Multi-matrices and arithmetical operations with multi-matrices

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Abstract. The multi-space structure has been defined by Fl Smarandache as a union spaces with some additional conditions hold. The mathematician L. Mao wrote a series of works in which he introduces the concepts of multi-group, multi-ring, multivector - space etc. In [1] (Smarandache Multi-Space Theory (I)), at open problems section, he suggests the introduction of a theory of matrices and applications defined on the multi-linear spaces. This paper will give an example of a multi-ring structure, introduces the notion of multi-matrix and defines the multi-matrix addition and multiplication.

Keywords: multi-matrix, multi-group, multi-ring

1. Introduction

Definition 1.1 [1]
For any integer $n, i, n \geq 2$ let A_i be a set with the ensemble of laws L_i and the intersection of k sets $A_{i_1}, A_{i_2}, ..., A_{i_k}$ of them with the laws $I(A_{i_1}, A_{i_2}, ..., A_{i_k})$ Then the union $\tilde{A}$

$$\tilde{A} = \bigcup_{i=1}^{n} A_i$$

is called a multi-space

Remark [1]. In Def. 1.1.are premised the following three situations:
(i) $A_1 = A_2 = ... = A_n$, n laws on a set;
(ii) $L_1 = L_2 = ... = L_n$, n sets with one law;
(iii) integers $s_1, s_2, ..., s_l$ exist, so that $I(s_j) = \emptyset, 1 \leq j \leq l$, there may not exist laws on intersections

Definition 1.2 [1] *A multi-space $\tilde{A}$ is called complete if between any two elements exists a binary operation with result in space $\tilde{A}$.*

Definition 1.3. [1]. *Let* $\tilde{G}$, $\tilde{G} = \bigcup_{i=1}^{n} G_i$ *be a complete multi-space with a set of operations*

$O(\tau) = \{\times_i, 1 \le i \le n\}$. *If* $(G_i, +_i)$ *is a group* $\forall i = 1, 2, ..., n$ *and for*
$\forall i, j \in \{1, 2, ...n\}, i \ne j$ *and* $\forall x, y, z \in \tilde{G}$

$$x \times_i (y \times_j z) = (x \times_i y) \times_j z$$

then $\tilde{G}$ *is called complete multi-group*

Definition 1.4. [1] *Let* $\tilde{R} = \bigcup_{i=1}^{m} R_i$ *be a complete multi-space with two binary operations*

$O(\tilde{R}) = \{(+_i, \times_i), 1 \le i \le m\}$

If for any integer $i, j, i \ne j, 1 \le i, j \le m, (R_i; +_i, \times_i)$ *is a ring and for* $\forall x, y, z \in \tilde{R}$

$x +_i (y +_j z) = (x +_i y) +_j z \qquad x \times_i (y \times_j z) = (x \times_i y) \times_j z$

and

$x \times_i (y +_j z) = x \times_i y +_j x \times_i z$
$(y +_j z) \times_i x = y \times_i x +_j z \times_i x$

On condition that all these operations exist, then $\tilde{R}$ *is called complete multi-ring space.*

If for any integer $1 \le i \le m, (R_i; +_i, \times_i)$ *is a field, then* $\tilde{R}$ *is called multi-field space.*

2. An example of multi-ring space

Let $R_1 = (\mathbb{Z}_m; +_m, \cdot_m)$, $R_2 = (\mathbb{Z}_n; +_n, \cdot_n)$, $R_3 = (\mathbb{Z}_d; +_d, \cdot_d)$ be integer rings modulo m, n, d, where $m, n, d \in \mathbb{N}^*, d = (m, n) \ne 1, m \le n$. Extending operations $+_m, \cdot_m, +_n, \cdot_n, +_d, \cdot_d$ on $\tilde{R} = R_1 \bigcup R_2 \bigcup R_3$ so that:

$\widehat{a}_i +_d \widehat{b}_j = \widehat{a + b}_d$, $\widehat{a}_i \cdot_d \widehat{b}_j = \widehat{ab}_d$, where $i, j \in \{m, n, d\}$

$\widehat{a}_i +_m \widehat{b}_j = \widehat{a + b}_d$, $\widehat{a}_i \cdot_m \widehat{b}_j = \widehat{ab}_d$, where $i, j \in \{m, n\}, i \ne j$

$\widehat{a}_i +_n \widehat{b}_j = \widehat{a + b}_d$, $\widehat{a}_i \cdot_n \widehat{b}_j = \widehat{ab}_d$, where $i, j \in \{m, n\}, i \ne j$

Theorem 2.1 $\tilde{R}$ is a multi-ring space

Proof. Because R_1, R_2, R_3 are rings, it is sufficient to prove:

$(\widehat{x}_i +_j \widehat{y}_k) +_l \widehat{z}_r = \widehat{x}_i +_j (\widehat{y}_k +_l \widehat{z}_r)$ (2.1)

$(\widehat{x}_i \cdot_j \widehat{y}_k) \cdot_l \widehat{z}_r = \widehat{x}_i \cdot_j (\widehat{y}_k \cdot_l \widehat{z}_r)$ (2.2)

$\widehat{x}_i \cdot_j (\widehat{y}_k +_l \widehat{z}_r) = \widehat{x}_i \cdot_j \widehat{y}_k +_l \widehat{x}_i \cdot_j \widehat{z}_r$ (2.3)

$(\widehat{y}_k +_l \widehat{z}_r) \cdot_j \widehat{x}_i = \widehat{y}_k \cdot_j \widehat{x}_i +_l \widehat{z}_r \cdot_j \widehat{x}_i$ (2.4) where $i, j, k, l, r \in \{m, n, p\}$

$(\widehat{x}_i +_i \widehat{y}_i) +_i \widehat{z}_i = x \widehat{+ y +} z_i = \widehat{x}_i +_i (\widehat{y}_i +_i \widehat{z}_i)$ where $i \in \{m, n, p\}$
$(\widehat{x}_i +_j \widehat{y}_k) +_l \widehat{z}_r = x \widehat{+ y +} z_d = \widehat{x}_i +_j (\widehat{y}_k +_l \widehat{z}_r)$ where $i, j, k, l, r \in \{m, n, p\}$ and at least two indexes are different, from where results relation (2.1). Similarly can be shown (2.2),too. For (2.3) , $\widehat{x}_i \cdot_i (\widehat{y}_i +_i \widehat{z}_i) = \widehat{x(y+z)}_i = \widehat{xy}_i +_i \widehat{xz}_i = \widehat{x}_i \cdot_i \widehat{y}_i +_i \widehat{x}_i \cdot_i \widehat{z}_i$ where $i \in \{m, n, p\}$

$\widehat{x}_i \cdot_j (\widehat{y}_k +_l \widehat{z}_r) = \widehat{x(y+z)}_d = \widehat{x}_i \cdot_j \widehat{y}_k +_l \widehat{x}_i \cdot_j \widehat{z}_r$ if $i, j, k, l, r \in \{m, n, p\}$ and at least two indexes are different

Analogue can be shown (2.4) too.

3. Multi-matrices and operations with multi-matrices

Definition 3.1 *Let* $\tilde{R} = \bigcup_{i=1}^{n} R_i, O(\tilde{R}) = \{(+_i, \times_i), 1 \le i \le n\}$ *be a complete multi-ring space with the numbering of the rings fixed* $l, m \in \mathbb{N}^*$ *and* $T \in \mathcal{M}_{l,m}(\{1, 2, ..., n\})$ *a matrix with l rows , m columns and with elements from set*$\{1, 2, ..., n\}$. *Application*

$A \colon \{1, 2, ..., l\} \times \{1, 2, ..., m\} \to \tilde{R}$ *is called multi-matrix of* $(l \times m, T)$ *type with elements from* $\tilde{R}$
We will note down $A(i, j)$ with A_{ij} and $A_{ij} \in R_{T_{ij}}$

Definition 3.2. *Let* $a \in \tilde{R}$. *Is called the type of element* a, *the lowest index of the ring whom a belongs to. Application* $t \colon \tilde{R} \to 1, 2, ..., n\}, t(a) = k$ *, k being type of* a, *is called type map*
We will define addition and multiplication of multi-matrices. We will note down with $\mathcal{M}_{l,m,T}(\tilde{R})$,the set of multi-matrices with l rows, m columns, type of elements T and elements from $\tilde{R}$

Definition 3.3. *Let* $A \in \mathcal{M}_{l,m,T_1}(\tilde{R})$ *and* $B \in \mathcal{M}_{l,m,T_2}(\tilde{R})$*be. Multi-matrix* $C \in \mathcal{M}_{l,m,T_3}(\tilde{R})$ *with elements* $C = (c_{ij})_{i=\overline{1,l}, j=\overline{1,m}}$, *where* $c_{ij} = a_{ij} +_k b_{ij}$ *and* T_3 *is the type of elements* c_{ij}, $T_{3ij} = t(c_{ij})$, *is called the sum of* A *and* B, $k = t(a_{ij})$ *or equivalent* $k = T_{1ij}$

We will exemplify the definition for multi-matrix with elements from ring $\tilde{R}$ defined in ξ2, in particular case m= 4 and n=6 We will note the elements from $\tilde{R}$ with $\widehat{a}_i$ index meaning the ring whom the element belongs to: 1 for R_6, 2 for R_4 respective 3 for R_2. Addition operations will be noted with $+_i$, $1 \le i \le 3$ ($+_1$ addition from R_6, $+_2$ addition from R_4 si $+_3$ addition from R_2

Let $A \in \mathcal{M}_{2,3,T_1}(\tilde{R})$ be, $A = \begin{bmatrix} \widehat{3}_1 & \widehat{2}_2 & \widehat{1}_3 \\ \widehat{2}_2 & \widehat{0}_3 & \widehat{5}_1 \end{bmatrix}$ where $T_1 = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{bmatrix}$ and $B \in \mathcal{M}_{2,3,T_2}(\tilde{R})$

$B = \begin{bmatrix} \widehat{5}_1 & \widehat{3}_2 & \widehat{0}_3 \\ \widehat{1}_3 & \widehat{4}_1 & \widehat{5}_1 \end{bmatrix}$, $T_2 = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix}$ Sum $C = A + B$ is the multi-matrix

$C = \begin{bmatrix} \widehat{3}_1 +_1 \widehat{5}_1 & \widehat{2}_2 +_2 \widehat{3}_2 & \widehat{1}_3 +_3 \widehat{0}_3 \\ \widehat{2}_2 +_2 \widehat{1}_3 & \widehat{0}_3 +_3 \widehat{4}_1 & \widehat{5}_1 +_1 \widehat{5}_1 \end{bmatrix} = \begin{bmatrix} \widehat{2}_1 & \widehat{1}_2 & \widehat{1}_3 \\ \widehat{1}_3 & \widehat{0}_3 & \widehat{4}_1 \end{bmatrix}$ of type $(2\text{x}3, T_3)$ where

$T_3 = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 3 & 1 \end{bmatrix}$

Theorem 3.1. *Let* $\tilde{R} = \bigcup_{i=1}^{n} R_i, O(\tilde{R}) = \{(+_i, \times_i), 1 \leq i \leq n\}$ *be a complete multi-ring space. The set of multi-matrices* $\tilde{R} = \bigcup_{i=1}^{n} R_i, O(\tilde{R}) = \{(+_i, \times_i), 1 \leq i \leq n\}$ *determines a group structure together with the addition operation.*

Proof. We will check the axioms of the group :

- associativity $\forall A, B, C \in \mathcal{M}_{l,m,T}(\tilde{R})$

  $(A + B) + C = A + (B + C)$

From $(a_{ij} +_k b_{ij}) +_k c_{ij} = a_{ij} +_k (b_{ij}) +_k c_{ij}), \forall i \in \{1, ..., l\}, \forall j \in \{1, ..., m\}$, $k = t_{ij}$ and $a_{ij}, b_{ij}, c_{ij} \in R_k$ results the associativity .

- zero element Exists the multi-matrix $O \in \mathcal{M}_{l,m,T}(\tilde{R}), O_{ij} = O_k$ where $k = T_{ij}$ and O_k is the zero element in R_k.

- symmetrical element $\forall A \in \mathcal{M}_{l,m,T}(\tilde{R})$, exists the multi-matrix noted with $-A$ formed with elements $-a_{ij}, i \in \{1, 2, ..., l\}, j \in \{1, 2, ..., m\}$ the opposite of the elements $a_{ij} \in R_k, k = T_{ij}$

We will note down $\mathcal{T}_{l,m}$ the set of all matrices T with l rows , m columns and with elements from set $\{1, 2, ..., n\}$

Theorem 3.2. *Let* $\tilde{R}=\bigcup_{i=1}^{n} R_i, O(\tilde{R})=\{(+_i,\times_i),1\le i\le n\}$*be a complete multi-ring space. The set of multi-matrices* $\tilde{\mathcal{M}}_{l,m}(\tilde{R})=\bigcup_{T\in\mathcal{T}}\mathcal{M}_{l,m,T}(\tilde{R})$ *with* $O(\tilde{M}_{l,m}(\tilde{R}))=\{+_T, T\in\mathcal{T}\}$ *determines a structure of multi-group (* $+_T$ *means the addition of multi-matrices of type* (l,m,T)

Proof. From Theorem 3.1. results that $(\mathcal{M}_{l,m,T}(\tilde{R}),+_T)$ determines a group-structure, $\forall T\in\mathcal{T}$**.** For any $T_1,\ T_2\in\mathcal{T}_{l,m}\ T_1\neq T_2$ and $\forall\ A,B,C\in\tilde{M}_{l,m,T}(\tilde{R})$ we prove that $(A\ +_{T_1}\ B)\ +_{T_2}\ C=A\ +_{T_1}\ (B\ +_{T_2}\ C)$. Let $E_1=(A\ +_{T_1}\ B)\ +_{T_2}\ C$ be and $E_2=A\ +_{T_1}\ (B\ +_{T_2}\ C)$, results that $E_{1ij}=(A_{ij}\ +_{k_1}\ B_{ij})\ +_{k_2}\ C_{ij}=A_{ij}\ +_{k_1}\ (B_{ij}\ +_{k_2}\ C_{ij})=E_{2ij}$ because $\tilde{R}$ is a complete multi-ring, $k_1=t(A_{ij}),\ k_2=t(B_{ij}),\ +_{k_1},\ +_{k2}$ are operations from $O(\tilde{R})$

Definition 3.4. *Let* $A\in\mathcal{M}_{l,m,T_1}(\tilde{R})$ *and* $B\in\mathcal{M}_{l,m,T_2}(\tilde{R})$ *be , A= B if and only if* $l=l_1, m=m_1, T=T_1$ *and* $A_{ij}=B_{ij},\forall i\in\{1,...,l\},\ \forall j\in\{1,...,m\}$

We will define the multiplication of multi-matrices

Definition 3.5. *Let* $A\in\mathcal{M}_{l,m,T_1}(\tilde{R})$, $B\in\mathcal{M}_{m,p,T_2}(\tilde{R})$*be . The multi-matrix* $C\in\mathcal{M}_{l,p,T_3}(\tilde{R})$ *with the elements* $C_{ij}=A_{i1}\times_{k_1}B_{1j}+_{d_1}A_{i2}\times_{k_2}B_{2j}+_{d_2}...+_{d_{m-1}}A_{im}\times_{k_{m-1}B_{mj}}$ *is called the product of multi-matrices A and B and is noted down with AB.* T_3 *is the type of elements of the multi-matrix C* $T_{3_{ij}}=t(C_{ij})$

Remark. The multiplication of multi-matrices is dependent of the type of elements of the two multi-matrices which multiply.

To simplify the expression we will note down

$$C_{ij}=\tilde{\sum}_{k=1}^{m}A_{ik}B_{kj}$$

We will exemplify the definition for multi-matrices with elements from the ring $\tilde{R}$ defined in ξ2, for the particular case m= 4 and n=6 We will note down the elements from $\tilde{R}$ with $\widehat{a}_i$ the index meaning the ring whom the element belongs to: 1 for R_6, 2 for R_4 respective 3 for R_2. The operations of addition will be noted down $+_i$, $1\le i\le 3$ ($+_1$ addition from R_6, $+_2$ addition from R_4 and $+_3$ addition from R_2

Let $A\in\mathcal{M}_{2,3,T_1}(\tilde{R})$ be, $A=\begin{bmatrix}\widehat{3}_1 & \widehat{2}_1 & \widehat{1}_1\\ \widehat{2}_2 & \widehat{0}_3 & \widehat{5}_1\end{bmatrix}$ where $T_1=\begin{bmatrix}1 & 1 & 1\\ 2 & 3 & 1\end{bmatrix}$ and $B\in\mathcal{M}_{3,2,T_2}(\tilde{R})$

$$B=\begin{bmatrix}\widehat{5}_1 & \widehat{3}_2 & \widehat{0}_3\\ \widehat{1}_1 & \widehat{4}_1 & \widehat{5}_1\\ \widehat{4}_1 & \widehat{2}_2 & \widehat{1}_3\end{bmatrix},\ T_2=\begin{bmatrix}1 & 2 & 3\\ 1 & 1 & 1\\ 1 & 2 & 3\end{bmatrix}$$ The product $C=AB$ is the multi-matrix

$$C=\begin{bmatrix}\widehat{3}_1 & \widehat{1}_3 & \widehat{1}_3\\ \widehat{0}_3 & \widehat{0}_3 & \widehat{1}_3\end{bmatrix}$$ of type $(2\text{x}3, T_3)$ where $$T_3=\begin{bmatrix}1 & 3 & 3\\ 3 & 3 & 3\end{bmatrix}$$

Theorem 3.3. *Let* $\tilde{R}=\bigcup_{i=1}^{n} R_i, O(\tilde{R})=\{(+_i,\times_i),1\le i\le n\}$*be a complete multi-ring space.*
The set of multi-matrices $\mathcal{M}_{l,l,T}(\tilde{R})$ *with the operations* $+_T$ *and* $._T$ *determines a ring-structure* $(l\in\{1,2,...n\},T\in\mathcal{M}_{l,l}(\{1,2,...,n\}))$

Proof. It has been demonstrated in Theorem 3.1. that $\mathcal{M}_{l,l,T}(\tilde{R})$ determines a multi-group structure to addition ($+_T$). We will demonstrate the associativity and distribution of the multiplication to addition We will note down "." instead of "$._T$". Let $E=(AB)C$ be and $F=A(BC)$, $A,B,C\in\mathcal{M}_{l,l,T}(\tilde{R})$ any

$E_{ij}=\tilde{\sum}_{k=1}^{l}(\tilde{\sum}_{t=1}^{l}A_{it}B_{tj})C_{kj}=\tilde{\sum}_{t=1}^{l}A_{it}(\tilde{\sum}_{k=1}^{l}B_{tk}C_{kj})=F_{ij}$, $\forall i,j\in\{1,2,...,\}$ because $\tilde{R}$ is a complete multi-ring space, resulting that $(AB)C=A(BC)$Let $A,B,C\in\mathcal{M}_{l,l,T}(\tilde{R})$ be any and $E=A($

$E_{ij}=\tilde{\sum}_{k=1}^{l}A_{ik}(B_{kj}+_T C_{kj})=\tilde{\sum}_{k=1}^{l}A_{ik}B_{kj}+_T\tilde{\sum}_{k=1}^{l}A_{ik}C_{kj}=F_{ij}$, $\forall i,j\in\{1,2,...,n\}$ because $\tilde{R}$ is a complete multi-ring space. It results that $A(B+_T C)=AB+_T AC$ si $\mathcal{M}_{l,l,T}(\tilde{R})$

is a ring

Theorem 3.4. *Let* $\tilde{R}=\bigcup_{i=1}^{n} R_i, O(\tilde{R})=\{(+_i,\times_i),1\le i\le n\}$ *be a complete multi-ring space*
The set of multi-matrices $\tilde{\mathcal{M}}_{l,l}(\tilde{R})=\bigcup_{T\in\mathcal{T}}\mathcal{M}_{l,l,T}(\tilde{R})$ *with* $O(\tilde{M}_{l,l}(\tilde{R}))=\{+_T,T\in\mathcal{T}\}$

determines a complete multi-ring space

Proof. From Theorem 3.3 results that $\mathcal{M}_{l,l,T}(\tilde{R})$ are rings to operations $O(\tilde{M}_{l,l}(\tilde{R})) = \{+_T, T \in \mathcal{T}\}$, and the other proprieties result from the proprieties of operations in multi-ring space $\tilde{R}$.

Definition 3.6. *Let* $\tilde{R} = \bigcup_{i=1}^{n} R_i, O(\tilde{R}) = \{(+_i, \times_i), 1 \leq i \leq n\}$ *be a complete multi-ring space ,*R_i *unitary rings with the proprieties:*

1) $0_1 = 0_2 = ... = 0_n = 0$

2) $1_1 = 1_2 = ... = 1_n = 1$

The multi-matrix $\tilde{O} \in \mathcal{M}_{l,m,T}(\tilde{R})$ *with all the elements 0 is called the zero multi-matrix (or null) of type* $(l \times m, T)$*.*

The multi-matrix $\tilde{I}_l \in \mathcal{M}_{l,l,T}(\tilde{R})$ *with* $\tilde{I}_{lii} = 1$ *and* $\tilde{I}_{lij} = 0$*,* $\forall i, j \in \{1, 2, ..., l\} i \neq j$*, is called the multi-matrix unity of* l *order*

Theorem 3.5. *The following proprieties take place*:

i) $\tilde{O} + A = A + \tilde{O} = A, \forall A \in \tilde{\mathcal{M}}_{l,l}(\tilde{R})$

ii) $\tilde{O}.A = A.\tilde{O} = \tilde{O}, \forall A \in \tilde{\mathcal{M}}_{l,l}(\tilde{R})$

iii) $\tilde{I}.A = A.\tilde{I} = A, \forall A \in \tilde{\mathcal{M}}_{l,l}(\tilde{R})$

Proof. The proprieties i)- iii) result from the proprieties of elements 0 and 1 from rings R_i $(0._k a_{ij} = 0, 0 +_k a_{ij} = a_{ij}, 1._k a_{ij} = a_{ij})$

Definition 3.7. Let $\tilde{F} = \bigcup_{i=1}^{n} F_i, O(\tilde{F}) = \{(+_i, \times_i), 1 \leq i \leq n\}$be a complete multi-field.

A multi-matrix $A \in \mathcal{M}_{l,m,T}(\tilde{F})$ is called invertible if there exists a multi-matrix $A_1 \in \mathcal{M}_{l,m,T_1}(\tilde{F})$ so that $A.A_1 = \tilde{I}, A_1.A = \tilde{I}$

We will note down the inverse multi- matrix of the multi-matrix A with A^{-1}

Theorem 3.5. *If a multi-matrix* $A \in \mathcal{M}_{l,m,T}(\tilde{F})$ *is invertible, then the inverse multi- matrix is unique.*

Proof Let A_1, A_2 be the inverse matrix for A, $A.A_1 = \tilde{I}, A_1.A = \tilde{I}$, $A.A_2 = \tilde{I}, A_2.A = \tilde{I}$

$A_1 = A_1.\tilde{I} = A_1(A.A_2) = (A_1.A)A_2 = \tilde{I}A_2 = A_2$

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