Survey on Singularities and Differential Algebras of Generalized Functions : A Basic Dichotomic Sheaf Theoretic Singularity Test

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Dedicated to Marie-Louise Nykamp

Abstract

It is shown how the infinity of differential algebras of generalized functions is naturally subjected to a basic dichotomic singularity test regarding their significantly different abilities to deal with large classes of singularities. In this respect, a review is presented of the way singularities are dealt with in four of the infinitely many types of differential algebras of generalized functions. These four algebras, in the order they were introduced in the literature are : the nowhere dense, Colombeau, space-time foam, and local ones. And so far, the first three of them turned out to be the ones most frequently used in a variety of applications. The issue of singularities is naturally not a simple one. Consequently, there are different points of view, as well as occasional misunderstandings. In order to set aside, and preferably, avoid such misunderstandings, two fundamentally important issues related to singularities are pursued. Namely, 1) how *large* are the sets of singularity points of various generalized functions, and 2) how are such generalized functions allowed to *behave* in the neighbourhood of their point of singularity. Following such a *two fold* clarification on singularities, it is further pointed out that, once one represents generalized functions - thus as well a large class of usual singular functions - as elements of suitable differential algebras of generalized functions, one of the main advantages is the resulting *freedom* to perform *globally* arbitrary algebraic and differential operations on such functions, simply as if they did not have any singularities at all. With the same freedom from singularities, one can perform globally operations such as limits, series, and so on, which involve infinitely many generalized functions. The property of a space of generalized functions of being a *flabby sheaf* proves to be essential in being able to deal with large classes of singularities. The first and third type of the mentioned differential algebras of generalized functions are flabby sheaves, while the second type fails to be so. The fourth type has not yet been studied in this regard.

Contents

0. Brief Overview

The following is obviously not quite a briefest review ...
And yet, given the inherent complexities of the subject, a genuine attempt was made to avoid mentioning but those issues which could indeed be necessary for a better understanding of what may go by the name of the *algebraic nonlinear theory of generalized functions*, as represented by a large variety of *differential algebras of generalized functions*, see 46F30 in the AMS Mathematical Subject Classification.

0.1. Five Questions on Singularities

Five questions will direct the presentation that follows in this paper :

First Question : *Why* are singularities so important in the context of generalized functions ?

Simply because the essential aim in introducing generalized functions, be they within a merely linear theory, or in fact within a nonlinear one, is precisely to deal with singularities of what are considered to be usual functions. And therefore, the larger the class of singularities that can be dealt with by generalized functions, the better ...

Second Question : *How* do various spaces of generalized functions deal with singularities ?

One of the typical ways is by the *regularization of the singularities* of usual functions, a process which leads to generalized functions upon which algebraic and differential operations can be performed *globally* on the whole of their domain of definition, that is, in a *singularity free* manner ...
Schematically, we have therefore

$$(0.1) \qquad \begin{array}{l} f \;\; \text{function with} \\ \quad\;\; \text{singularities} \\ \quad\;\; \text{on } \Gamma \subset \Omega \end{array} \xmapsto{\substack{\textit{singularity} \\ \textit{regularization}}} F \text{ generalized function on } \Omega$$

where $\Omega \subseteq \mathbb{R}^n$ is any domain of definition given by an open subset, with a suitable subset $\Gamma \subset \Omega$ of singularity points of the usual function f.
Here we note that the usual function is defined on $\Omega \setminus \Gamma$, or even at some of its singularity points in Γ.

Remark 0.1.

What is meant by a singularity of a usual function $f : \Omega \longrightarrow \mathbb{R}$ is a customary concept in analysis. Here we shall mostly mean a point $x \in \Omega$ at which f is not defined, or it is not $\mathcal{C}^m$-smooth, where in various instances, we may have $0 \leq m \leq \infty$.

As for the generalized functions F in (0.1), by singularities we shall mean the singularities of the usual functions f from which they were obtained by regularization of the singularities of the latter, or in general, points $x \in \Omega$ at which F is not $\mathcal{C}^m$-smooth, where again, in various instances we may have $0 \leq m \leq \infty$.

$\square$

Third Question : *which* is the extent various spaces of generalized functions can deal with singularities ?

The answer will be given by a *basic dichotomic singularity test* which, in short, is as follows :

- spaces of generalized functions which are flabby sheaves can deal with very large classes of singularities,
- spaces of generalized functions which are not flabby sheaves can only deal with restricted classes of singularities.

The further **Two Questions** relate to the possible *nature* of singularities of usual functions f which the generalized functions F can deal with through the singularity regularizations that led to their construction, namely :

- what is the allowed **SIZE** of the sets $\Gamma \subset \Omega$ of singularity points of usual and generalized functions, as well as

- what is the allowed **BEHAVIOUR** of the usual and generalized functions in the neighbourhood of their sets $\Gamma \subset \Omega$ of singularity points.

Here let us point out once again the meaning of the expression that the generalized functions F can *deal* with such singularities : one can perform upon these generalized functions all the algebraic and differential operations on the whole of Ω, that is, *globally*, and do so *singularity free*, that is, as if they did not have any singularities at all.
For the sake of clarity, this fact will be illustrated in sections 4 and 6, in the simple and well known example of the Heaviside function, an example which is typical for the above, be it in the case of the linear, or nonlinear theories of generalized functions.

0.2. Complexity of the Algebraic Nonlinear Theory

And now, let us recall several important facts related to the study of various generalized functions and their singularities in the nonlinear context which is based on the algebraic approach.

In this regard it may be relevant to recall the introduction in the early 1990s of the order completion method in solving large classes of nonlinear PDEs, [8,46,58,60,67,69,72,81,101,102,104,158-164], which will be mentioned in the sequel.

Earlier, in the 1960s, an algebraic method - reviewed here - was introduced in the nonlinear theory of generalized functions.

Nowadays, with hindsight, one may say that, when compared with the order completion method, the algebraic nonlinear theory, which is the subject of this review, may turn out to be deeply permeated with several important conflicts or incompatibilities between it various aims pursued. And as a consequence, it may exhibit a pronounced complexity.
Certainly, such may appear to be the case, when compared with the significantly more powerful order completion method which, not surprisingly in view of the relative simplicity of the concept of order, is considerably simpler, being free from all kind of complicated internal

conflicts or incompatibilities.

The algebraic nonlinear theory of generalized functions is indeed a rather complex edifice as it brings together a variety of often conflicting or incompatible ideas, concerns, aims and facts. Among them are :

- the basic, simple and purely algebraic conflict between
  - singularities
  - multiplication
  - differentiation
- inevitable interplay between stability, generality and exactness .

Furthermore, it also makes use of mathematical disciplines beyond the obviously involved analysis. Such disciplines are, for instance :

- algebra with its ring theory,
- topology with the theory of rings of continuous functions, Stone-Cech compactification, and also pseudo-topologies,
- familiarity with mathematical logic, and in particular, with reduced power and ultrapower algebras,
- sheaf theory, with an accent on flabby sheaves, and of course,
- various PDE studies, mainly nonlinear, and possibly, of the general, type independent kind.

It is in this way that the nonlinear theory of generalized functions can benefit a lot from an approach which brings not only focus on various details, an approach customary among analysts, but one which is also able to see the big picture in which more basic ideas have to be pursued with their interactions and within important inevitable constraints that may be quite new in the usual mathematical experience of the so called "working mathematicians"...

Consequently, a more appropriate overview of the subject can hardly be brief, even if one may try quite hard one's best to keep to such an

attractive promise ...
Here, indeed, we have but one of the examples of how the nonlinear theory of generalized functions is bound to navigate in realms with strong conflicting, or rather incompatible tendencies ...

What has so far been a guiding, and in fact inspiring, as well as moderating influence on that theory is the fact that, mostly, it has been created and developed in view of solving large classes of PDEs.
And as is known, ever since Newton's Calculus, PDEs have been the most general and rigorous mathematical models for the basic laws of Physics.
It is worth noting in this regard that during the last half a century or so, with the exception of the Schrödinger equation in Quantum Mechanics, and hardly at all other equations, by far most of the PDEs of interest have been nonlinear.

The origin of the modern linear theory of generalized functions goes back to the 1930s, with the introduction of Sobolev spaces, and then it reaches its full development following the introduction in the 1940s of the Schwartz distributions.
The two obvious limitations of that linear theory, usually called the theory of distributions, are :

- the considerable difficulties in dealing with nonlinear PDEs, with the resulting methods based mostly on adhoc approaches, due to the fact that, hardly without exception, the various spaces of distributions are not algebras, but only vector spaces,

- the considerable restrictions on the singularities which can be dealt with by various spaces of distributions, due to the fact that hardly any of them is a flabby sheaf.

Systematic nonlinear theories of generalized functions started to emerge in the 1960s, [16,17,22,23,3,4,6,7,38,39,43,45,47, 50,9,53-57,59,61, 70-72,97-99,105,141,143,165,172-178], by the time a considerable amount of applications of the distribution theory had taken place in solving linear and nonlinear PDEs. As it happened, however, those who contributed in the early stages to the development of the nonlinear theory of generalized functions were, with few exceptions, [3,4,6], doing so upon motivations, aims and ideas which were hardly at all related to linear, let alone, nonlinear PDEs. Instead, they were of a rather theoretical nature, trying to eliminate certain limitations of what by then was a well developed linear theory of generalized functions, that is, of distributions. One of such limitations which received a special attention was the celebrated and long misunderstood 1954 result of Schwartz, [245], claiming erroneously to prove the impossibility to multiply distributions.

The fact remains, however, that as much due the requirements of its own theoretical development, as due to those of its applications to the solution of linear, and especially nonlinear PDEs, the nonlinear theory of generalized functions is at the confluence of a variety of rather different motivations, aims and ideas, or for that matter, interpretations. Also lately, that theory is no longer applied alone to solving PDEs, as its applications have branched out into differential geometry, and in particular, general relativity, [177], as well as other basic disciplines of mathematics and physics, among them Lie groups, quantum gravity, and so on, [53-57,70-72,167,168-170], with the consequent further diversity of motivations, aims and ideas which contribute to its development.

0.3. Consensus, and also Some Controversy ...

The consensus which has emerged from the beginning is that spaces of generalized functions should be *differential algebras*, and not merely vector spaces of infinitely differentiable generalized functions, such as is the case, for instance, with the spaces of Schwartz distributions, and in particular, the Sobolev spaces.

There is also a consensus that the differential algebras of generalized functions should be large enough in order to contain some of the more frequently used linear spaces of distributions.

Further, there is some less clear consensus about the need to be able to deal with sufficiently large classes of singularities in the differential algebras of generalized functions.

More precisely, and rather strangely at that, there are quite different views - even if often tacitly accepted - about what kind of singularities are important to be dealt with, and on the other hand, what kind of singularities may be left outside of a nonlinear theory of generalized functions.
In this regard, it may appear that the respective lack of consensus is a consequence of an insufficient clarity among many of those involved in the nonlinear theory of generalized functions about the considerable ranges and variety of singularities which can occur in mathematical analysis and its various applications.
A possible reason for such a lack of clarity comes from the fact that attention is focused on the rather limited classes of singularities the Schwartz distributions and Sobolev spaces can deal with. And compared to them, the emerging nonlinear theory of generalized functions can, of course, do considerably better. That fact alone, however, need not - and actually it does not - mean that all presently developed and used differential algebras of generalized functions are quite equally good at dealing with singularities. Indeed, as seen is the sequel, some of such algebras can easily deal with surprisingly large classes of singularities, while other ones are note much better in this regard than the usual spaces of distributions.

And the remarkable fact in this regard - seemingly not yet realized by many - is that the respective dichotomy between various differential algebras of generalized functions goes parallel with the dichotomy between being, or on the contrary, failing to be a *flabby sheaf*. Indeed, flabbiness proves to be the very property which guarantees the ability to deal with truly large classes of singularities.

In this regard, one may conclude that the nonlinear theory of generalized functions, in its present form of infinitely many differential algebras of generalized functions - which correspond to the inevitable infinite branching of multiplication above certain levels of singularities, [4, pp. 118,119], [143], thus to infinitely many different multiplications of generalized functions with singularities, see section 14 - is subjected to a

Dichotomic Sheaf Theoretic Singularity Test :

> Those algebras which are flabby sheaves can handle very large classes of singularities.
>
> Those algebras which fail to be flabby sheaves cannot handle but limited classes of singularities.

This dichotomy, however, does not disqualify the differential algebras of generalized functions, among them the Colombeau algebras, for instance, which fail to be flabby sheaves. Indeed, hardly without exception, the various vector spaces of distributions in the linear theory of generalized functions also fail to be flabby sheaves. Yet their utility is clearly unquestionable in solving various ranges of PDEs.

Nevertheless, this dichotomy comes fully to the fore - and does so even in the linear, and not only nonlinear case - when dealing with large classes of singularities is involved.

The present lack of a sufficient understanding of the inevitable infinite branching of multiplication above certain levels of singularities has had several further negative consequences. Among them is a rather strange divergence of views which has emerged about an issue which - within the linear theory of distributions - had for long been clearly and definitely settled, ever since its very beginnings, that is, with the introduction of various Sobolev spaces in the 1930s.
Namely, it is one of the remarkable and obvious features of the linear theory of generalized functions that a large variety of vector spaces of distributions have been developed and used, each of them with its specific advantages, and of course, limitations as well. And as mentioned, that trend started from the very beginning with the natural variety of Sobolev spaces. Later, even in the case of the very large vector spaces of Schwartz distributions, two rather different such vector spaces proved to have fundamental importance, namely, the spaces $\mathcal{D}'$ and $\mathcal{S}'$, the first being a vector space of distributions useful in a considerable variety of situations, while the second being a vector space tailor made for the use of Fourier transforms.

And needless to say, a further large variety of vector spaces of distributions has over the decades been considered and used, as is well documented in the respective literature.

In this way, within the linear theory of generalized functions there has never been an issue which particular vector space of distributions should alone be declared canonical and of alleged universal and exclusive use, and then have the whole linear theory of distributions restricted to it.
Instead, there has always been a clear and strong awareness about the relative advantages and disadvantages of each vector space of distributions, and therefore, about the manifest convenience to develop and use the various spaces according to their specific features.

What further makes the mentioned divergence of views strange is the following *essential*, yet hardly noted fact typical for differential algebras of generalized functions.
Namely, as seen in the linear theory of generalized functions, the *addition* of such generalized functions can easily and naturally be extended in a well defined *unique* manner to considerably large vector spaces of generalized functions.
On the other hand, and in sharp contradistinction to the case of addition, the *multiplication* of generalized functions does *inevitably branches* into a large number of different possibilities when it occurs above certain levels of singularities. And the reasons for such an infinite branching are most simple, thus fundamental and unavoidable, namely, of a purely algebraic, more precisely, ring theoretic nature, see section 14.

A fundamental - and among those involved in the nonlinear theory of generalized functions, still hardly realized, let alone understood - consequence of this infinite branching of multiplication is that :

- while the operations of addition and scalar multiplication on singularities of generalized functions remain the same, whenever they can be performed, in various spaces of distributions, Sobolev spaces or differential algebras of generalized functions,

- on the other hand, and in sharp contradistinction, multiplication, and in general, nonlinear operations, as well as differentiation of singularities of generalized functions lead to significantly *different* results, depending on the mentioned spaces or algebras in which they are performed.

This fact alone is, therefore, sufficient to lead to the inevitability - thus as well necessity and convenience - to consider, develop and use a larger variety of differential algebras of generalized functions.

As for the present lack of understanding of that inevitable branching of multiplication, some amusing aspects cannot so easily be missed.

First, and as mentioned, the reason for such an inevitable infinite branching of multiplication is very simple, namely, purely of a *ring theoretic* nature. Nevertheless, certain leading scholars in mathematics and physics still miss on realizing the very existence, let alone, the inevitability of that branching, [240].

Added to such a state of affairs comes the fact that most of those presently involved in the nonlinear theory of generalized functions come from a background of mathematical analysis, with its well known particular point of view focusing on a rather limited variety of spaces, and pursuing their analysis to the extremes. In this regard one can note that, unlike analysts, even functional analysts are accustomed to a variety of rather different spaces. However, the deficiency there is that much of functional analysis has its strength in the linear, rather than nonlinear realms.

And needless to say, the technical complications which are inherent in a nonlinear theory of the generalized functions can attract the attention of analysts, and do so even to the extent of seemingly preventing a deeper awareness and understanding of important aspects of the complex conceptual underpinning of the emerging nonlinear version of the theory, aspects not so familiar to usual analytic thinking ...

In other words, it appears that, for many of the present practitioners of the emerging nonlinear theory of generalized functions, it may still be early days for a better understanding of such crucially important aspects as

- what are really large and important classes of singularities,
- the inevitability of infinite branching of multiplication above certain levels of singularities.

In this regard, the extent to which the nonlinear theory of generalized functions is rather complex in its nature, being at its best an interplay of a variety of motivations, aims, ideas, methods, applications, and so on, which come from a number of rather different branches of mathematics, and possibly physics as well, is illustrated among other by the fact that two major branches of mathematics, typically unfamiliar to analysts, play an obvious role in that theory, namely, model theory, which is a modern branch of mathematical logic, and sheaf theory.
Model theory is, indeed, present in the very way the various differential algebras of generalized functions are constructed, a construction closely related to what is called *reduced powers*.
As for sheaf theory, the issue of the *flabbiness*, or otherwise, of various differential algebras of generalized functions turns out to be fundamentally related to the extent such algebras can, or for that matter, cannot deal with large classes of singularities. Indeed, those algebras which are not flabby sheaves fail significantly when facing larger classes of singularities.
These two mathematical concepts, namely, reduced powers and flabby sheaves, let alone their deeper understanding in the context of their own respective mathematical theories, happen to be so often quite strange to analysts.
No wonder, therefore, that their crucial importance is not fully realized within the nonlinear theory of generalized functions ...

As for the very strong and direct connection between the ability to deal with considerably large classes of singularities and the flabbiness of the respective differential algebras of generalized functions, this is indeed a most fortunate mathematical fact.
Certainly, from the point of view of its strictly technical or formal aspect, the concept of flabby sheaf is surprisingly simple.
On the other hand, the issue of singularities, let alone, of large such classes, is notoriously difficult.
Nevertheless, in the case of certain sheaves of functions or generalized

functions on topological spaces, and among them the differential algebras of generalized functions, the concept of flabbiness turns out to be just about the perfect one in order to characterize the mentioned ability to deal with very large classes of singularities.

0.4. Far Too Many Differential Algebras of Generalized Functions ?

Here in this paper we shall limit the attention mostly to the issue of *singularities*. And we shall present various approaches to them in the following five types of differential algebras of generalized functions, types considered in the order they have been introduced in the literature :

- the nowhere dense algebras, see sections 7, 8 and [22,23,3,4,6,7,38, 39,43,45,47,49,9,53-57,59,61,70,71, 97-99,105,168-170,175]
- chains of algebras, see section 15 and [3,4,6,7]
- the Colombeau algebras, see section 9 and [172,173]
- the space-time foam algebras, see section 10 and[55-57,59,61,97, 165,168-170]
- the local algebras, see section 11 and [141]

It is important to note that, in fact, the natural approach to a thorough enough nonlinear theory of generalized functions is not by studying on its own one or another differential algebra among the infinitely many possible ones, a possibility resulting directly from the infinite branching of multiplication above certain levels of singularities. Indeed, a more careful study of differentiation, see sections 12 - 14 and [4, pp. 88-97], [6, pp. 287-347], [7, pp. 1-99] in this regard, shows that whole *chains of algebras*, instead of any single one, are actually the natural framework, chains recalling the classical one $\mathcal{C}^\infty \subset \ldots \subset \mathcal{C}^n \subset \ldots \subset \mathcal{C}^0$. And the reason for that, a most simple, basic, and purely algebraic one, is the inevitable presence of the *three conflicting* issues of, see section 12

- singularities,

- multiplication, and
- differentiation

which, needless to say, have to be brought together within any more appropriate nonlinear theory of generalized functions. Therefore, in order to help such a clarification, it is useful to recall in section 12 what appear to be the most simple and basic facts - of a purely *algebraic* nature - which happen to underlie differentiation in any algebraically based generalized context.

Amusingly, so far, the mentioned chains of algebras of generalized functions have not proved to be popular ...
It is, indeed, far more cozy for many to limit oneself to one single such algebra, and then, try to develop endlessly one's respective specialization ...
After all, in mathematics, as much as in science in general, and in fact, in all human ventures, so many decisions, no matter how consequential they may be, are done upon rather strong and instant emotional reasons ...
If the term "emotional reason" may actually make any sense at all ...
But then, we have that age old French saying, according to which "the heart can have reasons which the mind can never comprehend" ...

Last, but not least, it should be mentioned that, in view of the inevitable branching of multiplication pointed out above, it was in [4] for the first time that all possible differential algebras of generalized functions were identified and constructed, see also [6,7,38,39,43,45,47,50,9, 53-57,59,61,70,97-99,105] for further details.
Consequently, the first four types of differential algebras mentioned above, and which will be considered in the sequel with respect to their specific ability to deal with singularities, are but particular cases of the differential algebras of generalized functions introduced in [4].

0.5. The Classical Schwartz and Lewy "Impossibilities" ...

One cannot, however, end such an overview without mentioning the following.

Two crucial "impossibility" results happened to arise rather early in the development of the linear theory of generalized functions, or in other words, of the Schwartz distributions.

The first one, already mentioned, was the 1954 result of Schwartz, [245], which even more than two decades later kept being misinterpreted by leading specialists as allegedly proving the impossibility of multiplying distributions, [215].

Soon after, in 1957, came the Hans Lewy "impossibility" result, [229], showing that very simple first order linear smooth - and in fact, first degree polynomial - coefficient PDEs in three real variables do not have any distribution solutions in any neighbourhood of any point in $\mathbb{R}^3$.
And a special gravity of that impossibility result came from the fact that the respective PDEs were not, so to say, merely invented as a counter-example, but resulted from certain studies in functions of several complex variables.

The reaction - over the last more than half a century - of the community of the linear theory of generalized functions to these two "impossibility" events turned out to be rather radically different.

The first event got simply and quite instantly misinterpreted, as it was turned into a sort of slogan : "one cannot multiply distributions". After that, quite everybody returned to their usual ventures in developing and using distributions, mainly with a view to solving PDEs.
The few exceptions in research which kept up the interest in the Schwartz impossibility turned out, as mentioned, to lead to the emergence of the nonlinear theory of generalized functions ...

As for the second event, it was quickly forgotten, never to be mentioned again, simply as if it had never happened ...
After all, it was far too painful and obvious a warning about the rather dramatic insufficiency of distribution theory even in the study of linear PDEs ...
There were, fortunately, two studies which contributed to a significant clarification of the Lewy impossibility, [215,251]. However, successive

generations of specialists using distributions turned out to be hardly at all aware of that dramatic warning ...

After all, and for better or worse, mathematics is not quite history, and of course, neither is history mathematics ...
But then, history is quite a lot of sociology ...
And sociology may not be mathematics ...
But on the other hand, a lot of mathematics is, as it happens, mere sociology ...
And how about science ?
Well, it certainly has to be taken ... modulo ... sociology, and why not, perhaps even history ...

As mentioned, the nonlinear theory of generalized functions, with the respective variety of differential algebras, has completely clarified the Schwartz "impossibility", by showing that there is absolutely no any kind of impossibility in multiplying distributions, when done within such algebras.

Regarding the Lewy "impossibility", the situation has remained less clear within the nonlinear theory of generalized functions. Indeed, there are results on solving large classes of linear smooth coefficient PDEs. However, those differential algebras of generalized functions in which such solutions are found have, so far, to be subjected to rather complicated adhoc modifications with respect to the operation of differentiation, [172,173].

And the morale of the story of the nonlinear theory of generalized functions ?

0.6. Two Powerful Nonlinear Challenges in Dealing with Singularities

Well, for nearly two decades by now, and ever since the early 1990s, [8], it happens that the nonlinear theory of generalized solutions - based on various differential algebras of generalized functions - is no longer the most powerful and general theory for solving large classes of linear and nonlinear PDEs. A few indications in this regard are presented next in subsection 0.7.

A second and no less powerful nonlinear method for dealing with large classes of singularities was introduced by A Mallios also in the 1990s, under the name of Abstract Differential Geometry, or ADG, [168-170]. Several related details can be found below in subsection 0.8.

0.7. A Few Words on the Order Completion Method

Indeed, the *order completion method*, [8], and its subsequent considerable improvements, [58,60,67,69,158-164], turned out to be able to provide solutions for linear and nonlinear PDEs which in *two* important respects are far beyond what certain leading PDE specialists, [181,198], consider ever to be possible in mathematics, namely :

- Very general nonlinear systems of PDEs with possibly associated initial and/or boundary value problems can always be solved, the respective equations being of the form

$$(0.2) \qquad F(x, U(x), \ldots, D^p U(x), \ldots) = f(x), \quad x \in \Omega \subseteq \mathbb{R}^n$$

  where F and f are arbitrary continuous functions. And in fact, the functions F and f can have discontinuities on closed, nowhere dense subsets, [8].

- One can always obtain solutions which are no longer generalized functions, but instead are usual measurable functions on Ω, and in fact, are even more regular, being Hausdorff continuous. Furthermore, in case smoothness conditions are satisfied by F and f, corresponding smoothness properties are obtained for the solutions.

When it comes to proving the *existence* of solutions of nonlinear PDEs, it is important to *compare* the power of this order completion method, OCM, with that of the linear, or for that matter, nonlinear theory of generalized functions, NTGF. In this regard, the following are already obvious :

- The OCM proves the *existence* of solutions for nonlinear systems of PDEs with possibly associated initial and/or boundary value problems of such a generality to which NTGF cannot come anywhere near at present. In this regard OCM proves to be *type independent*, that is, delivers the existence of solutions in ways which do not depend on the particular type of the PDEs involved. Such type independence is, so far, unprecedented in the whole range of known solution methods for PDEs, except for the method improved upon in [114], which however, is more restrictive.

- Furthermore, OCM gives a blanket, universal, type independent *regularity* for the solutions obtained, since such solutions can be assimilated with usual measurable functions, thus they do no longer require the consideration of any kind of generalized functions. Also, if certain smoothness conditions are assumed on the PDEs, then similar smoothness conditions are obtained for the solutions.

- The OCM proves to be able to deal with initial and/or boundary value problems in the same way as it deals with the solution of PDEs without such associated problems. This is in sharp contradistinction with the considerable technical difficulties encountered in NTGF, let alone, the linear distribution theory, when dealing with such associated problems.

- The proofs of existence of solutions for PDEs obtained so far in NTGF are rather similarly adhoc with those in the linear theory of distributions and Sobolev spaces. Thus they are highly type dependent.

Needless to say, this order completion solution method, [8], was the first - and so far, it is the only one - to overcome the Lewy impossibility. And it did so with a very wide nonlinear margin.

And the reason for such a surprising power in the order completion solution method ?

Quite likely, the answer is in the fact that the concept of partial order - which at first appears to be so very simple - is considerably more powerful than it is so often credited for, when compared with algebraic or topological methods. In this regard, it is worth recalling the 1936 Freudenthal Spectral Theorem, [232, chapter 6], which is formulated exclusively in terms of partial order, and also proved exclusively in the same terms, yet it has as consequences the spectral theorem of normal operators in Hilbert spaces, the Radon-Nykodim theorem in measure theory, and the existence of solutions of certain Poisson PDEs.

As it happens however, such theorems are hardly known nowadays ... As if they did not even belong to mathematics or to the history of mathematics ...
Yes indeed, the interplay between mathematics and history appears to be far more subtle, mysterious, and also mystifying than most of us may realize, or could even imagine ...

0.8. ADG, or the Abstract Differential Geometry of Mallios

As it happened, no less than two new general nonlinear theories dealing with large classes of singularities have emerged in the 1990s. The first one was already mentioned in subsection 0.7., while the second is the Abstract Differential Geometry, or ADG, established by A Mallios, see [168-170] for some of the more relevant references in this regard.

These two approaches started from significantly different points of view, motivations and aims.

The first, as mentioned, was motivated by the limitations of both the linear distribution theory of Schwartz, as well as of the nonlinear theory of generalized functions as represented by various differential algebras of generalized functions, when attempted to be applied to the solution of the largest possible classes of nonlinear systems of PDEs.

The second aimed to eliminate the issue of singularities in usual smooth differential geometry, an issue which is essentially involved in the applications of that mathematical discipline to various disciplines of physics, among them general relativity, quantum gravity, and so on.

As it happened, however, the second nonlinear approach turned out to be essentially based on the use of the differential algebras of generalized functions, and especially of those which are flabby sheaves, [54,55,57,70,168-170].

Indeed, ADG is from its very first steps heavily involving sheaf theory. And as already mentioned, and further elaborated in the sequel, the sheaf point of view is most naturally and intimately connected with the issue of singularities. And to add to that, flabby sheaves turn out to be the inevitable natural setup when dealing with large classes of singularities.

Is this considerable role played by sheaves - and specifically by flabby sheaves - to delay for longer the involvement of analysts ?
To become sufficiently familiar, and in fact, quite proficient in the relevant aspects of sheaf theory, it is more than enough to read the remarkably well written first few respective chapters in [168] ...
And reading those chapters can in fact be a joy in itself, not to mention that it may prove useful to the readers in studying quite a few other branches of modern mathematics ...
In this regard, it is worth noting that more than two decades ago, Kaneko, [219], found it appropriate to present hyperfunctions in the context of sheaf theory. And reading that presentation, even hard core analysts may have to wonder whether, indeed, sheaf theory can be kept outside of their subject for longer in our times ...
Anyhow, it is quite clear by now that sheaf theoretical methods can, and do, give a significant competitive edge to those who are brave enough to know them, and thus, can use them ...
A shorter presentation of the required aspects of sheaf theory, one that follows the exposition in [168], can be found in [136].
And then, why should we not have analysts as well among such advantaged competitors ?

By the way, as it happens, the order completion method also benefits significantly from the sheaf theoretic point of view ...
And in fact, the spaces delivered by that method, spaces within which very large classes of nonlinear systems of PDEs, with possibly associated initial and/or boundary value problems, always have solutions, would be rather meaningless, unless those spaces would be sheaves, as they in fact turn out to be ...

0.9. Can Mathematics Deliver General PDE Solution Methods ?

As for the present understanding by certain leading specialists of what may, or for that matter, may not be possible in mathematics regarding the solution of PDEs, the following two citations can be instructive :

The 2004 edition of the Springer Universitext book "Lectures on PDEs" by V I Arnold, [181], starts on page 1 with the statement :

> "In contrast to ordinary differential equations, there is *no unified theory* of partial differential equations. Some equations have their own theories, while others have no theory at all. The reason for this complexity is a more complicated geometry ..." (italics added)

The 1998 edition of the book "Partial Differential Equations" by L C Evans, [198], starts his Examples on page 3 with the statement :

> "There is no general theory known concerning the solvability of all partial differential equations. Such a theory is *extremely unlikely* to exist, given the rich variety of physical, geometric, and probabilistic phenomena which can be modeled by PDE. Instead, research focuses on various particular partial differential equations ..." (italics added)

So much, therefore, for the differential algebras of generalized functions, or for the order completion method in solving large classes of nonlinear PDEs ...

Last, but not least, two further issues about singularities and differential algebras of generalized functions.

0.10. And What About the Schwartz Distributions ?

It seems that history has its heavy hand influencing for long the issue of singularities ...
First, the instant and long ongoing misinterpretation of the 1954

Schwartz "impossibility" entrenched the view that singularities, that is, those dealt with by generalized functions, and specifically, by distributions, could not possibly be subjected to nonlinear operations in a general and systematic manner, namely, within an appropriate wide ranging nonlinear theory. Consequently, a large range of adhoc methods were introduced, specifically when solving nonlinear PDEs. As for the application of distributions to such essentially nonlinear realms as differential geometry, and in particular, general relativity, no attempts worth mentioning have been developed, since the unsuitability of any linear theory - among them of the distributions as well - proved to be quite glaringly clear from the start ...

Nevertheless, when the nonlinear theory of generalized functions started to emerge, there was a somewhat lingering feeling that the Schwartz distributions should by all means be contained in the respective differential algebras of generalized functions, as a kind of guarantee for the legitimacy, relevance, interest in, and value of the emerging nonlinear theory ...
Thus amusingly, it seemed to be forgotten all about the severity of the limitations on singularities which the distributions were able to deal with ...
It seemed also to be forgotten the fact that the various Sobolev spaces are quite far from containing all the Schwartz distributions. And in fact, most of the usual ones do not contain even the celebrated Dirac delta distribution ...

And of course, not much awareness was present about the inevitable infinite branching of multiplication above certain levels of singularities, branching which would, among others, lead to multiplications rather different from those encountered in any adhoc methods developed within the linear theory of distributions ...

Remarkably in this regard, it took a mathematician of such wide and deep knowledge and understanding of its various disciplines like M Hazewinkel to ask, as if rather naively and innocently, why indeed must all the algebras of generalized functions contain all the Schwartz distributions, [16,17,3] ?
And as it happened, very few present at the respective informal 1997

seminar at the Erwin Schrödinger Institute for Mathematical Physics in Vienna, Austria, managed to grasp the timely relevance of that question ...
After all, it was only about half a century since distributions had been introduced ...
And as not seldom in science, the wheels of history can happen to turn rather slowly ...

Now, as soon as one realizes that there is no need for such a most close connection between the Schwartz distributions, and on the other hand, the various differential algebras of generalized functions, the issue of the embedding of the whole vector space $\mathcal{D}'$ of distributions into all differential algebras of generalized functions acquires a secondary importance. And then, inevitably, even more does so the issue of having such an embedding preserve the differentiation of distributions.
In this regard one may note that, until the early 2000s, it was thought that the Colombeau algebras were the only differential algebras of generalized functions in which the vector space $\mathcal{D}'$ of distributions could be embedded with the preservation of the differentiation of distributions. Then it was shown in [253] that it is rather easy to construct differential algebras of generalized functions with such a stronger embedding property for distributions. And furthermore, the respective differential algebras of generalized functions constructed in [253] allow a supplementary large class of nonlinear operations on generalized functions which cannot be performed in the Colombeau algebras.

Consequently, the issue of such a stronger embedding of the Schwartz distributions, regarding differentiation, lost considerably from what appeared to be its initial importance, and did so independently of the above challenging, if not in fact, troubling question of M Hazewinkel, or for that matter, of the answer one may give to that question ...

Furthermore, as mentioned in subsection 12.4.5., the basic, simple, and purely algebraic incompatibility between insufficient smoothness, multiplication and differentiation leads to the inevitable fact that, by requiring the preservation of distributional differentiation in algebras of generalized functions, one has to pay a price. Namely, one ends up in such algebras with multiplications which are different from the usual ones in the case of insufficiently smooth functions in $\mathcal{C}^m \setminus \mathcal{C}^{m+1}$, with $0 \leq m < \infty$.

0.11. Again and Again that Algebraic Conflict ...

A further issue worth recalling is the basic and most elementary *algebraic conflict* which is at the root of the so called Schwartz "impossibility", namely, the conflict between *discontinuity, multiplication* and *differentiation*, see section 12. In this regard, it happened that from the very beginning the formulation, as well as proof of the Schwartz "impossibility" created the impression that it was about an issue by no means less involved than analysis, if not in fact, functional analysis. A more careful consideration, however, clearly showed that it was indeed about noting more than a rather simple and quite basic algebraic conflict.

In conclusion, the emergence and development of the nonlinear theory of generalized functions has its own history, as well as sociology ...
As often in science, emerging theories may start at the margin of some established ones, and remain for a while longer on the margins ...
By the late 1990s, however, the nonlinear theory of generalized functions got a certain recognition being listed under 46F30 in the Mathematical Subject Classification of the American Mathematical Society.

Yet several specific features of that theory, in conjunction with its sociology, still seem to have a certain limiting effect on its further development.

In this regard, as mentioned above, two factors may seem to be important : the considerable complexity of the theory, not least due to its essentially nonlinear nature, and the fact that most of those involved in it at present come from a background of analysis.
In addition, as so often in science, and thus not specifically only to the nonlinear theory of generalized functions, come strong widespread tendencies of getting into, and then staying for ever more in the comfort zone of one or another particular, and thus inevitably narrow view of the subject ...

As for the significant challenge to the nonlinear theory of generalized functions by the order completion method in solving large classes of nonlinear systems of PDEs, there are mathematical disciplines where nonlinear approaches to generalized functions can still have a significant importance. Among such disciplines are, of course, differential geometry where the treatment of singularities through the classical approaches is utterly inadequate. In this regard, that is, of a differential geometry which can easily deal with large classes of singularities, remarkable recent developments have been made in [168-170], by the earlier mentioned ADG, see also [54,55,57,70] for related further contributions.

1. Singularities of Usual Functions

Let us start by considering some basic spaces of usual functions with various extent of smoothness, or for that matter, lack of smoothness, which can offer the customary set up in mathematical analysis for encountering singularities, and which, since the 1930s, have in fact led to a considerable amount of diverse spaces of generalized functions, and did so mainly in the study of solutions of linear and nonlinear PDEs. For simplicity, we shall often consider functions and generalized functions of one single real variable, and with real values. That however, will not exclude the consideration when appropriate of functions and generalized functions defined on arbitrary open subsets $\Omega \subseteq \mathbb{R}^n$.

Fortunately, most of the issues dealt with here and concerning singularities of generalized functions of one real variable can quite easily be extended to generalized functions of several real variables, and with real or complex value, and in fact, with values in Banach algebras as well, be they commutative or not.
It follows that the situation of singularities of generalized functions of several real variables can, in fact, be no less simple than those presented in the sequel, and which are already considerable.

As basic spaces of usual functions we can consider the following real valued ones, all of them defined on some open $\Omega \subseteq \mathbb{R}^n$, namely

$$(1.1) \qquad \mathcal{C}^\omega \underset{\neq}{\subsetneq} \mathcal{C}^\infty \underset{\neq}{\subsetneq} \ldots \underset{\neq}{\subsetneq} \mathcal{C}^n \underset{\neq}{\subsetneq} \ldots \underset{\neq}{\subsetneq} \mathcal{C}^0 \underset{\neq}{\subsetneq} \mathcal{L}_{loc}$$

where $\mathcal{C}^\omega$ and $\mathcal{L}_{loc}$ denote as usual the real analytic, respectively, locally integrable functions.

In such a setup, let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be a usual function, while $\mathcal{S}$ is one of the spaces in (1.1). Then f is usually considered to have a singularity at some point $a \in \Omega$, when considered in the context of $\mathcal{S}$, if

$$(1.2) \qquad f \notin \mathcal{S} \quad \text{on} \quad \Omega$$

however

(1.3) $\quad f \in \mathcal{S} \quad \text{on} \quad \Omega \setminus \{a\}$

A more general situation is that of a set $\Gamma \subseteq \Omega$ of singularity points of f, when again

(1.4) $\quad f \notin \mathcal{S} \quad \text{on} \quad \Omega$

nevertheless

(1.5) $\quad f \in \mathcal{S} \quad \text{on open subsets} \quad V \subseteq \Omega \setminus \Gamma$

It follows naturally that *two* aspects are important when considering the singularities of functions $f : \mathbb{R} \longrightarrow \mathbb{R}$ in the context of such spaces $\mathcal{S}$, namely

- SIZE : how large the singularity subsets Γ are,
- BEHAVIOUR : how the functions f behave in the neighbourhood of singularity points in Γ.

Let us give a few illustrations in this regard, and start with the *smallest* nontrivial size of singularity sets, namely

(1.6) $\quad \Gamma = \{a\} \subset \Omega$

Example 1.1.

One *best* possible behaviour in the neighbourhood of singularity is in the case of the Heaviside function

(1.7) $\quad H \in \mathcal{L}_{loc} \setminus \mathcal{C}^0 \quad \text{on} \quad \Omega = \mathbb{R}, \qquad H \in \mathcal{C}^\omega \quad \text{on} \quad \Omega = \mathbb{R} \setminus \{0\}$

since we have $\lim_{\,x \to 0,\ x < 0} H(x) = 0, \ \lim_{\,x \to 0,\ x > 0} H(x) = 1$.

$\square$

This situation has correspondents in higher cases of smoothness, as seen in

Example 1.2.

For $n \geq 1$, we have

$$(1.8) \qquad x_+^n \in \mathcal{C}^n \setminus \mathcal{C}^{n+1} \ \text{ on } \ \Omega = \mathbb{R}, \qquad x_+^n \in \mathcal{C}^\omega \ \text{ on } \ \Omega \setminus \{0\}$$

□

At the opposite end of behaviour in the neighbourhood of a singularitiy, but still for the smallest nontrivial size of singularities (1.6), we have, even in the case of analytic functions, a rather surprising fact, described by the Great Picard Theorem. Namely, in the neighbourhood of an essential singularity, an analytic function takes infinitely many times every complex value, except perhaps for a single one.
It follows that, in general, one can expect a rather arbitrary behaviour for functions $f \in \mathcal{C}^\infty$ on $\Omega \setminus \{a\}$ in the neigbourhood of their singularity point $a \in \Omega$. In particular, no given growth condition of any kind may ever describe by far most of such functions.

In our case of functions of real variable in (1.1), a situation similar with the behaviour in the Picard theorem can easily occur in the neighbourhood of a singularity even for $\mathcal{C}^\infty$ smooth functions, as seen in

Example 1.3.

Let $f : \mathbb{R} \setminus \{0\} \longrightarrow \mathbb{R}$ given by $f(x) = (\sin(1/x))/x$, for $x \in \mathbb{R} \setminus \{0\}$. Then obviously

$$(1.9) \qquad f \in \mathcal{C}^\omega \ \text{ on } \ \mathbb{R} \setminus \{0\}$$

And in the neighbourhood of its singularity $0 \in \mathbb{R}$, this function takes - this time without even one single exception - all real values infinitely many times.

□

The function in Example 1.3. obviously satisfies a polynomial growth condition in the neighbourhood of $0 \in \mathbb{R}$. However, with rather elementary means, one can construct $\mathcal{C}^\infty$ smooth functions which do not satisfy any such growth condition in the neighbourhood of their singularity, while at the same time, and again in a manner worse than described by the Picard theorem, take all real values infinitely many times, as seen in

Example 1.4.

Let $x_0 > x_1 > \ldots > x_n > \ldots > 0$, with $\lim_{n \to \infty} x_n = 0$. Further, let $c_0, c_, \ldots, c_n, \ldots \mathbb{R}$ be arbitrary given. Then there exist functions

$$(1.10) \qquad f \in \mathcal{C}^\infty \quad \text{on} \quad \mathbb{R} \setminus \{0\}$$

such that

$$(1.11) \qquad f(x_n) = c_n, \qquad n \in \mathbb{N}$$

Indeed, let $\eta \in \mathcal{C}^\infty$ on $\mathbb{R}$ be such that, [236]

$$(1.12) \qquad \begin{array}{rl} *) & 0 \leq \omega(x) \leq 1, \quad x \in \mathbb{R} \\ **) & \omega(x) = 1, \quad x \in \mathbb{R} \setminus (0,1) \\ ***) & \omega(x) = 0, \quad x \in [1/3, 2/3] \end{array}$$

Then we define $f : (0, \infty) \longrightarrow \mathbb{R}$ by

$$(1.13) \quad f(x) = \left| \begin{array}{ll} c_0\, \omega((x - x_1)/(x_0 - x_1)) & \text{if } \ (x_0 + x_1)/2 \leq x \\ c_1\, \omega((x - x_2)/(x_1 - x_2)) & \text{if } \ (x_1 + x_2)/2 \leq x \leq (x_0 + x_1)/2 \\ c_2\, \omega((x - x_3)/(x_2 - x_3)) & \text{if } \ (x_2 + x_3)/2 \leq x \leq (x_1 + x_2)/2 \\ \vdots & \end{array} \right.$$

$\square$

As for the case of *largest* possible sets of singularities, one has

Example 1.5.

Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ given by

$$(1.14) \qquad f(x) = \left| \begin{array}{l} 1 \ \text{ if } \ x \in \mathbb{R} \setminus \mathbb{Q} \\ \\ 0 \ \text{ if } \ x \in \mathbb{Q} \end{array} \right.$$

in which case

$$(1.15) \qquad f, (1 - f) \in \mathcal{L}_{loc} \setminus \mathcal{C}^0 \ \text{ on } \ \Omega = \mathbb{R}$$

and in fact f is not continuous at any point in $\mathbb{R}$.

□

However, the issue of large sets of singularities of usual functions is somewhat more involved, as seen next.

Example 1.6.

The Riemann function $r : \mathbb{R} \longrightarrow \mathbb{R}$ is defined by, [220, p.112]

$$(1.16) \qquad r(x) = \left| \begin{array}{l} 1/q \ \text{ if } \ x \in \mathbb{Q}, \ x = p/q, \ p, q \text{ are relatively prime} \\ \\ 1 \ \text{ if } \ x = 0 \\ \\ 0 \ \text{ if } \ x \in \mathbb{R} \setminus \mathbb{Q} \end{array} \right.$$

and r is continuous on $\mathbb{R} \setminus \mathbb{Q}$, while it is discontinuous on $\mathbb{Q}$.

□

But as it happens, due to a Baire Category argument, one has

Example 1.7.

There are *no* functions $f : \mathbb{R} \longrightarrow \mathbb{R}$ which would be continuous on $\mathbb{Q}$, while being discontinuous on $\mathbb{R} \setminus \mathbb{Q}$, [220, p.115].

□

Remark 1.1.

1) The above property in Example 1.7. is a consequence of the more general property that a function between a Baire space X and a topological space Y cannot have as continuity points a set $X \setminus E$, where $E \subset X$ is countable and dense in X, [220, p.115].

2) In view of Example 1.7., it is to be pointed out that in the space-time foam algebras, see section 10, one can have generalized functions on $\mathbb{R}$ whose set of singular points is $\mathbb{R} \setminus \mathbb{Q}$, while the set of non-singular points is $\mathbb{Q}$.

Indeed, for the set of singularity points $\Gamma \subset \Omega$ of generalized functions in such algebras given on arbitrary open sets $\Omega \subseteq \mathbb{R}^n$, the only restriction is that their complementaries $\Omega \setminus \Gamma$, that is, their set of non-singular points, should be dense in Ω, see (10.1).

2. The Inevitable, and still Unpopular Sheaf Point of View

As seen already in Example 1.1., namely, in such a simple instance as that of the Heaviside function, the issue - even in such a case of the smallest singularity set and a best kind of behaviour in its neighbourhood - is that :

- NO EXTESION : There are singular functions which *cannot* be extended to non-singular functions on larger domains that may contain the singularities of the initial, unextended functions.

This fundamentally important situation is simply and clearly described in a considerable generality by the powerful concepts of *sheaf* and *flabbiness* which we recall briefly, [8,136,219].

Let X be a topological space. Let $F(U)$ be a set, for every open subset $U \subseteq X$. Further, for pairs of open subsets $U \subseteq V \subseteq X$, let $\rho_{V,\, U} : F(V) \longrightarrow F(U)$ be mappings such that $\rho_{U,\, U}$ is the identity on $F(U)$, while $\rho_{V,\, U} \circ \rho_{W,\, V} = \rho_{W,\, U}$, for open subsets $U \subseteq V \subseteq W \subseteq X$. Then this structure is called a *sheaf* on X, iff it satisfies two more conditions which relate its local and global properties.

Definition 2.1.

A sheaf is called *flabby*, if and only if the mappings

$$\rho_{V,\, U} : F(V) \longrightarrow F(U)$$

are *surjective* for all pairs of open subsets $U \subseteq V \subseteq X$.

$\square$

The spaces in (1.1) are sheaves in the above sense. In the case, for instance, of the space $\mathcal{C}^0$ considered on any given open subset $X = \Omega \subseteq \mathbb{R}$, we define $F(U)$, for any open subset $U \subseteq X$, as being the set of $\mathcal{C}^0$ functions on U. And for open subsets $U \subseteq V \subseteq \Omega$, we define the mapping $\rho_{V,\, U} : F(V) \longrightarrow F(U)$ as the usual restriction of functions on V to functions on U.

Now, it is easy to see that *none* of the spaces in (1.1) is a flabby sheaf, no matter on which open subset $\Omega \subseteq \mathbb{R}$ they are considered. This is precisely the reason why the above NO EXTENSION property holds for all these spaces.

By the way, most of the usual spaces of generalized functions, such as the Schwartz distributions, or the Sobolev spaces fail to be flabby sheaves, [219], thus they are subjected to the above NO EXTENSION property.
The Colombeau algebras also *fail* to be flabby sheaves - due to the growth conditions in their definition - thus they as well are subjected to the above NO EXTENSION property.

On the other hand, the nowhere dense differential algebras of generalized functions, as well as those of space-time foam are flabby sheaves, [54,55,57,70].
Consequently, these two types of differential algebras of generalized functions do *not* suffer from the above inconvenient NO EXTENSION property when dealing with singularities.

3. The Natural Role of Closed, Nowhere Dense Subsets

Regarding the spaces

$$(3.1) \qquad \mathcal{C}^\infty \underset{\neq}{\subseteq} \ldots \underset{\neq}{\subseteq} \mathcal{C}^n \underset{\neq}{\subseteq} \ldots \underset{\neq}{\subseteq} \mathcal{C}^0$$

it is easy to see that each of them can naturally be embedded into a *smallest flabby sheaf*, [8], namely

$$(3.2) \qquad \mathcal{C}^n \underset{\neq}{\subseteq} \mathcal{C}^n_{nd}, \qquad 0 \leq n \leq \infty$$

where, for a given open subset $\Omega \subseteq \mathbb{R}$, we have

$$(3.3) \quad \mathcal{C}^n_{nd} = \left\{ f : \Omega \longrightarrow \mathbb{R} \;\middle|\; \begin{array}{l} \exists \;\; \Gamma \subset \Omega, \; \Gamma \;\; \text{closed, nowhere dense} \;\; : \\ \\ \quad f|_{\,\Omega \setminus \Gamma} \in \mathcal{C}^n \end{array} \right\}$$

And obviously, similar with (3.1), we have

$$(3.4) \qquad \mathcal{C}^\infty_{nd} \underset{\neq}{\subseteq} \ldots \underset{\neq}{\subseteq} \mathcal{C}^n_{nd} \underset{\neq}{\subseteq} \ldots \underset{\neq}{\subseteq} \mathcal{C}^0_{nd}$$

Remark 3.1.

It is important to point out that the closed, nowhere dense sets turn out to play a fundamental role also in the order completion method for solving large classes of nonlinear systems of PDEs, with possibly associated initial and/or boundary value problems, [8].

As is well known, [238], such closed, nowhere dense subsets in Euclidean spaces are "small" from topological point of view, since their complementaries are open and dense. However, from measure point of view they can be arbitrary "large". Namely, if $\Omega \subseteq \mathbb{R}^n$ is open, then for every $\epsilon > 0$, one can find $\Gamma \subset \Omega$ closed, nowhere dense, such that the Lebesgue measure of $\Omega \setminus \Gamma$ is less than ϵ.

Also, it should be recalled that closed, nowhere dense subsets play a basic role in the Baire Category argument, since sets of Baire Category I are precisely countable unions of such closed, nowhere dense sets.

4. Singularity Free and Global Algebraic and Differential Operations

One can note that the inclusions (3.2) and the flabbiness of their right hand terms do not help much in dealing with singularities, even if the spaces in (3.4) are in fact algebras, while $\mathcal{C}^{\infty}_{nd}$ are also differential algebras.
Indeed, let be given an open subset $\Omega \subseteq \mathbb{R}$. The problem with the spaces in (3.4) is that each function f in them has its own closed, nowhere dense set $\Gamma_f \subset \Omega$ of singularities, and none of these functions is defined globally on the whole of Ω, unless we are in the particular case of a function f with $\Gamma_f = \phi$.

Thus in *none* of the spaces in (3.4) can one perform *singularity free* and *globally* the usual algebraic and differential operations, without having all the time to keep track of and take into account the singularity sets of the functions involved in such operations.
Furthermore, a major obstacle appears with limits, series, and other such operations which involve infinitely many functions. Namely, countable, and in general, infinite unions of closed, nowhere dense sets need no longer be closed, nowhere dense. Thus when performing such operations on functions with closed, nowhere dense singularities, one can easily fall outside of the spaces in (3.4).

These two difficulties are among the reasons various spaces of generalized functions have been introduced, spaces in which one can operate *globally* on their elements, that is, the generalized functions, and do so *singularity free*, simply as if they did not have any singularities at all.

Remark 4.1.

Amusingly, and as seen for instance in the Appendix, the above is not yet quite well understood by some of those involved in the nonlinear theory of generalized functions.

□

One classical instance in the case of Schwartz distributions illustrates the above quite clearly. Let us, indeed, return to the Heaviside function in (1.7). This function belongs to the space of distribution $\mathcal{D}'(\mathbb{R})$, through the well known embedding

$$(4.1) \qquad \mathcal{L}_{loc}(\mathbb{R}) \ni f \longmapsto T_f \in \mathcal{D}'(\mathbb{R})$$

where

$$(4.2) \qquad \mathcal{D}(\mathbb{R}) \ni \varphi \longmapsto T_f(\varphi) = \int_{\mathbb{R}} f(x)\varphi(x)dx \in \mathbb{R}$$

Thus we obtain $T_H \in \mathcal{D}'(\mathbb{R})$, and since $\mathcal{D}'(\mathbb{R})$ is a vector space over $\mathbb{R}$, as well as a module over $\mathcal{C}^\infty(\mathbb{R})$, and furthermore, its elements are infinitely differentiable generalized functions, it follows that all the respective algebraic and differential operations can singularity free and globally be performed upon T_H, without any concern whatsoever about the singularities in the sense of the usual context of (1.1), or for that matter, (3.4), which may happen to be involved due possibly to other distributions in the respective operations, including the singularity at $0 \in \mathbb{R}$ of H itself, of course.

Let us recapitulate, in order to avoid possible misunderstandings. We note that we obviously have

$$(4.3) \qquad H \in \mathcal{C}^\infty_{nd}(\mathbb{R}) = \bigcup{}_{\Gamma \subset \mathbb{R},\ closed,\ nowhere\ dense}\ \mathcal{C}^\infty(\mathbb{R} \setminus \Gamma)$$

$$(4.4) \qquad T_H \in \mathcal{D}'(\mathbb{R})$$

$$(4.5) \qquad \mathcal{C}^\infty_{nd}(\mathbb{R}) \not\subseteq \mathcal{D}'(\mathbb{R})$$

with the last relation being a consequence of the fact that, unlike $\mathcal{C}^\infty_{nd}(\mathbb{R})$, the space $\mathcal{D}'(\mathbb{R})$ of Schwartz distributions is not a flabby sheaf.

And then, the rather immediate, albeit quite naive question may perhaps arise :

- Why should the Heaviside function H be considered in the rather involved distributional manner of T_H in (4.4), instead of being

dealt with in its easy, direct and natural setup in (4.3) ?
After all, $\mathcal{C}^\infty_{nd}(\mathbb{R})$ in (4.3) is a differential algebra, while $\mathcal{D}'(\mathbb{R})$ in (4.4) is not ?

The answer is obvious :

- Although $\mathcal{C}^\infty_{nd}(\mathbb{R})$ in (4.3) is a differential algebra, the usual algebraic and differential operations *cannot* be performed *singularity free* and *globally* on the functions which are its elements without having all the time to keep track of and take into account the singularity sets of the functions involved. Furthermore, operations such as limits, series, and other ones which involve infinitely many functions may easily fall outside of $\mathcal{C}^\infty_{nd}(\mathbb{R})$.

- On the other hand, even if $\mathcal{D}'(\mathbb{R})$ in (4.4) is not an algebra, all linear algebraic and differential operations can *globally* and *singularity free* be performed, without any regard to the singularities of the generalized functions involved. Furthermore, a large amount of operations, such as limits, series, and so on, can also be performed globally and singularity free, even if infinitely many generalized functions happen to be involved.

In conclusion

- the essential difference in (4.4) between dealing with the framework of usual functions in $\mathcal{C}^\infty_{nd}(\mathbb{R})$, or on the contrary, with the framework of distributions in $\mathcal{D}'(\mathbb{R})$, is that in the latter, one can perform the algebraic and differential operations *globally* and *singularity free*.

The deficiency with the framework in (4.4), however, is nevertheless *twofold*, namely :

- $\mathcal{D}'(\mathbb{R})$ is only a vector space, and *not* an algebra as well, thus it is severely restricted with respect to nonlinear operations,

- $\mathcal{D}'(\mathbb{R})$ is *not* a flabby sheaf, thus it is severely restricted with respect to the singularities it can deal with.

And so it comes to pass that :

5. We Need Differential Algebras of Generalized Functions, and They Can Easily Be Constructed ...

With hindsight, it may be said that it was quite a puzzling *oversight* that the first spaces of generalized functions, namely, those of Sobolev in the 1930s, and of Schwartz in 1940s, ended up being *only* vector spaces, and *not* as well algebras. Indeed, let us recall the case of the space $\mathcal{D}'(\Omega)$ of Schwartz distributions, where $\Omega \subseteq \mathbb{R}^n$ is any open subset. One of the first basic structural results about that space was the *quotient vector space* representation

$$(5.1) \qquad \mathcal{D}'(\Omega) = \mathcal{S}^\infty(\Omega)/\mathcal{V}^\infty(\Omega)$$

where

$$(5.2) \qquad \mathcal{V}^\infty(\Omega) \subset \mathcal{S}^\infty(\Omega) \subset (\mathcal{C}^\infty(\Omega))^{\mathbb{N}}$$

with $\mathcal{S}^\infty(\Omega)$ being the vector space of sequences of smooth functions $s = (s_\nu)_{\nu \in \mathbb{N}} \in (\mathcal{C}^\infty(\Omega))^{\mathbb{N}}$ which converge weakly distributionally - and thus also strongly strongly distributionally - to some distribution in $\mathcal{D}'(\Omega)$, while $\mathcal{V}^\infty(\Omega)$ is its vector subspace of those sequences which converge distributionally to zero.

Now clearly, $(\mathcal{C}^\infty(\Omega))^{\mathbb{N}}$ in (5.2) is a *differential algebra* with the termwise operations on sequences of smooth functions. Thus instead of (5.2), one could have from the beginning considered

$$(5.3) \qquad \mathcal{I}^\infty(\Omega) \subset \mathcal{A}^\infty(\Omega) \subseteq (\mathcal{C}^\infty(\Omega))^{\mathbb{N}}$$

where $\mathcal{A}^\infty(\Omega)$ is a *subalgebra* in $(\mathcal{C}^\infty(\Omega))^{\mathbb{N}}$, while $\mathcal{I}^\infty(\Omega)$ is an *ideal* in $\mathcal{A}^\infty(\Omega)$, in which case, instead of the vector space of generalized functions in (5.1), one would obtain an *algebra* of generalized functions

$$(5.4) \qquad \mathbb{A}(\Omega) = \mathcal{A}^\infty(\Omega)/\mathcal{I}^\infty(\Omega)$$

And by requesting that $\mathcal{A}^\infty(\Omega)$ and $\mathcal{I}^\infty(\Omega)$ in (5.3) be invariant under differentiation, one would so simply and easily obtain in (5.4) a *differential algebra of generalized functions* !

An immediate result relating to the fact that the algebras (5.4) are indeed algebras of *generalized functions*, that is, they contain the usual smooth functions, is in the following fact, [3,4,6,7,9] :

The *necessary* and *sufficient* condition for the existence of the embedding of differential algebras

(5.6) $\quad \mathcal{C}^\infty(\Omega) \subset \mathbb{A}(\Omega)$

through the mapping

(5.7) $\quad \mathcal{C}^\infty(\Omega) \ni f \longmapsto u(f) = (f, f, f, \ldots) + \mathcal{I}^\infty(\Omega) \in \mathbb{A}(\Omega)$

is the *off-diagonality* condition

(5.8) $\quad \mathcal{I}^\infty(\Omega) \bigcap \mathcal{U}^\infty(\Omega) = \{0\}$

where we denoted by $\mathcal{U}^\infty(\Omega)$ the *diagonal* in $(\mathcal{C}^\infty(\Omega))^{\mathbb{N}}$, that is, the subalgebra of all sequences of smooth functions $u(f) = (f, f, f, \ldots)$, with $f \in \mathcal{C}^\infty(\Omega)$.

What is further remarkable is that the off-diagonality condition (5.8) is also *sufficient* for the embedding of the Schwartz distributions in the algebras (5.4), namely, [3,4,6,7,9]

(5.9) $\quad \mathcal{D}'(\Omega) \subset \mathbb{A}(\Omega)$

Remark 5.1.

Amusingly, and quite unknown to most of those in linear distribution theory, or for that matter, to many in the nonlinear theory of generalized functions, the constructions in (5.1) - (5.4) above are all closely related to what are called *reduced powers* in model theory, which is a modern branch of mathematical logic.
And similar constructions occur in important other branches of mathematics. For instance, such are the Cauchy-Bolzano constructions of $\mathbb{R}$ from $\mathbb{Q}$, or more generally, the completion of a matric space, and in

fact, the completion of an uniform topological space.

□

A fundamental result related to the ability of various differential algebras of generalized functions to deal with singularities is presented in, [8]

Theorem 5.1. Basic Dichotomic Singularity Test

Given a differential algebra of generalized functions, see (5.4)

$$(5.10) \qquad \mathbb{A}(\Omega) = \mathcal{A}^\infty(\Omega)/\mathcal{I}^\infty(\Omega)$$

Then the necessary and sufficient condition for the existence of the embedding of differential algebras

$$(5.11) \qquad \mathcal{C}^\infty_{nd}(\Omega) \subset \mathbb{A}(\Omega)$$

is that $\mathbb{A}(\Omega)$ be a *flabby sheaf*.

Remark 5.2.

Amusingly, many in the nonlinear theory of generalized functions disregard the above basic dichotomic singularity test and prefer to work in algebras which fail to be flabby sheaves, thus having to give up dealing with large classes of singularities.
Two of the mentioned important problems which cannot even be formulated, let alone solved, in algebras which fail the above basic dichotomic singularity test are :

- the global Cauchy-Kovalevskaia theorem,
- arbitrary global Lie group actions on solutions of PDEs.

A typical case in this regard is that of the Colombeau algebras.

Once again, however, it is worth pointing out that each and every differential algebra of generalized functions may present advantages in certain regards. And in view of the inevitable infinite branching of

multiplication above certain levels of singularities, it is useful not to disregard any such algebra, be it flabby sheaf, or not.
What, nevertheless, should be avoided is to chase for certain results in the wrong algebras.
And to expect that large enough classes of singularities can be dealt with in algebras which fail to be flabby sheaves is but to fail to understand the mathematical structure underlying the above basic dichotomic singularity test.

6. Keep in Mind the Regularizations of Singularities !

Before going further, let us see what happens to the representation T_H in (4.4) of the Heaviside function H, when the representation (5.1) is used for the Schwartz distributions.

Well, the story is quite simple, indeed. Namely, one *regularizes the singularity* at $0 \in \mathbb{R}$ of the Heaviside function H, as follows.

One takes any smooth function $\chi \in \mathcal{C}^\infty(\mathbb{R})$ with the property, [3]

$$(6.1) \qquad \chi = 0 \;\text{ on }\; (-\infty, -1], \qquad \chi = 1 \;\text{ on }\; [1, \infty)$$

and defines a *regularization of singularity* at $0 \in \mathbb{R}$ of the Heaviside function H as being given by the sequence of smooth functions

$$(6.2) \qquad h = (h_\nu)_{\nu \in \mathbb{N}} \in (\mathcal{C}^\infty(\mathbb{R}))^{\mathbb{N}}$$

where

$$(6.3) \qquad h_\nu(x) = \chi(nx), \quad \nu \in \mathbb{R}, \; n \in \mathbb{N}$$

and thus, one obtains

$$(6.4) \qquad h = (h_\nu)_{\nu \in \mathbb{N}} \in \mathcal{S}^\infty(\mathbb{R})$$

as well as

$$(6.5) \qquad T_H = h + \mathcal{V}^\infty(\mathbb{R}) \in \mathcal{S}^\infty(\mathbb{R})/\mathcal{V}^\infty(\mathbb{R}) = \mathcal{D}'(\mathbb{R})$$

Remark 6.1.

The usual Heaviside function

$$(6.6) \qquad H \in \mathcal{C}^\infty_{nd}(\mathbb{R})$$

has therefore no less than *two* representations as a Schwartz distribution, namely

(6.7) $\quad T_H$ in (4.1), (4.2)

and on the other hand, through the *regularization* in (6.1) - (6.5)

(6.8) $\quad T_H$ in (6.5)

and clearly, the representation in (6.7) does not much seem to be amenable for a nonlinear theory of generalized functions, while on the other hand, as seen in (6.1) - (6.4), and in view of section 5, the representation (6.8) is naturally part of an approach to generalized functions which are elements of differential algebras.

And once again, in order to avoid possible misunderstandings, it should be noted that no one can claim that the representations (6.7), or (6.8) do not bring in anything new, and that they are in fact nothing else but the original form of the Heaviside function in (4.3).

Indeed, whenever one performs algebraic or differential operations with the latter form of the Heaviside function, one has to keep track of and take into account its singularity at $x = 0 \in \mathbb{R}$. Thus such operations are *not* singularity free and hence global.

On the other hand, with the representations (6.7), or (6.8) of the Heaviside function, one can in a *singularity free* manner perform algebraic and differential operations, and do so *globally* on $\mathbb{R}$, as if no singularities would exist at all. Furthermore, one can similarly *singularity free* and *globally* perform operations such as limits, series, and so on, in which infinitely many functions are involved.

In short, the essential difference between the Heaviside function in (4.3), and on the other hand, that in (6.5), (6.8) is precisely that in the latter a *regularization of singularities* was performed, and as a consequence, one can *globally* and in a *singularity free* manner perform all the algebraic and differential operations on that latter form of the Heaviside function.

7. Nowhere Dense Differential Algebras of Generalized Functions

These algebras are of the form, [4,6,7,9]

(7.1) $\quad \mathbb{A}_{nd}(\Omega) = (\mathcal{C}^\infty(\Omega))^{\mathbb{N}} / \mathcal{I}_{nd}^\infty(\Omega)$

where $\Omega \subseteq \mathbb{R}^n$ are open, while $\mathcal{I}_{nd}^\infty(\Omega)$ is the ideal in $(\mathcal{C}^\infty(\Omega))^{\mathbb{N}}$ consisting of all the sequences of smooth functions $w = (w_\nu)_{\nu \in \mathbb{N}}$ which satisfy the asymptotic vanishing condition

$$(7.2) \quad \begin{array}{l} \exists \ \ \Gamma \subset \Omega, \ \Gamma \text{ closed, nowehre dense } : \\ \\ \forall \ \ x \in \Omega \setminus \Gamma \ : \\ \\ \exists \ \ V \subseteq \Omega \setminus \Gamma, \ V \text{ neighbourhood of } x, \ \mu \in \mathbb{N} \ : \\ \\ \quad w_\nu = 0 \text{ on } V, \quad \nu \in \mathbb{N}, \ \nu \geq \mu \end{array}$$

We note that these ideals have the *off-diagonality* property

(7.3) $\quad \mathcal{I}_{nd}^\infty(\Omega) \bigcap \mathcal{U}^\infty(\Omega) = \{0\}$

where, we recall, we denoted by $\mathcal{U}^\infty(\Omega)$ the *diagonal* in $(\mathcal{C}^\infty(\Omega))^{\mathbb{N}}$, that is, the subalgebra of all sequences of smooth functions $u(f) = (f, f, f, \ldots)$, with $f \in \mathcal{C}^\infty(\Omega)$.

We note that the ideal $\mathcal{I}_{nd}^\infty(\Omega)$ is invariant under term-wise derivation, thus

(7.4) $\quad \mathbb{A}_{nd}(\Omega)$ is a differential algebra

Furthermore, in view of the off-diagonality characterization in (5.8), (5.9) of all such algebras, [3,4,6,7,9], it contains the Schwartz distributions, namely

(7.5) $\quad \mathcal{D}'(\Omega) \subset \mathbb{A}_{nd}(\Omega)$

Owing to (7.3), we also have the embedding of differential algebras

$$(7.6) \qquad \mathcal{C}^\infty(\Omega) \ni f \longmapsto u(f) + \mathcal{I}_{nd}^\infty(\Omega) \in \mathbb{A}_{nd}(\Omega)$$

An essential property of the nowhere dense algebras (7.1) is that they are *flabby sheaves*, [54].

Remark 7.1.

1) The essential feature of the nowhere dense differential algebras of generalized functions $\mathbb{A}_{nd}(\Omega)$ in (7.1) is due to the respective ideals $\mathcal{I}_{nd}^\infty(\Omega)$. And on their turn, the essential features of these ideals are the following :

- The sequences in these ideals have an *extreme dichotomic* property. Namely, outside of the respective singularity sets Γ they vanish asymptotically, while in the neighbourhood of the singularity sets Γ they can be arbitrary. Otherwise, the functions in these sequences must be smooth on the whole of the respective domains Ω.

- The conditions which define these ideals are purely in terms of the topology of Euclidean spaces, as they are induced on the respective domains of definition Ω.

Thus the nowhere dense differential algebras of generalized functions are defined exclusively through terms which are algebraic, more precisely, ring theoretic, and Euclidean topological.

2) The nowhere dense differential algebras of generalized functions can be defined in more general settings, namely, by taking $\mathcal{I}_{nd}^\infty(\Omega)$ as ideals algebras $(\mathcal{C}^\infty(\Omega))^\Lambda$, where Λ are arbitrary given infinite index sets which are right directed, [7].
However, as seen in [4,6,7,9], even the particular case when $\Lambda = \mathbb{N}$ is useful enough in order to lead to the solution of a number of important problems.

8. Again : Keep in Mind the Regularizations of Singularities !

As a consequence of the fact that the nowhere dense algebras (7.1) are flabby sheaves, as well as of (3.2), we have the embedding of differential algebras

$$(8.1) \qquad \mathcal{C}^\infty_{nd}(\Omega) \subset \mathbb{A}_{nd}(\Omega)$$

which are obtained according to a *regularization of singularities* which in its essence is identical to that of the Heaviside function H in (6.1) - (6.5).

Namely, let $\eta : \mathbb{R} \longrightarrow [0,1]$ be $\mathcal{C}^\infty$-smooth, such that, [236]

$$(8.2) \qquad \eta(x) \;=\; \left| \begin{array}{l} 0 \;\; \text{for} \;\; |\, x \,| \leq a \\ \\ 1 \;\; \text{for} \;\; |\, x \,| \geq b \end{array} \right.$$

for some $0 < a < b < \infty$.

Given now $f \in \mathcal{C}^\infty_{nd}(\Omega)$, let $\Gamma \subset \Omega$ be closed and nowhere dense, such that $f \in \mathcal{C}^\infty(\Omega \setminus \Gamma)$. Then there exists $\gamma : \mathbb{R} \longrightarrow \mathbb{R}$ and $\mathcal{C}^\infty$-smooth, such that, [236]

$$(8.3) \qquad \Gamma = \{\, x \in \Omega \mid \gamma(x) = 0 \,\}$$

We define now a *regularization of singularities* of the function f as being given by the sequence of smooth functions

$$(8.4) \qquad \psi = (\psi_\nu)_{\nu \in \mathbb{N}} \in (\mathcal{C}^\infty(\Omega))^{\mathbb{N}}$$

where, for $\nu \in \mathbb{N}$, we have

$$(8.5) \qquad \psi_\nu(x) \;=\; \left| \begin{array}{l} \eta((\nu+1)\gamma(x)f(x)) \;\; \text{for} \;\; x \in \Omega \setminus \Gamma \\ \\ 0 \;\; \text{for} \;\; x \in \Gamma \end{array} \right.$$

And now, the embedding (8.1) of differential algebras is obtained through

$$(8.6) \qquad \mathcal{C}^\infty_{nd}(\Omega) \ni f \longmapsto \psi + \mathcal{I}^\infty_{nd}(\Omega) \in \mathbb{A}_{nd}(\Omega)$$

where we note the independence of this mapping from the particular functions η in (8.2) and γ in (8.3), [4,6,7,9].

Remark 8.1.

1) As seen in the sequel, differential algebras of generalized functions $\mathbb{A}$ which are not flabby sheaves, among them the Colombeau algebras, do *not* allow embeddings $\mathcal{C}^\infty_{nd} \subset \mathbb{A}$, this fact having the negative consequences mentioned, for instance, in Remark 5.2. above.

An elementary an immediate test in this regard is to check whether algebras which fail to be flabby sheaves may contain such singular functions as those in Example 1.4., suggested by the Great Picard Theorem.

Certainly, and as seen below, the Colombeau algebras are not able to contain such functions.

On the other hand, the functions in Example 1.4. belong obviously to $\mathcal{C}^\infty_{nd}(\mathbb{R})$, thus in view of (8.1), they also belong to $\mathbb{A}_{nd}(\mathbb{R})$.

2) It is important to note that in the inclusion (8.1), the nowhere dense differential algebras of generalized functions $\mathbb{A}_{nd}(\Omega)$ are considerably *larger* than the differential algebras of singular functions $\mathcal{C}^\infty_{nd}(\Omega)$. This means, of course, that the former contain generalized functions with far *worse* singularities than those of the singular functions in the latter. To give a generic example in this regard, one obviously has generalized functions

$$(8.7) \qquad F = (f_\nu)_{\nu \in \mathbb{N}} + \mathcal{I}^\infty_{nd}(\Omega) \in \mathbb{A}_{nd}(\Omega)$$

where the sequence of smooth functions $(f_\nu)_{\nu \in \mathbb{N}} \in (\mathcal{C}^\infty(\Omega))^{\mathbb{N}}$ can be arbitrary.

Such a case, therefore, may be seen to correspond to the situation of maximum size of the set of singularity points, namely, the whole of Ω, of the set of singular points of the respective generalized functions F, as well as to the most arbitrary behaviour of such generalized functions in the neighbourhood of their singularities.

3) It should also be noted that, upon the usefulness of eliminating such extreme forms of singularities, one can easily consider appropriate differential subalgebras of $\mathbb{A}_{nd}(\Omega)$, as follows. Let $\mathcal{B}(\Omega)$ be any subalgebra in $(\mathcal{C}^\infty(\Omega))^{\mathbb{N}}$ which is invariant under differentiation, and it is such that

$$(8.8) \qquad \mathcal{I}^\infty_{nd}(\Omega) + \mathcal{U}^\infty(\Omega) \subset \mathcal{B}(\Omega)$$

Then obviously

$$(8.9) \qquad \mathbb{B}_{nd}(\Omega) = \mathcal{B}(\Omega)/\mathcal{I}^\infty_{nd}(\Omega)$$

will be a differential algebra of generalized functions which is a subalgebra of $\mathbb{A}_{nd}(\Omega)$.

8.1. Global Solutions in the Cauchy-Kovalevskaia Theorem

Let us briefly review the way the nowhere dense differential algebras of generalized functions $\mathbb{A}_{nd}(\Omega)$ allow not only a formulation of the Global Cauchy-Kovalevskaia Theorem, but also its solution, [6,7,39,61,99].

We consider the *general nonlinear analytic* partial differential operator

$$(8.10) \qquad T(x,D)U(x) \;=\; D_t^m U(t,y) - G(t,y,...,D_t^p D_y^q U(t,y),...)$$

where $U : \Omega \longrightarrow \mathbb{C}$ is the unknown function, while $x = (t,y) \in \Omega,\ t \in \mathbb{R},\ y \in \mathbb{R}^{n-1},\ p \in \mathbb{N},\ 0 \leq p < m,\ q \in \mathbb{N}^{n-1},\ p + |q| \leq m$, and G is arbitrary analytic in all of its variables.

Together with the analytic nonlinear PDE

$$(8.11) \qquad T(x,D)U(x) \;=\; 0, \quad x \in \Omega$$

we consider the non-characteristic analytic hypersurface

$$(8.12) \qquad S \;=\; \{\; x \;=\; (t,y) \in \Omega \;\mid\; t \;=\; t_0 \;\}$$

for any given $t_0 \in \mathbb{R}$, and on it, we consider the analytic initial value problem

$$(8.13) \qquad D_t^p U(t_0,y) \;=\; g_p(y), \quad 0 \leq p < m, \quad (t_0,y) \in S$$

Obviously, the analytic nonlinear partial differential operator $T(x,D)$ in (8.10) generates a mapping

$$(8.14) \qquad T(x,D) : \mathcal{C}^\infty(\Omega) \;\longrightarrow\; \mathcal{C}^\infty(\Omega)$$

and in view of (7.2), also a mapping, [6,7,39,61,99]

$$(8.15) \qquad T(x,D) : A_{nd}(\Omega) \;\longrightarrow\; A_{nd}(\Omega)$$

And then, within this setup, we obtain the *global existence* result in the nowhere dense differential algebras of generalized functions, given by

Theorem 8.1.

The analytic nonlinear PDE in (8.11), with the analytic non-characteristic initial value problem (8.12), (8.13) has *global* generalized solutions

$$(8.16) \qquad U \in A_{nd}(\Omega)$$

defined on the whole of Ω. These solutions U are *analytic* functions

$$(8.17) \qquad \psi : \Omega \setminus \Sigma \;\longrightarrow\; \mathbb{C}$$

when restricted to the *open dense* subsets $\Omega \setminus \Sigma$, where the singularity subsets

(8.18) $\quad \Sigma \subset \Omega, \quad \Sigma$ closed, nowhere dense in Ω

can be suitably chosen. Furthermore, one can choose Σ to have zero Lebesgue measure, namely

(8.19) $\quad \text{mes } \Sigma = 0$

Remark 8.2.

1) The proof of this theorem, [6,7,39,61,99], is in two steps :

- first, an analytic solution ψ, which in general is not unique, is constructed on $\Omega \setminus \Sigma$, thus we have for such solution $\psi \in \mathcal{C}^\infty_{nd}(\Omega)$,
- then, as the second step, ψ is subjected to a *regularization of singularities* according to (8.2) - (8.6), obtaining thus on the whole of Ω, the *global generalized solution* $U \in \mathbb{A}_{nd}(\Omega)$.

2) The mentioned singularity regularization process in step two above, see beginning of section 0, namely

(8.20) $\quad \mathcal{C}^\infty_{nd}(\Omega) \ni \psi \longmapsto U \in \mathbb{A}_{nd}(\Omega)$

gives a *global solution* precisely in the same way as the singularity regularizations for the Heaviside function H in (6.7), or for that matter, in (6.8), give within the vector space of Schwartz distributions $\mathcal{D}'(\mathbb{R})$ a *global solution* to the linear first order ODE in H, namely

(8.21) $\quad DH = \delta$

where δ is the Dirac distribution.

And such *singularity regularizations* are, as mentioned, precisely some of the main reasons for constructing spaces of generalized functions, be they merely vector spaces or algebras.

Consequently, the global solutions obtained in the above Cauchy-Kovalevskaia theorem happen to have the same legitimacy, interest and relevance with, for instance, the solution H of the linear ODE in

(8.21), or for that matter, with the known distribution solutions of any PDE.

3) As can be seen even in the simple case of analytic ODEs with finite time blow up in solutions, the respective solutions will always belong to $\mathcal{C}^\infty(\mathbb{R} \setminus \Sigma)$, for suitable closed, nowhere dense subsets $\Sigma \subset \mathbb{R}$. Thus in view of (8.1), they will always belong to the nowhere dense differential algebra of generalized functions $\mathbb{A}_{nd}(\mathbb{R})$.

On the other hand, such solutions may often fail to belong to various spaces of distributions, Sobolev spaces, or differential algebras of generalized functions which happen not to be flabby sheaves, as for instance is the case with the Colombeau algebras.
In view of the above, in Colombeau algebras, as much as in other differential algebras of generalized functions, or vector spaces of distributions and Sobolev spaces which fail to be flabby sheaves, it is not even possible to formulate a global version of the classical Cauchy-Kovalevskaia theorem.

8.2. Arbitrary Global Lie Group Actions, and a Complete Solution to Hilbert's Fifth Problem

The utility of the fact that the nowhere dense differential algebras of generalized functions are flabby sheaves was shown also in obtaining the following two results, both of them for the first time in the literature, [9]

- the construction of global action of arbitrary Lie groups on $\mathcal{C}^\infty(\Omega)$, with $\Omega \subseteq \mathbb{R}^n$ open,
- the complete solution of Hilbert's Fifth Problem.

And once again, these problems cannot even be formulated, let alone the respective results obtained, in vector spaces of distributions, Sobolev spaces, or differential algebras of generalized functions which fail to be flabby sheaves, among them, the Colombeau algebras.

8.3. General Nonlinear Operations on Nowhere Dense Algebras

Since the nowhere dense differential algebras of generalized functions $A_{nd}(\Omega)$ are in particular algebras, clearly, all polynomial nonlinear operations with coefficients in $\mathcal{C}^\infty(\Omega)$ can be performed in these algebras.

Furthermore, in view of (8.1), all polynomial operations with discontinuous coefficients in $\mathcal{C}^\infty_{nd}(\Omega)$ can also be performed in these algebras.

And one can even go beyond polynomial nonlinear operations. Namely, let $F : \mathbb{R}^m \longrightarrow \mathbb{R}$ be an arbitrary $\mathcal{C}^\infty$-smooth function. Further, for $1 \leq i \leq m$, let be given the generalized function

$$(8.22) \qquad f_i = (f_{i,\nu})_{\nu \in \mathbb{N}} + \mathcal{I}^\infty_{nd}(\Omega) \in A_{nd}(\Omega)$$

then one can define the generalized function

$$(8.23) \qquad F(f_1, \ldots, f_m) \in A_{nd}(\Omega)$$

by

$$(8.24) \qquad F(f_1, \ldots, f_m) = (f_\nu)_{\nu \in \mathbb{N}} + \mathcal{I}^\infty_{nd}(\Omega)$$

where

$$(8.25) \qquad f_\nu(x) = F(f_{1,\nu}(x), \ldots, f_{m,\nu}(x)), \quad \nu \in \mathbb{N}, \quad x \in \Omega$$

and in view of (7.2), $F(f_1, \ldots, f_m)$ in (8.23) does not depend on the representations of $f_1, \ldots, f_m$ in (8.22).

9. The Colombeau Algebras

We shall only deal with the most general version of these algebras, as introduced in [172]. As pointed out in [6,7], these algebras have a natural character from several important points of view.

However, dealing with the largest possible families of singularities turned out, in fact, not be an aim systematically enough pursued in their construction. Or even if it happened that it was, it was actually not achieved anywhere near to satisfaction, when sets of singularity points are considered from the two relevant points of view, namely, how large they are, and how freely can generalized functions behave in their neighbourhood.

Indeed, when it comes to the second criterion, the Colombeau algebras show a significant deficiency, since they are not flabby sheaves.

In this regard one has to note that the essential aim in introducing generalized functions, be they within a merely linear theory, or in fact within a nonlinear one, is precisely to deal with singularities of usual functions. And therefore, the larger the class of singularities that can be dealt with by generalized functions, the better ...
It is in this way rather strange that the flabbiness of spaces of generalized functions - so clearly and strongly related to their ability to deal with large classes of singularities - has not yet been sufficiently realized in wide enough circles ...

Let us now present briefly the respective facts about the Colombeau algebras, [172].

For $m \geq 1$, one denotes

$$(9.1) \quad \Phi_m = \left\{ \phi \in \mathcal{D}(\mathbb{R}^n) \;\middle|\; \begin{array}{l} *) \;\; \int_{\mathbb{R}^n} \phi(x)dx = 1 \\ \\ **) \;\; \int_{\mathbb{R}^n} x^p \phi(x)dx = 0, \quad p \in \mathbb{N}^n, \quad 1 \leq |\,p\,| \leq m \end{array} \right\}$$

and takes as index set

$$(9.2) \qquad \Phi \ = \ \Phi_1$$

Further, one takes

$$(9.3) \qquad \mathcal{I}(\mathbb{R}^n) \subset \mathcal{A}(\mathbb{R}^n) \subset (\mathcal{C}^\infty(\mathbb{R}^n))^\Phi$$

Here $\mathcal{A}(\mathbb{R}^n)$ is the subalgebra in $(\mathcal{C}^\infty(\mathbb{R}^n))^\Phi$ of elements $f = (f_\phi)_{\phi \in \Phi}$ for which

$$(9.4) \qquad \begin{array}{l} \forall \ \ K \subset \mathbb{R}^n, \ \ K \text{ compact}, \ \ p \in \mathbb{N}^n \ : \\[1ex] \exists \ \ m \in \mathbb{N}, \ \ m \geq 1 \ : \\[1ex] \forall \ \ \phi \in \Phi_m \ : \\[1ex] \exists \ \ \eta, \ \ c > 0 \ : \\[1ex] \forall \ \ x \in K, \ \ \epsilon \in (0, \eta) \ : \\[1ex] \qquad | \, D^p f_{\phi_\epsilon}(x) \, | \leq c/\epsilon^m \end{array}$$

where $\phi_\epsilon(x) = \phi(x/\epsilon)/\epsilon^n$.

Further, $\mathcal{I}(\mathbb{R}^n)$ is the ideal in $\mathcal{A}(\mathbb{R}^n)$ of elements $g = (g_\phi)_{\phi \in \Phi}$ for which

$$(9.5) \qquad \begin{array}{l} \forall \ \ K \subset \mathbb{R}^n, \ \ K \text{ compact}, \ \ p \in \mathbb{N}^n \ : \\[1ex] \exists \ \ k \in \mathbb{N}, \ \ k \geq 1, \ \ \beta \in B \ : \\[1ex] \forall \ \ m \in \mathbb{N}, \ \ m \geq k, \ \ \phi \in \Phi_m \ : \\[1ex] \exists \ \ \eta, \ \ c > 0 \ : \\[1ex] \forall \ \ x \in K, \ \ \epsilon \in (0, \eta) \ : \\[1ex] \qquad | \, D^p g_{\phi_\epsilon}(x) \, | \leq c \, \epsilon^{\beta(m) - k} \end{array}$$

where

$$(9.6) \qquad B = \left\{ \beta : \mathbb{N} \longrightarrow (0, \infty) \;\middle|\; \begin{array}{l} *) \;\; \beta \text{ increasing} \\ \\ **) \;\; \lim_{m \to \infty} \beta(m) = \infty \end{array} \right\}$$

Finally, the Colombeau differential algebra of generalized functions is given by

$$(9.7) \qquad \mathcal{G}(\mathbb{R}^n) = \mathcal{A}(\mathbb{R}^n) / \mathcal{I}(\mathbb{R}^n)$$

9.1. The Restrictive Effects of Growth Conditions

The growth conditions (9.4), (9.5) are a characteristic of the Colombeau algebras. And they have multiple restrictive consequences.

One of the first and most obvious ones is that, unlike for instance in the case of the nowhere dense differential algebras of generalized functions, see subsection 8.3. above, one *cannot* perform arbitrary smooth nonlinear operations on Colombeau generalized functions given by $\mathcal{C}^\infty$-smooth functions $F : \mathbb{R}^m \longrightarrow \mathbb{R}$. Instead, such functions F must be limited to *slowly increasing* ones, namely, those that satisfy the condition

$$(9.8) \qquad \begin{array}{l} \forall \;\; p \in \mathbb{N}^m \;\; : \\ \\ \exists \;\; a, \;\; c > 0 \;\; : \\ \\ \forall \;\; y \in \mathbb{R}^m \;\; : \\ \\ \qquad |\, D^p F(y) \,| \leq c(\, 1 + |\, y \,|\,)^a \end{array}$$

Regarding the restrictions on the class of singularities the Colombeau algebras can deal with, one obviously has

$$(9.9) \qquad \mathcal{C}^\infty_{nd}(\mathbb{R}^n) \not\subseteq \mathcal{G}(\mathbb{R}^n)$$

contrary to (8.1), in the case of the nowhere dense differential algebras of generalized functions.

In particular, the functions in Example 1.4. typically do not belong to the Colombeau Algebras.

Also, (9.9) has the effect that polynomial operations with discontinuous coefficients, such as for instance in $\mathcal{C}^\infty_{nd}$, cannot in general be performed in Colombeau algebras, unlike it happens in the case of the nowhere dense differential algebras of generalized functions, see subsection 8.3.

Furthermore, and as mentioned, neither the problem of the global version of the Cauchy-Kovalevskaia theorem, nor that of the global action of Lie groups on smooth functions can even be formulated, let alone solved, in Colombeau algebras.

Amusingly, analysts find it rather irresistible to get involved in mathematical structures which start with the assumption of certain more elaborate conditions, and then keep developing their various consequences for evermore ...
In the case of Colombeau algebras, for instance, the quite complicated polynomial type growth conditions obviously offer such possibilities. Not to mention that they recall the first contact which analysts have with Calculus, namely, elaborate manipulations with their much beloved ϵ ...
How fortunate that the hate of that "Calculus ϵ" is not shared by absolutely all of humankind ...
No, not at all !
After all, analysts do indeed love ϵ, and in fact, love it quite ... unconditionally ...
Even if ϵ only appears - and can in fact only appear - within the formulation of certain conditions ...
Such as for instance the growth conditions defining the Colombeua algebras ...

And then, in addition to the mentioned restrictive effect of the growth conditions upon the ranges of possible applications of Colombeau algebras, come also the technical "ϵ-chasing" complications in the situations when such algebras can be applied.

One example, for instance, is when defining differential algebras of generalized functions on manifolds, a necessity imposed by the need to deal with singularities in general relativity.
In such a situation, the complications involved by the use of Colombeau algebras can be seen, for instance, in [177].

On the other hand, as seen for instance in [54,55,57,59,97], the definition of nowhere dense, as well as space-time foam differential algebras of generalized functions on manifolds is rather immediate.

So much for the passion of chasing ϵ-s ...

10. The Space-Time Foam Algebras

We shall consider various families of singularities in a given open subset $\Omega \subseteq \mathbb{R}^n$, each such family being given by a corresponding set $\mathcal{S}$ of subsets $\Sigma \subset \Omega$, with each such subset Σ describing a possible set of singularities of a certain given generalized function.

The *largest* family of singularities $\Sigma \subset \Omega$ which we can consider is given by

$$(10.1) \qquad \mathcal{S}_{\mathcal{D}}(\Omega) \; = \; \{ \; \Sigma \subset \Omega \;\; | \;\; \Omega \setminus \Sigma \text{ is dense in } \Omega \; \}$$

In this way, the various families $\mathcal{S}$ of singularities $\Sigma \subset \Omega$ which we shall deal with, will each satisfy the condition $\mathcal{S} \subseteq \mathcal{S}_{\mathcal{D}}(\Omega)$.

Among other ones, two such families which will be of interest are the family of singularity sets

$$(10.2) \qquad \mathcal{S}_{nd}(\Omega) \; = \; \{ \; \Sigma \subset \Omega \;\; | \;\; \Sigma \text{ is closed and nowhere dense in } \Omega \; \}$$

which was already studied related to the nowhere dense differential algebras of generalized functions, see sections 7 and 8, as well as the considerably larger family of singularity sets

$$(10.3) \qquad \mathcal{S}_{Baire\ I}(\Omega) \; = \; \{ \; \Sigma \subset \Omega \;\; | \;\; \Sigma \text{ is of first Baire category in } \Omega \; \}$$

Obviously

$$(10.4) \qquad \mathcal{S}_{nd}(\Omega) \; \subset \; \mathcal{S}_{Baire\ I}(\Omega) \; \subset \; \mathcal{S}_{\mathcal{D}}(\Omega)$$

10.1. Families of Singularities and Asymptotically Vanishing Ideals

Let us now recall shortly the idea of the construction of the respective ideals introduced in [55-57,59,61,70,71,97,99,105,143,165]. There are *two* basic ingredients involved.

First, we take any family $\mathcal{S}$ of singularity sets $\Sigma \subset \Omega$, family which satisfies the following two conditions

(10.5) $\quad \forall \ \Sigma \in \mathcal{S} \ : \ \Omega \setminus \Sigma$ is dense in Ω

and

(10.6) $\quad \forall \ \Sigma, \Sigma' \in \mathcal{S} \ : \ \exists \ \Sigma'' \in \mathcal{S} \ : \ \Sigma \cup \Sigma' \subseteq \Sigma''$

Clearly, both families $\mathcal{S}_{nd}$ and $\mathcal{S}_{Baire\ I}$ satisfy the conditions (10.5) and (10.6).

Remark 10.1.

As seen below, given any family $\mathcal{S}$ of singularity sets $\Sigma \subset \Omega$ that satisfies the conditions (10.5), (10.6), the respective differential algebra of generalized functions will be able to deal with generalized functions having their sets of singularity points any such $\Sigma \subset \Omega$.

Here, therefore, it is instructive to recall Example 1.7. and Remark 1.1. Namely, it is *not* possible on Euclidean spaces to have functions whose continuity points are countable and dense.

On the other hand, in view of condition (10.5), in the space-time foam differential algebras of generalized functions defined below, it is possible to have generalized functions whose nonsingular points are merely countable and dense in their domains of definition.

$\square$

Now, as the second ingredient, and so far independently of $\mathcal{S}$ above, we take any right directed partial order $L = (\Lambda, \leq)$.

Although we shall only be interested in singularity sets $\Sigma \in \mathcal{S}_{\mathcal{D}}(\Omega)$, the following ideal can be defined for any $\Sigma \subseteq \Omega$. Indeed, let us denote by

(10.7) $\quad \mathcal{J}_{L,\Sigma}(\Omega)$

the *ideal* in $(\mathcal{C}^\infty(\Omega))^\Lambda$ of all the sequences of smooth functions indexed by $\lambda \in \Lambda$, namely, $w = (w_\lambda)_{\lambda \in \Lambda} \in (\mathcal{C}^\infty(\Omega))^\Lambda$, sequences which *outside* of the singularity set Σ will satisfy the *asymptotic vanishing* condition

$$
(10.8) \qquad
\begin{aligned}
& \forall \;\; x \in \Omega \setminus \Sigma \;\; : \\
& \exists \;\; \lambda \in \Lambda \;\; : \\
& \forall \;\; \mu \in \Lambda, \; \mu \geq \lambda \;\; : \\
& \forall \;\; p \in \mathbb{N}^n \;\; : \\
& \quad D^p w_\mu(x) = 0
\end{aligned}
$$

This means that the sequences of smooth functions $w = (w_\lambda)_{\lambda \in \Lambda}$ in the ideal $\mathcal{J}_{L,\,\Sigma}(\Omega)$ may in a way *cover* with their support the singularity set Σ, and at the same time, they *vanish asymptotically* outside of it, together with all their partial derivatives.
In this way, the ideal $\mathcal{J}_{L,\,\Sigma}(\Omega)$ carries in an *algebraic* manner the information on the singularity set Σ. Therefore, a *quotient* in which the factorization is made with such ideals may in certain ways *do away with* singularities, and do so through purely *algebraic* means, as see below.

We note that the assumption about $L = (\Lambda, \leq)$ being right directed is used in proving that $\mathcal{J}_{L,\,\Sigma}(\Omega)$ is indeed an ideal, more precisely that, for $w, w\,' \in \mathcal{J}_{L,\,\Sigma}(\Omega)$, we have $w + w\,' \in \mathcal{J}_{L,\,\Sigma}(\Omega)$.

Now, it is easy to see that for $\Sigma, \Sigma\,' \subseteq \Omega$, we have

$$
(10.9) \qquad \Sigma \;\subseteq\; \Sigma\,' \quad \Longrightarrow \quad \mathcal{J}_{L,\,\Sigma}(\Omega) \;\subseteq\; \mathcal{J}_{L,\,\Sigma\,'}(\Omega)
$$

in this way, in view of (10.6), it follows that

$$
(10.10) \qquad \mathcal{J}_{L,\,\mathcal{S}}(\Omega) \;=\; \bigcup_{\Sigma \in \mathcal{S}} \mathcal{J}_{L,\,\Sigma}(\Omega)
$$

is also an ideal in $(\mathcal{C}^\infty(\Omega))^\Lambda$.

10.2. Foam Algebras

In view of the above, for $\Sigma \subseteq \Omega$, we can define the algebra

$$(10.11) \qquad B_{L,\,\Sigma}(\Omega) \;=\; (\mathcal{C}^\infty(\Omega))^\Lambda / \mathcal{J}_{L,\,\Sigma}(\Omega)$$

However, we shall only be interested in singularity sets $\Sigma \in \mathcal{S}_{\mathcal{D}}(\Omega)$, that is, for which $\Omega \setminus \Sigma$ is *dense* in Ω. And in such a case the corresponding algebra $B_{L,\,\Sigma}(\Omega)$ will be called a *foam algebra.*

10.3 Multi-Foam Algebras

With the given family $\mathcal{S}$ of singularities, and based on (10.10), we can now associate the *multi-foam algebra*

$$(10.12) \qquad B_{L,\,\mathcal{S}}(\Omega) \;=\; (\mathcal{C}^\infty(\Omega))^\Lambda / \mathcal{J}_{L,\,\mathcal{S}}(\Omega)$$

10.4 Space-Time Foam Algebras

The foam algebras and the multi-foam algebras introduced above will for the sake of simplicity be called together *space-time foam algebras.*

Clearly, if the family $\mathcal{S}$ of singularities consists of one single singularity set $\Sigma \in \mathcal{S}_{\mathcal{D}}(\Omega)$, that is, $\mathcal{S} = \{\,\Sigma\,\}$, then conditions (10.5), (10.6)) are satisfied, and in this particular case the concepts of foam and multi-foam algebras are identical, in other words, $B_{L,\,\{\,\Sigma\,\}}(\Omega) = B_{L,\,Sigma}(\Omega)$. This means that the concept of multi-foam algebra is more general than that of foam algebra.

It is obvious from their quotient construction that the space-time foam algebras are associative and commutative. However, the above constructions can easily be extended to the case when, instead of real valued smooth functions, we use smooth functions with values in an arbitrary *normed algebra.* In such a case the resulting space-time foam algebras will still be associative, but in general they may be noncommutative.

10.5 Space-Time Foam Algebras as Algebras of Generalized Functions

The reason why we restrict ourself to singularity sets $\Sigma \in \mathcal{S}_{\mathcal{D}}(\Omega)$, that is, to subsets $\Sigma \subset \Omega$ for which $\Omega \setminus \Sigma$ is dense in Ω, is due to the following essential implication, see further details in [55-57,59,61,70,71,97,99,105,143,165], and for a full argument [4, chap. 3, pp. 65-119]

$$(10.13) \qquad \Omega \setminus \Sigma \text{ is dense in } \Omega \quad \Longrightarrow \quad \mathcal{J}_{L,\,\Sigma}(\Omega) \cap \mathcal{U}^{\infty}_{\Lambda}(\Omega) \;=\; \{\,0\,\}$$

where $\mathcal{U}^{\infty}_{\Lambda}(\Omega)$ denotes the *diagonal* of the power $(\mathcal{C}^{\infty}(\Omega))^{\Lambda}$, namely, it is the set of all $u(\psi) = (\psi_\lambda \,|\, \lambda \in \Lambda)$, where $\psi_\lambda = \psi$, for $\lambda \in \Lambda$, while ψ ranges over $\mathcal{C}^{\infty}(\Omega)$. In this way, we have the algebra isomorphism $\mathcal{C}^{\infty}(\Omega) \ni \psi \;\longmapsto\; u(\psi) \in \mathcal{U}^{\infty}_{\Lambda}(\Omega)$.

This implication (10.13) follows immediately from the asymptotic vanishing condition (10.8). Indeed, if $\psi \in \mathcal{C}^{\infty}(\Omega)$ and $u(\psi) \in \mathcal{J}_{L,\,\Sigma}(\Omega)$, then (1.8) implies that $\psi = 0$ on $\Omega \setminus \Sigma$, thus we must have $\psi = 0$ on Ω, since $\Omega \setminus \Sigma$ was assumed to be dense in Ω. It follows, therefore, that the ideal $\mathcal{J}_{L,\,\Sigma}(\Omega)$ is *off diagonal*.

The importance of (10.13) is that, for $\Sigma \in \mathcal{S}_{\mathcal{D}}(\Omega)$, it gives the following *algebra embedding* of the smooth functions into foam algebras

$$(10.14) \qquad \mathcal{C}^{\infty}(\Omega) \ni \psi \;\longmapsto\; u(\psi) + \mathcal{J}_{L,\,\Sigma}(\Omega) \in B_{L,\,\Sigma}(\Omega)$$

Now in view of (10.10), it is easy to see that (10.13) will as well yield the *off diagonality* property

$$(10.15) \qquad \mathcal{J}_{L,\,\mathcal{S}}(\Omega) \cap \mathcal{U}^{\infty}_{\Lambda}(\Omega) \;=\; \{\,0\,\}$$

and thus similar with (10.14), we obtain the *algebra embedding* of smooth functions into multi-foam algebras

$$(10.16) \qquad \mathcal{C}^{\infty}(\Omega) \ni \psi \;\longmapsto\; u(\psi) + \mathcal{J}_{L,\,\mathcal{S}}(\Omega) \in B_{L,\,\mathcal{S}}(\Omega)$$

The algebra embeddings (10.14), (10.16) mean that the foam and multi-foam algebras are in fact *algebras of generalized functions.* Also they mean that the foam and multi-foam algebras are unital, with the respective unit elements $u(1) + \mathcal{J}_{L,\Sigma}(\Omega),\ u(1) + \mathcal{J}_{L,\mathcal{S}}(\Omega)$.

Further, the asymptotic vanishing condition (10.8) also implies quite obviously that, for $\Sigma \subseteq \Omega$, we have

$$(10.17) \qquad D^p\, \mathcal{J}_{L,\Sigma}(\Omega) \ \subseteq\ \mathcal{J}_{L,\Sigma}(\Omega),\ \text{ for } p \in \mathbb{N}^n$$

where D^p denotes the termwise p-th order partial derivation of sequences of smooth functions, applied to each such sequence in the ideal $\mathcal{J}_{L,\Sigma}(\Omega)$.

Then again, in view of (10.10), we obtain

$$(10.18) \qquad D^p\, \mathcal{J}_{L,\mathcal{S}}(\Omega) \ \subseteq\ \mathcal{J}_{L,\mathcal{S}}(\Omega),\ \text{ for } p \in \mathbb{N}^n$$

Now (10.17), (10.18) mean that the the foam and multi-foam algebras are in fact *differential algebras*, namely

$$(10.19) \qquad D^p\, B_{L,\Sigma}(\Omega) \ \subseteq\ B_{L,\Sigma}(\Omega),\ \text{ for } p \in \mathbb{N}^n$$

where $\Sigma \in \mathcal{S}_{\mathcal{D}}(\Omega)$, and furthermore we also have

$$(10.20) \qquad D^p\, B_{L,\mathcal{S}}(\Omega) \ \subseteq\ B_{L,\mathcal{S}}(\Omega),\ \text{ for } p \in \mathbb{N}^n$$

In this way we obtain that the foam and multi-foam algebras are *differential algebras of generalized functions*.

Also, the foam and multi-foam algebras contain the Schwartz distributions, that is, we have the *linear embeddings* which respect the arbitrary partial derivation of smooth functions

$$(10.21) \qquad \mathcal{D}'(\Omega) \ \subset\ B_{L,\Sigma}(\Omega),\ \text{ for } \Sigma \in \mathcal{S}_{\mathcal{D}}(\Omega)$$

$$(10.22) \qquad \mathcal{D}'(\Omega) \ \subset\ B_{L,\mathcal{S}}(\Omega)$$

Indeed, let us recall the wide ranging purely *algebraic characterization* of all those quotient type algebras of generalized functions in which

one can embed linearly the Schwartz distributions, a characterization first given in 1978, see [3, pp. 1-32], as well as [4, pp. 75-88], [6, pp. 306-315], [7, pp. 234-244].
According to that characterization - which also contains the Colombeau algebras as a particular case - the *necessary and sufficient* condition for the existence of the linear embedding (10.21) is precisely the off diagonality condition in (10.13). Similarly, the necessary and sufficient condition for the existence of the linear embedding (10.22) is exactly the off diagonality condition (10.15).

One more property of the foam and multi-foam algebras will prove to be useful. Namely, in view of (10.10), it is clear that, for every $\Sigma \in \mathcal{S}$, we have the inclusion $\mathcal{J}_{L,\Sigma}(\Omega) \subseteq \mathcal{J}_{L,\mathcal{S}}$, and thus we obtain the *surjective algebra homomorphism*

$$(10.23) \qquad B_{L,\Sigma}(\Omega) \ni w + \mathcal{J}_{L,\Sigma}(\Omega) \longmapsto w + \mathcal{J}_{L,\mathcal{S}}(\Omega) \in B_{L,\mathcal{S}}(\Omega)$$

The general form of that property is the following one. Given families of singularities

$$(10.24) \qquad \mathcal{S} \subseteq \mathcal{S}\,' \subseteq \mathcal{S}_{\mathcal{D}}(\Omega)$$

such that $\mathcal{S}, \mathcal{S}\,'$ satisfy (10.5), (10.6). Then in view of (10.10), we have the *surjective algebra homomorphism*

$$(10.25) \qquad \alpha_{\mathcal{S},\ \mathcal{S}\,'} : B_{L,\mathcal{S}}(\Omega) \longrightarrow B_{L,\mathcal{S}\,'}(\Omega)$$

with

$$(10.26) \qquad \alpha_{\mathcal{S},\ \mathcal{S}\,'}(w + \mathcal{J}_{L,\mathcal{S}}(\Omega)) = w + \mathcal{J}_{L,\mathcal{S}\,'}(\Omega)$$

Furthermore, in view of (10.18), we also have

$$(10.27) \qquad \alpha_{\mathcal{S},\ \mathcal{S}\,'}(D^p F) = D^p \alpha_{\mathcal{S},\ \mathcal{S}\,'}(F), \qquad F \in B_{L,\mathcal{S}}(\Omega), \quad p \in \mathbb{N}^n$$

And as we shall see in the next subsection, (10.23) can naturally be interpreted as meaning that the typical generalized functions in $B_{L,\mathcal{S}\,'}(\Omega)$ are *more regular* than those in $B_{L,\mathcal{S}}\Omega$.

10.6. Regularity of Generalized Functions

One natural way to interpret (10.23) in the given context of generalized functions is the following. Given two spaces of generalized functions E and F, such as for instance

$$(10.28) \qquad \mathcal{C}^\infty(\Omega) \subset E \subset F$$

then the larger the space F the *less regular* its typical element can appear to be, when compared with those of E. By the same token, the it smaller the space E , the *more regular*, compared with those of F, one can consider its typical elements.

Similarly, given a *surjective* mapping

$$(10.29) \qquad E \longrightarrow F$$

one can again consider that the typical elements of F are at least as *regular* as those of E.

In this way, in view of (10.23), we can consider that, owing to the given *surjective* algebra homomorphism, the typical elements of the multi-foam algebra $B_{L,\mathcal{S}}(\Omega)$ can be seen as being *more regular* than the typical elements of the foam algebra $B_{L,\Sigma}(\Omega)$.

Furthermore, the algebra $B_{L,\mathcal{S}}(\Omega)$ is obtained by factoring the same $(\mathcal{C}^\infty(\Omega))^\Lambda$ as in the case of the algebra $B_{L,\Sigma}(\Omega)$, this time however by the significantly *larger* ideal $\mathcal{J}_{L,\mathcal{S}_L}(\Omega)$, an ideal which, unlike any of the individual ideals $\mathcal{J}_{L,\Sigma}(\Omega)$, can simultaneously deal with *all* the singularity sets $\Sigma \in \mathcal{S}_L$, some, or in fact, many of which can be *dense* in Ω.

Further details related to the connection between *regularization* in the above sense, and on the other hand, properties of *stability*, *generality* and *exactness* of generalized functions and solutions can be found in [4, pp. 12-18], [6, pp.195-230], [7, pp. 47-56].

This kind of interpretation can be used related to the global Cauchy-Kovalevskaia theorem in subsection 8.1. Indeed, in view of (8.16), (10.2), we have for the respective global solution of the Cauchy-Kovalevskaia theorem

$$(10.30) \qquad U \in B_{\mathbb{N},\,\mathcal{S}_{nd}}(\Omega)$$

therefore, if we take any family $\mathcal{S}$ of singularities which satisfies conditions (10.5), (10.6), and such that

$$(10.31) \qquad \mathcal{S}_{nd}(\Omega) \subseteq \mathcal{S} \subseteq \mathcal{S}_{\mathcal{D}}(\Omega)$$

then (10.24) gives

$$(10.32) \qquad V \;=\; \alpha_{\mathcal{S}_{nd},\;\mathcal{S}}(U) \in B_{\mathbb{N},\mathcal{S}}(\Omega)$$

and it is easy to see that V will again be a global solution of the Cauchy-Kovalevskaia theorem, this time in the algebra $B_{\mathbb{N},\mathcal{S}}(\Omega)$.

Thus V will be at least as *regular* a solution as is U.

10.7. A Singularity-Stability Property of Nowhere Dense Algebras

Recently, [165], it was shown that we have

$$(10.33) \qquad B_{\mathbb{N},\,\mathcal{S}_{nd}(\Omega)}(\Omega) \;=\; B_{\mathbb{N},\,\mathcal{S}_{Baire\ I}(\Omega)}(\Omega)$$

This is a rather surprising property, since the family of set of singularities $\mathcal{S}_{nd}(\Omega)$ is considerably smaller than $\mathcal{S}_{Baire\ I}(\Omega)$.

One way to see (10.33) is as a *stability* property of the nowhere dense algebras. And what it says, among others, is that - within the differential algebras of generalized functions - it is sufficient to take care of closed, nowhere dense singularities, in order to take in fact care of the far larger class of First Baire Category singularities as well. in this way, the *perturbation* of the family of closed, nowhere dense singularities by the considerably larger family of First Baire Category singularities still can be done within the nowhere dense algebras.

$$(10.34) \qquad \mathbb{A}_{nd}(\Omega) \;=\; B_{\mathbb{N},\mathcal{S}_{nd}(\Omega)}(\Omega)$$

with respect to the massive addition of singularity sets, namely, those in $\mathcal{S}_{Baire\ I}(\Omega)$.

Needless to say, beyond the family of singularity sets in $\mathcal{S}_{Baire\ I}(\Omega)$, and still within those in $\mathcal{S}_{\mathcal{D}}(\Omega)$, there are plenty of larger families of singularities than $\mathcal{S}_{Baire\ I}(\Omega)$. In other words, among the families of singularity sets $\mathcal{S}$ which satisfy (10.5), (10.6), as well as

$$(10.35) \qquad \mathcal{S}_{nd}(\Omega) \subseteq \mathcal{S} \subseteq \mathcal{S}_{\mathcal{D}}(\Omega)$$

there are many which are considerably larger than $\mathcal{S}_{Baire\ I}(\Omega)$.

In this regard, we can formulate the following

Open Problem

To what extent are the space-time foam algebras $B_{\mathbb{N},\mathcal{S}}(\Omega)$, with $\mathcal{S}$ satisfying (10.5), (10.6), (10.35), *singularity-stable* in the sense exemplified by (10.34) ?

Remark 10.2.

One can easily note that the family of singularity sets $\mathcal{S}_{\mathcal{D}}(\Omega)$ does *not* satisfy condition (10.6). Therefore, there is *no* largest family of singularities $\mathcal{S}$ for which (10.5), (10.6), as well as (10.35) hold.

It follows, than one can only have *maximal* families of singularity sets $\mathcal{S}_{\mathcal{D}}(\Omega)$ which satisfy (10.5), (10.6) and (10.35).

This fact obviously makes the above Open Problem less easy to solve.

10.8. Constrained Maximal Ideals Mean Largest Classes of Singularities

As seen in [98] and sections 7, 8, as well as earlier in the present section, the reduced power algebras (5.4) with the largest ideals $\mathcal{I}^\infty(\Omega)$ which satisfy the constraint of the off-diagonality condition (5.8), lead to those differential algebras of generalized functions which can deal with the largest classes of singularities.

In this regard, we still have the

Open Problem

Find the structure of the ideals

$$(10.36) \qquad \mathcal{I}^\infty(\Omega) \subset (\mathcal{C}^\infty(\Omega))^\Lambda$$

which are *maximal under the constraint* of the off-diagonality condition

$$(10.37) \qquad \mathcal{I}^\infty(\Omega) \bigcap \mathcal{U}^\infty(\Omega) = \{0\}$$

where Λ is an infinite set.

Of course, it may help to first solve the following

Open Problem

Find the structure of the ideals

$$(10.38) \qquad \mathcal{I}^0(\Omega) \subset (\mathcal{C}^0(\Omega))^\Lambda$$

which are *maximal under the constraint* of the off-diagonality condition

$$(10.39) \qquad \mathcal{I}^0(\Omega) \bigcap \mathcal{U}^0(\Omega) = \{0\}$$

One can note that the ideals (10.10) give in view of (10.15) already rather large ones which may help in solving the open problem (10.36), (10.37).

11. The Local Algebras

One can note from the above that there is a significant interest in *enlarging* evermore the classes of singularities that can be dealt with by differential algebras of generalized functions.
In this regard, an obvious limitation present in all the algebras mentioned so far is the use as building blocks of globally defined $\mathcal{C}^\infty$-smooth functions $f : \Omega \longrightarrow \mathbb{R}$, that is, defined on the whole domain $\Omega \subseteq \mathbb{R}^n$ of the respective generalized functions, as seen in (5.3), (5.4).

Here, therefore, we replace the use of such globally defined $\mathcal{C}^\infty$-smooth functions in (5.3), (5.4), with the use of a far larger class of functions which are only assumed to be *locally* smooth on the respective domains Ω.

Needless to say, the classes of singularities which can be dealt with by the resulting differential algebras of generalized functions will not be reduced in any way.

11.1. Basic Definitions

For the sake of generality, we shall start with functions defined locally on arbitrary sets X, instead of open subsets $\Omega \subseteq \mathbb{R}^n$. Also, the values of the local functions defined will be in arbitrary sets E. Typically, E can be a Banach algebra, either commutative, or not.

11.1.1. A Single Singularity Set

Let X be a nonvoid set and $\Sigma \subset X$ which will play the role of the subset of *singular* points of certain functions defined locally on X.

Definition 11.1.

A Σ-*local function* on X is every family

$$(11.1) \quad f = (f_{x,U_x} \mid x \in X \setminus \Sigma)$$

where $x \in U_x \subseteq X$, while $f_{x,U_x} : U_x \longrightarrow E$, and the following *compatibility* condition holds

$$(11.2) \quad \forall \, x, y \in X \backslash \Sigma \; : \; x \in U_y, \, y \in U_x \implies f_{x,U_x} = f_{y,U_y} \text{ on } U_x \cap U_y$$

Remark 11.1.

1) The sets X need not be topological spaces, and the subsets U_x need not be neighbourhoods of x, as they can be arbitrary, with the only restriction mentioned above, namely that $x \in U_x \subseteq X$.

2) We note that condition (11.2) is considerably weaker than condition

$$(11.3) \quad \forall \, x, y \in X \setminus \Sigma \; : \; U_x \cap U_y \neq \phi \implies f_{x,U_x} = f_{y,U_y} \text{ on } U_x \cap U_y$$

as seen in 2) in the following

Examples 11.1.

1) Let $f : X \longrightarrow \mathbb{R}$, then $(f|_{U_x} \mid x \in X \setminus \Sigma)$ is a Σ-local function on X, whenever $U_x \subseteq X$ are a neighbourhoods of $x \in X$.

2) Let $X = \mathbb{R}$, $\Sigma = \mathbb{R} \backslash \mathbb{Q}$, where as usual, $\mathbb{Q}$ denotes the set of rational numbers. We assume $X \setminus \Sigma = \{x_0, x_1, x_2, \ldots\}$ and take

$U_{x_0} = (x_0 - r_0, x_0 + r_0)$ with $r_0 > 0$

$U_{x_1} = (x_1 - r_1, x_1 + r_1)$ with $r_1 > 0$, such that $x_0 \notin U_{x_1}$

$U_{x_2} = (x_2 - r_2, x_2 + r_2)$ with $r_2 > 0$, such that $x_0, x_1 \notin U_{x_2}$

$U_{x_3} = (x_3 - r_3, x_3 + r_3)$ with $r_3 > 0$, such that $x_0, x_1, x_2 \notin U_{x_3}$

$\vdots$

Given now $c_0, c_1, c_2, \ldots \in \mathbb{R}$, we define $f_{x_n, U_{x_n}} = c_n$, for $n \in \mathbb{N}$. Then

(11.4) $\quad f = (f_{x, U_x} \mid x \in X \setminus \Sigma)$ is a Σ-local function on X

Indeed, let $n < m$. Then $x_0, x_1, \ldots, x_{m-1} \notin U_{x_m}$, hence $x_n \notin U_{x_m}$, therefore (11.2) is satisfied by default.

Clearly, for $n \in \mathbb{N}$, there are infinitely many $m \in \mathbb{N}$, such that

(11.5) $\quad U_{x_n} \cap U_{x_m} \neq \phi$

therefore, condition (11.3) is in general not satisfied.

3) The interest in the class of examples of local functions in (11.4) is in the following five facts :

- the set $X \setminus \Sigma$ is dense in X,
- the values $c_0, c_1, c_2, \ldots \in \mathbb{R}$ can be arbitrary,
- the sum $\sum_{n \in \mathbb{N}} r_n$ can be arbitrary small, thus so can be the measure of $\bigcup_{n \in \mathbb{N}} U_{x_n}$,
- the sets U_{x_x} are open neighbourhoods of the respective x, and
- the component functions $f_{x_n, U_{x_n}}$ are highly smooth or regular, being in fact constant.

□

We denote by

(11.6) $\quad \mathcal{B}_{lc,\Sigma}(X, E)$

the set of all Σ-local functions on X with values in the Banach algebra E. Clearly, $\mathcal{B}_{lc,\Sigma}(X, E)$ is a commutative, respectively, non-commutative unital algebra on $\mathbb{R}$, according to E being commutative or not.

Further, we define the *algebra embedding*

(11.7) $\quad E^X \ni f \longrightarrow lc(f) = (f_{x,\,X} \mid x \in X \setminus \Sigma) \in \mathcal{B}_{lc,\,\Sigma}(X, E)$

where $f_{x,\,X} = f$. Also, we denote by

(11.8) $\quad \mathcal{V}_{lc,\,\Sigma}(X, E)$

the subalgebra in $\mathcal{B}_{lc,\,\Sigma}(X, E)$ which is the *range* of the above algebra embedding (11.7).

Thus we have the *algebra isomorphism*

(11.9) $\quad E^X \ni f \longrightarrow lc(f) = (f_{x,\,X} \mid x \in X \setminus \Sigma) \in \mathcal{V}_{lc,\,\Sigma}(X, E)$

Given $Z \subseteq X$, with $Z \setminus \Sigma \neq \phi$, we denote by

(11.10) $\quad \mathcal{J}_{lc,\,\Sigma,\ Z}(X, E)$

the set of all $f = (f_{x,\,U_x} \mid x \in X \setminus \Sigma) \in \mathcal{B}_{lc,\,\Sigma}(X, E)$, such that

(11.11) $\quad \forall \ \ x \in Z \setminus \Sigma \ : \ f_{x,\,U_x}(x) = 0$

Obviously, $\mathcal{J}_{lc,\,\Sigma,\ Z}(X, E)$ is an *ideal* in $\mathcal{B}_{lc,\,\Sigma}(X, E)$.

Given now $\Sigma \subseteq \Sigma' \subset X$, we can define the mapping

(11.12) $\quad j_{\,\Sigma,\,\Sigma'} : \mathcal{B}_{lc,\,\Sigma}(X, E) \longrightarrow \mathcal{B}_{lc,\,\Sigma\,'}(X, E)$

by

(11.13) $\quad j_{\,\Sigma,\,\Sigma'}(f_{x,\,U_x} \mid x \in X \setminus \Sigma) = (f_{x,\,U_x} \mid x \in X \setminus \Sigma')$

And these mappings are *surjective algebra homomorphisms* which have the properties

(11.14) $\quad j_{\,\Sigma,\,\Sigma} = id_{\mathcal{B}_{lc,\,\Sigma}(X,E)}$, for $\Sigma \subset X$

(11.15) $\quad j_{\,\Sigma',\,\Sigma''} \circ j_{\,\Sigma,\,\Sigma'} = j_{\,\Sigma,\,\Sigma''}$, for $\Sigma \subseteq \Sigma' \subseteq \Sigma'' \subset X$

11.1.2. Families of Singularity Sets

Let us now turn to the case when instead of one single subset $\Sigma \subset X$ of singularities, we have a whole family $\mathcal{S} \subseteq \mathcal{P}(X)$ of such singularity subsets $\Sigma \in \mathcal{S}$. In this regard, we shall suppose in the sequel that

(11.16) $\quad X \notin \mathcal{S}$

(11.17) $\quad \forall \;\; \Sigma, \Sigma' \in \mathcal{S} \; : \; \exists \;\; \Sigma'' \in \mathcal{S} \; : \; \Sigma \cup \Sigma' \subseteq \Sigma''$

Obviously, (11.16) is equivalent with $\Sigma \subset X$, for $\Sigma \in \mathcal{S}$.

In view of (11.17), it follows that $(\mathcal{S}, \subseteq)$ is a *directed* partially ordered set.

Clearly, in the particular case when $\mathcal{S} = \{\Sigma\}$, that is, when we have one single subset $\Sigma \subset X$ of singularities, then the conditions (11.16), (11.17) are satisfied.

We consider now in the general case of (11.16), (11.17), the set

(11.18) $\quad \mathcal{B}_{lc,\mathcal{S}}(X, E) = \bigcup_{\Sigma \in \mathcal{S}} \mathcal{B}_{lc,\Sigma}(X, E)$

as well as

(11.19) $\quad \mathcal{V}_{lc,\mathcal{S}}(X, E) = \bigcup_{\Sigma \in \mathcal{S}} \mathcal{V}_{lc,\Sigma}(X, E)$

which is obviously a subset of $\mathcal{B}_{lc,\mathcal{S}}(X, E)$.

Further, let $Z \subseteq X$, such that

(11.20) $\quad \forall \;\; \Sigma \in \mathcal{S} \; : \; Z \setminus \Sigma \neq \phi$

Then we define

(11.21) $\quad \mathcal{J}_{lc,\mathcal{S},Z}(X, E) = \bigcup_{\Sigma \in \mathcal{S}} \mathcal{J}_{lc,\Sigma,Z}(X, E)$

which is obviously a subset of $\mathcal{B}_{lc,\,\mathcal{S}}(X, E)$.

Now we consider the *direct limit*

(11.22) $\quad \mathcal{A}_{lc,\,\mathcal{S}}(X, E) = \underrightarrow{\lim}_{\,\Sigma \in \mathcal{S}}\, \mathcal{B}_{lc,\,\Sigma}(X, E)$

It follows that

(11.23) $\quad \mathcal{A}_{lc,\,\mathcal{S}}(X, E) = \mathcal{B}_{lc,\,\mathcal{S}}(X, E) / \approx_{\mathcal{S}}$

where the equivalence relation $\approx_{\mathcal{S}}$ on $\mathcal{B}_{lc,\,\mathcal{S}}(X, E)$ is defined for $(f_{x,\,U_x} \mid x \in X \setminus \Sigma) \in \mathcal{B}_{lc,\,\Sigma}(X, E)$, $(g_{y,\,V_y} \mid y \in X \setminus \Sigma') \in \mathcal{B}_{lc,\,\Sigma'}(X, E)$, with $\Sigma, \Sigma' \in \mathcal{S}$, by

(11.24) $\quad (f_{x,\,U_x} \mid x \in X \setminus \Sigma) \approx_{\mathcal{S}} (g_{y,\,V_y} \mid y \in X \setminus \Sigma')$

if and only if there exist $\Sigma'' \in \mathcal{S}$, with $\Sigma \cup \Sigma' \subseteq \Sigma''$, as well as $(h_{z,\,W_z} \mid z \in X \setminus \Sigma'') \in \mathcal{B}_{lc,\,\Sigma''}(X, E)$, such that $j_{\Sigma,\,\Sigma''}(f_{x,\,U_x} \mid x \in X \setminus \Sigma) = j_{\Sigma',\,\Sigma''}(g_{y,\,V_y} \mid y \in X \setminus \Sigma') = (h_{z,\,W_z} \mid z \in X \setminus \Sigma'')$.

Similarly, one defines the *direct limit*

(11.25) $\quad \mathcal{U}_{lc,\,\mathcal{S}}(X, E) = \underrightarrow{\lim}_{\,\Sigma \in \mathcal{S}}\, \mathcal{V}_{lc,\Sigma}(X, E)$

and obtains

(11.26) $\quad \mathcal{U}_{lc,\,\mathcal{S}}(X, E) = \mathcal{V}_{lc,\,\mathcal{S}}(X, E)) / \approx_{\mathcal{S}}$

Clearly, we have the *injective* mapping

(11.27) $\quad E^X \ni f \longrightarrow (lc(f))_{\approx_{\mathcal{S}}} \in \mathcal{A}_{lc,\,\mathcal{S}}(X, E)$

where $(g)_{\approx_{\mathcal{S}}}$ denotes the $\approx_{\mathcal{S}}$ equivalence class of the element $g \in \mathcal{B}_{lc,\,\mathcal{S}}(X, E)$.

We note that, if $\mathcal{S} = \{\Sigma\}$, then $\mathcal{A}_{lc,\,\mathcal{S}}(X, E) = \mathcal{A}_{lc,\,\Sigma}(X, E) = \mathcal{B}_{lc,\,\Sigma}(X, E)$ and $\mathcal{U}_{lc,\,\mathcal{S}}(X, E) = \mathcal{U}_{lc,\,\Sigma}(X, E) = \mathcal{V}_{lc,\,\Sigma}(X, E)$.

Lastly, given $Z \subseteq X$ for which (11.20) holds, one defines the *direct limit*

$$(11.28) \quad \mathcal{I}_{lc,\,\mathcal{S},\,Z}(X,E) = \varinjlim_{\Sigma \in \mathcal{S}} \mathcal{J}_{lc,\,\Sigma,\,Z}(X,E)$$

and obtains

$$(11.29) \quad \mathcal{I}_{lc,\,\mathcal{S},\,Z}(X,E) = \mathcal{J}_{lc,\,\mathcal{S},\,Z}(X,E))/\approx_{\mathcal{S}}$$

Obviously, in view of (11.21), we have

$$(11.30) \quad \mathcal{I}_{lc,\,\mathcal{S},\,Z}(X,E) \subseteq \mathcal{A}_{lc,\,\mathcal{S}}(X,E)$$

We can also note that, if $\mathcal{S} = \{\Sigma\}$, then $\mathcal{I}_{lc,\,\mathcal{S},\,Z}(X,E) = \mathcal{I}_{lc,\,\Sigma,\,Z}(X,E) = \mathcal{J}_{lc,\,\Sigma,\,Z}(X,E)$

Let us summarize. Given $\Sigma \in \mathcal{S}$ as above, and $Z \subseteq X$ as in (11.20), we have the *commutative* diagram of mappings

$$(11.31) \quad \begin{array}{ccccccc}
E^X & \stackrel{*}{\longrightarrow} & \mathcal{V}_{lc,\,\Sigma}(X,E) & \longrightarrow & \mathcal{B}_{lc,\,\Sigma}(X,E) & \longleftarrow & \mathcal{J}_{lc,\,\Sigma,\,Z}(X,E) \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & \mathcal{V}_{lc,\,\mathcal{S}}(X,E) & \longrightarrow & \mathcal{B}_{lc,\,\mathcal{S}}(X,E) & \longleftarrow & \mathcal{J}_{lc,\,\mathcal{S},\,Z}(X,E) \\
 & & \downarrow * & & \downarrow * & & \downarrow * \\
E^X & \stackrel{*}{\longrightarrow} & \mathcal{U}_{lc,\,\mathcal{S}}(X,E) & \longrightarrow & \mathcal{A}_{lc,\,\mathcal{S}}(X,E) & \longleftarrow & \mathcal{I}_{lc,\,\mathcal{S},\,Z}(X,E)
\end{array}$$

where all the mappings are *injective*, except for the three mappings "$\downarrow *$" which are *surjective*, while the two mappings "$\stackrel{*}{\longrightarrow}$" are in fact *bijective*.

11.2. Properties

The following result can be obtained by direct, even if a somewhat elaborate verification :

Theorem 11.1.

$\mathcal{A}_{lc,\mathcal{S}}(X,E)$ is a unital algebra, and $\mathcal{U}_{lc,\mathcal{S}}(X,E)$ is a subalgebra in it, and it is the range of the mapping (11.27) which is an *algebra embedding*, namely, we have the *algebra isomorphism*

$$E^X \ni f \longrightarrow (lc(f))_{\approx_{\mathcal{S}}} \in \mathcal{U}_{lc,\,\mathcal{S}}(X,E) \subset \mathcal{A}_{lc,\,\mathcal{S}}(X,E) \tag{11.32}$$

The algebra $\mathcal{A}_{lc,\,\mathcal{S}}(X,E)$ is commutative, if and only if the Banach algebra E is commutative.

Furthermore, given $Z \subseteq X$ for which (11.20) holds, then $\mathcal{I}_{lc,\,\mathcal{S},\,Z}(X,E)$ is an *ideal* in the algebra $\mathcal{A}_{lc,\,\mathcal{S}}(X,E)$.

$\square$

For $0 \leq l \leq \infty$, we denote by, see (11.22), (11.23)

$$\mathcal{A}^l_{lc,\,\mathcal{S}}(X,E) \tag{11.33}$$

the set of all $(f)_{\approx_{\mathcal{S}}}$ where $f = (f_{x,\,U_x} \mid x \in X \setminus \Sigma) \in \mathcal{B}_{lc,\,\Sigma}(X,E)$, for some $\Sigma \in \mathcal{S}$, such that

$$f_{x,\,U_x} \in \mathcal{C}^l(U_x), \quad \text{for} \quad x \in X \setminus \Sigma \tag{11.34}$$

Further, we denote

$$\mathcal{U}^l_{lc,\,\mathcal{S}}(X,E) = \mathcal{U}_{lc,\,\mathcal{S}}(X,E) \bigcap \mathcal{A}^l_{lc,\,\mathcal{S}}(X,E) \tag{11.35}$$

$$\mathcal{I}^l_{lc,\,\mathcal{S},\,Z}(X,E) = \mathcal{I}_{lc,\,\mathcal{S},\,Z}(X,E) \bigcap \mathcal{A}^l_{lc,\,\mathcal{S}}(X,E) \tag{11.36}$$

Theorem 11.2.

Suppose that

$$\forall \;\; \Sigma \in \mathcal{S} \;:\; X \setminus \Sigma \;\; \text{is dense in} \;\; X \tag{11.37}$$

Then, for $0 \leq l \leq \infty$, the ideal $\mathcal{I}^l_{lc,\,\mathcal{S}}(X,E)$ in the algebra $\mathcal{A}^l_{lc,\,\mathcal{S}}(X,E)$ satisfies the *off-diagonality* condition

(11.38) $\quad \mathcal{I}_{lc,\mathcal{S}}(X,E) \bigcap \mathcal{U}_{lc,\mathcal{S}}(X,E) = \{0\}$

Proof

It follows from the density condition (11.37) and the continuity of the functions involved. Indeed, let, see (11.29)

(11.39) $\quad (f)_{\approx_\mathcal{S}} \in \mathcal{I}_{lc,\mathcal{S}}(X,E)$

where for some $\Sigma \in \mathcal{S}$, we have $f = (f_{x,U_x} \mid x \in X \setminus \Sigma) \in \mathcal{B}_{lc,\Sigma}(X,E)$. Then $(f)_{\approx_\mathcal{S}} \in \mathcal{U}_{lc,\mathcal{S}}(X,E)$ implies in view of (11.26) that, see (11.8)

(11.40) $\quad f = (f_{x,U_x} \mid x \in X \setminus \Sigma) \in \mathcal{V}_{lc,\Sigma}(X,E)$

Now (11.40), (11.8) give

(11.41) $\quad f_{x,U_x} = f, \quad x \in X$

for some $f \in \mathcal{C}^0(X,E)$.

On the other hand, (11.39), (11.11), (11.40) give

$$f(x) = f_{x,U_x}(x) = 0, \quad x \in X$$

thus indeed $f = 0$.

11.3. Differential Algebras with Dense Singularities

In [55-57,59,61,70,71,97,99], large classes of differential algebras of generalized functions which allow their elements to have singularities on *dense* subsets of their domain of definition, and *without* any restrictions on the respective generalized functions in the neighbourhood of their singularities, have been introduced, and these algebras have been applied to solving large classes of systems of nonlinear PDEs, as well as in highly singular problems in Lie group theory, differential geometry, and with respective applications in modern physics, including general relativity and quantum gravity, [168-170].

The algebras are of the form (0.1), (0.2), thus are built upon $\mathcal{C}^\infty$-smooth functions, namely, their elements are classes of equivalence modulo the respective ideals $\mathcal{I}$.

These ideals play a fundamental role in dealing with dense singularities, and so without any restrictions on the respective generalized functions in the neighbourhood of their singularities. Indeed, the power of the method consists precisely in the fact that the singularities, although possibly so many as to constitute dense subsets in the domain of definition of generalized functions, are dealt with exclusively *algebraic*, that is, ring theoretic means.
And here it should be mentioned that the singularities can forms sets which have a *larger* cardinal then the set of regular, that is, non-singular points. For instance, if the generalized functions are defined on $X = \mathbb{R}$, then the set of singularities can be given by all irrational numbers, thus the set of regular, non-singular points can be reduced to the set of rational numbers.

As argued in section 0, there is a major interest in extending such algebras by replacing the $\mathcal{C}^\infty$-smooth functions upon which they are built with considerably larger classes of functions, and specifically in this paper, with functions which are *locally* smooth, see (11.22), (11.23).

When proceeding with such an extension, the main issue is to extend in appropriate ways the definition of the large class of ideals $\mathcal{I}$ in such a way that they still can handle *dense* singularities, and do so *without* any restrictions on the respective generalized functions in the neighbourhood of their singularities.

Let us, therefore, recall for convenience the definition of the large class of ideals $\mathcal{I}$, see section 10, in the case when, as supposed, X is a domain in and Euclidean space $\mathbb{R}^n$.

First we recall that we supposed a directed partial order $\leq$ on the infinite set of indices Λ. Further, let, as in (11.16), (11.17), $\mathcal{S} \subseteq \mathcal{P}(X)$ be a family of singularity subsets $\Sigma \subset X$.

Now for given $\Sigma \in \mathcal{S}$, we considered, see section 10, the ideal

$$(11.42) \quad \mathcal{I}_\Sigma(X)$$

in $(\mathcal{C}^\infty(X))^\Lambda$, given by all the sequences of smooth functions $w = (w_\lambda \mid \lambda \in \Lambda) \in (\mathcal{C}^\infty(X))^\Lambda$, such that

$$(11.43) \quad \begin{array}{l} \forall \ \ x \in X \setminus \Sigma \ : \\ \exists \ \ \lambda \in \Lambda \ : \\ \forall \ \ \mu \in \Lambda, \ \mu \geq \lambda \ : \\ \forall \ \ p \in \mathbb{N}^n \ : \\ \quad D^p w_\mu(x) = 0 \end{array}$$

Further, we defined the ideal in $(\mathcal{C}^\infty(X))^\Lambda$, given by

$$(11.44) \quad \mathcal{I}_\mathcal{S}(X) = \bigcup_{\Sigma \in \mathcal{S}} \mathcal{I}_\Sigma(X)$$

which played the role of the ideals $\mathcal{I}$ in (0.1), (0.2).

In order to extend these ideals to the case of *locally* smooth functions, first we extend (11.43) according to (11.1) - (11.7). Namely, we denote by

$$(11.45) \quad \mathcal{N}_{lc,\mathcal{S}}(X, E)$$

the set of all the sequences $\widetilde{w}$, where for a suitable $\Sigma \in \mathcal{S}$, we have $\widetilde{w} = ((w_\lambda)_{\approx_\mathcal{S}} \mid \lambda \in \Lambda) \in (\mathcal{B}^\infty_{lc,\Sigma}(X, E))^\Lambda$, such that

$$(11.46) \quad \begin{array}{l} \forall \ \ x \in X \setminus \Sigma \ : \\ \exists \ \ \lambda \in \Lambda \ : \\ \forall \ \ \mu \in \Lambda, \ \mu \geq \lambda \ : \\ \forall \ \ p \in \mathbb{N}^n \ : \\ \quad D^p w_\mu(x) = 0 \end{array}$$

Theorem 11.3.

$\mathcal{N}_{lc,\mathcal{S}}(X, E)$ is an ideal in $(\mathcal{A}^\infty_{lc,\mathcal{S}}(X, E))^\Lambda$.

Proof.

It follows by direct verification, based on (11.33), (11.34) and (11.45), (11.46).

$\square$

At last, we can now arrive at the *main construction* in this paper, namely, the *reduced power algebras*

(11.47) $\quad A_{lc,\mathcal{S}}(X,E) = (\mathcal{A}^{\infty}_{lc,\mathcal{S}}(X,E))^{\Lambda}/\mathcal{N}_{lc,\mathcal{S}}(X,E)$

which are in fact *differential algebras of generalized functions.* Furthermore, they contain all the earlier differential algebras of generalized functions, and in particular, those with closed, nowhere dense, or even dense singularities, as well as the Colombeau algebras, and therefore, also the linear vector spaces of Schwartz distributions.

Indeed, it follows from a direct, albeit elaborate verification that the algebras (11.47) satisfy the corresponding conditions (11.48) - (11.51) next.

We recall, see section 5, that the structure of all the differential algebras of generalized functions is of the same form of *reduced powers*, namely

(11.48) $\quad A = \mathcal{A}/\mathcal{I}$

where

(11.49) $\quad \mathcal{I} \subset \mathcal{A} \subseteq (\mathcal{C}^{\infty}(X,E))^{\Lambda}$

with Λ a suitable infinite set of indices with a directed partial order, while $\mathcal{A}$ is a subalgebra in $(\mathcal{C}^{\infty}(X,E))^{\Lambda}$, and $\mathcal{I}$ is an ideal in $\mathcal{A}$.

The fact that the algebras (0.1) are *differential* algebras results easily, since as seen in sections 7-10, the following two conditions can be satisfied in a large variety of situations

$$(11.48) \quad D^p \mathcal{A} \subseteq \mathcal{A}, \quad p \in \mathbb{N}^n$$

$$(11.49) \quad D^p \mathcal{I} \subseteq \mathcal{I}, \quad p \in \mathbb{N}^n$$

in which case the partial derivative operators on the algebras (0.1) can of course be defined by

$$(11.50) \quad A \ni F = f + \mathcal{I} \longmapsto D^p F = D^p f + \mathcal{I} \in A, \quad p \in \mathbb{N}^n$$

where $f = (f_\lambda \mid \lambda \in \Lambda) \in (\mathcal{C}^\infty(X,E))^\Lambda$, while $D^p f = (D^p f_\lambda \mid \lambda \in \Lambda) \in (\mathcal{C}^\infty(X,E))^\Lambda$.

The fact that the corresponding differential algebras contain the Schwartz distributions, thus are algebras of *generalized functions* follows from the easy choice of their ideals $\mathcal{I}$ required to satisfy the *off-diagonality* condition, see sections 5, 7-10

$$(11.51) \quad \mathcal{I} \bigcap \mathcal{U}_\Lambda = \{0\}$$

where $\mathcal{U}_\Lambda$ is the *diagonal* in $(\mathcal{C}^\infty(X,E))^\Lambda$, that is, the subalgebra of all constant sequences $f = (f_\lambda \mid \lambda \in \Lambda) \in (\mathcal{C}^\infty(X,E))^\Lambda$, where $f_\lambda = f \in \mathcal{C}^\infty(X,E)$, for $\lambda \in \Lambda$.

11.4. Comments

1) The model theoretic, [186], construction of *reduced power*, although hardly known as such among so called working mathematicians, happens nevertheless to appear in quite a number of important places in mathematics at large. For a sample of them, one can note the following. The Cauchy-Bolzano construction of the field $\mathbb{R}$ of usual real numbers is in fact a reduced power of the rational numbers $\mathbb{Q}$. More generally, the completion of any metric space is a reduced power of that space. Furthermore, this is but a particular case of the fact that the completion of any uniform topological space is a reduced power of that space. Also, in a rather different direction, the field $^*\mathbb{R}$ of nonstandard real numbers can be obtained as a reduced power of the usual field $\mathbb{R}$ of real numbers.

In view of the above, the use of reduced powers in the construction of differential algebras of generalized functions should not be seen as much else but a further application of that basic construction in model theory, this time to the study of large classes of singularities.

As for dealing with singularities, there is a strongly entrenched trend to approach them with nothing else but methods of analysis, functional analysis, topology, complex functions. And this trend is particularly manifest in various theories of generalized functions.

On the other hand, as seen in the previous sections, the issue of singularities of generalized functions boils down to a rather simple and basic *algebraic conflict*. Consequently, the study of singularities of generalized functions through methods which are primarily of analysis, functional analysis, topology, or complex functions has the *double disadvantage* of

- unnecessarily complicating the situation,
- missing the root of the problem.

The Colombeau algebras do to a good extent fall under the above double disadvantage, and as result, mentioned in sections 0, 9, they can only deal with a rather limited class of singularities of generalized functions.

2) An important property of many reduced powers is the presence of *infinitesimals*, [186,138]. This fact, as it happens, has not yet been given its due consideration in the study of differential algebras of generalized functions.

3) A large class of *scalars* which most likely may have a considerable relevance in physics is given by reduced power algebras built upon the field $\mathbb{R}$ of usual real numbers. Indications in this regard can be found in [63,80,87,108,110,113,116,120,121,127,131-133,136,140,144].

12. Conflict between Singularities, Multiplication and Differentiation

As it can happen in mathematics, the true message of the so called 1954 Schwartz impossibility, [245], message long disregarded, and in fact, misunderstood, [215], is that there is a most basic, simple, and purely *algebraic conflict* between

- singularities, in particular, discontinuities,
- multiplication, and in general, nonlinear operations, and
- differentiation.

12.1. Is the Dirac δ Distribution Identically Zero ?

Indeed, this conflict is so basic and simple that it can already be seen in its purely algebraic nature with the Heaviside function $H : \mathbb{R} \longrightarrow \mathbb{R}$.

Clearly, with the usual multiplication of functions, we have

$$H^m = H, \quad m \in \mathbb{N}, \; m \geq 1 \tag{12.1}$$

Now the usual derivation

$$D : \mathcal{C}^1(\mathbb{R}) \longrightarrow \mathcal{C}^0(\mathbb{R}) \tag{12.2}$$

is a *linear* operator which satisfies the *Leibnitz* rule of product derivative, namely

$$D(fg) = (Df)g + f(Dg), \quad f, g \in \mathcal{C}^1(\mathbb{R}) \tag{12.3}$$

And if we want to *extend* D to $H \notin \mathcal{C}^1(\mathbb{R})$ with the *preservation* of its linearity and Leibnitz rule, then for $m \in \mathbb{N}, \; m \geq 2$, we have

$$\begin{aligned} D(H^m) = D(HH^{m-1}) = (DH)H^{m-1} + H(D(H^{m-1})) = \\ = (DH)H + (D(H^{m-1}))H \end{aligned} \tag{12.4}$$

assuming the *commutativity* of multiplication, wherever the range of the extended D would be. Repeating the above, we eventually obtain

$$(12.5) \qquad D(H^m) = m(DH)H$$

thus in view of (12.1), it follows that

$$(12.6) \qquad DH = m(DH)H, \quad m \in \mathbb{N}, \ m \geq 2$$

and therefore, if we suppose that the range of the extended D is a module over $\mathbb{Q}$, then

$$(12.7) \qquad (1/m' - 1/m)DH = 0, \quad m, m' \in \mathbb{N}, \ m, m' \geq 2$$

Assuming further that for elements F in the range of the extended D we have the simplification property

$$(12.8) \qquad kF = 0 \quad \Longrightarrow F = 0$$

for any given $k \in \mathbb{Q}, \ k \neq 0$, the relation (12.7) yields

$$(12.9) \qquad DH = 0$$

Here, however, we recall that in various applicative instance, one desires the relation

$$(12.10) \qquad DH = \delta$$

where δ is the Dirac distribution. Thus (12.9) becomes

$$(12.11) \qquad \delta = 0$$

which, again, is not desirable in a variety of applications of the Dirac distribution.

Remark 12.1.

And why the Heaviside function ?

Simple indeed : the equation

$$(12.12) \qquad \xi^m = \xi, \quad m \in \mathbb{N},\ m \geq 2$$

has in $\mathbb{R}$ the solutions

$$(12.13) \qquad \xi = 0 \ \text{ and } \ \xi = 1$$

which are precisely the only two values of the Heaviside function

$$(12.14) \qquad H = \chi_{[0,\infty)}$$

where for any sets $A \subseteq X$, we denote by χ_A the *characteristic* function of A.

And a discontinuous function on $\mathbb{R}$ has to have at least two different values. Thus the Heaviside function H is a simplest such discontinuous function since it has only one discontinuity, namely at $x = 0 \in \mathbb{R}$.

12.2. The Algebra of Universal Differentiation

Algebra turns out to be a natural setup for a *Universal Differentiation*. Indeed, a rather general, thus basic and simple, *purely algebraic* setup for differentiation is as follows, see Lang S : Algebra (Third Edition). Addison-Wesley, New York, 1993, pp. 746-749.

Let R be a commutative ring, A a commutative R-algebra, and M an A-module. A mapping

$$(12.15) \qquad D : A \longrightarrow M$$

is called an *R-differentiation*, if and only if

$$(12.16) \qquad D \ \text{ is } R\text{-linear}$$

and satisfies the Leibniz rule of product derivative, namely

(12.17) $\quad D(ab) = a(Db) + (Da)b, \quad a, b \in A$

Clearly

(12.18) $\quad D(R) = 0$

We also note that the set

(12.19) $\quad Der_R(A, M)$

of such differentiations is an A-module, with the multiplication

(12.20) $\quad A \times Der_R(A, M) \ni (a, D) \longmapsto aD \in Der_R(A, M)$

where

(12.21) $\quad (aD)(b) = a(Db), \quad b \in A$

Definition 12.1.

Let R be a commutative ring, A a commutative R-algebra. A *universal R-differentiation* over A is any R-differentiation Δ on A, namely

(12.22) $\quad \Delta : A \longrightarrow \Omega$

where Ω is an A-module, such that for every R-differentiation $D : A \longrightarrow M$, there exists a unique A-homomorphism $f : \Omega \longrightarrow M$, that gives the commutative diagram

(12.23)
$$\begin{tikzcd} A \arrow[rr, "\Delta"] \arrow[dr, "D"'] & & \Omega \arrow[dl, "f"] \\ & M & \end{tikzcd}$$

$\square$

And as with universal properties, it follows easily that a universal R-differentiation Δ on A is *uniquely* determined, up to a respective unique isomorphism. Thus, in terms of category theory, we have the functorial isomorphism

$$(12.24) \qquad Der_R(A, M) \approx Hom_A(\Omega, M)$$

Theorem 12.1.

Let R be a commutative ring, A a commutative R-algebra. Then

$$(12.25) \qquad \Delta : A \longrightarrow \Omega = J/J^2$$

is a universal differentiation on A, where

$$(12.26) \qquad J = ker m_A$$

for the multiplication homomorphims

$$(12.27) \qquad m_a : A \bigotimes A \ni a \bigotimes b \longmapsto ab \in A$$

while

$$(12.28) \qquad \Delta : A \ni a \longmapsto 1 \bigotimes a - a \bigotimes 1 \in J/J^2$$

Consequently, (12.24) becomes

$$(12.29) \qquad Der_R(A, M) \approx Hom_A(J/J^2, M)$$

$\square$

It is worth noting that the above universal differentiation further leads to a de Rham complex, as well as other applications, see Lange.

12.3. The Natural Setup for Differentiation

A conclusion of *fundamental importance* for the nonlinear theory of generalized functions based on differential algebras of generalized functions, see 46F30, is that the natural set up is *not* within any single such algebra A with a respective differentiation

$$(12.30) \qquad D : A \longrightarrow A$$

but rather with the differentiation acting between *different algebras*

$$(12.31) \qquad D : A \longrightarrow \widetilde{A}$$

Indeed, as seen in more detail in section 15, the simplifying assumption (12.30) does in fact lead to the *tacit and very strong* requirement that the elements such algebras are no less than *infinitely differentiable*, thus they are in certain ways similar to the $\mathcal{C}^\infty$ functions. On the other hand, what is actually the natural setup, namely in (12.31) leads to algebras which only allow a *finite differentiation* of their elements, thus they are similar to the various spaces of functions $\mathcal{C}^m$, with $0 \leq m < \infty$.

And clearly, the natural setup (12.31) leads inevitably to *chains of algebras of generalized functions*, instead of one or another single such algebra, as seen in section 15.

Now the essential difference between the strongly restrictive setup (12.30), and on the other hand, the natural setup (12.31) is that the former can only allow certain variants of multiplication of functions and generalized functions with singularities, as detailed in section 15.

In this way, it may be considered that the more true message of the 1954 Schwartz impossibility result is the need to deal with singularities - and even more so in a nonlinear context - within what proves to be the natural framework for differentiation, namely, (12.31), and thus, indeed within *chains of algebras of generalized functions*.

As it happens however, due to various reasons, mainly as it appears the present preponderance of analysts among those involved in the nonlinear theory of generalized functions, the above essential difference between the setups in (12.30), and on the other hand, the natural one on (12.31), is hardly at all understood, with the exclusive focus still being on the particular and strongly restrictive setup in (12.30).

12.4. Examples of Multiplication of Singularities

We shall illustrate the considerably different ways the above frameworks (12.30) and (12.31) can operate by presenting a few corresponding results on the multiplication of generalized functions with singularities, see [7, pp. 1-20].

It is convenient to consider the mentioned purely algebraic conflict within the somewhat larger context of

- insufficient smoothness
- multiplication
- differentiation

and note that the mentioned conflict can be seen as a *limitation upon compatibility* between the algebraic and differential structures, when they are considered beyond the classical framework of smooth functions, thus are supposed to contain entities that are in certain sense generalized functions.

Let us start with the continuous functions

$$x_+,\ x_- \in \mathcal{C}^0(\mathbb{R}) \setminus \mathcal{C}^1(\mathbb{R}) \tag{12.32}$$

given by

$$x_+ = \left| \begin{array}{l} x \ \text{ if } \ x \geq 0 \\ \\ 0 \ \text{ if } \ x < 0 \end{array} \right. \qquad x_- = \left| \begin{array}{l} x \ \text{ if } \ x \leq 0 \\ \\ 0 \ \text{ if } \ x > 0 \end{array} \right. \tag{12.33}$$

Then within the algebra $\mathcal{C}^0(\mathbb{R})$, we have the relation

$$x_+ + x_- = x = id_{\mathbb{R}} \tag{12.34}$$

and in particular, the multiplication

$$(12.35) \qquad xx_+ = (x_+)^2, \qquad xx_- = (x_-)^2$$

Now obviously

$$(12.36) \qquad (x_+)^2,\ (x_-)^2 \in \mathcal{C}^1(\mathbb{R}) \setminus \mathcal{C}^2(\mathbb{R})$$

and with the usual differentiation, we have

$$(12.37) \qquad Dx = 1, \quad D((x_+)^2) = 2x_+, \quad D((x_-)^2) = 2x_-$$

On the other hand, with the distributional differentiation in $\mathcal{D}'(\mathbb{R})$, we have

$$(12.38) \qquad Dx_+ = H, \quad D^2 x_+ = DH = \delta$$

12.4.1. An Algebra of Continuous Functions

We shall consider a rather simple and small algebra which need contain only a few continuous functions, yet on which, within the restrictive framework of (12.30), there is already an incompatibility between lack of enough smoothness, multiplication and differentiation. It follows that discontinuities are, in fact, not necessary to lead to such an incompatibility.

Let us, indeed, consider any commutative unital algebra A on $\mathbb{R}$ which has the following properties

$$(12.39) \qquad 1,\ x,\ x_+,\ x_- \in A$$

(12.40) $\quad$ 1 is the unit element in A and the relations (12.34), (12.35) hold in A

furthermore, there is a differentiation on A, see (12.30)

$$(12.41) \qquad D : A \longrightarrow A$$

thus D is linear and satisfies the Leibniz rule of product derivative.

Then the following relations hold in A

$$(12.42) \qquad x D x_+ = x_+, \quad x D x_- = x_-$$

$$(12.43) \qquad x_+ D x_+ = x_+, \quad x_- D x_- = x_-$$

$$(12.44) \qquad x_+ D x_- = x_- D x_+ = 0$$

$$(12.45) \qquad x D^2 x_+ = x D^2 x_- = 0$$

$$(12.46) \qquad (Dx_+)(Dx_-) = x_+ D^2 x_- = x_- D^2 x_+ =$$

$$= x_+ D^2 x_+ = x_- D^2 x_- = 0$$

$$(12.47) \qquad (Dx_+)^2 = Dx_+, \quad (Dx_-)^2 = Dx_-$$

$$(12.48) \qquad D^2 x_+ = D^2 x_- = 0$$

Remark 12.2.

1) The interest in the above multiplication results is that they are conflicting already with the differentiation of Schwartz distributions. Indeed, in the latter, we have $D^2 x_+ = DH = \delta \neq 0$, while (12.48) contradicts that relation.

2) The above algebra need not contain less smooth functions than continuous ones.

12.4.2. An Algebra with a Singular Function

Let us now allow a singular generalized function to belong to the algebra A. More precisely, we suppose that a generalized function having some of the well known properties of the Dirac δ distribution belongs to A. For convenience, we shall denote that generalized function again by δ, and we shall suppose about it only that

(12.49) $\quad \delta \in A, \quad \delta \neq 0 \in A$

as well as, see (12.51) below

(12.50) $\quad x\delta = 0 \in A$

Further, about A, let us suppose that

(12.51) $\quad x^m \in A, \quad m \in \mathbb{N}$

(12.52) the multiplication in A induces
on the monomials (12.51) the usual multiplication

(12.53) 1 is the unit element in the algebra A

(12.54) D applied to the monomials (12.51)
is the usual derivative

Then the following relations hold in A

(12.55) $\quad x^p D^q \delta = 0 \in A, \quad p, q \in \mathbb{N},\ p > q$

(12.56) $\quad (p+1)D^p \delta + x^{p+1}\delta = 0 \in A$

(12.57) $\quad x^p (D^p \delta)^q = 0 \in A$

(12.58) $\quad (\delta)^2 = \delta D\delta = 0 \in A$

Remark 12.3.

The degeneracy result (12.58) is in conflict with many adhoc distribution multiplications used among others in physics, [175], as well as with the multiplication in Colombeau algebras, according to which $(\delta)^2 \neq 0$.

12.4.3. Another Algebra with a Singular Function

Let us relax the conditions required on the algebra A in the previous subsection, but still keeping the singular generalized function δ in the algebra. Namely, we suppose that (12.49), (12.51), (12.53) and (12.54) hold. Therefore, we are still within the restrictive framework of (12.30).

If we now suppose that, contrary to (12.58), we have

$$(12.59) \qquad \delta, \delta^2, \delta^3, \ldots \neq 0 \in A$$

then

$$(12.60) \qquad x^m \delta \neq 0 \in A, \qquad m \in \mathbb{N}$$

Remark 12.4.

The relevance of (12.60) above is as follows.

Within the Schwartz distributions, the Dirac δ distribution is known to have the property, see (12.50)

$$(12.61) \qquad x\delta = 0 \in \mathcal{D}'(\mathbb{R})$$

This clearly means that the singularity of δ at $x = 0 \in \mathbb{R}$ is *less* than that of the function $f(x) = 1/x, \; x \in \mathbb{R}, \; x \neq 0$.

On the other hand, (12.60) implies that in the respective algebras A, the singularity of δ at $x = 0 \in \mathbb{R}$ is in fact *higher* than that of all the functions $f_m(x) = 1/(x^m), \; x \in \mathbb{R}, \; x \neq 0$, with $m \in \mathbb{N}$.

12.4.4. Differentiations of the type $D : A \longrightarrow \widetilde{A}$

The above differentiations in subsections 12.4.1. - 12.4.3. were within the restrictive framework of (12.30), where the differentiation operates within the same single algebra, namely, $D : A \longrightarrow A$.

Here instead, we briefly consider the *natural* framework (12.31) where differentiation may operate between two *different* algebras, namely, $D : A \longrightarrow \widetilde{A}$.

Once again, it is worth pointing out that a differentiation $D : A \longrightarrow A$ renders the elements of the algebra A *infinitely differentiable*, and does so either we like it or not. And the inevitable effect is that it leads to *particular* case of multiplication in the case of generalized functions with singularities.

On the other hand, as seen in subsection 12.2., the natural, and in fact, *universal* setup for differentiation in algebras is of differentiations $D : A \longrightarrow \widetilde{A}$ between possibly different algebras. And as the related universality property shows it, such a differentiation does *not* introduce any restrictions on multiplication, and in particular, on the multiplication of generalized functions with singularities, since in the commutative diagrams (12.23), the differentiations $D : A \longrightarrow M$ can not only be arbitrary, but furthermore, M can be modules over A, and not only algebras.

Nevertheless, as seen in the sequel, even within such an unrestricted and natural framework (12.30), one cannot so easily reproduce the multiplication of functions in $\mathcal{C}^m \setminus \mathcal{C}^{m+1}$, with $0 \leq m < \infty$, unless one conflicts with some of the most simple and customary products allowed within the Schwartz distributions and involving the Dirac δ distribution.

Let, indeed, be given three algebras on $\mathbb{R}$, namely

$$(12.62) \qquad A^2, \;\; A^1, \;\; A^0$$

together with linear operators called differentiations

$$(12.63) \qquad A^2 \xrightarrow{D} A^1 \xrightarrow{D} A^0$$

which satisfy the following weaker version of the Leibniz rule of product derivative

$$(12.64) \qquad D(fg) = (Df)g + f(Dg), \quad f, g \in A^{i+1} \cap A^i, \;\; 0 \leq i \leq 1$$

We shall suppose that

(12.65) $\quad x, \quad x(ln|\,x\,| - 1), \quad x^2(ln|\,x\,| - 1) \in A^2$

(12.66) $\quad 1, \quad x, \quad x(ln|\,x\,| - 1) \in A^1$

(12.67) $\quad 1, \quad x \in A^0$

We note that $x(ln|\,x\,| - 1) \in \mathcal{C}^0(\mathbb{R}) \setminus \mathcal{C}^1(\mathbb{R})$, while $x^2(ln|\,x\,| - 1) \in \mathcal{C}^1(\mathbb{R}) \setminus \mathcal{C}^2(\mathbb{R})$.

Further, the multiplication in A^2 is such that

(12.68) $\quad (x(ln|\,x\,| - 1))(x) = x^2(ln|\,x\,| - 1)$

The differentiation $D : A^2 \longrightarrow A^1$ is the same with the usual one for the functions

(12.69) $\quad x, \quad x^2(ln|\,x\,| - 1)$

The differentiation $D : A^1 \longrightarrow A^0$ is the same with the usual one for the function

(12.70) $\quad x$

(12.71) $\quad 1$ is the unit element in the algebras A^1 and A^0

(12.72) $\quad A^0$ is associative

Then, [4, p. 111], [6, p. 31], [7, p.268], there *cannot* exist $\delta \in A^0, \ \delta \neq 0$, such that in A^0 we have

(12.73) $\quad x\delta = 0 \in A^0$

12.4.5. Conclusions on Incompatibility

The result in (12.73) is in fact stronger than the similar result in the 1954 Schwartz impossibility, since the above requirements for the former are weaker, see [4, pp. 289-292], [6, pp. 27-30], [7, pp. 7-9]. In particular, the algebras A^2, A^1, A^0 need not relate to one another by any inclusion relation. Further, none of these algebras is supposed to be associative or commutative. And last, only the algebra A^0 is supposed to be associative.

The results in subsections 12.4.1 - 12.4.4 related to multiplications that involve generalized functions with singularities show a further limitation of the interest in the particular kind of embedding of the $\mathcal{D}'$ distributions into algebras of generalized functions, embeddings which preserve the distributional derivatives, [253]. Indeed, as seen above, there is an inevitable price to pay for such embeddings, namely, one has to accept multiplications for insufficiently smooth functions in $\mathcal{C}^m \setminus \mathcal{C}^{m+1}$, with $0 \leq m < \infty$, which are quite different from the usual multiplication of such functions.

Consequently, one simply cannot avoid the most basic and purely algebraic incompatibility between the algebra structure, and on the other hand, the differentiation, when it comes to insufficiently smooth functions in $\mathcal{C}^m \setminus \mathcal{C}^{m+1}$, with $0 \leq m < \infty$. And all one may be able to do is to require more on one of the two sides of that incompatibility, and then inevitably, pay the prize on the other side.

The embeddings in [253] are in a way maximal with respect to the properties of differentiation, and thus they lose a lot with respect to the multiplication of insufficiently smooth functions in $\mathcal{C}^m \setminus \mathcal{C}^{m+1}$, with $0 \leq m < \infty$.

It is therefore a mere matter of preference how one may decide to deal with the mentioned incompatibility. And needless to say, such a preference can be influenced by features of the specific applications of the algebraic nonlinear theory of generalized functions.

12.5. Open Problem : Do We Need the Leibniz Rule ?

The above incompatibility, with its inevitable trade-off between preserving differentiation, or on the contrary, multiplication for insufficiently smooth functions, and other functions or generalized functions with singularities, may raise the question whether one should consider differentiations which do no longer satisfy the Leibniz rule of product derivative, and instead, may satisfy a modified or weaker form of it ?

13. Stability, Generality and Exactness

The reduced power structure of differential algebras of generalized functions leads naturally to the following three basic concepts, and thus to the corresponding inevitable *interplay* between them, as well as a range of properties related to them, [4, pp. 13-16], [6, pp. 224-229], [7, pp. 50-54], namely

- stability
- generality
- exactness

And the respective concepts arise already in a more general context as seen next. Let $\Omega \subseteq \mathbb{R}^n$ be any nonvoid open set, and a partial differential operator act according to

$$(13.1) \qquad T(x, D) : \mathbb{E} = \mathcal{S}/\mathcal{V} \longrightarrow \mathbb{A} = \mathcal{A}/\mathcal{I}$$

where

$$(13.2) \qquad \mathcal{V} \subset \mathcal{S} \subseteq (\mathcal{C}^\infty(\mathbb{R}^n))^{\mathbb{N}}$$

are vector spaces, while

$$(13.3) \qquad \mathcal{I} \subset \mathcal{A} \subseteq (\mathcal{C}^\infty(\mathbb{R}^n))^{\mathbb{N}}$$

with $\mathcal{A}$ a subalgebra, while $\mathcal{I}$ is an ideal in $\mathcal{A}$.

Now, in the above setup, any solution $U \in \mathbb{E}$ of the PDE

$$(13.4) \qquad T(x, D)U = 0$$

has the form

$$(13.5) \qquad U = s + \mathcal{V} \in \mathcal{S}/\mathcal{V} = \mathbb{E}$$

for suitable $s \in \mathcal{S}$.

Let us now consider the relationship between U and s above. Clearly, the same U may correspond to different $s \in \mathcal{S}$. More precisely, instead of s, we can take any $t \in \mathcal{S}$ for which $t - s \in \mathcal{V}$.

It follows that the *maximal stability* of U in the above setup means

$$(13.6) \qquad \text{maximal} \quad \mathcal{V}$$

Further, since we may typically deal with generalized solutions U, when it comes to the issue of *existence* of such solutions, it is useful at that stage to have the space $\mathbb{E}$ *large*, which obviously means

$$(13.7) \qquad \text{large} \quad \mathcal{S} \quad \text{and small} \quad \mathcal{V}$$

Lastly, the PDE in (13.4) is satisfied in $\mathbb{A} = \mathcal{A}/\mathcal{I}$. That is, the equality in the PDE in (13.4) takes place in $\mathbb{A} = \mathcal{A}/\mathcal{I}$. And under rather general conditions, [3,4,6,7], we have

$$(13.8) \qquad T(x, D)U = T(x, D)(s + \mathcal{V}) = w + \mathcal{I} \in \mathcal{A}/\mathcal{I} = \mathbb{A}$$

with $w \in \mathcal{A}$. And then (13.4) is equivalent with

$$(13.9) \qquad w \in \mathcal{I}$$

and since $\mathcal{I}$ is an ideal in $\mathcal{A}$, it is further equivalent with

$$(13.10) \qquad w\mathcal{A} \subseteq \mathcal{I}$$

which we call the *exactness* property of the solution U. And then clearly, a better exactness means

$$(13.11) \qquad \text{large} \quad \mathcal{A} \quad \text{and small} \quad \mathcal{I}$$

Here it is important to note that the above three conditions of stability, generality and exactness are in general *conflicting*. Therefore, in various specific instances of PDEs, one has to decide according to the corresponding interests about the particular ways of *interplay* between these three conditions, in order to satisfy them simultaneously to some extent.

We can also note that both stability and generality refer exclusively to the space $\mathbb{E} = \mathcal{S}/\mathcal{V}$, and as such, do not involve the partial differential operator $T(x, D)$, or any solution of the corresponding PDE.

On the other hand, the exactness involves both spaces and the partial differential operator in (13.1). However, it still does not involve any solution of the corresponding PDE.

14. The Inevitable Infinite Branching of Multiplication

Let us briefly review several relevant facts. Singularities appear in numerous important mathematical models used in physics. And in most of such cases singularities are involved in essentially nonlinear contexts. As mentioned, for more than four decades by now, a general enough nonlinear theory of singularities has been developed through the introduction of a large variety of differential algebras of generalized functions.

A critically important related feature is that, above certain levels of singularities, the operation of multiplication, and in general, nonlinear operations on such singularities, do inevitably branch in infinitely many ways, without the possibility for the existence of some unique, natural or canonical way such nonlinear operations may be performed. Consequently, the choice in such branchings has to come from extraneous considerations.

Singularities have been present in mathematics ever since the simple and natural looking issue of dividing by zero. And with the mathematization of modern physics, they are causing major difficulties in quite a number of disciplines of that field of science. Needless to say, various branches of engineering, as well as other fields of science and technology encounter similar difficulties due to singularities in the mathematical models employed.

Here, as before, we shall consider singularities of functions $f : \Omega \longrightarrow \mathbb{R}$, where Ω is some Euclidean domain. In this way, and as mentioned, the following *two fundamental features* of singularities will be of concern, namely

- SIZE : the extent of the set of singularities as a subset in the domain of definition of functions or generalized functions,

- BEHAVIOUR : the behaviour of functions or generalized functions in the neighbourhood of singularities.

The main conclusion obtained will be as follows :

- In case no limitations are imposed on the above two features of singularities, as soon as multiplication, and in general, nonlinear operations are effectuated with generalized functions, there is an *inevitable infinite branching* in the way such operations can be defined. In other words, there is *no* canonical, natural or unique way multiplication, and in general, nonlinear operations can be defined for generalized functions. Consequently, the specific choice of the result of multiplication, and in general, of nonlinear operations on generalized functions has to be made based on *extraneous* considerations.

A first systematic and far reaching mathematical approach to singularities was given in the 19th century by the theory of functions of one complex variable. That was the time which led, among others and not necessarily in a manner related to singularities, to the celebrated and yet unsolved Riemann Hypothesis, which shows the depth of the respective theory.
As for singularities in the context of functions of one complex variable, one should not forget the Great Picard Theorem, according to which an analytic function in the neighbourhood of an isolated singularity point that is an essential singularity will take on all possible complex values infinitely often, with at most a single exception.

Consequently, in the neighbourhood even of one single and isolated singularity, one can expect a rather arbitrary behaviour, when one deals with more general functions than analytic ones. Not to mention the situation when the singularity points form a considerably larger subset in the domain of definition of a function. Therefore, the consideration of the above two aspects related to singularities is indeed appropriate, and in fact, necessary in the case of a deeper going and more wide ranging approach.

Beyond the confines of analytic functions, it was first the linear theory of Sobolev spaces, and then, in a more clear and systematic manner,the linear theory of Schwartz distributions which, starting with the mid 20th century, gave in certain respects a considerably more powerful and general treatment of singularities.

The major limitation of the Schwartz approach is in its essential confinement to linear situations, and its consequent inability to deal with singularities in a nonlinear context, and do so in a convenient, general and systematic manner, without the recourse to what often are merely adhoc approaches. This linear limitation has over the years become quite clear in view of the inability of the distributions to deal with singularities in nonlinear contexts such as those of general relativity, or more widely, differential geometry.
Such a state of affairs contrast sharply with the remarkable natural ease and clarity the earlier complex function theory was applicable within a nonlinear context, restricting itself, as it did naturally, to singularities that occur in analytic situations.
Furthermore, as a rather unfortunate event, the more notorious than celebrated 1954 paper of Schwartz [245] claimed to prove that a nonlinear theory of distributions would altogether be undesirable, if not in fact, impossible ...
Amusingly, this claim has attained a wider acceptance, [215], and consequently has for long distorted the perception of the situation concerning singularities within a general enough nonlinear context ...

History, nevertheless, still seems to hang somewhat uneasily upon the issue of singularities ...
Indeed, due to the remarkable generality, clarity and power of the Schwartz linear distributional approach, the issue of algebraic operations on singularities has naturally and inevitably been *restricted* to addition alone, without a similarly general consideration of the operation of *multiplication*, an operation which ended up being in fact implicitly excommunicated form any possible suitable and general enough theory of singularities, in view of the mentioned 1954 Schwartz paper and its long ongoing misinterpretations.

And the *surprising* and hardly yet noted fact here is as follows. Although multiplication is closely related to addition, being in certain ways but a repeated addition, when it nevertheless comes to singularities, an *essential* difference appears between these two basic algebraic operations. Namely

- addition can be extended from usual functions, be they regular or with singularities, to all sort of generalized functions, and such an extension appears to be naturally done in a unique, canonical manner,

while on the other hand

- due to most simple algebraic, more precisely, ring theoretic reasons, the extension of multiplication from usual functions, be they regular or with singularities, to generalized functions does *no* longer have such a naturally unique canonical way.

Here is, therefore, the root of what is in fact no less than an *infinite branching* of the ways multiplications can naturally be defined for generalized functions.
And no wonder, this root has so far been mostly missed due to the mentioned implicit omission to consider multiplication of singularities within a general enough, and not merely adhoc, context ...

Within the general nonlinear treatment of singularities in 46F30, so far three classes of differential algebras of generalized functions have been used in a variety of problems, mainly for the solution of large classes of nonlinear systems of partial differential equations.
It is instructive to recall the way these three classes of algebras relate to the above two fundamental features of singularities.
In this regard, all these algebras are able to deal with the singularities the Schwartz linear theory of distributions can, since each of these algebras contains all the Schwartz distributions.

The issue, therefore, is to what extent these algebras are able to deal with additional singularities. Let us then consider these algebras defined on any given Euclidean open set $\Omega \subseteq \mathbb{R}^n$. Consequently, the generalized functions will extend various classes of usual functions $f : \Omega \longrightarrow \mathbb{R}$.

Here it should be pointed out again that the class of admissible singularities is large in no less than *two* significantly useful ways :

- First, the singularities of the functions $f : \Omega \longrightarrow \mathbb{R}$ considered can be given by arbitrary subsets $\Sigma \subset \Omega$, subject to the only condition that their complementary $\Omega \setminus \Sigma$, that is, the set of regular, or in other words, non-singular points, be dense in Ω. For instance, if $\Omega = \mathbb{R}^n$ is an Euclidean space, then the set $\Sigma \subset \Omega$ of singularities can be the set of all points with at least one irrational coordinate. Indeed, in this case the set $\Omega \setminus \Sigma$ of non-singular, or regular points is the set of points with all coordinates rational numbers, thus it is dense in Ω. A relevant and rather remarkable fact to note in this case is that the cardinal of the singularity set Σ is *strictly larger* than the cardinal of the set of non-singular points, namely, $\Omega \setminus \Sigma$.

- Second, there is no restriction on the behaviour of functions $f : \Omega \longrightarrow \mathbb{R}$ in the neighbourhood of points in their singularity sets $\Sigma \subset \Omega$.

Related to this second freedom in dealing with singularities, one should not forget its significant importance in applications. Indeed, as stated in the mentioned Picard Great Theorem, an analytic function in the neighbourhood of an isolated singularity point which is an essential singularity takes on all possible complex values infinitely often, with at most a single exception.
Consequently, in the neighbourhood of a singularity, one can expect a rather arbitrary behaviour when one deals with more general functions than analytic ones.

The *first* class of algebras of generalized functions was aimed to deal with singularities within a systematic and as widely applicable as possible nonlinear theory, [16,17,22,23,3,4,6,7,39,43,45,47,50,53-57,59,61, 70,71,98,99, 105,143,165]. This class contains as *particular cases* all the subsequent classes of differential algebras of generalized functions constructed so far.

Within this largest class, a special subclass - of so called *nowhere dense* algebras - was developed from the beginning, class which is able to deal with arbitrary closed nowhere dense subsets $\Gamma \subset \Omega$ of singularities, while no restrictions whatsoever are imposed on the behaviour of generalized functions in the neigbourhood of singularities.

The *second* class of algebras, [172-178], requires polynomial growth conditions on generalized functions in the neigbourhood of singulari-

ties.
In this regard, and as mentioned, these algebras of generalized functions - which are but a particular case of the infinite variety of all possible differential algebras of generalized functions introduced in [16,17,3,4] - suffer from a severe limitation. Namely, in the neighbourhood of singularities of their generalized functions, these algebras require a polynomial type growth condition, thus they cannot deal even with isolated singularities such as essential singularities of analytic functions.

The *third* class of algebras, [3,4,6,7,9,55-57,59,61,70,71], is much more powerful than the class of so called *nowhere dense* ones, since these algebras are able to deal with arbitrary subsets $\Sigma \subset \Omega$ of singularities, subject to the mild condition that the respective complementary subsets $\Omega \setminus \Sigma$ be dense in Ω, while again, no restrictions whatsoever are imposed on the behaviour of generalized functions in the neigbourhood of singularities in Σ.
An important fact to note here is that the subsets Σ of singularities can have a cardinal *larger* than that of the subsets $\Omega \setminus \Sigma$ of nonsingular or regular points, since the condition that $\Omega \setminus \Sigma$ be dense in Ω can be satisfied even when $\Omega \setminus \Sigma$ is merely a dense countable subset of Ω, in which case Σ must of course be uncountable.

As for the nonlinear operations on singularities, the nowhere dense algebras and those in the third class allow arbitrary smooth such operations, while in the second class only smooth operations with polynomial growth are possible.

As a consequence of its restriction upon singularities, as well as upon operations on singularities, the second class of algebras cannot deal with a number of important problems which are easily treated within the nowhere dense algebras, or those in the third class. Among such problems are the following.

The global version of the classical Cauchy-Kovalevskaia theorem for solutions of analytic systems of nonlinear partial differential equations cannot even be formulated, let alone solved, within the second class of algebras.

On the other hand, the first class of algebras is already able to produce such a global version on the existence of solutions, [6,7,39].

Arbitrary Lie group actions, which are of major importance in the solution of partial differential equations cannot be defined within the second class of algebras.
Here again, the nowhere dense algebras are already enough to define globally arbitrary Lie group actions.
And as one of the consequences, one can for the first time obtain the complete solution of Hilbert's Fifth Problem, [9].

This problem, again, cannot even be formulated, let alone solved, within the second class of algebras due to the polynomial mentioned type growth conditions which they require.

Also, when defining differential algebras of generalized functions in the case of domains Ω which are arbitrary finite dimensional smooth manifolds, the algebras in the first and third classes allow for considerably simpler constructions than those in the second class.

At a deeper analysis, however, one that is done in terms of sheaf theory, the essential difference between the nowhere dense algebras or those in the third class, and on the other hand, the algebras in the second class, is that the former are *flabby sheaves*, while the latter fail to be so, as mentioned earlier. And as is known, [219], the lack of the flabbiness property in the case of spaces of functions or generalized functions is an essential indicator of their limitations in dealing with singularities.

Lastly, it should be noted that, in [141], the study of a *fourth* class which is far larger then the above third class of algebras has been initiated.

14.1. Inclusion Diagrams and Reduced Power Algebras with the corresponding Ideals

It is an elementary property of the linear vector space $\mathcal{D}'(\Omega)$ of Schwartz distributions that it can be represented as the *quotient* vector space

$$(14.1) \quad \mathcal{D}'(\Omega) = \mathcal{S}^\infty(\Omega)/\mathcal{V}^\infty(\Omega)$$

of the *vector subspaces*

$$(14.2) \quad \mathcal{V}^\infty(\Omega) \ \longrightarrow \ \mathcal{S}^\infty(\Omega) \ \longrightarrow \ (\mathcal{C}^\infty(\Omega))^{\mathbb{N}}$$

with the arrows "$\longrightarrow$" representing usual inclusions "$\subseteq$", and

$$(14.3) \quad \mathcal{S}^\infty(\Omega) \ = \ \{ \ s = (\psi_\nu)_{\nu \in \mathbb{N}} \in (\mathcal{C}^\infty(\Omega))^{\mathbb{N}} \mid s \text{ converges weakly in } \mathcal{D}'(\Omega) \ \}$$

$$(14.4) \quad \mathcal{V}^\infty(\Omega) \ = \ \{ \ v = (\chi_\nu)_{\nu \in \mathbb{N}} \in \mathcal{S}^\infty(\Omega) \mid v \text{ converges weakly to } 0 \text{ in } \mathcal{D}'(\Omega) \ \}$$

The remarkable fact, which has always been there in (14.1) - (14.4), is that in the right hand term of (14.2) we have the *differential algebra* $(\mathcal{C}^\infty(\Omega))^{\mathbb{N}}$, yet in (14.2) one uses only vector subspaces of it. Indeed, it takes little imagination to try to replace (14.2) with

$$(14.5) \quad \mathcal{I}(\Omega) \ \longrightarrow \ \mathcal{A}(\Omega) \ \longrightarrow \ (\mathcal{C}^\infty(\Omega))^{\mathbb{N}}$$

where $\mathcal{A}(\Omega)$ is a subalgebra in $(\mathcal{C}^\infty(\Omega))^{\mathbb{N}}$, while $\mathcal{I}(\Omega)$ is an ideal in $\mathcal{A}(\Omega)$, and thus instead of the quotient vector space in (14.1), obtain the quotient *algebra*

$$(14.6) \quad A(\Omega) = \mathcal{A}(\Omega)/\mathcal{I}(\Omega)$$

which may allow a *nonlinear* theory of generalized functions, thus of singularities as well.

Indeed, for that purpose, it may be convenient to have the *inclusion*, that is, *linear embedding*

$$(14.7) \quad \mathcal{D}'(\Omega) \ \longrightarrow \ A(\Omega)$$

and of course, also suitable partial derivations on $A(\Omega)$, which in some convenient manner may extend the distributional partial derivations on $\mathcal{D}'(\Omega)$. Clearly, such partial derivations can easily be obtained on $A(\Omega)$, in case $\mathcal{A}(\Omega)$ and $\mathcal{I}(\Omega)$ are *invariant* under the natural termwise partial derivations

$$(14.8) \quad (\mathcal{C}^\infty(\Omega))^{\mathbb{N}} \ni s = (\psi_\nu)_{\nu \in \mathbb{N}} \longmapsto D^p s = (D^p \psi_\nu)_{\nu \in \mathbb{N}} \in (\mathcal{C}^\infty(\Omega))^{\mathbb{N}}$$

with $p \in \mathbb{N}^n$. Namely, if one has

$$(14.9) \quad D^p \mathcal{I}(\Omega) \subseteq \mathcal{I}(\Omega), \quad D^p \mathcal{A}(\Omega) \subseteq \mathcal{A}(\Omega), \quad p \in \mathbb{N}^n$$

then one can simply define for $p \in \mathbb{N}^n$, the corresponding partial derivation

$$(14.10) \quad A(\Omega) \ni s + \mathcal{I}(\Omega) \longmapsto D^p s + \mathcal{I}(\Omega) \in A(\Omega)$$

Let us for the moment, however, deal only with the algebraic aspects of (14.5) - (14.7). An obvious immediate and simple way to obtain (14.5) - (14.7) would be to construct *inclusion diagrams* of the form, [16,17,22,23,3,4,6,7,9,39,43,45,47,50,53-57,59,61,70,71,97-99,105,165]

$$(14.11) \quad \begin{tikzcd} \mathcal{I}(\Omega) \arrow[r] & \mathcal{A}(\Omega) \arrow[r] & (\mathcal{C}^\infty(\Omega))^{\mathbb{N}} \\ \mathcal{V}^\infty(\Omega) \arrow[u] \arrow[r] & \mathcal{S}^\infty(\Omega) \arrow[u] & \end{tikzcd}$$

which satisfy the condition

$$(14.12) \quad \mathcal{I}(\Omega) \bigcap \mathcal{S}^\infty(\Omega) = \mathcal{V}^\infty(\Omega)$$

a condition which is both *necessary* and *sufficient* for the existence of the *linear embedding*

$$(14.13) \quad \mathcal{S}^\infty(\Omega)/\mathcal{V}^\infty(\Omega) \ni s + \mathcal{V}^\infty(\Omega) \longmapsto s + \mathcal{I}(\Omega) \in \mathcal{A}(\Omega)/\mathcal{I}(\Omega)$$

thus equivalently, for (14.7).

Unfortunately however, inclusion diagrams (14.11) *cannot* be constructed in view of the simple fact that, [16,17,3,4]

$$(14.14) \qquad (\ \mathcal{V}^\infty(\Omega)\ .\ \mathcal{V}^\infty(\Omega)\ ) \bigcap \mathcal{S}^\infty(\Omega) \nsubseteqq \mathcal{V}^\infty(\Omega)$$

as simple counterexamples can show it. Indeed, it is easy to construct sequences $v = (\chi_\nu)_{\nu \in \mathbb{N}} \in \mathcal{V}^\infty(\Omega)$, such that $v^2 \in \mathcal{S}^\infty(\Omega)$, yet $v^2 \notin \mathcal{V}^\infty(\Omega)$. For instance, when $\Omega = \mathbb{R}$, one can take $\chi_\nu(x) = \cos(\nu x)$, and obtain indeed that $v \in \mathcal{V}^\infty(\Omega)$, $v^2 \in \mathcal{S}^\infty(\Omega)$, and furthermore v^2 converges weakly to $1/2$ in $\mathcal{D}'(\Omega)$, thus clearly $v^2 \notin \mathcal{V}^\infty(\Omega)$.

Consequently, one can turn to the immediately more involved inclusion diagrams, [16,17,3,4]

$$(14.15) \qquad \begin{array}{ccccc} \mathcal{I}(\Omega) & \longrightarrow & \mathcal{A}(\Omega) & \longrightarrow & (\mathcal{C}^\infty(\Omega))^{\mathbb{N}} \\ \uparrow & & \uparrow & & \\ \mathcal{V} & \longrightarrow & \mathcal{S} & & \\ \downarrow & & \downarrow & & \\ \mathcal{V}^\infty(\Omega) & \longrightarrow & \mathcal{S}^\infty(\Omega) & & \end{array}$$

where $\mathcal{V}$ and $\mathcal{S}$ are vector subspaces, such that the following three conditions hold

$$(14.16) \qquad \mathcal{I}(\Omega) \bigcap \mathcal{S} \ = \ \mathcal{V}$$

$$(14.17) \qquad \mathcal{V}^\infty(\Omega) \bigcap \mathcal{S} \ = \ \mathcal{V}$$

$$(14.18) \qquad \mathcal{V}^\infty(\Omega) + \mathcal{S} \ = \ \mathcal{S}^\infty(\Omega)$$

which, as it is easy to see, are both *necessary* and *sufficient* for the existence of the *linear embeddings*

$$(14.19) \quad \mathcal{D}'(\Omega) S \;\longmapsto\; s + \mathcal{V}^\infty(\Omega) \in \mathcal{S}^\infty(\Omega)/\mathcal{V}^\infty(\Omega)$$

$$(14.20) \quad \mathcal{S}/\mathcal{V} \ni s + \mathcal{V} \;\longmapsto\; s + \mathcal{V}^\infty(\Omega) \in \mathcal{S}^\infty(\Omega)/\mathcal{V}^\infty(\Omega)$$

$$(14.21) \quad \mathcal{S}/\mathcal{V} \ni s + \mathcal{V} \;\longmapsto\; s + \mathcal{I}(\Omega) \in A(\Omega) = \mathcal{A}(\Omega)/\mathcal{I}(\Omega)$$

where the mappings (14.19), (14.20) are in fact *vector space isomorphisms.*

Now clearly, (14.19) - (14.21) give the desired *linear embedding* (14.7) of the Schwartz distributions into algebras of generalized functions, namely

$$(14.22) \quad \mathcal{D}'(\Omega) \;\longrightarrow\; A(\Omega) \;=\; \mathcal{A}(\Omega)/\mathcal{I}(\Omega)$$

In view of (14.5), (14.6), the algebras of generalized functions $A(\Omega)$ in (14.7), (14.22) are nothing else but *reduced powers* of the algebra $\mathcal{C}^\infty(\Omega)$ of smooth functions on Ω.

The general model theoretic, [186], construction of reduced powers, although hardly known as such among so called working mathematicians, happens nevertheless to appear in quite a number of important places in Mathematics at large. For a sample of them, one can note the following. The Cauchy-Bolzano construction of the field $\mathbb{R}$ of usual real numbers is in fact a reduced power of the rational numbers $\mathbb{Q}$. More generally, the completion of any metric space is a reduced power of that space. Furthermore, this is but a particular case of the fact that the completion of any uniform topological space is a reduced power of that space. Also, in a rather different direction, the field ${}^*\mathbb{R}$ of nonstandard real numbers can be obtained as a reduced power of the usual field $\mathbb{R}$ of real numbers.

In view of the above, the use of reduced powers in the construction of algebras of generalized functions should not be seen as much else but a further application of that basic construction in Model Theory, this time to the study of large classes of singularities, and as such, to the solution of rather general nonlinear systems of partial differential equations, among others.

As seen in the sequel, the *ideals* $\mathcal{I}(\Omega)$ in (14.22) play the essential role in the inevitable infinite branching which occurs when multiplying generalized functions that have singularities above a certain level.

14.2. Off-Diagonality Characterization

A fundamental result in the nonlinear algebraic theory of generalized functions, see 46F30, is the simple and purely algebraic *characterization* of the algebras of generalized functions (14.7) which are constructed upon inclusion diagrams (14.15) - (14.21). In this regard, first we note that these inclusion diagrams can be simplified as follows. In the inclusion diagrams (14.15) there are *four* spaces to be chosen, namely $\mathcal{I}(\Omega), \mathcal{A}(\Omega), \mathcal{V}$ and $\mathcal{S}$. However, it is easy to see, [3,4,6,7], that such inclusion diagrams can be reduced to the simpler form depending only on *two* spaces $(\mathcal{V}, \mathcal{S}\,')$, namely

$$(14.23)\quad \begin{tikzcd} \mathcal{I}(\mathcal{V},\mathcal{S}\,') \arrow[r] & \mathcal{A}(\mathcal{V},\mathcal{S}\,') \arrow[r] & (\mathcal{C}^\infty(\Omega))^{\mathbb{N}} \\ \mathcal{V} \arrow[u] \arrow[r] \arrow[d] & \mathcal{V}\oplus\mathcal{S}\,' \arrow[u] \arrow[d] & \mathcal{U}^\infty(\Omega) \arrow[l] \\ \mathcal{V}^\infty(\Omega) \arrow[r] & \mathcal{S}^\infty(\Omega) & \end{tikzcd}$$

where $\mathcal{V}, \mathcal{S}\,'$ are vector subspaces in $\mathcal{S}^\infty(\Omega)$, while

$$(14.24)\quad \mathcal{U}^\infty(\Omega) \;=\; \{\, u_\psi = (\psi,\psi,\psi,\ldots) \mid \psi \in \mathcal{C}^\infty(\Omega) \,\} \subset \mathcal{S}^\infty(\Omega)$$

is the *diagonal* in the Cartesian product $(\mathcal{C}^\infty(\Omega))^{\mathbb{N}}$.

As for the conditions (1-16) - (14.18), they now become

$$(14.25) \quad \mathcal{V} \bigcap \mathcal{S}\,' \ = \ \{0\}$$

$$(14.26) \quad \mathcal{I}(\mathcal{V}, \mathcal{S}\,') \bigcap \mathcal{S}\,' \ = \ \{0\}$$

$$(14.27) \quad \mathcal{V}^\infty(\Omega) \oplus \mathcal{S}\,' \ = \ \mathcal{S}^\infty(\Omega)$$

The mentioned fundamental result regarding the *characterization* of algebras of generalized functions (14.7) constructed upon inclusion diagrams (14.15) - (14.21) as simplified in (14.23) - (14.27) is the following :

Theorem 14.1. [12-29]

Within a large class of inclusion diagrams (14.23) - (14.27), the conditions (14.25) - (14.27) are equivalent with the following *off-diagonality* one

$$(14.28) \quad \mathcal{I}(\mathcal{V}, \mathcal{S}\,') \bigcap \mathcal{U}^\infty(\Omega) \ = \ \{0\}$$

Remark 14.1.

It is both theoretically and practically important to note that, as seen in [12-29], there are *infinitely* many different inclusion diagrams (14.23) - (2-5). Moreover, they give *infinitely* many different corresponding algebras of generalized functions

$$(14.29) \quad A \ = \ \mathcal{A}(\mathcal{V}, \mathcal{S}\,')/\mathcal{I}(\mathcal{V}, \mathcal{S}\,')$$

which, see (14.22), contain the vector space $\mathcal{D}'(\Omega)$ of Schwartz distributions.

14.3. Inevitable Infinite Branching in the Multiplication of Singularities

And now, we can come to the main issue in this section, namely, to indicate the reason for the *inevitable* infinite possibilities in defining multiplication of generalized functions in case their singularities are above certain levels.

The remarkable fact in this regard is that the respective reason is of a simple *algebraic* nature, namely, it is a direct consequence of the off-diagonality characterization in Theorem 14.23. above of the algebras of generalized functions (14.29)constructed through the method of reduced powers.
This fact, as mentioned, was first elaborated upon in [4, pp. 118,119].

Indeed, for a given pair $\mathcal{V}, \mathcal{S}\,'$ and a corresponding subalgebra $\mathcal{A}(\mathcal{V}, \mathcal{S}\,') \subseteq (\mathcal{C}^\infty(\Omega))^{\mathbb{N}}$ in an inclusion diagram (14.23) - (14.27), let us denote by

$$(14.30) \qquad \mathbf{ID}(\Omega, \mathcal{V}, \mathcal{S}\,', \mathcal{A}(\mathcal{V}, \mathcal{S}\,'))$$

the set of all ideals $\mathcal{I}$ in $\mathcal{A}(\mathcal{V}, \mathcal{S}\,')$ which can occur in such inclusion diagrams (14.23) - (14.27). This means, therefore, that for every such ideal $\mathcal{I} \in \mathbf{ID}(\Omega, \mathcal{V}, \mathcal{S}\,', \mathcal{A}(\mathcal{V}, \mathcal{S}\,'))$, there exists a corresponding algebra of generalized functions

$$(14.31) \qquad A = \mathcal{A}(\mathcal{V}, \mathcal{S}\,')/\mathcal{I}$$

which, in view of (14.22), contains the vector space $\mathcal{D}'(\Omega)$ of Schwartz distributions.

Now, the essential point regarding the multiplication of singularities is that, in view of the inclusion

$$(14.32) \qquad \mathcal{V} \oplus \mathcal{S}\,' \supset \mathcal{U}^\infty(\Omega)$$

in (14.23), it follows easily that the multiplication in each of the algebras of generalized functions A in (14.31) preserves the usual multiplication of smooth functions in $\mathcal{C}^\infty(\Omega)$.

On the other hand, regarding the multiplication of generalized functions that are not smooth - therefore, are elements in $A \setminus \mathcal{C}^\infty(\Omega)$ - it is well known that in general they do no longer preserve even the usual multiplication of continuous functions, this being one of the immediate implications of the 1954 so called Schwartz impossibility result, [3,4,6,7].

Furthermore, in view of Remark 14.1. above, there are *infinitely* many ways according to which multiplication ends up being done, ways corresponding to the various algebras A of generalized functions. And the possibility of this infinite branching of multiplication is manifested as soon as the generalized functions which are the factors in multiplication belong to $A \setminus \mathcal{C}^\infty(\Omega)$, and as such, are farther and farther removed from $\mathcal{C}^\infty(\Omega)$, or even form $\mathcal{C}^0(\Omega)$, that is, have *higher levels* of singularity.

Let us illustrate the above with a simple example. For that purpose, let us fix the pair $\mathcal{V}, \mathcal{S}\,'$ and take as a corresponding subalgebra $\mathcal{A}(\mathcal{V}, \mathcal{S}\,') = (\mathcal{C}^\infty(\Omega))^{\mathbb{N}}$ in the inclusion diagram (14.23) - (2-5).
In such a case, one may expect a natural or canonical multiplication, if one would be able to *single out* in a suitable manner a certain ideal $\mathcal{I} \in \mathbf{ID}(\Omega, \mathcal{V}, \mathcal{S}\,', (\mathcal{C}^\infty(\Omega))^{\mathbb{N}})$, and thus obtain the corresponding algebra of generalized functions A in (14.31).

Here however, the off-diagonality condition (14.28) interferes, leading to a rather involved structure for the set of ideals which satisfy that condition, as seen in

Proposition 14.1. [4, p. 118]

There is *no* largest ideal in the set

(14.33) $\quad \mathbf{ID}(\Omega)$

of ideals $\mathcal{I}$ in $(\mathcal{C}^\infty(\Omega))^{\mathbb{N}}$ which satisfy the off-diagonality condition

(14.34) $\quad \mathcal{I} \bigcap \mathcal{U}^\infty(\Omega) \;=\; \{0\}$

Proof.

Let us again take $\Omega = \mathbb{R}$, together with $v\,' = (\chi\,'_\nu)_{\nu \in \mathbb{N}}, v\,'' = (\chi\,''_\nu)_{\nu \in \mathbb{N}} \in (\mathcal{C}^\infty(\Omega))^{\mathbb{N}}$, where

$$\chi'_\nu(x) = 1 + \sin(\nu x), \quad \chi''_\nu(x) = 1 + \cos(\nu x), \quad \nu \in \mathbb{N}, x \in \Omega$$

Then it follows easily that

$$\mathcal{I}' = v'(\mathcal{C}^\infty(\Omega))^{\mathbb{N}},\ \mathcal{I}'' = v''(\mathcal{C}^\infty(\Omega))^{\mathbb{N}} \in \mathbf{ID}(\Omega)$$

However

$$\mathcal{I} \ = \ \mathcal{I}' + \mathcal{I}''$$

is an ideal in $(\mathcal{C}^\infty(\Omega))^{\mathbb{N}}$ which fails to satisfy the off-diagonality condition (14.34), therefore

$$(14.35) \quad \mathcal{I} \notin \mathbf{ID}(\Omega)$$

Indeed

$$v = v' + v'' \in \mathcal{I}$$

and $v = (\psi_\nu)_{\nu \in \mathbb{N}}$, where

$$\psi_\nu(x) = 2 + \sin(\nu x) + \cos(\nu x) > 0, \quad \nu \in \mathbb{N}, x \in \Omega$$

Consequently

$$\mathcal{I} \ = \ (\mathcal{C}^\infty(\Omega))^{\mathbb{N}}$$

thus it is not a proper ideal in $(\mathcal{C}^\infty(\Omega))^{\mathbb{N}}$, and in particular, it does not satisfy condition (14.34).

It follows that $\mathbf{ID}(\Omega)$ does not contain ideals which may contain both ideals $\mathcal{I}'$ and $\mathcal{I}''$.

$\square$

In view of the above, it is obvious that there are infinitely many ideals in $\mathbf{ID}(\Omega)$, and in fact, also in $\mathcal{I} \in \mathbf{ID}(\Omega, \mathcal{V}, \mathcal{S}', (\mathcal{C}^\infty(\Omega))^{\mathbb{N}})$. Furthermore, various ideals in these sets clearly lead to significantly different multiplications in the corresponding algebras of generalized functions (14.31).

Let us however, consider in some more detail this issue in the next subsection.

14.4. Which Are the Maximal Off-Diagonal Ideals ?

Let us consider the better known case of ideals in rings of continuous functions, [204], instead of ideals in rings of $\mathcal{C}^\infty$-smooth functions. And for further convenience, let us suppose that $\Omega = \mathbb{R}^n$.

We shall be interested, therefore, in the largest possible ideals $\mathcal{I}$ in the algebra $(\mathcal{C}(\mathbb{R}^n))^\Lambda$, where Λ is any given infinite set.

The ideals $\mathcal{I}$ of interest will have to satisfy both the *off diagonality* condition

$$(14.36) \qquad \mathcal{I} \bigcap \mathcal{U}_\Lambda(\mathbb{R}^n) \;=\; \{\ 0\ \}$$

as well as the following *group invariance* one

(14.37) $\quad \mathcal{I}$ is invariant under the transformations

$$(14.38) \qquad \mathbb{R}^n \ni x \longmapsto ax + b \in \mathbb{R}^n, \quad a \in \mathbb{R},\ a \neq 0,\ b \in \mathbb{R}^n$$

Let us clarify the above notation, as well as the meaning of conditions (14.37), (14.38).

First, $\mathcal{C}(\mathbb{R}^n)$ is the set of all real valued continuous functions on $\mathbb{R}^n$, while Λ is the mentioned arbitrary given *infinite* set. Consequently, $(\mathcal{C}(\mathbb{R}^n))^\Lambda$ is the Cartesian product of Λ copies of $\mathcal{C}(\mathbb{R}^n)$, thus it can be identified with $\mathcal{C}(\Lambda \times \mathbb{R}^n)$, that is, with the set of real valued continuous functions on $\Lambda \times \mathbb{R}^n$, where Λ is taken with the discrete topology. Clearly, $(\mathcal{C}(\mathbb{R}^n))^\Lambda$ is a commutative unital algebra over $\mathbb{R}$, and we have the algebra embedding

$$(14.39) \qquad \mathcal{C}(\mathbb{R}^n) \ni \psi \;\longmapsto\; u(\psi) \in (\mathcal{C}(\mathbb{R}^n))^\Lambda$$

where $u(\psi) = (\ \psi_\lambda \mid \lambda \in \Lambda\ )$, with $\psi_\lambda = \psi$, for $\lambda \in \Lambda$. In this way, the unit element in $(\mathcal{C}(\mathbb{R}^n))^\Lambda$ is $u(1)$, where $1 \in \mathcal{C}(\mathbb{R}^n)$ denotes the constant function with value 1 defined on $\mathbb{R}^n$.

Further, $\mathcal{U}_\Lambda(\mathbb{R}^n)$ denotes the image of $\mathcal{C}(\mathbb{R}^n)$ in $(\mathcal{C}(\mathbb{R}^n))^\Lambda$ through the algebra embedding (14.39), thus

$$(14.40) \quad \mathcal{U}_\Lambda(\mathbb{R}^n) \;=\; \{\; u(\psi) \mid \psi \in \mathcal{C}(\mathbb{R}^n) \;\}$$

is a subalgebra in $(\mathcal{C}(\mathbb{R}^n))^\Lambda$, and through (14.39), it is isomorphic with $\mathcal{C}(\mathbb{R}^n)$.
With the above, the meaning of (14.36) becomes clear, recalling that $\{ \; 0 \; \}$ in its right hand term denotes the trivial zero ideal in $(\mathcal{C}(\mathbb{R}^n))^\Lambda$.

In this way $\mathcal{U}_\Lambda(\mathbb{R}^n)$ is in fact the *diagonal* in the Cartesian product $(\mathcal{C}(\mathbb{R}^n))^\Lambda$. Thus (14.36) is indeed an *off diagonality* condition on the respective ideals $\mathcal{I}$ in $(\mathcal{C}(\mathbb{R}^n))^\Lambda$.

As seen in section 5, the interest in maximal ideals satisfying the off diagonality condition (14.36) comes from the fact that such ideals lead to the effective construction of differential algebras of generalized functions which can handle the largest classes of *singularities*.

Before going further, let us briefly point to the mathematical nontriviality of the problem in (14.36). Indeed, as mentioned, $(\mathcal{C}(\mathbb{R}^n))^\Lambda$ can be identified with $\mathcal{C}(\Lambda \times \mathbb{R}^n)$, thus as is well known, [204], the problem of the structure of *maximal ideals* $\mathcal{I}$ in $(\mathcal{C}(\mathbb{R}^n))^\Lambda$ is closely related to the Stone-Čech compactification $\beta(\Lambda \times \mathbb{R}^n)$ of $\Lambda \times \mathbb{R}^n$, which in itself is a rather involved problem even in the simplest case of interest above, namely, when $\Lambda = \mathbb{N}$. One of the reasons which makes $\beta(\Lambda \times \mathbb{R}^n)$ not easy to deal with is that, in general, for two completely regular topological spaces X and Y, the spaces $\beta(X \times Y)$ and $\beta X \times \beta Y$ are different. Furthermore, the space $\beta\mathbb{N}$ alone is known to be highly nontrivial.

On the other hand, in (14.36) - (14.38) one asks the yet more difficult problem of finding the maximal ideals $\mathcal{I}$ in $(\mathcal{C}(\mathbb{R}^n))^\Lambda$ which satisfy the respective *additional condition*, thus they can *no longer* be maximal in $(\mathcal{C}(\mathbb{R}^n))^\Lambda$. Therefore, their structure is quite likely still more complex,

[204].

Lastly, by the group invariance property (14.37), (14.38) we mean that, for $w = (w_\lambda)_{\lambda \in \Lambda} \in \mathcal{I}$ and $a \in \mathbb{R},\ a \neq 0,\ b \in \mathbb{R}^n$, we have

$$(14.41) \qquad w \circ \tau_{a,\,b} \in \mathcal{I}$$

where

$$(14.42) \qquad w \circ \tau_{a,\,b} = (w_\lambda \circ \tau_{a,\,b})_{\lambda \in \Lambda}$$

while

$$(14.43) \qquad \mathbb{R}^n \ni x \longmapsto \tau_{a,\,b}(x) = ax + b \in \mathbb{R}^n$$

Remark 14.2.

1) The transformations (14.43), with $a \in \mathbb{R},\ a \neq 0,\ b \in \mathbb{R}^n$, obviously form a non-commutative group.

2) The meaning of the group invariance conditions (14.37), (14.38) is obvious. Namely, the corresponding algebra of generalized functions

$$(14.44) \qquad \mathbb{A} = (\mathcal{C}(\mathbb{R}^n))^\Lambda / \mathcal{I}$$

will have the group invariance property

$$(14.45) \qquad \mathbb{A} \ni F \longmapsto \tau_{a,\,b}(F) \in \mathbb{A}, \quad a \in \mathbb{R},\ a \neq 0,\ b \in \mathbb{R}^n$$

where for

$$(14.46) \qquad F = (f_\lambda)_{\lambda \in \Lambda} + \mathcal{I} \in \mathbb{A}$$

we have

$$(14.47) \qquad F \circ \tau_{a,\,b} = ((f_\lambda)_{\lambda \in \Lambda}) \circ \tau_{a,\,b} + \mathcal{I} = (f_\lambda \circ \tau_{a,\,b})_{\lambda \in \Lambda} + \mathcal{I}$$

□

We can obtain an idea about how large can be the ideals $\mathcal{I}$ in $(\mathcal{C}(\mathbb{R}^n))^\Lambda$ which satisfy (14.36) - (14.38), from the following

Example 14.1.

Let $L = (\Lambda, \leq)$ be a directed partial order on the infinite set Λ. We shall consider the following continuous version of the ideals in (10.10). Namely, let $\mathcal{S}$ be any family of singularity sets $\Sigma \subset \mathbb{R}^n$ which satisfies conditions (10.5), (10.6), with $\Omega = \mathbb{R}^n$. Then we define

$$\mathcal{I}_{L,\,\mathcal{S}} \tag{14.48}$$

as the set of all sequences of continuous functions $w = (w_\lambda)_{\lambda \in \Lambda} \in (\mathcal{C}(\mathbb{R}^n))^\Lambda$, which satisfy the condition

$$\begin{array}{ll} & \exists \;\; \Sigma \in \mathcal{S} \;\; : \\ & \forall \;\; x \in \mathbb{R}^n \;\; : \\ (14.49) & \exists \;\; \lambda \in \Lambda \;\; : \\ & \forall \;\; \mu \in \Lambda, \;\; \mu \geq \lambda \;\; : \\ & \quad w_\mu(x) = 0 \end{array}$$

It is easy to see that the ideals $\mathcal{I}_{L,\,\mathcal{S}}$ in $(\mathcal{C}(\mathbb{R}^n))^\Lambda$ do indeed satisfy the conditions (14.36) - (14.38), provided that the family $\mathcal{S}$ of singularities is such that

$$\mathcal{S} \ni \Sigma \longmapsto \tau_{a,\,b}(\Sigma) \in \mathcal{S}, \quad a \in \mathbb{R}, \;\; a \neq 0, \;\; b \in \mathbb{R}^n \tag{14.50}$$

And clearly, the families of singularities (10.2) and (10.3), for instance, satisfy (14.50).

Remark 14.3.

In fact, the ideals $\mathcal{I}_{L,\,\mathcal{S}}$ in (14.48), have a far stronger *semigroup invariance* property. Namely, let

$$(14.51) \qquad \tau : \mathbb{R}^n \longrightarrow \mathbb{R}^n$$

any continuous mapping, such that

$$(14.52) \qquad \begin{array}{l} \forall \ \ \Sigma \in \mathcal{S} \ : \\ \exists \ \ \Sigma' \in \mathcal{S} \ : \\ \qquad \tau(\mathbb{R}^n \setminus \Sigma) \subseteq \mathbb{R}^n \setminus \Sigma' \end{array}$$

then

$$(14.53) \qquad \mathcal{I}_{L,\mathcal{S}} \ni w \longmapsto w \circ \tau \in \mathcal{I}_{L,\mathcal{S}}$$

Clearly, if τ is *injective*, and is such that

$$(14.54) \qquad \mathcal{S} \ni \Sigma \longmapsto \tau(\Sigma) \in \mathcal{S}$$

then it satisfies (14.52).

14.5. On Isomorphisms of Algebras

Let us consider the issue to what extent can different reduced power algebras (5.4) be isomorphic. And for simplicity, we shall address that issue in the following more general case.

Let E be a unital commutative algebra over $\mathbb{R}$, while Λ is an infinite set. We consider two algebras

$$(14.55) \qquad \mathbb{A} = \mathcal{A}/\mathcal{I}, \qquad \mathbb{B} = \mathcal{B}/\mathcal{J}$$

where

$$(14.56) \qquad \mathcal{I} \subseteq \mathcal{A} \subseteq E^\Lambda, \qquad \mathcal{J} \subseteq \mathcal{B} \subseteq E^\Lambda$$

with $\mathcal{A}, \mathcal{B}$ being subalgebras, while $\mathcal{I}, \mathcal{J}$ are ideals in the respective subalgebras.

Let us define the *algebra embedding*

$$(14.57) \qquad E \ni a \longmapsto u(a) \in E^\Lambda$$

where

$$(14.58) \qquad u(a) : \Lambda \ni \lambda \longmapsto a \in E$$

We denote by

$$(14.59) \qquad \mathcal{U}$$

the image in E^Λ of the algebra embedding (14.57). Thus we have the *algebra isomorphism*

$$(14.60) \qquad E \ \approx \ \mathcal{U}$$

As for the off-diagonality condition (5.8) which is typical for the differential algebras of generalized functions, we shall assume that the ideals $\mathcal{I}, \mathcal{J}$ satisfy the condition

$$(14.61) \qquad \mathcal{I} \bigcap \mathcal{U} = \mathcal{J} \bigcap \mathcal{U} = \{0\}$$

Finally, we shall assume that

$$(14.62) \qquad \mathcal{U} \subseteq \mathcal{A} \bigcap \mathcal{B}$$

which means that we have the *algebra embeddings*

$$(14.63) \qquad \begin{array}{l} E \ni a \longmapsto u(a) + \mathcal{I} \in \mathcal{A}/\mathcal{I} = \mathbb{A} \\ \\ E \ni a \longmapsto u(a) + \mathcal{J} \in \mathcal{B}/\mathcal{J} = \mathbb{B} \end{array}$$

Regarding now the possible isomorphism of the algebras (14.55), namely

$$(14.64) \qquad \mathbb{A} \ \approx \ \mathbb{B}$$

there are the following two situations of interest :

First, we can have

(14.65) $\quad \mathcal{I} \subseteq \mathcal{J}, \quad \mathcal{A} \subseteq \mathcal{B}$

In this case, we obviously have the *algebra homomorphism*

(14.66) $\quad \mathbb{A} \ni s + \mathcal{I} \longmapsto s + \mathcal{J} \in \mathbb{B}$

and in view of (14.63), the *commutative* diagram follows

(14.67)
$$\begin{tikzcd}
\mathbb{A} \ni s + \mathcal{I} \arrow[r] & s + \mathcal{J} \in \mathbb{B} \\
E \arrow[u] \arrow[r, "id_E"'] & E \arrow[u]
\end{tikzcd}$$

Clearly, in this case the mapping (14.66) is an *algebra isomorphism*, thus (14.64) holds, if and only if

(14.68) $\quad \mathcal{I} = \mathcal{J}, \quad \mathcal{A} = \mathcal{B}$

And now, we consider the second case, namely, the general situation in (14.55) - (14.63). We note that we have the inclusion diagram where each arrow $\longrightarrow$ means an inclusion $\subseteq$, namely

(14.69)
$$\begin{tikzcd}
 & \mathcal{I} \arrow[r] & \mathcal{A} \\
\mathcal{I} \bigcap \mathcal{J} \arrow[ur] \arrow[dr] & & \\
 & \mathcal{J} \arrow[r] & \mathcal{B}
\end{tikzcd}$$

Here, we can apply the *Second and Third Isomorphism Theorems*, and obtain

$$(14.70) \qquad \begin{aligned} \mathbb{A} \;&\approx\; [\mathcal{A}/(\mathcal{I} \bigcap \mathcal{J})]/[\mathcal{I}/(\mathcal{I} \bigcap \mathcal{J})] \;\approx\; [\mathcal{A}/(\mathcal{I} \bigcap \mathcal{J})]/[(\mathcal{I}+\mathcal{J})/\mathcal{J}] \\ \mathbb{B} \;&\approx\; [\mathcal{B}/(\mathcal{I} \bigcap \mathcal{J})]/[\mathcal{J}/(\mathcal{I} \bigcap \mathcal{J})] \;\approx\; [\mathcal{B}/(\mathcal{I} \bigcap \mathcal{J})]/[(\mathcal{I}+\mathcal{J})/\mathcal{I}] \end{aligned}$$

The algebra isomorphisms of interest between $\mathbb{A}$ and $\mathbb{B}$ are supposed to have again the diagram (14.67) commute. Thus among others, we have the following *three* possibilities for an algebra isomorphism

$$(14.71) \qquad \mathbb{A} = \mathcal{A}/\mathcal{I} \ni s + \mathcal{I} \longmapsto s + \mathcal{J} \in \mathcal{B}/\mathcal{J} = \mathbb{B}$$

$$(14.72) \qquad \mathbb{A} \ni (s + (\mathcal{I} \bigcap \mathcal{J})) + (v + (\mathcal{I} \bigcap \mathcal{J})) \longmapsto$$

$$\longmapsto (s + (\mathcal{I} \bigcap \mathcal{J})) + (w + (\mathcal{I} \bigcap \mathcal{J})) \in \mathbb{B}$$

and

$$(14.73) \qquad \mathbb{A} \ni (s + (\mathcal{I} \bigcap \mathcal{J})) + (v + w + \mathcal{J}) \longmapsto$$

$$\longmapsto (s + (\mathcal{I} \bigcap \mathcal{J})) + (v + w + \mathcal{I}) \in \mathbb{B}$$

where $s \in \mathcal{A},\ v \in \mathcal{I},\ w \in \mathcal{J}$.

Now the bijectivity of (14.71) implies that

$$(14.74) \qquad \mathcal{B} \subseteq \mathcal{A} + \mathcal{J}, \qquad \mathcal{A} \subseteq \mathcal{B} + \mathcal{I}$$

Similarly, the bijectivity of (14.72) and (14.73) imply (14.74).

And from (14.74), it follows that

$$(14.75) \qquad \begin{aligned} &\mathbb{B} = \mathcal{B}/\mathcal{J} \subseteq (\mathcal{A}+\mathcal{J})/\mathcal{J} \;\approx\; \mathcal{A}/(\mathcal{J} \bigcap \mathcal{A}) \;\approx \\ &\qquad \mathcal{A}/(\mathcal{I} \bigcap \mathcal{J}) \;\approx\; \mathcal{I}/(\mathcal{I} \bigcap \mathcal{J}) \\ &\mathbb{A} = \mathcal{A}/\mathcal{I} \subseteq (\mathcal{B}+\mathcal{I})/\mathcal{I} \;\approx\; \mathcal{B}/(\mathcal{I} \bigcap \mathcal{B}) \;\approx \\ &\qquad \mathcal{B}/(\mathcal{I} \bigcap \mathcal{J}) \;\approx\; \mathcal{J}/(\mathcal{I} \bigcap \mathcal{J}) \end{aligned}$$

In this way, one obtains

$$(14.76) \qquad \mathcal{I}/(\mathcal{I} \bigcap \mathcal{J}) \;\approx\; \mathcal{J}/(\mathcal{I} \bigcap \mathcal{J})$$

which can be seen as the bijection

$$(14.77) \qquad \mathcal{I}/(\mathcal{I} \bigcap \mathcal{J}) \ni v + (\mathcal{I} \bigcap \mathcal{J}) \longmapsto v + (\mathcal{I} \bigcap \mathcal{J}) \in \mathcal{J}/(\mathcal{I} \bigcap \mathcal{J})$$

that obviously leads to

$$(14.78) \qquad \mathcal{I} = \mathcal{J}$$

and thus as well to

$$(14.79) \qquad \mathcal{A} = \mathcal{B}$$

In this way, the assumption that the algebras $\mathbb{A}$ and $\mathbb{B}$ are isomorphic may under the condition (14.67) lead to their equality.

Clearly, in the general case of (14.55) - (14.63), the above is rather a heuristic sketch of the implication

$$(14.80) \qquad \mathbb{A} \;\approx\; \mathbb{B} \;\Longrightarrow\; \mathbb{A} \;=\; \mathbb{B}$$

whose more detailed study can further be pursued.

However, as seen next, for our purpose, namely, to show the existence of *infinitely* many different differential algebras of generalized functions, such a detailed study is not necessary.

Indeed, let us only consider the case of the space-time foam differential algebras of generalized functions, see section 10. Then we can take

$$(14.81) \qquad E = \mathcal{C}^\infty(\Omega), \qquad \Lambda = \mathbb{N}$$

Further, let $\mathcal{S}$, $\mathcal{T}$ be two families of singularity sets on Ω which satisfy (10.5), (10.6). Then the corresponding space-time foam differential algebras of generalized functions are, see (10.12)

$$(14.82) \qquad \mathbb{A} = (\mathcal{C}^\infty(\Omega))^{\mathbb{N}} / \mathcal{J}_{L,\mathcal{S}}(\Omega), \qquad \mathbb{B} = (\mathcal{C}^\infty(\Omega))^{\mathbb{N}} / \mathcal{J}_{L,\mathcal{T}}(\Omega)$$

thus

$$(14.83) \qquad \mathcal{I} = \mathcal{J}_{L,\mathcal{S}}(\Omega), \qquad \mathcal{J} = \mathcal{J}_{L,\mathcal{T}}(\Omega), \qquad \mathcal{A} = \mathcal{B} = (\mathcal{C}^\infty(\Omega))^{\mathbb{N}}$$

Now obviously, if one has

$$(14.84) \qquad \mathcal{S} \subseteq \mathcal{T}$$

then in view of (10.8), (10.10), it follows that

$$(14.85) \qquad \mathcal{I} \subseteq \mathcal{J}$$

therefore, we are in the situation (14.65), which means that

$$(14.86) \qquad \mathbb{A} \approx \mathbb{B} \iff \mathbb{A} = \mathbb{B} \iff \mathcal{S} = \mathcal{T}$$

And as seen in the example next, there are *infinitely* many different families $\mathcal{S}$, $\mathcal{T}$ which satisfy (10.5), (10.6) and (14.84), thus in view of (14.86), give *different* space-time foam differential algebras of generalized functions.

Example 14.2.

Let $\Omega = \mathbb{R}$ and $\Lambda = \mathbb{N}$. Further, let $X = \{ x_0, x_1, x_2, \ldots, x_m, \ldots \} \subset \Omega$ which is dense in Ω. Then we define

$$
(14.87) \qquad \begin{aligned}
&\mathcal{S}_0 = \{ \ \Sigma \subset \Omega \mid X \subseteq \Sigma \ \} \\
&\mathcal{S}_1 = \{ \ \Sigma \subset \Omega \mid (\ X \setminus \{ \ x_0 \ \} \ ) \subseteq \Sigma \ \} \\
&\mathcal{S}_2 = \{ \ \Sigma \subset \Omega \mid (\ X \setminus \{ \ x_0, x_1 \ \} \ ) \subseteq \Sigma \ \} \\
&\mathcal{S}_3 = \{ \ \Sigma \subset \Omega \mid (\ X \setminus \{ \ x_0, x_1, x_2 \ \} \ ) \subseteq \Sigma \ \} \\
&\vdots \\
&\mathcal{S}_{m+1} = \{ \ \Sigma \subset \Omega \mid (\ X \setminus \{ \ x_0, x_1, x_3, \ldots, x_m \ \} \ ) \subseteq \Sigma \ \} \\
&\vdots
\end{aligned}
$$

Clearly

$$
(14.88) \qquad \mathcal{S}_0 \subsetneqq \mathcal{S}_1 \subsetneqq \mathcal{S}_2 \subsetneqq \mathcal{S}_3 \subsetneqq \ \ldots \ \subsetneqq \mathcal{S}_m \subsetneqq \ \ldots
$$

and each of them satisfies the conditions (10.5), (10.6). Consequently, in view of (10.12), we have the space-time foam differential algebras of generalized functions

$$
(14.89) \qquad \mathbb{A}_m \ = \ (\mathcal{C}^\infty(\Omega))^{\mathbb{N}} / \mathcal{J}_{\mathbb{N}, \mathcal{S}_m}(\Omega), \qquad m \in \mathbb{N}
$$

with the respective *commutative* diagrams of algebra embeddings

$$
(14.90) \qquad \begin{tikzcd}
\mathbb{A}_m \ni s + \mathcal{J}_{\mathbb{N}, \mathcal{S}_m}(\Omega) \arrow[r] & s + \mathcal{J}_{\mathbb{N}, \mathcal{S}_{m++}}(\Omega) \in \mathbb{A}_{m+1} \\
\mathcal{C}^\infty(\Omega) \arrow[u] \arrow[r, "id_{\mathcal{C}^\infty(\Omega)}"'] & \mathcal{C}^\infty(\Omega) \arrow[u]
\end{tikzcd}
$$

15. Chains of Algebras of Generalized Functions

Let us start by summarizing the most relevant features involved in the ways singularities can be dealt with by the algebraic nonlinear theory of generalized functions, that is, by the respective variety of differential algebras of generalized functions.

In this regard, we have seen

- the inevitable infinite branching of multiplication above certain levels of singularities

This phenomenon, still not sufficiently in the awareness of many involved in nonlinear aspects of generalized functions, is responsible for

- the conflict between discontinuity, multiplication and differentiation

or in more general terms

- the incompatibility between insufficient smoothness, nonlinear operations and differentiation

Further, we have also seen

- the conflict between stability, generality and exactness

As always, a better way to deal with such conflicts or incompatibilities is not by simply choosing one or another fixed position and thereafter confine oneself to it for evermore, but rather, by engaging in the *interplay* which naturally opens up in such situations between the aspects in conflict, or which are incompatible.

In the case of differential algebras of generalized functions, this comes down to no less than *two departures* from what may appear as normal practice to many presently involved, namely

- one has not only to explore a large variety of different differential algebras of generalized functions,

- but even more, when dealing with PDEs, for instance, the respective partial differential operators should not be confined to acting within the same algebra,

- and instead, they could also be considered to act between different algebras.

In short, the basic entities in the algebraic nonlinear theory of generalized functions are not so much one or another such algebra among the infinitely many possible ones. Instead, the basic entities are *chains* of such algebras, chains which recall the classical situation with the spaces

$$(15.1) \qquad \mathcal{C}^\infty \subset \ldots \subset \mathcal{C}^m \subset \ldots \subset \mathcal{C}^1 \subset \mathcal{C}^0$$

And the main reason for dealing with such chains is that, except for the leftmost in them, they do not assume the infinite differentiability of their elements, therefore, they can allow multiplications of insufficiently smooth functions to be nearer to their usual multiplication.

And the lack of infinite differentiability need not in fact be a problem, as long as the PDEs considered are of finite order.

And now, to some of the details, with the more full picture being presented in [4,6,7].

16. A Latest Surprise

In recent work of Jan Harm van der Walt, it turns out rather surprisingly that there is a close connection between certain differential algebras of generalized functions and the order completion method.

The respective differential algebras happen to be of yet another type than those mentioned above, and thus used so far in the literature.

The interest of this latest development is that the respective strengths of the two methods, namely of differential algebras of generalized functions and of the order completion method, can now be transferred from one method to the other.
In this regard, what is so far the unprecedented power of the order completion method to deliver the existence of solutions to very large classes on nonlinear systems of PDEs with possibly associated initial and/or boundary value problems, and guarantee the Hausdorff continuity of those solutions, may benefit the method of differential algebras of generalized functions. Conversely, the algebra and differential structure of differential algebras of generalized functions may further enrich the order completion method.

Results related to the above are to be presented elsewhere.

17. Appendix

The still persisting and rather frequent misinterpretations related to singularities among some of those involved in the algebraic nonlinear theory of generalized functions are illustrated by the recent example of a review of a paper submitted by one of my young colleagues.
As it happens, the paper was eventually accepted for publication.
However, the sort of arguments of the referee are not quite a singular event ...
Therefore, the interest in the following ...

17.1. Review of a Recent Paper

Here are briefly presented in their essence the points in some of the objections and comments of the anonymous referee :

1) The paper shows that the nowhere dense algebras are the same with the space-time foam ones which correspond to Baire category I singularities. This is mathematically nontrivial and it shows an obstruction in the algebras introduced by Rosinger which are general almost to the absurd.

2) The nowhere dense algebras and the space-time foam algebras do not give a deep understanding of singularities, or of the structures in quantum gravity.

3) The Global Cauchy-Kovaleveskaia Theorem is not deep, since it only gives in fact local solutions. Thus it is not intellectually honest to call them global solutions. Indeed, the differential algebra

$$A_\Omega = \bigcup_{\Gamma \subset \Omega,\ \Gamma \text{ closed, nowhere dense}} \mathcal{C}^\infty(\Omega \setminus \Gamma) / \sim$$

where $\sim$ is the equivalence relation defined by $u \sim v$, iff there exists some closed and nowhere dense $\Gamma \subset \Omega$, such that $u = v$ on $\Omega \setminus \Gamma$, is almost a direct mathematical translation of the nowhere dense algebra, since one simply considers smooth functions off a closed nowhere dense set and ask by definition that equality in the algebra, the so-called global equality, is equality off a closed nowhere dense set.

4) The algebra A_Ω satisfies nearly all the properties of the nowhere dense algebras.

5) The paper seems to consider the statement "an algebra of generalized functions can deal with large singularities" as equivalent with "the algebra neglects large singularity sets".

6) The generalized function in the Colombeau algebra with representative $(sin(x/\epsilon))/\epsilon$ is everywhere singular.

7) Among the other similar requests for revision of the paper, the referee asks that the paper should not call solutions in the nowhere dense algebra or the space-time foam algebras as being global solutions, and also not state that other algebras of generalized functions, among them the Colombeau algebras, cannot deal with large singularity sets.

17.2. Comments on the Above Review

1) There are inevitably infinitely many different differential algebras of generalized functions due to the inevitable infinite branching of multiplication above certain levels of singularities, as mentioned several times in this paper.
Consequently, there is nothing untoward, let alone absurd in exploring more of these algebras. After all, we shall never be able to explore all of them.
In short, no one has to introduce any of those algebras, since they are already there due to the mentioned inevitable infinite branching of multiplication.

Also, it is worth recalling that in the usual linear theory of distributions no one calls superfluous the large variety of Sobolev spaces, for instance.

Neither is anyone complaining that most of the very same Sobolev spaces do not contain the Dirac δ distribution ...

One may, however, emphasize somewhat with those who, due to various subjective reasons, may find it rather unsettling to have new and new differential algebras of generalized functions come to be studied before the earlier ones could have been sufficiently exhausted by those who have limited their attention to them ...
Certainly, new algebras do bring in different multiplications, all of which may make some feel less than comfortable ...

However, these new algebras may allow the treatment of larger classes of singularities ...
And such singularities can be far more general than that exhibited by the reviewer's example $(sin(x/\epsilon))/\epsilon$ in one of the Colombeau algebras. Indeed, as seen in (7.1) in the case of nowhere dense algebras, a generalized function F belonging to them can be of the form $F = (f_\nu)_{\nu \in \mathbb{N}} + \mathcal{I}^\infty_{nd}$, where f_ν are arbitrary $\mathcal{C}^\infty$-smooth functions. Thus indeed, there is hardly any limit on the possible singularities of F.

It is precisely due to such a possibility that the singularities considered are specified by two clearly formulated criteria, namely, what is the allowed SIZE of the sets of points of singularity, and what is the BEHAVIOUR of the generalized functions in the neigbourhood of singularities.

On the other hand, and as mentioned, the various vector spaces of distributions or Sobolev spaces always brought in the very same addition, and fortunately for those less than comfortable with algebras, did not bring in any, let alone, ever new multiplications ...
Anyhow, since the early 1990s, there is an alternative to all such algebras, an alternative which is so much more simple and also powerful in solving large classes of nonlinear PDEs, namely, the order completion method ...

2) The result in the paper of my young colleague under review is indeed of interest. However, it does not show any obstruction related to the infinitely many possible different differential algebras of generalized functions, since that inevitable infinity of different differential algebras of generalized functions remains an essential - even if rather elementary ring theoretic - fact, see sections 0, 12-15 above.

What that paper does show instead is what could possibly be seen as a certain kind of *singularity-stability* property of nowhere dense algebras, see subsection 10.7. above.

3) No one has yet defined precisely enough what does "deep" mean in mathematics.
And before anyone may venture to do so, the following could be useful to consider carefully enough :
In the time of D'Alembert, for instance, the result that every algebraic equation of degree n has n roots was considered to be the fundamental theorem of algebra, and it was "deep" among others due to its long proof. Not much later, functions of one complex variable were considered. And in that context, the mentioned theorem is a trivial case of the Rouché Lemma. In general, it is a theorem of mathematical logic that in a given axiomatic system, one can choose equivalent axioms in such a way that the proof of any given theorem becomes arbitrary long, or on the contrary, arbitrary short, [129]. Does that mean that theorems with long proofs are "deep" ? Or rather it means that the respective theory was not developed properly from the point of view of the proof of that given theorem ?
Most likely, it simply means that one is pursuing what is not the best theory in obtaining the respective result ...
Here, it may be useful to recall the saying that "old theorems never die, they just become definitions" ...
Indeed, appropriate definitions can make an immense difference in turning formerly "deep" theorems into rather trivial ones, as it happened to what once used to be the fundamental theorem of algebra ...
Consequently, to talk about "deep" in mathematics recalls talking about beauty which, as is well known, is in the eyes of the beholder ...
And not seldom of a beholder with a rather narrow view ...

4) The global Cauchy-Kovalevskaia theorem can only be formulated, let alone be obtained, in differential algebras of generalized functions which are *flabby sheaves*, as mentioned in sections 0 - 3, and seen in sections 7, 8 and 10. And the nowhere dense, or the space-time foam differential algebras of generalized functions are flabby sheaves, [54,55]. Therefore, the global Cauchy-Kovalevskaia theorem can not only be formulated, but it can also be proved in the nowhere dense, as well as

the space-time foam differential algebras of generalized functions, as seen in sections 7, 8 and 10, with the resulting global solutions.

On the other hand, the Colombeau algebras, as well as various vector spaces of distributions fail to be flabby sheaves. Thus it happens that the global Cauchy-Kovalevskaia theorem can not even be formulated, let alone proved in the Colombeau algebras or in various spaces of distributions which are not flabby sheaves.

That sharp dichotomy between differential algebras of generalized functions which are flabby sheaves, and on the other hand, those which fail to be so has other important consequences as well. For instance, arbitrary Lie group actions, important in the study of linear and nonlinear PDEs, can only be defined globally in the former, and cannot be defined in the latter. And such a definition of global Lie group actions in the nowhere dense differential algebras of generalized functions has led to the first time complete solution of Hilbert's fifth problem, [9].

5) The topmost problem in PDEs is, of course, to prove *existence* of solutions. Certainly, as long as we do not know that a solution exists, there is no point in discussing regularity or other properties of assumed to exist solutions.
Consequently, to comment negatively on solutions obtained by certain methods, while the method one chooses to be limited to is unable not only to find alternative solutions, but simply cannot even allow the formulation of the problem of finding respective solutions, is at best amusing ...
In this regard, one may recall the well known saying "those who do not play the game, do not make the rules" ...

Now of course, after proving the existence of solutions, and only after that, can come the issue of regularity, as well as of other properties of solutions already proved to exist.
As mentioned, the global Cauchy-Kovalevskaia theorem gives solutions in the nowhere dense, as well as the space-time foam differential algebras of generalized functions, see section 7, 8 and 10.
On the other hand, in those differential algebras of generalized functions which fail to be flabby sheaves, one cannot even formulate the

global Cauchy-Kovalevskaia theorem, let alone obtain the corresponding solutions.
In this way, it is indeed rather amusing, and also strange, to find fault with the solutions obtained for the global Cauchy-Kovalevskaia theorem in the nowhere dense, or the space-time foam differential algebras of generalized functions, when the reviewer is not able to show absolutely anything comparable in the Colombeau algebras, or for that matter, in any other differential algebra of generalized functions which fails to be a flabby sheaf. Actually, and as mentioned, in the algebras used by the reviewer one cannot even formulate the global Cauchy-Kovalevskaia theorem ...

In short, those who choose to work in differential algebras of generalized functions which fail to be flabby sheaves place themselves in a strange position when complaining about solutions of PDEs in other algebras which are flabby sheaves, solutions which they can never get in their non-flabby algebras. And such is the case, among others, with the solution of the global Cauchy-Kovalevskaia theorem in the nowhere dense, as well as the space-time foam differential algebras of generalized functions.

6) The claim that the algebra

$$(17.1) \qquad A_\Omega = \bigcup{}_{\Gamma \subset\ \Omega,\ \Gamma\ \mathit{closed,\ nowhere\ dense}}\ \mathcal{C}^\infty(\Omega \setminus \Gamma) / \sim$$

is almost the same with the nowhere dense differential algebras of generalized functions is simply completely wrong on the following two counts :

First, the nowhere dense algebras are considerably larger than the above one in (17.1), as seen easily from their definition in (7.1).

Second, the above claim misses an elementary, yet essential fact pointed out up front in (0.1) in section 0, and which is obvious already in (4.3),(4.5), (6.1) - (6.8), (8.1) - (8.6) about the Heaviside function H and its two usual representations as a distribution, as well as its representation in a nowhere dense differential algebra of generalized functions.

Namely, it misses the process of *regularization of singularities*, a process which is a main reason why generalized functions, be they linear or not, have ever been introduced in mathematics in the first place.
A process whose very result are the various spaces of generalized functions, be they vector spaces or algebras.

Indeed, as an element in the differential algebra (17.1), H - just as is the case with any other function from $\mathcal{C}^\infty_{nd}$ - is considered only on $\Omega \setminus \Gamma = \mathbb{R} \setminus \Gamma$, where the closed, nowhere dense subset $\Gamma = \{0\}$ is the set of its points of singularity.
However, in this way, one cannot perform algebraic and differential operations with H globally and singularity free, since one has to take into account all the time its nonvoid singularity set Γ.
This is precisely why, even in the case of the linear theory of generalized functions, one does not stop at (4.3), when dealing with H.
On the other hand, once one considers H as a distribution (4.4), or in a nowhere dense algebra, as in (6.8), the situation changes completely, as one can perform all the algebraic and differential operations on H *singularity free*, and therefore *globally* on $\Omega = \mathbb{R}$.
In short, the essential difference between the differential algebras (17.1), and on the other hand, the nowhere dense differential algebras is that the elements of the latter - that is, the respective generalized functions - are obtained by a *regularization of singularities* which is perfectly *identical* with that of the Heaviside function H, namely, the regularization that leads to its distribution representation T_H in (6.8).

7) In view of 5) and 6) above, it is obvious that the global solutions of the Cauchy-Kovalevskaia theorem obtained in the nowhere dense algebras are *not* elements in the differential algebras (17.1).
On the contrary, the solutions in (17.1) are subjected to a process of *regularization of singularities*, process which is *perfectly* identical with that used, for instance, for the Heaviside function H in (6.1) - (6.5), in order to obtain its representation T_H as a distribution in (6.8).
And the consequent essential difference between (17.1), and on the other hand, the nowhere dense algebras, is illustrated, among others, by the fact that - with the algebraic and differential operations in the respective nowhere dense algebras - such solutions in these algebras, solutions obtained by the mentioned regularization of singularities,

satisfy the respective PDEs on the whole of the domain of definition Ω of the respective equations, and do so with no regard to any singularities at all.
This is precisely the meaning of such solutions being *global* on the domain of definition of the respective PDEs.
Certainly, if seen only as elements in the differential algebras (17.1), such solutions do not satisfy the respective PDEs on the sets of singularities of the respective solutions.
After all, as mentioned in subsection 0.1., this difference is precisely one of the main reasons spaces of generalized functions have been introduced, be they distributions, or elements in differential algebras of generalized functions.
In short, what is missed is, indeed, a most simple fact, namely, the essential difference which the *regularization of singularities* brings about between the elements of the differential algebras (17.1), and on the other and, the generalized functions which are elements in the nowhere dense differential algebras of generalized functions.

18. In conclusion :

The possible reasons for this kind of most trivial misunderstanding are not so easy to guess ...
And for as long as such, and similar misunderstandings persist, well, all we can do is to live with them ...

And the ... culprit ... to blame ?
Well, clearly, it is ... **multiplication** ... !

More precisely, what appears to be the so very hard to notice and assimilate, namely, the inevitable infinite branching of multiplication above certain levels of singularity ...
After all, the unfortunate slogan that "one cannot multiply distributions", which arose as an instant misinterpretation of the so called 1954 Schwartz impossibility result, seems to have survived, even if in the modified form :

> "many do not yet understand the existence of infinitely many multiplications of generalized functions" ...

Yet, upon a better consideration, we may find that there are reasons to be cheerful :

In number theory, since Gödel's 1931 Incompleteness Theorem, it turned out - so much contrary to most ancient, deep and widespread conviction - that even addition in Peano arithmetics is not well understood, and in fact, it could never ever be completely understood, since arithmetics can branch into no less than infinitely many different theories ...

But then, until a few centuries ago, even the most learned and wise most firmly believed that Planet Earth is flat and immobile at the very center of Creation ...

Well, in the nonlinear theory of generalized functions, so far we have not had any problems whatsoever with addition ...
Or for that matter, with Planet Earth not being flat and immobile ...
And we only had - and still have, as the above in this paper indicate

- some problems with ... multiplication ...

In this regard, it may be somewhat hard not to recall that the very first commandment in the Old Testament is :

"Be fruitful, and multiply ..."

And why not, multiply perhaps also distributions and generalized functions ...

As for a more proper and rather general understanding of such multiplications, well, we can possibly wait somewhat longer ...

Meanwhile, let us not extinguish the passion with too hard commentaries, that passion which can also be seen in the above quite off the mark review of a paper of one of my young colleagues, the passion which - like all passion - not seldom may end up being misdirected, that passion which however is so much needed, and thus most welcome, in any emerging new and hopefully better theory ...

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