Spinor gravity

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Abstract

A theory of gravitation based upon a spinor connection and solder form with a nonvanishing torsion tensor is constructed and a mapping to Einstein Cartan theory is made. As applications, the flat Friedmann and Schwarzschild solutions are derived.

1 Introduction

We construct a theory of gravitation in which both the spacetime metric and connection are composite objects constructed from a solder form σ and spinor connection $\Gamma^A_{\mu B}$. The underlying idea is that two spinors are more fundamental objects than (co)vectors since the latter are squares of the former; also all fundamental particles excluding the gauge bosons turn out to be spinorial. Therefore it seems more natural to start from a theory with a local $SL(2,C)$ connection as well as a covariantly constant σ and derive the corresponding spacetime quantities from there. Doing this, it turns out that the spacetime connection has a nonvanishing torsion tensor which satisfies its own equations of motion. As an application of the theory, the flat Friedmann and Schwarzschild solutions are derived.

2 Special relativity revisited.

This aim of this section is to quickly present the two spinor formalism [1] in the context of special relativity. It is generally accepted that the relevant kinematical group in physics is $SL(2,C)$, the universal cover of the orthochronous Lorentz group $O^+(1,3)$. Let W be a two dimensional complex vector space with basis e_A and volume form ϵ_{AB}. In the literature, one puts $e_1 = o$ and $e_2 = n$: $\epsilon(o,o) = \epsilon(n,n) = 0$ and $\epsilon(o,n) = 1$. The subgroup of $GL(2,C)$ which leaves ϵ_{AB} invariant is isomorphic to $SL(2,C)$. The complex conjugation sends W to $\overline{W}$, that is $v \in W \to \bar{v} \in \overline{W}$ which is spanned by $\bar{o}, \bar{n}$ with a volume form

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represented by $\bar{\epsilon}$. $W \otimes \overline{W}$ is four dimensional over C and one is interested in its real subspace. The latter is spanned (over the real numbers) by

$$\begin{aligned}
\hat{t} &:= \frac{1}{\sqrt{2}}\left(o \otimes \bar{o} + n \otimes \bar{n}\right) \\
\hat{x} &:= \frac{1}{\sqrt{2}}\left(o \otimes \bar{n} + n \otimes \bar{o}\right) \\
\hat{y} &:= \frac{i}{\sqrt{2}}\left(o \otimes \bar{n} - n \otimes \bar{o}\right) \\
\hat{z} &:= \frac{1}{\sqrt{2}}\left(o \otimes \bar{o} - n \otimes \bar{n}\right)
\end{aligned}$$

It is easily computed that $\{\hat{t}, \hat{x}, \hat{y}, \hat{z}\}$ is orthonormal with respect to $\omega \otimes \overline{\omega}$ and that the signature is $(+---)$. This suggests an identification with some inertial system (t, x, y, z) by means a "solder" form $\sigma = t\hat{t} + x\hat{x} + y\hat{y} + z\hat{z}$. Employing the usual isomorphism between the the tensor product $W \otimes \overline{W}$ and the 2×2 complex matrix algebra, σ can be written as

$$\sigma = \frac{1}{\sqrt{2}}\left(t1 + x\sigma_1 + y\sigma_2 + z\sigma_3\right)$$

where the σ_i are the usual Pauli matrices. This allows one to write $\sigma_{\mu}^{AB'}$ where primed Latin capital letters are indices with respect to the complex conjugate basis in $\overline{W}$. As is very well known, each complex vector η determines a real null vector ${\sigma^{\mu}}_{AA'}\eta^A\bar{\eta}^{A'}$, the sum of such two null vectors ${\sigma^{\mu}}_{AA'}\left(\eta^A\bar{\eta}^{A'} + \chi^A\bar{\chi}^{A'}\right)$ is always timelike and the currents

$${\sigma^{\mu}}_{AA'}\left(\eta^A\bar{\chi}^{A'} + \chi^A\bar{\eta}^{A'}\right)$$

and

$$i{\sigma^{\mu}}_{AA'}\left(\eta^A\bar{\chi}^{A'} - \chi^A\bar{\eta}^{A'}\right)$$

are all spacelike and *orthogonal* to both currents ${\sigma^{\mu}}_{AA'}\eta^A\bar{\eta}^{A'}$ and ${\sigma^{\mu}}_{AA'}\chi^A\bar{\chi}^{A'}$. The latter, again, is a very strong feature of the kinematical variables at hand : the "signature" of interaction terms is fixed by construction. Moreover, it turns out that all timelike vectors constructed in this way point in the same half of the light cone.

3 The gravitational field.

We will be occupied in this section with formulating a unique two spinor valued closed field theory which recuperates some relevant features of general relativity. Inspired by the previous section, we should first wonder what the relevant dynamical variables are. Since spinors X^A are the fundamental objects, we need a spinor one form valued connection $\Gamma^A_{\mu B}$ and in order to define second derivatives we need a usual space time connection $\Gamma^{\kappa}_{\ \mu\nu}$. A non degenerate spinor volume form is required ϵ_{AB} and the dynamics is such that the latter is covariantly[1] constant $\nabla_\mu \epsilon_{AB} = 0$. The volume form allows us to raise and lower spinor indices, eg. $X_A = \epsilon_{BA} X^B$ where the contraction is by convention over the *first* index. Therefore, the connection satisfies $\Gamma_{A\mu B} - \Gamma_{B\mu A} = 0$, in either it is symmetric in the spinor indices. Next, we must examine the role of $\sigma_\mu^{AB'}$ which determines a physical coupling between Hermitian spinor currents and vector particles. $\sigma_\mu^{\ AB'}$ is a dynamical variable and we demand the connection to be such that it is covariantly constant, eg. $\nabla_\mu \sigma_\nu^{AB'} = 0$. This allows one to solve the spacetime connection

$$\Gamma^{\kappa}_{\ \mu\nu} = \sigma^{\kappa}_{\ AB'} \left(\partial_\mu \sigma^{\ AB'}_{\nu} + \Gamma^{A}_{\ \mu C} \sigma^{\ CB'}_{\nu} + \overline{\Gamma}^{B'}_{\ \mu C'} \sigma^{\ AC'}_{\nu} \right)$$

and the torsion of this *real* connection is defined as usual $T^{\kappa}_{\ \mu\nu} = 2\Gamma^{\kappa}_{\ [\mu\nu]}$. Hence, we have two dynamical fields $\sigma^{AB'}_{\mu}$ and $\Gamma^{A}_{\ \mu B}$. In order to better understand the geometry, we calculate $\left(\nabla_\mu \nabla_\nu - \nabla_\nu \nabla_\mu \right) X^A$, the latter equals:

$$\left(\nabla_\mu \nabla_\nu - \nabla_\nu \nabla_\mu \right) X^A = R^{A}_{\ B\mu\nu} X^B - T^{\kappa}_{\ \mu\nu} \nabla_\kappa X^A$$

where $R^{A}_{\ B\mu\nu} = 2 \left(\partial_{[\mu} \Gamma^{A}_{\ \nu]B} + \Gamma^{A}_{\ [\mu|C|} \Gamma^{C}_{\ \nu]B} \right)$. It is easily seen that $R_{AB\mu\nu}$ is symmetric in A and B and therefore

$$R^{A}_{\ A\mu\nu} = 0.$$

We now define the "bundle Ricci tensor and scalar" as follows $R_{B'A\nu} = R^{C}_{\ B\mu\nu} \sigma^{\mu}_{\ CB'}$ and $R = R^{B'A}_{\nu} \sigma^{\nu}_{\ AB'}$ where the raising of the indices has been done by means of ϵ^{AB}. Clearly, the Ricci tensor is also symmetric. The second Bianchi identities become:

$$R^{A}_{\ B[\mu\nu;\kappa]} + T^{\lambda}_{\ [\mu\nu} R^{A}_{\ |B\lambda|\kappa]} = 0$$

and

$$T^{\lambda}_{\ [\nu\kappa;\mu]} - T^{\gamma}_{\ [\nu\kappa} T^{\lambda}_{\ \mu]\gamma} - R^{\lambda}_{\ [\nu\kappa\mu]} = 0.$$

In order to define the gravitational action principle, we still need the totally antisymmetric symbol :

$$e_{AA'\,BB'\,CC'\,DD'} = i \left(\epsilon_{AB} \epsilon_{CD} \overline{\epsilon}_{A'C'} \overline{\epsilon}_{B'D'} - \overline{\epsilon}_{A'B'} \overline{\epsilon}_{C'D'} \epsilon_{AC} \epsilon_{BD} \right)$$

With $\sigma^{AA'} = \sigma^{AA'}_{\mu} dx^\mu$, the Ricci type gauge invariant action is given by

$$\mathcal{S} = \int e_{AA'\,BB'\,CC'\,DD'} \, \sigma^{AA'} \wedge \sigma^{BB'} \wedge \sigma^{CC'} \wedge \sigma^{DD'} R$$

or in more familiar form

$$\mathcal{S} = \int d^4x \ \mathrm{det}_e \left(\sigma^{AA'}_{\mu} \right) R.$$

[1]This constancy of ϵ_{AB} is not strictly necessary; it is sufficient to demand that $\nabla_\mu \epsilon_{AB} = i A_\mu \epsilon_{AB}$ with $A_\mu dx^\mu$ a real valued one form.

Variation of $\mathcal{S}$ produces:

$$\begin{aligned}\delta\mathcal{S} &= \int d^4x\,\det{}_\omega\left(\sigma^{BB'}_{\nu}\right)\Big[\left(2R^{A'A}_{\mu}-R\sigma^{AA'}_{\mu}\right)\delta\sigma^{\mu}{}_{AA'}+\Big[T^{\kappa}_{\mu\nu}\sigma^{\mu}{}_{AA'}\sigma^{\nu}{}_{CC'}\overline{\epsilon}^{A'C'}\epsilon^{BC}\\ &\quad +2T^{\mu}_{\mu\nu}\sigma^{\nu}{}_{(A|A'}\sigma^{\kappa}{}_{C)C'}\epsilon^{BC}\overline{\epsilon}^{A'C'}\Big]\delta\Gamma^{A}{}_{\kappa B}\Big].\end{aligned}$$

Variation with respect to $\sigma^{\mu}{}_{AA'}$ produces the usual "Einstein" equations, the other equations[2] are

$$T^{\kappa}_{\mu\nu}\sigma^{\mu}{}_{AA'}\sigma^{\nu}{}_{CC'}\overline{\epsilon}^{A'C'}+2T^{\mu}_{\mu\nu}\sigma^{\nu}{}_{(A|A'|}\sigma^{\kappa}{}_{C)C'}\overline{\epsilon}^{A'C'}=0.$$

Contracting the second Bianchi identity with $\sigma^{\mu}{}_{AA'}\overline{\epsilon}^{A'C'}\epsilon^{BC}\sigma^{\kappa}{}_{CC'}$ delivers:

$$2G^{A'A}_{\nu;\kappa}\sigma^{\kappa}{}_{AA'}-2T^{\lambda}_{\nu\mu}R^{A'A}_{\lambda}\sigma^{\mu}{}_{AA'}+T^{\lambda}_{\kappa\mu}\sigma^{\kappa}{}_{CC'}\sigma^{\mu}{}_{AA'}\overline{\epsilon}^{A'C'}R^{AC}_{\lambda\nu}=0.$$

In the sequel, it is convenient to know that

$$\delta\Gamma^{\kappa}{}_{\mu\nu}=-\sigma^{AA'}_{\nu}\nabla_{\mu}\delta\sigma^{\kappa}{}_{CC'}$$

and

$$\int d^4x\,\nabla_{\mu}\left(\det\left(\sigma^{AA'}_{\nu}\right)W^{\mu}\right) = \int d^4x\,\det\left(\sigma^{AA'}_{\nu}\right)T^{\mu}_{\mu\kappa}W^{\kappa}.$$

4 The Friedmann universe.

The aim of this section is to construct the class of homogeneous and isotropic universes on a R^4 topology for Ricci gravity as well as to introduce computational tools which facilitate the algebra involved. A more general treatment - that is an embedding of our theory into the more standard one - is left for the subsequent section. Our aim is to reproduce the metric $dt^2 - a^2(t)\left(dx^2+dy^2+dz^2\right)$. Denote the timelike "vector"

$$\frac{1}{\sqrt{2}}\left(\begin{pmatrix}1\\0\end{pmatrix}\otimes\begin{pmatrix}1\\0\end{pmatrix}+\begin{pmatrix}0\\1\end{pmatrix}\otimes\begin{pmatrix}0\\1\end{pmatrix}\right)$$

by the ket $|0>$. Denote by $\sigma_j \equiv \sigma_j\otimes 1$ the standard Pauli matrices which act upon W; that is $\sigma_j\sigma_k=-i\epsilon_{jkl}\sigma_l$, $\sigma_j^*=\sigma_j$ and $(\sigma_j)^2=1$ where the star denotes the Hermitian conjugate. Obviously, the real vectors in $W\otimes\overline{W}$ are spanned (over the reals) by $\sigma_j|0>$. Therefore, the solder form (which we denote by $\widetilde{\sigma}$) is given by $\widetilde{\sigma}_0=|0>$ and $\widetilde{\sigma}_j=a\,\sigma_j|0>$. The inverse symbol $\widetilde{\sigma}^{\mu}$ is constructed from the bra $<0|$ given by

$$<0|=\frac{1}{\sqrt{2}}\left(\begin{pmatrix}1 & 0\end{pmatrix}\otimes\begin{pmatrix}1 & 0\end{pmatrix}+\begin{pmatrix}0 & 1\end{pmatrix}\otimes\begin{pmatrix}0 & 1\end{pmatrix}\right).$$

[2]Remark that variation of the connection $\Gamma^{A}{}_{\mu B}$ is constrained : one has $\delta\Gamma^{A}{}_{\mu A}=0$. However, the trace of the associated equations of motion vanishes by antisymmetry of the torsion tensor; therefore, everything remains consistent.

It is easily verified that $< 0|0 >= 1$, $< 0|\sigma_i|0 >= 0$, $\widetilde{\sigma}^0 =< 0|$ and

$$\widetilde{\sigma}^j = a^{-1} < 0|\sigma_j.$$

In the sequel we denote Γ_μ for the "matrix" $\Gamma^A_{\mu B}$; demanding that the torsion tensor $T^\kappa_{\mu\nu}$ vanishes leads to the following equations :

$$\left(\Gamma_{[j}\sigma_{k]} \otimes 1 + \sigma_{[k} \otimes \overline{\Gamma}_{j]}\right)|0 >= 0$$
$$\left(\dot{a}\sigma_j \otimes 1 + a\left(\Gamma_0\sigma_j \otimes 1 + \sigma_j \otimes \overline{\Gamma}_0\right) - \Gamma_j\sigma_0 \otimes 1 - \sigma_0 \otimes \overline{\Gamma}_j\right)|0 >= 0$$

Putting $\Gamma_j = -\frac{\alpha\dot{a}}{2}\sigma_j$ with $\alpha \in R$ and using that $(\sigma_k \otimes 1)|0 >= (1 \otimes \overline{\sigma}_k)|0 >$ results in the satisfaction of the first equations since $\Gamma_{[j}\sigma_{k]} = i\frac{\alpha\dot{a}}{2}\epsilon_{jkl}\sigma_l$ is anti Hermitian. We shall drop the use of tensor products since it should be clear from the context which factor is referred to. Solving the second equations results in:

$$\Gamma_0 = -\frac{(1+\alpha)\dot{a}}{2a}\sigma_0.$$

Enforcing the constraints $\nabla_\mu \epsilon^{AB}$ leads to

$$(\Gamma_\mu \otimes 1 + 1 \otimes \Gamma_\mu)\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} - \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix}\right) = 0$$

implying that $\alpha = -1$. Since $R_{\mu\nu} = 2\left(\partial_{[\mu}\Gamma_{\nu]} + \Gamma_{[\mu}\Gamma_{\nu]}\right)$. It is easy to calculate that

$$R_{0j} = \frac{\ddot{a}}{2}\sigma_j$$
$$R_{jk} = \frac{\dot{a}^2}{4}[\sigma_j, \sigma_k] = -\frac{i\dot{a}^2}{2}\epsilon_{jkl}\sigma_l$$

At this moment, one can check that the second Bianchi identity $R_{[\mu\nu;\kappa]} = 0$ is satisfied. Calculating the Ricci tensor $R_{B'B\nu} = \widetilde{\sigma}^\mu{}_{AB'} R^A{}_{B\mu\nu}$ is equivalent to computing $< 0|\widetilde{\sigma}^\mu R_{\mu\nu}$. An elementary calculation learns that:

$$< 0|\widetilde{\sigma}^\mu R_{\mu j} = \frac{1}{2}\left(\frac{2\dot{a}^2}{a} + \ddot{a}\right) < 0|\sigma_j$$
$$< 0|\widetilde{\sigma}^\mu R_{\mu 0} = -\frac{3\ddot{a}}{2a} < 0|.$$

In order to calculate the Ricci scalar, we have to raise the two indices of the Ricci tensor. In our language, this boils down to taking the transpose and multiplying by a matrix $\omega \otimes \omega$ from the left. It is easy to calculate that $\omega\overline{\sigma}_j = -\sigma_j\omega$; therefore:

$$R = - < 0|\frac{3\ddot{a}}{2a}|0 > + \frac{1}{2a}\left(\frac{2\dot{a}^2}{a} + \ddot{a}\right) < 0|\sigma_j(\omega \otimes \omega)\sigma_j^T|0 > .$$

Further calculation - using that $(\omega \otimes \omega)|0 >= |0 >$ - results in

$$R = -\frac{3\ddot{a}}{a} - 3\left(\frac{\dot{a}}{a}\right)^2$$

We are now ready to compute the "Einstein tensor" $G_{\mu}^{A'A}$, the latter equals:

$$G_0 = \frac{3}{2}\left(\frac{\dot{a}}{a}\right)^2 |0>$$
$$G_j = \left(\frac{\dot{a}^2}{2a} + \ddot{a}\right)\sigma_j|0>$$

The "energy momentum" tensor is given by $T_0 = \rho\,|0>$ and $T_j = ap\,\sigma_j|0>$; the components of the space time connection are given by:

$$\Gamma^0{}_{00} = \Gamma^0{}_{0j} = \Gamma^0{}_{j0} = \Gamma^j{}_{00} = 0$$
$$\Gamma^0{}_{kj} = \delta_{kj}\,\dot{a}a$$
$$\Gamma^j_{k0} = \Gamma^j{}_{0k} = \frac{\dot{a}}{a}\,\delta^j_k$$
$$\Gamma^j_{kl} = 0$$

From this one can verify that $G^{A'A}_{\mu;\nu}\sigma^{\nu}{}_{AA'} = 0$ as it should since the torsion tensor vanishes. The Einstein equations are :

$$\rho = \frac{3}{2}\left(\frac{\dot{a}}{a}\right)^2$$
$$p = \left(\frac{\dot{a}^2}{2a^2} + \frac{\ddot{a}}{a}\right)$$

which are identical to the standard Friedmann equations in the flat case $k = 0$. The equation of motion for the fluid is

$$\dot{\rho} + 3(\rho - p)\frac{\dot{a}}{a} = 0.$$

5 Correspondence with Einstein Cartan theory.

The goal of this section is to study the connection with general relativity. As is customary in Einstein Cartan theory, we split the connection in the usual Levi Civita and torsion part. This is achieved by noticing that

$$0 = \nabla_{\mu}g_{\nu\kappa} = \partial_{\mu}g_{\nu\kappa} - \Gamma\ {}_{\kappa(\mu\nu)} - \Gamma_{\nu(\mu\kappa)} - \frac{1}{2}T_{\kappa\mu\nu} - \frac{1}{2}T_{\nu\mu\kappa}$$

where lowering and raising of the indices occurs with $g_{\mu\nu}$ and $g^{\mu\nu}$ respectively. By taking the combination $\nabla_{\mu}g_{\nu\kappa} + \nabla_{\nu}g_{\mu\kappa} - \nabla_{\kappa}g_{\mu\nu}$ one arrives at

$$\Gamma^{\kappa}{}_{(\mu\nu)} = \widetilde{\Gamma}^{\kappa}{}_{\mu\nu} + \frac{1}{2}\left(T_{\mu}{}^{\kappa}{}_{\nu} + T_{\nu}{}^{\kappa}{}_{\mu}\right)$$

and therefore

$$\Gamma^{\kappa}{}_{\mu\nu} = \widetilde{\Gamma}^{\kappa}{}_{\mu\nu} + K^{\kappa}{}_{\mu\nu}$$

where $K^{\kappa}_{\ \mu\nu} = \frac{1}{2}\left(T_{\mu\ \nu}^{\ \kappa} + T_{\nu\ \mu}^{\ \kappa} + T^{\kappa}_{\ \mu\nu}\right)$ is the contorsion tensor and

$$\widetilde{\Gamma}^{\kappa}_{\ \mu\nu} = \frac{1}{2} g^{\kappa\alpha}\left(\partial_\mu g_{\alpha\nu} + \partial_\nu g_{\mu\alpha} - \partial_\alpha g_{\mu\nu}\right)$$

the usual Levi Civita connection. The contorsion tensor satisfies $K_{\kappa\mu\nu} + K_{\nu\mu\kappa} = 0$; given that

$$\left(\nabla_\mu \nabla_\nu - \nabla_\nu \nabla_\mu\right) X^\alpha = R^{\alpha}_{\ \beta\mu\nu} X^\beta - T^{\beta}_{\ \mu\nu} \nabla_\beta X^\alpha$$

with $R^{\alpha}_{\ \beta\mu\nu} = 2\left(\partial_{[\mu}\Gamma^{\alpha}_{\ \nu]\beta} + \Gamma^{\alpha}_{\ [\mu|\kappa|}\Gamma^{\kappa}_{\ \nu]\beta}\right)$, one notices that the latter contains derivatives of the contorsion tensor and can be decomposed as

$$R^{\alpha}_{\ \beta\mu\nu} = \widetilde{R}^{\alpha}_{\ \beta\mu\nu} + 2\widetilde{\nabla}_{[\mu} K^{\alpha}_{\ \nu]\beta} + 2K^{\alpha}_{\ [\mu|\kappa|} K^{\kappa}_{\ \nu]\beta}.$$

It follows from the symmetry of the contorsion tensor that $R_{\alpha\beta\mu\nu} = R_{[\alpha\beta][\mu\nu]}$; however, the first Bianchi identity $R_{\alpha[\mu\nu\kappa]} = 0$ does not hold. As before, the second Bianchi identity is

$$R^{\alpha}_{\ \beta[\mu\nu;\kappa]} + T^{\gamma}_{\ [\mu\nu} R^{\alpha}_{\ |\beta|\kappa]\gamma} = 0$$

and the contracted identity equals

$$\nabla^\alpha G_{\alpha\nu} + T^{\gamma}_{\nu\kappa} R^{\kappa}_{\ \gamma} + \frac{1}{2} T^{\gamma}_{\alpha\kappa} R^{\alpha\kappa}_{\ \ \ \gamma\nu} = 0.$$

We now proceed by calculating the space time curvatures in terms of our new dynamical variables. The Riemann tensor is particularly simple

$$R^{\alpha}_{\ \beta\mu\nu} = \sigma^{\alpha}_{\ AB'} R^{A}_{\ C\mu\nu} \sigma^{CB'}_{\beta} + \sigma^{\alpha}_{\ AB'} \overline{R}^{B'}_{\ C'\mu\nu} \sigma^{AC'}_{\beta}$$

therefore, the Ricci tensor equals $R_{\mu\nu} = R_{A'A\nu}\sigma^{AA'}_{\mu} + \overline{R}_{AA'\nu}\sigma^{AA'}_{\mu}$ and the Ricci scalar is given by $R + \overline{R}$. In terms of the Levi Civita connection it is given by

$$R = \widetilde{R} + 2\widetilde{\nabla}_{[\alpha} K^{\alpha\ \beta}_{\ \beta]} + 2K^{\alpha}_{\ [\alpha|\kappa|} K^{\kappa\ \beta}_{\ \beta]}.$$

6 The matter part

Consider the matter Lagrangian for a single spinor field

$$\mathcal{S}_M \quad = \quad i\alpha \int d^4x \det\left(\sigma^{BB'}_{\kappa}\right)\left(\overline{\chi}^{A'} \sigma^{\mu}_{\ AA'} \nabla_\mu \chi^A - \nabla_\mu \overline{\chi}^{A'} \sigma^{\mu}_{\ AA'} \chi^A\right).$$

Variation of $\mathcal{S}_M$ results in:

$$\delta\mathcal{S}_M = i\alpha \int d^4x \det{}_e\left(\sigma^{CC'}_{\kappa}\right)\Big[\left(\overline{\chi}^{A'}\nabla_\mu\chi^A - \nabla_\mu\overline{\chi}^{A'}\chi^A - \sigma^{AA'}_{\mu}\left(\overline{\chi}^{B'}\sigma^{\nu}_{BB'}\nabla_\nu\chi^B - \nabla_\nu\overline{\chi}^{B'}\sigma^{\nu}_{BB'}\chi^B\right)\right)\delta\sigma^{\mu}_{AA'}$$
$$+\overline{\chi}^{A'}\sigma^{\mu}_{AA'}\chi^B\delta\Gamma^{A}_{\mu B} - \overline{\chi}^{B'}\sigma^{\mu}_{AA'}\chi^A\delta\overline{\Gamma}^{A'}_{\mu B'}\Big]$$

and the equation of motion for matter is given by

$$2\sigma^{\mu}_{AA'}\nabla_{\mu}\chi^{A} - T^{\mu}_{\mu\kappa}\sigma^{\kappa}_{AA'}\chi^{A} = 0.$$

Therefore[3],

$$G^{A'A}_{\mu} + \overline{G}^{A'A}_{\mu} = -\frac{i\,\alpha}{2}\left(\overline{\chi}^{A'}\nabla_{\mu}\chi^{A} - \nabla_{\mu}\overline{\chi}^{A'}\chi^{A}\right)$$

and

$$T^{\mu}_{\nu\kappa}\sigma^{\nu}{}_{AA'}\sigma^{\kappa}{}_{CC'}\overline{\epsilon}^{A'C'}\epsilon^{BC} + 2T^{\nu}_{\nu\kappa}\sigma^{\kappa}_{(A|A'}\sigma^{\mu}_{C)C'}\epsilon^{BC}\overline{\epsilon}^{A'C'} = -i\,\alpha\overline{\chi}^{A'}\sigma^{\mu}{}_{AA'}\chi^{B}.$$

It follows that

$$R = G^{A'A}_{\mu}\sigma^{\mu}{}_{AA'} = 0$$

but the Levi Civita curvature scalar is generically nonvanishing.

7 The Schwarzschild solution

The torsion tensor $T^{\kappa}_{\mu\nu}$ fixes the connection $\Gamma^{A}_{\mu B}$ given $\sigma^{AA'}_{\mu}$ since it determines 24 real independent equations in 24 (12 complex) real variables. It could be possible for gauge inequivalent solder forms to exist, determining the same spacetime metric but different connection theories with identical torsion tensor. Also, there could be a distinction between the complex and real vacuum theories. In this section we start from a particular choice of solder forms spanning the metric

$$g^2(r)dt^2 - f^2(r)dr^2 - r^2 d\Omega^2$$

and show that the vanishing of the torsion tensor fixes the gauge connection as it should. Moreover, the complex and real Ricci flatness conditions are the same and our solution is the static Schwarzschild black hole. Using the notation from section four, we pick

$$\begin{aligned}
\sigma_t &= g(r)|0> \\
\sigma_r &= f(r)\sigma_1|0> \\
\sigma_\theta &= r\sigma_2|0> \\
\sigma_\phi &= r\sin\theta\sigma_3|0>
\end{aligned}$$

A lengthy calculation shows that the vanishing of the torsion tensor implies that

$$\begin{aligned}
\Gamma_t &= \frac{g'(r)}{2f(r)}\sigma_1 \\
\Gamma_r &= 0 \\
\Gamma_\theta &= \frac{i}{2f(r)}\sigma_3 \\
\Gamma_\phi &= \frac{i\cos\theta}{2}\sigma_1 - \frac{i\sin\theta}{2f(r)}\sigma_2
\end{aligned}$$

[3]For the gravitational part we have to use the real action $\mathcal{S} + \overline{\mathcal{S}}$.

The vanishing of the *complex* Ricci tensor leads to the following equations:

$$\begin{aligned} \frac{g'(r)}{f(r)} &= \frac{a}{r^2} \\ \frac{1}{g(r)}\partial_r\left(\frac{g'(r)}{2f(r)}\right) + \frac{1}{r}\partial_r\left(\frac{1}{f(r)}\right) &= 0 \\ \frac{1}{f(r)}\partial_r\left(\frac{1}{f(r)}\right) - \frac{1}{r} + \frac{1}{rf^2(r)} + \frac{g'(r)}{g(r)f^2(r)} &= 0 \end{aligned}$$

where $a \geq 0$. From hereon, one can derive that

$$\frac{2}{f(r)}\partial_r\left(\frac{1}{f(r)}\right) - \frac{1}{r} + \frac{1}{rf^2(r)} = 0$$

which is easily solved to

$$f(r) = \left(1 + \frac{b}{r}\right)^{-\frac{1}{2}}$$

and

$$g(r) = -\frac{2a}{b}\left(1 + \frac{b}{r}\right)^{\frac{1}{2}}.$$

References

[1] R.M. Wald, General relativity, Chicago University Press, 1984.