The differential coordinate transformation in the general relativity theory

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ABSTRACT

:In the general relativity theory's the gravity field,in the vacuum, save the motive particle's coordinate systems' the differential coordinate transformation.And study the velocity add formula in the general relativity theory

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I.Introduction

In the general relativity theory, Schwarzschild solution is

$$d\tau^2 = (1-\frac{2GM}{rc^2})dt^2 - \frac{1}{c^2}[\frac{dr^2}{1-\frac{2GM}{rc^2}} + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2] \quad (1)$$

$$d\tau^2 = (1-\frac{2GM}{rc^2})dt^2 - \frac{1}{c^2}[\frac{dr^2}{1-\frac{2GM}{rc^2}} + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2]$$

$$= (1-\frac{2GM}{rc^2})dt^2[1-\frac{1}{c^2}\{(\frac{dr}{dt})^2 \frac{1}{(1-\frac{2GM}{rc^2})^2} + r^2(\frac{d\theta}{dt})^2 \frac{1}{(1-\frac{2GM}{rc^2})}$$

$$+ r^2 \sin^2\theta (\frac{d\phi}{dt})^2 \frac{1}{(1-\frac{2GM}{rc^2})}\}]$$

$$= (1-\frac{2GM}{rc^2})dt^2[1-\frac{1}{c^2}(\overline{V}_r^{\,2} + \overline{V}_\theta^{\,2} + \overline{V}_\phi^{\,2})/c^2]$$

$$= (1-\frac{2GM}{rc^2})dt^2[1-\overline{V}^2/c^2],$$

$$\overline{V}_r = \frac{dr}{dt}\frac{1}{(1-\frac{2GM}{rc^2})}, \overline{V}_\theta = r\frac{d\theta}{dt}\frac{1}{\sqrt{1-\frac{2GM}{rc^2}}}, \overline{V}_\phi = r\sin\theta\frac{d\phi}{dt}\frac{1}{\sqrt{1-\frac{2GM}{rc^2}}}$$

$$\overline{V}^2 = \overline{V}_r^{\,2} + \overline{V}_\theta^{\,2} + \overline{V}_\phi^{\,2} \quad (2)$$

But the real velocity V is

$$d\tau^2 = (1-\frac{2GM}{rc^2} - \frac{V^2}{c^2})dt^2 \quad ,$$

$$V^2 = (\frac{ds}{dt})^2 = \frac{1}{1-\frac{2GM}{rc^2}}(\frac{dr}{dt})^2 + r^2(\frac{d\theta}{dt})^2 + r^2\sin^2\theta(\frac{d\phi}{dt})^2$$

$$= V_r^{\,2} + V_\theta^{\,2} + V_\phi^{\,2}$$

$$V_r = \frac{dr}{dt}\frac{1}{\sqrt{1-\frac{2GM}{rc^2}}}, V_\theta = r\frac{d\theta}{dt}, V_\phi = r\sin\theta\frac{d\phi}{dt}$$

$$ds^2 = \frac{1}{1-\frac{2GM}{rc^2}} dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2$$

Hence

$$\overline{V} = \frac{V}{\sqrt{1-\frac{2GM}{rc^2}}} \tag{3}$$

II.Additional chapter

In this case, think that a particle move.And think the coordinate systems by the particle's point and velocity.In this time, think the coordinate systems' transformation likely the special relativity theory.In the formula (2), on the coordinate system $S(t,r)$, move the other coordinate system $S'(t',r')$ by the real velocity component u_r, u_θ, u_ϕ,exist a differential coordinate transformation of the coordinate system $S(t,r)$ and the other coordinate system $S'(t',r')$ .In this time, the differential coordinate transformation of the coordinate system is likely the Lorentz transformation.

$$\sqrt{1-\frac{2GM}{rc^2}}dt = \gamma(\sqrt{1-\frac{2GM}{r'c^2}}dt' + \frac{\overline{u}_r}{c^2}\frac{dr'}{\sqrt{1-\frac{2GM}{r'c^2}}} + \frac{\overline{u}_\theta}{c^2}r'd\theta' + \frac{\overline{u}_\phi}{c^2}r'\sin\theta' d\phi'),$$

$$\frac{dr}{\sqrt{1-\frac{2GM}{rc^2}}} = \frac{dr'}{\sqrt{1-\frac{2GM}{r'c^2}}} + \gamma\overline{u}_r dt'\sqrt{1-\frac{2GM}{r'c^2}}$$

$$-(1-\gamma)\frac{1}{\overline{u}^2}(\overline{u}_r\frac{dr'}{\sqrt{1-\frac{2GM}{r'c^2}}} + \overline{u}_\theta r'd\theta' + \overline{u}_\phi r'\sin\theta' d\phi')\overline{u}_r,$$

$$rd\theta = r'd\theta' + \gamma\overline{u}_\theta dt'\sqrt{1-\frac{2GM}{r'c^2}}$$

$$-(1-\gamma)\frac{1}{\overline{u}^2}(\overline{u}_r\frac{dr'}{\sqrt{1-\frac{2GM}{r'c^2}}} + \overline{u}_\theta r'd\theta' + \overline{u}_\phi r'\sin\theta' d\phi')\overline{u}_\theta,$$

$$r\sin\theta d\phi = r'\sin\theta d\phi' + \gamma\overline{u}_\phi dt'\sqrt{1-\frac{2GM}{r'c^2}}$$

$$-(1-\gamma)\frac{1}{\overline{u}^2}(\overline{u}_r\frac{dr'}{\sqrt{1-\frac{2GM}{r'c^2}}} + \overline{u}_\theta r'd\theta' + \overline{u}_\phi r'\sin\theta' d\phi')\overline{u}_\phi,$$

$$\overline{u}_r = u_r/\sqrt{1-\frac{2GM}{rc^2}}, \overline{u}_\theta = u_\theta/\sqrt{1-\frac{2GM}{rc^2}}, \overline{u}_\phi = u_\phi/\sqrt{1-\frac{2GM}{rc^2}}, \overline{u} = u/\sqrt{1-\frac{2GM}{rc^2}}$$

$$\bar{u}^2 = \bar{u}_r^{\ 2} + \bar{u}_\theta^{\ 2} + \bar{u}_\phi^{\ 2} \ , \ u^2 = u_r^{\ 2} + u_\theta^{\ 2} + u_\phi^{\ 2}, \ \gamma = \frac{1}{\sqrt{1-\frac{\bar{u}^2}{c^2}}} \qquad (4)$$

The formula (4) insert the formula (1),

$$d\tau^2 = (1-\frac{2GM}{rc^2})dt^2 - \frac{1}{c^2}[\frac{dr^2}{1-\frac{2GM}{rc^2}} + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2]$$

$$= \gamma^2(\sqrt{1-\frac{2GM}{r'c^2}}dt' + \frac{\bar{u}_r}{c^2}\frac{dr'}{\sqrt{1-\frac{2GM}{r'c^2}}} + \frac{\bar{u}_\theta}{c^2}r'd\theta' + \frac{\bar{u}_\phi}{c^2}r'\sin\theta' d\phi')^2$$

$$-\frac{1}{c^2}[\frac{dr'}{\sqrt{1-\frac{2GM}{r'c^2}}} + \gamma\bar{u}_r dt'\sqrt{1-\frac{2GM}{r'c^2}}$$

$$-(1-\gamma)\frac{1}{\bar{u}^2}(\bar{u}_r\frac{dr'}{\sqrt{1-\frac{2GM}{r'c^2}}} + \bar{u}_\theta r'd\theta' + \bar{u}_\phi r'\sin\theta' d\phi')\bar{u}_r]^2$$

$$-\frac{1}{c^2}[r'd\theta' + \gamma\bar{u}_\theta dt'\sqrt{1-\frac{2GM}{r'c^2}}$$

$$-(1-\gamma)\frac{1}{\bar{u}^2}(\bar{u}_r\frac{dr'}{\sqrt{1-\frac{2GM}{r'c^2}}} + \bar{u}_\theta r'd\theta' + \bar{u}_\phi r'\sin\theta' d\phi')\bar{u}_\theta]^2$$

$$-\frac{1}{c^2}[r'\sin\theta d\phi' + \gamma\bar{u}_\phi dt'\sqrt{1-\frac{2GM}{r'c^2}}$$

$$-(1-\gamma)\frac{1}{\bar{u}^2}(\bar{u}_r\frac{dr'}{\sqrt{1-\frac{2GM}{r'c^2}}} + \bar{u}_\theta r'd\theta' + \bar{u}_\phi r'\sin\theta' d\phi')\bar{u}_\phi]^2$$

$$= (1-\frac{2GM}{r'c^2})dt'^2 - \frac{1}{c^2}[\frac{dr'^2}{1-\frac{2GM}{r'c^2}} + r'^2 d\theta'^2 + r'^2\sin^2\theta' d\phi'^2]$$

$$\bar{u}_r = u_r/\sqrt{1-\frac{2GM}{rc^2}}, \bar{u}_\theta = u_\theta/\sqrt{1-\frac{2GM}{rc^2}}, \bar{u}_\phi = u_\phi/\sqrt{1-\frac{2GM}{rc^2}}, \bar{u} = u/\sqrt{1-\frac{2GM}{rc^2}}$$

$$\bar{u}^2 = \bar{u}_r^{\ 2} + \bar{u}_\theta^{\ 2} + \bar{u}_\phi^{\ 2} \ , \ u^2 = u_r^{\ 2} + u_\theta^{\ 2} + u_\phi^{\ 2}, \ \gamma = \frac{1}{\sqrt{1-\frac{\bar{u}^2}{c^2}}} \qquad (5)$$

If $d\theta = d\phi = 0$ is in the formula (1), treat the light,

$$d\tau^2 = (1-\frac{2GM}{rc^2})dt^2 - \frac{1}{c^2}\frac{dr^2}{1-\frac{2GM}{rc^2}} = 0\,, d\theta = d\phi = 0$$

$$cdt = \frac{dr}{(1-\frac{2GM}{rc^2})},\ \ ct = r + \frac{2GM}{c^2}\ln|r - \frac{2GM}{c^2}| \quad (6)$$

In the r-axis, in the motive coordinate system $S(t,r)$ and $S'(t',r')$, Energy-momentum transformation is by the formula (4)

$$E = m_0c^2\frac{dt}{d\tau},\ \ E' = m_0c^2\frac{dt'}{d\tau}$$

$$p^r = m_0\frac{dr}{d\tau},\ \ p'^r = m_0\frac{dr'}{d\tau},\ \ p^\theta = m_0 r\frac{d\theta}{d\tau}, p'^\theta = m_0 r'\frac{d\theta'}{d\tau}$$

$$p^\phi = m_0 r\sin\theta\frac{d\phi}{d\tau},\qquad p'^\phi = m_0 r'\sin\theta'\frac{d\phi'}{d\tau}$$

$$\sqrt{1-\frac{2GM}{rc^2}}E = \gamma(\sqrt{1-\frac{2GM}{r'c^2}}E' + \frac{\bar{u}_r p'^r}{\sqrt{1-\frac{2GM}{r'c^2}}} + \bar{u}_\theta p'^\theta + \bar{u}_\phi p'^\phi),$$

$$\frac{p^r}{\sqrt{1-\frac{2GM}{rc^2}}} = \frac{p'^r}{\sqrt{1-\frac{2GM}{r'c^2}}} + \gamma\frac{\bar{u}_r}{c^2}E'\sqrt{1-\frac{2GM}{r'c^2}}$$

$$-(1-\gamma)\frac{1}{\bar{u}^2}(\bar{u}_r\frac{p'^r}{\sqrt{1-\frac{2GM}{r'c^2}}} + \bar{u}_\theta p'^\theta + \bar{u}_\phi p'^\phi)\bar{u}_r,$$

$$p^\theta = p'^\theta + \gamma\frac{\bar{u}_\theta}{c^2}E'\sqrt{1-\frac{2GM}{r'c^2}}$$

$$-(1-\gamma)\frac{1}{\bar{u}^2}(\bar{u}_r\frac{p'^r}{\sqrt{1-\frac{2GM}{r'c^2}}} + \bar{u}_\theta p'^\theta + \bar{u}_\phi p'^\phi)\bar{u}_\theta,$$

$$p^\phi = p'^\phi + \gamma\frac{\bar{u}_\phi}{c^2}E'\sqrt{1-\frac{2GM}{r'c^2}}$$

$$-(1-\gamma)\frac{1}{\bar{u}^2}(\bar{u}_r\frac{p'^r}{\sqrt{1-\frac{2GM}{r'c^2}}} + \bar{u}_\theta p'^\theta + \bar{u}_\phi p'^\phi)\bar{u}_\phi,$$

$$\bar{u}_r = u_r/\sqrt{1-\frac{2GM}{rc^2}}\,, \bar{u}_\theta = u_\theta/\sqrt{1-\frac{2GM}{rc^2}}\,, \bar{u}_\phi = u_\phi/\sqrt{1-\frac{2GM}{rc^2}}\,, \bar{u} = u/\sqrt{1-\frac{2GM}{rc^2}}$$

$$\bar{u}^2 = \bar{u}_r^{\ 2} + \bar{u}_\theta^{\ 2} + \bar{u}_\phi^{\ 2} \ , \ u^2 = u_r^{\ 2} + u_\theta^{\ 2} + u_\phi^{\ 2}, \ \gamma = \frac{1}{\sqrt{1-\frac{\bar{u}^2}{c^2}}} \quad (7)$$

$$(1-\frac{2GM}{rc^2})E^2 - \frac{1}{1-\frac{2GM}{rc^2}}(p^r)^2c^2 - (p^\theta)^2c^2 - (p^\phi)^2c^2$$

$$= (1-\frac{2GM}{rc^2})m_0^{\ 2}c^4(\frac{dt}{d\tau})^2 - \frac{1}{1-\frac{2GM}{rc^2}}m_0^{\ 2}c^2(\frac{dr}{d\tau})^2 - m_0^{\ 2}c^2r^2(\frac{d\theta}{d\tau})^2$$

$$- m_0^{\ 2}c^2r^2\sin^2\theta(\frac{d\phi}{d\tau})^2$$

$$= m_0^{\ 2}c^4[(1-\frac{2GM}{rc^2})dt^2 - \frac{1}{c^2}\{\frac{dr^2}{1-\frac{2GM}{rc^2}} + r^2d\theta^2 + r^2\sin^2\theta d\phi^2\}]/d\tau^2$$

$$= m_0^{\ 2}c^4 \ , E = m_0c^2\frac{dt}{d\tau}, p^r = m_0\frac{dr}{d\tau}, p^\theta = m_0 r\frac{d\theta}{d\tau}, p^\phi = m_0 r\sin\theta\frac{d\phi}{d\tau} \quad (8)$$

$$m_0^{\ 2}c^4 = (1-\frac{2GM}{rc^2})E^2 - \frac{1}{1-\frac{2GM}{rc^2}}(p^r)^2c^2 - (p^\theta)^2c^2 - (p^\phi)^2c^2$$

$$= \gamma^2(\sqrt{1-\frac{2GM}{r'c^2}}E' + \frac{\bar{u}_r p'^r}{\sqrt{1-\frac{2GM}{r'c^2}}} + \bar{u}_\theta p'^\theta + \bar{u}_\phi p'^\phi)^2$$

$$- c^2[\frac{p'^r}{\sqrt{1-\frac{2GM}{r'c^2}}} + \gamma\frac{\bar{u}_r}{c^2}E'\sqrt{1-\frac{2GM}{r'c^2}}$$

$$- (1-\gamma)\frac{1}{\bar{u}^2}(\bar{u}_r\frac{p'^r}{\sqrt{1-\frac{2GM}{r'c^2}}} + \bar{u}_\theta p'^\theta + \bar{u}_\phi p'^\phi)\bar{u}_r]^2$$

$$- c^2[p'^\theta + \gamma\frac{\bar{u}_\theta}{c^2}E'\sqrt{1-\frac{2GM}{r'c^2}}$$

$$- (1-\gamma)\frac{1}{\bar{u}^2}(\bar{u}_r\frac{p'^r}{\sqrt{1-\frac{2GM}{r'c^2}}} + \bar{u}_\theta p'^\theta + \bar{u}_\phi p'^\phi)\bar{u}_\theta]^2,$$

$$- c^2[p'^\phi + \gamma\frac{\bar{u}_\phi}{c^2}E'\sqrt{1-\frac{2GM}{r'c^2}}$$

$$-(1-\gamma)\frac{1}{\bar{u}^2}(\bar{u}_r\frac{p'^r}{\sqrt{1-\frac{2GM}{r'c^2}}}+\bar{u}_\theta p'^\theta+\bar{u}_\phi p'^\phi)\bar{u}_\phi]^2$$

$$=(1-\frac{2GM}{r'c^2})E'^2-\frac{1}{1-\frac{2GM}{r'c^2}}(p'^r)^2c^2-(p'^\theta)^2c^2-(p'^\phi)^2c^2$$

$$E=m_0c^2\frac{dt}{d\tau},p^r=m_0\frac{dr}{d\tau},p^\theta=m_0r\frac{d\theta}{d\tau},p^\phi=m_0r\sin\theta\frac{d\phi}{d\tau}$$

$$\bar{u}_r=u_r/\sqrt{1-\frac{2GM}{rc^2}},\bar{u}_\theta=u_\theta/\sqrt{1-\frac{2GM}{rc^2}},\bar{u}_\phi=u_\phi/\sqrt{1-\frac{2GM}{rc^2}},\bar{u}=u/\sqrt{1-\frac{2GM}{rc^2}}$$

$$\bar{u}^2=\bar{u}_r^{\ 2}+\bar{u}_\theta^{\ 2}+\bar{u}_\phi^{\ 2}\ ,\ u^2=u_r^{\ 2}+u_\theta^{\ 2}+u_\phi^{\ 2},\ \gamma=\frac{1}{\sqrt{1-\frac{\bar{u}^2}{c^2}}}\qquad(9)$$

In the r-axis, in the motive coordinate system $S(t,r)$ and $S'(t',r')$, by the formula (4) that if $d\theta=d\phi=0,d\theta'=d\phi'=0,u_r=u,\bar{u}_r=\bar{u},u_\theta=u_\phi=0$ is, the velocity add formula is

$$\frac{dr}{\sqrt{1-\frac{2GM}{rc^2}}}=\gamma(\frac{dr'}{\sqrt{1-\frac{2GM}{r'c^2}}}+\bar{u}dt'\sqrt{1-\frac{2GM}{r'c^2}})$$

$$\sqrt{1-\frac{2GM}{rc^2}}dt=\gamma(\sqrt{1-\frac{2GM}{r'c^2}}dt'+\frac{\bar{u}}{c^2}\frac{dr'}{\sqrt{1-\frac{2GM}{r'c^2}}})$$

$$\bar{u}=u/\sqrt{1-\frac{2GM}{rc^2}}\qquad,\gamma=\frac{1}{\sqrt{1-\frac{\bar{u}^2}{c^2}}}\quad,$$

$$d\theta=d\phi=0,\ d\theta'=d\phi'=0,u_r=u,\bar{u}_r=\bar{u},u_\theta=u_\phi=0$$

$$\frac{dr}{dt}\frac{1}{(1-\frac{2GM}{rc^2})}=\frac{\frac{dr'}{\sqrt{1-\frac{2GM}{r'c^2}}}+\bar{u}dt'\sqrt{1-\frac{2GM}{r'c^2}}}{dt'\sqrt{1-\frac{2GM}{r'c^2}}+\frac{\bar{u}}{c^2}\frac{dr'}{\sqrt{1-\frac{2GM}{r'c^2}}}}$$

$$=\frac{\dfrac{dr'}{\sqrt{1-\dfrac{2GM}{r'c^2}}}+\dfrac{u}{\sqrt{1-\dfrac{2GM}{rc^2}}}dt'\sqrt{1-\dfrac{2GM}{r'c^2}}}{dt'\sqrt{1-\dfrac{2GM}{r'c^2}}+\dfrac{1}{c^2}\dfrac{u}{\sqrt{1-\dfrac{2GM}{rc^2}}}\dfrac{dr'}{\sqrt{1-\dfrac{2GM}{r'c^2}}}}$$

$$=\frac{\dfrac{dr'}{\sqrt{1-\dfrac{2GM}{r'c^2}}}+\dfrac{u/\sqrt{1-\dfrac{2GM}{rc^2}}}{\sqrt{1-\dfrac{2GM}{r'c^2}}}dt'(1-\dfrac{2GM}{r'c^2})}{dt'\dfrac{(1-\dfrac{2GM}{r'c^2})}{\sqrt{1-\dfrac{2GM}{r'c^2}}}+\dfrac{1}{c^2}\dfrac{dr'}{\sqrt{1-\dfrac{2GM}{r'c^2}}}u/\sqrt{1-\dfrac{2GM}{rc^2}}}$$

$$=\frac{dr'+\dfrac{u}{\sqrt{1-\dfrac{2GM}{rc^2}}}dt'(1-\dfrac{2GM}{r'c^2})}{dt'(1-\dfrac{2GM}{r'c^2})+\dfrac{dr'}{c^2}\dfrac{u}{\sqrt{1-\dfrac{2GM}{rc^2}}}}$$

$$d\theta = d\phi = 0\,, d\theta' = d\phi' = 0\,, u_r = u\,, \overline{u}_r = \overline{u}\,, u_\theta = u_\phi = 0 \quad (10)$$

III.Conclusion

Hence, in the general relativity theory, the velocity add formula is that if $d\theta = d\phi = 0$ $, d\theta' = d\phi' = 0\,, u_r = u\,, u_\theta = u_\phi = 0$ is

$$\overline{V}_r = \overline{V} = \frac{V_r}{\sqrt{1-\dfrac{2GM}{rc^2}}} = \frac{V}{\sqrt{1-\dfrac{2GM}{rc^2}}} = \frac{dr}{dt}\frac{1}{(1-\dfrac{2GM}{rc^2})}$$

$$=\frac{\dfrac{dr'}{dt'}\dfrac{1}{(1-\dfrac{2GM}{r'c^2})}+\dfrac{u}{\sqrt{1-\dfrac{2GM}{rc^2}}}}{1+\dfrac{1}{c^2}\cdot\dfrac{dr'}{dt'}\dfrac{1}{(1-\dfrac{2GM}{r'c^2})}\dfrac{u}{\sqrt{1-\dfrac{2GM}{rc^2}}}}$$

$$=\frac{\dfrac{V'}{\sqrt{1-\dfrac{2GM}{r'c^2}}}+\dfrac{u}{\sqrt{1-\dfrac{2GM}{rc^2}}}}{1+\dfrac{1}{c^2}\cdot\dfrac{V'}{\sqrt{1-\dfrac{2GM}{r'c^2}}}\dfrac{u}{\sqrt{1-\dfrac{2GM}{rc^2}}}} \quad (11)$$

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