Proving the Theorem of Wigner by an Exercise in Simple Geometry

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Abstract

The leading idea of this paper is to prove the theorem of Wigner with concepts and methods inspired by geometry. The exercise mentionned in the title has two functions: On the one hand it can serve as a pedagogical text in order to make the reader acquainted with the essential features of the theorem and its proof. On the other hand it will turn out to be the core of the general proof.

1 Introduction

The theorem of Wigner [1] is an important result of quantum mechanics, and this is its message: If one describes a physical system by the states of a Hilbert space and afterwards changes to another representation, then the invariance of transition probability is not only necessary for the correspondence with the experimental results but also sufficient for the existence of a transformation of the Hilbert space, which is either linear and unitary or antilinear and antiunitary. Apart from this physical meaning the theorem of Wigner can be understood as a purely mathematical theorem being concerned with Hilbert spaces. In this sense it is presented in this paper. In section 2 some notions are introduced, before in section 3 the assumptions and assertions of the theorem of Wigner can be written down. The exercise already announced in the title will be carried out in section 4. It is referred to affine geometry in three dimensions and will give an introduction to the essential ideas of the theorem. On the other hand it is the core of the general proof given in section 5. In section 6 the proof is compared with two other proofs that can be found in the literature.

2 Some conceptual preliminaries

Rays in a Hilbert space are one dimensional subspaces. A ray mapping σ is a bijective mapping from the set of all rays onto itself. In order to define the ray function u for two rays r and s first of all the expression

$$u(e,f) := \frac{<e,f><f,e>}{<e,e><f,f>}$$

is defined for two elements e and f generating the two rays r and s. Since $u(e,f)$ is independent of the special choice of e and f, the ray function u is defined for the rays r and s by setting

$$u(r,s) := u(e,f)$$

Orthogonality between two rays r and s is defined by the condition

$$u(r,s) = 0.$$

3 Assumptions and assertions of the theorem

The theorem of Wigner is presented in the following version:

Let be given a Hilbert space H over the field of complex numbers and a ray mapping σ. Moreover in H there may be an orthonormal basis B over a set I of indices. Then the following assertions are valid.

(a) If σ as well as its inverse mapping σ^{-1} conserves the orthogonality of rays, then there is a mapping φ from H to itself that can be described by the relations

$$x_i' = r_i f(x_i) \qquad i \,\epsilon\, I$$

between the coordinates x_i of an element x of H and the coordinates x_i' of the image x' of x under φ. The factor r_i is a complex number and f an automorphism on the field of complex numbers.

(c) If additionally the invariance of the ray function u is assumed, then

$$x_i' = f(x_i) \qquad i \,\epsilon\, I$$

In both cases the ray mapping σ is reproduced by the mapping φ. If the automorphism f is the identity, then φ is linear and unitary. If f is the transition to the complex conjugate, then φ is antilinear and antiunitary.

4 An exercise in elementary geometry as the kernel of the proof

4.1 A construction based on the conservation of orthogonality

In this section H may be a three dimensional vector space over the field of real numbers. The three basis vectors e_1, e_i, e_j of B generate three rays k_1, k_i, k_j that are constituting an orthonormal coordinate system K with coordinates x_1, x_i, x_j. If orthogonality is conserved under the ray mapping σ and its inverse σ^{-1}, then K is mapped by σ into a corresponding coordinate system K'. On the rays k'_1, k'_i, k'_j of K' three unit vectors e'_1, e'_i, e'_j are chosen, arbitrarily for the time being. The choice of the sign factors of the e'_i shall be held open until to the end of the proof. The coordinates in K' may be x'_1, x'_i, x'_j.

Now a construction in H can be performed.

Let E be the plane with the defining equation $x_1 = 1$. In K' the plane E' may be defined by $x'_1 = 1$. Then a mapping φ from E to E' is introduced by the following prescription: An arbitrary element e of E is connected with the origin of K by the ray s. Since s is not orthogonal to k_1, the image s' of s cannot be orthogonal to k'_1, because otherwise σ^{-1} would not conserve orthogonality. Thus s' intersects E' in an element e'. This element shall be the image e' of e. Vice versa to each element e' of E' uniquely an element e can be constructed. Hence φ is bijective.

4.2 Investigation of the mapping φ

In order to investigate the mapping φ a straight line g on E may be given by the equation

$$a_i x_i + a_j x_j + a = 0 \quad \text{with} \quad a_i^2 + a_j^2 > 0$$

The plane h given by the equation

$$a_i x_i + a_j x_j + a x_1 = 0$$

is connecting g with the origin of K. On the other hand h is orthogonal to the ray generated by the vector

$$v = a_i e_i + a_j e_j + a e_1$$

and thus consists of all rays orthogonal to v. Hence h is mapped by σ into a plane h' intersecting E' in a line g', which yields the equation

$$a'_i x'_i + a'_j x'_j + a' = 0 \quad \text{with} \quad a_i'^2 + a_j'^2 > 0$$

By construction line g' is the image of g under φ. Hence φ is a collineation from E to E'.

Variation of the parameter a will lead to a class of straight lines g all of whom being parallel to one another. The corresponding lines g' in E' are also parallel to one another, because otherwise a contradiction to the bijectivity of σ could be inferred. Hence the mapping φ is a collineation, which respects parallelity. Since the same is true for the inverse mapping φ^{-1}, three points can be found in E that don't lie on a straight line, such that their images are not lying in a straight line, too.

4.3 Investigation of the mapping φ to be continued

On the base of these propositions the fundamental theorem of affine geometry could be applied. But as, perhaps, the reader might not be familiar with this theorem, the single steps of the argumentation are given explicitly.

On a line g' parallel to the axis k'_i all elements have the same coordinate x'_i. The original line g of g' is parallel to k_i. All elements on it have the same coordinate x_i. Thus x'_i only depends on x_i and likewise x'_j only on x_j. Hence

$$\varphi: \; x'_i = f_i(x_i) \qquad x'_j = f_j(x_j)$$

with two functions f_i and f_j.

The line g' connecting the points $a'(1,0,0)$ and $b'(1, r_i, r_j)$ in E', which are the images of $a(1,0,0)$ and $b(1,1,1)$ in E, satisfy the relation

$$y_i = x'_i / r_i = x'_j / r_j = y_j$$

while the original g of g' connecting the points $a(1,0,0)$ and $b(1,1,1)$ yields the equation

$$x_i = x_j$$

Hence

$$\varphi: \; y_i = f(x_i) \qquad y_j = f(x_j) \quad \text{with} \quad y_i = x'_i / r_i \quad y_j = x'_j / r_j$$

with a sole function f. Because of $a' = \varphi(a)$ and $b' = \varphi(b)$ two special values of f are $f(0) = 0$ and $f(1) = 1$. The straight line with the equation $x_j = \lambda x_i + \mu$ is passing over to the straight line with an equation that can be written as $y_j = \lambda' y_i + \mu'$. Hence

$$f(\lambda x_i + \mu) = \lambda' f(x_i) + \mu'$$

Special cases are

$$f(\mu) = \mu' \quad \text{for} \quad x_i = 0$$

$$f(\lambda) = \lambda' \quad \text{for} \quad x_i = 1 \quad \mu = 0$$

with the result

$$f(\lambda x_i + \mu) = f(\lambda) f(x_i) + f(\mu)$$

Using the variables a und b this can be written as

$$f(a + b) = f(a) + f(b) \quad \text{for} \quad x_i = 1 \quad a = \lambda \quad b = \mu$$

$$f(a \cdot b) = f(a) \cdot f(b) \quad \text{for} \quad \mu = 0 \quad a = \lambda \quad b = x_i$$

Hence f is an automorphism of the field of real numbers.

Now the mapping φ can be extended to the whole space H by the simple prescription:

$$\varphi \,:\, e(x_1, x_i, x_j) \to e'(r_1 f(x_1), r_i f(x_i), r_j f(x_j)) \quad \text{with} \quad r_1 = 1$$

Of cause φ is an affine mapping, which reproduces σ.

4.4 The final part of the exercise

This all can be inferred only from the invariance of orthogonality thus justifying the assertion (a) of section 3 in a special case. In order to prove assertion (b) additionally the invariance of the function u is assumed. Application to $u(e_i, e_1 + x_i e_i)$ will yield

$$|x_i|^2/(1 + |x_i|^2) = |r_i f(x_i)|^2/(1 + |r_i f(x_i)|^2)$$

Thus

$$|r_i f(x_i)| = |x_i|$$

For the special case $x_i = 1$ this means

$$|r_i f(1)| = |1|$$

Hence the condition reduces to $|r_i| = 1$. One can get rid of the remaining sign factor by multiplying e'_i with a suited factor.

5 The general proof of the theorem

In order to prove the theorem of Wigner only a few parts of the exercise in section 3 are to be generalized.

First of all the field of real numbers, which was until now only preferred for the sake of illustration, can be substituted by the field of complex numbers. For this purpose one only needs to substitute 'real number' by 'complex number' and 'sign factor' by 'phase factor'.

In the Hilbert space H the elements e_i of the orthonormal basis B generate rays k_i playing the role of coordinate axes in a coordinate system K with coordinates x_i. The rays k_i are mutually orthogonal, and hence their images k'_i under σ, too. On each axis k'_i a unit vector e'_i is chosen. All e'_i together constitute a coordinate system K' with coordinates x'_i. The 'plane' E shall be the set of all elements of H with $x_1 = 1$ and 'plane' E' the set of all elements with $x'_1 = 1$.

A ray s connecting an arbitrary element e of E with the origin of K is not orthogonal to k_1. The image s' of s cannot be orthogonal to k'_1, because otherwise σ^{-1} would not conserve orthogonality. Thus s' intersects E' in an element e'. This element shall be the image e' of e.

If the ray mapping σ is confined to rays s that are generated by linear combinations of the three basis elements e_1, e_i, e_j, then these rays are orthogonal to all other elements of the basis B. Hence the image s' of s is generated by a linear combination of e'_1, e'_i, e'_j. That is to say, σ is a ray mapping between two vector spaces of dimension three. Although these spaces might be different from each other, the exercise of the last section can be applied in order to investigate the mapping φ. The result is

$$\varphi \,:\, x'_i = f(x_i)$$

with an automorphism f of the field of cmplex numbers. If the automorphism f is the identity, then φ is linear and unitary. If f is the transition to the complex conjugate, then φ is antilinear and antiunitary. In both cases the ray mapping σ is reproduced by the mapping φ. This completes the proof.

6 Comparison with other proofs

The beginning of the proof given in the last two sections is similar to the corresponding part in the proof of S. Weinberg [2]. But in contrast to the strategy that was pursued here the proof of Weinberg determines the phase factors already at an early stage. Thus Weinberg has to fight with a lot of problems as for instance with the discrimination and investigation of several cases. If the text written down in [2] is taken together with all footnotes and all calculations, whose explications are lacking, then the proof is rather complicated.

The proof of K.J. Keller et al. [4] and the ansatz persued in the present paper have in common that they associate an investigation of Hilbert spaces with analytical geometry and not, as usual, with functional analysis. But the combination with projective geometry in arbitrary dimensions seems to be a detour, especially because the main theorem of projective geometry can be reduced to the corresponding theorem of affine geometry. In the present paper it has been shown that simple geometry in three dimensions is sufficient for a proof of Wigner's theorem.

References

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