Logistic equation of population growth or exhaustion of main resources: generalization to the case of reactive environment, reduction to *Abel* ODE, asymptotic solution for final Human population prognosis.

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Abstract:

Here are presented a key points of new universal model for evolution of population in reactive environment: 1) generalization of the Logistic equation to the case of reactive environment for models of population dynamics in biology (also, for the model of exhaustion of main resources in geology, or for filling of an ecological niches in ecology, or for modeling of the capacities of a proper markets in economics), 2) new type of asymptotic solution for such equation (*which is tested on human population growth*), 3) reduction of such an equation to Abel ODE in general case.

Due to a very special character of Abel ODE, it's general solution is proved to have a proper gap of components of such a solution.

It means an existence of continuous general solution only at some definite, restricted range of time-parameter, or a possibility of sudden gradient catastrophe in regard to the components of solution (*population growth or exhaustion of main resources*), at the definite moment of time-parameter.

In accordance with [1-3], logistic equation describes how population evolves over time. Such an equation actually determines a linear dependence of self-similar rate of evolution process (*or dynamics of population*) in regard to the proper residual capacity of non-filled part of niche. The last is assumed to be proportional to the "difference of the potentials", defining a proper rate of population dynamics, as below [4]:

$$\left(\frac{d\,N\,/\,d\,t}{N}\right) \;=\; b\cdot(K-N)\;,$$

- here t – is the time-parameter, N – is the population total (or a proper level of niche saturation, or the total of main resources), $N = N\,(t)$; $b = b\,(t)$ – is the Malthusian parameter (the rate of maximum population growth); K – is the carrying capacity (i.e., the maximum sustainable population at the total saturation of a proper niche), $K = K\,(t)$.

Besides [4], a key point in modeling of such a population dynamics processes - is to take into consideration the moment of reactive environment. In this case, the function of resistance of environment should be presented in a form below:

$$R_{\,active} + R_{\,reactive}\;,$$

- where $R_{\,active}$ – is the active, constant resistance of environment, or a proper resistance of environment due to a saturation of niche by extraneous elements (*or in modeling a process of an exhaustion of main resources, which means the exhaustion of the niche in regard to it's own elements*); $R_{\,reactive}$ - is the reactive, non-constant resistance of environment, i.e. the proper resistance of environment as a reaction in regard to increasing of the extraneous elements, incorporated into this population $N\,(t)$ (*or into the non-filled part of a proper niche*), where $R_{\,reactive} = R_{\,reactive}\,(N)$.

Active resistance $R_{\,active}$ above does not depend on the amounts of elements (i.e., it has a stable, constant value). For example, in the case of human population dynamics, such an active resistance $R_{\,active}$ means the world-wide accidents which unfortunately take place every year in the world ~ like aviation-accidents or technical catastrophes,

catastrophes of natural or other character (as a result, we have a decreasing of human population by millions every year).

We should also take into consideration that the level of saturation of a proper niche is known to be determined by the level of demographic pressure [4]:

$$P(t) = N(t) / K(t) ,$$

- besides, we know about the principle "*counteraction is to be like action*":

$$R_{reactive}(N) \sim N(t) ,$$

- the total of environment resistance should be defined as below:

$$R(t) \sim R_0(t) + (N(t) / K(t)) .$$

Summarizing all the assumptions above, we could formulate below a new universal principle which is to define the dynamics of such a processes of evolution:

> *Self-similar rate of evolution process is to be directly proportional to the residual capacity of non-filled part of the proper niche, besides it should simultaneously be in inverse proportion to the function of resistance of environment.*

Taking into consideration the universal principle above, we should represent the logistic equation of evolution in a form below:

$$\left(\frac{d\,N / d\,t}{N} \right) = b \cdot \left(\frac{1 - (N / K)}{R + (N / K)} \right) ,$$

- here R – is the function of active, constant resistance of environment $R_0(t)$, but for simplicity we will denote it as $R(t)$. Besides, let us represent the last equation as below:

$$(K \cdot R + N) \cdot \frac{dN}{dt} = b \cdot (K \cdot N - N^2) \qquad (1.1)$$

- we have obtained the proper *Abel* ordinary differential equation of the 2-nd kind [5], in regard to the function $N(t)$.

Due to a very special character of *Abel* equations, it's general solution is known to have a proper gap of function $N(t)$ at some moment t_o [6]. It means the existence of continuous solution only at some definite, restricted range of parameter t, or possibility of sudden gradient catastrophe [7] at some moment t_o.

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Let us make the proper change of variables: $N(t) + K \cdot R = 1 / y(t)$ in equation (1.1), then we obtain appropriate *Abel* ordinary differential equation of the 1-st kind [5]:

$$y' = b \cdot \left(K^2 \cdot R \cdot (1+R) \cdot y^3 - \left(K \cdot (1+2R) + \frac{(K \cdot R)'}{b} \right) \cdot y^2 + y \right) \qquad (1.2)$$

If we assume: $K(t) = const = K$, $b(t) = const = b$, $R(t) = const = R$, equation (1.2) could be reduced as below:

$$y' = b \cdot \left(K^2 \cdot R \cdot (1+R) \cdot y^3 - K \cdot (1+2R) \cdot y^2 + y \right) .$$

- which has a proper analytical solution [5].

Besides, let us assume $R = 1 / K$; it means that we consider a case when the level of resistance of the environment should be directly proportional to the carrying capacity. Such an assumption should properly simplify the right part of equation (1.2) above:

$$y' = b \cdot \left(\frac{1}{R} \cdot (1+R) \cdot y^3 - \frac{1}{R} \cdot (1+2R) \cdot y^2 + y \right) ,$$

- or, in other form:

$$\left(\frac{R}{1+R}\right)\cdot\int\frac{d\,y}{(y-1)\cdot y\cdot\left(y\;-\;\frac{R}{R+1}\right)}\;=\;b\cdot\int d\,t\;\;,$$

- but if we make the change of variables $ln\ y = z$, equation above could be easily solved:

$$\frac{y\cdot(y-1)^{\,R}}{\left(y\;-\;\frac{R}{1+R}\right)^{(1+R)}}\;=\;e^{\,b\cdot\Delta t}\;\;,$$

- where, if we express the function: $y\,(t) = 1\,/\,(N\,(t) + 1)$, we should obtain

$$\left(\frac{N\,(t)}{R\cdot N\,(t)-1}\right)^{(1+R)}\cdot\left(-\frac{1}{N\,(t)}\right)\;=\;\frac{e^{\,b\cdot\Delta t}}{(1\,+R)^{\,(1+R)}}\;\;,$$

Let us represent equality above in other form ($R\;=1\,/\,K$):

$$\left(\frac{1}{K}\;-\;\frac{1}{N\,(t)}\right)^{\left(1\,+\frac{1}{K}\right)}\cdot\,\bigl(-N\,(t)\bigr)\;=\;\left(1\,+\frac{1}{K}\right)^{\left(1\,+\frac{1}{K}\right)}\cdot\,e^{\,-b\cdot\Delta t}\;\;,$$

- here we have obtained the proper expression for $N\,(t)$.

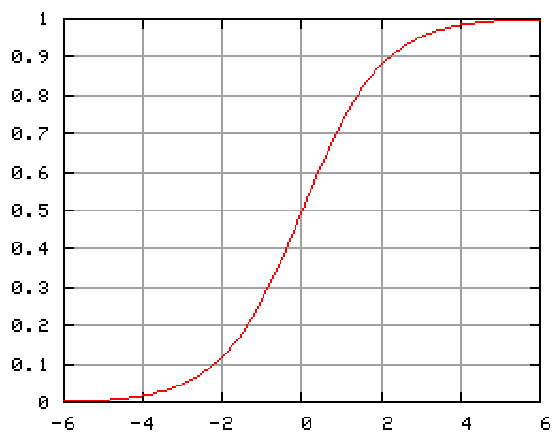
Besides, if we take into consideration that carrying capacity K in the case of modeling of human population is a large enough (K ~ *18 billions of persons* [4]), the last equality could be easily simplified under the appropriate condition $(1 + (1/K)) \rightarrow 1$:

$$N\,(t)\;=\;\left(1\;-\;e^{\,-b\cdot\Delta t}\right)\cdot K\;\;.$$

Thus, we have obtained the simple asymptotic solution for final prognosis of Human population (or for prognosis of population dynamics in biology, etc.).

Finally, let us especially note the analytical representation of solution of equation (1.2), when variables are separated if $R = 1 / K = const$, $b(t) \neq const$, $(1 / K) \rightarrow 1$:

$$N(t) = \left(1 - e^{-\int b(t)dt}\right) \cdot K$$



Pic.1. Logistic curve.

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