Universal topological Quantum Manifolds

Johan Noldus, Relativity Group, Department of Mathematical Analysis,
University of Gent, Belgium.

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Abstract

In quantum gravity, one looks for alternative structures to spacetime physics than ordinary real manifolds. In this first part, we propose an alternative universal construction containing the latter as an equilibrium state under the action of the universal diffeomorphism group. Our theory contains many other previous proposals in the literature as special cases. However, the crucial point we make is that those have to be appreciated in the universal context developed here.

1 Introduction

In modern physics, people question the very fabric of spacetime from many different vantage points of view. As explained in great detail in the upcoming book of the author [1], the superposition principle cannot be applied to spacetime which implies that spacetime cannot be quantized in the operational sense. This indicates that an observer lives in one "spacetime" and since no a priori discreteness can be imposed, the author [2] reached the conclusion that any approach to spacetime had to be based on the continuum given that the notion of locality is only canonically defined in that context. This appears to imply that even in the ontological sense, a standard real manifold is the only *natural* candidate for a spacetime structure. So, the only question is whether there exists a *universal* construction based on the continuum allowing for more generic possibilities? The answer to this question surprisingly is yes and the difference lies in the statistical density matrix approach to quantum mechanics and the normal textbook state approach. We can, and will look at spacetime in the quantum statistical sense with the standard notions of locality inherited from $\mathbb{R}^4$. The approach we will take is the algebraic one by means of $W^\star$ algebras; this is just a temporary step and we are aware that more exotic avenues will have to be taken as explained in [1]. The problematic aspect of all non-commutative approaches so far is that the diffeomorphism group has no natural place in the formalism and indeed, imposing algebraic relations by hand breaks diffeomorphism invariance of the single algebra. The answer to this problem is to consider all possible algebras and modeling one manifold on a particular one. Hence, a diffeomorphism will map one manifold into another and the only fixed manifolds are the abelian and free ones. Moreover, the abelian continuum spacetimes have the largest symmetry group and therefore they are preferred from the point of view of internal symmetries. Therefore, any quantum spacetime dynamics should be based upon the fact that a maximal internal symmetry group, as a subgroup of the free diffeomorphisms, determines the only stable ground state. Hence, we conjecture that the theory developed in [1] describes the ground state of a much larger one which allows for small scale granularity as quantum fluctuations at sufficiently small scales.

This paper is organized as follows: first we introduce topological manifolds and give some examples. However, a deeper understanding of topological manifolds emerges from the development of (first order) differential calculus; this is accomplished in section 3 and some nontrivial insights are provided. The construction of higher differential operators, the curvature of the quantum connection and the general definition of differentiable manifolds is postponed to future work. Although this approach to quantum spacetime has been developed independently, the most valuable personal contact in this regard has been with Shahn Majid, some of whose writings and ideas regarding C^* algebraic representations of Hopf $\star$ spacetime algebras have been useful. In particular, I recommend [3] and [4]. Also, in retrospect, some of the ideas in this paper resemble those of Grothendieck topology in the sense that the open algebras form a sheaf over an ordinary topological space and the immersions of the open algebras associated to the open subsets in the covering of charts are prime examples of what category theorists call a sieve. However, there is also more information to it which is given in the definition of the local algebras attached to the generators of the coordinate structure [5]. Therefore, our construction is more restricted than the one of Grothendieck topology (since we can also do analysis) and it might be helpful to see if there exists a more category theoretical definition for which our manifolds constitute particular representations. There is a very slight resemblance to the standard Haag Kleinert axioms of Quantum Field Theory which also works with local algebras over $\mathbb{R}^4$ but the correspondence does not carry very far in the sense that no Minkowski causality or anything like that is implemented.

2 Topological Quantum Manifolds

Basically, the universal complex (or real) algebra in n variables $\widehat{x}_i$ is the free one $\mathcal{F}_n^{\infty}$; we shall also be concerned with the free algebra of finite words $\mathcal{F}_n$ which is equipped with a canonical involution $\star$ which simply reverses the order of the words and conjugates the complex numbers. Hence, every generator is Hermitian and therefore has a real spectrum if one restricts to $W^\star$ algebraic representations. Besides $\mathcal{F}_n^r$, there is the totally commutative algebra $\mathcal{C}_n^r$ in n variables x_i and we denote by $\phi : \mathcal{F}_n^r \to \mathcal{C}_n^r : \widehat{x}_i \to x_i$ the canonical homomorphisms where $r \in \{\emptyset, \infty\}$. Moreover, we adjoin all algebras with an identity element and restrict to unital $\star$ homomorphisms. The idea is to represent $\mathcal{F}_n$ in unital $W^\star$ algebras $\mathcal{A}$ equipped with a trace functional $\omega_{\mathcal{A}}$. Therefore let $\pi : \mathcal{F}_n \subset \text{Dom}(\pi) \subset \mathcal{F}_n^\infty \to \mathcal{A}$ be a unital, maximal, star homomorphism (where $\text{Dom}(\pi)$ is a subalgebra) with a dense image and denote by $\sigma(i, \pi, \mathcal{A})$ the spectrum of $\pi(\widehat{x}_i)$ in $\mathcal{A}$; then it is natural to construct the compact and bounded "cube"

$$\mathcal{O}(\pi, \mathcal{A}) = \times_{i=1}^n \sigma(i, \pi, \mathcal{A}) \; .$$

Likewise, one can restrict the variables in $\mathcal{C}_n$ to $\mathcal{O}(\pi, \mathcal{A})$. Because of the spectral decomposition theorem, for every n vector $\vec{\alpha}$ in the cube, index i and $\epsilon_i > 0$, one has a unique Hermitian spectral operator $P^{\epsilon_i}_{\alpha_i}$ which is by definition a shorthand for

$$P^i((\alpha_i - \epsilon_i, \alpha_i + \epsilon_i)) \; .$$

The operators have the usual intersection properties. Hence for every resolution $\vec{\epsilon}$, we may define an event $P^{\vec{\epsilon}}(\vec{\alpha})$ in the algebra $\mathcal{A}$ as the maximal Hermitian projection operator which is smaller than all $P^{\epsilon_i}_{\alpha_i}$ (notice that this projection operator may become zero if the resolution becomes to high, that is ϵ_i too small) and it is formally denoted by

$$P^{\vec{\epsilon}}(\vec{\alpha}) = \wedge_{i=1}^n P^i((\alpha_i - \epsilon_i, \alpha_i + \epsilon_i)).$$

Now, it is easy to see that if one were to cover a cube by smaller cubes (arbitrary overlaps are allowed), take the projection operators associated to those and consider the smallest projection operator which majorizes all of these, then, by the superposition principle, the latter is smaller or equal to the projection operator of the full cube. This is a very quantum mechanical idea where we acknowledge that the whole is more than the sum of its parts and therefore we have to give up the idea of a classical partition. Hence, for any relative open subset $\mathcal{W}$; there exists a unique smallest projection operator which majorizes all projection operators attached to subcoverings of $\mathcal{W}$ by relative open cubes (a subcovering simply is a set of relative open cubes contained in $\mathcal{W}$). Hence, there is a natural almost everywhere weakly continuous[1] mapping $\kappa_{(\pi,\mathcal{A})}$ from relative open subsets $\mathcal{W}$ of $\mathcal{O}(\pi, \mathcal{A})$ to $\mathcal{A}$ given by

$$\kappa_{(\pi,\mathcal{A})}(\mathcal{W}) = P(\mathcal{W}) \; .$$

For disjoint $\mathcal{W}_j$ one obtains that

$$P(\mathcal{W}_1)\, P(\mathcal{W}_2) = 0 \; ,$$

meaning that the coherence of the theory depends upon the scale you are observing at. Concretely, if you zoom into the region $\mathcal{W}_1$ you will be oblivious to the entanglement with the region $\mathcal{W}_2$; however, looking at both together gives a very different picture. If the *dynamics* itself were scale dependent in this way, then it might explain why we see a local world on our scales of observation and above, while the macroscopic world would seem to be completely entangled.

[1]We shall explain this notion later on.

This picture would offer a complete relativization of physics where giants would look to us as if we were electrons. Also,

$$P(\mathcal{W}_1) \prec P(\mathcal{W}_2)$$

of $\mathcal{W}_1 \subset \mathcal{W}_2$, which means that zooming in is a consistent procedure. Now, we can go on and construct several forms of equivalence, going from ultra strong to ultra weak. Two representations $\pi_i : \mathcal{F}_n \subset \mathrm{Dom}(\pi_i) \subset \mathcal{F}_n^{\infty} \to \mathcal{A}_i$ are ultra strongly isomorphic if and only if there exists a $C^\star$ isomorphism $\gamma : \mathcal{A}_1 \to \mathcal{A}_2$ such that $\pi_2 = \gamma \circ \pi_1$ and $\mathrm{Dom}(\pi_1) = \mathrm{Dom}(\pi_2)$. They are called strongly isomorphic it is only demanded that γ is a unital star isomorphism from $\pi_1(\mathrm{Dom}(\pi_1))$ to $\pi_2(\mathrm{Dom}(\pi_2))$. We say, moreover, that they are weakly isomorphic when equality is supposed to only hold on $\mathrm{Dom}(\pi_1) \cap \mathrm{Dom}(\pi_2)$ and finally we define them to be ultra weakly equivalent if and only if γ is a star isomorphism from $\pi_1(\mathcal{F}_n)$ to $\pi_2(\mathcal{F}_n)$ and equality only holds on $\mathcal{F}_n$. In the case of real manifolds, ultra weak covariance is the only notion which applies and we continue now to investigate it. Now, we are ready to go over to an atlas construction; a topological space $\mathcal{M}$ is said to be a real, n-dimensional, non-commutative manifold if there exists a covering of $\mathcal{M}$ by open sets $\mathcal{V}_\beta$, a homeomorphism ϕ_β from $\mathcal{V}_\beta$ to a relative open subset of the cube $\mathcal{O}(\pi_\beta, \mathcal{A}_\beta)$ associated to some representation $\pi_\beta : \mathrm{Dom}(\pi_\beta) \to \mathcal{A}_\beta$ of the free algebra in n letters. This homeomorphism canonically lifts to the algebra on the open subsets $\mathcal{W} \subset \mathcal{V}_\beta$ by stating that $\widehat{\phi}_\beta(\mathcal{W}) = \kappa_{(\pi_\beta, \mathcal{A}_\beta)}(\phi_\beta(\mathcal{W}))$. Hence, a single chart is a tuple $(\mathcal{V}_\beta, \pi_\beta, \mathcal{A}_\beta, \phi_\beta)$ and we proceed now to construct an atlas by demanding compatibility.

Definition 1 *Two charts $\mathcal{V}_{\beta_j}$ with some non zero overlap $\mathcal{V}_{\beta_1} \cap \mathcal{V}_{\beta_2} \neq \emptyset$ are said to be compatible if and only if the canonical mapping between the normed subsets*

$$\{P_{\beta_j}(\widehat{\phi}_{\beta_j}(\mathcal{W})) \,|\, \mathcal{W} \subset \mathcal{V}_{\beta_1} \cap \mathcal{V}_{\beta_2}\}$$

induces a star isomorphism between the normed algebras generated by them; the latter preserves the trace functionals $\omega_{\mathcal{A}_\beta}$.

We now proceed by giving some examples.

We start by the most trivial thing and show that ordinary real manifolds have a natural place in this setup. Let $\mathcal{M}$ be an n-dimensional real manifold and consider the coordinate chart $(\mathcal{V}, \psi)$. Define now the Hilbert space $\mathrm{L}^2(\overline{\psi(\mathcal{V})}, \mathrm{d}^n x)$ and the multiplication operators x_i. Define $\mathcal{A}$ to be the W^* subalgebra generated by the x_i of the full $W^\star$ algebra of bounded operators, then $\pi : \mathcal{F}_n \to \mathcal{A} : \widehat{x}_i \to x_i$ has a unique maximal extension. The spectrum of each of these multiplication operators is continuous and varies between $a_i < b_i$ and the canonical mapping ϕ is given by $\phi(v) = \psi(v)$. Then, the canonical projectors associated to $\mathcal{W} \subset \mathcal{V}$ are given by $P(\mathcal{W}) = \chi_{\phi(\mathcal{W})}$ where the latter is the characteristic function on $\mathcal{W}$. Clearly, a coordinate transformation induces a $W^\star$ algebraic isomorphism between these commutative projection operators. By the same arguments, one sees that any commutative n-dimensional measure space is represented in this framework; so we are left with presenting a non abelian example. A very simple example is a double sheeted manifold constructed from the Hilbert space $\mathrm{L}^2(\mathbb{R}^4, \mathrm{d}^4x) \otimes \mathbb{C}^2$ and consider the algebra generated by the operators $x^\mu \otimes \sigma^\mu(x)$ where the $\sigma^\mu(x)$ are automorphic to the standard spacetime Pauli algebra $(\sigma^\mu) = (1, \sigma^i)$. That is $\sigma^\mu(x) = U(x)\,\sigma^\mu U^\dagger(x)$ for $U(x)$ some 2×2 complex unitary matrix. The whole manifold structure depends upon $U(x)$, since suppose $U(x) = 1$, then the cube is $\mathbb{R}^4$ and the set of basic projection operators is given by

$$
\begin{aligned}
P^\epsilon_t &= \chi_{[t-\epsilon,t+\epsilon]} \otimes 1 \\
P^\epsilon_x &= \frac{1}{2}\left[\chi_{[x-\epsilon,x+\epsilon]} \otimes |1,1\rangle\langle 1,1| + \chi_{[-x-\epsilon,-x+\epsilon]} \otimes |1,-1\rangle\langle 1,-1|\right] \\
P^\epsilon_y &= \frac{1}{2}\left[\chi_{[y-\epsilon,y+\epsilon]} \otimes |i,1\rangle\langle i,1| + \chi_{[-y-\epsilon,-y+\epsilon]} \otimes |-i,1\rangle\langle -i,1|\right] \\
P^\epsilon_z &= \frac{1}{2}\left[\chi_{[z-\epsilon,z+\epsilon]} \otimes |0,1\rangle\langle 0,1| + \chi_{[-z-\epsilon,-z+\epsilon]} \otimes |1,0\rangle\langle 1,0|\right].
\end{aligned}
$$

Hence, the operators $P^\epsilon_{(t,x,y,z)}$ vanish as soon as at least *two* of the spatial coordinates have modulus greater or equal to ϵ. Therefore, if one is far away in two coordinates from the origin, one sees nothing except on the scales of the distances to the origin itself. If only one coordinate, say z has a modulus greater than ϵ, then the projection operator is given by

$$
\begin{aligned}
P^\epsilon_{(t,x,y,z)} = \; & \tfrac{1}{2}\,\chi_{[|x|-\epsilon,-|x|+\epsilon]\times[|y|-\epsilon,-|y|+\epsilon]\times[z-\epsilon,z+\epsilon]} \otimes |0,1\rangle\langle 0,1| \,+ \\
& \tfrac{1}{2}\,\chi_{[|x|-\epsilon,-|x|+\epsilon]\times[|y|-\epsilon,-|y|+\epsilon]\times[-z-\epsilon,-z+\epsilon]} \otimes |1,0\rangle\langle 1,0|\,,
\end{aligned}
$$

and the reader is invited to work out the projection operator for a case in which all spatial coordinates have a modulus smaller than ϵ. Therefore, one obtains an axial structure where any of the coordinate axes are privileged as well as a neighborhood of the origin. In a forthcoming publication, we shall work out an example for more generic $U(x)$.

The reader may well have noticed that we still have to say something about dimension since dimensional collapse is possible; indeed any real n dimensional manifold is a m dimensional noncommutative one if and only if $m \geq n$. On the other hand, discrete manifolds do not necessarily have a one dimensional representation due to the algebraic relations (so we have some kind of entanglement dimension). Therefore, one might be tempted to declare the dimension of a manifold to be the minimal one; it is for now a matter of taste whether one allows for collapse or not and we leave this to the discretion of the reader.

3 Canonical Differentiable Structure

Before we define a differential structure, we have to identify the natural class of functions on a local chart $(V_\beta, \pi_\beta, \mathcal{A}_\beta, \phi_\beta)$. The thing is that points and functions are simply unified in the algebraic context; they just are elements of

$\mathcal{A}_\beta$. Indeed, a function is nothing than some limit of a finite polynomial in the $\pi_\beta(\widehat{x}_i)$ and the natural question is how we should define the function on an open set $\mathcal{W} \subset \mathcal{V}_\beta$. There are two natural candidates for local functions which we call the entangled and unentangled one for obvious reasons. The former forgets how an element $A \in \mathcal{A}_\beta$ arises from the fundamental building blocks and maps $A \to \widehat{A}$, where the latter is defined as

$$\widehat{A}(\mathcal{W}) = P_\beta(\mathcal{W}) A P_\beta(\mathcal{W}) \ ,$$

and obviously $\widehat{A}$ maps distinct regions to orthogonal operators; moreover, $\widehat{A}$ preserves the order relation in the sense that

$$\widehat{\widehat{A}(\mathcal{W}_2)}(\mathcal{W}_1) = \widehat{A}(\mathcal{W}_1)$$

for $\mathcal{W}_1 \subset \mathcal{W}_2$. However, this transformation does not erase entanglement with regions outside $\mathcal{W}$ as the reader may easily verify and obviously, this ansatz is not a suitable candidate for defining a differential since it does not "feel" the order in which the elementary variables occur. Let us start with finite polynomials in unity and the preferred variables $\pi_\beta(\widehat{x}_i)$, then one meets a rarity which might seem to be a lethal problem at first sight but really is nothing but a manifestation of what breaking of entanglement means. That is let $A = Q(1, \pi_\beta(\widehat{x}_i))$, where Q is some polynomial of finite degree, then we define

$$\widehat{Q}(\mathcal{W}) = Q(P_\beta(\mathcal{W}), P_\beta(\mathcal{W})\pi_\beta(\widehat{x}_i)P_\beta(\mathcal{W}))$$

as the local unentangled realization of Q. Now, it is possible for two polynomials Q_1 and Q_2 to determine identical elements in $\mathcal{A}_\beta$, but the local realizations $\widehat{Q}_j$ differ; also, the reader is invited to construct some examples on this. All this implies that we have to define nets of polynomials and declare equivalence with respect to the resolution one is measuring which removes the absolutism from $\mathcal{A}_\beta$; that is,

$$\widehat{Q}_1 \sim_{\mathcal{W}} \widehat{Q}_2$$

if and only if $\widehat{Q}_1(\mathcal{W}) = \widehat{Q}_2(\mathcal{W})$. One verifies moreover that the local unentangled $\widehat{A}$ has the same inclusion and disjoint properties than the entangled one. Therefore, consider a natural directed net $(Q_i, i \in \mathbb{N})$ of finite polynomials in the fundamental variables $\widehat{x}_i$ and unity, then we say that the domain $\text{Dom}((Q_i, i \in \mathbb{N}), (\pi_\beta, \mathcal{A}_\beta))$ of this net relative to the chart $(\pi_\beta, \mathcal{A}_\beta)$ is given by the set of relative opens $\mathcal{W} \subset \mathcal{O}(\pi_\beta, \mathcal{A}_\beta)$ so that $\widehat{Q}_i(\mathcal{W})$ is a weakly convergent series of operators. For the general reader, the weak topology on a $W^\star$ algebra is the locally convex topology generated by the continuous complex linear functionals $\psi_\beta : \mathcal{A}_\beta \to \mathbb{C}$. Now in order to define continuity and differentiability of such functions, we need to equip the relative open sets with a canonical topology, that is the Vietoris topology which is defined by the relative open subsets $(\mathcal{O}, \mathcal{V})(\mathcal{W})$ where $\overline{\mathcal{V}} \subset \mathcal{W} \subset \overline{\mathcal{W}} \subset \mathcal{O}$ and $(\mathcal{O}, \mathcal{V})(\mathcal{W})$ is the set of all open sets $\mathcal{Z}$ satisfying $\mathcal{V} \subset \mathcal{Z} \subset \mathcal{O}$.

Definition 2 *Therefore, the net* $(Q_i, i \in \mathbb{N})$ *is of bounded variation relative to* $(\pi_\beta, \mathcal{A}_\beta)$ *in* $\mathcal{W} \in Dom((Q_i, i \in \mathbb{N}), (\pi_\beta, \mathcal{A}_\beta))$ *if and only if for every* $\epsilon > 0$ *and continuous functional* ψ_β*, there exists an open set containing* $\mathcal{W}$ *such that for any open* $\mathcal{Z}$ *contained in it we have that*

$$|\psi_\beta((\widehat{Q}_i, i \in \mathbb{N})(\mathcal{W}) - (\widehat{Q}_i, i \in \mathbb{N})(\mathcal{Z}))| < \epsilon \, .$$

In order to define directional continuity, partial differential operators and finite difference operators, we need the notion of directional displacement. Therefore, let $\vec{e}$ be a unit vector in $\mathbb{R}^n$ and δ; then the translation $T_{(\delta\vec{e})}$ canonically lifts as a continuous map to the space of all open sets by the prescription

$$T_{(\delta\vec{e})}(\mathcal{W}) = \mathcal{W} + \delta\vec{e} \, .$$

We need also need to lift the translations to homomorphisms between the local algebras $\mathcal{A}^{loc}_\beta(\mathcal{W})$ which requires the use of a quantum connection. Here, $\mathcal{A}^{loc}_\beta(\mathcal{W})$ is the $W^\star$ subalgebra of $\mathcal{A}_\beta$ generated by $P_\beta(\mathcal{W})\pi_\beta(\widehat{x}_i)P_\beta(\mathcal{W})$ and $P_\beta(\mathcal{W})$ which is not the same as $P_\beta(\mathcal{W})\mathcal{A}_\beta P_\beta(\mathcal{W})$ (which is also a Von Neumann algebra) as explained before. The reason why we need a connection is because at some resolution ϵ, $P_\beta(\mathcal{W})$ will not majorize, nor commute with the $P^i((\alpha_i - \epsilon, \alpha_i + \epsilon))$ so that the projection operators will not be projection operators anymore but twisted depending upon the region $\mathcal{W}$ and spectral operator at hand. This does of course not happen in the abelian case where everything remains trivial. Also, it is generally not so that for $\mathcal{V} \subset \mathcal{W}$ one obtains that

$$\mathcal{A}^{loc}_\beta(\mathcal{V}) \subset \mathcal{A}^{loc}_\beta(\mathcal{W})$$

and the reason is that fine grained projections can add a twist where coarser grained projections do not. Of course, this inclusion property does hold when we do not cut entanglement, that is

$$P_\beta(\mathcal{V})\mathcal{A}_\beta P_\beta(\mathcal{V}) \subset P_\beta(\mathcal{W})\mathcal{A}_\beta P_\beta(\mathcal{W})$$

for $\mathcal{V} \subset \mathcal{W}$. Let us give some example confirming these facts, consider the following discrete four dimensional quantum manifold

$$\begin{aligned} t &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\ x &= \begin{pmatrix} 0 & \sigma_1 \\ \sigma_1 & 0 \end{pmatrix} \\ y &= \begin{pmatrix} 0 & \sigma_2 \\ \sigma_2 & 0 \end{pmatrix} \\ z &= \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} . \end{aligned}$$

A little algebra reveals that $[t,x]=[t,y]=0$, $\{x,y\}=0$ and $t^2=x^2=y^2=1$. Also, one notices that y and z do not commute nor anticommute. The spectrum of t,x,y is $\{-1,1\}$ and both eigenspaces have dimension two; for z it clearly is $\{0,1,2\}$ and therefore the cube consists out of 24 points. Associate $\mathcal{V}$ to that subset of the cube with arbitrary values for t,x and $y=1=z$ and $\mathcal{W}$ to arbitrary values for t,x,z and $y=1$, then clearly $\mathcal{V}\subset\mathcal{W}$. One computes that

$$P(\mathcal{V}) = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & -i & 0 \\ 0 & i & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

and $P(\mathcal{W})=\frac{1}{2}(1+y)$. We compute $\mathcal{A}^{loc}(\mathcal{W})$ and show that $P(\mathcal{V})$ does not belong to it. Elementary algebra shows that

$$\begin{aligned} P(\mathcal{W})tP(\mathcal{W}) &= \frac{1}{2}\begin{pmatrix} \sigma_2 & 1 \\ 1 & \sigma_2 \end{pmatrix} \\ P(\mathcal{W})xP(\mathcal{W}) &= 0 \\ P(\mathcal{W})yP(\mathcal{W}) &= P(\mathcal{W}) \\ P(\mathcal{W})zP(\mathcal{W}) &= P(\mathcal{W}) \end{aligned}$$

even though $P(\mathcal{W})$ does not commute with z. It is now easy to show that $\mathcal{A}^{loc}(\mathcal{W})$ is two dimensional and that $P(\mathcal{V})$ is not in it. Finally, we compute the dimension of $P(\mathcal{W})\mathcal{A}P(\mathcal{W})$; the latter is four as can be easily seen by starting from the expression

$$\frac{3}{2}z-\frac{1}{2}z^2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \alpha$$

and notice that $P(\mathcal{W})\alpha P(\mathcal{W})-\frac{1}{2}P(\mathcal{W})\sim P(\mathcal{V})$.

From the weak continuity of κ "almost everywhere" one deduces that the local algebra's $\mathcal{A}^{loc}(\mathcal{W})$ almost never jump when we move $\mathcal{W}$ around. Therefore, what one could call quasilocal algebra's are basically the same as the local ones. Hence, we define a connection, or parallel transport, $\Gamma_\beta(\mathcal{V},\mathcal{W})$ as a bifunction of two relatively open sets which map to a star homomorphism between the respective local algebra's; that is,

$$\Gamma_\beta(\mathcal{V},\mathcal{W}):\mathcal{A}^{loc}_\beta(\mathcal{V})\to\mathcal{A}^{loc}_\beta(\mathcal{W})$$

where a path dependence is possible in the composition and we could at most look for rules of intersection and inclusion. For $\mathcal{V}\subset\mathcal{W}$, one has that when a spectral projector $P\prec P(\mathcal{V})$ or $P(\mathcal{V})PP(\mathcal{V})=P$ then the same is true for $P(\mathcal{W})$ and we demand $\Gamma(\mathcal{V},\mathcal{W})$ to preserve these fixpoints. Other principles of this kind are not possible, it might be that P commutes with $P(\mathcal{V})$ but not with $P(\mathcal{W})$ and vice versa. We might still ask however for the connection to be optimal which means that the homomorphisms cannot be majorized. Therefore, in case the local algebra's are isomorphic, $\Gamma(\mathcal{V},\mathcal{W})$ is an isomorphism too. Also, we demand the connection to be unital, meaning that $\Gamma(\mathcal{W},\mathcal{W})$ is equal to the identity. There will be two further requirements on the connection which is that the basic functions $\pi_\beta(\widehat{x}_i)$ are weakly continuous or differentiable wherever $\kappa_{(\pi_\beta,\mathcal{A}_\beta)}$ is in all or some directions $\vec{e}$. The latter is a huge constraining between the analytical and $W^\star$ algebraic aspects of $\mathcal{A}_\beta$.

We have two different notions of continuity and differentiability because $\kappa_{(\pi_\beta,\mathcal{A}_\beta)}$ has a peculiar and natural status within our construction. First of all, we say that $\kappa_{(\pi_\beta,\mathcal{A}_\beta)}$ is weakly continuous in a point $\mathcal{W}$ in the Vietoris topology when for all $\epsilon > 0$ and continuous functionals ψ_β, there exists an open neighborhood $\mathcal{O}$ in the Vietoris topology such that for any $\mathcal{Z} \in \mathcal{O}$ we have that

$$|\psi_\beta\left(\kappa_{(\pi_\beta,\mathcal{A}_\beta)}(\mathcal{W}) - \kappa_{(\pi_\beta,\mathcal{A}_\beta)}(\mathcal{Z})\right)| < \epsilon.$$

Likewise, we say that $\kappa_{(\pi_\beta,\mathcal{A}_\beta)}$ is continuous in the direction $\vec{e}$ at $\mathcal{W}$ when for any $\epsilon > 0$ and ψ_β, there exists a $\delta > 0$ so that for any $|h| < \delta$

$$|\psi_\beta\left(\kappa_{(\pi_\beta,\mathcal{A}_\beta)}(\mathcal{W}) - \kappa_{(\pi_\beta,\mathcal{A}_\beta)}(T_{(h\vec{e})}(\mathcal{W}))\right)| < \epsilon.$$

Concerning the notion of weak differentiability of $\kappa_{(\pi_\beta,\mathcal{A}_\beta)}$, there exist several and we have to find out if some of them are equivalent or not. Let me first start by examining the abelian case in sufficient detail and then generalize to the nonabelian setting. In the Schrodinger like setting explained before, the projection operators are just characteristic functions and in one dimension, the computations simplify considerably (however, there is no problem generalizing this to higher dimensions as the reader may try to do) while the results are universal. Naively, one would think we have to calculate the limit of

$$\frac{1}{\delta}\left(\chi_{(a+\delta,b+\delta)} - \chi_{(a,b)}\right)$$

for $0 < \delta \to 0$. If one would restrict to the continuous functions as a separating *subalgebra* of the L^2 functions (at least on a compact measure space), then this limit exists in the weak sense and it is $\delta(b) - \delta(a)$ which is outside the algebra since it is not well defined on the whole Hilbert space. Now, if again, we would only restrict to the continuous functions, then the limit

$$\frac{1}{\delta^{1-\gamma}}\left(\chi_{(a+\delta,b+\delta)} - \chi_{(a,b)}\right)$$

is zero and independent of $\gamma > 0$. However, if one were to go over to the full Hilbert space, then it is necessary and sufficient that $\gamma > \frac{1}{2}$ in which case the limit is also zero. Therefore, we say that $\kappa_{(\pi_\beta,\mathcal{A}_\beta)}$ is γ-weakly differentiable with respect to a separating[2] subset $\Psi_\beta(\gamma)$ of continuous functionals in the direction

[2]Separating means that for all distinct $A, B \in \mathcal{A}_\beta$ there exists a $\psi_\beta \in \Psi_\beta(\gamma)$ such that $\psi_\beta(A) \neq \psi_\beta(B)$.

$\vec{e}$ at $\mathcal{W}$ if there exists an element $\partial^{\gamma}_{\vec{e}}\, \kappa_{(\pi_\beta, \mathcal{A}_\beta)}(\mathcal{W})$ such that for all $\epsilon > 0$ and $\psi_\beta \in \Psi_\beta(\gamma)$, there exists a $\delta > 0$ such that for all $0 < h < \delta$ we have that

$$\left| \psi_\beta \left(\frac{1}{h^{1-\gamma}} \left(\kappa_{(\pi_\beta, \mathcal{A}_\beta)}(T_{(h\vec{e})}(\mathcal{W})) - \kappa_{(\pi_\beta, \mathcal{A}_\beta)}(\mathcal{W}) \right) - \partial^{\gamma}_{\vec{e}}\, \kappa_{(\pi_\beta, \mathcal{A}_\beta)}(\mathcal{W}) \right) \right| < \epsilon.$$

Similarly, one could forget about $\Psi_\beta(\gamma)$ and demand that $\gamma > \frac{1}{2}$. This attitude could lead to very different algebra's and we will not even start its investigation in this short paper. An obvious property is that if $\kappa_{(\pi_\beta, \mathcal{A}_\beta)}$ is differentiable with respect to $(\gamma_1, \Psi_\beta(\gamma_1))$, then it is also the case for $(\gamma_2, \Psi_\beta(\gamma_1))$ where $\gamma_2 < \gamma_1$ and the differential is exactly zero.

We now turn to continuity and differentiability of nets $(\widehat{Q}_i, i \in \mathbb{N})$ of finite polynomials on their relative domain (with respect to $(\pi_\beta, \mathcal{A}_\beta)$). Define now

$$\widehat{T}_{(\delta\vec{e})}(\mathcal{W}) = \Gamma(\mathcal{W}, T_{(\delta\vec{e})}(\mathcal{W}))$$

then we say that $(\widehat{Q}_i, i \in \mathbb{N})$ differentiable at $\mathcal{W}$ in the interior of its relative domain in the direction of $\vec{e}$ if and only if for any ψ_β, there exists a unique element

$$\partial_{\vec{e}}\, (\widehat{Q}_i, i \in \mathbb{N})(\mathcal{W}) \in \mathcal{A}^{loc}(\mathcal{W})$$

such that

$$\begin{aligned} & \psi_\beta \left(\partial_{\vec{e}}\, (\widehat{Q}_i, i \in \mathbb{N})(\mathcal{W}) \right) \\ = & \lim_{\delta \to 0} \frac{1}{\delta}\, \psi_\beta \left(\widehat{T}_{-\delta\vec{e}} \left[(\widehat{Q}_i, i \in \mathbb{N})(T_{(\delta\vec{e})}(\mathcal{W})) \right] - (\widehat{Q}_i, i \in \mathbb{N})(\mathcal{W}) \right). \end{aligned}$$

So, the differential operator is only defined if some translates of $\mathcal{W}$ belong to the relative domain of $(\widehat{Q}_i, i \in \mathbb{N})$ for arbitrarily small δ. Therefore, partial differentials are not defined for directions in which the set at hand is isolated. Of course, if one looks only at larger scales, then jumps may be accomplished and the difference operators are canonically defined. One could also resort here to notions of $(\gamma, \Psi_\beta(\gamma))$ differentiability, but I see no stringent need to do it at this point.

Before we give some examples, let us proceed by defining the holonomy groups attached to the connection; for any $\mathcal{W}$, we define $H(\mathcal{W})$ as the group of homomorphisms from $\mathcal{A}^{loc}(\mathcal{W})$ to itself generated by finite compositions of the kind

$$\Gamma(\mathcal{W}_n, \mathcal{W})\Gamma(\mathcal{W}_{n-1}, \mathcal{W}_n) \ldots \Gamma(\mathcal{W}_1, \mathcal{W}_2)\Gamma(\mathcal{W}, \mathcal{W}_1).$$

We say that a connection is flat when all the holonomy groups are equal to the identity. Consider as before the trivial example of a real n dimensional manifold, then the translation mappings induce a canonical flat connection on the pairs of opens differing by a translate as follows: every spectral operator $P^i((\alpha_i - \epsilon, \alpha_i + \epsilon) \cap \mathcal{W}) = P(\mathcal{W})P^i((\alpha_i - \epsilon, \alpha_i + \epsilon))P(\mathcal{W})$ gets mapped to

$$P^i((\alpha_i - \epsilon + \delta e_i, \alpha_i + \epsilon + \delta e_i) \cap T_{(\delta\vec{e})}(\mathcal{W})).$$

Actually, this is all we need to calculate differentials and so on, but the reader might wish to extend this definition in a canonical way to generic pairs. For $\mathcal{W}$ of compact closure and real differentiable function f with $\overline{\mathcal{W}} \subset \mathrm{Dom}(f)$ one associates a unique algebra element $\widetilde{f}$ (in the commutative case we do not need the nets). It is easy to calculate that the new differential

$$\partial_{\vec{e}} \widehat{\widetilde{f}}(\mathcal{W}) = \widehat{\partial_{\vec{e}} \widetilde{f}} \, \chi_{\mathcal{W}} \ ,$$

reduces to the old one and that the latter even exists in the norm topology in this case.

All these results allow us now to obtain a better insight into the nature of noncommutative n dimensional manifolds. Before we engage in this discussion we still need to solve some questions:

- We have demanded that for overlapping charts the algebra's of local projection operators (with respect to these charts) are isomorphic; how does this algebra relate to the local algebra with respect to that chart?
- We have seen that for $\mathcal{V} \subset \mathcal{W}$, it does not necessarily hold that $\mathcal{A}^{loc}(\mathcal{V}) \subset \mathcal{A}^{loc}(\mathcal{W})$. However, does there exist an isomorphism of $\mathcal{A}^{loc}(\mathcal{V})$ into a subalgebra of $\mathcal{A}^{loc}(\mathcal{W})$?
- Finally, say that $\mathcal{W}$ contains r components with respect to $\mathcal{V}_\beta$; does the spectrum of the local algebra $\mathcal{A}^{loc}(\mathcal{W})$ contain at least r components ?

As a response to the first question, we already know that the algebra of local projection operators is not contained in the local algebra and the question is whether the inverse holds. But before we treat these questions in generality, let us see how they are answered in the our previous example. Concerning the first question, we notice that the only nonzero projection operators (apart from $P(\mathcal{V})$ and $P(\mathcal{W})$) arise from $y = 1$ and $t = \pm 1$; they are given by

$$\begin{aligned} P(t = 1 = y) &= \frac{1}{4} \begin{pmatrix} 1 & i & 1 & i \\ -i & 1 & -i & 1 \\ 1 & i & 1 & i \\ -i & 1 & -i & 1 \end{pmatrix} \\ P(t = -1 = -y) &= \frac{1}{4} \begin{pmatrix} 1 & -i & -1 & i \\ i & 1 & -i & -1 \\ -1 & i & 1 & -i \\ -i & -1 & i & 1 \end{pmatrix} . \end{aligned}$$

It is most easily seen that $P(\mathcal{W}) t P(\mathcal{W}) = 2P(t = 1 = y) - P(\mathcal{W})$ which shows that $\mathcal{A}^{loc}(\mathcal{W})$ is a subalgebra of the algebra generated by the local projection operators $P(\mathcal{V})$ with $\mathcal{V} \subseteq \mathcal{W}$. The second question is answered in the *negative* since $\mathcal{A}^{loc}(\mathcal{V})$ is generated by $P(\mathcal{V})$ and

$$P(\mathcal{V})tP(\mathcal{V}) = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & i \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -i & 0 & 0 & 1 \end{pmatrix}$$

and it is easy to verify that this algebra is not isomorphic to $\mathcal{A}^{loc}(\mathcal{W})$. Therefore, the answer to the second question is inconclusive since in the commutative case $\mathcal{A}^{loc}(\mathcal{V}) \subseteq \mathcal{A}^{loc}(\mathcal{W})$. Regarding the third issue, $\mathcal{W}$ contains 12 points and the cube of $\mathcal{A}^{loc}(\mathcal{W})$ contains also 12 of them[3]. However, all projection operators vanish in the former case while in the latter exactly 3 of them are nonzero. Therefore, the question appears to hold on the ontological as well as the empirical level.

Let us start with some mathematical preliminaries.

Theorem 1 *Let P and Q be two (noncommuting) Hermitian projection operators then the projection operators $P \wedge Q$ and $P \vee Q$ belong to $\mathcal{M}' \cap \mathcal{M}$, where $\mathcal{M}'$ is the commutant in $\mathcal{A}_\beta$ of the Von Neumann algebra $\mathcal{M}$ generated by P and Q. In particular, any Hermitian projection operator which is smaller than $P \wedge Q$ or larger than $P \vee Q$ belongs to $\mathcal{M}'$.*

Proof : Represent P and Q on a Hilbert space $\mathcal{H}$ and consider the smallest closed subspace $\mathcal{H}'$ which is left invariant by both of them. Then this $\mathcal{H}'$ has $P \vee Q$ as identity operator and we have to show that it is generated by P and Q. For the intersection, the proof is easy: $\frac{1}{2}(PQ + QP) = P \wedge Q + A$ where $(P \wedge Q)A = 0$, $A^\star = A$, $||A|| \leq 1$ but 1 does not belong to the discrete spectrum, and therefore

$$P \wedge Q = \lim_{n \to \infty} \left(\frac{1}{2}(PQ + QP) \right)^n$$

in the weak sense. Replacing Q by $Q' = Q - PQ - QP + PQP$, we see that it is zero if and only if $Q = P$; moreover, $PQ' = Q'P = 0$ and Q' as a mapping from $(1 - P)\mathcal{H}'$ to $(1 - P)\mathcal{H}'$ does not contain 0 in its discrete spectrum. Otherwise, there would exist a vector $v \in (1 - P)\mathcal{H}'$ such that $(1 - P)Qv = 0$ or $Qv = PQv$ which is impossible unless v is in the intersection of both hyperspaces which implies it must be the zero vector. In the finite dimensional case, it easy to construct polynomials $f_\alpha(x)$ with $f_\alpha(0) = 0$ such that

$$f_\alpha(Q') = P_\alpha$$

where $\alpha \in \sigma(Q')$ and P_α is its spectral operator. Therefore, one can recuperate the identity $P \vee Q - P$ on $(1 - P)\mathcal{H}'$ in the algebra of Q' only. In the infinite dimensional case, this technique fails since the polynomials will start to oscillate heavily which has a detrimental effect on the continuous spectrum. However, if

[3] One calculates that the spectrum of $P(\mathcal{W})tP(\mathcal{W})$ is $\{-2, 2, 0\}$ and the projection operator on the zero eigenvalue is $\frac{1}{2}(1 - y)$.

one considers the algebra generated by $1, P, Q$ a similar argument holds due to the Stone Weierstrass and spectral theorem.

Concerning the first question, let us elaborate on whether given a cube P_1, P_2 where $P_1 = P + Q$ with $PQ = 0$ and corresponding to distinct discrete eigenvalues, it is true that

$$(P_1 \wedge P_2)P(P_1 \wedge P_2) = \alpha P_1 \wedge P_2 + (1-\alpha) P \wedge P_2 - \alpha Q \wedge P_2$$

for some $\alpha \in \mathbb{R}$ (actually the reader can check that any linear combination of these operators has to be of this form). It is easily seen that this statement is false, since consider the orthonormal unit vectors e_i, $i : 1 \ldots 5$, and the following subspaces:

$$\begin{aligned}
\mathcal{P} &= \text{Span}\{\cos(\theta)e_1 + \sin(\theta)e_2, \cos(\psi)e_3 + \sin(\psi)e_4\} \\
\mathcal{Q} &= \text{Span}\{\sin(\theta)e_1 - \cos(\theta)e_2, \sin(\psi)e_3 - \cos(\psi)e_4\} \\
\mathcal{P}_2 &= \text{Span}\{e_2, e_3, e_5\}.
\end{aligned}$$

Then, one has the following identities:

$$\begin{aligned}
P_1 \wedge P_2 &= |e_2\rangle\langle e_2| + |e_3\rangle\langle e_3| \\
P \wedge P_2 &= 0 \\
Q \wedge P_2 &= 0 \\
PQ &= 0.
\end{aligned}$$

However, one easily calculates that

$$(P_1 \wedge P_2)P(P_1 \wedge P_2) = \sin^2(\theta)|e_2\rangle\langle e_2| + \cos^2(\psi)|e_3\rangle\langle e_3|$$

which is not a multiple of $P_1 \wedge P_2$. Therefore, one has that $P(\mathcal{W})P^i P(\mathcal{W})$ is in general not in the algebra generated by $P(\mathcal{V})$ where $\mathcal{V} \subseteq \mathcal{W}$. It is now easy to pick $\pi_\beta(\widehat{x}_i) = P + \mu R$ where $R = |e_5\rangle\langle e_5|$ to conclude that

$$(P_1 \wedge P_2)\pi_\beta(\widehat{x}_i)(P_1 \wedge P_2)$$

is not in the algebra generated by the $P(\mathcal{V})$. This shows that $\mathcal{A}^{loc}(\mathcal{W})$ and the $W^\star$ algebra $\mathcal{A}^{open}(\mathcal{W})$ generated by the $P(\mathcal{V})$ where $\mathcal{V} \subseteq \mathcal{W}$ have no relation to one and another.

Definition 3 *We call the chart* $(\mathcal{V}_\beta, \pi_\beta, \mathcal{A}_\beta, \phi_\beta)$ pointed *when for all* $\mathcal{W}$*,*

$$\mathcal{A}^{loc}(\mathcal{W}) \subseteq \mathcal{A}^{open}(\mathcal{W}).$$

We now proceed to answer the third question which intuitively means that if you zoom in you see more and more disconnected components. Now, it is obvious that this property does not even hold in the commutative case where on large scales one may see many isles but on small scales all one sees is one of them.

However, a refinement of the question is nevertheless interesting and one might want to look for manifolds which have only one component on a given scale and where the number of components grows polynomially (or even exponentially) in the inverse scaling $\frac{1}{\lambda}$.

We now have obtained a better view on how we should do function theory on a noncommutative topological manifold although we are confronted with an apparent dilemma. On one side $\mathcal{A}^{open}_{\beta}(\mathcal{W})$ is the natural algebra we should use to compare overlapping charts, but $\mathcal{A}^{loc}_{\beta}(\mathcal{W})$ is the natural algebra for function theory. What we learned is that they have generically little to do with one and another; therefore, this begs the question of how to even define *algebraic* functions on the entire manifold. It is here that the (trace) functionals $\omega_{\mathcal{A}_\beta}$ come into play in the following sense: let $\mathcal{M}$ be a noncommutative manifold, then $F : \tau(\mathcal{M}) \to \mathbb{C}$, where $\tau(\mathcal{M})$ is the set of open subsets of $\mathcal{M}$ equipped with the Vietoris topology, is an algebraic function if and only if for any chart $(\mathcal{V}_\beta, \pi_\beta, \mathcal{A}_\beta, \phi_\beta)$, there exists a net of polynomials $(Q^{\beta}_i, i \in \mathbb{N})$ such that

$$F(\mathcal{W}) = \omega_{\mathcal{A}_\beta}\left(\left(\widehat{Q}^{\beta}_i, i \in \mathbb{N}\right)(\mathcal{W})\right).$$

Continuity of F is obviously defined with respect to the Vietoris topology. We call F nuclear if and only if for any $\mathcal{V}, \mathcal{W}$, one has that

$$F(\mathcal{V} \cup \mathcal{W}) = F(\mathcal{W}) + F(\mathcal{V}) - F(\mathcal{V} \cap \mathcal{W}).$$

Obviously, the standard continuous functions on a real n dimensional manifold with a volume element induce nuclear continuous functions by putting the trace functional equal to the n dimensional integral. We can define higher order algebraic functions as follows

$$F(\mathcal{W}, \mathcal{V}_1, \ldots \mathcal{V}_m) = \omega_{\mathcal{A}_\beta}\left(P_\beta(\mathcal{V}_1) \ldots P_\beta(\mathcal{V}_m)\left(\widehat{Q}^{\beta}_i, i \in \mathbb{N}\right)(\mathcal{W})\right)$$

where $\mathcal{V}_j \subset \mathcal{W}$. The gluing conditions ensure us that the identity element in $\mathcal{F}_n$ canonically defines a set of (higher order) algebraic functions.

Let us finish by commenting upon the very act of pasting together "algebraic charts". We have learned two ways of cutting entanglement, which was by going over to local and open $W^\star$ algebra's associated to open subsets of $\mathcal{M}$; also, the $W^\star$ algebraic framework forces us in the cauldron of relatively open subsets of $\mathbb{R}^n$. This implies that in order to generate a nontrivial topology (with respect to a continuum background) some sort of "decoherence" has to occur. Indeed, saying that two charts are described by separate $W^\star$ algebra's really means that the points in both charts do not "entangle" in some sense. Whether or not this is a desirable conclusion remains to be seen.

4 Conclusions

We have made first steps with universal n dimensional manifolds in the context of $W^\star$ algebra's by defining topological noncommutative manifolds, clarifying the (lack of) relationships between different local $W^\star$ algebra's and by making first steps with functional calculus. What remains to be done it to treat higher differential calculus and define general differentiable nonabelian manifolds. From thereon, one can construct vector and tensor calculus and define noncommutative geometry. It would be instructive to construct explicit realizations of Hopf $\star$ algebra's as our construction should allow for this and much more; this would offer a concrete interpretational framework for amongst others kappa Minkowski spacetime.

Hence, the we have reached the conclusion that what we *see* depends upon the scale that we are looking at, but the continuum $\mathbb{R}^n$ background always *is* and constitutes the very backbone of the entire construction. Therefore, spacetime *is* grounded in the continuum albeit we may perceive it in an atomistic way. This is precisely the conclusion the author advocated in [2].

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