SOME CHARACTERISATIONS OF α-CUT IN INTUITIONISTIC FUZZY SET THEORY

*[1] V.VEERAMANI, [2] ROQUE BATULAN

Abstract:
This paper contains the Basic Definitions of an Intuitionstic Fuzzy Set theory and operations on it. Mainly we discussed the basic concepts of α - cut with examples and Characterisations.

2008 AMS subject Classification: 03F55 08A72 20N25 06DF

Key words:
Intuitionistic fuzzy sets, **α** - cut in intuitionistic fuzzy set, support and core of a fuzzy set

INTRODUCTION:

After an introduction of fuzzy sets by L.A. Zadeh several researchers explored on the generalization of the notion of fuzzy set. The concept of intuitionistic fuzzy set was introduced K.T. Atanassov[2] as a generalization of the notion of a fuzzy set.

1.1 Definition:

An Intuitionistic Fuzzy Set (IFS) **A** in X is defined as an object of the form

A = {<x, $\mu_A(x), \gamma_A(x)$ >/x ∈ X} where $\mu_A : X \to [0,1]$ and $\gamma_A : X \to [0,1]$ define the degree of membership and the degree of non-membership of the elements x ∈ X, respectively and for every x∈ X, satisfying $0 \le \mu_A(x) + \gamma_A(x) \le 1$

1.2 Definition:

For an IFS **A**, we define the Support of **A** (denoted by supp(**A**)) to be the set of all those elements of an universal set whose membership and non-membership grades in A are greater than zero. It is simply the set of all those elements whose membership grades are not equal to zero and non-membership grades are greater than or equal to zero.

In other words,

$$\text{Supp}(\mathbf{A}) = \{x \in X / \mu_A(x) > 0, \gamma_A(x) \ge 0, \mu_A(x) + \gamma_A(x) \le 1\}$$

1.3 Example:

Let X = {a, b, c, d} and **A** = {(a, 0.3, 0.4), (b, 0.4, 0), (c, 0.5, 0), (d, 0, 0.2)}

Then supp(**A**) = {a, b, c}

1.4 Definition:

For an IFS, we define the core of **A** (denoted by core(**A**)) to be the set of all those elements of an universal set whose membership grades in A are equal to one and non-membership grades in A are equal to zero.

In other words , Core(**A**)= $\{x \in X / \mu_A(x) = 1 \text{ and } \gamma_A(x) = 0\}$

1.5 Example :

Let X = {a, b, c, d} and **A** = {(a, 0.2, 0.6), (b, 1, 0), (c, 0, 1), (d, 0.5, 0.5)} then

Core (**A**) = {b}

1.6 Definition:

For an IFS **A**, the maximum value attained by $\mu_A(x)$ is referred to as the height of A is denoted by ht(A). If ht($\mu_A(x)$) = 1 and ht($\gamma_A(x)$) = 0 , then the IFS A is said to be Normal. Otherwise **A** is said to be Subnormal.

1.7 Example:

Let X = {a, b, c, d} and

A = {(a, 1, 0), (b, 0.5, 0.3), (c, 0.2, 0.5), (d, 0, 1)} then A is normal

1.8 Example :

Let X = {a, b, c, d} and

A = {(a, 0.8, 0.1), (b, 0.7, 0.1), (c, 0.8, 0), (d, 0.6, 0.3)} then A is subnormal

Since ht(**A**) = 0.8

1.9 Definition:

For an IFS **A**, the α-cut of **A** is defined by

$\alpha_A = \{x \in X / \mu_A(x) \geq \alpha, \gamma_A(x) \geq 0 \text{ such that } \mu_A(x) + \gamma_A(x) \leq 1\}$

2.0 Definition :

For an IFS **A**, the strong α – cut of **A** is defined by

$\alpha^+{}_A = \{x \in X / \mu_A(x) > \alpha, \gamma_A(x) \geq 0 \text{ such that } \mu_A(x) + \gamma_A(x) \leq 1\}$

For any IFS **A**, we have

$0_A = \{x \in X / \mu_A(x) \geq 0, \gamma_A \geq 0\} = X$

$0^+{}_A = \{x \in X / \mu_A(x) > 0, \gamma_A \geq 0\} = \sup(A)$

$1_A = \{x \in X / \mu_A(x) \geq 1, \gamma_A \geq 0\} = core(A)$

$1^+{}_A = \{x \in X / \mu_A(x) > 1, \gamma_A \geq 0\} = \phi$

2.1 Example :

X = {a, b, c, d, e, f, g}

A = {(a, 1, 0), (b, 0, 1), (c, 0.1, 0.5), (d, 0.3, 0.7), (e, 0.5, 0.2), (f, 0.7. 0.1), (g, 0.4, 0)}

0_A={a,b,c,d,e,f,g} and X= {a,b,c,d,e,f,g}

$\therefore 0_A = X$

Now, 0_A^+ = {a,c,d,e,f,g} and sup(A) = {a,c,d,e,f,g}

$\therefore 0^+{}_A$ = sup(**A**)

1_A = {a} and core(**A**) = {a}

$\therefore 1_A$ = core(**A**)

Clearly $1^+{}_A = \phi$

2.2 Definition :

The set of all levels α ∈ [0,1] that represent distinct α-cuts of a given IFS **A** is called a Level set of **A**

In other words $\Lambda(A) = \{\alpha / \mu_A(x) = \alpha, \gamma_A(x) \geq 0$ for some $x \in X\}$

Where Λ denotes the level set of IFS **A** defined on X

2.3 *Theorem:*

For any two IF sets **A** and **B** and for any two numbers α and β in [0,1] following are true:

(i) $\alpha^+{}_A \subseteq \alpha_A$

(ii) $\alpha \leq \beta \Rightarrow \beta_A \subseteq \alpha_A$

(iii) $\alpha_{(A \cup B)} = \alpha_A \cup \alpha_B$

(iv) $\alpha_{(A \cap B)} = \alpha_A \cap \alpha_B$

(v) $\alpha^+{}_{A \cup B} = \alpha_A \cap \alpha_B$

(vi) $\alpha^+{}_{(A \cap B)} = \alpha^+{}_A \cap \alpha^+{}_B$

(vii) $\alpha_{A^C} \neq \left((1-\alpha)^+{}_A\right)^C$

PROOF:

(i) If $x \in \alpha_A^+$, then $\mu_A(x) > \alpha, \gamma_A(x) \geq 0$ which means $\mu_A(x) \geq \alpha, \gamma_A(x) \geq 0$ proving that $x \in \alpha_A^+$

$\therefore \alpha_A^+ \subseteq \alpha_A$

(ii) If $x \in \beta_A$ then $\mu_A(x) \geq \beta$ and $\gamma_A(x) \geq 0$

Clearly $\mu_A(x) \geq \alpha$ since $\alpha \leq \beta$

$x \in \alpha_A^+$

$\therefore \alpha \leq \beta \Rightarrow \beta_A \subseteq \alpha_A$

(iii) Note that $A \cup B$ is the union of two IF sets and hence is an IFS $\alpha_{(A \cup B)}$ denotes the α-cut of this IFS and hence is a crisp set. $\alpha_A \cup \alpha_B$ denotes the union of two crisp sets and hence is a crisp set. This property says that these two crisp sets are equal.

Suppose $x \in \alpha_{A \cup B}$ then $(A \cup B)(x) \geq \alpha$

i.e., $\max\{\mu_A(x), \mu_B(x)\} \geq \alpha$ and $\gamma_A(x) \geq 0$ and $\gamma_B(x) \geq 0$ which means either $\mu_A(x) \geq 0$ and $\gamma_A(x) \geq 0$ or $\mu_B(x) \geq \alpha$ and $\gamma_B(x) \geq 0$

Thus $x \in \alpha_A$ or $x \in \alpha_B$

$\Rightarrow x \in \alpha_A \cup \alpha_B$

$\Rightarrow \alpha_{A \cup B} \subseteq \alpha_A \cup \alpha_B \rightarrow (1)$

Conversely, assume that $x \in \alpha_A \cup \alpha_B$ then $x \in \alpha_A$ or $x \in \alpha_B$

$\Rightarrow \mu_A(x) \geq \alpha$ and $\gamma_A(x) \geq 0$ or $\Rightarrow \mu_B(x) \geq \alpha$ and $\gamma_B(x) \geq 0$

$\Rightarrow \max\langle \mu_A(x), \mu_B(x) \rangle \geq \alpha$ and $\gamma_A(x) \geq 0$, $\gamma_B(x) \geq 0$

$\Rightarrow (A \cup B)(x) \geq \alpha$

$\Rightarrow x \in \alpha_{A \cup B}$

$\therefore \alpha_A \cup \alpha_B \subseteq \alpha_{A \cup B} \rightarrow (2)$

from (1) and (2) we get $\alpha_{A \cup B} = \alpha_A \cup \alpha_B$

(iv) Suppose

$x \in \alpha_{A \cap B}$

$\Rightarrow (A \cap B)(x) \geq \alpha$

ie., $\min\{\mu_A(x), \mu_B(x)\} \geq \alpha$ and $\gamma_A(x) \geq 0$, $\gamma_B(x) \geq 0$

$\Rightarrow \mu_A(x) \geq \alpha$ and $\gamma_A(x) \geq 0$ and $\mu_B(x) \geq \alpha$, $\gamma_B(x) \geq 0$

$\Rightarrow x \in \alpha_A$ and $x \in \alpha_B$

$\Rightarrow x \in \alpha_A \cap \alpha_B$

$\therefore \alpha_{A \cap B} \subseteq \alpha_A \cap \alpha_B \rightarrow (1)$

Conversely assume that $x \in \alpha_A \cap \alpha_B$

$\Rightarrow x \in \alpha_A$ and $x \in \alpha_B$

$\Rightarrow \mu_A(x) \geq \alpha$ and $\gamma_A(x) \geq 0$ and $\mu_B(x) \geq \alpha$ and $\gamma_B(x) \geq 0$

$\Rightarrow \min\{\mu_A(x), \mu_B(x)\} \geq \alpha$ and $\gamma_A(x) \geq 0$ and $\gamma_B(x) \geq 0$

$\Rightarrow (A \cap B)(x) \geq \alpha$

$\Rightarrow x \in \alpha_{A \cap B}$

$\therefore \alpha_A \cap \alpha_B \subseteq \alpha_{A \cap B} \rightarrow (2)$

from (1) and (2) we get $\alpha_{A \cap B} = \alpha_A \cap \alpha_B$

Proof of (v) and (vi) are exactly similar to those of (iii) and (iv). Only difference is instead of $\geq$ inequality here we will have > inequality

Result: In any fuzzy set $\alpha_{A^C} = \left((1-\alpha)^+{}_A\right)^C$ but in IFS $\alpha_{A^C} \neq \left((1-\alpha)^+{}_A\right)^C$

2.4 Example:

X = {a, b, c, d, e}

A = {(a, 0.3,0.2), (b, 0.1, 0.4), (c, 0.2, 0.6), (d, 1, 0), (e, 0.6, 0.5)}

B = {(a, 0.1,0.2), (b, 0.2, 0.5), (c, 0.2, 0.6), (d, 0.4,0.6), (e, 0.3, 0.1)}

α_A^C = {b,c,e} and 1- α = 0.7 (α = 0.3)

$(1-\alpha)_A^+ = \{\Phi\}$

$\left((1-\alpha)_A^+\right)^C = \{a, b, c, d, e\}$

Hence in IFS, $\alpha_{A^C} \neq \left((1-\alpha)^+{}_A\right)^C$

2.5 Theorem:

(i) for any IFS **A** and **B** $A \subseteq B$ if and only if $\alpha_A \subseteq \alpha_B$ for all α ∈ [0,1]

(ii) For any two IFSs **A** and **B** $A \subseteq B$ if and only if $\alpha_A^+ \subseteq \alpha_B^+$ for all α ∈ [0,1]

(iii) For any two IFSs **A** and **B**, **A** = **B** if and only if $\alpha_A = \alpha_B$ for all α ∈ [0,1]

<u>Proof:</u>

Assume that $A \subseteq B$ we will prove that $\alpha_A \subseteq \alpha_B$ for all α ∈ [0,1]

Suppose there is $\beta \in [0,1]$ such that $\beta_A \not\subset \beta_B$

This means there is an x in β_A such that $x \notin \beta_B$

Then $\mu_A(x) \geq \beta > \mu_B(x)$ and $\gamma_A(x) \geq 0$ & $\gamma_B(x) \geq 0$, a contradiction since $A \subseteq B$

Conversely, assuming that $\alpha_A \subseteq \alpha_B$ for all α ∈ [0,1]

We will prove that $A \subseteq B$

For this we have to prove that $\mu_A(x) \le \mu_B(x) and \gamma_A(x) \ge \gamma_B(y)$ for all x

Suppose there is a y such that $f = \mu_A(y) > \mu_B(y)$

Clearly f ∈ [0,1] and clearly $\gamma_A(y) < \gamma_B(y)$

$\Rightarrow f = \mu_A(y) > \mu_B(y)$ and $\gamma_A(y) \ge 0$ and $\gamma_B(y) \ge 0$

$\Rightarrow y \in \delta_A$ but $y \notin \delta_B$

$\Rightarrow \delta_A \not\subset \delta_B$

A contradiction to the fact that $\alpha_A \subseteq \alpha_B$ for all α ∈ [0,1]

$\therefore A \subseteq B$

(ii) Assuming that $A \subseteq B$

We will prove that $\alpha_A^+ \subseteq \alpha_B^+$ for all α ∈ [0,1]

Suppose there is a $\beta \in [0,1]$ such that $\beta_A^+ \not\subset \beta_B^+$

This means there is an x in β_A^+ such that $x \notin \beta_B^+$

Then $\mu_A(c) > \beta \ge \mu_B(x)$ and $\gamma_A(x) \ge 0$ and $\gamma_B(x) \ge 0$

A contradiction since $A \subseteq B$

$\therefore \alpha_A^+ \subseteq \alpha_B^+$

Conversely assuming that $\therefore \alpha_A^+ \subseteq \alpha_B^+$ for all α ∈ [0,1]

We will prove that $A \subseteq B$

For this, we have to prove that $\mu_A(x) \le \mu_B(x)$ and $\gamma_A(x) \ge \gamma_B(x)$ for all x

Suppose there is a y such that $\mu_A(y) > \mu_B(y)$ and $\gamma_A(y) < \gamma_B(y)$

Take δ such that $\mu_A(y) > \delta > \mu_B(y)$ clearly $\delta \in [0,1]$

$\gamma_A(y) \ge 0$ and $\gamma_B(y) \ge 0$

$\Rightarrow y \in \delta_A^+$ but $y \notin \delta_B^+$ so that $\delta_A^+ \not\subset \delta_B^+$

A contradiction to the fact that $\alpha_A^+ \subseteq \alpha_B^+ \ \forall \alpha \in [0,1]$

$\therefore A \subseteq B$

(iii) Proof is trivial.

REFERENCES:

[1] ATANASSOV.K.T., Intuitionistic fuzzy sets, fuzzy sets and systems, 20(1) (1986) 87-96.

[2]. Fuzzy set and Fuzzy logic by George J. Klir/ Bo Yuan

[3]. Nero Fuzzy and Soft computing by J.S.R. Jang, C.T. Sun, E. Mizutani

1 **Lecturer in Mathematics, Shinas College of Technology, PO Box 77 PC 324, Shinas, Sultanate of Oman, Email: jayaveer73@gmail.com**

2 **Head of Section in Mathematics, Shinas College of Technology, PO Box 77, PC 324, Shinas, Sultanate of Oman**
Email: rox_batulan@yahoo.com