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A mathematical model of the quark and lepton mixing angles (2011 update)

J. S. Markovitch
P.O. Box 752
Batavia, IL 60510
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A single mathematical model encompassing both quark and lepton mixing is described. This model exploits the fact that when a 3×3 rotation matrix whose elements are squared is subtracted from its transpose, a matrix is produced whose non-diagonal elements have a common absolute value, where this value is an intrinsic property of the rotation matrix. For the traditional CKM quark mixing matrix with its second and third rows interchanged (i.e., c – t interchange), this value equals one-third the corresponding value for the leptonic matrix (roughly, 0.05 versus 0.15). By imposing this and two additional related constraints on mixing, and letting leptonic φ_{23} equal 45°, a framework is defined possessing just two free parameters. A mixing model is then specified using values for these two parameters that derive from an equation that reproduces the fine structure constant. The resultant model, which possesses no constants adjusted to fit experiment, has mixing angles of $\theta_{23} = 2.367445^\circ$, $\theta_{13} = 0.190987^\circ$, $\theta_{12} = 12.920966^\circ$, $\varphi_{23} = 45^\circ$, $\varphi_{13} = 0.013665^\circ$, and $\varphi_{12} = 33.210911^\circ$. A fourth, newly-introduced constraint of the type described above produces a Jarlskog invariant for the quark matirx of 2.758×10^{-5}. Collectively these achieve a good fit with the experimental quark and lepton mixing data. The model predicts the following CKM matrix elements: $|V_{us}| = \sqrt{0.05} = 2.236 \times 10^{-1}$, $|V_{ub}| = 3.333 \times 10^{-3}$, and $|V_{cb}| = 4.131 \times 10^{-2}$. For leptonic mixing the model predicts $\sin^2 \varphi_{12} = 0.3$, $\sin^2 \varphi_{23} = 0.5$, and $\sin^2 \varphi_{13} = 5.688 \times 10^{-8}$. At the time of its 2007 introduction the model's values for $|V_{us}|$ and $|V_{ub}|$ had disagreements with experiment of an improbable 3.6σ and 7.0σ, respectively, but 2010 values from the same source now produce disagreements of just 2.4σ and 1.1σ, the absolute error for $|V_{us}|$ having been reduced by 53%, and that for $|V_{ub}|$ by 78%.

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Contents

I. INTRODUCTION

The phenomenon of mixing [1–3] has been explored with the aid of a wide variety of physical models [4]. However, an alternative approach to understanding mixing is available: the mathematics of rotation matrices, on the one hand, and the quark and lepton mixing data, on the other, can be analyzed apart from the Standard Model to see what they can reveal about each other. As this article will show, even this limited approach can produce results worthy of note. Specifically, we will demonstrate that if a 3×3 rotation matrix whose elements are squared is subtracted from its transpose, a matrix is produced whose non-diagonal elements possess a common absolute value; this value is an intrinsic property of the rotation matrix. For the mixing matrices determined by experiment this value in the leptonic sector measures three times its value in the quark sector in four independent ways.

Generally speaking, however, the above property has not been discussed in the mixing literature. Why should this be so? It is perhaps because it is only if one builds the CKM quark mixing matrix with its c- and t-quarks interchanged relative to convention that this distinguishing property for leptons (equaling roughly 0.15) [2] is readily seen as three times that for quarks (equaling roughly 0.05) [3]. When the quark mixing matrix is built in the traditional manner no such obvious relation presents itself. Of course, the above relation might be coincidental, but given that the traditional assignment of the c-quark to the 2^{nd} generation and the t-quark to the 3^{rd} is arbitrary, there is no reason such an exchange should not be made.

In this article a single mathematical model encompassing both quark and lepton mixing will be described. The model is defined with the aid of, and distinguished by, four constraints that each exploit the property described above. This will allow the model's six angles and two phases to arise naturally within a common mathematical framework. This article is an update to articles from 2007 [5] and 2009 [6], which made no attempt to model CP violating phases. The purpose here is, in part, to incorporate phase into the mixing model, but also to see how well the model has fared experimentally since 2007. As will be seen, a value close to experiment for the CKM mixing matrix's Jarlskog invariant arises naturally when a constraint of the above type is imposed on the quark and lepton phases (see Section VIII). A comparison of the model's 2007 predictions against the most recent quark mixing data is offered by Table I, while Table II provides the corresponding comparison for leptonic data.

II. MIXING MATRICES AND THE PRIMARY COUPLING CONSTRAINT

The quark and lepton mixing matrices [2, 3] without their phases are each merely a 3×3 rotation matrix. Let

$$R = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} \tag{2.1}$$

be such a matrix, so that squaring its elements gives

$$\begin{bmatrix} r_{11}^2 & r_{12}^2 & r_{13}^2 \\ r_{21}^2 & r_{22}^2 & r_{23}^2 \\ r_{31}^2 & r_{32}^2 & r_{33}^2 \end{bmatrix} \quad .$$

Given that a rotation matrix with its elements squared has rows and columns that sum to one, the above matrix can also be written

$$\begin{bmatrix} r_{11}^2 & 1 - r_{11}^2 - r_{13}^2 & r_{13}^2 \\ 1 - r_{11}^2 - r_{31}^2 & r_{11}^2 + r_{13}^2 + r_{31}^2 + r_{33}^2 - 1 & 1 - r_{13}^2 - r_{33}^2 \\ r_{31}^2 & 1 - r_{31}^2 - r_{33}^2 & r_{33}^2 \end{bmatrix} \quad .$$

When this matrix is subtracted from its transpose it gives

$$\begin{bmatrix} 0 & r_{31}^2 - r_{13}^2 & r_{13}^2 - r_{31}^2 \\ r_{13}^2 - r_{31}^2 & 0 & r_{31}^2 - r_{13}^2 \\ r_{31}^2 - r_{13}^2 & r_{13}^2 - r_{31}^2 & 0 \end{bmatrix} \quad ,$$

a matrix whose non-diagonal elements all equal

$$\pm \left(r_{31}^2 - r_{13}^2 \right) \quad .$$

It follows that $|r_{31}^2 - r_{13}^2|$ is an intrinsic property of the rotation matrix R.

Now assume that R is produced by rotations through the angles ψ_{23}, ψ_{13}, and ψ_{12}, so that

$$R = \begin{bmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13} & c_{12}c_{23} - s_{12}s_{23}s_{13} & s_{23}c_{13} \\ s_{12}s_{23} - c_{12}c_{23}s_{13} & -c_{12}s_{23} - s_{12}c_{23}s_{13} & c_{23}c_{13} \end{bmatrix} \quad , \tag{2.2}$$

where $s_{12} \equiv \sin\psi_{12}$, $c_{12} \equiv \cos\psi_{12}$, etc. For R define

$$\begin{aligned} \Delta P_{\psi_{23},\psi_{13},\psi_{12}} &= |r_{31}^2 - r_{13}^2| \\ &= |\,(s_{12}s_{23} - c_{12}c_{23}s_{13})^2 - (s_{13})^2| \\ &= |(s_{12}s_{23})^2 + (-c_{12}c_{23}s_{13})^2 - 2s_{12}c_{23}c_{12}s_{23}s_{13} - (s_{13})^2| \quad , \end{aligned} \tag{2.3}$$

its degree of *coupling asymmetry.* (Note that Eq. (8.2) will later deal with the changes in coupling asymmetry caused by phase.)

Also define the quark mixing matrix

$$V = \begin{bmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13} & c_{12}c_{23} - s_{12}s_{23}s_{13} & s_{23}c_{13} \\ s_{12}s_{23} - c_{12}c_{23}s_{13} & -c_{12}s_{23} - s_{12}c_{23}s_{13} & c_{23}c_{13} \end{bmatrix} \quad , \tag{2.4}$$

where $s_{12} \equiv \sin\theta_{12}$, $c_{12} \equiv \cos\theta_{12}$, etc., and where θ_{23}, θ_{13}, and θ_{12} are the quark mixing angles.

And define the leptonic mixing matrix

$$U = \begin{bmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13} & c_{12}c_{23} - s_{12}s_{23}s_{13} & s_{23}c_{13} \\ s_{12}s_{23} - c_{12}c_{23}s_{13} & -c_{12}s_{23} - s_{12}c_{23}s_{13} & c_{23}c_{13} \end{bmatrix} \quad , \tag{2.5}$$

where $s_{12} \equiv \sin\varphi_{12}$, $c_{12} \equiv \cos\varphi_{12}$, etc., and where φ_{23}, φ_{13}, and φ_{12} are the leptonic mixing angles. Note that both of these matrices are in the usual form, except that phase is omitted for both [2][3].

With the aid of the above definitions it is now possible to illustrate how $\Delta P_{\psi_{23},\psi_{13},\psi_{12}}$ will be exploited in this article. Note that in the remainder of this section the experimental values of all matrix elements will be approximate.

Consider first the mixing matrix that arises if the traditional CKM quark mixing matrix [3] with its elements squared

$$\begin{array}{c|c} & \begin{matrix} d & s & b \end{matrix} \\ \begin{matrix} u \\ c \\ t \end{matrix} & \begin{bmatrix} 0.95 & 0.05 & 0 \\ 0.05 & 0.95 & 0 \\ 0 & 0 & 1.00 \end{bmatrix} \end{array}$$

has its second and third rows (i.e., its c- and t-quarks) interchanged

$$\begin{array}{c|c} & \begin{matrix} d & s & b \end{matrix} \\ \begin{matrix} u \\ t \\ c \end{matrix} & \begin{bmatrix} 0.95 & 0.05 & 0 \\ 0 & 0 & 1.00 \\ 0.05 & 0.95 & 0 \end{bmatrix} \end{array}$$

(equivalent to applying an $\pi/2$ offset to θ_{23}). Subtracting this second matrix from its transpose gives

$$\begin{bmatrix} 0.95 & 0.05 & 0 \\ 0 & 0 & 1.00 \\ 0.05 & 0.95 & 0 \end{bmatrix} - \begin{bmatrix} 0.95 & 0 & 0.05 \\ 0.05 & 0 & 0.95 \\ 0 & 1.00 & 0 \end{bmatrix} = \begin{bmatrix} 0 & +0.05 & -0.05 \\ -0.05 & 0 & +0.05 \\ +0.05 & -0.05 & 0 \end{bmatrix} \tag{2.6}$$

so that

$$\Delta P_{\frac{\pi}{2}+\theta_{23},\theta_{13},\theta_{12}} = 0.05 \quad . \tag{2.7}$$

For the traditional leptonic mixing matrix [2] with its elements squared

$$\begin{array}{c|c} & \begin{matrix} \nu_1 & \nu_2 & \nu_3 \end{matrix} \\ \begin{matrix} \nu_e \\ \nu_\mu \\ \nu_\tau \end{matrix} & \begin{bmatrix} 0.70 & 0.30 & 0 \\ 0.15 & 0.35 & 0.50 \\ 0.15 & 0.35 & 0.50 \end{bmatrix} \end{array}$$

no equivalent interchange will be required. Subtracting it from its transpose gives

$$\begin{bmatrix} 0.70 & 0.30 & 0 \\ 0.15 & 0.35 & 0.50 \\ 0.15 & 0.35 & 0.50 \end{bmatrix} - \begin{bmatrix} 0.70 & 0.15 & 0.15 \\ 0.30 & 0.35 & 0.35 \\ 0 & 0.50 & 0.50 \end{bmatrix} = \begin{bmatrix} 0 & +0.15 & -0.15 \\ -0.15 & 0 & +0.15 \\ +0.15 & -0.15 & 0 \end{bmatrix} \tag{2.8}$$

so that

$$\Delta P_{\varphi_{23}-\frac{\pi}{2},\varphi_{13},\varphi_{12}} = 0.15 \quad . \tag{2.9}$$

Note that, for reasons of symmetry with Eq. (2.7) an offset of $\pi/2$ is also applied to φ_{23} in Eq. (2.9). In what follows this will lead to φ_{23} equaling 135° rather than the usual 45° (see Eq. (7.6)).

In this way the calculation of experimental ΔP for quark and lepton mixing leads to Eqs. (2.7) and (2.9), which are forms of what will be termed the *primary coupling asymmetry* for the mixing matrices. These equations, in turn, combine to form

$$\Delta P_{\varphi_{23}-\frac{\pi}{2},\varphi_{13},\varphi_{12}} = 3 \times \Delta P_{\frac{\pi}{2}+\theta_{23},\theta_{13},\theta_{12}} \quad , \tag{2.10}$$

which this article will maintain constitutes a precise physical law.

III. THE MIXING MODEL: IMPOSING SIX CONSTRAINTS ON THE SIX MIXING ANGLES

It will now be shown how two mixing matrices and their six mixing angles can be specified by imposing on them six independent constraints.

Firstly, let

$$\varphi_{23} = \frac{\pi}{4} + \frac{\pi}{2} \quad . \tag{3.1}$$

Secondly, following Eq. (2.10), let

$$\Delta P_{\varphi_{23}-\frac{\pi}{2},\varphi_{13},\varphi_{12}} = 3 \times \Delta P_{\frac{\pi}{2}+\theta_{23},\theta_{13},\theta_{12}} \tag{3.2}$$

the *primary coupling constraint*. (Ultimately, a total of four such constraints with be imposed on the quark and lepton mixing matrices: the remaining three being the two *secondary coupling constraints* of Eqs. (4.1) and (4.2)—implied by the four equations immediately below—and the *phase constraint* of Eq. (8.3), introduced in Section VIII.)

And, finally, generate four mixing angles with the aid of g_{12} and g_{13}, model parameters whose possible values will be examined in detail later. These subscripts are chosen because g_{12} helps define the mixing angles φ_{12} and θ_{12}, whereas g_{13} helps define the mixing angles φ_{13} and θ_{13}:

$$\sin\varphi_{12} = \sqrt{3g_{12}} \quad , \tag{3.3}$$

$$\sin\theta_{13} = \sqrt{g_{13}/3} \quad , \tag{3.4}$$

$$\sin\theta_{12}/\sin\varphi_{23} = \sqrt{g_{12}} \quad , \tag{3.5}$$

$$\sin\varphi_{13}/\sin\theta_{23} = \sqrt{g_{13}} \quad . \tag{3.6}$$

Equations (3.1)–(3.6) together supply the six constraints needed to determine the six quark and lepton mixing angles and comprise the *mixing model specification*. Given that φ_{23} has its value explicitly assigned by Eq. (3.1) it is easy to calculate the three angles specified by Eqs. (3.3)–(3.5); in contrast, because Eq. (3.6) must be solved simultaneously with Eq. (3.2) the angles φ_{13} and θ_{23} are not so easily computed.

Also make note that the expressions whose square roots occupy the right sides of Eqs. (3.3)–(3.6)

$$\begin{aligned} &3g_{12} \quad , \\ &g_{13}/3 \quad , \\ &g_{12} \quad , \\ &g_{13} \end{aligned}$$

will be seen later in Eq. (6.17) in connection with the fine structure constant.

IV. THE TWO SECONDARY COUPLING CONSTRAINTS

It is important to recognize that Eqs. (3.3)–(3.6) were chosen only because they automatically impose the following additional two constraints on what will be termed *secondary coupling asymmetry*

$$\Delta P_{\varphi_{23},0,\varphi_{12}} = 3 \times \Delta P_{\frac{\pi}{2},0,\theta_{12}} \quad , \tag{4.1}$$

$$\Delta P_{-\frac{\pi}{2},\varphi_{13},0} = 3 \times \Delta P_{\theta_{23},\theta_{13},0} \quad . \tag{4.2}$$

It is these *secondary coupling constraints* in combination with the primary coupling constraint of Eq. (3.2) that constitute the conceptual key to the parameterization described by this article. Below, these three constraints are expressed in a list that makes it easier to compare the angles they employ. Observe, particularly, that the angles of the first row equal the sum of the angles of the second and third rows:

$$\Delta P\left(\varphi_{23} - \frac{\pi}{2}\,,\quad \varphi_{13}\,,\quad \varphi_{12}\right) = 3 \times \Delta P\left(\frac{\pi}{2} + \theta_{23}\,,\quad \theta_{13}\,,\quad \theta_{12}\right) \quad , \tag{4.3}$$

$$\Delta P\left(\varphi_{23}\,,\quad 0\,,\quad \varphi_{12}\right) = 3 \times \Delta P\left(\frac{\pi}{2}\,,\quad 0\,,\quad \theta_{12}\right) \quad , \tag{4.4}$$

$$\Delta P\left(-\frac{\pi}{2}\,,\quad \varphi_{13}\,,\quad 0\right) = 3 \times \Delta P\left(\theta_{23}\,,\quad \theta_{13}\,,\quad 0\right) \quad . \tag{4.5}$$

(See Eqs. (10.1)–(10.4) for these constraints summarized along with the phase constraint, mentioned above.)

Finally, consider that the net electric charge of the leptonic sector equals three times that of the quark sector

$$-1 + 0 = 3 \times \left(\frac{1}{3} + -\frac{2}{3}\right) \quad , \tag{4.6}$$

where three is the number of quark colors. Considering that in the above list the value for ΔP in the leptonic sector consistently equals three times ΔP in the quark sector it is reasonable to conjecture that the number of quark colors may fulfill an equivalent role for coupling asymmetry: that is to say, they may balance the net amount of coupling asymmetry possessed by the leptonic sector against that possessed by the quark sector.

V. DERIVATION OF THE SECONDARY COUPLING CONSTRAINTS

To see how Eqs. (4.1) and (4.2) derive from Eqs. (3.3)–(3.6), consider that according to Eq. (2.3)

$$\Delta P_{\psi_{23},\psi_{13},\psi_{12}} = \left(\sin\psi_{12}\sin\psi_{23} - \cos\psi_{12}\cos\psi_{23}\sin\psi_{13}\right)^2 - \sin^2\psi_{13} \quad . \tag{5.1}$$

Substitution reveals that

$$\Delta P_{\varphi_{23},0,\varphi_{12}} = \sin^2\varphi_{12}\sin^2\varphi_{23} \quad , \tag{5.2}$$

so that Eqs. (3.3) and (5.2) give

$$\Delta P_{\varphi_{23},0,\varphi_{12}} = 3g_{12}\sin^2\varphi_{23} \quad ; \tag{5.3}$$

and substitution reveals that

$$\Delta P_{\frac{\pi}{2},0,\theta_{12}} = \sin^2\theta_{12} \quad , \tag{5.4}$$

$$\tag{5.5}$$

so that Eqs. 3.5) and (5.4) give

$$3 \times \Delta P_{\frac{\pi}{2},0,\theta_{12}} = 3g_{12}\sin^2\varphi_{23} \quad . \tag{5.6}$$

Combining Eqs. (5.3) and (5.6) recovers Eq. (4.1).

Substitution reveals that

$$\begin{aligned}\Delta P_{\theta_{23},\theta_{13},0} &= (-\cos\theta_{23}\sin\theta_{13})^2 - \sin^2\theta_{13} \qquad (5.7)\\ &= \cos^2\theta_{23}\sin^2\theta_{13} - \sin^2\theta_{13}\\ &= (\cos^2\theta_{23} - 1)\sin^2\theta_{13}\\ &= -\sin^2\theta_{23}\sin^2\theta_{13} \quad ,\end{aligned} \tag{5.8}$$

so that (3.4) and (5.7) give

$$3\times\Delta P_{\theta_{23},\theta_{13},0} = -g_{13}\sin^2\theta_{23} \quad ; \tag{5.9}$$

and substitution reveals that

$$\Delta P_{-\frac{\pi}{2},\varphi_{13},0} = -\sin^2\varphi_{13} \quad , \tag{5.10}$$

so that Eqs. (3.6) and (5.10) give

$$\Delta P_{-\frac{\pi}{2},\varphi_{13},0} = -g_{13}\sin^2\theta_{23} \quad . \tag{5.11}$$

Combining Eqs. (5.9) and (5.11) recovers Eq. (4.2). In this way Eqs. (3.3)–(3.6) assure that various g_{12} and g_{13} produce mixing angles that fulfill the secondary coupling constraints described by Eqs. (4.1) and (4.2).

VI. MIXING MODEL PARAMETERS g_{12} AND g_{13} AND THE MIXING MODEL "NEXUS"

Of course the values for g_{12} and g_{13} can simply be adjusted in an attempt to fit the five mixing angles that are not explicitly assigned a value in the model. In and of itself, such a fit, if successful, would be no small achievement, given that just two values must be adjusted to fit five.

However, it is more interesting—and predictive—to derive the values for g_{12} and g_{13} from the following nearly-symmetric equation

$$\begin{aligned}\frac{(M-g_{13})^3}{N^3} + (M-g_{13})^2 &= \frac{\left(M^3-g_{12}^3\right)}{N^3} + \left(M^2-g_{12}^3\right)\\ &= \frac{1}{\alpha} \quad .\end{aligned} \tag{6.1}$$

Here the constants M and N are the smallest positive integers that solve Eq. (6.1), where g_{12} and g_{13} are variables fulfilling

$$\frac{dg_{13}}{dg_{12}} \approx g_{12}^3 \tag{6.2}$$

at

$$g_{12} = \frac{1}{M} \quad . \tag{6.3}$$

The reasons for employing Eqs. (6.1)–(6.3) are purely pragmatic. They define values for g_{12} and g_{13} that:

- Reproduce the six mixing angles within their limits of experimental error.
- Reproduce the fine structure constant α to within a few parts per billion (ppb).

To see how, observe that if some higher-order terms are ignored Eq. (6.1) can be solved for g_{13} as follows:

$$\frac{3M^2g_{13}}{N^3} + 2Mg_{13} \approx \frac{g_{12}^3}{N^3} + g_{12}^3 \tag{6.4}$$

$$3M^2g_{13} + 2N^3Mg_{13} \approx g_{12}^3 + N^3g_{12}^3 \tag{6.5}$$

$$Mg_{13}(3M+2N^3) \approx g_{12}^3(1+N^3) \tag{6.6}$$

$$g_{13} \approx \frac{g_{12}^3}{M} \times \frac{1+N^3}{3M+2N^3} \tag{6.7}$$

$$g_{13} \approx \frac{g_{12}^3}{3M} \times \frac{1+N^3}{M+\frac{2}{3}N^3} \quad . \tag{6.8}$$

Now, if

$$M = \frac{N^3}{3} + 1 \quad , \tag{6.9}$$

then Eq. (6.8) simplifies to

$$g_{13} \approx \frac{g_{12}^3}{3M} \quad . \tag{6.10}$$

Taking the derivative of each side yields

$$\frac{dg_{13}}{dg_{12}} \approx \frac{g_{12}^2}{M} \quad , \tag{6.11}$$

and substituting from Eq. (6.3) gives

$$\frac{dg_{13}}{dg_{12}} \approx \frac{g_{12}^2}{1/g_{12}} \approx g_{12}^3 \quad , \tag{6.12}$$

which recovers Eq. (6.2). The smallest positive integers that solve Eq. (6.9) are

$$M = 10 \tag{6.13}$$

and

$$N = 3 \quad . \tag{6.14}$$

It follows that:

Equations (6.1), (6.3), (6.13), and (6.14) determine that

$$g_{13} = \frac{1}{29999.932116} \quad . \tag{6.15}$$

Equations (6.3), (6.13), and (6.14) determine that the right side of Eq. (6.1) equals

$$\frac{10^3 - 10^{-3}}{3^3} + 10^2 - 10^{-3} = \frac{1}{\alpha} = 137.036 \quad . \tag{6.16}$$

And Eqs. (6.13)–(6.15) determine that the left side of Eq. (6.1) equals

$$\begin{aligned} &\left[\frac{10}{3} - \frac{1}{3 \times 29999.932116} \right]^3 + \left[10 - \frac{1}{29999.932116} \right]^2 = \\ &\left[\frac{1}{3g_{12}} - \frac{g_{13}}{3} \right]^3 + \left[\frac{1}{g_{12}} - g_{13} \right]^2 = \frac{1}{\alpha} = 137.036 \quad . \end{aligned} \tag{6.17}$$

In this way, as claimed earlier, Eq. (6.1) reproduces the precisely-known fine structure constant α to within a few ppb. Specifically, the 2006 CODATA value for the fine structure constant inverse $1/\alpha$ equals 137.035 999 679 [7], which is within just 2.3 ppb of 137.036.

More important for the issue of mixing, however, is that the four expressions

$$3g_{12} = 3/10 \quad , \tag{6.18}$$

$$g_{13}/3 = 1/(3 \times 29999.932116) \quad , \tag{6.19}$$

$$g_{12} = 1/10 \quad , \tag{6.20}$$

$$g_{13} = 1/29999.932116 \tag{6.21}$$

that appear in Eq. (6.17) reappear in the square roots that occupy the right sides of Eqs. (3.3)–(3.6). All this suggests a common mathematical substructure shared by the mixing model angles and the fine structure constant, *however obscure its physical origin.* The above four expressions and their values will be termed the *mixing model nexus*, as they tie the fine structure constant to the mixing model. It only remains to demonstrate that the above values for g_{12} and g_{13} "accurately reproduce," as also claimed earlier, the experimental quark and lepton mixing angles. Accordingly, the model's closeness of fit will be assessed in the next section.

It is important to recognize that the above values for g_{12} and g_{13} have *not* been chosen to fit either the mixing angles or the fine structure constant: instead they arise independently in connection with the study of nearly-symmetric Eq. (6.1), an issue that has already been examined in four distinct ways:

- See [8] for a more general analysis of equations taking the form of Eq. (6.1).
- See [9] for a brute-force computer search for compact approximations of the fine structure constant, one that independently arrives at Eq. (6.16).
- See [10] for equations tying Eq. (6.17) to the muon-, neutron-, and proton-electron mass ratios.
- Also relevant is [11], which argues that the constants 3, 10, and 41/10 relate to the quark and lepton mass ratios.

VII. MIXING MODEL PREDICTIONS WITHOUT PHASE

Equations (3.1)–(3.6) in combination with the assignments of the previous section

$$g_{12} = \frac{1}{10} \quad , \tag{7.1}$$

$$g_{13} = \frac{1}{29999.932116} \tag{7.2}$$

produce the following mixing angles

$$\theta_{23} = 2.367445^\circ \quad , \tag{7.3}$$
$$\theta_{13} = 0.190987^\circ \quad , \tag{7.4}$$
$$\theta_{12} = 12.920966^\circ \quad , \tag{7.5}$$
$$\varphi_{23} = 135^\circ \quad , \tag{7.6}$$
$$\varphi_{13} = 0.013665^\circ \quad , \tag{7.7}$$
$$\varphi_{12} = 33.210911^\circ \quad . \tag{7.8}$$

(Note that, although $g_{13} = 1/29999.932116$ is used above, whereas $g_{13} = 1/30000$ was used in 2007 [5], this change is too small to affect the assessment of the model's fit of experiment presented in Tables I and II.)

The above angles, in turn, produce mixing matrices *without phase* that are close to those determined by experiment. Specifically, the calculated quark mixing matrix equals

$$V_{g_{12}=1/10,\ g_{13}=1/29999.932116} = \begin{array}{c} \begin{array}{ccc} d & s & b \end{array} \\ \begin{bmatrix} 0.974674 & 0.223606 & 0.003333 \\ 0.005991 & 0.041007 & 0.999141 \\ 0.223550 & 0.973817 & 0.041308 \end{bmatrix} \end{array} \begin{array}{c} \\ u \\ t \\ c \end{array} \quad , \tag{7.9}$$

whereas the corresponding leptonic matrix equals

$$U_{g_{12}=1/10,\ g_{13}=1/29999.932116} = \begin{array}{c} \begin{array}{ccc} \nu_1 & \nu_2 & \nu_3 \end{array} \\ \begin{bmatrix} 0.836660 & 0.547723 & 0.000238 \\ 0.387439 & 0.591516 & 0.707107 \\ 0.387157 & 0.591700 & 0.707107 \end{bmatrix} \end{array} \begin{array}{c} \\ \nu_e \\ \nu_\mu \\ \nu_\tau \end{array} \quad . \tag{7.10}$$

(See Eqs. (9.1) and (9.3), respectively, for the calculated quark and lepton mixing matrices *with* phase.)

For quark mixing the model predicts the following sines squared

$$\sin^2\theta_{12} = 1/20 = g_{12} \times \sin^2\varphi_{23} \quad ,$$
$$\sin^2\theta_{23} = 1.706346 \times 10^{-3} \quad ,$$
$$\sin^2\theta_{13} = 1/(3 \times 29999.932116) = g_{13}/3 \quad ,$$

while also predicting these CKM matrix elements

TABLE I: Quark mixing data compared against 2007 predictions.

Year	$\lvert V_{us}\rvert$	$\lvert V_{ub}\rvert$	$\lvert V_{cb}\rvert$	J_Q
2007 Prediction	0.2236	0.003333	0.04131	2.758×10^{-5} [d]
2010 [a]	$0.2253^{+0.0007}_{-0.0007}$	$0.00347^{+0.00016}_{-0.00012}$	$0.0410^{+0.0011}_{-0.0007}$	$2.91^{+0.19}_{-0.11} \times 10^{-5}$
Error in SD	2.4	1.1	0.3	1.4
2008 [b]	$0.2257^{+0.0010}_{-0.0010}$	$0.00359^{+0.00016}_{-0.00016}$	$0.0415^{+0.0010}_{-0.0011}$	$3.05^{+0.19}_{-0.20} \times 10^{-5}$
Error in SD	2.1	1.6	0.2	1.5
2006 [c]	$0.2272^{+0.0010}_{-0.0010}$	$0.00396^{+0.00009}_{-0.00009}$	$0.04221^{+0.0001}_{-0.0008}$	$3.08^{+0.16}_{-0.18} \times 10^{-5}$
Error in SD	3.6	7.0	1.1	1.8

[a] Ref. [3]. Particle Data Group 1σ global fit.
[b] Ref. [14]. Particle Data Group 1σ global fit.
[c] Ref. [13]. Particle Data Group 1σ global fit.
[d] Although this value for the Jarlskog invariant was unambiguously implied by the model angles of 2007, it was not explicitly introduced until this article (see Eq. (8.6) in Section VIII).

$$|V_{us}| = 2.236 \times 10^{-1} \quad , \qquad |V_{ub}| = 3.333 \times 10^{-3} \quad , \qquad |V_{cb}| = 4.131 \times 10^{-2}$$

each of which is unaffected by phase, and each of which derives primarily from a different mixing angle. Given that the model described in this article was first introduced in 2007 [5] it is logical to begin by comparing these predictions against 2006 CKM mixing data [13], which was then current. As it turns out, the model differs from these 2006 data

$$|V_{us}| = 2.272^{+0.01}_{-0.01} \times 10^{-1} \quad , \qquad |V_{ub}| = 3.96^{+0.09}_{-0.09} \times 10^{-3} \quad , \qquad |V_{cb}| = 4.221^{+0.010}_{-0.080} \times 10^{-2}$$

by 3.6, 7.0, and 1.1 standard deviations, respectively, a large discrepancy. However, this same source also offers 2010 data [3]

$$|V_{us}| = 2.253^{+0.007}_{-0.007} \times 10^{-1} \quad , \qquad |V_{ub}| = 3.47^{+0.16}_{-0.12} \times 10^{-3} \quad , \qquad |V_{cb}| = 4.10^{+0.11}_{-0.07} \times 10^{-2}$$

from which the model differs by just 2.4, 1.1, and 0.3 standard deviations, respectively. It is particularly striking that the discrepancy regarding $|V_{us}|$ is reduced from 3.6 to 2.4 standard deviations, and that for $|V_{ub}|$ from 7.0 to 1.1, though for this second case the reduction occurs in part because of a widening of its lower error bar from -0.09 to -0.12. That the absolute error for $|V_{us}|$ is reduced by 53%, and that for $|V_{ub}|$ by 78%, is also good evidence for the correctness of the model (see summary in Table I).

The 2010 leptonic data is much less telling, as there has always been general agreement between the model and experiment. The model's leptonic sines squared equal

$$\begin{aligned} \sin^2\varphi_{12} &= 0.3 = 3g_{12} \quad , \\ \sin^2\varphi_{23} &= 0.5 \quad , \\ \sin^2\varphi_{13} &= 5.687834 \times 10^{-8} \quad . \end{aligned}$$

These can be compared against these 2010 experimental values [12]

$$\begin{aligned} \sin^2\varphi_{12} &= 0.318^{+0.019}_{-0.016} \quad , \\ \sin^2\varphi_{23} &= 0.50^{+0.07}_{-0.06} \quad , \\ \sin^2\varphi_{13} &= 0.013^{+0.013}_{-0.009} \quad , \end{aligned}$$

from which they differ by 1.2, 0, and 1.4 standard deviations, respectively (see summary in Table II).

TABLE II: Lepton mixing data compared against 2007 predictions.

Year	$\sin^2 \varphi_{12}$	$\sin^2 \varphi_{23}$	$\sin^2 \varphi_{13}$	J_L
2007 Prediction	0.3	0.5	5.688×10^{-8}	5.465×10^{-5} [e]
2010 [a]	$0.318^{+0.019}_{-0.016}$	$0.50^{+0.07}_{-0.06}$	$0.013^{+0.013}_{-0.009}$	no data
Error in SD	1.1	0	1.4	
2008 [b]	$0.304^{+0.022}_{-0.016}$	$0.50^{+0.07}_{-0.06}$	$0.010^{+0.016}_{-0.011}$	no data
Error in SD	0.25	0	0.9	
2006 [c]	$0.300^{+0.020}_{-0.030}$	$0.50^{+0.08}_{-0.07}$	≤ 0.025 [d]	no data
Error in SD	0	0		

[a] Ref. [12]. A 1σ global fit. This source includes an update containing 2010 data.
[b] Ref. [12]. A 1σ global fit.
[c] Ref. [15]. A 1σ global fit.
[d] Ref. [15]. A 2σ global fit.
[e] Although this value for the Jarlskog invariant was unambiguously implied by the model angles of 2007, it was not explicitly introduced until this article (see Eq. (8.7) in Section VIII).

VIII. THE PHASE CONSTRAINT AND THE JARLSKOG INVARIANT

The following matrix

$$R = \begin{bmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta} & s_{23}c_{13} \\ s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta} & -c_{12}s_{23} - s_{12}c_{23}s_{13}e^{i\delta} & c_{23}c_{13} \end{bmatrix} \tag{8.1}$$

employs a phase δ in order to model mixing with CP violation [2, 3] (as earlier, $s_{12} \equiv \sin\psi_{12}$, $c_{12} \equiv \cos\psi_{12}$, etc., while now $e^{i\delta} = \cos\delta + i\sin\delta$).

How does phase affect the squares of the magnitudes of this matrix? The short answer is: by its effect on the four complex matrix elements at its lower left. The key point is that when one of these elements is multiplied by its complex conjugate to compute the square of its magnitude, imaginary cross-terms are produced that cancel, thereby erasing some fraction of the expression

$$2s_{12}c_{23}c_{12}s_{23}s_{13} \quad .$$

Accordingly, when the above four complex matrix elements each possess a zero phase the above expression contributes its full value to the square of the elements' magnitudes (as it does in Eq. (2.3)); e.g.,

$$(s_{12}s_{23})^2 + (-c_{12}c_{23}s_{13})^2 - 2s_{12}c_{23}c_{12}s_{23}s_{13}\cos 0^\circ \quad ;$$

whereas when these four elements each possess a non-zero phase δ the above expression contributes only a fraction of its value to the square of the elements' magnitudes; e.g.,

$$(s_{12}s_{23})^2 + (-c_{12}c_{23}s_{13})^2 - 2s_{12}c_{23}c_{12}s_{23}s_{13}\cos\delta \quad ;$$

whereas when these four elements each possess a 90° phase the above expression contributes nothing to the square of the elements' magnitudes; e.g.,

$$(s_{12}s_{23})^2 + (-c_{12}c_{23}s_{13})^2 - 2s_{12}c_{23}c_{12}s_{23}s_{13}\cos 90^\circ \quad .$$

It is particularly logical and instructive to explore the effect that such a 90° phase has on leptonic magnitudes, given that it neatly equalizes the second and third rows of the model's leptonic matrix (a consequence of $\varphi_{23} = 135°$; see Eq. (9.3)). With this is mind, let the earlier

$$\Delta P_{\psi_{23},\psi_{13},\psi_{12}}$$

notation, introduced with Eq. (2.3), be adapted for use with the matrix of Eq. (8.1), so that by definition

$$\Delta P^{\delta}_{\psi_{23},\psi_{13},\psi_{12}} = |\ -2s_{12}c_{23}c_{12}s_{23}s_{13}(1 - \cos\delta)\ | \quad . \tag{8.2}$$

(Here the expression $1 - \cos\delta$ reflects that fraction of $-2s_{12}c_{23}c_{12}s_{23}s_{13}$ that is *lost* because of phase δ, so that $\Delta P^{\delta}_{\psi_{23},\psi_{13},\psi_{12}}$ expresses the resultant *change* to a matrix element's magnitude squared, caused by this phase.)

Now, with the aid of this notation, let

$$\Delta P^{90^\circ}_{\varphi_{23}-\frac{\pi}{2},\varphi_{13},\varphi_{12}} = 3 \times \Delta P^{\delta}_{\frac{\pi}{2}+\theta_{23},\theta_{13},\theta_{12}} \quad , \tag{8.3}$$

the *phase constraint* promised in Section III, which constrains the quark and lepton phases in the same way that Eqs. (3.2), (4.1), and (4.2) constrain the quark and lepton mixing angles. Equation (8.3) expands to

$$\frac{\Delta P^{90^\circ}_{\varphi_{23}-\frac{\pi}{2},\varphi_{13},\varphi_{12}}}{\Delta P^{\delta}_{\frac{\pi}{2}+\theta_{23},\theta_{13},\theta_{12}}} = \frac{|\sin\varphi_{12}\cos(\varphi_{23}-\pi/2)\cos\varphi_{12}\sin(\varphi_{23}-\pi/2)\sin\varphi_{13}(1-\cos 90^\circ)|}{|\sin\theta_{12}\cos(\pi/2+\theta_{23})\cos\theta_{12}\sin(\pi/2+\theta_{23})\sin\theta_{13}(1-\cos\delta)|} = 3 \quad , \tag{8.4}$$

which, for the model angles of Eqs. (7.3)–(7.8), determines

$$\delta = 66.889573^\circ \quad . \tag{8.5}$$

This phase and the quark mixing angles of Eqs. (7.3)–(7.5) produce a Jarlskog invariant [16] of

$$\begin{aligned} J_Q &= |\sin\theta_{12}\cos(\pi/2+\theta_{23})\cos\theta_{12}\sin(\pi/2+\theta_{23})\sin\theta_{13}\cos^2\theta_{13}\sin\delta| \\ &= |-2.757743 \times 10^{-5}| \\ &= 2.757743 \times 10^{-5} \quad , \end{aligned} \tag{8.6}$$

which compares well against its 2010 experimental value [3] of

$$J_Q = 2.91^{+0.19}_{-0.11} \times 10^{-5} \quad . \tag{8.7}$$

This calculated J_Q differs from its experimental counterpart by just 1.4 standard deviations (see Table I). And finally, a 90° leptonic phase and the leptonic mixing angles of Eqs. (7.6)–(7.8) produce a leptonic Jarlskog invariant of

$$\begin{aligned} J_L &= |\sin\varphi_{12}\cos(\varphi_{23}-\pi/2)\cos\varphi_{12}\sin(\varphi_{23}-\pi/2)\sin\varphi_{13}\cos^2\varphi_{13}\sin 90^\circ| \\ &= 5.464533 \times 10^{-5} \quad , \end{aligned} \tag{8.8}$$

which is as yet unmeasured.

IX. MIXING MODEL PREDICTIONS WITH PHASE

Earlier the mixing matrices were calculated, but without the aid of phase. Here this defect will be remedied. Below, the quark phase of 66.889573° from Eq. (8.5) helps generate this quark mixing matrix

$$V_{g_{12}=1/10,\ g_{13}=1/29999.932116,\ \delta=66.889573^\circ} = \begin{array}{cc} \begin{array}{ccc} d & s & b \end{array} & \\ \begin{bmatrix} 0.974674 & 0.223606 & 0.003333 \\ 0.008504 & 0.040560 & 0.999141 \\ 0.223469 & 0.973835 & 0.041308 \end{bmatrix} & \begin{array}{c} u \\ t \\ c \end{array} \end{array} \quad , \tag{9.1}$$

whereas the 2010 best fit CKM matrix magnitudes [3] are as follows

$$V_{CKM} = \begin{array}{cc} \begin{array}{ccc} d & s & b \end{array} & \\ \begin{bmatrix} 0.97428^{+0.00015}_{-0.00015} & 0.2253^{+0.0007}_{-0.0007} & 0.00347^{+0.00016}_{-0.00012} \\ 0.00862^{+0.0007}_{-0.0007} & 0.0403^{+0.0011}_{-0.0007} & 0.999152^{+0.000030}_{-0.000045} \\ 0.2252^{+0.00026}_{-0.00020} & 0.97345^{+0.00015}_{-0.00016} & 0.0410^{+0.0011}_{-0.0007} \end{bmatrix} & \begin{array}{c} u \\ t \\ c \end{array} \end{array} \quad . \tag{9.2}$$

Similarly, a 90° leptonic phase helps generate this leptonic mixing matrix

$$U_{g_{12}=1/10,\ g_{13}=1/29999.932116,\ \delta=90^\circ} = \begin{array}{cc} \begin{array}{ccc} \nu_1 & \nu_2 & \nu_3 \end{array} & \\ \begin{bmatrix} 0.836660 & 0.547723 & 0.000238 \\ 0.387298 & 0.591608 & 0.707107 \\ 0.387298 & 0.591608 & 0.707107 \end{bmatrix} & \begin{array}{c} \nu_e \\ \nu_\mu \\ \nu_\tau \end{array} \end{array} \quad , \tag{9.3}$$

where, notably, the 90° phase equalizes the matrix's second and third rows. (See Eqs. (7.9) and (7.10), respectively, for the quark and lepton mixing matrices *without* phase.)

X. SUMMARY OF THE FOUR CONSTRAINTS ON COUPLING ASYMMETRY

The following four equations

$$\Delta P^{90^\circ}\left(135^\circ - \frac{\pi}{2},\ \varphi_{13},\ \varphi_{12}\right) = 3 \times \Delta P^{\delta}\left(\frac{\pi}{2} + \theta_{23},\ \theta_{13},\ \theta_{12}\right), \tag{10.1}$$

$$\Delta P\left(135^\circ - \frac{\pi}{2},\ \varphi_{13},\ \varphi_{12}\right) = 3 \times \Delta P\left(\frac{\pi}{2} + \theta_{23},\ \theta_{13},\ \theta_{12}\right), \tag{10.2}$$

$$\Delta P\left(135^\circ,\ 0,\ \varphi_{12}\right) = 3 \times \Delta P\left(\frac{\pi}{2},\ 0,\ \theta_{12}\right), \tag{10.3}$$

$$\Delta P\left(-\frac{\pi}{2},\ \varphi_{13},\ 0\right) = 3 \times \Delta P\left(\theta_{23},\ \theta_{13},\ 0\right), \tag{10.4}$$

which assume a leptonic phase of 90° and a φ_{23} of 135°, derive their unusual form from Eqs. (4.3)–(4.5). Together they summarize the four constraints this article's mixing model imposes on quark and lepton coupling asymmetry. These equations map over to earlier equations as follows:

$$\begin{array}{ccc} 10.1 & \longleftrightarrow & 8.3 \\ 10.2 & \longleftrightarrow & 3.2 \\ 10.3 & \longleftrightarrow & 4.1 \\ 10.4 & \longleftrightarrow & 4.2 \end{array}$$

The CKM phase of Eq. (8.5) and the model angles of Eqs. (7.3)–(7.8) provide a good fit to experiment; they also fulfill all four of the above constraints. The model's angles, in turn, derive from g_{12}, g_{13} and Eqs. (3.3)–(3.6), where the values for g_{12} and g_{13} derive from a special solution to Eq. (6.1). Substituting the model angles of Eqs. (7.3)–(7.8) and the CKM phase δ of Eq. (8.5) into Eqs. (10.1)–(10.4) gives

$$\Delta P^{90^\circ}\left(45^\circ,\ 0.013665^\circ,\ 33.210911^\circ\right) = 3 \times \Delta P^{66.889573^\circ}\left(92.367445^\circ,\ 0.190987^\circ,\ 12.920966^\circ\right), \tag{10.5}$$

$$\Delta P\left(45^\circ,\ 0.013665^\circ,\ 33.210911^\circ\right) = 3 \times \Delta P\left(92.367445^\circ,\ 0.190987^\circ,\ 12.920966^\circ\right), \tag{10.6}$$

$$\Delta P\left(135^\circ,\ 0^\circ,\ 33.210911^\circ\right) = 3 \times \Delta P\left(90^\circ,\ 0^\circ,\ 12.920966^\circ\right), \tag{10.7}$$

$$\Delta P\left(-90^\circ,\ 0.013665^\circ,\ 0^\circ\right) = 3 \times \Delta P\left(2.367445^\circ,\ 0.190987^\circ,\ 0^\circ\right). \tag{10.8}$$

XI. SUMMARY OF HOW THE MODEL HAS FARED AGAINST EXPERIMENT SINCE 2007

It is in its divergences from experiment that a model is most interesting:

- Will the model continue at odds with experiment indefinitely?
- Or will experiment accommodate the model?

At the time of its introduction in 2007 the model's values for $|V_{us}|$ and $|V_{ub}|$ had disagreements with experiment of an improbable 3.6σ and 7.0σ, respectively. Because it lacked free parameters the model could not then be "adjusted to fit experiment." In any case, this has proven unnecessary: the 2010 values from the same source now produce disagreements of just 2.4σ and 1.1σ, the absolute error for $|V_{us}|$ having been reduced by 53%, and that for $|V_{ub}|$ by 78%. For $|V_{us}|$ a narrowing of its error bars leaves its predicted value still in disagreement with experiment by an uncomfortable 2.4σ (see Table I). This disagreement suggests a new prediction: The quark and lepton mixing angle that should now undergo the greatest adjustment is the Cabibbo angle. This angle is associated with $|V_{us}|$, which should shrink from its 2010 value of 0.2253, toward its predicted value of 0.2236.

XII. CONCLUSION

Why has the 2007 mixing model proven so prescient?

Apparently, because it possesses no constants adjusted to fit experiment. Instead, its predictions arise from the study of the intrinsic properties of pairs of rotation matrices. One would expect that this total lack of "wiggle room" would make the model easy to refute experimentally. On the contrary, it has almost completely healed its earlier serious conflict with experiment and, with the exception of the Cabibbo angle, finds itself in excellent accord with all mixing data.

The mixing model described here exploits a non-traditional version of the CKM quark mixing matrix, a version in which its second and third rows are interchanged. It also exploits the fact that when a 3×3 rotation matrix whose elements are squared is subtracted from its transpose the matrix produced has non-diagonal elements that possess a common absolute value. For the above non-traditional CKM matrix this value equals one-third the corresponding value for the leptonic matrix. By a building a framework of four such constraints and assuming a leptonic CP violating phase of $90°$ and a leptonic φ_{23} of $45°$, the quark and lepton mixing matrices with their phases can be specified with just two free parameters. Using values for these two parameters that derive from an equation that precisely reproduces the fine structure constant, a specific mixing model is then generated. The resultant model is, therefore, purely mathematical in origin, but, importantly, it is also entirely devoid of free parameters. Given the model's lack of freely adjusted parameters, its excellent fit of the mixing data, and its correct prediction of the magnitude and direction of the most recent changes to experimental $|V_{us}|$ and $|V_{ub}|$, its further study appears justified.

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