The differential coordinate transformation in the general relativity theory

Sangwha-Yi
Department of Math , Taejon University 300-716

ABSTRACT

:In the general relativity theory's the gravity field,in the vacuum, save the motive particle's coordinate systems' the differential coordinate transformation.And study the velocity add formula in the general relativity theory

PACS Number:**04,04.90.+e,04.20.-q**

Key words:The general relativity theory,

The differential coordinate transformation

e-mail address:sangwha1@**nate.com**

Tel:051-624-3953

I.Introduction

In the general relativity theory, Schwarzschild solution is

$$d\tau^2 = (1-\frac{2GM}{rc^2})dt^2 - \frac{1}{c^2}[\frac{dr^2}{1-\frac{2GM}{rc^2}} + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2] \quad (1)$$

$$d\tau^2 = (1-\frac{2GM}{rc^2})dt^2 - \frac{1}{c^2}[\frac{dr^2}{1-\frac{2GM}{rc^2}} + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2]$$

$$= (1-\frac{2GM}{rc^2})dt^2[1-\frac{1}{c^2}\{(\frac{dr}{dt})^2 \frac{1}{(1-\frac{2GM}{rc^2})^2} + r^2(\frac{d\theta}{dt})^2 \frac{1}{(1-\frac{2GM}{rc^2})}$$

$$+ r^2 \sin^2\theta (\frac{d\phi}{dt})^2 \frac{1}{(1-\frac{2GM}{rc^2})}\}]$$

$$= (1-\frac{2GM}{rc^2})dt^2[1-\frac{1}{c^2}(\overline{V}_r^{\,2} + \overline{V}_\theta^{\,2} + \overline{V}_\phi^{\,2})/c^2]$$

$$= (1-\frac{2GM}{rc^2})dt^2[1-\overline{V}^2/c^2],$$

$$\overline{V}_r = \frac{dr}{dt}\frac{1}{(1-\frac{2GM}{rc^2})}, \quad \overline{V}_\theta = r\frac{d\theta}{dt}\frac{1}{\sqrt{1-\frac{2GM}{rc^2}}}, \quad \overline{V}_\phi = r\sin\theta\frac{d\phi}{dt}\frac{1}{\sqrt{1-\frac{2GM}{rc^2}}}$$

$$\overline{V}^2 = \overline{V}_r^{\,2} + \overline{V}_\theta^{\,2} + \overline{V}_\phi^{\,2} \quad (2)$$

But the real velocity V is

$$d\tau^2 = (1-\frac{2GM}{rc^2} - \frac{V^2}{c^2})dt^2 \quad ,$$

$$V^2 = (\frac{ds}{dt})^2 = \frac{1}{1-\frac{2GM}{rc^2}}(\frac{dr}{dt})^2 + r^2(\frac{d\theta}{dt})^2 + r^2\sin^2\theta(\frac{d\phi}{dt})^2$$

$$= V_r^{\,2} + V_\theta^{\,2} + V_\phi^{\,2}$$

$$V_r = \frac{dr}{dt}\frac{1}{\sqrt{1-\frac{2GM}{rc^2}}}, V_\theta = r\frac{d\theta}{dt}, V_\phi = r\sin\theta\frac{d\phi}{dt}$$

$$ds^2 = \frac{1}{1-\frac{2GM}{rc^2}}dr^2 + r^2 d\theta^2 + r^2\sin^2\theta d\phi^2$$

Hence

$$\overline{V} = \frac{V}{\sqrt{1-\frac{2GM}{rc^2}}} \qquad (3)$$

II.Additional chapter

In this case, think that a particle move.And think the coordinate systems by the particle's point and velocity.In this time, think the coordinate systems' transformation likely the special relativity theory.In the formula (2), on the coordinate system $S(t,r)$, move the other coordinate system $S'(t',r')$ by the real velocity u 's the component, u_r, u_θ, u_ϕ ,exist a differential coordinate transformation of the coordinate system $S(t,r)$ and the other coordinate system $S'(t',r')$ .In this time, the differential coordinate transformation of the coordinate system is likely the Lorentz transformation.

$$\sqrt{1-\frac{2GM}{rc^2}}dt = \gamma\left(\sqrt{1-\frac{2GM}{r'c^2}}dt' + \frac{\overline{u}_r}{c^2}\frac{dr'}{\sqrt{1-\frac{2GM}{r'c^2}}} + \frac{\overline{u}_\theta}{c^2}r'd\theta' + \frac{\overline{u}_\phi}{c^2}r'\sin\theta' d\phi'\right),$$

$$\frac{dr}{\sqrt{1-\frac{2GM}{rc^2}}} = \frac{dr'}{\sqrt{1-\frac{2GM}{r'c^2}}} + \gamma\overline{u}_r dt'\sqrt{1-\frac{2GM}{r'c^2}}$$
$$-(1-\gamma)\frac{1}{\overline{u}^2}\left(\overline{u}_r\frac{dr'}{\sqrt{1-\frac{2GM}{r'c^2}}} + \overline{u}_\theta r'd\theta' + \overline{u}_\phi r'\sin\theta' d\phi'\right)\overline{u}_r,$$

$$rd\theta = r'd\theta' + \gamma\overline{u}_\theta dt'\sqrt{1-\frac{2GM}{r'c^2}}$$
$$-(1-\gamma)\frac{1}{\overline{u}^2}\left(\overline{u}_r\frac{dr'}{\sqrt{1-\frac{2GM}{r'c^2}}} + \overline{u}_\theta r'd\theta' + \overline{u}_\phi r'\sin\theta' d\phi'\right)\overline{u}_\theta,$$

$$r\sin\theta d\phi = r'\sin\theta d\phi' + \gamma\overline{u}_\phi dt'\sqrt{1-\frac{2GM}{r'c^2}}$$
$$-(1-\gamma)\frac{1}{\overline{u}^2}\left(\overline{u}_r\frac{dr'}{\sqrt{1-\frac{2GM}{r'c^2}}} + \overline{u}_\theta r'd\theta' + \overline{u}_\phi r'\sin\theta' d\phi'\right)\overline{u}_\phi,$$

$$\overline{u}_r = u_r/\sqrt{1-\frac{2GM}{rc^2}},\ \overline{u}_\theta = u_\theta/\sqrt{1-\frac{2GM}{rc^2}},\ \overline{u}_\phi = u_\phi/\sqrt{1-\frac{2GM}{rc^2}},\ \overline{u} = u/\sqrt{1-\frac{2GM}{rc^2}}$$

$$\overline{u}^2 = \overline{u}_r^{\ 2} + \overline{u}_\theta^{\ 2} + \overline{u}_\phi^{\ 2},\ u^2 = u_r^{\ 2} + u_\theta^{\ 2} + u_\phi^{\ 2},\ \gamma = \frac{1}{\sqrt{1-\frac{\overline{u}^2}{c^2}}} \qquad (4)$$

The formula (4) insert the formula (1),

$$d\tau^2 = (1-\frac{2GM}{rc^2})dt^2 - \frac{1}{c^2}[\frac{dr^2}{1-\frac{2GM}{rc^2}} + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2]$$

$$= \gamma^2(\sqrt{1-\frac{2GM}{r'c^2}}dt' + \frac{\overline{u}_r}{c^2}\frac{dr'}{\sqrt{1-\frac{2GM}{r'c^2}}} + \frac{\overline{u}_\theta}{c^2}r'd\theta' + \frac{\overline{u}_\phi}{c^2}r'\sin\theta' d\phi')^2$$

$$-\frac{1}{c^2}[\frac{dr'}{\sqrt{1-\frac{2GM}{r'c^2}}} + \gamma\overline{u}_r dt'\sqrt{1-\frac{2GM}{r'c^2}}$$

$$-(1-\gamma)\frac{1}{\overline{u}^2}(\overline{u}_r\frac{dr'}{\sqrt{1-\frac{2GM}{r'c^2}}} + \overline{u}_\theta r'd\theta' + \overline{u}_\phi r'\sin\theta' d\phi')\overline{u}_r]^2$$

$$-\frac{1}{c^2}[r'd\theta' + \gamma\overline{u}_\theta dt'\sqrt{1-\frac{2GM}{r'c^2}}$$

$$-(1-\gamma)\frac{1}{\overline{u}^2}(\overline{u}_r\frac{dr'}{\sqrt{1-\frac{2GM}{r'c^2}}} + \overline{u}_\theta r'd\theta' + \overline{u}_\phi r'\sin\theta' d\phi')\overline{u}_\theta]^2$$

$$-\frac{1}{c^2}[r'\sin\theta d\phi' + \gamma\overline{u}_\phi dt'\sqrt{1-\frac{2GM}{r'c^2}}$$

$$-(1-\gamma)\frac{1}{\overline{u}^2}(\overline{u}_r\frac{dr'}{\sqrt{1-\frac{2GM}{r'c^2}}} + \overline{u}_\theta r'd\theta' + \overline{u}_\phi r'\sin\theta' d\phi')\overline{u}_\phi]^2$$

$$= (1-\frac{2GM}{r'c^2})dt'^2 - \frac{1}{c^2}[\frac{dr'^2}{1-\frac{2GM}{r'c^2}} + r'^2 d\theta'^2 + r'^2\sin^2\theta' d\phi'^2]$$

$$\overline{u}_r = u_r/\sqrt{1-\frac{2GM}{rc^2}}, \overline{u}_\theta = u_\theta/\sqrt{1-\frac{2GM}{rc^2}}, \overline{u}_\phi = u_\phi/\sqrt{1-\frac{2GM}{rc^2}}, \overline{u} = u/\sqrt{1-\frac{2GM}{rc^2}}$$

$$\overline{u}^2 = \overline{u}_r^2 + \overline{u}_\theta^2 + \overline{u}_\phi^2, \quad u^2 = u_r^2 + u_\theta^2 + u_\phi^2, \quad \gamma = \frac{1}{\sqrt{1-\frac{\overline{u}^2}{c^2}}} \quad (5)$$

If $d\theta = d\phi = 0$ is in the formula (1), treat the light,

$$d\tau^2 = (1-\frac{2GM}{rc^2})dt^2 - \frac{1}{c^2}\frac{dr^2}{1-\frac{2GM}{rc^2}} = 0, d\theta = d\phi = 0$$

$$cdt = \frac{dr}{(1-\frac{2GM}{rc^2})}\,,\ ct = r + \frac{2GM}{c^2}\ln|r - \frac{2GM}{c^2}| \quad (6)$$

In the r-axis, in the motive coordinate system $S(t,r)$ and $S'(t',r')$, by the formula (4) that if $d\theta = d\phi = 0$, $d\theta' = d\phi' = 0$, $u_r = u$, $\bar{u}_r = \bar{u}$, $u_\theta = u_\phi = 0$ is, the velocity add formula is

$$\frac{dr}{\sqrt{1-\frac{2GM}{rc^2}}} = \gamma\left(\frac{dr'}{\sqrt{1-\frac{2GM}{r'c^2}}} + \bar{u}dt'\sqrt{1-\frac{2GM}{r'c^2}}\right)$$

$$\sqrt{1-\frac{2GM}{rc^2}}dt = \gamma\left(\sqrt{1-\frac{2GM}{r'c^2}}dt' + \frac{\bar{u}}{c^2}\frac{dr'}{\sqrt{1-\frac{2GM}{r'c^2}}}\right)$$

$$\bar{u} = u/\sqrt{1-\frac{2GM}{rc^2}}\qquad,\quad \gamma = \frac{1}{\sqrt{1-\frac{\bar{u}^2}{c^2}}}$$

$$d\theta = d\phi = 0,\ d\theta' = d\phi' = 0,\ u_r = u,\ \bar{u}_r = \bar{u},\ u_\theta = u_\phi = 0 \quad (7)$$

$$\frac{dr}{dt}\frac{1}{(1-\frac{2GM}{rc^2})} = \frac{\frac{dr'}{\sqrt{1-\frac{2GM}{r'c^2}}} + \bar{u}dt'\sqrt{1-\frac{2GM}{r'c^2}}}{dt'\sqrt{1-\frac{2GM}{r'c^2}} + \frac{\bar{u}}{c^2}\frac{dr'}{\sqrt{1-\frac{2GM}{r'c^2}}}}$$

$$= \frac{\frac{dr'}{\sqrt{1-\frac{2GM}{r'c^2}}} + \frac{u}{\sqrt{1-\frac{2GM}{rc^2}}}dt'\sqrt{1-\frac{2GM}{r'c^2}}}{dt'\sqrt{1-\frac{2GM}{r'c^2}} + \frac{1}{c^2}\frac{u}{\sqrt{1-\frac{2GM}{rc^2}}}\frac{dr'}{\sqrt{1-\frac{2GM}{r'c^2}}}}$$

$$= \frac{\frac{dr'}{\sqrt{1-\frac{2GM}{r'c^2}}} + \frac{u/\sqrt{1-\frac{2GM}{rc^2}}}{\sqrt{1-\frac{2GM}{r'c^2}}} dt'(1-\frac{2GM}{r'c^2})}{dt'\frac{(1-\frac{2GM}{r'c^2})}{\sqrt{1-\frac{2GM}{r'c^2}}} + \frac{1}{c^2}\frac{dr'}{\sqrt{1-\frac{2GM}{r'c^2}}} u/\sqrt{1-\frac{2GM}{rc^2}}}$$

$$= \frac{dr' + \frac{u}{\sqrt{1-\frac{2GM}{rc^2}}} dt'(1-\frac{2GM}{r'c^2})}{dt'(1-\frac{2GM}{r'c^2}) + \frac{dr'}{c^2}\frac{u}{\sqrt{1-\frac{2GM}{rc^2}}}} \text{,} \bar{u} = u/\sqrt{1-\frac{2GM}{rc^2}}$$

$$d\theta = d\phi = 0 \,, d\theta' = d\phi' = 0 \,, u_r = u \,, \bar{u}_r = \bar{u} \,, u_\theta = u_\phi = 0 \quad (8)$$

III.Conclusion

Hence, in the general relativity theory, the velocity add formula is that if $d\theta = d\phi = 0$ $, d\theta' = d\phi' = 0 \,, u_r = u \,, u_\theta = u_\phi = 0$ is

$$\bar{V}_r = \bar{V} = \frac{V_r}{\sqrt{1-\frac{2GM}{rc^2}}} = \frac{V}{\sqrt{1-\frac{2GM}{rc^2}}} = \frac{dr}{dt}\frac{1}{(1-\frac{2GM}{rc^2})}$$

$$= \frac{\frac{dr'}{dt'}\frac{1}{(1-\frac{2GM}{r'c^2})} + \frac{u}{\sqrt{1-\frac{2GM}{rc^2}}}}{1+\frac{1}{c^2}\cdot\frac{dr'}{dt'}\frac{1}{(1-\frac{2GM}{r'c^2})}\frac{u}{\sqrt{1-\frac{2GM}{rc^2}}}}$$

$$= \frac{\frac{V'_r}{\sqrt{1-\frac{2GM}{r'c^2}}} + \frac{u}{\sqrt{1-\frac{2GM}{rc^2}}}}{1+\frac{1}{c^2}\cdot\frac{V'_r}{\sqrt{1-\frac{2GM}{r'c^2}}}\frac{u}{\sqrt{1-\frac{2GM}{rc^2}}}} = \frac{\frac{V'}{\sqrt{1-\frac{2GM}{r'c^2}}} + \frac{u}{\sqrt{1-\frac{2GM}{rc^2}}}}{1+\frac{1}{c^2}\cdot\frac{V'}{\sqrt{1-\frac{2GM}{r'c^2}}}\frac{u}{\sqrt{1-\frac{2GM}{rc^2}}}}$$

$$\frac{V'_r}{\sqrt{1-\frac{2GM}{r'c^2}}} = \frac{V'}{\sqrt{1-\frac{2GM}{r'c^2}}} = \frac{dr'}{dt'}\frac{1}{(1-\frac{2GM}{r'c^2})} \quad (9)$$

Reference

[1]S.Weinberg,Gravitation and Cosmology(John wiley & Sons,Inc,1972)

[2]C.H.Lee,The genaral relativity theory(Minumsa pub. Co.Inc.,Daewoo Academic book Publishing Natural science 38,Korea,Seoul,1986),Chapter 1

[3]C.Misner, K,Thorne and J. Wheeler, Gravitation(W.H.Freedman & Co.,1973)

[4]S.Hawking and G. Ellis,The Large Scale Structure of Space-Time(Cam-bridge University Press,1973)

[5]R.Adler,M.Bazin and M.Schiffer,Introduction to General Relativity(McGraw-Hill,Inc.,1965)

[6]A.Miller,Albert Einstein’s Special theory of Relativity(Addison-Wesley Publishing Co.,Inc.,1981)

[7]W.Rindler, Special Relativity(2nd ed.,Oliver and Boyd,Edinburg,1966)