta}{dt} = \frac{1}{3}[\eta_1 AB - B^2(-\lambda_1\sin 3\theta + \eta_1\cos 3\theta)],$$

λ_0, λ_1 and η_1 in the equation (9.22) and (9.25) are the same. If $a_1 = a_2 = 1$, then $\lambda_0 = 3, \lambda_1 = \eta_1 = 0$.

From the equation (9.25) we obtain a special solution as follows

$$A = \frac{A(0)}{A(0) + (1 - A(0))e^{-t}}. \tag{9.26}$$

We now discuss stability and instability of the nonlinear terms of $\frac{dA}{dt}$ in the equation (9.25)

$$\left(\frac{dA}{dt}\right)_{A^2} = -\frac{\lambda_0}{3}A^2. \tag{9.27}$$

For $\lambda_0 > 0$, it is stable.

$$\left(\frac{dA}{dt}\right)_{B^2} = -\frac{2}{3}\lambda_1 B^2. \tag{9.28}$$

If $\lambda_1 > 0$, namely $a_1 + a_2 < 2$, it is stable. If $\lambda_1 < 0$, namely $a_1 + a_2 > 2$, it is unstable. The results obtained from the equations (9.22) and (9.28) are the same. Studying stability and instability of the equation (9.18) is transformed into studying stability and instability of the nonlinear terms of

$$\frac{dA}{dt} = \frac{dN_1}{dt} + \frac{dN_2}{dt} + \frac{dN_3}{dt}$$

at the equation (9.25). This method allows us to study general nonlinear equations including the Lorenz equation which produces the chaotic manifolds and nonlinear dynamics.

A mechanism of the spiral chaos. By using the above idea we study the following equation

$$dN_i/dt = N_i\Big(1 - \sum_{j=1}^{3} a_{ij}N_j), \tag{9.29}$$

where

$$(a_{ij}) = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

(a_{ij}) denotes interaction coefficients.

The fixed point solutions may be expressed as points in the 3-dimensional space: they are $(0, 0, 0)$ and (N_{10}, N_{20}, N_{30}). By substituting the equations (9.29) and (9.23) into the equation (9.24) we obtain

$$\frac{dA}{dt} = A - \frac{1}{9}\{\beta_0 A^2 + B^2[\beta_3 - \beta_4 + (\beta_2 - \beta_3 - \beta_4)\cos 2\theta - (\eta_2 + \eta_3 + \eta_4)\sin 2\theta]$$
$$+[(\beta_1 + \beta_3 + \beta_4)\cos\theta - (\eta_1 + \eta_3 + \eta_4)\sin\theta]AB\},$$

$$\frac{dB}{dt} = B - \frac{1}{18}\{[\beta_1 \cos\theta - \eta_1 \sin\theta]A^2 + [(\beta_2 + \beta_3 + \beta_4)\cos\theta$$
$$+(\eta_2 - \eta_3 - \eta_4)\sin\theta + (\beta_3 - \beta_4)\cos 3\theta + (\eta_3 - \eta_4)\sin 3\theta]B^2 + [\beta_5 + \beta_6$$
$$+(\beta_5 - \beta_6)\cos 2\theta - (\eta_5 + \eta_6)\sin 2\theta]AB\}, \tag{9.30}$$
$$B\frac{d\theta}{dt} = \frac{1}{18}\{(\beta_1 \sin\theta + \eta_1 \cos\theta)A^2 + [(\beta_2 - \beta_3 - \beta_4)\sin\theta$$
$$-(\eta_2 - \eta_3 - \eta_4)\cos\theta + (\beta_3 - \beta_4)\sin 3\theta + (-\eta_3 + \eta_4)\cos 3\theta]B^2$$
$$+[(-\eta_5 + \eta_6) + (\beta_5 - \beta_6)\sin 2\theta + (\eta_5 + \eta_6)\cos 2\theta]AB\}$$

where

$$\beta_0 = a_{11} + a_{22} + a_{33} + a_{12} + a_{21} + a_{13} + a_{31} + a_{23} + a_{32},$$
$$\beta_1 = 2a_{11} + 2a_{12} + 2a_{13} - a_{21} - a_{22} - a_{23} - a_{31} - a_{32} - a_{33},$$
$$\eta_1 = \sqrt{3}(a_{21} + a_{22} + a_{23} - a_{31} - a_{32} - a_{33}),$$
$$\beta_2 = 4a_{11} - 2a_{12} - 2a_{13} + a_{21} - 2a_{22} + a_{23} + a_{31} + a_{32} - 2a_{33},$$
$$\eta_2 = \sqrt{3}(a_{21} - 2a_{22} + a_{23} - a_{31} - a_{32} + 2a_{33}),$$
$$\beta_3 = \frac{1}{2}(4a_{11} - 2a_{12} - 2a_{13} + a_{21} + a_{22} - 2a_{23} + a_{31} - 2a_{32} + a_{33}),$$
$$\eta_3 = \frac{\sqrt{3}}{2}(a_{21} + a_{22} - 2a_{23} - a_{31} + 2a_{32} - a_{33}),$$
$$\beta_4 = \frac{3}{2}(a_{21} - a_{22} + a_{31} - a_{33}),$$
$$\eta_4 = \frac{\sqrt{3}}{2}(2a_{12} - 2a_{13} - a_{21} + a_{22} + a_{31} - a_{33}),$$
$$\beta_5 = \frac{1}{2}(8a_{11} + 2a_{12} + 2a_{13} - a_{21} + 2a_{22} + 2a_{23} - a_{31} + 2a_{32} + 2a_{33}),$$
$$\eta_5 = \frac{\sqrt{3}}{2}(2a_{12} - 2a_{13} - a_{21} - 2a_{22} + a_{31} + 2a_{33}),$$
$$\beta_6 = \frac{3}{2}(a_{21} + 2a_{22} + a_{31} + 2a_{33}),$$
$$\eta_6 = \frac{\sqrt{3}}{2}(a_{21} - 2a_{22} - 2a_{23} - a_{31} + 2a_{32} + 2a_{33}).$$

Now we discuss stability and instability of the nonlinear terms of $\frac{dA}{dt}$ in the equation (9.30)

$$\left(\frac{dA}{dt}\right)_{A^2} = -\frac{1}{9}\beta_0 A^2 \tag{9.31}.$$

For $\beta_0 > 0, (\frac{dA}{dt})_{A^2} < 0$, it is stable.

$$\left(\frac{dA}{dt}\right)_{B^2} = -\frac{1}{9}\Big[\beta_3 - \beta_4 + (\beta_2 - \beta_3 - \beta_4)\cos 2\theta - (\eta_2 + \eta_3 + \eta_4)\sin 2\theta\Big]B^2. \quad (9.32)$$

If $[\beta_3 - \beta_4 + (\beta_2 - \beta_3 - \beta_4)\cos 2\theta - (\eta_2 + \eta_3 + \eta_4)\sin 2\theta] > 0$, then $(\frac{dA}{dt})_{B^2} < 0$. It is stable.

If $[\beta_3 - \beta_4 + (\beta_2 - \beta_3 - \beta_4)\cos 2\theta - (\eta_2 + \eta_3 + \eta_4)\sin 2\theta] < 0$, then $(\frac{dA}{dt})_{B^2} > 0$. It is unstable.

$$\left(\frac{dA}{dt}\right)_{AB} = -\frac{1}{9}[(\beta_1 + \beta_3 + \beta_4)\cos\theta - (\eta_1 + \eta_3 + \eta_4)\sin\theta]AB. \quad (9.33)$$

If $[(\beta_1 + \beta_3 + \beta_4)\cos\theta - (\eta_1 + \eta_3 + \eta_4)\sin\theta]AB > 0$, then $(\frac{dA}{dt})_{AB} < 0$. It is stable. If $[(\beta_1 + \beta_3 + \beta_4)\cos\theta - (\eta_1 + \eta_3 + \eta_4)\sin\theta]AB < 0$, then $(\frac{dA}{dt})_{AB} > 0$. It is unstable.

From the above analysis we come to a conclusion that the sign changes of $\sin\theta$, $\cos\theta$, $\sin 2\theta$ and $\cos 2\theta$ of the nonlinear terms of $\frac{dA}{dt}$ in the equation (9.30) lead to the unstable manifolds (sources) and stable manifolds (sinks) at the fixed points in space (A, B, θ). It is a mechanism for producing the spiral chaos.

We consider the Lorenz equation

$$\begin{aligned} \frac{dN_1}{dt} &= \sigma(N_1 - N_2), \\ \frac{dN_2}{dt} &= \rho N_1 - N_2 - N_1 N_3, \\ \frac{dN_3}{dt} &= -\beta N_1 + N_1 N_2. \end{aligned} \quad (9.34)$$

It contains three constants: σ (the Prandtl number), ρ (the Rayleigh number), and β (an aspect ratio).

For $\rho > 1$, there are two nontrivial fixed points, at

$$(N_1, N_2, N_3) = (\pm\sqrt{\beta(\rho - 1)}, \pm\sqrt{\beta(\rho - 1)}, (\rho - 1)). \quad (9.35))$$

For $\rho < 1$, there is one trivial fixed point, at

$$(N_1, N_2, N_3) = (0, 0, 0). \quad (9.36)$$

From (9.23), (9.24) and (9.34) we obtain a chaotic equation

$$\frac{dA}{dt} = \frac{1}{3}[(\rho - 1 - \beta)A + (3\sigma + 2\rho - 2\beta + 1)B\cos\theta$$
$$+\sqrt{3}(\sigma + 1)\sin\theta] - \frac{(2\sqrt{3})}{9}[AB\sin\theta + B^2\sin 2\theta], \quad (9.37).$$

We now discuss the stability and instability of nonlinear terms in (9.37). We have

$$\left(\frac{dA}{dt}\right)_{AB} = -\frac{2\sqrt{3}}{9}AB\sin\theta, \quad \left(\frac{dA}{dt}\right)_{B^2} = -\frac{2\sqrt{3}}{9}B^2\sin 2\theta. \qquad (9.38).$$

For $\rho > 1$, we have two nontrivial fixed points, at

$$(A, B, \theta) = ((\pm 2\sqrt{\beta(\rho-1)} + \rho - 1), (\pm\sqrt{\beta(\rho-1)} - (\rho-1)), -\frac{\pi}{3}).$$

For $\rho < 1$, we have one nontrivial fixed point, at

$$(A, B, \theta) = (0, 0, 0)$$

For $\rho = 1$, we have one nontrivial fixed point, at

$$(A, B, \theta) = (0, 0, -\frac{\pi}{3}).$$

If $(\frac{dA}{dt})_{AB} < 0$, it is stable. If $(\frac{dA}{dt})_{AB} > 0$, it is unstable. If $(\frac{dA}{dt})_{B^2} < 0$, it is stable. If $(\frac{dA}{dt})_{B^2} > 0$, it is unstable. Chaos occurs when stable and unstable manifolds cross at nontrivial fixed points.

For $p = 5$ from the equation (9.10) we obtain

$$dN_i/dt = N_i(1 - \sum_{j=1}^{5} \bar{a}_{ij}N_j), \qquad (9.39)$$

$$(\bar{a}_{ij}) = \begin{bmatrix} 1 & a_1 & a_2 & a_3 & a_4 \\ a_4 & 1 & a_1 & a_2 & a_3 \\ a_3 & a_4 & 1 & a_1 & a_2 \\ a_2 & a_3 & a_4 & 1 & a_1 \\ a_1 & a_2 & a_3 & a_4 & 1 \end{bmatrix}$$

where N_i is the number of individuals of the i-th species.

The fixed points solutions may be expressed as points in the 5-dimensional space: they are the point $(0, 0, 0, 0, 0)$; 5 single-point solutions of the form $(1, 0, 0, 0, 0)$; a nontrivial fixed point $(1, 1, 1, 1, 1)/\lambda_0$, where

$$\lambda_0 = 1 + a_1 + a_2 + a_3 + a_4$$

Now we study the local solutions of the equation (9.39) at fixed point $(1, 1, 1, 1, 1)/\lambda_0$. The equation (9.39) may be written into

$$\frac{d\ln N_i}{dt} = 1 - \sum_{j=1}^{5} \bar{a}_{ij}N_j. \qquad (9.40)$$

Setting $\lambda_0 N_j = e^{y_j}$, then the equation (9.40) is transformed into

$$\frac{dy_i}{dt} = 1 - \frac{1}{\lambda_0}\sum_{j=1}^{5} \bar{a}_{ij} e^{y_j}. \qquad (9.41).$$

We study the linear solutions of the equation (9.41). Setting $e^{y_j} = 1 + y_i$, the equation (9.41) is transformed into

$$\frac{dy_i}{dt} = -\frac{1}{\lambda_0}\sum_{j=1}^{5} \bar{a}_{ij} y_j. \qquad (9.42)$$

The equation (9.42) has the following exact solutions

$$\begin{aligned}
y_1 &= \frac{1}{5}\left[e^{-t} + 2e^{\frac{-\lambda_1}{\lambda_0}t}\cos\left(\frac{\eta_1}{\lambda_0}t\right) + 2e^{\frac{-\lambda_2}{\lambda_0}t}\cos\left(\frac{\eta_2}{\lambda_0}t\right)\right],\\
y_2 &= \frac{1}{5}\left[e^{-t} - 2e^{\frac{-\lambda_1}{\lambda_0}t}\cos\left(\frac{\eta_1}{\lambda_0}t_1 - \frac{\pi}{5}\right) + 2e^{\frac{-\lambda_2}{\lambda_0}t}\cos\left(\frac{\eta_2}{\lambda_0}t + \frac{2\pi}{5}\right)\right],\\
y_3 &= \frac{1}{5}\left[e^{-t} + 2e^{\frac{-\lambda_1}{\lambda_0}t}\cos\left(\frac{\eta_1}{\lambda_0}t - \frac{2\pi}{5}\right) + 2e^{\frac{-\lambda_2}{\lambda_0}t}\cos\left(\frac{\eta_2}{\lambda_0}t + \frac{4\pi}{5}\right)\right], \qquad (9.43)\\
y_4 &= \frac{1}{5}\left[e^{-t} - 2e^{\frac{-\lambda_1}{\lambda_0})t}\cos(\frac{\eta_1}{\lambda_0}t - \frac{3\pi}{5}) + 2e^{\frac{-\lambda_2}{\lambda_0}t}\cos(\frac{\eta_2}{\lambda_0}t + \frac{6\pi}{5}\right)\right],\\
y_5 &= \frac{1}{5}\left[e^{-t} + 2e^{\frac{-\lambda_1}{\lambda_0}t}\cos(\frac{\eta_1}{\lambda_0}t - \frac{4\pi}{5}) + 2e^{(\frac{-\lambda_1}{\lambda_0})t}\cos(\frac{\eta_2}{\lambda_0}t + \frac{8\pi}{5}\right)\right],
\end{aligned}$$

where

$$\begin{aligned}
\lambda_0 &= 1 + a_1 + a_2 + a_3 + a_4,\\
\lambda_1 &= 1 - (a_1 + a_2)\cos\frac{\pi}{5} + (a_2 + a_3)\cos\frac{2\pi}{5},\\
\eta_1 &= (a_1 - a_4)\sin\frac{\pi}{5} - (a_2 - a_3)\sin\frac{2\pi}{5},\\
\lambda_2 &= 1 + (a_1 + a_2)\cos\frac{2\pi}{5} + (a_2 + a_3)\cos\frac{4\pi}{5},\\
\eta_2 &= (a_1 - a_4)\sin\frac{2\pi}{5} + (a_2 - a_3)\sin\frac{4\pi}{5}.
\end{aligned}$$

A necessary and sufficient condition for stability of the equation (9.43) is $\lambda_1 > 0$ and $\lambda_2 > 0$. If $\lambda_1 < 0$ and $\lambda_2 < 0; \lambda_1 < 0$ and $\lambda_2 > 0$; and $\lambda_1 > 0$ and $\lambda_2 < 0$, then it is unstable.

Now we study the global stability of the equation (9.39) by using the 5-chaotic function. From the equation (9.7) we obtain

$$N_1 = \frac{1}{5}[A + 2B_1\cos\theta_1 + 2B_2\cos\theta_2],$$

$$\begin{aligned}
N_2 &= \frac{1}{5}[A - 2B_1\cos(\theta_1 - \frac{\pi}{5}) + 2B_2\cos(\theta_2 + \frac{2\pi}{5})],\\
N_3 &= \frac{1}{5}[A + 2B_1\cos(\theta_1 - \frac{2\pi}{5}) + 2B_2\cos(\theta_2 + \frac{4\pi}{5})],\\
N_4 &= \frac{1}{5}[A - 2B_1\cos(\theta_1 - \frac{3\pi}{5}) + 2B_2\cos(\theta_2 + \frac{6\pi}{5})],\\
N_5 &= \frac{1}{5}[A + 2B_1\cos(\theta_1 - \frac{4\pi}{5}) + 2B_2\cos(\theta_2 + \frac{8\pi}{5})].
\end{aligned} \tag{9.44}$$

From the equation (9.44) we obtain

$$\frac{dA}{dt} = \sum_{i=1}^{5} \frac{dN_i}{dt},$$

$$\begin{aligned}
\frac{dB_1}{dt} &= \sum_{i=0}^{4}(-1)^i\cos(\theta_1 - \frac{i\pi}{5})\frac{dN_{i+1}}{dt},\\
B_1\frac{d\theta_1}{dt} &= \sum_{i=0}^{4}(-1)^{i+1}\sin(\theta_1 - \frac{i\pi}{5})\frac{dN_{i+1}}{dt},\\
\frac{dB_2}{dt} &= \sum_{i=0}^{4}\cos(\theta_2 + \frac{2\pi i}{5})\frac{dN_{i+1}}{dt},\\
B_2\frac{d\theta_2}{dt} &= -\sum_{i=0}^{4}\sin(\theta_2 + \frac{2\pi i}{5})\frac{dN_{i+1}}{dt}.
\end{aligned} \tag{9.45}$$

By substituting the equation (9.39) and (9.44) into the equation (9.45), we obtain the 5-chaotic equation

$$\begin{aligned}
\frac{dA}{dt} &= A - \frac{1}{5}[\lambda_0 A^2 + 2\lambda_1 B_1^2 + 2\lambda_2 B_2^2],\\
\frac{dB_1}{dt} &= B_1 - \frac{1}{5}\{(\lambda_0+\lambda_1)AB_1 + B_2^2[\lambda_2\cos(2\theta_2-\theta_1) + \eta_2\sin(2\theta_2-\theta_1)]\\
&\quad +B_1B_2[(\lambda_1+\lambda_2)\cos(2\theta_1+\theta_2) + (\eta_1+\eta_2)\sin(2\theta_1+\theta_2)]\},\\
B_1\frac{d\theta_1}{dt} &= \frac{1}{5}\{\eta_1 AB_1 - B_2^2[\lambda_2\sin(2\theta_2-\theta_1) - \eta_2\cos(2\theta_2-\theta_1)]\\
&\quad +B_1B_2[(\lambda_1+\lambda_2)\sin(2\theta_1+\theta_2) - (\eta_1+\eta_2)\cos(2\theta_1+\theta_2)]\},\\
\frac{dB_2}{dt} &= B_2 - \frac{1}{5}\{(\lambda_0+\lambda_2)AB_2 + B_1^2[\lambda_1\cos(2\theta_1+\theta_2) + \eta_1\sin(2\theta_1+\theta_2)]\\
&\quad +B_1B_2[(\lambda_1+\lambda_2)\cos(2\theta_2-\theta_1) + (-\eta_1+\eta_2)\sin(2\theta_2-\theta_1)]\},
\end{aligned} \tag{9.46}$$

$$B_2\frac{d\theta_2}{dt} = \frac{1}{5}\{\eta_2 AB_2 + B_1^2[\lambda_1 \sin(2\theta_1+\theta_2) - \eta_1\cos(2\theta_1+\theta_2)]$$
$$+B_1B_2[(\lambda_1+\lambda_2)\sin(2\theta_2-\theta_1) + (\eta_1-\eta_2)\cos(2\theta_2-\theta_1)]\}.$$

$\lambda_0, \lambda_1, \lambda_2, \eta_1$ and η_2 in both equations (9.43) and (9.46) are same. If $a_i = 1$, where $i = 1, 2, 3, 4$ and 5, then $\lambda_0 = 5$, and $\lambda_1 = \lambda_2 = \eta_1 = \eta_2 = 0$. From the equation (9.46) we obtain a special solution

$$A = \frac{A(0)}{A(0) + (1 - A(0))e^{-t}}. \tag{9.47}$$

Now we discuss stability and instability of the nonlinear terms of $\frac{dA}{dt}$ in the equation (9.46).

$$\left(\frac{dA}{dt}\right)_{A^2} = -\frac{1}{5}\lambda_0 A^2. \tag{9.48}$$

It is stable for $\lambda_0 > 0$.

$$\left(\frac{dA}{dt}\right)_{B_1^2} = -\frac{2}{5}\lambda_1 B_2^2. \tag{9.49}$$

If $\lambda_1 > 0$, then $(\frac{dA}{dt})_{B_1^2} < 0$. It is stable. If $\lambda_1 < 0$, then $(\frac{dA}{dt})_{B_1^2} > 0$. It is unstable.

$$\left(\frac{dA}{dt}\right)_{B_2^2} = -\frac{2}{5}\lambda_2 B_2^2. \tag{9.50}$$

If $\lambda_2 > 0$, then $(\frac{dA}{dt})_{B_2^2} < 0$. It is stable. If $\lambda_2 < 0$, then $(\frac{dA}{dt})_{B_2^2} > 0$. It is unstable. The results obtained from the equations (9.43) and (9.44) and (9.30) are the same. Studying stability and instability of the equation (9.39) is transformed into studying stability and instability of the nonlinear terms of $\frac{dA}{dt}$ in the equation (9.46).

9.2. 2p-Nonlinear Dynamics

In order to study solutions of the $2p$-nonlinear equation from (3.1) we define the $2p$-chaotic function

$$N_i = \frac{1}{2p}\left[A_1 + 2\sum_{j=1}^{(p-1)/2}(-1)^{(i-1)j}B_j\cos(\theta_j + (-1)^j\frac{(i-1)j\pi}{p})\right]$$
$$+\frac{(-1)^{(i-1)}}{2p}[A_2 + 2\sum_{j=1}^{(p-1)/2}(-1)^{(i-1)j}D_j\cos(\phi_j + (-1)^{j+1}\frac{(i-1)j\pi}{p})]. \tag{9.51}$$

where $i = 1, 2, \cdots, 2p$.

From the equation (9.51) we obtain its inverse transformation

$$A_1 = \sum_{i=1}^{2p} N_i, A_2 = \sum_{i=1}^{2p} N_i(-1)^{1+i},$$

$$B_j \cos\theta_j = N_1 + \sum_{i=1}^{2p-1} (-1)^{ij} N_{1+i} \cos\frac{ij\pi}{p},$$

$$B_j \sin\theta_j = (-1)^{j+1} \sum_{i=1}^{2p-1} N_{1+i}(-1)^{ij} \sin\frac{ij\pi}{p}, \tag{9.52}$$

$$D_j \cos\phi_j = N_1 + \sum_{i=1}^{2p-1} N_{1+i}(-1)^{(j-1)i} \cos\frac{ij\pi}{p},$$

$$D_j \sin\phi_j = (-1)^j \sum_{i=1}^{2p-1} N_{1+i}(-1)^{(j-1)i} \sin\frac{ij\pi}{p}.$$

The equations (9.51) and (9.52) are a nonlinear transformation group. From the equation (9.52) we obtain

$$\frac{dA_1}{dt} = \sum_{i=1}^{2p} \frac{dN_i}{dt},$$

$$\frac{dB_j}{dt} = \sum_{i=1}^{2p} \frac{dN_i}{dt}(-1)^{(i-1)j} \cos(\theta_j + (-1)^j \frac{(i-1)j\pi}{p}),$$

$$B_j\frac{d\theta_j}{dt} = (-1)^{j+1} \sum_{i=1}^{2p} \frac{dN_i}{dt}(-1)^{ij} \sin(\theta_j + (-1)^j \frac{(i-1)j\pi}{p}),$$

$$\frac{dA_2}{dt} = \sum_{i=1}^{2p}(-1)^{1+i}\frac{dN_i}{dt},$$

$$\frac{dD_j}{dt} = \sum_{i=1}^{2p}(-1)^{(j-1)(i+1)}\frac{dN_i}{dt} \cos(\phi_j + (-1)^{j+1}\frac{(i-1)j\pi}{p}),$$

$$D_j\frac{d\phi_j}{dt} = (-1)^j \sum_{i=1}^{2p}(-1)^{(j-1)i}\frac{dN_i}{dt} \sin(\phi_j + (-1)^{j+1}\frac{(i-1)j\pi}{p}). \tag{9.53}$$

We define the $2p$-nonlinear equations

$$dN_i/dt = N_i(1 - \sum_{j=1}^{2p} \bar{a}_{ij}N_i), \tag{9.54}$$

where

$$(\bar{a}^{ij}) = \begin{bmatrix} 1 & a_1 & \cdots & a_{2p-1} \\ a_{2p-1} & 1 & \cdots & a_{2p-2} \\ \vdots & \vdots & \vdots & \vdots \\ a_1 & a_2 & \cdots & 1 \end{bmatrix}$$

The fixed point solutions of the equation (9.54) may be expressed as points in the $2p-$dimensional space: they are the point $(0, 0, \cdots, 0)$; $2p$ single-point solutions of the form $(1, 0, \cdots, 0)$; a nontrivial fixed point $(1, 1, \cdots, 1)/\lambda_{A_1}$, where

$$\lambda_{A_1} = 1 + \sum_{j=1}^{2p-1} a_j.$$

Now we study the local solutions of Eq. (9.54) at the fixed point $(1, 1, \cdots, 1)/\lambda_{A_1}$. The equation (9.54) may be written into

$$\frac{d \ln N_i}{dt} = 1 - \sum_{j=1}^{2p} \bar{a}_{ij} N_i, \tag{9.55}$$

$\frac{d \ln N_i}{dt}$ is the average growth rate of i-th species. Setting $\lambda_{A_1} N_i = e^{y_i}$, the equation (9.55) is transformed into

$$\frac{dy_i}{dt} = 1 - \sum_{j=1}^{2p} \bar{a}_{ij} \frac{1}{\lambda_{A_1}} e^{y_j}. \tag{9.56}$$

We study the linear solutions of the equation (9.56). Setting $e^{y_j} = 1 + y_j$, the equation (9.56) is transformed into

$$\frac{dy_i}{dt} = -\frac{1}{\lambda_{A_1}} \sum_{j=1}^{p} \bar{a}_{ij} y_j. \tag{9.57}$$

The equation (9.57) has the following exact solutions

$$y_i = \frac{1}{2p} \Big[e^{-t} + 2 \sum_{j=1}^{(p-1)/2} (-1)^{(i-1)j} \exp\Big(- \frac{\lambda_{B_J}}{\lambda_{A_1}} t \Big)$$

$$\times \cos\Big(\frac{\eta_{\theta_j}}{\lambda_{A_1}} t + (-1)^j \frac{(i-1)j\pi}{p} \Big) \Big]$$

$$+ \frac{(-1)^{(i-1)}}{2p} \Big[\exp\Big(- \frac{\lambda_{A_2}}{\lambda_{A_1}} t \Big) + 2 \sum_{j=1}^{(p-1)/2} (-1)^{(i-1)j}$$

$$\times \exp\left(-\frac{\lambda_{D_j}}{\lambda_{A_1}}t\right)\cos\left(\frac{\eta_{\phi_j}}{\lambda_{A_1}}t+(-1)^{j+1}\frac{(i-1)j\pi}{p}\right)\Bigg], \tag{9.58}$$

where

$$\lambda_{A_1}=1+\sum_{i=1}^{2p-1}a_i, \lambda_{A_2}=1+\sum_{i=1}^{2p-1}(-1)^i a_i,$$

$$\lambda_{B_j}=1+\sum_{i=1}^{2p-1}(-1)^{ij}a_i\cos\frac{ij\pi}{p},$$

$$\eta_{\theta_j}=(-1)^{j+1}\sum_{i=1}^{2p-1}a_i(-1)^{ij}\sin\frac{ij\pi}{p},$$

$$\lambda_{D_j}=1+\sum_{i=1}^{2p-1}a_i(-1)^{(j-1)i}\cos\frac{ij\pi}{p},$$

$$\eta_{\phi_j}=(-1)^{j}\sum_{i=1}^{2p-1}a_i(-1)^{(j-1)i}\sin\frac{ij\pi}{p}.$$

A necessary and sufficient condition for stability of the equation (9.58) is $\lambda_{B_j}>0$, $\lambda_{A_2}>0$, and $\lambda_{D_j}>0$, where $j=1,2,\cdots,(p-1)/2$. If $\lambda_{B_j}<0, \lambda_{A_2}<0$, and $\lambda_{D_j}<0$, then their solutions are unstable.

Now we study the global stability of the equation (9.54) by using the $2p-$chaotic function. By substituting both equations (9.52) and (9.54) into (9.53) we obtain the $2p-$ chaotic function

$$\frac{dA_1}{dt}=A_1-\frac{1}{2p}\left[\lambda_{A_1}A_1^2+\lambda_{A_2}A_2^2+2\sum_{j=1}^{(p-1)/2}\lambda_{B_j}B_j^2+2\sum_{J=1}^{(p-1)/2}\lambda_{D_j}D_j^2\right]. \tag{9.59}$$

The $\lambda_{A_1}, \lambda_{A_2}, \lambda_{B_J}$ and λ_{D_j} in both equations (9.58) and (9.59) are identical.

We will discuss stability and instability of the nonlinear equation (9.59). We have

$$\left(\frac{dA_1}{dt}\right)_{A_1^2}=-\frac{1}{2p}\lambda_{A_1}A_1^2. \tag{9.60}$$

For $\lambda_{A_1}>0$, then $(dA_1/dt)_{A_1^2}<0$, so it is stable. Also

$$\left(\frac{dA_1}{dt}\right)_{A_2^2}=-\frac{1}{2p}\lambda_{A_2}A_2^2. \tag{9.61}$$

If $\lambda_{A_2}>0$, then $(dA_1/dt)_{A_2^2}<0$, so it is stable; if $\lambda_{A_2}<0$, then $(dA_1/dt)_{A_2^2}>0$, so it is unstable. Also

$$\left(\frac{dA_1}{dt}\right)_{B_j^2}=-\frac{1}{p}\lambda_{B_j}B_j^2. \tag{9.62}$$

If $\lambda_{B_j} > 0$, then $(dA_1/dt)_{B_j^2} < 0$, so it is stable; if $\lambda_{B_J} < 0$, then $(dA_1/dt)_{B_j^2} > 0$, so it is unstable

$$\left(\frac{dA_1}{dt}\right)_{D_j^2} = -\frac{1}{p}\lambda_{D_j} D_j^2. \tag{9.63}$$

If $\lambda_{D_j} > 0$, then $(dA_1/dt)_{D_j^2} < 0$, so it is stable; if $\lambda_{D_j} < 0$, then $(dA_1/dt)_{D_j^2} > 0$, so it is unstable.

From the above analysis we come to the conclusion that studying the stability and instability of the nonlinear equation (9.54) is transformed into studying the stability and instability of the nonlinear terms of dA_1/dt in the equation (9.59). The study of the nonlinear equations are greatly simplified by using this method. It provides a useful tool for studying the nonlinear equations.

For $2p = 6$, from the equation (9.54) we obtain

$$\frac{dN_i}{dt} = N_i(1 - \sum_{j=1}^{6} \bar{a}_{ij} N_j), \tag{9.64}$$

where $i = 1,2,3,4,5,6$, $j = 1,2,3,4,5,6$, and

$$(\bar{a}_{ij}) = \begin{bmatrix} 1 & a_1 & a_2 & a_3 & a_4 & a_5 \\ a_5 & 1 & a_1 & a_2 & a_3 & a_4 \\ a_4 & a_5 & 1 & a_1 & a_2 & a_3 \\ a_3 & a_4 & a_5 & 1 & a_1 & a_2 \\ a_2 & a_3 & a_4 & a_5 & 1 & a_1 \\ a_1 & a_2 & a_3 & a_4 & a_5 & 1 \end{bmatrix}$$

$N_i(t)$ is the number of individuals of the i-th species, and $(\bar{a}_{ij})$ denotes the interaction matrix. The fixed-point solutions may be expressed as points in the 6-dimensional space: they are the point (0, 0, 0, 0, 0, 0); six single-point solutions of the form (1, 0, 0, 0, 0, 0); and a nontrivial fixed point $(1,1,1,1,1,1)/\lambda_{A_1}$, where $\lambda_{A_1} = 1 + a_1 + a_2 + a_3 + a_4 + a_5$.

We study the local solutions of Eq. (9.64) at the fixed point $(1,1,1,1,1,1)/\lambda_{A_1}$. The equation (9.64) may be written as

$$\frac{d \ln N_i}{dt} = 1 - \sum_{j=1}^{6} \bar{a}_{ij} N_j. \tag{9.65}$$

Setting $\lambda_{A_1} N_i = e^{y_i}$, the equation (9.65) is transformed into

$$\frac{dy_i}{dt} = 1 - \frac{1}{\lambda_{A_1}} \sum_{j=1}^{6} \bar{a}_{ij} e^{y_j}. \tag{9.66}$$

We study the linear solutions of the equation (9.66). Setting $e^{y_j} = 1 + y_j$, it is transformed into

$$\frac{dy_i}{dt} = -\frac{1}{\lambda_{A_1}} \sum_{j=1}^{6} \bar{a}_{ij} y_j. \tag{9.67}$$

The equation (9.67) has the following exact solutions

$$y_i = \frac{1}{6}\Big[e^{-t} + 2(-1)^{i-1} \exp\Big(-\frac{\lambda_B}{\lambda_{A_1}}t\Big)\cos(\frac{\eta_\theta}{\lambda_{A_1}}t - \frac{(i-1)\pi}{3}\Big)\Big]$$

$$+\frac{(-1)^{(i-1)}}{6}\left[\exp\left(-\frac{\lambda_{A_2}}{\lambda_{A_1}}t\right)+2(-1)^{i-1}\exp\left(-\frac{\lambda_D}{\lambda_{A_1}}t\right)\cos\left(\frac{\eta_\phi}{\lambda_{A_1}}t+\frac{(i-1)\pi}{3}\right)\right], \tag{9.68}$$

$$\lambda_{A_1} = 1 + a_1 + a_2 + a_3 + a_4 + a_5,$$

$$\lambda_{A_2} = 1 - a_1 + a_2 - a_3 + a_4 - a_5,$$

$$\lambda_B = 1 + a_3 - \frac{a_1 + a_2 + a_4 + a_5}{2},$$

$$\eta_\theta = \frac{\sqrt{3}}{2}(-a_1 + a_2 - a_4 + a_5),$$

$$\lambda_D = 1 - a_3 + \frac{a_1 - a_2 - a_4 + a_5}{2},$$

$$\eta_\phi = \frac{\sqrt{3}}{2}(-a_1 - a_2 + a_4 + a_5).$$

A necessary and sufficient condition for stability of the equation (9.68) is $\lambda_{A_1} > 0$, $\lambda_{A_2} > 0$, $\lambda_B > 0$, and $\lambda_D > 0$. If $\lambda_{A_2} < 0$, $\lambda_B < 0$, and $\lambda_D < 0$; $\lambda_{A_2} > 0$, $\lambda_B > 0$, and $\lambda_D < 0$; $\lambda_{A_2} > 0$, $\lambda_B < 0$, and $\lambda_D > 0$; or $\lambda_{A_2} < 0$, $\lambda_B > 0$, and $\lambda_D > 0$, their solutions are unstable.

We study the global stability of the equation (9.64) by using the 6-chaotic function. From the equation (9.51) we obtain the chaotic functions

$$N_1 = \frac{1}{6}[A_1 + 2B\cos\theta + A_2 + 2D\cos\phi],$$

$$N_2 = \frac{1}{6}[A_1 - 2B\cos(\theta - \frac{1}{3}\pi) - A_2 + 2D\cos(\phi + \frac{1}{3}\pi)],$$

$$N_3 = \frac{1}{6}[A_1 + 2B\cos(\theta - \frac{2}{3}\pi) + A_2 + 2D\cos(\phi + \frac{2}{3}\pi)],$$

$$N_4 = \frac{1}{6}[A_1 + 2B\cos\theta - A_2 - 2D\cos\phi],$$

$$N_5 = \frac{1}{6}[A_1 - 2B\cos(\theta - \frac{1}{3}\pi) + A_2 - 2D\cos(\phi + \frac{1}{3}\pi)],$$

$$N_6 = \frac{1}{6}[A_1 + 2B\cos(\theta - \frac{2}{3}\pi) - A_2 - 2D\cos(\phi + \frac{2}{3}\pi)]. \tag{9.69}$$

From the equation (9.69) we obtain

$$\frac{dA_1}{dt} = \sum_{j=1}^{6} \frac{dN_j}{dt},$$

$$\frac{dB}{dt} = \sum_{j=0}^{2} (-1)^j \cos(\theta - \frac{j\pi}{3}) \times (\frac{dN_{j+1}}{dt} + \frac{dN_{4+j}}{dt}),$$

$$B\frac{d\theta}{dt} = \sum_{j=0}^{2} (-1)^{j+1} \sin(\theta - \frac{j\pi}{3}) \times (\frac{dN_{j+1}}{dt} + \frac{dN_{4+j}}{dt}),$$

$$\frac{dA_2}{dt} = \sum_{j=1}^{6} (-1)^{j+1} \frac{dN_j}{dt},$$

$$\frac{dD}{dt} = \sum_{j=0}^{2} \cos(\phi + \frac{j\pi}{3}) \times (\frac{dN_{j+1}}{dt} - \frac{dN_{4+j}}{dt}),$$

$$D\frac{d\phi}{dt} = -\sum_{j=0}^{2} \sin(\phi + \frac{j\pi}{3}) \times (\frac{dN_{j+1}}{dt} - \frac{dN_{4+j}}{dt}). \tag{9.70}$$

By substituting both equations (9.64) and (9.69) into the equation (9.70) we obtain the 6-chaotic equations

$$\frac{dA_1}{dt} = A_1 - \frac{1}{6}(\lambda_{A_1} A_1^2 + \lambda_{A_2} A_2^2 + 2\lambda_B B^2 + 2\lambda_D D^2),$$

$$\frac{dB}{dt} = B - \frac{1}{6}\{(\lambda_{A_1} + \lambda_B) A_1 B + [(\lambda_{A_2} + \lambda_D)\cos(\theta + \phi) + \eta_\phi \sin(\theta + \phi)] A_2 D$$
$$+ (\lambda_B \cos 3\theta + \eta_\theta \sin 3\theta) B^2 + [\lambda_D \cos(\theta - 2\phi) - \eta_\phi \sin(\theta - 2\phi)] D^2\},$$

$$B\frac{d\theta}{dt} = \frac{1}{6}\{\eta_\theta A_1 B + [(\lambda_{A_2} + \lambda_D)\sin(\theta + \phi) - \eta_\phi \cos(\theta + \phi)] A_2 D +$$
$$(\lambda_B \sin 3\theta - \eta_\theta \cos 3\theta) B^2 + [\lambda_D \sin(\theta - 2\phi) + \eta_\phi \cos(\theta - 2\phi)] D^2\},$$

$$\frac{dA_2}{dt} = A_2 - \frac{1}{6}\{(\lambda_{A_1} + \lambda_{A_2}) A_1 A_2 + [(\lambda_B + \lambda_D)\cos(\theta + \phi) - (\eta_\theta + \eta_\phi)\sin(\theta + \phi)] BD\},$$

$$\frac{dD}{dt} = D - \frac{1}{6}\{(\lambda_{A_1} + \lambda_D) A_1 D + [(\lambda_{A_2} + \lambda_B)\cos(\theta + \phi) + \eta_\theta \sin(\theta + \phi)] A_2 B$$
$$+ [(\lambda_B + \lambda_D)\cos(\theta - 2\phi) - (\eta_\theta - \eta_\phi)\sin(\theta - 2\phi)] BD\},$$

$$D\frac{d\phi}{dt} = \frac{1}{6}\{\eta_\phi A_1 D + [(\lambda_{A_2} + \lambda_B)\sin(\theta+\phi) - \eta_\theta\cos(\theta+\phi)]A_2 B$$
$$+[(\lambda_B + \lambda_D)\sin(\theta - 2\phi) + (\eta_\theta - \eta_\phi)\cos(\theta - 2\phi)]BD\}. \quad (9.71)$$

The $\lambda_{A_1}, \lambda_{A_2}, \lambda_B$ and λ_D in both equations (9.68) and (9.71) are identical. If $a_1 = a_2 = a_3 = a_4 = a_5 = 1$, then $\lambda_{A_1} = 6, \lambda_{A_2} = \lambda_B = \lambda_D = \eta_0 = \eta_\phi = 0$. From the equation (9.71) we obtain a special solution

$$A_1 = \frac{A_1(0)}{A_1(0) + (1 - A_1(0))e^{-t}}. \quad (9.72)$$

We now discuss the stability and instability of the nonlinear terms of dA_1/dt in the equation (9.71). We have

$$\left(\frac{dA_1}{dt}\right)_{A_1^2} = -\frac{1}{6}\lambda_{A_1}A_1^2. \quad (9.73)$$

If $\lambda_{A_1} > 0$, then $(\frac{dA_1}{dt})_{A_1^2} < 0$, so it is stable.

Next,

$$\left(\frac{dA_1}{dt}\right)_{A_2^2} = -\frac{1}{6}\lambda_{A_2}A_2^2. \quad (9.74)$$

If $\lambda_{A_2} > 0$, then $(dA_1/dt)_{A_2^2} < 0$, so it is stable. If $\lambda_{A_2} < 0$, then $(dA_1/dt)_{A_2^2} > 0$, so it is unstable.

Next,

$$\left(\frac{dA_1}{dt}\right)_{B^2} = -\frac{1}{3}\lambda_B B^2. \quad (9.75)$$

If $\lambda_B > 0$, then $(\frac{dA_1}{dt})_{B^2} < 0$, so it is stable. If $\lambda_B < 0$, then $(\frac{dA_1}{dt})_{B^2} > 0$, so it is unstable.

Finally,

$$\left(\frac{dA_1}{dt}\right)_{D^2} = -\frac{1}{3}\lambda_D D^2. \quad (9.76)$$

If $\lambda_D > 0$, then $(dA_1/dt)_{D^2} < 0$, so it is stable. If $\lambda_D < 0$, then $(dA_1/dt)_{D^2} > 0$, so it is unstable.

9.3. $4m$-Nonlinear Dynamics

In order to study solutions of the $4m$-nonlinear equation from (4.1) we define the $4m$-chaotic function

$$N_i = \frac{1}{4m}\left[A_1 + 2H\cos\left(\beta + \frac{(i-1)\pi}{2}\right) + 2\sum_{j=1}^{m-1} B_j\left(\theta_j + \frac{(i-1)j\pi}{2m}\right)\right]$$

$$+\frac{(-1)^{i-1}}{4m}\left[A_2+2\sum_{j=1}^{m-1}D_j\cos\left(\phi_j-\frac{(i-1)j\pi}{2m}\right)\right]. \tag{9.77}$$

where $i=1,2,\cdots,4m$.

From the equation (9.77) we obtain its inverse transformation

$$A_1=\sum_{i=1}^{4m}N_i,\quad A_2=\sum_{i=1}^{4m}N_i(-1)^{i+1},$$

$$H\cos\beta=\sum_{i=1}^{2m}N_{2i-1}(-1)^{i+1},\quad H\sin\beta=\sum_{i=1}^{2m}N_{2i}(-1)^{i},$$

$$B_j\cos\theta_j=N_1+\sum_{i=1}^{4m-1}N_{1+i}\cos\frac{ij\pi}{2m}, B_j\sin\theta_j=-\sum_{i=1}^{4m-1}N_{1+i}\sin\frac{ij\pi}{2m},$$

$$D_j\cos\phi_j=N_1+\sum_{i=1}^{4m-1}N_{1+i}(-1)^i\cos\frac{ij\pi}{2m}, D_j\sin\phi_j=\sum_{i=1}^{4m-1}N_{1+i}(-1)^i\sin\frac{ij\pi}{2m}. \tag{9.78}$$

The equations (9.77) and (9.78) constitute a nonlinear transformation group. From the equation (9.78) we obtain

$$\frac{dA_1}{dt}=\sum_{i=1}^{4m}\frac{dN_i}{dt}, \frac{dA_2}{dt}=\sum_{i=1}^{4m}(-1)^{1+i}\frac{dN_i}{dt},$$

$$\frac{dH}{dt}=\sum_{i=1}^{4m}\cos\left(\beta+\frac{(i-1)\pi}{2}\right)\frac{dN_i}{dt}, H\frac{d\beta}{dt}=-\sum_{i=1}^{4m}\sin\left(\beta+\frac{(i-1)\pi}{2}\right)\frac{dN_i}{dt},$$

$$\frac{dB_j}{dt}=\sum_{i=1}^{4m}\cos\left(\theta_j+\frac{(i-1)j\pi}{2m}\right)\frac{dN_i}{dt}, B_j\frac{d\theta_j}{dt}=-\sum_{i=1}^{4m}\sin\left(\theta_j+\frac{(i-1)j\pi}{2m}\right)\frac{dN_i}{dt},$$

$$\frac{dD_j}{dt}=\sum_{i=1}^{4m}(-1)^{i-1}\cos\left(\phi_j-\frac{(i-1)j\pi}{2m}\right)\frac{dN_i}{dt}, D_j\frac{d\phi_j}{dt}=\sum_{i=1}^{4m}$$

$$(-1)^i\sin\left(\phi_j-\frac{(i-1)j\pi}{2m}\right)\frac{dN_i}{dt}. \tag{9.79}$$

We define the $4m$-nonlinear equations

$$\frac{dN_i}{dt}=N_i\Big(1-\sum_{j=1}^{4m}\bar{a}_{ij}N_j\Big), \tag{9.80}$$

where

$$(\bar{a}_{ij}) = \begin{bmatrix} 1 & a_1 & \cdots & a_{4m-1} \\ a_{4m-1} & 1 & \cdots & a_{4m-2} \\ \vdots & \vdots & \vdots & \vdots \\ a_1 & a_2 & \cdots & 1 \end{bmatrix}.$$

The fixed point solutions of the equation (9.80) may be expressed as points in the $4m$-dimensional space: they are the point $(0, 0, \cdots, 0)$;$4m$ single-point solutions of the form $(1, 0, \cdots, 0)$; a nontrivial fixed point $(1, 1, \cdots, 1)/\lambda_{A_1}$ where

$$\lambda_{A_1} = 1 + \sum_{j=1}^{4m-1} a_j.$$

Now we study the local solutions of Eq. (9.80) at the fixed point $(1, 1, \cdots, 1)/\lambda_{A_1}$. The equation (9.80) may be written as

$$\frac{d \ln N_i}{dt} = 1 - \sum_{j=1}^{4m} \bar{a}_{ij} N_j, \tag{9.81}$$

where $d \ln N_i/dt$ is the average growth rate of the i-th species. Setting $\lambda_{A_1} N_i = e^{y_i}$, the equation (9.81) is transformed into

$$\frac{dy_i}{dt} = 1 - \sum_{j=1}^{4m} \bar{a}_{ij} \frac{1}{\lambda_{A_1}} e^{y_j}. \tag{9.82}$$

We study the linear solutions of the equation (9.82). Setting $e^{y_j} = 1 + y_j$, the equation (9.82) is transformed into

$$\frac{dy_i}{dt} = -\frac{1}{\lambda_{A_1}} \sum_{j=1}^{4m} \bar{a}_{ij} y_j. \tag{9.83}$$

The equation (9.83) has the following exact solutions

$$y_i = \frac{1}{4m}\Big[e^{-t} + 2\exp\Big(-\frac{\lambda_H t}{\lambda_{A_1}}\Big)\cos\Big(\frac{\eta_\beta t}{\lambda_{A_1}} + \frac{(i-1)\pi}{2}\Big)$$

$$+2\sum_{j=1}^{m-1}\exp\Big(-\frac{\lambda_{B_j} t}{\lambda_{A_1}}\Big)\cos\Big(\frac{\lambda_{\theta_j} t}{\lambda_{A_1}} + \frac{(i-1)j\pi}{2m}\Big)\Big]$$

$$+\frac{(-1)^{i-1}}{4m}\Big[\exp\Big(-\frac{\lambda_{A_2}}{\lambda_{A_1}}t\Big) + 2\sum_{j=1}^{m-1}\exp(-\frac{\lambda_{D_j} t}{\lambda_{A_1}})\cos\Big(\frac{\lambda_{\phi_j} t}{\lambda_{A_1}} - \frac{(i-1)j\pi}{2m}\Big)\Big]. \tag{9.84}$$

where

$$\lambda_{A_1} = 1 + \sum_{i=1}^{4m-1} a_i, \quad \lambda_{A_2} = 1 + \sum_{i=1}^{4m-1} (-1)^i a_i,$$

$$\lambda_H = 1 + \sum_{i=1}^{2m-1} (-1)^i a_{2i}, \quad \lambda_\beta = \sum_{i=1}^{2m} (-1)^i a_{2i-1},$$

$$\lambda_{B_j} = 1 + \sum_{i=1}^{4m-1} a_i \cos\frac{ij\pi}{2m}, \quad \lambda_{\theta_j} = -\sum_{i=1}^{4m-1} a_i \sin\frac{ij\pi}{2m},$$

$$\lambda_{D_j} = 1 + \sum_{i=1}^{4m-1} a_i (-1)^i \cos\frac{ij\pi}{2m}, \quad \lambda_{\phi_j} = \sum_{i=1}^{4m-1} a_i (-1)^i \sin\frac{ij\pi}{2m}.$$

A necessary and sufficient condition for stability of the equation (9.84) is $\lambda_H > 0, \lambda_{B_j} > 0$, and $\lambda_{D_j} > 0$, where $j = 1, 2, \cdots, m-1$. If $\lambda_H < 0$, $\lambda_{B_j} < 0$, and $\lambda_{D_j} < 0$, then their solutions are unstable.

Now we study the global stability of the equation (9.80) by using the $4m$-chaotic function. By substituting both equations (9.77) and (9.80) into (9.79) we obtain the $4m$-chaotic function

$$\frac{dA_1}{dt} = A_1 - \frac{1}{4m}\left[\lambda_{A_1} A_1^2 + \lambda_{A_2} A_2^2 + \lambda_H H^2 + 2\sum_{j=1}^{m-1} \lambda_{B_j} B_j^2 + 2\sum_{j=1}^{m-1} \lambda_{D_j} D_j^2\right]. \quad (9.85)$$

The $\lambda_{A_1}, \lambda_{A_2}, \lambda_H, \lambda_{B_j}$ and λ_{D_j} in both equations (9.84) and (9.85) are identical. We will discuss stability and instability of the nonlinear equation (9.85). We have

$$\left(\frac{dA_1}{dt}\right)_{A_1^2} = -\frac{1}{4m}\lambda_{A_1} A_1^2. \quad (9.86)$$

For $\lambda_{A_1} > 0$, then $(\frac{dA_1}{dt})_{A_1^2} < 0$, so it is stable.Also

$$\left(\frac{dA_1}{dt}\right)_{A_2^2} = -\frac{1}{4m}\lambda_{A_2} A_2^2. \quad (9.87)$$

If $\lambda_{A_2} > 0$, then $(\frac{dA_1}{dt})_{A_2} < 0$, so it is stable; if $\lambda_{A_2} < 0$, then $\left(\frac{dA_1}{dt}\right)_{A^2} > 0$, so it is unstable. Also,

$$\left(\frac{dA_1}{dt}\right)_{H^2} = -\frac{1}{4m}\lambda_H H^2. \quad (9.88)$$

If $\lambda_H > 0$, then $(\frac{dA_1}{dt})_{H^2} < 0$, so it is stable; if $\lambda_H < 0$, then $\left(\frac{dA_1}{dt}\right) H^2 > 0$, so it is unstable. Also,

$$\left(\frac{dA_1}{dt}\right)_{B_j^2} = -\frac{1}{2m}\lambda_{B_j} B_j^2. \quad (9.89)$$

If $\lambda_{B_j} > 0$, then $(\frac{dA_1}{dt})_{B_j^2} < 0$, so it is stable; if $\lambda_{B_j} < 0$, then $\left(\frac{dA_1}{dt}\right)_{B_j^2} > 0$, so it is unstable.

$$\left(\frac{dA_1}{dt}\right)_{D_j^2} = -\frac{1}{2m}\lambda_{D_j} D_j^2. \tag{9.90}$$

If $\lambda_{D_j} > 0$, then $(\frac{dA_1}{dt})_{D_j^2} < 0$, so it is stable; if $\lambda_{D_j} < 0$, then $(\frac{dA_1}{dt})_{D_j^2} > 0$, so it is unstable.

From the above analysis we come to the conclusion that chaos occurs when stable and unstable manifolds cross at nontrivial fixed points.

For $4m = 4$, from the equation (9.80) we obtain

$$\frac{dN_i}{dt} = N_i\left(1 - \sum_{j=1}^{4} \bar{a}_{ij} N_j\right), \tag{9.91}$$

where $i = 1, 2, 3, 4$, $j = 1, 2, 3, 4$ and

$$(\bar{a}_{ij}) = \begin{bmatrix} 1 & a_1 & a_2 & a_3 \\ a_3 & 1 & a_1 & a_2 \\ a_2 & a_3 & 1 & a_1 \\ a_1 & a_2 & a_3 & 1 \end{bmatrix}$$

where N_i is the number of individuals of the i-th species.

The fixed-point solutions may be expressed as points in the 4-dimensional space: they are the point (0, 0, 0, 0); 4 single-point solutions of the form (1, 0, 0, 0); and a nontrivial fixed point $(1, 1, 1, 1)/\lambda_{A_1}$, where

$$\lambda_{A_1} = 1 + a_1 + a_2 + a_3.$$

Now we study the local solutions of Eq. (9.91) at the fixed point $(1, 1, 1, 1)/\lambda_{A_1}$. The equation (9.91) may be written into

$$\frac{d \ln N_i}{dt} = 1 - \sum_{j=1}^{4} \bar{a}_{ij} N_j. \tag{9.92}$$

Setting $\lambda_{A_1} N_j = e^{y_j}$, the equation (9.92) is transformed into

$$\frac{dy_i}{dt} = 1 - \frac{1}{\lambda_{A_1}} \sum_{j=1}^{4} \bar{a}_{ij} e^{y_j}. \tag{9.93}$$

We study the linear solutions of the equation (9.93). Setting $e^{y_j} = 1 + y_j$, the equation (9.93) is transformed into

$$\frac{dy_i}{dt} = -\frac{1}{\lambda_{A_1}} \sum_{j=1}^{4} \bar{a}_{ij} y_j. \tag{9.94}$$

The equation (9.94) has the following exact solutions

$$y_1 = \frac{1}{4}\Big[e^{-t} + 2\exp\Big(-\frac{\lambda_H t}{\lambda_{A_1}}t\Big)\cos\Big(\frac{\lambda_\beta t}{\lambda_{A_1}}\Big) + \exp\Big(-\frac{\lambda_{A_2}}{\lambda_{A_1}}\Big)\Big],$$
$$y_2 = \frac{1}{4}\Big[e^{-t} - 2\exp\Big(-\frac{\lambda_H t}{\lambda_{A_1}}t\Big)\sin\Big(\frac{\lambda_\beta t}{\lambda_{A_1}}\Big) - \exp\Big(-\frac{\lambda_{A_2}}{\lambda_{A_1}}t\Big)\Big],$$
$$y_3 = \frac{1}{4}\Big[e^{-t} - 2\exp\Big(-\frac{\lambda_H t}{\lambda_{A_1}}t\Big)\cos\Big(\frac{\lambda_\beta t}{\lambda_{A_1}}t\Big) + \exp\Big(-\frac{\lambda_{A_2}}{\lambda_{A_1}}t\Big)\Big],$$
$$y_4 = \frac{1}{4}\Big[e^{-t} + 2\exp\Big(-\frac{\lambda_H t}{\lambda_{A_1}}\Big)\sin\Big(\frac{\lambda_\beta t}{\lambda_{A_1}}t\Big) - \exp\Big(-\frac{\lambda_{A_2}}{\lambda_{A_1}}t\Big)\Big]. \tag{9.95}$$

where $\lambda_{A_1} = 1 + a_1 + a_2 + a_3$, $\lambda_{A_2} = 1 - a_1 + a_2 - a_3$, $\lambda_H = 1 - a_2$, $\lambda_\beta = -a_1 + a_3$.

A necessary and sufficient condition for stability of the equation (9.95) is $\lambda_H > 0$ and $\lambda_{A_2} > 0$. If $\lambda_H < 0$ and $\lambda_{A_2} > 0$, then the solutions are unstable.

For $4m = 4$ from (9.77) we have

$$\begin{bmatrix} N_1 \\ N_2 \\ N_3 \\ N_4 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & -1 & 0 & -1 \\ 1 & 1 & -1 & 0 \\ 1 & -1 & 0 & 1 \end{bmatrix}\begin{bmatrix} A_1 \\ A_2 \\ 2H\cos\beta \\ 2H\sin\beta \end{bmatrix}. \tag{9.96}$$

From (9.96) we have

$$\begin{bmatrix} dA_1/dt \\ dA_2/dt \\ dH/dt \\ Hd\beta/dt \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ \cos\beta & -\sin\beta & -\cos\beta & \sin\beta \\ -\sin\beta & -cos\beta & \sin\beta & \cos\beta \end{bmatrix}\begin{bmatrix} dN_1/dt \\ dN_2/dt \\ dN_3/dt \\ dN_4/dt \end{bmatrix}. \tag{9.97}$$

By substituting (9.91) and (9.96) into (9.97) we obtain the 4-chaotic equations

$$\frac{dA_1}{dt} = A_1 - \frac{1}{4}(\lambda_{A_1}A_1^2 + \lambda_{A_2}A_2^2 + 2\lambda_H H^2),$$
$$\frac{dA_2}{dt} = A_2 - \frac{1}{4}[(\lambda_{A_1} + \lambda_{A_2})A_1A_2 + 2(\lambda_H\cos 2\beta + \lambda_\beta\sin 2\beta)H^2],$$
$$\frac{dH}{Hdt} = 1 - \frac{1}{4}(\lambda_{A_2} + \lambda_H)A_1 + \frac{1}{4}[(\lambda_{A_2} + \lambda_H)\cos 2\beta - \lambda_\beta\sin 2\beta]A_2,$$
$$\frac{d\beta}{dt} = \frac{1}{4}\lambda_\beta A_1 + \frac{1}{4}[(\lambda_{A_2} + \lambda_H)\cos 2\beta - \lambda_\beta\sin 2\beta]A_2. \tag{9.98}$$

The $\lambda_{A_1}, \lambda_{A_2}, \lambda_H$ and λ_β in the equations (9.95) and (9.98) are the same. We will discuss the stability and instability of the nonlinear terms of dA_1/dt in the equation (9.98). We have

$$\Big(\frac{dA_1}{dt}\Big)_{A_1^2} = -\frac{1}{4}\lambda_{A_1}A_1^2. \tag{9.99}$$

For $\lambda_{A_1} > 0$, it is stable.

$$\left(\frac{dA_1}{dt}\right)_{A_2^2} = -\frac{1}{4}\lambda_{A_2}A_2^2. \tag{9.100}$$

If $\lambda_{A_2} > 0$, it is stable. If $\lambda_{A_2} < 0$, it is unstable.

$$\left(\frac{dA_1}{dt}\right)_{H^2} = -\frac{\lambda_H H^2}{2}. \quad (9.101).$$

If $(\frac{dA_1}{dt})_{H^2} > 0$, it is stable; if $(\frac{dA_1}{dt})_{H^2} < 0$, it is unstable.
For $4m = 4$, we have

$$\frac{dN_i}{dt} = N_i\left(1 - \sum_{j-1}^{4} a_{ij}N_j\right). \tag{9.102}$$

where

$$a_{ij} = \begin{bmatrix} a_{11} & a_{21} & a_{31} & a_{41} \\ a_{12} & a_{22} & a_{32} & a_{42} \\ a_{13} & a_{23} & a_{33} & a_{43} \\ a_{14} & a_{24} & a_{34} & a_{44} \end{bmatrix}$$

In the same way we have a 4-chaotic equation

$$\frac{dA_1}{dt} = A_1 - \frac{A_1^2}{16}(a_{11} + a_{22} + a_{33} + a_{44} + a_{12}$$

$$+a_{13} + a_{14} + a_{21} + a_{23} + a_{24} + a_{31} + a_{32} + a_{34} + a_{41} + a_{42} + a_{43})$$

$$-\frac{A_2^2}{16}(a_{11} + a_{22} + a_{33} + a_{44} + a_{13} + a_{31} + a_{24} + a_{42}$$

$$-a_{12} - a_{21} - a_{14} - a_{41} - a_{23} - a_{32} - a_{34} - a_{43}) - \frac{A_1A_2}{8}$$

$$(a_{11} + a_{33} + a_{13} + a_{31} - a_{22} - a_{44} - a_{24} - a_{42})$$

$$-\frac{H^2}{8}[a_{11} + a_{22} + a_{33} + a_{44} - a_{13} - a_{31} - a_{24} - a_{42}$$

$$+\cos 2\beta(a_{11} + a_{33} - a_{22} - a_{44} - a_{13} - a_{31} + a_{24} + a_{42})]$$

$$-\frac{HA_1}{8}[\cos\beta(2a_{11} - 2a_{33} + a_{12} + a_{21} + a_{14} + a_{41}$$

$$-a_{23} - a_{32} - a_{34} - a_{43}) + \sin\beta(-2a_{22} + 2a_{44} + a_{12} + a_{21}$$

$$+a_{14} + a_{41} - a_{23} - a_{32} - a_{34} - a_{43})]$$

$$-\frac{HA_2}{8}[\cos\beta(2a_{11} - 2a_{33} + a_{12} + a_{21} + a_{14} + a_{41} - a_{23} - a_{32} - a_{34} - a_{43})$$

$$+\sin\beta(2a_{22}-2a_{44}-a_{12}-a_{21}+a_{14}+a_{41}-a_{23}-a_{32}+a_{34}+a_{43})]$$
$$-\frac{H^2\sin 2\beta}{8}(-a_{12}-a_{21}+a_{14}+a_{41}+a_{23}+a_{32}-a_{34}-a_{43}). \quad (9.103)$$

If $\frac{dA_1}{dt}<0$, it is stable; if $\frac{dA_1}{dt}>0$, it is unstable. Chaos occurs when stable and unstable manifolds cross at nontrivial fixed points.

For $4m=8$, we have

$$\frac{dN_i}{dt}=N_i(1-\sum_{j=1}^{8}\bar{a}_{ij}N_j). \quad (9.104)$$

where

$$\bar{a}_{ij}=\begin{bmatrix} 1 & a_1 & a_2 & \cdots & a_7 \\ a_7 & 1 & a_1 & \cdots & a_6 \\ a_6 & a_7 & 1 & \cdots & a_5 \\ \vdots & \vdots & \vdots & \vdots & \\ a_1 & a_2 & a_3 & \cdots & 1 \end{bmatrix}$$

In the same way we have a 8-chaotic equation

$$\frac{dA_1}{dt}=A_1-\frac{A_1^2}{8}(1+a_1+a_2+a_3+a_4+a_5+a_6+a_7)$$
$$-\frac{A_2^2}{8}(1-a_1+a_2-a_3+a_4-a_5+a_6-a_7)$$
$$-\frac{H^2}{4}(1-a_2+a_4-a_6)$$
$$-\frac{B^2}{8}[2+\sqrt{2}(a_1+a_7)-2a_4-\sqrt{2}(a_3+a_5)]$$
$$-\frac{D^2}{8}[2-\sqrt{2}(a_1+a_7)-2a_4+\sqrt{2}(a_3+a_5)]. \quad (9.105)$$

If $\frac{dA_1}{dt}<0$, it is stable; if $\frac{dA_1}{dt}>0$, it is unstable. Chaos occurs when stable and unstable manifolds cross at nontrivial fixed points.

In the same way we may introduce the chaotic functions of the second kind from (5.1), (6.1) and (7.1).

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Chapter 4

THE PROOFS OF BINARY GOLDBACH'S THEOREM USING ONLY PARTIAL PRIMES

When the answers to a mathematical problem cannot be found, then the reason is frequently the fact that we have not recognized the general idea, from which the given problem appears only as a single link in a chain of related problems.

David Hilbert

In 1994 we discovered the new arithmetic function $J_2(\omega)$. Using it we proved the binary Goldbach's theorem [1]. In this chapter we yield the more detailed proofs of the binary Goldbach's theorem using only partial primes.

Definition. We define the arithmetic progressions [1–4]

$$E_{p_\alpha}(K) = \omega K + p_\alpha, \tag{1}$$

where $K = 0, 1, 2, ..., \omega = \prod_{2 \le p \le p_i} p, (p_\alpha, \omega) = 1, p_i < p_\alpha = p_1, p_2, ..., p_{\phi(\omega)} = \omega + 1, \phi(\omega) = \prod_{2 \le p \le p_i}(p-1)$ being Euler totient function.

$E_{p_\alpha}(K)$ can constitute all the primes and composites except the numbers of factors: $2, 3, ..., p_i$. We define the primes and the composites by K below.

Theorem 1. If there exist the infinitely many primes p such that $ap + b$ is also a prime, then $ap + b$ must satisfy three necessary and sufficient conditions:

(I) Let $ap + b$ be an irreducible polynomial satisfying $ab \neq 0, (a, b) = 1, 2|ab$.

(II) There exists an arithmetic function $J_2(\omega)$ which denotes the number of subequations. It is also the number of solutions for

$$(aE_{p_\alpha}(K)+b,\omega)=(ap_\alpha+b,\omega)=1. \tag{2}$$

From (2) defining $J_2(\omega)$ can be written in the form

$$J_2(\omega)=\sum_{\alpha=1}^{\phi(\omega)}\left[\frac{1}{(ap_\alpha+b,\omega)}\right]=\prod_{3\le p\le p_i}(p-1-\chi(p)). \tag{3}$$

(III) t_α is independent of p_α [5], where t_α denotes the number of primes K_p less than n in $aE_{p_\alpha}(K_p)+b=p''$. Taking $t_1=t_\alpha$, where $\alpha=1,...,J_2(\omega).t_\alpha$ seem to be equally distributed among the $J_2(\omega)$. We have

$$\pi_2(N,2)=|\{p:p\le N, ap+b=p'\}|=\sum_{\alpha=1}^{J_2(\omega)}t_\alpha\sim J_2(\omega)t_1. \tag{4}$$

First we deal with a subequation $aE_{p_1}(K)+b=p''$. We define the sequence

$$K=0,1,2,...,n. \tag{5}$$

We take the average value

$$t_1=|\{K_p:K_p\le n, aE_{p_1}(K_p)+b=p''\}|\sim\frac{(\pi_1(\omega n))^2}{n}, \tag{6}$$

where $\pi_1(\omega n)$ denotes the number of primes K_p less than n in $E_{p_1}(K)$. We show that t_1 is independent of p_1, because $\pi_1(\omega n)$ is independent of p_1 [5].

Let $N=\omega n$ and $\pi_1(N)\sim\frac{N}{\phi(\omega)\log N}$. Substituting it into (6) and then (6) into (4) we have

$$\pi_2(N,2)=|\{p:p\le N, ap+b=p'\}|\sim\frac{J_2(\omega)\omega}{\phi^2(\omega)}\frac{N}{\log^2 N}. \tag{7}$$

From (2) we have

$$ap_\alpha+b\equiv 0\ (\mathrm{mod}\ \ p). \tag{8}$$

Every $p_\alpha>p$ can be expressed in the form

$$p_\alpha=pm+q, \tag{9}$$

where $q=1,2,...,p-1$.

Substituting (9) into (8) we have

$$aq+b\equiv 0\ (\mathrm{mod}\ \ p). \tag{10}$$

If $p|ab$, then (10) has no solutions. We define $\chi(p) = 0$. If $p \not| ab$, then (10) has a solution. We define $\chi(p) = 1$. Substituting it into (3) we have

$$J_2(\omega) = \prod_{3 \le p \le p_i} (p-2) \prod_{p|ab, 3 \le p \le p_i} \frac{p-1}{p-2} \ne 0, \tag{11}$$

Since $J_2(\omega) \to \infty$ as $\omega \to \infty$, there exist the infinitely many primes p such that $ap+b$ is also a prime. It is a generalization of Euler proof of the existence of the infinitely many primes [1–4].

Substituting (11) into (7) we have

$$\pi_2(N,2) = |\{p : p \le N, ap+b = p'\}| \sim 2 \prod_{3 \le p \le p_i} \left(1 - \frac{1}{(p-1)^2}\right) \prod_{p|ab, 3 \le p \le p_i} \frac{p-1}{p-2} \frac{N}{\log^2 N}. \tag{12}$$

The Prime Twins Theorem. Let $a = 1$ and $b = 2$. From (11) we have

$$J_2(\omega) = \prod_{3 \le p \le p_i} (p-2) \ne 0, \tag{13}$$

Since $J_2(\omega) \to \infty$ as $\omega \to \infty$, there exist the infinitely many primes p such that $p+2$ is also a prime.

From (12) we have

$$\pi_2(N,2) = |\{p : p \le N, p+2 = p'\}| \sim 2 \prod_{3 \le p \le p_i} \left(1 - \frac{1}{(p-1)^2}\right) \frac{N}{\log^2 N}. \tag{14}$$

(14) is the best asymptotic formula conjectured by Hardy and Littlewood [6].

The Binary Goldbach's Theorem [1-4]. Let $a = -1$ and $b = N$. From (11) we have

$$J_2(\omega) = \prod_{3 \le p \le p_i} (p-2) \prod_{p|N, 3 \le p \le p_i} \frac{p-1}{p-2} \ne 0, \tag{15}$$

Since $J_2(\omega) \to \infty$ as $\omega \to \infty$, every even number N greater than 4 is the sum of two primes.

From (12) we have

$$\pi_2(N,2) = |\{p : p \le N, N-p = p'\}| \sim 2 \prod_{3 \le p \le p_i} \left(1 - \frac{1}{(p-1)^2}\right) \prod_{p|N} \frac{p-1}{p-2} \frac{N}{\log^2 N}. \tag{16}$$

(16) is the best asymptotic formula conjectured by Hardy and Littlewood [6].

To understand the binary Goldbach's theorem, we yield the more detailed proofs below.

Corollary 1. Let $p_i = 5$ and $\omega = 30$. From (1) we have [1-2]

$$E_{p_\alpha}(K) = 30K + p_\alpha, \tag{17}$$

where $K = 0, 1, 2, \cdots; p_\alpha = 7, 11, 13, 17, 19, 23, 29, 31$.

All the even numbers N greater than 16 can be expressed as

$$N = 30m + h, \tag{18}$$

where $m = 0, 1, 2, \cdots; h = 18, 20, 22, 24, 26, 28, 30, 32, 34, 36, 38, 40, 42, 44, 46$.

From (17) and (18) we have

$$N = 30m + h = E_{p_1}(K_1) + E_{p_2}(K_2). \tag{19}$$

From (19) we have

$$m = K_1 + K_2, \quad h \equiv p_1 + p_2 \pmod{30}. \tag{20}$$

$m = K_1 + K_2$ is called Yu's mathematical problem, namely, integer m greater than 1 is the sum of primes K_1 and K_2. To prove $m = K_1 + K_2$ is transformed into studying $N = E_{p_1}(K_1) + E_{p_2}(K_2)$.

If $3|N$ from (15) we have $J_2(30) = 6$. From (17) and (19) we have six subequations: $6 = 2 + 2 + 2$

$$N = 30m + 18 = E_7(K_1) + E_{11}(K_2) = E_{17}(K_1) + E_{31}(K_2)$$
$$= E_{19}(K_1) + E_{29}(K_2), \tag{21}$$
$$N = 30m + 24 = E_7(K_1) + E_{17}(K_2) = E_{11}(K_1) + E_{13}(K_2)$$
$$= E_{23}(K_1) + E_{31}(K_2), \tag{22}$$
$$N = 30m + 36 = E_7(K_1) + E_{29}(K_2) = E_{13}(K_1) + E_{23}(K_2)$$
$$= E_{17}(K_1) + E_{19}(K_2), \tag{23}$$
$$N = 30m + 42 = E_{11}(K_1) + E_{31}(K_2) = E_{13}(K_1) + E_{29}(K_2)$$
$$= E_{19}(K_1) + E_{23}(K_2). \tag{24}$$

If $5|N$ from (15) we have $J_2(30) = 4$. From (17) and (19) we have four subequations: $4 = 2 + 2$

$$N = 30m + 20 = E_7(K_1) + E_{13}(K_2) = E_{19}(K_1) + E_{31}(K_2), \tag{25}$$
$$N = 30m + 40 = E_{11}(K_1) + E_{29}(K_2) = E_{17}(K_1) + E_{23}(K_2). \tag{26}$$

If $3, 5 \not| N$ from (15) we have $J_2(30) = 3$. From (17) and (19) we have three subequations: $3 = 2 + 1$

$$N = 30m + 22 = E_{23}(K_1) + E_{29}(K_2) = E_{11}(K_1) + E_{11}(K_2), \tag{27}$$
$$N = 30m + 26 = E_7(K_1) + E_{19}(K_2) = E_{13}(K_1) + E_{13}(K_2), \tag{28}$$
$$N = 30m + 28 = E_{11}(K_1) + E_{17}(K_2) = E_{29}(K_1) + E_{29}(K_2), \tag{29}$$
$$N = 30m + 32 = E_{13}(K_1) + E_{19}(K_2) = E_{31}(K_1) + E_{31}(K_2), \tag{30}$$
$$N = 30m + 34 = E_{11}(K_1) + E_{23}(K_2) = E_{17}(K_1) + E_{17}(K_2), \tag{31}$$

$N = 30m + 38 = E_7(K_1) + E_{31}(K_2) = E_{19}(K_1) + E_{19}(K_2),$ (32)
$N = 30m + 44 = E_{13}(K_1) + E_{31}(K_2) = E_7(K_1) + E_7(K_2),$ (33)
$N = 30m + 46 = E_{17}(K_1) + E_{29}(K_2) = E_{23}(K_1) + E_{23}(K_2).$ (34)

If $3, 5|N$ from (15) we have $J_2(30) = 8$. From (17) and (19) we have eight subequations: $8 = 2 + 2 + 2 + 2$

$$N = 30m + 30 = E_7(K_1) + E_{23}(K_2) = E_{11}(K_1) + E_{19}(K_2)$$
$$= E_{13}(K_1) + E_{17}(K_2) = E_{29}(K_1) + E_{31}(K_2). \quad (35)$$

We can prove the binary Goldbach's theorem using only thirty subequations:

$N = 30m + 18 = E_7(K_1) + E_{11}(K_2)$, $N = 30m + 20 = E_7(K_1) + E_{13}(K_2)$,
$N = 30m + 22 = E_{23}(K_1) + E_{29}(K_2)$, $N = 30m + 24 = E_7(K_1) + E_{17}(K_2)$,
$N = 30m + 26 = E_7(K_1) + E_{19}(K_2)$, $N = 30m + 28 = E_{11}(K_1) + E_{17}(K_2)$,
$N = 30m + 30 = E_7(K_1) + E_{23}(K_2)$, $N = 30m + 32 = E_{13}(K_1) + E_{19}(K_2)$,
$N = 30m + 34 = E_{11}(K_1) + E_{23}(K_2)$, $N = 30m + 36 = E_7(K_1) + E_{29}(K_2)$,
$N = 30m + 38 = E_7(K_1) + E_{31}(K_2)$, $N = 30m + 40 = E_{11}(K_1) + E_{29}(K_2)$,
$N = 30m + 42 = E_{19}(K_1) + E_{23}(K_2)$, $N = 30m + 44 = E_{13}(K_1) + E_{31}(K_2)$,
$N = 30m + 46 = E_{17}(K_1) + E_{29}(K_2)$. (36)

For every equation we have the arithmetic function

$$J_2(\omega > 30) = \prod_{7 \le p \le p_i} (p-2) \prod_{p|N, 7 \le p \le p_i} \frac{p-1}{p-2} \neq 0. \quad (37)$$

Since $J_2(\omega > 30) \to \infty$ as $\omega \to \infty$, we prove Yu's mathematical problem. We prove also the binary Goldbach's theorem using the partial primes.

Substituting (37) into (7) we have the best asymptotic formula

$$\pi_2(N, 2) = \sum_{m=K_1+K_2} 1 = \sum_{N=E_{p_1}(K_1)+E_{p_2}(K_2)} 1 \sim$$
$$\frac{15}{32} \prod_{7 \le p \le p_i} \left(1 - \frac{1}{(p-1)^2}\right) \prod_{p|N} \frac{p-1}{p-2} \frac{N}{\log^2 N}. \quad (38)$$

Corollary 2. Let $p_i = 7$ and $\omega = 210$. From (1) we have

$$E_{p_\alpha}(K) = 210K + p_\alpha, \quad (39)$$

where $K = 0, 1, 2, \cdots, (210, p_\alpha) = 1, p_\alpha = 11, 41, 71, 101, 131, 191;$
$13, 43, 73, 103, 163, 193; 17, 47, 107, 137, 167, 197; 19, 79, 109, 139, 169, 199;$
$23, 53, 83, 113, 143, 173; 29, 59, 89, 149, 179, 209; 31, 61, 121, 151, 181, 211;$
$37, 67, 97, 127, 157, 187.$

All the even numbers N greater than 38 can be expressed as

$$N = 210m + h, \quad (40)$$

where $m = 0, 1, 2, \cdots, h = 40, 42, \cdots, 248$.

From (39) and (40) we have

$$N = 210m + h = E_{p_1}(K_1) + E_{p_2}(K_2). \tag{41}$$

From (41) we have

$$m = K_1 + K_2, \quad h \equiv p_1 + p_2 (\text{mod } \ 210). \tag{42}$$

From (39), (41) and (42) we have the 2304 subequations as follows.

$40 \equiv 11 + 29 \equiv 41 + 209 \equiv 191 + 59 \equiv 71 + 179 \equiv 101 + 149;$
$40 \equiv 17 + 23 \equiv 197 + 53 \equiv 167 + 83 \equiv 107 + 143 \equiv 137 + 113.$
$42 \equiv 11 + 31 \equiv 41 + 211 \equiv 191 + 61 \equiv 71 + 181 \equiv 101 + 151 \equiv 131 + 121;$
$42 \equiv 13 + 29 \equiv 43 + 209 \equiv 193 + 59 \equiv 73 + 179 \equiv 103 + 149 \equiv 163 + 89;$
$42 \equiv 19 + 23 \equiv 199 + 53 \equiv 79 + 173 \equiv 169 + 83 \equiv 109 + 143 \equiv 139 + 113.$
$44 \equiv 13 + 31 \equiv 43 + 211 \equiv 193 + 61 \equiv 73 + 181 \equiv 103 + 151;$
$44 \equiv 67 + 187 \equiv 97 + 157 \equiv 127 + 127.$
$46 \equiv 17 + 29 \equiv 47 + 209 \equiv 197 + 59 \equiv 167 + 89 \equiv 107 + 149;$
$46 \equiv 23 + 23 \equiv 83 + 173 \equiv 113 + 143.$
$48 \equiv 11 + 37 \equiv 191 + 67 \equiv 71 + 187 \equiv 101 + 157 \equiv 131 + 127;$
$48 \equiv 17 + 31 \equiv 47 + 211 \equiv 197 + 61 \equiv 107 + 151 \equiv 137 + 121;$
$48 \equiv 19 + 29 \equiv 199 + 59 \equiv 79 + 179 \equiv 169 + 89 \equiv 109 + 149.$
$50 \equiv 13 + 37 \equiv 193 + 67 \equiv 73 + 187 \equiv 163 + 97 \equiv 103 + 157;$
$50 \equiv 19 + 31 \equiv 199 + 61 \equiv 79 + 181 \equiv 109 + 151 \equiv 139 + 121.$
$52 \equiv 23 + 29 \equiv 53 + 209 \equiv 83 + 179 \equiv 173 + 89 \equiv 113 + 149.$
$52 \equiv 11 + 41 \equiv 71 + 191 \equiv 131 + 131.$
$54 \equiv 11 + 43 \equiv 41 + 13 \equiv 71 + 193 \equiv 191 + 73 \equiv 101 + 163;$
$54 \equiv 17 + 37 \equiv 197 + 67 \equiv 167 + 97 \equiv 107 + 157 \equiv 137 + 127;$
$54 \equiv 23 + 31 \equiv 53 + 211 \equiv 83 + 181 \equiv 113 + 151 \equiv 143 + 121.$
$56 \equiv 13 + 43 \equiv 73 + 193 \equiv 103 + 163;$
$56 \equiv 19 + 37 \equiv 199 + 67 \equiv 79 + 187 \equiv 169 + 97 \equiv 109 + 157 \equiv 139 + 127.$
$58 \equiv 11 + 47 \equiv 41 + 17 \equiv 71 + 197 \equiv 101 + 167 \equiv 131 + 137;$
$58 \equiv 29 + 29 \equiv 59 + 209 \equiv 89 + 179.$
$60 \equiv 13 + 47 \equiv 43 + 17 \equiv 73 + 197 \equiv 103 + 167 \equiv 163 + 107;$
$60 \equiv 19 + 41 \equiv 199 + 71 \equiv 79 + 191 \equiv 169 + 101 \equiv 139 + 131;$
$60 \equiv 23 + 37 \equiv 83 + 187 \equiv 173 + 97 \equiv 113 + 157 \equiv 143 + 127;$
$60 \equiv 29 + 31 \equiv 59 + 211 \equiv 209 + 61 \equiv 89 + 181 \equiv 149 + 121.$
$62 \equiv 19 + 43 \equiv 199 + 73 \equiv 79 + 193 \equiv 169 + 103 \equiv 109 + 163;$
$62 \equiv 31 + 31 \equiv 61 + 211 \equiv 121 + 151.$
$64 \equiv 11 + 53 \equiv 41 + 23 \equiv 191 + 83 \equiv 101 + 173 \equiv 131 + 143;$
$64 \equiv 17 + 47 \equiv 107 + 167 \equiv 137 + 137.$
$66 \equiv 13 + 53 \equiv 43 + 23 \equiv 193 + 83 \equiv 103 + 173 \equiv 163 + 113;$
$66 \equiv 19 + 47 \equiv 79 + 197 \equiv 169 + 107 \equiv 109 + 167 \equiv 139 + 137;$
$66 \equiv 29 + 37 \equiv 209 + 67 \equiv 89 + 187 \equiv 179 + 97 \equiv 149 + 127.$
$68 \equiv 31 + 37 \equiv 211 + 67 \equiv 181 + 97 \equiv 121 + 157 \equiv 151 + 127;$
$68 \equiv 79 + 199 \equiv 109 + 169 \equiv 139 + 139.$
$70 \equiv 11 + 59 \equiv 41 + 29 \equiv 71 + 209 \equiv 191 + 89 \equiv 101 + 179 \equiv 131 + 149;$
$70 \equiv 17 + 53 \equiv 47 + 23 \equiv 197 + 83 \equiv 107 + 173 \equiv 167 + 113 \equiv 137 + 143.$

$72 \equiv 11 + 61 \equiv 41 + 31 \equiv 71 + 211 \equiv 101 + 181 \equiv 131 + 151;$
$72 \equiv 13 + 59 \equiv 43 + 29 \equiv 73 + 209 \equiv 103 + 179 \equiv 193 + 89;$
$72 \equiv 19 + 53 \equiv 109 + 173 \equiv 169 + 113 \equiv 139 + 143 \equiv 199 + 83.$
$74 \equiv 13 + 61 \equiv 43 + 31 \equiv 73 + 211 \equiv 103 + 181 \equiv 163 + 121;$
$74 \equiv 37 + 37 \equiv 97 + 187 \equiv 127 + 157.$
$76 \equiv 17 + 59 \equiv 47 + 29 \equiv 197 + 89 \equiv 107 + 179 \equiv 137 + 149;$
$76 \equiv 23 + 53 \equiv 113 + 173 \equiv 143 + 143.$
$78 \equiv 11 + 67 \equiv 41 + 37 \equiv 191 + 97 \equiv 101 + 187 \equiv 131 + 157;$
$78 \equiv 17 + 61 \equiv 47 + 31 \equiv 107 + 181 \equiv 167 + 121 \equiv 137 + 151;$
$78 \equiv 19 + 59 \equiv 79 + 209 \equiv 199 + 89 \equiv 109 + 179 \equiv 139 + 149.$
$80 \equiv 13 + 67 \equiv 43 + 37 \equiv 193 + 97 \equiv 103 + 187 \equiv 163 + 127;$
$80 \equiv 19 + 61 \equiv 79 + 211 \equiv 109 + 181 \equiv 169 + 121 \equiv 139 + 151.$
$82 \equiv 11 + 71 \equiv 41 + 41 \equiv 101 + 191.$
$82 \equiv 23 + 59 \equiv 53 + 29 \equiv 83 + 209 \equiv 113 + 179 \equiv 143 + 149.$
$84 \equiv 11 + 73 \equiv 71 + 13 \equiv 41 + 43 \equiv 109 + 193 \equiv 191 + 103 \equiv 131 + 163;$
$84 \equiv 17 + 67 \equiv 47 + 37 \equiv 107 + 187 \equiv 167 + 127 \equiv 137 + 157 \equiv 197 + 97.$
$84 \equiv 23 + 61 \equiv 53 + 31 \equiv 83 + 211 \equiv 113 + 181 \equiv 173 + 121 \equiv 143 + 151.$
$86 \equiv 19 + 67 \equiv 109 + 187 \equiv 139 + 157 \equiv 169 + 127 \equiv 199 + 97;$
$86 \equiv 13 + 73 \equiv 43 + 43 \equiv 103 + 193.$
$88 \equiv 17 + 71 \equiv 47 + 41 \equiv 107 + 191 \equiv 167 + 131 \equiv 197 + 101;$
$88 \equiv 29 + 59 \equiv 89 + 209 \equiv 149 + 149.$
$90 \equiv 11 + 79 \equiv 71 + 19 \equiv 101 + 199 \equiv 131 + 169 \equiv 191 + 109;$
$90 \equiv 17 + 73 \equiv 47 + 43 \equiv 107 + 193 \equiv 137 + 163 \equiv 197 + 103;$
$90 \equiv 23 + 67 \equiv 53 + 37 \equiv 113 + 187 \equiv 143 + 157 \equiv 173 + 127;$
$90 \equiv 29 + 61 \equiv 59 + 31 \equiv 89 + 211 \equiv 149 + 151 \equiv 179 + 121.$
$92 \equiv 13 + 79 \equiv 73 + 19 \equiv 103 + 199 \equiv 163 + 139 \equiv 193 + 109;$
$92 \equiv 31 + 61 \equiv 121 + 181 \equiv 151 + 151.$
$94 \equiv 11 + 83 \equiv 71 + 23 \equiv 41 + 53 \equiv 131 + 173 \equiv 191 + 113;$
$94 \equiv 47 + 47 \equiv 107 + 197 \equiv 137 + 167.$
$96 \equiv 13 + 83 \equiv 43 + 53 \equiv 73 + 23 \equiv 163 + 143 \equiv 193 + 113;$
$96 \equiv 17 + 79 \equiv 107 + 199 \equiv 137 + 169 \equiv 167 + 139 \equiv 197 + 109;$
$96 \equiv 29 + 67 \equiv 59 + 37 \equiv 149 + 157 \equiv 179 + 127 \equiv 209 + 97.$
$98 \equiv 31 + 67 \equiv 61 + 37 \equiv 121 + 187 \equiv 151 + 157 \equiv 181 + 127 \equiv 211 + 97;$
$98 \equiv 19 + 79 \equiv 109 + 199 \equiv 139 + 169.$
$100 \equiv 11 + 89 \equiv 41 + 59 \equiv 71 + 29 \equiv 101 + 209 \equiv 131 + 179;$
$100 \equiv 17 + 83 \equiv 47 + 53 \equiv 137 + 173 \equiv 167 + 143 \equiv 197 + 113.$
$102 \equiv 13 + 89 \equiv 73 + 29 \equiv 43 + 59 \equiv 103 + 209 \equiv 163 + 149;$
$102 \equiv 19 + 83 \equiv 79 + 23 \equiv 139 + 173 \equiv 169 + 143 \equiv 199 + 113;$
$102 \equiv 31 + 71 \equiv 61 + 41 \equiv 121 + 191 \equiv 181 + 131 \equiv 211 + 101.$
$104 \equiv 31 + 73 \equiv 61 + 43 \equiv 121 + 193 \equiv 151 + 163 \equiv 211 + 103;$
$104 \equiv 37 + 67 \equiv 127 + 187 \equiv 157 + 157.$
$106 \equiv 17 + 89 \equiv 47 + 59 \equiv 107 + 209 \equiv 137 + 179 \equiv 167 + 149;$
$106 \equiv 23 + 83 \equiv 53 + 53 \equiv 143 + 173.$
$108 \equiv 11 + 97 \equiv 71 + 37 \equiv 41 + 67 \equiv 131 + 187 \equiv 191 + 127;$
$108 \equiv 19 + 89 \equiv 79 + 29 \equiv 109 + 209 \equiv 139 + 179 \equiv 169 + 149;$
$108 \equiv 47 + 61 \equiv 107 + 211 \equiv 137 + 181 \equiv 167 + 151 \equiv 197 + 121.$
$110 \equiv 13 + 97 \equiv 73 + 37 \equiv 43 + 67 \equiv 163 + 157 \equiv 193 + 127;$
$110 \equiv 31 + 79 \equiv 121 + 199 \equiv 151 + 169 \equiv 181 + 139 \equiv 211 + 109.$
$112 \equiv 23 + 89 \equiv 83 + 29 \equiv 53 + 59 \equiv 113 + 209 \equiv 143 + 179 \equiv 173 + 149;$

$112 \equiv 11 + 101 \equiv 41 + 71 \equiv 131 + 191.$
$114 \equiv 11 + 103 \equiv 41 + 73 \equiv 71 + 43 \equiv 101 + 13 \equiv 131 + 193;$
$114 \equiv 17 + 97 \equiv 47 + 67 \equiv 137 + 187 \equiv 167 + 157 \equiv 197 + 127;$
$114 \equiv 31 + 83 \equiv 61 + 53 \equiv 151 + 173 \equiv 181 + 143 \equiv 211 + 113.$
$116 \equiv 19 + 97 \equiv 79 + 37 \equiv 139 + 187 \equiv 169 + 157 \equiv 199 + 127;$
$116 \equiv 13 + 103 \equiv 43 + 73 \equiv 163 + 163.$
$118 \equiv 11 + 107 \equiv 71 + 47 \equiv 101 + 17 \equiv 131 + 197 \equiv 191 + 137;$
$118 \equiv 29 + 89 \equiv 59 + 59 \equiv 149 + 179.$
$120 \equiv 11 + 109 \equiv 41 + 79 \equiv 101 + 19 \equiv 131 + 199 \equiv 191 + 139;$
$120 \equiv 13 + 107 \equiv 73 + 47 \equiv 103 + 17 \equiv 163 + 167 \equiv 193 + 137;$
$120 \equiv 31 + 89 \equiv 61 + 59 \equiv 121 + 209 \equiv 151 + 179 \equiv 181 + 149;$
$120 \equiv 23 + 97 \equiv 53 + 67 \equiv 83 + 37 \equiv 143 + 187 \equiv 173 + 157.$
$122 \equiv 13 + 109 \equiv 43 + 79 \equiv 103 + 19 \equiv 163 + 169 \equiv 193 + 139;$
$122 \equiv 61 + 61 \equiv 121 + 211 \equiv 151 + 181.$
$124 \equiv 11 + 113 \equiv 41 + 83 \equiv 71 + 53 \equiv 101 + 23 \equiv 191 + 143;$
$124 \equiv 17 + 107 \equiv 137 + 197 \equiv 167 + 167.$
$126 \equiv 13 + 113 \equiv 43 + 83 \equiv 73 + 53 \equiv 103 + 23 \equiv 163 + 173 \equiv 193 + 143;$
$126 \equiv 17 + 109 \equiv 47 + 79 \equiv 107 + 19 \equiv 137 + 199 \equiv 167 + 169 \equiv 197 + 139;$
$126 \equiv 29 + 97 \equiv 59 + 67 \equiv 89 + 37 \equiv 149 + 187 \equiv 179 + 157 \equiv 209 + 127.$
$128 \equiv 31 + 97 \equiv 61 + 67 \equiv 151 + 187 \equiv 181 + 157 \equiv 211 + 127;$
$128 \equiv 19 + 109 \equiv 139 + 199 \equiv 169 + 169.$
$130 \equiv 17 + 113 \equiv 47 + 83 \equiv 107 + 23 \equiv 167 + 173 \equiv 197 + 143;$
$130 \equiv 29 + 101 \equiv 59 + 71 \equiv 89 + 41 \equiv 149 + 191 \equiv 209 + 131.$
$132 \equiv 11 + 121 \equiv 71 + 61 \equiv 101 + 31 \equiv 131 + 211 \equiv 191 + 151;$
$132 \equiv 19 + 113 \equiv 79 + 53 \equiv 109 + 23 \equiv 169 + 173 \equiv 199 + 143;$
$132 \equiv 29 + 103 \equiv 59 + 73 \equiv 89 + 43 \equiv 149 + 193 \equiv 179 + 163.$
$134 \equiv 13 + 121 \equiv 73 + 61 \equiv 103 + 31 \equiv 163 + 181 \equiv 193 + 151.$
$134 \equiv 37 + 97 \equiv 67 + 67 \equiv 157 + 187.$
$136 \equiv 29 + 107 \equiv 89 + 47 \equiv 149 + 197 \equiv 179 + 167 \equiv 209 + 137;$
$136 \equiv 23 + 113 \equiv 53 + 83 \equiv 173 + 173.$
$138 \equiv 11 + 127 \equiv 41 + 97 \equiv 71 + 67 \equiv 101 + 37 \equiv 191 + 157;$
$138 \equiv 17 + 121 \equiv 107 + 31 \equiv 137 + 211 \equiv 167 + 181 \equiv 197 + 151;$
$138 \equiv 29 + 109 \equiv 59 + 79 \equiv 149 + 199 \equiv 179 + 169 \equiv 209 + 139.$
$140 \equiv 13 + 127 \equiv 43 + 97 \equiv 73 + 67 \equiv 103 + 37 \equiv 163 + 187 \equiv 193 + 157;$
$140 \equiv 19 + 121 \equiv 79 + 61 \equiv 109 + 31 \equiv 139 + 211 \equiv 169 + 151 \equiv 199 + 151.$
$142 \equiv 29 + 113 \equiv 59 + 83 \equiv 89 + 53 \equiv 179 + 173 \equiv 209 + 143;$
$142 \equiv 11 + 131 \equiv 41 + 101 \equiv 71 + 71.$
$144 \equiv 13 + 131 \equiv 43 + 101 \equiv 73 + 71 \equiv 103 + 41 \equiv 163 + 191;$
$144 \equiv 17 + 127 \equiv 47 + 97 \equiv 107 + 37 \equiv 167 + 187 \equiv 197 + 157;$
$144 \equiv 23 + 121 \equiv 83 + 61 \equiv 113 + 31 \equiv 143 + 211 \equiv 173 + 181.$
$146 \equiv 19 + 127 \equiv 79 + 67 \equiv 109 + 37 \equiv 169 + 187 \equiv 199 + 157;$
$146 \equiv 43 + 103 \equiv 73 + 73 \equiv 163 + 193.$
$148 \equiv 11 + 137 \equiv 41 + 107 \equiv 101 + 47 \equiv 131 + 17 \equiv 191 + 167;$
$148 \equiv 59 + 89 \equiv 149 + 209 \equiv 179 + 179.$
$150 \equiv 11 + 139 \equiv 41 + 109 \equiv 71 + 79 \equiv 131 + 19 \equiv 191 + 169;$
$150 \equiv 13 + 137 \equiv 43 + 107 \equiv 103 + 47 \equiv 163 + 197 \equiv 193 + 167;$
$150 \equiv 23 + 127 \equiv 53 + 97 \equiv 83 + 67 \equiv 113 + 37 \equiv 173 + 187;$
$150 \equiv 29 + 121 \equiv 89 + 61 \equiv 149 + 211 \equiv 179 + 181 \equiv 209 + 151.$
$152 \equiv 13 + 139 \equiv 43 + 109 \equiv 93 + 79 \equiv 163 + 199 \equiv 193 + 169;$

$152 \equiv 31 + 121 \equiv 151 + 211 \equiv 181 + 181.$
$154 \equiv 11 + 143 \equiv 41 + 113 \equiv 71 + 83 \equiv 101 + 53 \equiv 131 + 23 \equiv 191 + 173;$
$154 \equiv 17 + 137 \equiv 47 + 107 \equiv 167 + 197.$
$156 \equiv 13 + 143 \equiv 43 + 113 \equiv 73 + 83 \equiv 103 + 53 \equiv 193 + 173;$
$156 \equiv 17 + 139 \equiv 47 + 109 \equiv 137 + 19 \equiv 167 + 199 \equiv 197 + 169;$
$156 \equiv 29 + 127 \equiv 59 + 97 \equiv 89 + 67 \equiv 179 + 187 \equiv 209 + 157.$
$158 \equiv 31 + 127 \equiv 97 + 61 \equiv 121 + 37 \equiv 181 + 187 \equiv 211 + 157;$
$158 \equiv 19 + 139 \equiv 79 + 79 \equiv 169 + 199.$
$160 \equiv 11 + 149 \equiv 71 + 89 \equiv 101 + 59 \equiv 131 + 29 \equiv 191 + 179;$
$160 \equiv 17 + 143 \equiv 47 + 113 \equiv 107 + 53 \equiv 137 + 23 \equiv 197 + 173.$
$162 \equiv 11 + 151 \equiv 41 + 121 \equiv 101 + 61 \equiv 131 + 31 \equiv 191 + 181;$
$162 \equiv 13 + 149 \equiv 73 + 89 \equiv 103 + 59 \equiv 163 + 209 \equiv 193 + 179;$
$162 \equiv 19 + 143 \equiv 79 + 83 \equiv 109 + 53 \equiv 139 + 23 \equiv 199 + 173.$
$164 \equiv 13 + 151 \equiv 43 + 121 \equiv 103 + 61 \equiv 163 + 211 \equiv 193 + 181;$
$164 \equiv 37 + 127 \equiv 67 + 97 \equiv 187 + 187.$
$166 \equiv 17 + 149 \equiv 107 + 59 \equiv 137 + 29 \equiv 167 + 209 \equiv 197 + 179;$
$166 \equiv 23 + 143 \equiv 53 + 113 \equiv 83 + 83.$
$168 \equiv 11 + 157 \equiv 41 + 127 \equiv 71 + 97 \equiv 101 + 67 \equiv 131 + 37 \equiv 191 + 187;$
$168 \equiv 17 + 151 \equiv 47 + 121 \equiv 107 + 61 \equiv 137 + 31 \equiv 167 + 211 \equiv 197 + 181;$
$168 \equiv 19 + 149 \equiv 79 + 89 \equiv 109 + 59 \equiv 139 + 29 \equiv 169 + 209 \equiv 199 + 179.$
$170 \equiv 13 + 157 \equiv 43 + 127 \equiv 73 + 97 \equiv 103 + 67 \equiv 193 + 181;$
$170 \equiv 19 + 151 \equiv 109 + 61 \equiv 139 + 31 \equiv 169 + 211 \equiv 199 + 181.$
$172 \equiv 23 + 149 \equiv 83 + 89 \equiv 113 + 59 \equiv 143 + 29 \equiv 173 + 209;$
$172 \equiv 41 + 131 \equiv 71 + 101 \equiv 191 + 191.$
$174 \equiv 11 + 163 \equiv 71 + 103 \equiv 101 + 73 \equiv 131 + 43 \equiv 191 + 193;$
$174 \equiv 17 + 157 \equiv 47 + 127 \equiv 107 + 67 \equiv 137 + 37 \equiv 197 + 187;$
$174 \equiv 23 + 151 \equiv 53 + 121 \equiv 113 + 61 \equiv 143 + 31 \equiv 173 + 211.$
$176 \equiv 19 + 157 \equiv 79 + 97 \equiv 109 + 67 \equiv 139 + 37 \equiv 199 + 187;$
$176 \equiv 13 + 163 \equiv 73 + 103 \equiv 193 + 193.$
$178 \equiv 11 + 167 \equiv 41 + 137 \equiv 107 + 71 \equiv 131 + 47 \equiv 191 + 197;$
$178 \equiv 29 + 149 \equiv 89 + 89 \equiv 179 + 209.$
$180 \equiv 11 + 169 \equiv 41 + 139 \equiv 71 + 109 \equiv 101 + 79 \equiv 191 + 199;$
$180 \equiv 13 + 167 \equiv 43 + 137 \equiv 73 + 107 \equiv 163 + 17 \equiv 193 + 197;$
$180 \equiv 23 + 157 \equiv 53 + 127 \equiv 83 + 97 \equiv 113 + 67 \equiv 143 + 37;$
$180 \equiv 29 + 151 \equiv 59 + 121 \equiv 149 + 31 \equiv 179 + 211 \equiv 209 + 181.$
$182 \equiv 13 + 169 \equiv 43 + 139 \equiv 73 + 109 \equiv 103 + 79 \equiv 163 + 19 \equiv 193 + 199;$
$182 \equiv 31 + 151 \equiv 61 + 121 \equiv 181 + 211.$
$184 \equiv 11 + 173 \equiv 41 + 143 \equiv 71 + 113 \equiv 101 + 83 \equiv 131 + 53;$
$184 \equiv 17 + 167 \equiv 47 + 137 \equiv 197 + 197.$
$186 \equiv 13 + 173 \equiv 43 + 143 \equiv 73 + 113 \equiv 103 + 83 \equiv 163 + 23;$
$186 \equiv 17 + 169 \equiv 47 + 139 \equiv 107 + 79 \equiv 167 + 19 \equiv 197 + 199;$
$186 \equiv 29 + 157 \equiv 59 + 127 \equiv 89 + 97 \equiv 149 + 37 \equiv 209 + 187.$
$188 \equiv 31 + 157 \equiv 61 + 127 \equiv 121 + 67 \equiv 151 + 37 \equiv 211 + 187;$
$188 \equiv 19 + 169 \equiv 79 + 109 \equiv 199 + 199.$
$190 \equiv 11 + 179 \equiv 41 + 149 \equiv 101 + 89 \equiv 131 + 59 \equiv 191 + 209;$
$190 \equiv 17 + 173 \equiv 47 + 143 \equiv 107 + 83 \equiv 137 + 53 \equiv 167 + 23.$
$192 \equiv 11 + 181 \equiv 41 + 151 \equiv 71 + 121 \equiv 131 + 61 \equiv 191 + 211;$
$192 \equiv 13 + 179 \equiv 43 + 149 \equiv 103 + 89 \equiv 163 + 29 \equiv 193 + 209;$
$192 \equiv 23 + 169 \equiv 53 + 139 \equiv 83 + 109 \equiv 113 + 79 \equiv 173 + 19.$

$194 \equiv 13 + 181 \equiv 43 + 151 \equiv 73 + 121 \equiv 163 + 31 \equiv 193 + 211;$
$194 \equiv 37 + 157 \equiv 69 + 127 \equiv 97 + 97.$
$196 \equiv 17 + 179 \equiv 47 + 149 \equiv 107 + 89 \equiv 167 + 29 \equiv 197 + 209 \equiv 137 + 59;$
$196 \equiv 23 + 173 \equiv 53 + 143 \equiv 83 + 113.$
$198 \equiv 11 + 187 \equiv 41 + 157 \equiv 71 + 127 \equiv 101 + 97 \equiv 131 + 67;$
$198 \equiv 17 + 181 \equiv 47 + 151 \equiv 137 + 61 \equiv 167 + 31 \equiv 197 + 211;$
$198 \equiv 19 + 179 \equiv 109 + 89 \equiv 139 + 59 \equiv 169 + 29 \equiv 199 + 209.$
$200 \equiv 13 + 187 \equiv 43 + 157 \equiv 73 + 127 \equiv 103 + 97 \equiv 163 + 37;$
$200 \equiv 19 + 181 \equiv 79 + 121 \equiv 139 + 61 \equiv 169 + 31 \equiv 199 + 211.$
$202 \equiv 23 + 179 \equiv 53 + 149 \equiv 113 + 89 \equiv 143 + 59 \equiv 173 + 29;$
$202 \equiv 11 + 191 \equiv 71 + 131 \equiv 101 + 101.$
$204 \equiv 11 + 193 \equiv 41 + 163 \equiv 101 + 103 \equiv 131 + 73 \equiv 191 + 13;$
$204 \equiv 17 + 187 \equiv 47 + 157 \equiv 107 + 97 \equiv 137 + 67 \equiv 167 + 37;$
$204 \equiv 23 + 181 \equiv 53 + 151 \equiv 83 + 121 \equiv 143 + 61 \equiv 173 + 31.$
$206 \equiv 19 + 187 \equiv 79 + 127 \equiv 109 + 97 \equiv 139 + 67 \equiv 169 + 37;$
$206 \equiv 13 + 193 \equiv 43 + 163 \equiv 103 + 103.$
$208 \equiv 11 + 197 \equiv 41 + 167 \equiv 71 + 137 \equiv 101 + 107 \equiv 191 + 17;$
$208 \equiv 29 + 179 \equiv 59 + 149 \equiv 209 + 209.$
$210 \equiv 11 + 199 \equiv 41 + 169 \equiv 71 + 139 \equiv 101 + 109 \equiv 131 + 79 \equiv 191 + 19;$
$210 \equiv 13 + 197 \equiv 43 + 167 \equiv 73 + 137 \equiv 103 + 107 \equiv 163 + 47 \equiv 193 + 17;$
$210 \equiv 23 + 187 \equiv 53 + 157 \equiv 83 + 127 \equiv 113 + 97 \equiv 143 + 67 \equiv 173 + 37;$
$210 \equiv 29 + 181 \equiv 59 + 151 \equiv 89 + 121 \equiv 149 + 61 \equiv 179 + 31 \equiv 209 + 211.$
$212 \equiv 13 + 199 \equiv 43 + 169 \equiv 73 + 139 \equiv 103 + 109 \equiv 193 + 19;$
$212 \equiv 31 + 181 \equiv 61 + 151 \equiv 211 + 211.$
$214 \equiv 23 + 191 \equiv 83 + 131 \equiv 113 + 103 \equiv 143 + 71 \equiv 173 + 41;$
$214 \equiv 17 + 197 \equiv 47 + 167 \equiv 107 + 107.$
$216 \equiv 17 + 199 \equiv 47 + 169 \equiv 107 + 109 \equiv 137 + 79 \equiv 197 + 19;$
$216 \equiv 23 + 193 \equiv 53 + 163 \equiv 113 + 103 \equiv 143 + 73 \equiv 173 + 43;$
$216 \equiv 29 + 187 \equiv 59 + 157 \equiv 89 + 127 \equiv 149 + 67 \equiv 179 + 37.$
$218 \equiv 31 + 187 \equiv 61 + 157 \equiv 121 + 97 \equiv 151 + 67 \equiv 181 + 37;$
$218 \equiv 19 + 199 \equiv 79 + 139 \equiv 109 + 109.$
$220 \equiv 11 + 209 \equiv 41 + 179 \equiv 71 + 149 \equiv 131 + 89 \equiv 191 + 29;$
$220 \equiv 23 + 197 \equiv 53 + 167 \equiv 83 + 137 \equiv 113 + 107 \equiv 173 + 47.$
$222 \equiv 11 + 211 \equiv 41 + 181 \equiv 71 + 151 \equiv 101 + 121 \equiv 191 + 31;$
$222 \equiv 13 + 209 \equiv 43 + 179 \equiv 73 + 149 \equiv 163 + 59 \equiv 193 + 29;$
$222 \equiv 23 + 199 \equiv 53 + 169 \equiv 83 + 139 \equiv 113 + 109 \equiv 143 + 79.$
$224 \equiv 13 + 211 \equiv 43 + 181 \equiv 73 + 151 \equiv 103 + 121 \equiv 163 + 61 \equiv 193 + 31;$
$224 \equiv 37 + 187 \equiv 67 + 157 \equiv 97 + 127.$
$226 \equiv 17 + 209 \equiv 47 + 179 \equiv 137 + 89 \equiv 167 + 59 \equiv 197 + 29;$
$226 \equiv 53 + 173 \equiv 83 + 143 \equiv 113 + 113.$
$228 \equiv 17 + 211 \equiv 47 + 181 \equiv 107 + 121 \equiv 167 + 61 \equiv 197 + 31;$
$228 \equiv 19 + 209 \equiv 79 + 149 \equiv 139 + 89 \equiv 169 + 59 \equiv 199 + 29;$
$228 \equiv 37 + 191 \equiv 97 + 131 \equiv 127 + 101 \equiv 157 + 71 \equiv 187 + 41.$
$230 \equiv 19 + 211 \equiv 79 + 151 \equiv 109 + 121 \equiv 169 + 61 \equiv 199 + 31;$
$230 \equiv 37 + 193 \equiv 67 + 163 \equiv 127 + 103 \equiv 157 + 73 \equiv 187 + 43.$
$232 \equiv 23 + 209 \equiv 53 + 179 \equiv 83 + 149 \equiv 143 + 89 \equiv 173 + 59;$
$232 \equiv 41 + 191 \equiv 101 + 131 \equiv 11 + 11.$
$234 \equiv 23 + 211 \equiv 53 + 181 \equiv 83 + 151 \equiv 113 + 121 \equiv 173 + 61;$
$234 \equiv 37 + 197 \equiv 67 + 167 \equiv 97 + 137 \equiv 127 + 107 \equiv 187 + 47;$

$234 \equiv 41 + 193 \equiv 191 + 43 \equiv 71 + 163 \equiv 131 + 103 \equiv 11 + 23.$
$236 \equiv 37 + 199 \equiv 67 + 169 \equiv 97 + 139 \equiv 127 + 109 \equiv 157 + 79;$
$236 \equiv 43 + 193 \equiv 73 + 163 \equiv 13 + 13.$
$238 \equiv 41 + 197 \equiv 71 + 167 \equiv 101 + 137 \equiv 131 + 107 \equiv 191 + 47 \equiv 11 + 17;$
$238 \equiv 29 + 209 \equiv 59 + 179 \equiv 89 + 149.$
$240 \equiv 29 + 211 \equiv 59 + 181 \equiv 89 + 151 \equiv 179 + 61 \equiv 209 + 31;$
$240 \equiv 41 + 199 \equiv 71 + 169 \equiv 101 + 139 \equiv 131 + 109 \equiv 11 + 19;$
$240 \equiv 43 + 197 \equiv 73 + 167 \equiv 103 + 137 \equiv 193 + 47 \equiv 13 + 17;$
$240 \equiv 23 + 187 \equiv 173 + 67 \equiv 83 + 157 \equiv 143 + 87 \equiv 113 + 127.$
$242 \equiv 43 + 199 \equiv 73 + 169 \equiv 163 + 97 \equiv 103 + 139 \equiv 13 + 19;$
$242 \equiv 31 + 211 \equiv 61 + 181 \equiv 121 + 121.$
$244 \equiv 53 + 191 \equiv 173 + 71 \equiv 143 + 101 \equiv 113 + 131 \equiv 11 + 13;$
$244 \equiv 47 + 197 \equiv 107 + 137 \equiv 17 + 17.$
$246 \equiv 37 + 209 \equiv 181 + 59 \equiv 67 + 179 \equiv 157 + 89 \equiv 97 + 149;$
$246 \equiv 47 + 199 \equiv 167 + 79 \equiv 107 + 139 \equiv 137 + 109 \equiv 17 + 19;$
$246 \equiv 53 + 193 \equiv 173 + 73 \equiv 83 + 163 \equiv 143 + 103 \equiv 13 + 23.$
$248 \equiv 37 + 211 \equiv 187 + 61 \equiv 67 + 181 \equiv 97 + 151 \equiv 127 + 121;$
$248 \equiv 79 + 169 \equiv 109 + 139 \equiv 19 + 19.$

For studying the binary Goldbach's theorem we discuss only 210 subequations:

$$N = 210m + 40 = E_{11}(K_1) + E_{29}(K_2), \cdots, N = 210m + 248 = E_{79}(K_1) + E_{169}(K_2). \tag{43}$$

For every equation we have the arithmetic function

$$J_2(\omega > 210) = \prod_{11 \le p \le p_i} (p-2) \prod_{p|N} \frac{p-1}{p-2} \ne 0. \tag{44}$$

Since $J_2(\omega > 210) \to \infty$ as $\omega \to \infty$ every even number N from some point onward can be expressed as the sum of two primes using only partial primes.

Substituting (44) into (7) we have the best asymptotic formula

$$\pi_2(N,2) = \sum_{m=K_1+K_2} 1 = \sum_{N=E_{p_1}(K_1)+E_{p_2}(K_2)} 1 \sim$$

$$\frac{35}{384} \prod_{11 \le p \le p_i} \left(1 - \frac{1}{(p-1)^2}\right) \prod_{p|N} \frac{p-1}{p-2} \frac{N}{\log^2 N}. \tag{45}$$

Corollary 3. Let $p_i = 11$ and $\omega = 2310$. From (1) we have

$$E_{p_\alpha}(K) = 2310K + P_\alpha, \tag{46}$$

where $K = 0, 1, 2, ...;$ $\quad (2310, p_\alpha) = 1;$ $\quad p_\alpha = 13, 43, 73, ..., 2263, 2293; 17, 47, 107, ..., 2267, 2273; 19, 79, 109, ..., 2239, 2269; 23, 53, 83, ..., 2243, 2273; 29, 59, 89, ..., 2279, 2309;$ $31, 61, 151, ..., 2281, 2311; 37, 67, 97, ..., 2257, 2287; 41, 71, 101, ..., 2231, 2291.$

All the even numbers N greater than 126 can be expressed as

$$N = 2310m + h, \tag{47}$$

where $m = 0, 1, 2, ...; h = 128, 130, ..., 2436$.

From (46) and (47) we have

$$N = 2310m + h = E_{p_1}(K_1) + E_{p_2}(K_2). \tag{48}$$

From (48) we have

$$m = K_1 + K_2, \quad h \equiv p_1 + p_2 (\text{mod } 2310). \tag{49}$$

From (48) we have the $(480)^2$ subequations as follows:

$$\begin{gathered} N = 2310m + 128 = E_{31}(K_1) + E_{97}(K_2) = E_{61}(K_1) + E_{67}(K_2) = \cdots, \\ \cdots \quad \cdots \\ N = 2310m + 2436 = E_{13}(K_1) + E_{113}(K_2) = E_{43}(K_1) + E_{83}(K_2) = \cdots. \end{gathered} \tag{50}$$

For studying the binary Goldbach's theorem we discuss only 2310 subequations:

$$N = 2310m+128 = E_{31}(K_1)+E_{97}(K_2), ..., N = 2310m+2436 = E_{13}(K_1)+E_{113}(K_2). \tag{51}$$

For every equation we have the arithmetic function

$$J_2(\omega > 2310) = \prod_{13 \le p \le p_i} (p-2) \prod_{p|N} \frac{p-1}{p-2} \neq 0. \tag{52}$$

Since $J_2(\omega > 210) \to \infty$ as $\omega \to \infty$ every even natural number N from some point onward can be expressed as the sum of two primes using only partial primes.

Substituting (52) into (7) we have the best asymptotic formula

$$\pi_2(N, 2) = \sum_{m=K_1+K_2} 1 = \sum_{N=E_{p_1}(K_1)+E_{p_2}(K_2)} 1 \sim$$

$$\frac{77}{7680} \prod_{13 \le p \le p_i} \left(1 - \frac{1}{(p-1)^2}\right) \prod_{p|N} \frac{p-1}{p-2} \frac{N}{\log^2 N}. \tag{53}$$

Corollary 4. Let $p_i = 13$ and $\omega = 30030$. From (1) we have

$$E_{p_\alpha}(K) = 30030K + P_\alpha, \tag{54}$$

where $K = 0, 1, 2, ...;\ (30030, p_\alpha) = 1;\ \ p_\alpha = 17, 47, , ..., 29987; 19, 79, ..., 29989;$ $23, 53, ..., 29993;\ 29, 59, ...30029; 31, 61, ..., 30031; 37, 67, ..., 30007;\ 41, 71, ...,$ $30011;\ 43, 73, ..., 30013.$

All the even numbers N greater than 254 can be expressed as

$$N = 30030m + h, \tag{55}$$

where $m = 0, 1, 2, ...; h = 256, 258, ..., 30284$.
From (54) and (55) we have

$$N = 30030m + h = E_{p_1}(K_1) + E_{p_2}(K_2). \tag{56}$$

From (56) we have

$$m = K_1 + K_2, \quad h \equiv p_1 + p_2 (\text{mod } 30030). \tag{57}$$

From (56) we have the $(5760)^2$ subequations as follows:

$$\begin{gathered} N = 30030m + 256 = E_{17}(K_1) + E_{239}(K_2) = \cdots, \\ \cdots \quad \cdots \\ N = 30030m + 30284 = E_{31}(K_1) + E_{223}(K_2) = \cdots. \end{gathered} \tag{58}$$

For studying the binary Goldbach's theorem we discuss only 30030 subequations:

$$N = 30030m + 256 = E_{17}(K_1) + E_{239}(K_2),$$

$$N = 30030m + 258 = E_{17}(K_1) + E_{241}(K_2),$$

$$N = 30030m + 260 = E_{19}(K_1) + E_{241}(K_2), \cdots,$$

$$N = 30030m + 30282 = E_{19}(K_1) + E_{233}(K_2),$$

$$N = 30030m + 30284 = E_{31}(K_1) + E_{233}(K_2).$$

For every equation we have the arithmetic function

$$J_2(\omega > 30030) = \prod_{17 \le p \le p_i} (p-2) \prod_{p|N} \frac{p-1}{p-2} \neq 0. \tag{59}$$

Since $J_2(\omega > 30030) \to \infty$ as $\omega \to \infty$ every even number N from some point onward can be expressed as the sum of two primes using only partial primes.
Substituting (59) into (7) we have the best asymptotic formula

$$\pi_2(N, 2) = \sum_{m=K_1+K_2} 1 = \sum_{N=E_{p_1}(K_1)+E_{p_2}(K_2)} 1 \sim$$

$$\frac{1001}{1105920} \prod_{17 \le p \le p_i} \left(1 - \frac{1}{(p-1)^2}\right) \prod_{p|N} \frac{p-1}{p-2} \frac{N}{\log^2 N}. \tag{60}$$

Corollary 5. Let $p_i = 17$ and $\omega = 510510$. From (1) we have

$$E_{p_\alpha}(K) = 510510K + P_\alpha, \tag{61}$$

where $K = 0, 1, 2, ...;\quad (510510, p_\alpha) = 1;\quad p_\alpha = 19, ..., 510469; 23, ..., 510473;$ $29, ..., 510509; 31, ..., 510511; 37, ..., 510487; 41, ..., 510491;\quad 43, ..., 510463;$ $47, ..., 510467.$

All the even numbers N greater than 510 can be expressed as

$$N = 510510m + h, \tag{62}$$

where $m = 0, 1, 2, ...; h = 512, 514, ..., 511020.$

From (61) and (62) we have

$$N = 510510m + h = E_{p_1}(K_1) + E_{p_2}(K_2). \tag{63}$$

From (63) we have

$$m = K_1 + K_2, \quad h \equiv p_1 + p_2 (\text{mod } 510510). \tag{64}$$

From (63) we have the $(92160)^2$ subequations as follows:

$$N = 510510m + 512 = E_{73}(K_1) + E_{439}(K_2) = E_{103}(K_1) + E_{409}(K_2) = \cdots,$$

$$\cdots \quad \cdots \tag{65}$$

$$N = 510510m + 511020 = E_{19}(K_1) + E_{491}(K_2) = E_{79}(K_1) + E_{431}(K_2) = \cdots.$$

For studying the binary Goldbach's theorem we discuss only 510510 subequations:

$$N = 510510m + 512 = E_{73}(K_1) + E_{439}(K_2), ...,$$

$$N = 510510m + 511020 = E_{19}(K_1) + E_{491}(K_2). \tag{66}$$

For every equation we have the arithmetic function

$$J_2(\omega > 510510) = \prod_{19 \le p \le p_i} (p-2) \prod_{p|N} \frac{p-1}{p-2} \neq 0. \tag{67}$$

Since $J_2(\omega > 510510) \to \infty$ as $\omega \to \infty$ every even natural number N from some point onward can be expressed as the sum of two primes using only partial primes.

Substituting (67) into (7) we have the best asymptotic formula

$$\pi_2(N, 2) = \sum_{m=K_1+K_2} 1 = \sum_{N=E_{p_1}(K_1)+E_{p_2}(K_2)} 1 \sim$$

$$\frac{17017}{283115520} \prod_{19 \le p \le p_i} \left(1 - \frac{1}{(p-1)^2}\right) \prod_{p|N} \frac{p-1}{p-2} \frac{N}{\log^2 N}. \tag{68}$$

Corollary 6. Let $p_i = p_g$ and $\omega_g = \prod_{2\le p\le p_g} p$. From (1) we have

$$E_{p_\alpha}(K) = \omega_g K + P_\alpha, \tag{69}$$

where $K = 0, 1, 2, ...; (\omega_g, p_\alpha) = 1,\ p_g < p_\alpha = p_1, \cdots, p_{\phi(\omega_g)} = \omega_g + 1$. All the even numbers N greater than H can be expressed as

$$N = \omega_g m + h, \tag{70}$$

where $m = 0, 1, 2, ...; h = H + 2, H + 4, \cdots, \omega_g + H, H$ being an even number.

From (69) and (70) we have

$$N = \omega_g m + h = E_{p_1}(K_1) + E_{p_2}(K_2). \tag{71}$$

From (71) we have

$$m = K_1 + K_2, \quad h \equiv p_1 + p_2 (\text{mod } \omega_g). \tag{72}$$

From (71) we have the $\phi^2(\omega_g)$ subequations. For studying the binary Goldbach's theorem we discuss only ω_g subequations among them. For every equation we have the arithmetic function

$$J_2(\omega > \omega_g) = \prod_{p_g<p\le p_i} (p-2) \prod_{p|N} \frac{p-1}{p-2} \ne 0. \tag{73}$$

Since $J_2(\omega > \omega_g) \to \infty$ as $\omega \to \infty$ every even natural number N from some point onward can be expressed as the sum of two primes using only partial primes.

Substituting (73) into (7) we have the best asymptotic formula

$$\pi_2(N, 2) = \sum_{m=K_1+K_2} 1 = \sum_{N=E_{p_1}(K_1)+E_{p_2}(K_2)} 1 \sim$$

$$\frac{\omega_g}{\phi^2(\omega_g)} \prod_{p_g<p\le p_i} \left(1 - \frac{1}{(p-1)^2}\right) \prod_{p|N} \frac{p-1}{p-2} \frac{N}{\log^2 N}. \tag{74}$$

From $\omega_g = 6$ and (74) we have

$$m_0 = \exp\left(\frac{3\sqrt{c}\phi(\omega_g)\log\omega_g - \omega_g \log 6}{\omega_g - 3\sqrt{c}\phi(\omega_g)}\right), \tag{75}$$

where

$$c = \prod_{3<p\le p_g} \left(1 - \frac{1}{(p-1)^2}\right).$$

From (75) we have $\omega_g = 30, m_0 = 42; \omega_g = 210, m_0 = 141; \omega_g = 2310, m_0 = 946$. The integer m greater than m_0 is the sum of primes K_1 and K_2, that is every even number N greater than $\omega_g m_0$ can be expressed as the sum of two primes using only partial primes. It is $\omega_g/\phi^2(\omega_g)$ of the the total primes. In the same way we can prove the prime twins theorem and other problems using only partial primes. We will establish the additive prime theory with partial primes.

Theorem 2. $p_1 = p + 6,\ p_2 = N - p$.
We have the arithmetic function

$$J_2(\omega) = \prod_{3|N}(p-1) \prod_{5\le p\le p_i}(p-3) \prod_{p|N, p|(N+6)} \frac{p-2}{p-3} \neq 0. \tag{76}$$

Since $J_2(\omega) \to \infty$ as $\omega \to \infty$ every even number N from some point onward can be expressed as the sum of two primes satisfying that $p + 6$ is a prime.
We have exact asymptotic formula

$$\pi_3(N, 2) \sim \frac{J_2(\omega)\omega^2}{\phi^3(\omega)} \frac{N}{\log^3 N}. \tag{77}$$

Theorem 3. $p_1 = p + 6,\ p_2 = p + 12,\ p_3 = N - p$.
We have the arithmetic function

$$J_2(\omega) = \prod_{3|N}(p-1) \prod_{5\le p\le p_i}(p-4) \prod_{p|N, p|(N+6), p|(N+12)} \frac{p-3}{p-4} \neq 0. \tag{78}$$

Since $J_2(\omega) \to \infty$ as $\omega \to \infty$ every even number N from some point onward can be expressed as the sum of two primes satisfying that $p + 6$ and $p + 12$ are primes.
We have exact asymptotic formula

$$\pi_4(N, 2) \sim \frac{J_2(\omega)\omega^3}{\phi^4(\omega)} \frac{N}{\log^4 N}. \tag{79}$$

Theorem 4. $p_1 = p + 2,\ \ p_2 = N - p$.
We have that $J_2(\omega) \neq 0$ if $3 \nmid (N - 2)$; $J_2(3) = 0$ if $3|(N - 2)$.

Theorem 5. $p_1 = p + 4,\ \ p_2 = N - p$.
We have that $J_2(\omega) \neq 0$ if $3 \nmid (N - 1)$; $J_2(3) = 0$ if $3|(N - 1)$.

Theorem 6. $p_1 = p + 30,\ p_2 = N - p$.
We have the arithmetic function

$$J_2(\omega) = \prod_{3,5|N}(p-1) \prod_{5\nmid N}(p-2) \prod_{7\le p\le p_i}(p-3) \prod_{p|N, p|(N+30)} \frac{p-2}{p-3} \neq 0. \tag{80}$$

Since $J_2(\omega) \to \infty$ as $\omega \to \infty$, there exist primes p such that $p + 30$ and $N - p$ are primes for every even number N from some point onward.

We have exact asymptotic formula

$$\pi_3(N,2) \sim \frac{J_2(\omega)\omega^2}{\phi^3(\omega)} \frac{N}{\log^3 N}. \tag{81}$$

Theorem 7. $p_1 = p + 30,\ p_2 = p + 60,\ \ p_3 = N - p.$
We have the arithmetic function

$$J_2(\omega) = \prod_{3,5|N} (p-1) \prod_{5\nmid N} (p-2) \prod_{7\le p\le p_i} (p-4) \prod_{p|N, p|(N+30), p|(N+60)} \frac{p-3}{p-4} \neq 0. \tag{82}$$

Since $J_2(\omega) \to \infty$ as $\omega \to \infty$, there exist primes p such that $p+30,\ p+60$ and $N-p$ are primes for every even number N from some point onward.
We have exact asymptotic formula

$$\pi_4(N,2) \sim \frac{J_2(\omega)\omega^3}{\phi^4(\omega)} \frac{N}{\log^4 N}. \tag{83}$$

Theorem 8. $p_1 = p^2 + 30,\ p_2 = N - p.$
We have the arithmetic function

$$J_2(\omega) = \prod_{3,5|N} (p-1) \prod_{3,5\nmid N} (p-2) \prod_{7\le p\le p_i} \left(p - 3 - (\frac{-30}{p}) - \chi(p)\right) \neq 0. \tag{84}$$

where $\chi(p) = -1$ if $p|N;\ \chi(p) = 0$ if $p \nmid N$.
We have exact asymptotic formula

$$\pi_3(N,2) \sim \frac{J_2(\omega)\omega^2}{2\phi^3(\omega)} \frac{N}{\log^3 N}. \tag{85}$$

Theorem 9. $p_1 = p^2 + 210,\ p_2 = N - p.$
We have the arithmetic function

$$J_2(\omega) = \prod_{3,5,7|N} (p-1) \prod_{3,5,7\nmid N} (p-2) \prod_{11\le p\le p_i} \left(p - 3 - (\frac{-210}{p}) - \chi(p)\right) \neq 0. \tag{86}$$

where $\chi(p) = -1$ if $p|N;\ \chi(p) = 0$ if $p \nmid N$.
We have exact asymptotic formula

$$\pi_3(N,2) \sim \frac{J_2(\omega)\omega^2}{2\phi^3(\omega)} \frac{N}{\log^3 N}. \tag{87}$$

Theorem 10. $p_1 = p^2 + p + 41,\ p_2 = N - p.$

We have the arithmetic function

$$J_2(\omega) = \prod_{3\le p\le p_i} (p - 3 - (\frac{-163}{p}) - \chi(p)) \neq 0, \quad (88)$$

where $\chi(p) = -1$ if $p|N$; $\chi(p) = 0$ if $p \not| N$; $(\frac{-163}{163}) = 0$.

We have exact asymptotic formula

$$\pi_3(N, 2) \sim \frac{J_2(\omega)\omega^2}{2\phi^3(\omega)} \frac{N}{\log^3 N}. \quad (89)$$

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Chapter 5

SANTILLI'S ISOCRYPTOGRAPHIC THEORY

Fundamental open problems in science, such as biological structures, irreversibility, classical treatment of antimatter, grand-unification, constituents of hadrons definable in our spacetime, and others, can only be solved via basically new mathematics which, in turn, can only be really new if based on new numbers.

Ruggero Maria Santilli

1. Introduction

In Appendix 2C of the second edition of monograph [1] written in 1995, Santilli introduced a basically new class of cryptograms. The main idea is that of lifting *any* existing numerical cryptogram based on the trivial unit +1 and the trivial associative product *ab initio* a form based on isonumbers of the first kind with isounit $\hat{I} = 1/\hat{T}$ and isoproduct $a\hat{\times}b = a\hat{T}b$ where $\hat{I}$ is an arbitrary non-zero number generally outside the original set.

This lifting produces the new cryptograms, today known as *Santilli's isocryptograms*, whose evident advantages are the following:

1) The availability of an *infinite number of basic units*, with consequential evident increase of the difficulties for the resolution of the cryptogram, assuming that it can

be resolved in a finite period of time, particularly when using a series of cryptograms all with different isounits.

2) The capability of computerizing the change of the isounit on a periodical if not continuous basis *without* any need to alter the basic cryptograms themselves, with additional dramatic increase of security; and

3) A significant decrease of costs while increasing security, since Santilli's isotopic lifting can dramatically increase the security of simple cryptograms, while today a lesser security is achieved via extreme complex cryptograms with consequential high costs.

The reader should be aware that Santilli also proposed the much more complex *genocryptograms* and *hypercryptograms* based on ordered units and products which are single-valued and multi-valued, respectively. The consequential increase of complexity as compared to isocryptograms is evident and so are the difficulties for their resolution and the increased security.

In this final chapter we shall mainly outline the foundations of Santilli's isocryptographic theory and provide only a few comments on the more complex geno- and hyper-cryptograms.

2. Secret-Key Isocryptography

There are essentially two different types of isocryptograms: *secret-key isocryptographic system* and *public-key isocryptographic system.*

Before discussing these two types of different isocryptosystems, we present the following notations:

The message space M is a set of strings (plaintext messages) over some alphabet, that needs using a series of isocryptograms all with different isounits. Ciphertext space C: a set of strings (ciphertext messages) over some alphabet, that has been encrypted.

Key space K: a set of strings (keys) over some alphabet, which includes the encryption key $\hat{I}$ and the decryption key $\hat{T}$.

The encryption process (algorithm) $E : \hat{I}M = C$.

The decryption process (algorithm) $D : \hat{T}C = M$.

The algorithms E and D must have the property that

$$\hat{T}C = \hat{T}\hat{I}M = M. \tag{2.1}$$

The sender uses an invertible transformation f defined by

$$f : M \stackrel{\hat{I}}{\longrightarrow} C \tag{2.2}$$

to produce the ciphertext

$$C = \hat{I}M, \tag{2.3}$$

and transmits it over the public insecure channel to the receiver. The key $\hat{I}$ should also be transmitted to the legitimate receiver for decryption but via a secure channel. Since the receiver knows the key $\hat{T}$, he can decrypt C by a transformation f^{-1} defined by

$$f^{-1} : C \xrightarrow{\hat{T}} M \tag{2.4}$$

and obtain $\hat{T}C = \hat{T}\hat{I}M = M$ the original plaintext message.

(1) Character ciphers. Let us define the numerical equivalents of the 26 English letters, since our operations will be on the numerical equivalents of letters, rather than the letters themselves. We have encryption

$$C \equiv \hat{I}m + b(\text{mod } 26) \tag{2.5}$$

with $\hat{I}, b \in Z$ the key, $0 \leq \hat{I}, b, m \leq 26$ and $\gcd(\hat{I}, 26) = 1$

We have decryption

$$m \equiv \hat{T}(C - b)(\text{mod } 26), \tag{2.6}$$

where $\hat{T}$ is the multiplicative inverse of $\hat{I}(\text{mod } 26)$.

(2) Block ciphers. Block ciphers can be made more secure by splitting the plaintext into groups of letters (rather than a single letter) and then performing the encryption and decryption on these groups of letters. This block technique is called block ciphering. Translate the letters into their numerical equivalents and form the ciphertext

$$C_i \equiv \hat{I}M_i(\text{mod } 26), \tag{2.7}$$

where

$$C_i = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}, M_i = \begin{bmatrix} m_1 \\ \vdots \\ m_n \end{bmatrix}$$

and $\hat{I}$ is an invertible $n \times n$ matrix which is called isounit matrix.

For decryption we have

$$M_i \equiv \hat{T}C_i \equiv \hat{T}\hat{I}M_i(\text{mod } 26). \tag{2.8}$$

We have

$$\hat{T}\hat{I}(\text{mod } 26) = E_1, \tag{2.9}$$

where E_1 is a unit matrix.

Remark: Santilli's isomatrix theory. Suppose that

$$A^{\hat{0}} = \hat{I}, \tag{2.10}$$

where A is an invertible $n \times n$ matrix, $\hat{0}$ is an isozero and $\hat{I}$ is an invertible $n \times n$ matrix which is called isounit matrix.

From (2.10) we define the matrix isomultiplication ($\hat{\times} = \hat{T}\times$) and matrix isodivision $\hat{\div} = \hat{I} \div$. We have

$$A\hat{\times}B = A\hat{T}B, A\hat{\div}B = A\hat{I}B^{-1}, \hat{I}^{\hat{n}} = \hat{I}, \hat{I}\hat{T} = E_1,$$

where $A, B, \hat{I}$, and $\hat{T}$ are the invertible $n \times n$ matrices and E_1 is unit matrix.

One important problem is to find the better isounit matrix. Using isounit matrix $\hat{I}$ the Santilli's isomatrix theory may be established. It has a wide application in various fields.

(3) Exponentiation ciphers. The exponentiation cipher may be described as follows. Let p be a prime number, M the numerical equivalent of the plaintext, where each letter of the plaintext is replaced its two digit equivalents. Subdivide M into block M_i such that $0 < M_i < p$. Let $\hat{I}$ be an integer with $0 < \hat{I} < p$ and $\gcd(\hat{I}, p-1) = 1$. Then the encryption transformation for M_i is defined by

$$C_i \equiv M_i^{\hat{I}} (\text{mod } p), \tag{2.11}$$

and the decryption transformation by

$$M_i = C_i^{\hat{T}} \equiv M_i^{\hat{I}\hat{T}} \equiv M_i (\text{mod } p). \tag{2.12}$$

where $\hat{I}\hat{T} \equiv 1(\text{mod } p-1)$.

(4) The encryption process E :

$$\hat{I}_j M = C. \tag{2.13}$$

The decryption process D :

$$\hat{T}_j C = \hat{T}_j \hat{I}_j M = M. \tag{2.14}$$

If $j = 1, 2$, then (2.13) and (2.14) are the Santilli's genocryptographic theory in secret-key isocryptography. If $j = 1, 2, 3, \ldots$, then (2.13) and (2.14) are the Santilli's hypercryptographic theory in secret-key isocryptography.

Public-Key Isocryptography.

Let A and B be finite sets. A one-way function

$$f : A \longrightarrow B \tag{2.15}$$

is an invertible function satisfying (i) f is easy to compute, that is, given $x \in A, y = f(x)$ is easy to compute; (ii) f^{-1}, the inverse function of f, is difficult to compute, that is $y \in B, x = f^{-1}(y)$ is difficult to compute; (iii) f^{-1} is easy to compute when a trapdoor (*i.e.*, a secret string of information associated with the function) become available. A function f satisfying only first two conditions is also a one-way function. If f satisfies further the third condition, it is called a trapdoor one-way function.

(1) The RSA scheme. The idea is to use a product of two primes, $n = pq$ as the modulus. We choose an integer $\hat{I}$ such that $\gcd(\hat{I}, \phi(n)) = 1$, where $\phi(n) = (p-1)(q-1)$. The enciphering process is

$$C \equiv m^{\hat{I}} (\text{mod } n). \tag{2.16}$$

To determine the decryption process, we compute $\hat{T}$ such that

$$\hat{I}\hat{T} \equiv 1(\text{mod}\phi(\text{n})). \tag{2.17}$$

Then the decryption operation is given

$$m = C^{\hat{T}} (\text{mod } n). \tag{2.18}$$

We assume that $\hat{I}$ and n are publicly known but p, q and $\phi(n)$ are not. It is hard to compute $m = C^{\hat{T}}(\text{mod } n)$ without knowing $\hat{T}$. The knowledge of p, q makes it easy to compute $\hat{T}$. For public-key isocryptography, one needs large composite numbers of the form pq, where p, q are in turn large prime numbers. Since integer factorization is a computationally intractable problem.

(2) Three primes system. Let $n = p_1p_2p_3$ and $\phi(n) = (p_1-1)(p_2-1)(p_3-1)$. We have $\gcd(\hat{I}, \phi(n)) = 1$. The enciphering process is

$$C \equiv m^{\hat{I}} (\text{mod } n). \tag{2.19}$$

To determine the decryption process, we compute $\hat{T}$ such that

$$\hat{I}\hat{T} \equiv 1(\text{mod } n). \tag{2.20}$$

The decryption process is given

$$m = C^{\hat{T}} (\text{mod } n). \tag{2.21}$$

We assume that $\hat{I}$ and n are publicly known but p_1, p_2, p_3 and $\phi(n)$ are not. It is hard to compute $m = C^{\hat{T}}(\text{mod } n)$ without knowing $\hat{T}$. The knowledge of p_1, p_2 and p_3 makes it easy to compute $\hat{T}$. For public-key isocryptography,one needs large composite numbers of the form $p_1p_2p_3$, where p_1, p_2 and p_3 are large prime numbers. Since integer factorization is a computationally intractable problem.

(3) Two primes system. Let $n = pq$ and $\phi(n) = (p-1)(q-1)$. We have $\gcd(\hat{I}, \phi(n)) = 1$. The enciphering process is

$$C \equiv m^{\hat{I}_j} (\text{mod } n). \tag{2.22}$$

To determine the decryption process, we compute $\hat{T}_j$ such that

$$\hat{I}_j\hat{T}_j \equiv 1(\text{mod } \phi(n)). \tag{2.23}$$

The decryption process is given

$$m = C^{\hat{T}_j}(\text{mod } n). \tag{2.24}$$

We assume that $\hat{I}_j$ and n are publicly known but p, q and $\phi(n)$ are not. If $j = 1, 2$ then (2.22)-(2.24) are the Santilli's genocryptographic theory in public-key isocryptography. If $j = 1, 2, \cdots$, then (2.22)-(2.24) are the Santilli's hypercryptographic theory in public-key isocryptography.

(4) Large prime problem. In RSA one important point has to do with the choice of the primes p and q. If they are small, then the system is easy to break. We need better techniques to find much large prime numbers.

1) The fundamental prime problem. There exist the infinitely many primes. We have the arithmetic function

$$J_2(\omega) = \prod_{3\le p\le p_i}(p - 1 - \chi(p)), \tag{2.25}$$

where $\chi(p)$ is the number of solutions of congruence

$$(p-1)! \equiv 0(\text{mod } p) \tag{2.26}$$

From (26) we have $\chi(p) = 0$. Substituting it into (2.25) we have

$$J_2(\omega) = \prod_{3\le p\le p_i}(p - 1), \tag{2.27}$$

Since $J_2(\omega) \to \infty$ as $\omega \to \infty$, we prove that there exist the infinitely many prime numbers. Using $J_2(\omega)$ we proved many prime theorems and can find much large prime numbers.

2) Suppose that

$$p_1 = \frac{(p-1)^{p_0} - 1}{p-2}, \tag{2.28}$$

where p_0 is an odd prime.

We have the arithmetic function

$$J_2(\omega) = \prod_{3\le p\le p_i}(p - 1 - \chi(p)) \neq 0, \tag{2.29}$$

where $\chi(p_0) = 1; \chi(p) = p_0 - 1$ if $p \equiv 1(\text{mod } p_0); \chi(p) = 0$ if $p \not\equiv 1(\text{mod } p_0)$.

We have the asymptotic formula of the number of primes p less than N

$$\pi_2(N, 2) \sim \frac{J_2(\omega)\omega}{(p_0 - 1)\phi^2(\omega)}\frac{N}{\log^2 N}. \tag{2.30}$$

Since $J_2(\omega) \neq 0$, there exist the infinitely many primes p such that p_1 is a prime. Therefore from (28) one may find the much large primes. When $p = 3$, numbers $p_1 = 2^{p_0} - 1$ of this form are called Mersenne number, and primes of this form are called Mersenne primes. From (2.30) we have $\pi_2(3,2) \to 0$ as $p_0 \to \infty$. We prove that there exist the finite Mersenne primes. When $p = 11$, numbers $p_1 = \frac{10^{p_0}-1}{9}$ of this form are called repunits, and primes of this form are called prime repunits. From (2.30) we have $\pi_2(11,2) \to 0$ as $p_0 \to \infty$. We prove that there exist the finite prime repunits. We point out that Mersenne primes and prime repunits are useless in finding much large primes.

3) Suppose that

$$p_1 = 2(p-1)^n - 1, \tag{2.31}$$

where n is an integer.

We have the arithmetic function

$$J_2(\omega) = \prod_{3 \le p \le p_i} (p - 1 - \chi(p)) \neq 0, \tag{2.32}$$

where $\chi(p)$ is the number of solutions of congruence

$$2(q-1)^n - 1 \equiv 0 (\text{mod } p), \tag{2.33}$$

where $q = 1, 2, \cdots, p-1$.

We have the asymptotic formula of the number of primes p less than N

$$\pi_2(N,2) \sim \frac{J_2(\omega)\omega}{n\phi^2(\omega)} \frac{N}{\log^2 N}. \tag{34}$$

Since $J_2(\omega) \neq 0$, there exist the infinitely many primes p such that p_1 is a prime. Therefore from (31) one may find the much large primes. When $p = 11$, numbers $p_1 = 2(10)^n - 1$ of this form are called Santilli's numbers, and primes of this form are called Santilli's primes. When $n = 1, 2, 3, p_1 = 19, 199, 1999$ are Santilli's primes. From (2.34) we have $\pi_2(11,2) \to 0$ as $n \to \infty$. We prove that there exist the finite Santilli's primes.

In the same way we prove that primes of these forms: $3 \times 2^n \pm 1, 5 \times 2^n \pm 1, 7 \times 2^n \pm 1, \cdots, 3 \times 10^n \pm 1, 6 \times 10^n \pm 1, 9 \times 10^n \pm 1, 5 \times 10^n - 1, 8 \times 10^n - 1, 4 \times 10^n + 1, 7 \times 10^n + 1$ are finite.

4) A Cunningham chains. A Cunningham chain of length k is a finite set of primes $p_1, p_2, \cdots, p_k$ such that $p_{i+1} = 2p_i + 1$ or $p_{i+1} = 2p_i - 1$ for $i = 1, 2, \cdots, k-1$. It can apply to generate the large primes. For example, given the primes p, q and r, where $p = 2q + 1$ and $q = 2r + 1$. A public key infrastructure can be established. One may find the large primes. We establish the Santilli's isoprime m-chains: $p_{i+1} = mp_i + m - 1$ or $p_{i+1} = mp_i - m + 1$ for $i = 1, 2, \cdots, k-1$. We consider that $p_{i+1} = 16p_i + 15$ or $p_{i+1} = 16p_i - 15$ can generate the large primes. In Chapter 2 we proved many prime theorems which can produce much large primes. This book is the much large primebank in Santilli's isocryptographic theory.

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