

Equivalence of classical Klein-Gordon field theory to correspondence-principle first quantization of the spinless relativistic free particle

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Abstract

It has recently been shown that the classical electric and magnetic fields which satisfy the source-free Maxwell equations can be linearly mapped into the real and imaginary parts of a transverse-vector wave function which in consequence satisfies the time-dependent Schrödinger equation whose Hamiltonian operator is physically appropriate to the free photon. The free-particle Klein-Gordon equation for scalar fields modestly extends the classical wave equation via a mass term. It is physically untenable for complex-valued wave functions, but has a sound nonnegative conserved-energy functional when it is restricted to real-valued classical fields. Canonical Hamiltonization and a further canonical transformation maps the real-valued classical Klein-Gordon field and its canonical conjugate into the real and imaginary parts of a scalar wave function (within a constant factor) which in consequence satisfies the time-dependent Schrödinger equation whose Hamiltonian operator has the natural correspondence-principle relativistic square-root form for a free particle, with a mass that matches the Klein-Gordon field theory's mass term. Quantization of the real-valued classical Klein-Gordon field is thus second quantization of this natural correspondence-principle first-quantized relativistic Schrödinger equation. Source-free electromagnetism is treated in a parallel manner, but with the classical scalar Klein-Gordon field replaced by a transverse vector potential that satisfies the classical wave equation. This reproduces the previous first-quantized results that were based on Maxwell's source-free electric and magnetic field equations.

Introduction

The classical Hamiltonian for the spinless relativistic free particle is $(|c\mathbf{p}|^2 + m^2c^4)^{\frac{1}{2}}$, which, from the correspondence principle, unequivocally implies that its first-quantized description is via state vectors $|\psi(t)\rangle$ that satisfy the time-dependent Schrödinger equation,

$$i\hbar d|\psi(t)\rangle/dt = (|c\hat{\mathbf{p}}|^2 + m^2c^4)^{\frac{1}{2}}|\psi(t)\rangle. \quad (1)$$

In momentum representation the relativistic free-particle energy operator $(|c\hat{\mathbf{p}}|^2 + m^2c^4)^{\frac{1}{2}}$ is transparently diagonal, but in configuration representation it is a nonlocal integral operator. This was regarded as concerning by early quantum mechanics pioneers, *not* for any *physical* reason, but because they feared it would pose unpalatable *calculational* issues. Perhaps they were not mindful, in this regard, that relativistic corrections to the hydrogen atom are expected to be of order one percent, and that Schrödinger had invented the bound state perturbation theory. That notwithstanding, Klein, Gordon and Schrödinger decided to *iterate* Eq. (1) in order to rid it of its “vexing” square root, which maneuver produces the Klein-Gordon equation [1],

$$(c^{-2}d^2/dt^2 + |\hat{\mathbf{p}}/\hbar|^2 + \mu^2)|\psi(t)\rangle = 0, \quad (2)$$

where $\mu \stackrel{\text{def}}{=} ((mc)/\hbar)$. In configuration representation, where $\hat{\mathbf{p}} = -i\hbar\nabla$, it follows that $|\hat{\mathbf{p}}/\hbar|^2 = -\nabla^2$, so that aside from its mass term, the Klein-Gordon equation matches the classical wave equation. For every solution of Eq. (1) that has a definite momentum and positive energy, Eq. (2) has another, *completely extraneous*, partner solution of the same momentum, but energy of the *opposite sign*. Each such pair of energy-partner solutions *fail* to be mutually orthogonal because they have the same momentum, but this nonorthogonality of solutions with two different energies contradicts a fundamental characteristic of quantum theory that is necessary for its probability interpretation. It is thus not surprising that the Klein-Gordon theory gives rise to unacceptable negative probabilities [1]. We have seen that this is related to its negative-energy solutions, which are not a feature of Eq. (1). The associated negative energies as well have the physically problematic trait of being *unbounded below*. Since the Klein-Gordon equation does not directly involve a Hamiltonian operator, but only the *square* of such an operator, it cannot be related to the Heisenberg picture, the Heisenberg equations of motion or the Ehrenfest theorem. Thus it does not properly correspond to quantum mechanics at all. The straightforward and certainly physically most sensible response to its defective nature is to unhesitatingly discard the Klein-Gordon equation in favor of Eq. (1), which is both *mandated* by the correspondence principle and has *none* of the Klein-Gordon equation's deficiencies.

Some physicists have been unaccountably reluctant to simply *heed* these imperatives, and have cast about for a loophole which enables the Klein-Gordon equation to survive. Since the Klein-Gordon equation abjectly fails the tests of quantum mechanics, the fact that it resembles the classical wave equation has led to the notion that it too represents a classical field theory. To obtain any manner of quantum physics from the Klein-Gordon equation, then, would presumably involve quantizing the classical-field physics it represents. The idea of the Klein-Gordon equation as a classical field equation seems physically incongruous at first glance because that equation is associated with a *particle* which can perfectly well exist in a *vacuum*, whereas the only *familiar* classical field theory which does *not* describe collective motions of an *underlying medium* is the electromagnetic field. In fact, by way of shedding light on this issue of dynamical classical field theories which do *not* describe motions of some medium, it has recently been shown that the classical electric and magnetic fields which satisfy the source-free Maxwell equations can be linearly mapped into the real and imaginary parts of a transverse-vector wave function that, as a consequence, satisfies the time-dependent Schrödinger equation whose Hamiltonian operator is $|c\hat{\mathbf{p}}|$, which is, of course, physically appropriate to a massless free particle, i.e., the free photon [2]. Thus we have this electromagnetic example of *classical* field equations which in fact are *equivalent* to a physically appropriately related *first-quantized Schrödinger equation*—this equivalence has until recently unfortunately been effectively hidden by the unfamiliar and rather unusual linear mapping between the two equation systems. If *classical* Klein-Gordon field theory should *likewise* turn out to be *physically appropriate quantum mechanics* that is *merely disguised* by an unfamiliar linear mapping, then it would in fact be entirely acceptable on that basis. A caveat regarding this speculation is of course that the correspondence principle *pinpoints* the time-dependent Schrödinger equation of Eq. (1) as the *physically correct description of the quantum mechanics for this case of a spinless relativistic free particle*.

An extremely important consideration regarding *classical* Klein-Gordon field theory is that the *quantum-mechanical* Eq. (2) does *not* define a *classical* Klein-Gordon field. On physical measurement grounds, a spinless *classical* field must be *strictly real-valued in configuration representation*. Therefore a *classical* Klein-Gordon field is a *real-valued function* $\phi(\mathbf{r}, t)$ which satisfies the equation,

$$(c^{-2}\partial^2/\partial t^2 - \nabla^2 + \mu^2)\phi(\mathbf{r}, t) = 0. \quad (3)$$

The fact that $\phi(\mathbf{r}, t)$ is *real* will enable us to define a conserved energy for this field that is *nonnegative*, which effectively *eliminates* the unphysical properties that flow from the quantum-mechanical Klein-Gordon equation's *extraneous negative energies*.

Since our *goal* for this classical Klein-Gordon field ϕ is its *quantization*, we *must* cast it into *canonical Hamiltonian form*, which involves a field π that is canonically conjugate to ϕ , and also a corresponding conserved Hamiltonian from which Hamilton's equations yield Eq. (3). The first step in this direction is to find an action functional $S[\phi]$ that is stationary for those ϕ which satisfy Eq. (3). This can in fact be obtained in terms of a local *action density*, usually termed a Lagrangian density, \mathcal{L}_ϕ such that [3],

$$S[\phi] = \int \mathcal{L}_\phi d^3\mathbf{r} dt. \quad (4a)$$

For the classical Klein-Gordon field, \mathcal{L}_ϕ is conventionally taken to be [3],

$$\mathcal{L}_\phi = \frac{1}{2}(\dot{\phi}^2/c^2 - |\nabla\phi|^2 - \mu^2\phi^2), \quad (4b)$$

which yields for the functional derivative of $S[\phi]$ with respect to ϕ ,

$$\delta S[\phi]/\delta\phi = -\ddot{\phi}/c^2 + \nabla^2\phi - \mu^2\phi. \quad (4c)$$

Setting this first-order variation of $S[\phi]$ with respect to ϕ to zero indeed produces the classical Klein-Gordon equation of Eq. (3).

With this Lagrangian density \mathcal{L}_ϕ in hand, we readily obtain a field π that is canonically conjugate to ϕ [3],

$$\pi = \partial\mathcal{L}_\phi/\partial\dot{\phi} = \dot{\phi}/c^2, \quad (5a)$$

which implies that,

$$\dot{\phi} = c^2\pi. \quad (5b)$$

The Hamiltonian density which corresponds to ϕ and π is [3],

$$\mathcal{H}_{\phi,\pi} = \dot{\phi}\pi - \mathcal{L}_\phi \Big|_{\dot{\phi}=c^2\pi}. \quad (5c)$$

After combining this with Eq. (4b), the result in terms of ϕ and π is,

$$\mathcal{H}_{\phi,\pi} = \frac{1}{2}(|\nabla\phi|^2 + \mu^2\phi^2 + c^2\pi^2), \quad (5d)$$

which we note is indeed *nonnegative* for our *real-valued* classical ϕ and π fields. This *nonnegative* energy density is a *crucial* feature of the *classical* Klein-Gordon field theory with its *strictly real-valued fields*, as we have emphasized above. From the Hamiltonian density of Eq. (5d) and integration by parts we obtain the Hamiltonian functional,

$$H[\phi, \pi] = \int \mathcal{H}_{\phi,\pi} d^3\mathbf{r} = \frac{1}{2} \int [\phi(-\nabla^2 + \mu^2)\phi + c^2\pi^2] d^3\mathbf{r}. \quad (6a)$$

We now apply the standard prescriptions for Hamilton's equations of motion to this Hamiltonian functional to obtain,

$$\dot{\phi} = \delta H[\phi, \pi]/\delta\pi = c^2\pi, \quad (6b)$$

and,

$$\dot{\pi} = -\delta H[\phi, \pi]/\delta\phi = (\nabla^2 - \mu^2)\phi, \quad (6c)$$

which together clearly imply the classical Klein-Gordon equation of Eq. (3). Note that the canonically conjugate field π satisfies the Klein-Gordon equation as well.

Before going further with this now canonically Hamiltonized classical Klein-Gordon field theory, we wish to digress in order to demonstrate that, within a constant factor, *the real and imaginary parts of the configuration representation of the state vector* $|\psi(t)\rangle$ which satisfies the Schrödinger equation of Eq. (1) are *also canonically conjugate classical fields* whose equations of motion follow from the classical Hamiltonian functional which is given by the conserved mean free-particle energy $\langle\psi(t)|(|c\hat{\mathbf{p}}|^2 + m^2c^4)^{\frac{1}{2}}|\psi(t)\rangle$.

Schrödinger's equation from a classical Hamiltonian functional

It is apparent that the nonnegative conserved mean free-particle energy,

$$\langle\psi(t)|(|c\hat{\mathbf{p}}|^2 + m^2c^4)^{\frac{1}{2}}|\psi(t)\rangle,$$

is a nonnegative linear functional of *both* the wave function $\psi(\mathbf{r}, t)$ and its complex conjugate $\psi^*(\mathbf{r}, t)$ when it is expressed in the form,

$$H[\psi, \psi^*] = \int \psi^*(\mathbf{r}, t) \left[(|c\hat{\mathbf{p}}|^2 + m^2c^4)^{\frac{1}{2}} \psi \right] (\mathbf{r}, t) d^3\mathbf{r}. \quad (7a)$$

In configuration representation, the Hermitian operator $(|c\hat{\mathbf{p}}|^2 + m^2c^4)^{\frac{1}{2}}$ is a real, symmetric nonlocal *integral* operator whose kernel is, of course, given by,

$$\langle\mathbf{r}|(|c\hat{\mathbf{p}}|^2 + m^2c^4)^{\frac{1}{2}}|\mathbf{r}'\rangle = (2\pi)^{-3} \int e^{i\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')} (|c\hbar\mathbf{k}|^2 + m^2c^4)^{\frac{1}{2}} d^3\mathbf{k}, \quad (7b)$$

an integral whose result is distribution-valued, i.e., it is singular as $|\mathbf{r} - \mathbf{r}'| \rightarrow 0$, a feature which requires careful treatment akin to that required by the delta function. From Eq. (7a) and Eq. (1) it is clear that time-dependent Schrödinger equation for the configuration-space wave function ψ follows from the simple functional differential equation,

$$i\hbar\dot{\psi} = \delta H[\psi, \psi^*]/\delta\psi^*. \quad (7c)$$

The complex-valued fields ψ and ψ^* have the dimensions of probability density amplitude. From these we readily define two *strictly real-valued fields* ϕ_ψ and π_ψ which each have the dimensions of action density amplitude that is appropriate to their being canonically conjugate,

$$\phi_\psi \stackrel{\text{def}}{=} (\hbar/2)^{\frac{1}{2}}(\psi + \psi^*), \quad \pi_\psi \stackrel{\text{def}}{=} -i(\hbar/2)^{\frac{1}{2}}(\psi - \psi^*). \quad (8a)$$

In terms of ϕ_ψ and π_ψ we have that,

$$\psi = (\phi_\psi + i\pi_\psi)/(2\hbar)^{\frac{1}{2}}, \quad \psi^* = (\phi_\psi - i\pi_\psi)/(2\hbar)^{\frac{1}{2}}, \quad (8b)$$

which when substituted into Eq. (7a), bearing in mind that $\mu = ((mc)/\hbar)$ and that in configuration representation $\hat{\mathbf{p}} = -i\hbar\nabla$, yields,

$$H[\phi_\psi, \pi_\psi] = (c/2) \int \left[\phi_\psi(-\nabla^2 + \mu^2)^{\frac{1}{2}} \phi_\psi + \pi_\psi(-\nabla^2 + \mu^2)^{\frac{1}{2}} \pi_\psi \right] d^3\mathbf{r}. \quad (9a)$$

Since,

$$\delta H[\psi, \psi^*]/\delta\psi^* = (\delta H[\phi_\psi, \pi_\psi]/\delta\phi_\psi)(\partial\phi_\psi/\partial\psi^*) + (\delta H[\phi_\psi, \pi_\psi]/\delta\pi_\psi)(\partial\pi_\psi/\partial\psi^*),$$

we can readily obtain the real and imaginary parts of both the left and right hand sides of Eq. (7c) by application of Eqs. (8a) and (8b). The results of carrying out this decomposition are the two prescriptions for equations of motion,

$$\dot{\phi}_\psi = \delta H[\phi_\psi, \pi_\psi]/\delta\pi_\psi, \quad \dot{\pi}_\psi = -\delta H[\phi_\psi, \pi_\psi]/\delta\phi_\psi, \quad (9b)$$

which are identical to the standard prescriptions for Hamilton's equations of motion. This demonstrates that ϕ_ψ and π_ψ are *indeed* canonically conjugate fields which pertain to the Hamiltonian functional $H[\phi_\psi, \pi_\psi]$. It is further readily verified by applying Eqs. (8a) and (8b) that the prescriptions for Hamilton's equations of motion of Eq. (9b) for ϕ_ψ and π_ψ are indeed *equivalent* to Eq. (7c), and therefore *are equivalent as well* to the time-dependent Schrödinger equation of Eq. (1). Actual application of the prescriptions of Eq. (9b) for Hamilton's equations of motion to the Hamiltonian functional $H[\phi_\psi, \pi_\psi]$ of Eq. (9a) yields,

$$\dot{\phi}_\psi = c(-\nabla^2 + \mu^2)^{\frac{1}{2}} \pi_\psi, \quad \dot{\pi}_\psi = -c(-\nabla^2 + \mu^2)^{\frac{1}{2}} \phi_\psi. \quad (9c)$$

From Eq. (9c) and the first equation of Eq. (8b), we readily show that,

$$i\hbar\dot{\psi} = \hbar c(-\nabla^2 + \mu^2)^{\frac{1}{2}} \psi, \quad (9d)$$

which, since $\mu = ((mc)/\hbar)$ and $\hat{\mathbf{p}} = -i\hbar\nabla$ in configuration representation, is, of course, equivalent to Eq. (1).

It is therefore very clear indeed that the time-dependent Schrödinger equation of Eq. (1), which describes the solitary spinless relativistic free particle, is *equivalent* to the *classical Hamiltonian field system* that is described by the classical Hamiltonian functional $H[\phi_\psi, \pi_\psi]$ of Eq. (9a). A very notable feature of $H[\phi_\psi, \pi_\psi]$ is that it exhibits complete symmetry under the *interchange* of its canonically conjugate fields ϕ_ψ and π_ψ ; indeed these two fields as well have exactly the *same dimensions*, namely that of action density amplitude. We now *return* to the conventional Hamiltonian functional $H[\phi, \pi]$ of Eq. (6a) for the *classical* Klein-Gordon field ϕ and its canonical conjugate π .

Canonical transformation of the classical Klein-Gordon field

We note that the *form* of the Hamiltonian functional of Eq. (6a) is *nonsymmetrical* under the *interchange* of ϕ and π ; indeed ϕ and π *themselves* have *different dimensions*. It is straightforward to utilize fractional powers of the real, symmetric nonnegative operator $(-\nabla^2 + \mu^2)$ to devise a *canonical transformation* to *new* canonical fields that occur *symmetrically* in the Hamiltonian functional, and which *both* have the dimensions of action density amplitude. This *canonical transformation* is given by,

$$\phi_\psi = c^{-\frac{1}{2}}(-\nabla^2 + \mu^2)^{\frac{1}{4}} \phi, \quad \pi_\psi = c^{\frac{1}{2}}(-\nabla^2 + \mu^2)^{-\frac{1}{4}} \pi, \quad (10a)$$

which, of course, implies that,

$$\phi = c^{\frac{1}{2}}(-\nabla^2 + \mu^2)^{-\frac{1}{4}} \phi_\psi, \quad \pi = c^{-\frac{1}{2}}(-\nabla^2 + \mu^2)^{\frac{1}{4}} \pi_\psi. \quad (10b)$$

Upon substituting these expressions for ϕ and π in terms of their *canonical transforms* ϕ_ψ and π_ψ into the Hamiltonian of Eq. (6a), and also taking note of the symmetric-operator nature of powers of the operator $(-\nabla^2 + \mu^2)$, the new Hamiltonian functional $H[\phi_\psi, \pi_\psi]$ is seen to exhibit complete symmetry under the *interchange* of ϕ_ψ and π_ψ ,

$$H[\phi_\psi, \pi_\psi] = (c/2) \int \left[\phi_\psi (-\nabla^2 + \mu^2)^{\frac{1}{2}} \phi_\psi + \pi_\psi (-\nabla^2 + \mu^2)^{\frac{1}{2}} \pi_\psi \right] d^3\mathbf{r}. \quad (10c)$$

Indeed this *canonically transformed* Hamiltonian functional of the *classical* Klein-Gordon field theory is *identical* to that of Eq. (9a), whose equations of motion are, as we have noted above, *entirely equivalent* to the time-dependent Schrödinger equation of Eq. (1) for the solitary spinless relativistic free particle. Thus we have demonstrated the canonical equivalence of classical Klein-Gordon field theory to correspondence-principle first quantization of the spinless relativistic free particle.

We can, in particular, provide the precise linear mapping of the real-valued classical Klein-Gordon field ϕ and its canonical conjugate π into the complex-valued Schrödinger wave function ψ ,

$$\psi = (2\hbar c)^{-\frac{1}{2}}(-\nabla^2 + \mu^2)^{\frac{1}{4}} \phi + i(c/(2\hbar))^{\frac{1}{2}}(-\nabla^2 + \mu^2)^{-\frac{1}{4}} \pi, \quad (11a)$$

which mapping can, of course, be inverted,

$$\phi = ((\hbar c)/2)^{\frac{1}{2}}(-\nabla^2 + \mu^2)^{-\frac{1}{4}}(\psi + \psi^*), \quad \pi = -i(\hbar/(2c))^{\frac{1}{2}}(-\nabla^2 + \mu^2)^{\frac{1}{4}}(\psi - \psi^*). \quad (11b)$$

One can express the linear mapping of Eqs. (11) *entirely* in terms of the classical Klein-Gordon field ϕ *alone* by recalling from Eq. (5a) that $\pi = \dot{\phi}/c^2$. It then becomes an interesting exercise, which we leave to the reader, to verify that merely because ϕ satisfies the Klein-Gordon equation, which is second-order in time, the complex-valued wave-function construct ψ of Eq. (11a) satisfies the Schrödinger equation of Eq. (1), which is first-order in time! Nor is that all: the wave function construct ψ of Eq. (11a) has also been painstakingly composed to have the dimensions of probability density amplitude, which is appropriate to a wave function; in fact its detailed construction is such that when it is inserted into the Hamiltonian functional $H[\psi, \psi^*]$ of Eq. (7a), which is the first-quantized free particle's conserved mean energy, the result is the Hamiltonian functional $H[\phi, \pi]$ of Eq. (6a), which is, naturally enough, the energy of the corresponding *classical* Klein-Gordon field. The fact that a first-quantized *free* particle's mean energy is *nonnegative*, along with the *form* of the Klein-Gordon classical field's energy density that is given by Eq. (5d), together make it abundantly clear that the Klein-Gordon classical field is *restricted to being real-valued*, which gives us additional insight into the *completely unphysical nature* of the complex-valued quantum-mechanical Klein-Gordon wave function of Eq. (2). An alternative approach to verification of the Schrödinger equation of Eq. (1) from Eq. (11a) is, of course, to apply the Hamilton's equations of motion for ϕ and π that are given by Eqs. (6b) and (6c).

Classical field quantization via particle second quantization

The quantization of the classical Klein-Gordon field is by far best done in the Schrödinger wave-function picture, where it is merely *second quantization* of the spinless relativistic free particle, which makes both the physics and the mathematics completely transparent. This second quantization is achieved in the standard canonical fashion by promoting the wave function ψ and its complex conjugate ψ^* to become the *Hermitian conjugate operators* $\hat{\psi}$ and $\hat{\psi}^\dagger$ which have the commutation relation [3],

$$[\hat{\psi}(\mathbf{r}), \hat{\psi}^\dagger(\mathbf{r}')] = \delta^{(3)}(\mathbf{r} - \mathbf{r}'). \quad (12a)$$

With that, $\hat{\psi}^\dagger(\mathbf{r})$ is interpreted as the operator which creates a particular type of spinless relativistic free particle of mass m at the position \mathbf{r} , while $\hat{\psi}(\mathbf{r})$ is interpreted as the operator which annihilates such a particle at the position \mathbf{r} . Fourier transforms of these operators perform these same creation/annihilation functions in wave-vector (i.e., momentum) space.

The real Hamiltonian functional $H[\psi, \psi^*]$ of Eq. (7a) is thereby quantized as the *Hermitian* operator $\hat{H}[\hat{\psi}, \hat{\psi}^\dagger]$, which is then taken to be the Hamiltonian operator of the second-quantized spinless relativistic

free particle system (or quantized Klein-Gordon field system). In the Heisenberg picture which is defined by this Hamiltonian operator, the relativistic free-particle time-dependent Schrödinger equation of Eq. (1) continues to hold as a field-operator relation, i.e.,

$$i\hbar\partial\widehat{\psi}(\mathbf{r},t)/\partial t = \hbar c \left[(-\nabla^2 + \mu^2)^{\frac{1}{2}}\widehat{\psi}\right](\mathbf{r},t). \quad (12b)$$

By multiple applications of the creation operator, arbitrarily large numbers of this particular type of spinless relativistic free particle can be produced. Indeed the Hermitian operator $\int \widehat{\psi}^\dagger \widehat{\psi} d^3\mathbf{r}$ has the interpretation of particle number operator [3]. The vast underlying Hilbert space is called Fock space [3].

The relations of Eqs. (11) may be simply transcribed into this second-quantized regime by noting the ϕ and π become the *Hermitian operators* $\widehat{\phi}$ and $\widehat{\pi}$, and of course ψ and ψ^* become the operators $\widehat{\psi}$ and $\widehat{\psi}^\dagger$ with their fundamental canonical commutation relation and interpretation described above. These transcribed relations of Eqs. (11) capture the technical essence of the relationship between the quantized Klein-Gordon field theory and the second-quantized spinless relativistic free particle picture that, because of the just-mentioned interpretation of the precise role of the operators $\widehat{\psi}$ and $\widehat{\psi}^\dagger$, is obviously vastly more physically and computationally transparent.

Radiation-gauge vector potential equivalence to the first-quantized free photon

Maxwell's source-free equations for the transverse electric and and magnetic fields have previously been canonically Hamiltonized by a linear mapping that did not mix these fields, did but treat them in a highly symmetric manner [2]. This symmetric mapping of the source-free electric and magnetic fields into canonically conjugate counterparts turns out to be directly related to the real and imaginary parts of the free photon's first-quantized transverse-vector wave function. The traditional prescription for canonically Hamiltonizing source-free electromagnetism in order to quantize it, on the other hand, invariably eschews the electric and magnetic fields in favor of the vector potential [4], and therefore strongly parallels the treatment we have just presented of the classical Klein-Gordon field. The transverse part of the electromagnetic vector potential \mathbf{A}_T satisfies the two equations,

$$\nabla \cdot \mathbf{A}_T = 0, \quad \ddot{\mathbf{A}}_T/c^2 - \nabla^2 \mathbf{A}_T = \mathbf{j}_T/c, \quad (13a)$$

the second of which becomes simply the classical wave equation in the source-free case. In that case one may *also* use the *radiation gauge*, for which $A^0 = 0$ and $\nabla \cdot \mathbf{A} = 0$ [4], conditions which imply that the *only* nonvanishing part of the electromagnetic four-potential A^μ is precisely \mathbf{A}_T , and it, of course, satisfies the classical wave equation, which is *merely a special case of the Klein-Gordon equation*. For that reason one arrives at a Lagrangian density which strongly resembles the Lagrangian density of Eq. (4b),

$$\mathcal{L}_{\mathbf{A}} = \frac{1}{2}(|\dot{\mathbf{A}}|^2/c^2 - |\nabla \times \mathbf{A}|^2), \quad (13b)$$

where \mathbf{A} is, of course, constrained to be *transverse*, i.e., $\nabla \cdot \mathbf{A} = 0$ and $\nabla \cdot \dot{\mathbf{A}} = 0$. This Lagrangian density yields the canonical momentum $\mathbf{\Pi}_A = \dot{\mathbf{A}}/c^2$, and therefore we have that $\dot{\mathbf{A}} = c^2 \mathbf{\Pi}_A$. Clearly $\mathbf{\Pi}_A$ is *also* transverse, i.e., $\nabla \cdot \mathbf{\Pi}_A = 0$. With these results for the canonical momentum, the corresponding Hamiltonian density comes out to be,

$$\mathcal{H}_{\mathbf{A},\mathbf{\Pi}_A} = \frac{1}{2}(|\nabla \times \mathbf{A}|^2 + c^2|\mathbf{\Pi}_A|^2), \quad (13c)$$

which is nonnegative and strongly resembles the Hamiltonian density of Eq. (5d). Of course the Hamiltonian functional is the integral of the Hamiltonian density over three-dimensional space. Noting that $\nabla \cdot \mathbf{A} = 0$ and integrating by parts, one readily puts the Hamiltonian functional into a form which is in essence the same as that of Eq. (6a) for the classical Klein-Gordon field,

$$H[\mathbf{A},\mathbf{\Pi}_A] = \frac{1}{2} \int [\mathbf{A} \cdot (-\nabla^2 \mathbf{A}) + c^2|\mathbf{\Pi}_A|^2] d^3\mathbf{r}. \quad (14a)$$

This Hamiltonian functional has corresponding Hamilton's equations of motion that are very similar to those of Eqs. (6b) and (6c),

$$\dot{\mathbf{A}} = c^2 \mathbf{\Pi}_A, \quad \dot{\mathbf{\Pi}}_A = \nabla^2 \mathbf{A}. \quad (14b)$$

Eq. (14b) implies that both \mathbf{A} and $\mathbf{\Pi}_A$ *obey the classical wave equation*.

The nonsymmetry of $H[\mathbf{A},\mathbf{\Pi}_A]$ under interchange of \mathbf{A} and $\mathbf{\Pi}_A$ now motivates a *canonical transformation* which is completely analogous to the one made in Eq. (10a) for the classical Klein-Gordon field,

$$\Phi = c^{-\frac{1}{2}}(-\nabla^2)^{\frac{1}{4}} \mathbf{A}, \quad \Pi = c^{\frac{1}{2}}(-\nabla^2)^{-\frac{1}{4}} \mathbf{\Pi}_A. \quad (15a)$$

We note that both Φ and Π are transverse vector fields, i.e., $\nabla \cdot \Phi = \nabla \cdot \Pi = 0$. This canonical transformation results in a change to the form of the Hamiltonian functional, with a result that is completely analogous to the Hamiltonian functionals of Eqs. (10c) and (9a),

$$H[\Phi, \Pi] = \frac{1}{2} \int \left[\Phi \cdot \left(c(-\nabla^2)^{\frac{1}{2}} \Phi \right) + \Pi \cdot \left(c(-\nabla^2)^{\frac{1}{2}} \Pi \right) \right] d^3\mathbf{r}. \quad (15b)$$

The Hamiltonian functional $H[\Phi, \Pi]$ has corresponding Hamilton's equations of motion that are very similar to those of Eq. (9c),

$$\dot{\Phi} = c(-\nabla^2)^{\frac{1}{2}} \Pi, \quad \dot{\Pi} = -c(-\nabla^2)^{\frac{1}{2}} \Phi. \quad (15c)$$

If one now, in analogy with Eq. (8b), defines,

$$\Psi \stackrel{\text{def}}{=} (\Phi + i\Pi)/(2\hbar)^{\frac{1}{2}}, \quad (16a)$$

which has the dimensions of probability density amplitude appropriate to a wave function, one deduces from Eq. (15c) that its equation of motion is,

$$i\hbar\dot{\Psi} = \hbar c(-\nabla^2)^{\frac{1}{2}} \Psi, \quad (16b)$$

which is precisely that of the time-dependent Schrödinger equation for the free photon because, since $\hat{\mathbf{p}} = -i\hbar\nabla$ in configuration representation, $\hbar c(-\nabla^2)^{\frac{1}{2}} = |c\hat{\mathbf{p}}|$ in that representation. We further note that this complex-valued free-photon wave function Ψ of Eq. (16a) is of course a transverse vector field, i.e., $\nabla \cdot \Psi = 0$, and that its dimensions are indeed appropriate to a wave function.

We have thus demonstrated that in the case of source-free electromagnetism there exists a linear mapping of the electromagnetic transverse-vector canonically conjugate fields \mathbf{A} and Π_A into the real and imaginary parts of the free photon wave function Ψ . This linear mapping is, of course, highly analogous to that given in Eq. (11a) for the classical Klein-Gordon field,

$$\Psi = (2\hbar c)^{-\frac{1}{2}}(-\nabla^2)^{\frac{1}{4}} \mathbf{A} + i(c/(2\hbar))^{\frac{1}{2}}(-\nabla^2)^{-\frac{1}{4}} \Pi_A, \quad (17a)$$

and its inverse is,

$$\mathbf{A} = ((\hbar c)/2)^{\frac{1}{2}}(-\nabla^2)^{-\frac{1}{4}}(\Psi + \Psi^*), \quad \Pi_A = -i(\hbar/(2c))^{\frac{1}{2}}(-\nabla^2)^{\frac{1}{4}}(\Psi - \Psi^*). \quad (17b)$$

The Schrödinger equation of Eq. (16b) and the result for \mathbf{A} given by Eq. (17b) are exactly the same as those which were previously obtained via the symmetric canonical Hamiltonization of the electric and magnetic fields which satisfy Maxwell's source-free equations [2].

One can express the linear mapping of Eqs. (17) *entirely* in terms of the transverse vector potential \mathbf{A} *alone* by recalling that $\Pi_A = \dot{\mathbf{A}}/c^2$ (e.g., see Eq. (14b)). It then becomes an interesting exercise to verify that the time-dependent Schrödinger equation of Eq. (16b) for the free photon's wave function Ψ , as given by Eq. (17a) and supplemented by the formula $\Pi_A = \dot{\mathbf{A}}/c^2$, follows *merely* from the fact that the transverse vector potential \mathbf{A} satisfies the *classical wave equation*, notwithstanding that the classical wave equation is second-order in time, whereas the Schrödinger equation of Eq. (16b) is first-order in time! Of course an alternative approach to verification of the Schrödinger equation of Eq. (16b) from Eq. (17a) is to apply the Hamilton's equations of motion for \mathbf{A} and Π_A that are given by Eq. (14b).

Conclusion

The most striking results of this paper are the one-to-one linear mappings, given by Eqs. (11) and (17), of real-valued dynamical classical fields that are described by simple, second-order in time, wave equations, i.e., the classical Klein-Gordon equation and the classical wave equation, onto correspondence-principle first-quantized relativistic Schrödinger-equation wave functions for free particles, i.e., spinless relativistic free particles and free photons—particularly in light of the fact that such correspondence-principle first-quantized free-particle wave functions can be transparently converted in a flash into the free-particle annihilation and creation operators that are the very heart of the quantum many-free-particle description. These one-to-one linear mappings lend a truly gratifying dollop of theoretical concreteness to the heretofore vague notion of *complementarity* between dynamical classical wave fields and the corresponding particle quanta.

Fascinating though it is that classical Klein-Gordon field theory is equivalent to the elementary correspondence-principle first quantization of the spinless relativistic free particle, there is in fact no practical point

to the exercise. Elementary correspondence-principle first quantization is clearly enormously simpler and more physically transparent than is long-winded canonical Hamiltonization and canonical transformation of classical field equations that are second-order in time. Furthermore, elementary correspondence-principle *first* quantization is by far the most physically and mathematically transparent *gateway* to field quantization because its *second* quantization is so extraordinarily straightforward and simple. It is only for electromagnetism, where classical field theoretic methods have been completely entrenched for a century and a half, that discussion of the relationship of classical field theory to elementary correspondence-principle first quantization is at all worthwhile. Even there, field quantization definitely ought to be handled by proceeding directly from the photon's elementary correspondence-principle first quantization to its second quantization—there is clearly no alternative field-theory approach that is as physically transparent or computationally simple.

There can by now be no doubt at all that elementary correspondence-principle first- and second-quantization is physically correct in all cases [2, 5]. It has clearly proved its mettle vis-à-vis both classical electromagnetism and classical Klein-Gordon field theory. As for Dirac theory, it is even *more* unphysical (if such a thing is possible!) than is treating the classical Klein-Gordon field as a complex-valued wave function. Not merely does Dirac theory manifest the signature unbounded-below energies, its artificially introduced anticommuting matrices cause the commutators of some *very basic observables* to behave insanely. The commutator of any two *orthogonal* components of the free-particle Dirac *velocity operator* not only *fails* to vanish, its value *is not affected in the slightest* by taking the *classical limit* $\hbar \rightarrow 0$. And when the *nonrelativistic limit* $c \rightarrow \infty$ is taken, these orthogonal velocity-component commutators *diverge*! The *free-particle* Dirac theory as well violates Newton's first law of motion with mind-boggling spontaneous particle *acceleration* that is *inversely* proportional to Planck's constant and directly proportional to particle mass and the cube of the speed of light; it also features a *universal fixed particle speed* which is 73% *greater* than that of light, and it unaccountably manifests strong spontaneous spin-orbit coupling [5]. In light of the fact that elementary correspondence-principle first- and second-quantization has *no discernible pathologies whatsoever*, it is patently obvious what the physics status of Dirac theory ought to be.

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