the general science journal

PRODUCT OF DISTRIBUTIONS AND ZETA REGULARIZATION OF DIVERGENT INTEGRALS $\int_0^\infty x^{m-s}dx$ AND FOURIER TRANSFORMS

Jose Javier Garcia Moreta
Graduate student of Physics at the UPV/EHU (University of Basque country)
In Solid State Physics
Address: Address: Practicantes Adan y Grijalba 2 5 G
P.O 644 48920 Portugalete Vizcaya (Spain)
Phone: (00) 34 685 77 16 53
E-mail: josegarc2002@yahoo.es

- ABSTRACT: Using the theory of distributions and Zeta regularization we manage to give a definition of product for Dirac delta distributions, we show how the fact of one can be define a coherent and finite product of dDirac delta distributions is related to the regularization of divergent integrals $\int_0^\infty x^{m-s}dx$ and Fourier series, for a Fourier series making a Taylor substraction we can define a regular part $F_{reg}(u)$ defined as a function for every 'u' plus a dirac delta series $\sum_{i=0}^{N} c_i \delta^{(i)}(u)$, which is divergent for u=0 , we show then how $\delta^{(i)}(0)$ can be regularized using a combination of Euler-Mclaurin formula and analytic continuation for the series $\sum_{i=0}^{\infty} i^k = \zeta(-k)$

PRODUCT OF DIRAC DELTA DISTRIBUTIONS $\delta^{(m)}(x) \times \delta^{(n)}(x)$

One of the problems with distributions , as proved by Schartz (see ref [1]) is that we can not (in general) define a coherent product of distributions, for example

$$(\delta \times x) \times P\left(\frac{1}{x}\right) = 0 \qquad \delta \times \left(\delta \times P\left(\frac{1}{x}\right)\right) = 0 \qquad P\left(\frac{1}{x}\right)[\phi] = \lim_{\varepsilon \to 0} \int_{|x| \ge \varepsilon} \frac{\phi(x)}{x} dx \quad (1)$$

For the case of the product of a Heaviside step function H(x) with the derivatives of the Delta function (and its derivatives) we have to deal with the problem of divergent quantities, for example according to [2] we can define the product $H \times \delta^{(m)}$, with the aid of a test function $\phi(x) \in C^{\infty}(R)$ as the recurrence

$$\int_{-\infty}^{\infty} dx H(x)\delta^{(m)}(x)\phi(x) = \int_{0}^{\infty} dx \delta^{(m)}(x)\phi(x) = -\delta^{(m-1)}(0)\phi(0) - \int_{0}^{\infty} dx \delta^{(m-1)}(x)\phi'(x) \quad (2)$$

The case m=0 is just $H \times \delta = \frac{1}{2}\delta$, and comes from considering the Heaviside fucntion H(x) to be the derivative of $\delta(x)$, so $\int_{-\infty}^{\infty} dx H(x)\delta(x) = \frac{1}{2}\left(H^2(\infty) - H^2(-\infty)\right) = \frac{1}{2}$

If we use the ‘Convolution theorem’ [5] in a formal sense, so it can be regarded as valid even for the case that the Fourier transform are defined ONLY as distributions

$$(2\pi)^2 i^{m+n} D^m \delta(\omega) D^n \delta(\omega) = F_{\omega}\left(x^m * x^n\right) = AF_{\omega}\left\{\int_{-\infty}^{\infty} dt t^m (x-t)^n\right\} \quad (3)$$

Here ‘A’ is a normalization (finite) constant that depends on the definition you take for the Fourier transform, but it can not be dependent on m or n and $D = \frac{d}{dx}$. Unofrtunately (3) makes no sense since the integral over ‘t’ is DIVERGENT and needs to be regularized, if we use the Binomial theorem on $t^m (x-t)^n$ for m and n integres

$$i^{m+n} D^m \delta(\omega) D^n \delta(\omega) = \sum_{k=0}^{n} \binom{n}{k} i^{m+k} A D^{n-k} \delta(\omega)(-1)^k i^{n-k} D^{m+k} \delta(0) \quad (4)$$

The problem here is that $D^{m+k}\delta(0) = \frac{i^{m+k}}{2\pi} \int_{-\infty}^{\infty} x^{m+k} dx$ is infinite and would need to be regularizad in order to make sense inside (3) or (4) , for m+k being an Odd integer, using Cauchy’s principal value definition $P.v\left(\int_{-\infty}^{\infty} x^{2n+1} dx\right) = 0$ (this imposes the condition that only +1 or -1 can appear inside (4)) , the problem is that $2\int_{0}^{\infty} x^{2n} dx$ is still divergent , the same problem happened inside (2) where one needs to to regularize expressions $\delta^{(m-1)}(0)$ in order to define a coherente product of distributions involving Heaviside step-function and Dirac delta and its derivatives. In general (4) will be non-commutative so we can in general expect $\delta^{(m)}(u) \times \delta^{(n)}(u) \neq \delta^{(n)}(u) \times \delta^{(m)}(u)$ example

$$\delta(u) \times \delta^{(1)}(u) = \delta_{reg}(0)\delta^{(1)}(u) \quad \text{but} \quad \delta^{(1)}(u) \times \delta(u) = \delta(u)\delta_{reg}{}^{(1)}(0) = 0 \quad (5)$$

The last equality in (5) comes from the fact that $\delta'(0) = (2\pi)^{-1}\int_{-\infty}^{\infty} x dx$ is 0 by using Cauchy's principal value , the case m=n=0 is just the square of delta function $\delta \times \delta = \delta^2 = -\frac{A}{2\pi}\delta$, this can be obtained from the zeta regularization $\int_{-\infty}^{\infty} dx = 2\int_0^{\infty} dx = \left(2\sum_{n=0}^{\infty} 1\right)_{reg} = -1$ as we will see in the next section

- *Zeta regularization for divergent integrals:*

In our previosu paper [4] we used the Euler-Maclaurin summation formula with $f(x) = x^{m-s}$ in order to stablish

$$\int_a^{\infty} x^{m-s} dx = \frac{m-s}{2}\int_a^{\infty} x^{m-1-s} dx + \zeta(s-m) + a^{m-s} - \sum_{k=1}^{a} k^{m-s}$$
$$-\sum_{r=1}^{\infty} \frac{B_{2r}\Gamma(m-s+1)}{(2r)!\Gamma(m-2r+2-s)}(m-2r+1-s)\int_a^{\infty} x^{m-2r-s} dx \qquad a>0 \quad (6)$$

The idea is , given a fixed 'm' we define an s sufficiently large so the integral $\int_a^{\infty} x^{m-s} dx$ and the series $\zeta(s-m) = \sum_{i=0}^{\infty} i^{m-s}$ converge , and then use the analytic continuation to extend the definition of the sum as the negative value of the Riemann Zeta $\zeta(-m) = \sum_{i=0}^{\infty} i^m$, in order to regularize , using (5) the divergent integrals, if 'm' is an integer we can set a=0 and (5) becomes an easier expression

$$\int_0^{\infty} x^m dx = \frac{m}{2}\int_0^{\infty} x^{m-1} dx + \zeta(-m) - \sum_{r=1}^{\infty} \frac{B_{2r}\Gamma(m+1)}{(2r)!\Gamma(m-2r+2)}(m-2r+1)\int_0^{\infty} x^{m-2r} dx \qquad (7)$$

The case m-s=-1 inside (6) can not be regularized inmediatly due to the pole $\zeta(1) = \sum_{i=0}^{\infty} i^{-1} = \infty$, hence to regularize $\int_0^{\infty} \frac{dx}{x+a}$ we integrate with respect to 'a' to find $C_a + \int_0^{\infty} dx \log(x+a)$, using Euler-Maclaurin summation formula plus the regularization of Hurwitz Zeta function $\sum_{n=0}^{\infty} \log(x+a) = -\partial_s \zeta(0,a)$ and taking the derivative respect to 'a'

$$\int_0^{\infty} \frac{dx}{x+a} = \frac{1}{2a} - \frac{\partial^2 \zeta(0,a)}{\partial s \partial a} + \sum_{r=1}^{\infty} \frac{B_{2r}}{(2r)!}\frac{d^{2r-1}}{dx^{2r-1}}\left(\frac{1}{x+a}\right)_{x=0} \qquad (8)$$

The first three terms of the recurrence (7) are

$$-1/2=\int_0^\infty dx \qquad \frac{1}{2}\zeta(0)+\zeta(-1)=\int_0^\infty xdx \qquad \left(\frac{1}{2}\int_0^\infty dx+\zeta(-1)\right)-\frac{B_2}{2}a_{21}\int_0^\infty dx=\int_0^\infty x^2dx \quad (9)$$

With $a_{mr}=\frac{\Gamma(m+1)}{\Gamma(m-2r+2)}$ and $\frac{x}{e^x-1}=\sum_{n=0}^{\infty}\frac{B_n x^n}{n!}$ being the Bernoulli numbers $B_{2n+1}=0$, from the definition of our product of Dirac delta distributions given in (4) and since we want the identity $H\times\delta=\frac{1}{2}\delta$ to be true for every test function , we can identify $\pi\delta_{reg}(0)=\zeta(0)$, $\delta_{reg}{}'(0)=0$ and $-2\pi\delta_{reg}{}''(0)=\left(\zeta(0)+2\zeta(-1)\right)-B_2a_{21}\zeta\left(0\right)$ from the point of view of Zeta regularization. Although we have used only a definition for distributions on R , it can be generalized to R^n by using the definition of Dirac delta function and Heaviside function in several variables $\prod_{j=1}^{n}\delta(x_j) \quad \prod_{j=1}^{n}H(x_j)$, in any case we have chosen the regularization $2\pi i^m\delta^{(m)}(0)=\int_{-\infty}^{\infty}x^m dx$ for 'm' integer odd or even, other definition for the Fourier transform can make a factor different to 2π appear in (4) for example $\int_{-\infty}^{\infty}dxe^{2\pi iux}=\delta(u)$

REGULARIZATION OF FOURIER INTEGRAL USING DISTRIBUTIONS

Let be R^n , then we can regularize the Fourier transform $\int_{R^n}d^nke^{i\vec{u}.\vec{k}}f(k)=F(u)$ via a taylor series substraction with the definition $\left\langle e^{i\vec{u}.\vec{k}}\mid f(k)\right\rangle=\int_{R^n}d^nke^{i\vec{u}.\vec{k}}f(k)$ (see [6])

$$\left\langle e^{i\vec{u}.\vec{k}}\mid f(k)-\sum_{|\alpha|<N}\frac{k^\alpha}{\alpha!}\left(\partial_\alpha f\right)(0)\right\rangle+\left\langle e^{i\vec{u}.\vec{k}}\mid \sum_{|\alpha|<N}\frac{k^\alpha}{\alpha!}\left(\partial_\alpha f\right)(0)\right\rangle \quad (10)$$

$|\alpha|=\alpha_1+\alpha_2+....+\alpha_n \qquad \alpha!=\alpha_1!.\alpha_2!....\alpha_n! \qquad \partial^\alpha=\partial^{\alpha_1}.\partial^{\alpha_2}......\partial^{\alpha_n}$ is the multi-index notation to write down the definition of Taylor series (9)

The Taylor series is finite and is truncated after a given N so $\int_{R^n}d^nkf(k)\approx\Lambda^{N+1}$ (ultraviolet divergence cut-off) , this allows us to write down a regular part of the Fourier transform plus a distributional part for the Fourier transform

$$F_{reg}(u)=\int_{R^n}d^nke^{i\vec{u}.\vec{k}}\left(f(k)-\sum_{|\alpha|<N}\frac{k^\alpha}{\alpha!}\left(\partial_\alpha f\right)(0)\right) \qquad \sum_{|\alpha|<N}\frac{C_\alpha}{\alpha!}\left(\partial_\alpha f\right)(0)\left(-i\frac{\partial^\alpha}{\partial x^\alpha}\right)\delta(u) \quad (11)$$

(regularized part = function) (singular part = distribution)

The problem with (10) comes whenever the integral is divergent and we set u=0 , in this case we should have to evaluate $\delta^{(m)}(0)$ and other divergent quantities, also since two distributions can not in general be multiplied then $F(u)\times G(u)$ can NOT be defined, only the 'regular' parts of both F and G $F_{reg}(u)\times G_{reg}(u)$ or $F_{reg}(u)\times \delta^{(m)}(u)$, $G_{reg}(u)\times \delta^{(m)}(u)$ can be defined, here we find the problem of giving a regularized definition to $\delta^{(n)}(u)\times \delta^{(m)}(u)$ for integers (m,n) , this was discussed in (4) (5) (6) and (7) and (8) formulae including on how to deal with with the infinite terms $\delta^{(m)}(0)$ via Zeta-regularization , an small problem we find here is that depending on the definition of the Dirac delta function via Fourier transform an extra term proportional to 2π or similar could appear, this happens because usually the definition of the Fourier transform is not universal (up to a factor proprtional to 2π or square root of 2π) . So in general depending on the definition for the Fourier transform we should make the replacement $u \to 2\pi u$ to get the correct results.

PRODUCT OF DISTRIBUTIONS $\delta(u)\times P\left(\frac{1}{u}\right)$, $P\left(\frac{1}{u}\right)\times P\left(\frac{1}{u}\right), \delta'(u)\times P\left(\frac{1}{u}\right)$

Applying the convolution plus the zeta regularization algorithm and the Fourier transform for the Heaviside function $\int_{-\infty}^{\infty} dx H(x) e^{-iux} = \pi\delta(u) + iP\left(\frac{1}{u}\right)$ we can extend our definition of (regularized) product of distribution to include the Principal value distribution $P\left(\frac{1}{u}\right)$ related to Cauchy's principal value of the integral $P\left(\frac{1}{u}\right)[\phi] = pv\int_{-\infty}^{\infty} \frac{\phi(x)}{x}$, using again the Fourier transform convolution theorem

- Product of $\delta(u)\times P\left(\frac{1}{u}\right)$: in this case using the convolution definition

$$\delta(u)\times\left(\pi\delta(u) + iP\left(\frac{1}{u}\right)\right) = AF\left\{\int_{-\infty}^{\infty} dt H(x-t)\right\} = iA\delta'(u) + \zeta(0)A\delta(u) \quad (12)$$

- $P\left(\frac{1}{u}\right)\times\delta(u)$: using again the Fourier transform for H(x)

$$\left(\pi\delta(u) + iP\left(\frac{1}{u}\right)\right)\times\delta(u) = AF\left\{\int_{-\infty}^{\infty} dt H(t)\right\} = \frac{\zeta(0)A}{2\pi}\delta(u) \qquad (13)$$

- $P\left(\frac{1}{u}\right)\times P\left(\frac{1}{u}\right)$: this case is a far bit more complicated to obtain this product we need the identity $\int_{-\infty}^{\infty} dtH(t)H(x-t) = H(0)xH(x)$ H(0)= 1/2

$$\left(\pi\delta(u)+iP\left(\frac{1}{u}\right)\right)\times\left(\pi\delta(u)+iP\left(\frac{1}{u}\right)\right) = AH(0)i\pi\delta'(u)+AH(0)P\left(\frac{1}{u^2}\right) \quad (14)$$

- $i\delta'(u)\times\left(\pi\delta(u)+iP\left(\frac{1}{u}\right)\right)$ and $\left(\pi\delta(u)+iP\left(\frac{1}{u}\right)\right)\times i\delta'(u)$, again using the appropiate form of the convolution theorem

$$i\delta'(u)\times\left(\pi\delta(u)+iP\left(\frac{1}{u}\right)\right) = -AI_1\delta(u)-\frac{\delta''(u)}{2}A \quad (15)$$

$$\left(\pi\delta(u)+iP\left(\frac{1}{u}\right)\right)\times i\delta'(u) = iA\zeta(0)\delta'(u)-I_1\delta(u)A \quad (16)$$

- $\delta'(u)\times P\left(\frac{1}{u^2}\right)$ and $P\left(\frac{1}{u^2}\right)\times\delta'(u)$: using $\int_{-\infty}^{\infty} dxe^{-iux}H(x)x = \pi i\delta'(u)+P(u^{-2})$ and the convolution theorem we can write down

$$\left(\pi\delta'(u)+iP\left(\frac{1}{u^2}\right)\right)\times i\delta'(u) = i\delta'(u)\zeta(0)A-I_1\delta(u) \quad (17)$$

$$i\delta'(u)\times\left(i\pi\delta'(u)+P\left(\frac{1}{u^2}\right)\right) = -\frac{1}{2}\delta''(u)A-AI_1\delta(u) \quad (18)$$

- $P\left(\frac{1}{u^2}\right)\times P\left(\frac{1}{u^2}\right)$: using (14) (17) (18) and the product

$$\left(\pi i\delta'(u)+P\left(\frac{1}{u^2}\right)\right)\times\left(\pi i\delta'(u)+P\left(\frac{1}{u^2}\right)\right) = AF\left\{\int_{-\infty}^{\infty} dtH(x-t)H(t)(x-t)t\right\} \quad (19)$$

The last expression in (19) is just $\frac{-i\pi A\delta'''(u)}{6}+AP\left(\frac{1}{u^4}\right)$,again we have used the identity $\int_{-\infty}^{\infty} dtH(t)H(x-t) = H(0)xH(x)$ together with (4) and (5) in order to give a finite meaning for the product $P\left(\frac{1}{u^2}\right)\times P\left(\frac{1}{u^2}\right)$, note that in expressions (12-18) we need to evaluate products of the form $\delta^{(m)}(u)\times\delta^{(n)}(u)$ which need to be regularized by (4)

Depending on the order in which convolution is taken we may find H(x-t) or H(t) (x-t) or simply 't' inside (12-18) , here as always A is a number introduced by the definition taken for the convolution and $\frac{\zeta(0)}{2}+\zeta(-1)=\left(\int_0^\infty xdx\right)_{reg}=I_1$, $\zeta(0)=\left(\int_0^\infty dx\right)_{reg}$ are finite corrections (regularizations) for the divergent integrals that appear when we try to define a correct product of distributions , from these formulae above together with the Lebiniz formula (considered to be valid at least in a formal sense) $\frac{d\left(A\times B\right)}{du}=\frac{dA}{du}\times B+A\times\frac{dB}{du}$, we can define also $\delta\left(u\right)\times P\left(\frac{1}{u^2}\right)$ or similar products

- $\delta^{(m)}(u)\times P\left(\frac{1}{u}\right)$ and $P\left(\frac{1}{u}\right)\times\delta^{(m)}(u)$ for arbitrary 'm' , $H(u)\times P\left(\frac{1}{u}\right)$

$$i^m\delta^{(m)}(u)\times P\left(\frac{1}{u}\right)=AF\left\{\int_{-\infty}^{\infty}dtH(x-t)t^m\right\}=\delta(u)(-1)^m I_m+\frac{\delta^{(m+1)}(u)i^{m+1}}{m+1} \quad (20)$$

$$P\left(\frac{1}{u}\right)\times i^m\delta^{(m)}(u)=AF\left\{\int_{-\infty}^{\infty}dt(x-t)^m H(t)\right\}=A\sum_{k=0}^{m}\binom{m}{k}(-1)^{m-k}I_{m-k}i^k\delta^{(k)}(u) \quad (21)$$

Here $I_m=\left(\int_0^\infty t^m dt\right)_{reg}$ these integrals can be regularized via formula (6) or (7) However if we put m=-1 in order to evaluate $H\left(u\right)\times P\left(\frac{1}{u}\right)$ inside (20) and (21) we will find several oddities that prevent us from defining a coherent expression , however the derivative of this product of distribution satisfy

$$\frac{d}{dx}\left(H\times P\left(\frac{1}{u}\right)+P\left(\frac{1}{u}\right)\times H\right)=\delta\times P\left(\frac{1}{u}\right)-H\times P\left(\frac{1}{u^2}\right)+P\left(\frac{1}{u}\right)\times\delta-P\left(\frac{1}{u^2}\right)\times H \quad (22)$$

Another possibility is to define $H\left(u-a\right)\times P\left(\frac{1}{u}\right)=T_a(u)$ so its derivative $-\delta\left(u-a\right)\times P\left(\frac{1}{u}\right)=\frac{dT_a(u)}{dx}$, using the Taylor distributional series given in [2] $\sum_{n=0}^{\infty}\frac{(-a)^n}{n!}\delta^{(n)}(u)=\delta(u-a)$, using formulae (21) and (22) and integration with respect to 'a' we can get $H\left(u-a\right)\times P\left(\frac{1}{u}\right)=T_a(u)$ up to some constant C_a. Also if we knew how to multiply $H\left(u-a\right)\times P\left(\frac{1}{u}\right)=T_a(u)$ for some a >0 , (to avoid the singular point u=0) , using the Taylor distributional series $\sum_{n=1}^{\infty}\frac{(-a)^n}{n!}\delta^{(n-1)}(u)+H(u)=H(u-a)$ and then using (21) (22) . Although we have only considered the 1-D case, the Convolution theorem, Binomial theorem and similar can be defined also in R^n , also we must take into account that in general for divergent integrals a change of variable could not work $\int_{-\infty}^{\infty} x^m dx \int_{-\infty}^{\infty} y^n dy \ne \int_{0}^{\infty} dr \int_{0}^{2\pi} r^{m+n+1} d\alpha \sin^n \alpha \cos^m \alpha$, the best method would be to use a Feynmann parametrization to define the product of n integrals

$$\frac{1}{A_1 A_2 ....A_n} = (n-1)! \int_0^1 du_1 \int_0^1 du_2 .... \int_0^1 du_n \frac{\delta\left(u_1 + ...u_n - 1\right)}{\left(u_1 A_1 + ...u_n A_n\right)} \qquad (23)$$

With $A_n(\mu) = \int_0^{\infty} x^{\mu} dx$ being a divergent integral that can be regularized ($\mu \ne -1$) via Zeta-regularization

CONCLUSIONS AND FINAL REMARKS

Using the zeta regularization algorithm (6) (7) we have managed to give a finite (Non-commutative) product of dirac delta distributions $\delta^{(m)}(u) \times \delta^{(n)}(u)$, and $\delta^{(m)}(u) \times H(u)$, with 'H' being the Heaviside step-function , since the product is non-commutative we should also take care when taking the products $(\delta^{(m)} \times \delta^{(n)}) \times \delta^{(k)} \ne \delta^{(m)} \times \left(\delta^{(n)} \times \delta^{(k)}\right)$ so associativity will not always hold , using the Convolution theorem plus the use of Fourier transform, with the m-th and n-th powers of 'x' $F\left(x^m * x^n\right) = AF\left(\int_{-\infty}^{\infty} dt (x-t)^n t^m\right)$ A = normalization constant , will allow us to compute the product $\delta^{(m)} \times \delta^{(n)}$ up to several divergent quantities $\delta^{(m)}(0)$, which are proportional to the divergent integral $\int_{-\infty}^{\infty} x^m dx$, this integral can be regularized [4] using the zeta regularization algorithm in order to 'substract' finite quantities proportionals to $\zeta(-m)$ m=0,1,2,3,........ . Although we have only examinated the case of dirac delta and its derivatives , in several cases it could appear the distribution

$$\int_{-\infty}^{\infty} \frac{e^{iux}}{x} dx = \pi \sin g(u) = \pi \frac{d\,|u|}{du} = \pi H(u) - \pi H(-u) \qquad (24)$$

Although we have not mentioned the case $\int_0^{\infty} dx f(x) e^{iux}$, this integral can be reduced to the calculation of a Fourier integral by setting

$$f(x)H(x) + f(-x)H(-x) = f_+(x) + f_-(x) = g(x) \quad \frac{1}{2}\int_{-\infty}^{\infty} dx g(x) e^{iux} = \int_0^{\infty} dx f(x) e^{iux} \quad (25)$$

In this case we will encounter divergent terms $\delta^{(m)}(0)$, when using the Leibniz's formula to perform the Taylor substraction near x=0 $\frac{d^n}{dx^n}(f.H)=\sum_{k=0}^{n}\binom{n}{k}\frac{d^k f}{dx^k}.\frac{d^{n-k}H}{dx^{n-k}}$ since the derivative of an step funciton involves a dirac delta , again we will need formula (5) (6) and (7) to get some finite results.

If the integral of f(x) has some logarithmic divergence so $\int_0^{\Lambda} dxf(x)\approx \log\Lambda$, then we may have to regularize the distribution $H(x)x^{-1}$ as

$$\left\langle P.f\left(\frac{H}{x}\right)|\phi\right\rangle = -\phi(0)\log\varepsilon+\int_{\varepsilon}^{1}\frac{\phi(x)-\phi(0)}{x}dx+\int_{1}^{\infty}\frac{\phi(x)}{x}dx \qquad (26)$$

And then ignoring all the divergent terms proportional to $\log\varepsilon$ (via counterterms) inside (12) so only finite contributions will appear inside $\int_0^{\infty} dxf(x)$

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