Advances in Ternary and Octonionic Gauge Field Theories

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Abstract

A novel ternary gauge field theory is explicitly constructed based on a totally antisymmetric ternary-bracket structure associated with a 3-Lie algebra. Invariant actions including scalar fields are displayed. We proceed with the formulation of a nonassociative Octonionic ternary gauge field theory based on a ternary-bracket involving the octonion algebra and defined earlier by Yamazaki. The octonionic ternary bracket cannot be rewritten in terms of 2-brackets, $[A, B, C] \neq \frac{1}{4}[[A, B], C]$. Two different methods to construct gauge invariant actions are studied. In one of them it is found that gauge-invariant matter kinetic terms for an octonionic-valued scalar field can be introduced in the action if one recurs to an octonionic-valued rank-three antisymmetric field strength $F_{\mu\nu\rho} = \partial_\rho A_{\mu\nu} + [A_{\mu\nu}, A_\rho, \mathbf{g}]+$ permutations, and which is defined in terms of an antisymmetric tensor field of rank two $A_{\mu\nu} = A^a_{\mu\nu} e_a$ and the vector field $A_\mu = A^a_\mu e_a$. Some preliminary steps towards the construction of generalized ternary gauge field theories involving *both* 3-Lie algebras and octonions are discussed in the conclusion.

1 Introduction

Exceptional, Jordan, Division, Clifford, noncommutative and nonassociative algebras are deeply related and are essential tools in many aspects in Physics, see [1], [2], [3], [4], [7], [8], for references, among many others. A thorough discussion of the relevance of ternary and nonassociative structures in Physics has been provided in [5], [9], [10]. The earliest example of nonassociative structures in Physics can be found in Einstein's special theory of relativity. Only colinear velocities are commutative and associative, but in general, the addition of non-colinear velocities is non-associative and non-commutative.

Recently, tremendous activity has been launched by the seminal works of Bagger, Lambert and Gustavsson (BLG) [15], [16] who proposed a Chern-Simons type Lagrangian describing the world-volume theory of multiple $M2$-branes. The original BLG theory requires the algebraic structures of generalized Lie 3-algebras and also of nonassociative algebras. Later developments by [17] provided a $3D$ Chern-Simons matter theory with $\mathcal{N} = 6$ supersymmetry and with gauge groups $U(N) \times U(N)$, $SU(N) \times SU(N)$. The original construction of [17] did not require generalized Lie 3-algebras, but it was later realized that it could be understood as a special class of models based on Hermitian 3-algebras [18], [19].

A Nonassociative Gauge theory based on the Moufang S^7 loop product (not a Lie algebra) has been constructed by [20]. Taking the algebra of octonions with a unit norm as the Moufang S^7-loop, one reproduces a nonassociative octonionic gauge theory which is a generalization of the Maxwell and Yang-Mills gauge theories based on Lie algebras. $BPST$-like instantons solutions in $D = 8$ were also found. These solutions represented the physical degrees of freedom of the transverse 8-dimensions of superstring solitons in $D = 10$ preserving one and two of the 16 spacetime supersymmetries. Nonassociative deformations of Yang-Mills Gauge theories involving the left and right bimodules of the octonionic algebra were presented by [21].

The novel (to our knowledge) nonassociative octonionic ternary gauge theory developed in this work differs from the nonassociative gauge theories of [20] in many respects, mainly that it is based on a ternary bracket involving the octonion algebra that was proposed by Yamazaki [14]. It also differs from the work by [15], [16] in that our octonionic-valued gauge fields $A^a_\mu e_a$; $a = 0, 1, 2, ...7$ are not, and cannot be represented, in terms of matrices $\mathbf{A}_\mu = A^{ab}_\mu f_{ab}{}^{cd} = (\tilde{A}_\mu)^{cd}$, defined in terms of $f_{ab}{}^{cd}$ which are the structure constants of the 3-Lie algebra $[t_a, t_b, t^c] = f_{ab}{}^{cd} t_d$. This construction is not unlike writing the matrices $\mathbf{A}_\mu = A^a_\mu f_a{}^{bc} = (A_\mu)^{bc}$ of ordinary Yang-Mills gauge theory in terms of the adjoint representation of the gauge algebra : $[t_a, t_b] = f_{ab}{}^c t_c$. Furthermore, our field strengths $F_{\mu\nu}$ are explicitly defined in terms of a 3-bracket $[A_\mu, A_\nu, \mathbf{g}]$ involving an auxiliary octonionic-valued scalar field $\mathbf{g} = g^a e_a$ which plays the role of a "coupling" function. Whereas the definition of $F_{\mu\nu}$ by [15], [16] was based on the standard commutator of the matrices $(\tilde{A}_\mu)^a_c (\tilde{A}_\nu)^c_b - (\tilde{A}_\nu)^a_c (\tilde{A}_\mu)^c_b$.

The contents of this work are outlined as follows. In section **2** we develop a ternary gauge field theory formulation associated to a 3-Lie algebra and whose structure constants are totally antisymmetric in all their indices. An invariant action involving the 3-Lie algebra valued gauge field and scalar field is provided. In section **3** the Nonassociative Octonionic ternary gauge field theory is presented and it differs mainly from the prior formulation (besides nonassociativity) due to the fact that the structure constants are not totally antisymmetric in all their indices. Generalized ternary gauge field theories involving *both* 3-Lie algebras and octonions are briefly discussed in the conclusion.

2 3-Lie-Algebra-valued Gauge Field Theories

In this section we will construct a gauge field theory based on a 3-Lie algebra-valued gauge fields. As outlined by [15], one introduces a basis T^a for the 3-Lie algebra and one expands the gauge field $A_\mu = A^a_\mu T_a, a = 1, ..., N$, where N is the dimension of the 3-Lie algebra. The structure constants are introduced as

$$[\ T_a,\ T_b,\ T_c\ ]\ =\ f_{abc}{}^d\ T_d \tag{2.1}$$

such that $f_{abc}{}^d = f_{[abc]}{}^d$. The trace-form provides a metric $h^{ab} = Tr(T^a, T^b)$. that we can use to raise indices: $f^{abcd} = h^{de} f^{abc}{}_e$. On physical grounds one assume that h^{ab} is positive definite.

A bilinear positive symmetric product written as $< X, Y > = < Y, X > = Tr(X, Y) = Tr(Y, X)$ is required and such that that the ternary bracket/derivation obeys what is called the metric compatibility condition

$$< [u,\ v,\ x],\ y\ >\ =\ -\ <\ [u,\ v,\ y],\ x\ >\ =\ -\ <\ x,\ [u,\ v,\ y]\ >\ \Rightarrow$$

$$D_{u,v}\ <\ x,\ y\ >\ =\ <\ [u,\ v,\ x],\ y\ >\ +\ <\ x,\ [u,\ v,\ y]\ >\ =\ 0 \tag{2.2}$$

The symmetric product remains invariant under derivations. Since the ternay bracket is totally antisymmetric one can rewrite $< [u, v, x], y > = < [x, u, v], y >$, and from (2.2) one infers that the 3-Lie algebra admits a totally antisymmetric ternary product which satisfies

$$Tr(\ [A,\ B,\ C\ ],\ D\ )\ =\ -\ Tr(\ A,\ [\ B,\ C,\ D\ ]\ ) \tag{2.3}$$

The condition (2.3) on the trace-form implies that $f^{abcd} = -f^{dabc}$ and this further implies that the structure constants are totally *antisymmetric* in all their indices $f^{abcd} = f^{[abcd]}$, in analogy with the familiar result in Lie algebras.

If gauge symmetries act as a derivation

$$\delta(\ [\ X,\ Y,\ Z]\ )\ =\ [\ \delta X,\ Y,\ Z\ ]\ +\ [\ X,\ \delta Y,\ Z]\ +\ [\ X,\ Y,\ \delta Z\ ] \tag{2.4}$$

this leads to the fundamental identity

$$[\ U, V, [X, Y, Z]\ ]\ =\ [\ [U, V, X],\ Y,\ Z\ ] + [\ X,\ [U, V, Y],\ Z\ ] + [\ X,\ Y,\ [U, V, Z]\ ] \tag{2.5}$$

which plays a role analogous to the Jacobi identity in ordinary Lie algebras. Among the key properties we shall use are the fundamental identity (2.5) and the fact that the structure constants $f_{[abcd]}$ are totally *antisymmetric* in all of their indices.

The simplest nontrivial Lie 3-algebra is $\mathcal{A}_4$. It has 4 generators $T_a,\ a = 1, 2, 3, 4$. The ternary bracket is defined by $[T_a, T_b, T_c] = \epsilon_{abcd} T_d$. The invariant metric of $\mathcal{A}_4$ is δ_{ab}. The 3-Lie algebra $\mathcal{A}_4$ [11] is a natural generalization of the Lie algebra $su(2)$. It was conjectured in [12] and later proved in [13] that the only finite dimensional Lie 3-algebras with a positive-definite metric are the trivial algebra, $\mathcal{A}_4$, and their direct sums. On the other hand, it is possible to define many infinite dimensional Lie 3-algebras with positive-definite metrics. All the Nambu-Poisson algebras are of this kind [12].

We define the field strength in terms of the *ternary* bracket as

$$F_{\mu\nu} \;=\; \partial_\mu A_\nu \;-\; \partial_\nu A_\mu \;+\; [\; A_\mu,\; A_\nu,\; \mathbf{g} \;] \tag{2.6}$$

where $\mathbf{g} = g^a T_a$ is a 3-Lie-algebra valued "coupling" function which is inert under gauge transformations. Under the local gauge transformations the field transforms as

$$\delta(A^d_\mu \; T_d) \;=\; -\; (\partial_\mu \Lambda^d(x)) \; T_d \;+\; [\; \Lambda^a(x) \; T_a,\; A^b_\mu \; T_b,\; g^c \; T_c \;] \tag{2.7}$$

and the 3-Lie-algebra-valued coupling function is gauge invariant

$$\delta(g^d \; T_d) \;=\; [\Lambda^a(x) \; T_a,\; \mathbf{g},\; \mathbf{g}] = 0 \tag{2.8}$$

since the ternary brackets $[X, Y, Y] = 0$. After some straightforward algebra one can verify that the ternary field strength $F_{\mu\nu}$ defined in terms of the ternary-brackets (2.6) transforms properly (homogeneously) under the ternary gauge transformations because $(\partial_\mu\partial_\nu - \partial_\nu\partial_\mu)\Lambda^d = 0$, and

$$\big(\; (\partial_\mu\Lambda^a)A^b_\nu g^c - (\partial_\nu\Lambda^a)A^b_\mu g^c - (\partial_\mu\Lambda^a)A^b_\nu g^c - (\partial_\nu\Lambda^b)A^a_\mu g^c \;\big)\; f_{abc}{}^d \;=\; 0 \tag{2.9}$$

is *identically* zero. The second and fourth terms in (2.9) are symmetric under the exchange of indices $a \leftrightarrow b$ so they will cancel out due to the antisymmetry $f_{abcd} = -f_{bacd}$. This simply can be seen by rewriting the fourth term as

$$-\; (\partial_\nu\Lambda^b)\; A^a_\mu \; g^c \; f_{abc}{}^d \;=\; (\partial_\nu\Lambda^b)\; A^a_\mu \; g^c \; f_{bac}{}^d \;=\; (\partial_\nu\Lambda^a)\; A^b_\mu \; g^c \; f_{abc}{}^d \tag{2.10}$$

and relabeling the $a \leftrightarrow b$ indices in the last line of (2.10). Therefore $F_{\mu\nu}$ transforms *homogeneously* under the infinitesimal ternary gauge transformations as

$$\delta(F^d_{\mu\nu} \; T_d) \;=\; [\; \Lambda^a(x) \; T_a,\; F^b_{\mu\nu} \; T_b,\; g^c \; T_c \;] \;=\; \Lambda^a(x) \; F^b_{\mu\nu} \; g^c f_{abc}{}^d \; T_d \;\Rightarrow$$

$$\delta F^d_{\mu\nu} \;=\; \Lambda^a(x) \; F^b_{\mu\nu} \; g^c \; f_{abc}{}^d \tag{2.11}$$

The result (2.11) is a direct consequence of the fundamental identity; i.e the ternary bracket is a derivation with respect to the first two entries

$$[\; \Lambda,\; \mathbf{g},\; [A_\mu,\; A_\nu,\; \mathbf{g}] \;] \;=$$

$$[\; [\Lambda,\; \mathbf{g},\; A_\mu],\; A_\nu,\; \mathbf{g} \;] \;+\; [\; A_\mu,\; [\Lambda,\; \mathbf{g},\; A_\nu],\; \mathbf{g} \;] \;+\; [\; A_\mu,\; A_\nu,\; [\Lambda,\; \mathbf{g},\; \mathbf{g}] \;] \tag{2.12}$$

Because the ternary bracket is totally antisymmetric under the exchange of any pair of indices, one may exchange the entries in (2.12) as follows

$$[\ \Lambda,\ \mathbf{g},\ [A_\mu,\ A_\nu,\ \mathbf{g}]\ ]\ =\ -\ [\ \Lambda,\ [A_\mu,\ A_\nu,\ \mathbf{g}],\ \mathbf{g}\ ]$$

$$[\Lambda,\ \mathbf{g},\ A_\mu]\ =\ -\ [\Lambda,\ A_\mu,\ \mathbf{g}];\quad [\Lambda,\ \mathbf{g},\ A_\nu]\ =\ -\ [\Lambda,\ A_\nu,\ \mathbf{g}];....\qquad (2.13)$$

leading to the relation

$$[\ [\Lambda,\ A_\mu,\ \mathbf{g}],\ A_\nu,\ \mathbf{g}\ ]\ +\ [\ A_\mu,\ [\Lambda,\ A_\nu,\ \mathbf{g}],\ \mathbf{g}\ ]\ +\ [\ A_\mu,\ A_\nu,\ [\Lambda,\ \mathbf{g},\ \mathbf{g}]\ ]\ =$$

$$[\ \Lambda,\ [A_\mu,\ A_\nu,\ \mathbf{g}],\ \mathbf{g}\ ]\qquad (2.14)$$

which is precisely the relation required in order to show that $F_{\mu\nu}$ transforms *homogeneously* under the infinitesimal ternary gauge transformations.

The finite ternary gauge transformations can be obtained by "exponentiation" as follows

$$F'\ =\ F\ +\ \delta F\ +\ \frac{1}{2!}\delta(\delta F)\ +\frac{1}{3!}(\delta(\delta(\delta F)))\ +\ ....\qquad (2.15a)$$

where

$$\delta F\ =\ [\ \Lambda^a\, T_a,\ F^b_{\mu\nu}\, T_b,\ g^c\, T_c\ ];\ \ \delta(\delta F)\ =\ [\ \Lambda^m\, T_m,\ [\ \Lambda^a\, T_a,\ F^b_{\mu\nu}\, T_b,\ g^c\, T_c\ ],\ g^n\, T_n\ ];\ ....\qquad (2.15b)$$

A gauge invariant action under ternary infinitesimal gauge transformations in D-dim is given

$$S\ =\ -\ \frac{1}{4\kappa^2}\int d^Dx\ <\ F_{\mu\nu}\, F^{\mu\nu}\ >\qquad (2.16)$$

κ is a numerical parameter introduced to make the action dimensionless and it can be set to unity for convenience.

Under infinitesimal ternary gauge transformations of the ordinary quadratic action one has

$$\delta\ S\ =\ -\frac{1}{4}\int d^Dx\ <\ F_{\mu\nu}\ (\delta F^{\mu\nu})\ +\ (\delta F_{\mu\nu})\ F^{\mu\nu}\ >\ =$$

$$-\frac{1}{4}\int d^Dx\ <\ F^c_{\mu\nu}\ T_c\ [\ \Lambda^a T_a,\ F^{\mu\nu\ b}\ T_b,\ g^n\ T_n]\ >\ +$$

$$-\ \frac{1}{4}\int d^Dx\ <\ [\Lambda^a\ T_a,\ F^b_{\mu\nu}\ T_b,\ g^n\ T_n]\ F^{\mu\nu\ c}\ T_c\ >\ =$$

$$-\frac{1}{4}\int d^Dx\, \Lambda^a\, g^n\, F^b_{\mu\nu}\, F^{\mu\nu\ c}\ (\ <\ T_c\ f_{abn}{}^k\ T_k\ >\ +\ <\ f_{abn}{}^k\ T_k\ T_c\ >\ )\ =$$

$$-\ \frac{1}{2}\int d^Dx\ \Lambda^a\ g^n\ F^b_{\mu\nu}\ F^{\mu\nu\ c}\ f_{abnc}\ =\ 0\qquad (2.17)$$

the last terms terms inside the integrand in eq-(2.17) vanish identically due to the full antisymmetry of $f_{abnc} = -f_{acnb}$, and the symmetry of $F^b_{\mu\nu}\ F^{\mu\nu\ c}$ under the exchange of indices $b \leftrightarrow c$. Therefore, the action is invariant under infinitesimal ternary gauge transformations $\delta S = 0$.

The physical interpretation of the 3-Lie algebra-valued coupling $\mathbf{g} = g^a T_a$ deserves further investigation. As described by [15] one can augment the 3-Lie algebra by including an element T^0 that associates with everything, or more precisely, that satisfies $f^{0ab}{}_d = 0$. If we assume that $h^{0b} = 0$ if $b \neq 0$, one finds that $f_{abc0} = 0$. Since T_0 decouples from the ternary brackets $[A, B, g^0 T_0] = 0$, the physical coupling $g^0 = \textit{constant}$ can be incorporated into the field strength in the same fashion as it occurs in ordinary Yang-Mills. One may rewrite the physical coupling g^0 as a prefactor in front of the 3-bracket as $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + g^0 [A_\mu, A_\nu, \mathbf{g}]$, and reabsorb g^0 into the definition of the A_μ field as $F_{\mu\nu} = \frac{1}{g^0}\left(\partial_\mu(g^0 A_\nu) - \partial_\nu(g^0 A_\mu) + [g^0 A_\mu, g^0 A_\nu, \mathbf{g}]\right)$. Thus $F_{\mu\nu} \rightarrow \frac{1}{g^0} F_{\mu\nu}$ and the action is rescaled as $S \rightarrow \frac{1}{(g^0)^2} S$ as it is customary in the Yang-Mills action.

Having formulated a gauge invariant action (2.16) the next step is to introduce gauge invariant matter terms like $(D_\mu \Phi)^2$ where $\Phi = \Phi^a T_a$ is 3-Lie algebra-valued scalar and $D_\mu \Phi = \partial_\mu \Phi + [A_\mu,\ \Phi,\ \mathbf{g}]$. The derivative $D_\mu \Phi$ transforms homogeneously, when

$$\delta(\Phi^d T_d) \ = \ [\ \Lambda^a T_a,\ \Phi^b\ T_b,\ g^c\ T_c\ ] \ \Rightarrow\ \delta(D_\mu \Phi) \ = \ [\ \Lambda,\ D_\mu \Phi,\ \mathbf{g}\ ] \tag{2.18}$$

The action

$$S \ = \ \int d^4 x \ < \ -\frac{1}{2\ (g^0)^2}\ F_{\mu\nu}\ F^{\mu\nu} \ + \ \frac{1}{2}\ (D_\mu \Phi)^2 \ > \tag{2.19}$$

is invariant under the infinitesimal gauge transformations given by eqs-(2.11,2.18). To show invariance under finite gauge transformations via the "exponentiation" procedure in eqs-(2.15) is much more cumbersome. In the next section we shall analyze Nonassociative Octonionic Ternary Gauge Field Theories based on a ternary octonionic product with the fundamental difference, besides the nonassociativity, that the structure constants f_{abcd} are no longer totally antisymmetric in their indices.

3 Nonassociative Octonionic Ternary Gauge Field Theories

The nonassociative and noncommutative octonionic ternary gauge field theory is based on a ternary-bracket structure involving the octonion algebra. The ternary bracket obeys the fundamental identity (generalized Jacobi identity) and was developed earlier by Yamazaki [14]. Given an octonion $\mathbf{X}$ it can be expanded in a basis (e_o, e_m) as

$$\mathbf{X} = x^o\ e_o \ + \ x^m\ e_m, \quad m, n, p = 1, 2, 3, ...7. \tag{3.1}$$

where e_o is the identity element. The Noncommutative and Nonassociative algebra of octonions is determined from the relations

$$e_o^2 = e_o,\ e_o e_i = e_i e_o = e_i,\ e_i e_j = -\delta_{ij} e_o \ + c_{ijk} e_k,\ i,j,k = 1,2,3,....7. \quad (3.2)$$

where the fully antisymmetric structure constants c_{ijk} are taken to be 1 for the combinations $(124),(235),(346),(457),(561),(672),(713)$. The octonion conjugate is defined by $\bar{e}_o = e_o$, $\bar{e}_m = -e_m$

$$\bar{\mathbf{X}} = x^o\ e_o\ -\ x^m\ e_m. \quad (3.3)$$

and the norm is

$$N(\mathbf{X})\ =\ |\ <\ \mathbf{X}\ \mathbf{X}\ >\ |^{\frac{1}{2}}\ = |\ Real\ (\bar{\mathbf{X}}\ \mathbf{X})\ |^{\frac{1}{2}}\ = |\ (x_o\ x_o\ + x_k\ x_k)\ |^{\frac{1}{2}}. \quad (3.4)$$

The inverse

$$\mathbf{X}^{-1}\ =\ \frac{\bar{\mathbf{X}}}{<\ \mathbf{X}\ \mathbf{X}\ >},\quad \mathbf{X}^{-1}\mathbf{X} = \mathbf{X}\mathbf{X}^{-1} = 1. \quad (3.5)$$

The non-vanishing associator is defined by

$$(\mathbf{X},\mathbf{Y},\mathbf{Z}) = (\mathbf{X}\mathbf{Y})\mathbf{Z} - \mathbf{X}(\mathbf{Y}\mathbf{Z}) \quad (3.6)$$

In particular, the associator

$$(e_i,e_j,e_k)\ =\ (e_i e_j)e_k - e_i(e_j e_k)\ =\ 2\ d_{ijkl}\ e_l$$
$$d_{ijkl}\ =\ \frac{1}{3!}\ \epsilon_{ijklmnp}\ c^{mnp},\ i,j,k.... = 1,2,3,.....7 \quad (3.7)$$

Yamazaki [14] defined the three-bracket as

$$[\,u,\ v,\ x\,]\ \equiv\ D_{u,v}\,x\ =\ \frac{1}{2}\ (\ u(vx) - v(ux)\ +\ (xv)u\ -\ (xu)v\ +\ u(xv)\ -\ (ux)v\ ). \quad (3.8)$$

After a straightforward calculation when the indices span the imaginary elements $a,b,c,d = 1,2,3,......,7$, and using the relationship [24]

$$c_{abd}\ c_{dcm}\ =\ -\ d_{abcm}\ +\ \delta_{ac}\ \delta_{bm}\ -\ \delta_{bc}\ \delta_{am} \quad (3.9a)$$

the ternary bracket becomes

$$[\ e_a,\ e_b,\ e_c\ ]\ =\ f_{abcd}\ e_d\ =\ [\ d_{abcd}\ + 2\ \delta_{ac}\ \delta_{bd}\ - 2\ \delta_{bc}\ \delta_{ad}\ ]\ e_d \quad (3.9b)$$

whereas e_0 has a vanishing ternary bracket

$$[\ e_a,\ e_b,\ e_0\ ]\ =\ [\ e_a,\ e_0,\ e_b\ ]\ =\ [\ e_0,\ e_a,\ e_b\ ]\ =\ 0 \quad (3.9c)$$

It is important to note that $f_{abcd} \neq \pm\, c_{abd}\, c_{dcm}$ otherwise one would have been able to *rewrite* the ternary bracket in terms of ordinary 2-brackets as follows $[e_a,\ e_b,\ e_c] \sim \frac{1}{4}[\ [e_a,\ e_b],\ e_c\ ]$.

The ternary bracket (3.8) obeys the fundamental identity

$$[\ [x,\ u,\ v],\ y,\ z\ ]\ +\ [\ x,\ [y,\ u,\ v],\ z\ ]\ +\ [\ x,\ y,\ [z,\ u,\ v]\ ]\ =\ [\ [x,\ y,\ z],\ u,\ v\ ] \tag{3.10}$$

A bilinear positive symmetric product $< u, v > = < v, u >$ is required such that that the ternary bracket/derivation obeys what is called the metric compatibility condition

$$<\ [u,\ v,\ x],\ y\ >\ =\ -\ <\ [u,\ v,\ y],\ x\ >\ =\ -\ <\ x,\ [u,\ v,\ y]\ >\ \Rightarrow$$

$$D_{u,v}\ \ < x, y >\ =\ 0 \tag{3.11}$$

The symmetric product remains invariant under derivations. There is also the additional symmetry condition required by [14]

$$<\ [u,\ v,\ x],\ y\ >\ =\ <\ [x,\ y,\ u],\ v\ > \tag{3.12}$$

The ternary product provided by Yamazaki (3.8) *obeys* the key fundamental identity (3.10) and leads to the structure constants f_{abcd} that are *pairwise* antisymmetric but are *not* totally antisymmetric in all of their indices : $f_{abcd} = -f_{bacd} = -f_{abdc} = f_{cdab}$; however : $f_{abcd} \neq f_{cabd}$; and $f_{abcd} \neq -\ f_{dbca}$. The associator ternary operation for octonions $(x, y, z) = (xy)z - x(yz)$ *does not obey* the fundamental identity (3.10) as emphasized by [14]. For this reason we cannot use the associator to construct the 3-bracket.

We define the field strength in terms of the *ternary* bracket as before

$$F_{\mu\nu}\ =\ \partial_\mu A_\nu\ -\ \partial_\nu A_\mu\ +\ [\ A_\mu,\ A_\nu,\ \mathbf{g}\ ] \tag{3.13}$$

where $\mathbf{g} = g^a e_a$ is an octonionic-valued "coupling" function. Under the infinitesimal ternary gauge transformations $\delta(F^d_{\mu\nu}\ e_d) = [\Lambda^a(x) e_a, F^b_{\mu\nu} e_b, g^c e_c]$, the ordinary quadratic action

$$S\ =\ -\ \frac{1}{4\kappa^2}\int d^D x\ <\ F_{\mu\nu}\ F^{\mu\nu}\ > \tag{3.14}$$

is *not* invariant under ternary infinitesimal gauge transformations as we shall see next. The octonionic valued field strength is $F_{\mu\nu} = F^a_{\mu\nu}\ e_a$, and has *real valued* components $F^0_{\mu\nu}, F^i_{\mu\nu};\ i = 1, 2, 3, ..., 7$. The $<\ \ >$ operation extracting the e_0 part is defined as $< XY > = Real(\bar{X} Y) = < YX > = Real(\bar{Y} X)$. Under infinitesimal ternary gauge transformations of the ordinary quadratic action one has

$$\delta\ S\ =\ -\frac{1}{4}\int d^D x\ <\ F_{\mu\nu}\ (\delta F^{\mu\nu})\ +\ (\delta F_{\mu\nu})\ F^{\mu\nu}\ >\ =$$

$$-\frac{1}{4}\int d^D x\ <\ F^c_{\mu\nu}\ e_c\ [\ \Lambda^a e_a,\ F^{\mu\nu\ b}\ e_b,\ g^n e_n]\ >\ +$$

$$- \frac{1}{4} \int d^D x \; < \; [\Lambda^a e_a, \; F^b_{\mu\nu} \; e_b, \; g^n e_n] \; F^{\mu\nu \; c} \; e_c \; > \; =$$

$$- \frac{1}{4} \int d^D x \; \Lambda^a \; F^c_{\mu\nu} \; F^{\mu\nu \; b} \; (\; < \; e_c \; f_{abnk} \; e_k \; > \; + \; < \; f_{abnk} \; e_k \; e_c \; > \;) \; =$$

$$- \frac{1}{2} \int d^D x \; \Lambda^a \; g^n \; F^c_{\mu\nu} \; F^{\mu\nu \; b} \; f_{abnc} \; =$$

$$- \int d^D x \; \left(\; (\Lambda^a g_a) \; (F^b_{\mu\nu} F^{\mu\nu}_b) \; - \; (\Lambda^a F^{\mu\nu}_a) \; (g_c F^c_{\mu\nu}) \; \right) \; \neq \; 0 \qquad (3.15)$$

Hence, because

$$f_{abnc} \; = \; (\; d_{abnc} \; + \; 2 \; \delta_{an} \; \delta_{bc} \; - \; 2 \; \delta_{bn} \; \delta_{ac} \;) \qquad (3.16)$$

is *not* antisymmetric under the exchange of indices $b \leftrightarrow c$: $f_{abnc} \neq -f_{acnb}$, eq-(3.15) is *not* zero like it was in section **2**. Had f_{abnc} been fully antisymmetric then the variation δS would have been zero due to the fact that $F^c_{\mu\nu} \; F^{\mu\nu \; b}$ is symmetric under $b \leftrightarrow c$. Concluding, $\delta S \neq 0$.

If eq-(3.15) is set to zero it will impose a field-dependent condition on the gauge parameter $\Lambda^a(x)$ in terms of the field strength $F^a_{\mu\nu} e_a$ and the coupling function $g^c e_c$ given symbolically in the internal $7D$ space by $(\vec{\Lambda}.\vec{g})(\vec{F}.\vec{F}) = (\vec{\Lambda}.\vec{F})(\vec{g}.\vec{F})$, after the spacetime indices are contracted. In an internal $3D$ space, the vector equation equivalent to the latter geometric condition would have been $(\vec{\Lambda} \times \vec{F}).(\vec{g} \times \vec{F}) = 0$. In $7D$ there is an analog of the vector product and similar relations but with some crucial *modifications* due to the nonassociativity [25] of octonions. Therefore, the action would only be invariant under those *restricted* transformations where Λ^a becomes a field-dependent parameter leading to highly nonlinear gauge transformations.

To solve this problem we could try another octonion ternary product, like the totally antisymmetric Okubo, de Wit-Nicolai, Gurzey-Tze triple product but unfortunately it does not satisfy the fundamental identity, like it also happens to the associator. Instead we can *modify* the gauge transformations as follows

$$\delta(A^d_\mu \; e_d) \; = \; - \; (\partial_\mu \Lambda^d(x)) \; e_d \; + \; \Lambda^{ab}(x)[\; e_a, \; e_b, \; A^c_\mu \; e_c \;] \qquad (3.17)$$

$$\delta(g^d \; e_d) \; = \; \Lambda^{ab}(x)[\; e_a, \; e_b, \; g^c \; e_c] \qquad (3.18)$$

where one has introduced the additional gauge parameter $\Lambda^{ab}(x) = -\Lambda^{ba}(x)$ and now the coupling $g^c e_c$ is not inert under the transformations (3.18). Only the real part g^0 is inert. After some straightforward algebra one can verify that the ternary field strength $F_{\mu\nu}$ defined in terms of the 3-brackets transforms properly (homogeneously) under the new ternary gauge transformations (3.17, 3.18) if, and only if, the gauge parameters (functions) $\Lambda^d(x); \Lambda^{ab}(x)$ are field-dependent and obey the following relationship in terms of g^c and the gauge fields

$$[\; (\partial_{[\mu} \Lambda^{ab}) \; A^c_{\nu]} \; - \; (\partial_{[\mu} \Lambda^a) \; A^b_{\nu]} \; g^c \;] \; f_{abcd} \; = \; 0; \; d = 1, 2, 3, ..., 7 \qquad (3.19a)$$

The antisymmetrization of indices $[\mu\nu]$ is performed with unit weight. Since the first term in (3.19a) is antisymmetric in the internal a, b indices, one can have also an antisymmetric expression in these latter indices for the second term, by choosing properly $\partial_\mu \Lambda^a = A^a_\mu \Rightarrow \Lambda^a = \int A^a_\mu \, dx^\mu$. Inserting this solution for Λ^a back into (16a) leads to the field-dependent relationship for $\Lambda^{ab}(x)$

$$[\ (\partial_{[\mu} \Lambda^{ab})\ A^c_{\nu]}\ -\ A^a_{[\mu}\ A^b_{\nu]}\ g^c\ ]\ f_{abcd} =\ 0;\ d = 1, 2, 3, ..., 7 \tag{3.19b}$$

One may notice that $D = 3$ dimensions is very special because the number of independent components of Λ^{ab} is 21 which matches the number of equations in (3.19b) given by $7 \times 3 = 21$. The factor of 3 stems from the number of different $[\mu\nu]$ pairs in $D = 3$. Three dimensions corresponds to the world-volume dimension of the membrane. Therefore, due to the field-dependence of the gauge parameters, the gauge transformations themselves are now highly non-linear. This is one of the key differences to ordinary Yang-Mills theories. To sum up, if eq-(3.19a) is obeyed in general, $F_{\mu\nu}$ transforms *homogeneously* under the infinitesimal ternary gauge transformations as

$$\delta(F^m_{\mu\nu}\, e_m)\ =\ \Lambda^{ab}\, [\, e_a,\, e_b,\, F^c_{\mu\nu}\, e_c\,]\ =\ \Lambda^{ab}\, F^c_{\mu\nu}\, f_{abc}{}^m\, e_m\ \Rightarrow\ \delta F^m_{\mu\nu}\ =\ \Lambda^{ab}\, F^c_{\mu\nu}\, f_{abc}{}^m \tag{3.20}$$

The result (3.20) is a direct consequence of the fundamental identity (3.10) because the 3-bracket (3.8) is defined as a derivation

$$[\ [e_a,\ e_b,\ A_\mu],\ A_\nu,\ \mathbf{g}\ ]\ +\ [\ A_\mu,\ [e_a,\ e_b,\ A_\nu],\ \mathbf{g}\ ]\ +\ [\ A_\mu,\ A_\nu,\ [e_a,\ e_b,\ \mathbf{g}]\ ]\ =$$

$$[\ e_a,\ e_b,\ [A_\mu,\ A_\nu,\ \mathbf{g}]\ ] \tag{3.21}$$

The parameter (function) $\Lambda^0(x)$ involved in the transformation $\delta A^0_\mu = \partial_\mu \Lambda^0(x)$, corresponding to the real (identity) element e_0 of the octonion algebra leads to $\delta F^0_{\mu\nu} = 0$, where the field strength component is Abelian-Maxwell-like $F^0_{\mu\nu} = \partial_\mu A^0_\nu - \partial_\nu A^0_\mu$. The finite ternary transformations can be obtained by "exponentiation" as follows

$$F'\ =\ F\ +\ \delta F\ +\ \frac{1}{2!}\delta(\delta F)\ +\frac{1}{3!}(\delta(\delta(\delta F)))\ +\ .... \tag{3.22}$$

where $\delta(F^m_{\mu\nu}\ e_m) = \Lambda^{ab}[e_a, e_b, F^c_{\mu\nu} e_c]$; $\delta(\delta F) = \Lambda^{mn}[\ e_m, e_n, \Lambda^{ab}[e_a, e_b, F^c_{\mu\nu} e_c]\ ]$; ...... Given the octonionic valued field strength $F_{\mu\nu} = F^a_{\mu\nu}\ e_a$, with *real valued* components $F^0_{\mu\nu}, F^i_{\mu\nu};\ i = 1, 2, 3, ..., 7$, a gauge invariant action under ternary infinitesimal gauge transformations in D-dim is

$$S\ =\ -\ \frac{1}{4\kappa^2}\int d^D x\ <\ F_{\mu\nu}\ F^{\mu\nu}\ > \tag{3.23}$$

κ is a numerical parameter introduced to make the action dimensionless and it can be set to unity for convenience. The < > operation extracting the e_0 part is defined as $< XY >= Real(\bar{X}Y) =< YX >= Real(\bar{Y}X)$. Under infinitesimal ternary gauge transformations of the action one has

$$\delta\ S \ = \ -\frac{1}{4} \int d^D x \ < \ F_{\mu\nu}\ (\delta F^{\mu\nu}) \ + \ (\delta F_{\mu\nu})\ F^{\mu\nu} \ > \ =$$

$$-\frac{1}{4} \int d^D x \ < \ F^c_{\mu\nu}\ e_c\ \Lambda^{ab}\ [e_a,\ e_b,\ F^{\mu\nu\ n}\ e_n] \ > \ +$$

$$-\frac{1}{4} \int d^D x \ < \ \Lambda^{ab}\ [e_a,\ e_b,\ F^c_{\mu\nu}\ e_c]\ F^{\mu\nu\ n}\ e_n \ > \ =$$

$$-\frac{1}{4} \int d^D x\, \Lambda^{ab}\ F^c_{\mu\nu}\ F^{\mu\nu\ n}\ (\ <\ e_c\ f_{abnk}\ e_k\ >\ +\ <\ f_{abck}\ e_k\ e_n\ >\ )\ =\ 0. \tag{3.24}$$

since

$$< \ e_c\ f_{abnk}\ e_k \ > + < \ f_{abck}\ e_k\ e_n \ > \ = \ f_{abnk}\ \delta_{ck} + f_{abck}\ \delta_{kn} \ = \ f_{abnc} + f_{abcn} \ =$$

$$[\ d_{abnc} \ +2\ \delta_{an}\ \delta_{bc} \ -2\ \delta_{bn}\ \delta_{ac}\ ] + [\ d_{abcn} \ +2\ \delta_{ac}\ \delta_{bn} \ -2\ \delta_{bc}\ \delta_{an}\ ] \ = \ 0 \tag{3.25}$$

because $d_{abnc}+d_{abcn} = 0$; $d_{nabc}+d_{cabn} = 0$, due to the total antisymmetry of the associator structure constant d_{nabc} under the exchange of any pair of indices. Invariance $\delta S = 0$, only occurs if, and only if, $\delta F = \Lambda^{ab}[e_a, e_b, F^c e_c] \neq \Lambda^{ab}[F^c e_c, e_a, e_b]$. The ordering inside the 3-bracket is crucial. One can check that if one sets $\delta F = \Lambda^{ab}[F^c e_c, e_a, e_b]$, the variation δS leads to a term in the integral which is *not* zero. However, under $\delta F = \Lambda^{ab}[e_a, e_b, F^c e_c]$, the variation δS is indeed zero as shown. This is a consequence of the fact that $[e_a, e_b, e_c] \neq [e_c, e_a, e_b]$ when the 3-bracket is given by eq-(3.8).

To show that the action is invariant under finite ternary gauge transformations requires to follow a few steps. Firstly, one defines

$$< \ x\,y \ > \ \equiv \ Real\,[\,\bar{x}\,y\,] \ = \ \frac{1}{2}\,(\,\bar{x}\,y + \bar{y}\,x\,) \ \Rightarrow \ < \ x\,y \ > \ = \ < \ y\,x \ > \tag{3.26}$$

Despite nonassociativity, the *very special conditions*

$$x(\bar{x}u) \ = \ (x\bar{x})u; \quad x(u\bar{x}) \ = \ (xu)\bar{x}; \quad x(xu) \ = \ (xx)u; \quad x(ux) \ = \ (xu)x \tag{3.27}$$

are obeyed for octonions resulting from the Moufang identities. Despite that $(xy)z \neq x(yz)$ one has that their real parts obey

$$Real\ [\ (x\ y)\ z\ ] \ = \ Real\ [x\ (y\ z)\ ] \tag{3.28}$$

Due to the nonassociativity of the algebra, in general one has that $(UF)U^{-1} \neq U(FU^{-1})$. However, if and only if $U^{-1} = \bar{U} \Rightarrow \bar{U}U = U\bar{U} = 1$, as a result of the the *very special conditions* (3.27, 3.28) one has that $F' = (UF)U^{-1} = U(FU^{-1}) = UFU^{-1} = UF\bar{U}$ is *unambiguously* defined. One can equate the result of the exponentiation procedure in eq-(3.22) to the expression

$$F' = UFU^{-1} = UF\bar{U} = e^{\Sigma^k(\Lambda^a)e_k} \ (F^c \ t_c) \ e^{-\Sigma^k(\Lambda^a)e_k}; \quad k = 1,2,3,....,7. \tag{3.29}$$

where $\Sigma^k(\Lambda^a)e_k$ is a complicated function of Λ^{ab}. It yields the finite gauge transformations which agree with the infinitesimal *ternary* ones when $\Lambda^{ab}(x)$ are *infinitesimals*. For instance, to lowest order in Λ^{ab}, one has that Σ^k satisfies $2\Sigma^k c_{kcd} = \Lambda^{ab} f_{abcd}$ and which follows by comparing the transformations in (3.22) to those in (3.29), to lowest order.

In ordinary associative Yang-Mills involving 2-brackets, it is well known that the finite gauge transformations are

$$(F^n_{\mu\nu})' \ T_n = e^{i\Lambda^m T_m} \ F^n_{\mu\nu} T_n \ e^{-i\Lambda^m T_m}. \tag{3.30}$$

where T_m are the Hermitian Lie-algebra generators obeying the commutation relations $[T_m, T_n] = i f_{mnp} T_p$. It is a challenging work to derive the explicit functional dependence $\Sigma^k(\Lambda^{ab})e_k$ in eq-(3.29) that matches the transformation in eq-(3.22), to all orders in Λ^{ab}, for the ternary-brackets case.

Dropping the spacetime indices for convenience in the expressions for $F^{\mu\nu}, F_{\mu\nu}$, and by repeated use of eqs-(25,26), when $U^{-1} = \bar{U}$, the action density is also invariant under (unambiguously defined) gauge transformations of the form $F' = UFU^{-1} = UF\bar{U}$,

$$< F' \ F' > = Re \, [\bar{F}' \ F'] = Re \, [(U\bar{F}U^{-1}) \, (UFU^{-1})] = Re \, [(U\bar{F}) \, (\, U^{-1} \, (UF \, U^{-1}) \,)] =$$

$$Re \, [(U \ \bar{F}) \ (U^{-1} \ U) \ (FU^{-1})] = Re \, [(U\bar{F}) \ (FU^{-1})] = Re \, [(FU^{-1}) \ (U\bar{F})] =$$

$$Re \, [F \, (\, U^{-1} \, (U \, \bar{F}) \,)] = Re \, [F \, (U^{-1}U) \, \bar{F}] = Re \, [F \, \bar{F}] = Re \, [\bar{F} \, F] = < F \, F > . \tag{3.31}$$

If the action (3.23) is invariant under finite ternary gauge transformations one can impose the condition $S[A^a_\mu; g^a] = S[(A^a_\mu)'; (g^a)' = C^a]$, where $\mathbf{C} = C^a e_a$ is a constant octonionic-valued coupling which can be obtained from gauging the octonionic-valued coupling function $\mathbf{g}(x)$ to a constant $\mathbf{C}$. The physical interpretation of the octonionic-valued coupling $\mathbf{g} = g^a e_a$ deserves further investigation. The real part of the coupling g^0 can be set to a constant, since g^0 is inert under gauge transformations, and it decouples from the definition of the field strength $F_{\mu\nu}$ because e_0 has a vanishing 3-bracket with other elements of the octonion algebra. The coupling $g^0 = constant$ can be incorporated into the field strength in the same fashion as it occurs in ordinary Yang-Mills. One may rewrite the physical coupling g^0 as a prefactor in front of the 3-bracket as $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + g^0[A_\mu, A_\nu, \mathbf{g}]$, and reabsorb g^0 into the definition of the A_μ field as $F_{\mu\nu} = \frac{1}{g^0}\left(\partial_\mu(g^0 A_\nu) - \partial_\nu(g^0 A_\mu) + [g^0 A_\mu, g^0 A_\nu, \mathbf{g}]\right)$. Thus $F_{\mu\nu} \rightarrow \frac{1}{g^0} F_{\mu\nu}$ and the action is rescaled as $S \rightarrow \frac{1}{(g^0)^2} S$ as it is customary in the Yang-Mills action.

Having formulated a gauge invariant action (3.23) the next step is to introduce gauge invariant matter terms like $(D_\mu \Phi)^2$ where $\Phi = \Phi^a e_a$ is an octonionic-valued scalar and $D_\mu \Phi = \partial_\mu \Phi + [A_\mu, \Phi, \mathbf{g}]$. However, there is a caveat. If derivative $D_\mu \Phi$ transforms homogeneously, when $\delta \Phi = \Lambda^{ab}[e_a, e_b, \Phi]$, one arrives to the conditions $(\partial_\mu \Lambda^{ab})\Phi^c - (\partial_\mu \Lambda^a)\Phi^b g^c = 0$ which would impose *additional* constraints on the scalar field Φ. For this reason, gauge invariant matter terms in the action can be introduced if one starts instead with an octonionic-valued rank-three antisymmetric field strength

$$F_{\mu\nu\rho} = \partial_\rho A_{\mu\nu} + \partial_\mu A_{\nu\rho} + \partial_\nu A_{\rho\mu} +$$
$$[A_{\mu\nu}, A_\rho, \mathbf{g}] + [A_{\nu\rho}, A_\mu, \mathbf{g}] + [A_{\rho\mu}, A_\nu, \mathbf{g}] \quad (3.32)$$

defined in terms of the antisymmetric tensor field of rank two $A_{\mu\nu} = A^a_{\mu\nu} e_a$, the field $A_\mu = A^a_\mu e_a$, and the auxiliary coupling function $\mathbf{g} = g^a e_a$. Under the local gauge transformations

$$\delta(A^d_{\mu\nu} e_d) = \partial_{[\mu} \Lambda^d_{\nu]}(x)\, e_d + \Lambda^{ab}(x)\, [e_a, e_b, A^c_{\mu\nu} e_c] \quad (3.33a)$$

$$\delta(A^d_\mu e_d) = -(\partial_\mu \Lambda^d(x))\, e_d + \Lambda^{ab}(x)\, [e_a, e_b, A^c_\mu e_c];\ \delta(g^d e_d) = \Lambda^{ab}(x)\, [e_a, e_b, g^c e_c]; \quad (3.33b)$$

the antisymmetric field strength $F_{\mu\nu\rho}$ will transform homogeneously

$$\delta(F^d_{\mu\nu\rho} e_d) = \Lambda^{ab}\, [e_a, e_b, F^c_{\mu\nu\rho} e_c] \quad (3.33c)$$

if, and only if, the following conditions are met

$$[\, \partial_{[\mu} \Lambda^a_{\nu]}\, A^b_\rho\, g^c - A^a_{\mu\nu}\, (\partial_\rho \Lambda^b)\, g^c + (\partial_\rho \Lambda^{ab})\, A^c_{\mu\nu}\,]\, f_{abcd} = 0;\ d = 1, 2, 3, ..., 7 \quad (3.34)$$

in conjunction with similar equations obtained by a permutation of the space-time indices. A particular solution to the field-dependent conditions on the gauge parameters (3.34) is

$$\partial_\rho \Lambda^b = A^b_\rho \Rightarrow \Lambda^b(x) = \int_0^x A^b_\rho(x')\, dx'^\rho \quad (3.35a)$$

$$\partial_{[\mu} \Lambda^a_{\nu]}(x) = A^a_{\mu\nu}(x); \quad \Lambda^{ab} = constant \quad (3.35b)$$

Therefore, eqs-(3.35) determine the field-dependent behavior of $\Lambda^a(x), \Lambda^a_\nu(x)$ in terms of the gauge fields $A_\rho, A_{\mu\nu}$. One must emphasize that despite that $\Lambda^{ab} =$ *constant* in (3.35b) this does *not* mean that one has rigid *global* transformations for the gauge fields $A_\mu, A_{\mu\nu}$ in eqs-(3.33a, 3.33b), due to the fact that the gauge parameters $\Lambda^a(x), \Lambda^a_\nu(x)$ are explicitly x-dependent !. Therefore, one has truly local gauge transformations for the gauge fields. It is true, however, that the homogeneous transformation for the field strength $\delta(F^d_{\mu\nu\rho} e_d)$ given by eq-(3.33c) does exhibit a rigid global behavior when $\Lambda^{ab} =$ *constant*. There are other solutions to eq-(3.34) besides those in eqs-(3.35) that do not involve setting $\Lambda^{ab} =$ *constant*. In this case there is also a field dependence on the coupling function $g^c(x)$. For simplicity, we shall focus only in the solutions in eqs-(3.35).

Omitting internal indices, now one can introduce gauge invariant scalar matter by defining the covariant derivative in an explicit *nonlinear* manner as

$$D_\mu \Phi = \partial_\mu \Phi + l^2\, [A_{\mu\nu}, A^\nu, \Phi,] \quad (3.36)$$

the above nonlinear covariant derivative is defined both in terms of A_μ *and* $A_{\mu\nu}$. l is a parameter of length dimensions that must be introduced because $A_{\mu\nu}$ has dimensions of $length^{-2}$. One may verify that $D_\mu \Phi$ transforms homogeneously when $\delta\Phi = \Lambda^{ab}[e_a, e_b, \Phi]$ if, and only if, the same field-dependent conditions on the gauge parameters given by eqs-(3.35a,3.35b) are provided. In this case no additional constraints on the fields are introduced. Furthermore, an action

$$S \;=\; \int d^D x \;<\; -\frac{1}{2}\;\frac{1}{(3!\kappa^2)}\; F_{\mu\nu\rho}\; F^{\mu\nu\rho} \;+\; \frac{1}{2}\;(D_\mu\Phi)^2 \;> \tag{3.37}$$

is invariant under the gauge transformations given by eqs-(3.33); κ is a parameter of suitable dimensions introduced in order to render the action dimensionless.

Due to the conditions $\partial_{[\mu}\Lambda^a_{\nu]}(x) = A^a_{\mu\nu}(x)$, for all $\mu, \nu = 1, 2, ..., D$ indices in eq-(3.35b), the three terms in the first line of the right hand side of (3.32) vanish and $F_{\mu\nu\rho}$ reduces to $[A_{\mu\nu}, A_\rho, \mathbf{g}]+$ permutations; i.e. there is no dynamics for the $A_{\mu\nu}$ field in the action (3.37); while the scalar Φ has dynamics. Note that if eqs-(3.19) and eqs-(3.35) are used simultaneously, after setting $A^a_\mu = \partial_\mu \Lambda^a$, it would constrain A^a_μ to zero which is not acceptable. Instead, one may use eqs-(3.19) and eq-(3.34) *without* imposing the condition $\partial_\mu \Lambda^{ab} = 0$, so that $\Lambda^{ab} \neq constant$ and the gauge transformations of the field strength are no longer rigid (global). In this case, eqs-(3.19) and eq-(3.34) determine the field-dependence of $\Lambda^{ab}(x)$ and $\Lambda^a_\nu(x)$. We may then incorporate $F_{\mu\nu}$ simultaneously with $F_{\mu\nu\rho}$ in the action if one does *not* include the scalar matter terms. In this latter case there *is* dynamics for the $A^a_{\mu\nu}$ field because $\partial_{[\mu}\Lambda^a_{\nu]}$ is no longer constrained to be equal to $A^a_{\mu\nu}$; but A^a_μ has no dynamics because $\partial_\mu \Lambda^a = A^a_\mu \Rightarrow \partial_{[\mu} A^a_{\nu]} = 0$.

To summarize, one may build invariant actions if the gauge parameters are field dependent; i.e. actions which are invariant under restricted (non-linear) gauge transformations. Out of all the possibilities studied here, it is preferable to choose the geometric constraint described in the paragraph after eq-(3.16) : $(\vec{\Lambda}.\vec{g})(\vec{F}.\vec{F}) = (\vec{\Lambda}.\vec{F})(\vec{g}.\vec{F})$, in order to determine the field-dependent condition on $\Lambda^a(x)$, because : (i) there is only *one* single constraint to satisfy encompassing all of the components of Λ^a, g^c and $F^a_{\mu\nu}$; (ii) it does not involve differential equations.

The motivation in constructing an octonionic-valued field strength in terms of ternary brackets is because the ordinary 2-bracket does *not* obey the Jacobi identity

$$[\; e_i,\; [\; e_j,\; e_k\;]\;] \;+\; [\; e_j,\; [\; e_k,\; e_i\;]\;] \;+\; [\; e_k,\; [\; e_i,\; e_j\;]\;] \;=\; 3\; d_{ijkl}\; e_l \;\neq\; 0 \tag{3.38}$$

If one has the ordinary Yang-Mills expression for the field strength

$$F_{\mu\nu} \;=\; \partial_\mu A_\nu \;-\; \partial_\nu A_\mu \;+\; [\; A_\mu,\; A_\nu\;] \tag{3.39}$$

because the 2-bracket does *not* obey the Jacobi identity, one has an extra (spurious) term in the expression for

$$[\ D_\mu,\ D_\nu\ ]\ \Phi\ =\ [\ F_{\mu\nu},\ \Phi\ ]\ +\ (\ A_\mu,\ A_\nu,\ \Phi\ ) \tag{3.40}$$

given by the crucial contribution of the non-vanishing associator $(A_\mu, A_\nu, \Phi) = (A_\mu A_\nu)\Phi - A_\mu(A_\nu \Phi) \neq 0$. For this reason, due to the non-vanishing condition (3.38), the ordinary Yang-Mills field strength does *not* transform homogeneously under ordinary gauge transformations involving the parameters $\Lambda = \Lambda^a e_a$

$$\delta A_\mu = \ \partial_\mu \Lambda\ +\ [A_\mu, \Lambda] \tag{3.41}$$

and it yields an extra contribution of the form

$$\delta F_{\mu\nu}\ =\ [F_{\mu\nu}, \Lambda]\ +\ (\ \Lambda,\ A_\mu,\ A_\nu) \tag{3.42}$$

As a result of the additional contribution (Λ, A_μ, A_ν) in eq-(3.42), the ordinary Yang-Mills action $S = \int < F_{\mu\nu} F^{\mu\nu} >$ will *no* longer be gauge invariant. Under infinitesimal variations eqs-(3.41), the variation of the action is *no* longer zero but receives spurious contributions of the form $\delta S = -4 F^l_{\mu\nu} \Lambda^i A^{\mu j} A^{\nu k} d_{ijkl} \neq 0$ due to the non-associativity of the octonion algebra.

To finalize we discuss further constructions, like having an octonionic-valued and $SU(N)$-valued gauge field $\mathbf{A}_\mu = A^{am}_\mu (e_a \otimes T_m)$ involving the $SU(N)$ algebra generators $T_m, m = 1, 2, 3, ..., N^2 - 1$ and the octonion algebra generators $e_a, a = 0, 1, 2, 3, ..., 7$; i.e. one has octonionic-valued components for the $SU(N)$ gauge fields. The commutator is

$$[\ \mathbf{A}_\mu,\ \mathbf{A}_\nu\ ]\ =\ [\ A^{am}_\mu\ (e_a \otimes T_m),\ A^{bn}_\nu\ (e_b \otimes T_n)\ ]\ =$$

$$\frac{1}{2}\ A^{am}_\mu\ A^{bn}_\nu\ \{e_a,\ e_b\} \otimes [T_m,\ T_n]\ +\frac{1}{2}\ A^{am}_\mu\ A^{bn}_\nu\ [e_a,\ e_b] \otimes \{T_m,\ T_n\} \tag{3.43}$$

where

$$\{e_a,\ e_b\}\ =\ -\,2\ \delta_{ab}\ e_o, \quad [e_a,\ e_b]\ =\ 2\ c_{abc}\ e_c \tag{3.44}$$

and for the $SU(N)$ Hermitian generators one has

$$\{T_m,\ T_n\}\ =\ \frac{1}{N}\ \delta_{mn}\ +\ d_{mnp}\ T_p, \quad [T_m,\ T_n]\ =\ i\ f_{mnp}\ T_p \tag{3.45}$$

One may note that the r.h.s of (3.43) involves both commutators and anti-commutators. Due to the fact that the octonion algebra does not obey the Jacobi identities this will spoil the gauge invariance of typical Yang-Mills actions as described before. Let us have instead a ternary Lie algebra (3-Lie algebra) obeying the ternary commutation relations

$$[\ T_m,\ T_n,\ T_p\ ]\ =\ f_{mnpq}\ T_q \tag{3.46}$$

and such that the ternary-bracket structure-constants f_{mnpq} obey the fundamental identity. A 3-Lie-algebra and octonionic-valued field is defined by $\mathbf{A}_\mu \equiv A^{ma}_\mu\ (T_m \otimes e_a)$. However, the triple commutator

$$[\ \mathbf{A}_\mu,\ \mathbf{A}_\nu,\ \mathbf{A}_\rho\ ]\ =\ [\ A_\mu^{mi}\ (T_m \otimes e_i),\ A_\nu^{nj}\ (T_n \otimes e_j),\ A_\rho^{pk}\ (T_p \otimes e_k)\ ] \quad (3.47)$$

would furnish a very *complicated* expression for the r.h.s of eq-(3.47). To simplify matters one could define the ternary bracket as

$$[\ \mathbf{A}_\mu,\ \mathbf{A}_\nu,\ \mathbf{A}_\rho\ ]\ \equiv\ A_\mu^{mi}\ A_\nu^{nj}\ A_\rho^{pk}\ [T_m,\ T_n,\ T_p] \otimes [e_i, e_j, e_k]\ =$$

$$A_\mu^{mi}\ A_\nu^{nj}\ A_\rho^{pk}\ f_{mnpq}\ f_{ijkl}\ (T_q \otimes e_l) \quad (3.48)$$

so that one has closure in the r.h.s of eq-(3.48). It is warranted to explore further these generalized ternary gauge field theories involving 3-Lie algebras and octonions in M-theory.

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