A different approach to logic

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Contents

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Abstract

The paper is about an approach to logic that differs from the standard first-order logic and other known approaches. It should be a new approach the author has created proposing to obtain a general and unifying approach to logic and a faithful model of human mathematical deductive process. We list the most relevant features of the system. In first-order logic there exist two different concepts of term and formula, in place of these two concepts in our approach we have just one notion of expression. The set-builder notation is enclosed as an expression-building pattern. In our system we can easily express second-order and all-order conditions (the set to which a quantifier refers is explicitly written in the expression). The meaning of a sentence will depend solely on the meaning of the symbols it contains, it will not depend on external 'structures'. Our deductive system is based on a very simple definition of proof and provides a good model of human mathematical deductive process. The soundness and consistency of the system are proved, as well as the fact that our system is not affected by the most known types of paradox. The paper provides both the theoretical material and two fully documented examples of deduction. The author believes his aims have been achieved but obviously the reader is free to examine the system and get his own opinion about it.

1. Introduction

This paper outlines a system or approach to mathematical logic which is different from the standard one. By ‘the standard approach to logic’ I mean the one presented in chapter 2 of Enderton’s book ([1]) and there named ‘First-Order Logic’. The same approach is also outlined in chapter 2 of Mendelson’s book ([2]), where it is named ‘Quantification Theory’.

We now list the features of our system, pointing out the differences and improvements with respect to standard logic.

In first-order logic there exist two different concepts of term and formula, in place of these two concepts in our approach we have just one notion of expression. Each expression is referred to a certain ‘context’. A context is a (possibly empty) sequence of ordered pairs (x,φ), where x is a variable and φ is itself an expression. Given a context $k = (x_1,\varphi_1) .. (x_m,\varphi_m)$ we call a ‘state on k’ a function which assigns ‘allowable values’ (we’ll explain this later) to the variables $x_1, .., x_m$. If t is an expression with respect to context k and σ is a state on k, we’ll be able to define the meaning of t with respect to k and σ, which we’ll denote by $\#(k,t,\sigma)$.
Our approach requires to build all at the same time, contexts, expressions, states and meanings. We’ll call sentences those expressions which are related to an empty context and whose meaning is true or false. The meaning of a sentence depends solely on the meaning of the symbols it contains, it doesn’t depend on external ‘structures’.

In first-order logic we have terms and formulas and we cannot apply a predicate to one or more formulas, this seems a clear limitation. With our system we can apply predicates to formulas. We’ll see this allows in principle to give a rigorous construction of something similar to the liar paradox, but we can also give a fairly simple explanation of such paradox, which in the end is not a paradox (see section 8).

When we specify a set in mathematics we often use the ‘set-builder notation’. Examples of sets defined with this notation are $\{x\in\mathbb{N} \mid \exists y\in\mathbb{N}: x = 2y\}$, $\{x\in\mathbb{R} \mid x = x^2\}$, and so on. In our system the set-builder notation is enclosed as an expression-building pattern, and this is an advantage over standard logic.

Of course in our approach we allow connectives and quantifiers, but unlike first-order logic these are at the same level of other operators, such as equality, membership and more. While the set-builder notation is necessarily present with its role, connectives and quantifiers as ‘operators’ are not strictly mandatory and are part of a broader category. For instance the universal quantifier simply applies an operation of logical conjunction to a set of conditions, and so it can be classified as an operator.

In first-order logic variables range over individuals, but in mathematics there are statements in which both quantifiers over individuals and quantifiers over sets of individuals occur. One simple example is the following condition:

for each subset X of $\mathbb{N}$ and for each $x \in \mathbb{N}$ we have $x\in X$ or $x\notin X$.

Another example is the condition in which we state that every bounded, non empty set of real numbers has a supremum. Formalisms better suited to express such conditions are second-order logic and type theory, but these systems have a certain level of complexity and are based on different types of variable. In our system we can express the conditions we mentioned above, and we absolutely don't need different types of variables, the set to which the quantifier refers is explicitely written in the expression, this ultimately makes things easier and allows a more general approach. If we read the statement of a theorem in a mathematics book, usually in this statement some variables are introduced, and when introducing them often the set in which they are varying is explicitly specified, so from this point of view our approach is consistent with the actual processes of mathematics.

Let's examine how our system behaves when giving a meaning and possibly a truth value to expressions. Standard logic doesn't plainly associate meanings and truth values to formulas. It introduces some related notion as the concepts of 'structure' (defined in section 2.2 of Enderton's book), truth in a structure, validity, satisfiability. Within first-order logic a structure is used, first of all, to define the collection of things to which a quantifier refers to. Moreover, some symbols such as connectives and quantifiers have a fixed meaning, while for other symbols the meaning is given by the structure. In first-order logic there is a certain level of independence between the meaning of symbols and the language's set of formulas. For instance, if P is a 2-places predicate symbol and t_1, t_2 are terms then Pt_1t_2 is always a formula, and this doesn't depend on the meaning of P, t_1 and t_2. Anyway, what if P was a 3-places predicate? In this case Pt_1t_2 couldn't be a formula. This is just an example to show that the independence between the meaning of symbols and the set of formulas isn't absolute.

In our approach we do not ask, as a requirement, to have independence between the meaning of symbols and the set of expressions, nor do we take care to investigate what happens when changing the meaning of symbols. It wouldn't be easy to deal with this because, for example, you should determine the desired level of independence and variability. Also, I could not say whether trying to deal with this matters would produce any result or added value. For a first presentation of our approach, this topic doesn't seem a priority, it could be a subject of future studies.

Therefore if a symbol is in our system it has his own meaning, and we don't feature a notion of structure like the one of first-order logic. Also, the set of expressions in our language depends on the meaning of symbols. We'll simply speak of the meaning of an expression and when possible of the truth value of that meaning. As we've already said, the meaning of a sentence will depend solely on the meaning of the symbols it contains, it will not depend on external 'structures'.

Our deductive system seeks to provide a good model of human mathematical deductive process. The concept of proof we'll feature is probably the most simple and intuitive that comes to mind, we try to anticipate some of it.

If we define S as the set of sentences then an axiom is a subset of S, an n-ary rule is a subset of S^{n+1}. If φ is a sentence a proof of φ is a sequence $(\psi_1, .. , \psi_m)$ of sentences such that

- there exists an axiom A such that $\psi_1 \in A$;
- if m>1 then for each j=2..m one of the following hold
  - there exists an axiom A such that $\psi_j \in A$
  - there exist an n-ary rule R and $i_1, .., i_n < j$ such that $(\psi_{i(1)}, .. , \psi_{i(n)}, \psi_j) \in R$
- $\psi_m = \varphi$.

Our deductive system, in order to do its job, needs to track the various hypotheses we have introduced along our proof. In a fixed moment of our reasoning we have a sequence of active hypotheses, and we need to be able to apply one of our rules. To this end our axioms and rules need to be properly constructed.

As regards the soundness of the system, it is proved at the beginning of section 5. Consistency is a direct consequence of the soundness. We also discuss (in section 8) how the system relates with some well known paradoxes, it comes out that our system doesn't lead to this kind of inconsistencies. Actually (and obviously) I'm not aware of inconsistencies to which it would lead.

We have examined the main features of the system. If the reader will ask what is the basic idea behind a system of this type, in agreement with what I said earlier I could say that the principle is to provide something like a general and unifying approach to logic and a faithful model of human mathematical deductive process.

This statement about our system of course is not a mathematical statement, so I cannot give a mathematical proof of it. On the other hand, logic exists with the specific primary purpose of being a model to human deduction. In general, suppose we want to provide a mathematical model of some process or reality. The fairness of the model can be judged much more through experience than through mathematics. In fact, mathematics always has to do with models and not directly with reality.

This paper's purpose is to present an approach to logic, but clearly we cannot provide here all possible explanations and comparisons in any way related to the approach itself.
The author believes that this paper provides a fairly comprehensive presentation of the approach in question, this introduction includes significant elements of explanation, justification and comparison with the standard approach to logic. Other material in this regard is presented in the subsequent sections (for example in the final part of Section 5 and in Section 8).

First-order logic has been around for many decades, but to date no absolute evidence has been found that first-order logic is the best possible logic system. In this regard I may quote a stronger statement at the beginning of Josè Ferreiros' paper 'The road to modern logic – an interpretation' ([4]).

"It will be my contention that, contrary to a frequent assumption (at least among philosophers), First-Order Logic is *not* a 'natural unity', i.e. a system the scope and limits of which could be justified solely by rational argument."

Honestly, in my opinion, the approach to logic I propose seems to be a 'natural unity' much more than first-order logic is, and I did what I thought was reasonable to explain this.

Further investigations on this approach will be conducted, in the future, if and when possible, by the author and/or other people. If any claim of this introduction would seem inappropriate, the author is ready to reconsider and possibly fix it. In any case he believes the most important part of this paper is not in the introduction, but in subsequent sections.

The paper is quite long but you can get an idea of the content quickly enough. In fact, the author has chosen to include all the proofs, but quite often these are simple proofs. In addition, the most complex parts are the two definitions 2.1 and 4.6. These have a certain complexity, but at first reading it is not necessary to care of all the details.

2. Language: symbols, expressions and sentences, and their meaning

We begin to describe our language and then the expressions that characterize it. In the process of defining expressions we also define their meaning and the context to which the expression refers. The expressions of our language are constructed from some set of symbols according to certain rules. Expressions are sequences of symbols with meaning, ‘sentences’ are specific expression whose meaning has the property of being true or false. We begin by describing the sets of symbols we need.

First we need a set of symbols V. V members are also called ‘variables’ and just play the role of variables in the construction of our expressions (this implies that V members have no meaning associated).

In addition we need another set of symbols C. C members are also called ‘constants’ and have a meaning. For each c in C we denote by #(c) the meaning of c .

Let f be a member of C. Being f endowed with meaning, f is always an expression of our language. However, the meaning of f could also be a function. In this case f can also play the role of 'operator' in the construction of expressions that are more complex than the simple constant f.

Not all operators that we need, however, are identifiable as functions. Think to the logical connectives (logical negation, logical implication, quantifiers, etc..), but also to the predicate of membership ‘$\in$’ and the predicate of equality ‘=’. These operators are symbols without a precise meaning, therefore we don’t give them a precise meaning in our language, but we will need to give meaning to the application of the operator to objects, where the operator is applicable.

In mathematics and in the real world objects can have properties, such as having a certain color, or being true, or being false. A property is therefore something that can be assigned to an object, no object, more than one object. For example, with reference to color, one or more objects are red or have the property 'to be of red color'. But more generally one or more objects have a color. Suppose to indicate, for objects x that have a color, the color of x with C(x). So we can say that C is a property applicable to a class of objects. On the same object class we can indicate with R(x) the condition 'x has the red color'. R is in turn a property applicable to a class of objects, with the characteristic that for all x R(x) is true or false. A property with this additional feature can be called a 'predicate'.

The class of objects to which a property may be assigned may be called the domain of the property. The members of that domain may be individual objects or sequences of objects, for example, if x is an object and X is a set, the condition '$x\in X$' involves two objects, and then members of the domain of the membership property are the ordered pairs (x,X), where x is an object and X is a set.
Generally we are dealing with properties such that the objects of their domain are all individual objects, or all pairs. Theoretically there may also be properties such that the objects of their domain are sequences of more than two items or even the number of items in sequence may be different in different elements of the domain.

As mentioned above the concept of ‘property’ is similar to the concept of function, but in mathematics there are properties that are not functions. For example, the condition '$x\in X$' just introduced can be applied to an arbitrary object and an arbitrary set, so the ‘membership property’ has not a well determined domain and cannot be considered a function in a strict sense.

So to build our language we need another set of symbols F, where each f in F represents a property P_f. Symbols in F are also called operators or 'property symbols'. We will not assign a meaning to operators, because a property is not easily mappable to a consistent mathematical object (function or other). However, for each f we must know

- the condition $A_f(x_1,\ldots,x_n)$ (where x_1, .., x_n are variables that stay for an arbitrary finite number of arbitrary objects) that indicates if P_f is applicable to (x_1, .., x_n);
- the value of $P_f(x_1,..,x_x)$ (where x_1, .., x_n are variables representing an arbitrary finite number of arbitrary objects for which $A_f(x_1,\ldots,x_n)$ is true) .

Since the concept is subtle we immediately specify what are the most important operators that we may include in our language, providing for each of them the conditions $A_f(x_1,\ldots,x_n)$ and $P_f(x_1,..,x_n)$.

- Logical conjunction: it’s the symbol $\wedge$ and we have

  For $n\neq 2$ $A_\wedge(x_1,\ldots,x_n)$ = false
  $A_\wedge(x_1,x_2)$ = (x_1 true or x_1 false) and (x_2 true or x_2 false)
  $P_\wedge(x_1,x_2)$ = both x_1 and x_2 are true

- Logical disjunction: it’s the symbol $\vee$ and we have

  For $n\neq 2$ $A_\vee(x_1,\ldots,x_n)$ = false
  $A_\vee(x_1,x_2)$ = (x_1 true or x_1 false) and (x_2 true or x_2 false)
  $P_\vee(x_1,x_2)$ = at least one between x_1 and x_2 is true

- Logical implication: it’s the symbol $\rightarrow$ and we have

  For $n\neq 2$ $A_\rightarrow(x_1,\ldots,x_n)$ = false
  $A_\rightarrow(x_1,x_2)$ = (x_1 true or x_1 false) and (x_2 true or x_2 false)
  $P_\rightarrow(x_1,x_2)$ = (x_1 is false) or (x_2 is true)

- Logical negation: it’s the symbol $\neg$ and we have

  For $n>1$ $A_\neg(x_1,\ldots,x_n)$ = false
  $A_\neg(x_1)$ = true
  $P_\neg(x_1)$ = x_1 is false

- Universal quantifier: it’s the symbol $\forall$ and we have

  For $n>1$ $A_\forall(x_1,\ldots,x_n)$ = false
  $A_\forall(x_1)$ = x_1 is a set, for each x in x_1 (x is true or x is false)
  $P_\forall(x_1)$ = for each x in x_1 (x is true) .

- Existential quantifier: it’s the symbol $\exists$ and we have

  For $n>1$ $A_\exists(x_1,\ldots,x_n)$ = false
  $A_\exists(x_1)$ = x_1 is a set, for each x in x_1 (x is true or x is false)

$P_\exists(x_1)$ = there is x in x_1 such that (x is true) .

- Membership predicate: it's the symbol $\in$ and we have

  For $n \neq 2$ $A_\in(x_1,\ldots,x_n)$ = false
  $A_\in(x_1,x_2)$ = x_2 is a set
  $P_\in(x_1,x_2)$ = x_1 is a member of x_2

- Equality predicate: it's the symbol = and we have

  For $n \neq 2$ $A_=(x_1,\ldots,x_n)$ = false
  $A_=(x_1,x_2)$ = true
  $P_=(x_1,x_2)$ = x_1 is equal to x_2 .

We can think and use also other operators, for instance operations between sets such as union or intersection can be represented through an operator, etc. .

In the standard approach to logic, quantifiers are not treated like the other logical connectives, but in this system we mean to separate the operation of applying a quantifier from the operation whereby we build the set to which the quantifier is applied, and therefore the quantifier is treated as the other logical operators (altogether, the universal quantifier is simply an extension of logical conjunction, the existential quantifier is simply an extension of logical disjunction).

With regard to the operation of building a set, we need a specific symbol to indicate that we are doing this, this symbol is the symbol '{ }' which we will consider as a unique symbol.

Besides the set builder symbol, we need parentheses and commas to avoid ambiguity in the reading of our expressions; to this end we use the following symbols: left parenthesis '(', right parenthesis ')', comma ',' and colon ':'. We can indicate this further set of symbols with Z .

To avoid ambiguity in reading our expressions we require that the sets V, C, F and Z are disjoint. It's also requested that a symbol does not correspond to any chain of more symbols of the language. More generally, given $\alpha_1, \ldots, \alpha_n$ and $\beta_1, .., \beta_m$ symbols of our language, we assume that (supposed to use the symbol '||' to indicate the concatenation of characters or strings) equality of the two chains $\alpha_1 \parallel \ldots \parallel \alpha_n$ and $\beta_1 \parallel \ldots \parallel \beta_m$ is achieved when and only when m = n and for each i = 1..n $\alpha_i=\beta_i$.

While the set Z will be always the same, the sets V, C, F, may change according to what is the language that we describe. To describe our language it is required to know the sets V, F, C and the function # which associates a meaning to every element of C. In other words, our language is identified by the 4-tuple (V, F, C, #). Since the 'meaning' of an operator is not a mathematical object, operators must be seen as symbols which are tightly coupled with their meaning.

Before we can describe the process of constructing expressions we still need to introduce some notation about (finite) sequences of ordered pairs (briefly 'soops'). In fact in that process we'll use the notion of 'context' and the notion of 'state', both contexts and states will be defined as soops. We immediately agree to indicate the empty sequence with ε, a non-empty soop is clearly a sequence $((a_1,b_1), \ldots, (a_m,b_m))$ where m is a positive integer and a_i and b_i can be whatever object. Given two soops $\alpha = ((a_1,b_1), \ldots, (a_m,b_m))$ and $\gamma = ((c_1,d_1), \ldots, (c_n,d_n))$ we indicate with $\alpha \parallel \gamma$ the concatenation of α and γ, so

$\alpha \parallel \gamma = ((a_1,b_1), \dots , (a_m,b_m),(c_1,d_1), \dots , (c_n,d_n))$.

Given an ordered pair (a_1,b_1), $((a_1,b_1))$ is a soop. Often we will simply write (a_1,b_1) to mean $((a_1,b_1))$, and this will not be ambiguous.

For instance if $\alpha = ((a_1,b_1), \dots , (a_m,b_m))$ and we write $\alpha \parallel (c_1,d_1)$, the meaning of this expression is clearly $\alpha \parallel ((c_1,d_1))$, if we speak of context (a_1,b_1) we clearly refer to context $((a_1,b_1))$. Moreover, the soop $\alpha = ((a_1,b_1), \dots , (a_m,b_m))$ can also be indicated with $((a_1,b_1)) \parallel \dots \parallel ((a_m,b_m))$, or more sinthetically with $(a_1,b_1) \parallel \dots \parallel (a_m,b_m)$.

Given a soop $\alpha = (a_1,b_1) \parallel \dots \parallel (a_m,b_m)$ we can define dom(α) (the domain of α) as the set $\{a_1, .., a_m\}$. If for each i,j=1..m $i\neq j \rightarrow a_i\neq a_j$ then α is called a 'univocal soop' and for each i=1..m we can define $\alpha(a_i) = b_i$.
We define $R(\alpha)$ as the set of α's 'restrictions', so

$$R(\alpha) = \{\varepsilon\} \cup \{ (a_1,b_1) \parallel \dots \parallel (a_p,b_p) \mid p\leq m \} .$$

And of course $\mathrm{dom}(\varepsilon) = \varnothing$, $R(\varepsilon) = \{\varepsilon\}$.

If α and γ are soops, we write $\alpha \sqsubseteq \gamma$ to mean that $\alpha \in R(\gamma)$.

Given a univocal soop $\alpha = (a_1,b_1) \parallel \dots \parallel (a_m,b_m)$ and a set A in $\{\varnothing\} \cup \{ \{a_1,..,a_p\} \mid p\leq m \}$ there is exactly one $\gamma\in R(\alpha)$ such that dom(γ)=A, we will identify γ with α/A .

If α is a univocal soop and $\gamma\in R(\alpha)$ it is easy to see that α/dom(γ) = γ .

We also need some notation referred to generic strings, this notation will be useful when applied to our expressions, which are non-empty strings. If t is a string we can indicate with $\ell(t)$ t's length, i.e. the number of characters in t. If $\ell(t) > 0$ for each α in $\{1, .. , \ell(t)\}$ at position α within t there is a character, this symbol will be indicated with t[α]. We call 'depth of α within t' (briefly d(t,α)) the number which is obtained by subtracting the number of right round brackets ')' that occur in t before position α from the number of left round brackets '(' that occur in t before position α .
Let ϑ, φ, η be strings with $\ell(\vartheta)>0$, $\ell(\varphi)>0$, and let $t = \vartheta \parallel \varphi \parallel \eta$; let also α in $\{1, .. , \ell(\varphi)\}$. It seems clear enough that $d(t, \ell(\vartheta)+\alpha) = d(t, \ell(\vartheta)+1) + d(\varphi, \alpha)$.
We assume the 'space' or 'blank' character will never occur in our expressions (the expressions we'll build in definition 2.1). This character might occur in the representations of expressions just for the sake of readability, but formally we assume there are no blank characters.

We can now describe the process of constructing expressions for our language L. This is an inductive process in which not only we build expressions, but also we associate them with meaning, and in parallel also define the fundamental concept of 'context'. This process will be identified as 'Definition 2.1' although in reality it is a process in which we give the definitions and prove properties which are needed in order to set up those definitions.

A small note on notation: In general we use the □ symbol to indicate the end of a definition / lemma / theorem (especially when it may be unclear where the 'item' ends). In the case of big definitions as the following within the definition you may find titled paragraphs (which may also correspond to propositions or assumptions). If not clear on where the paragraph ends, we will use the ⊣ symbol to indicate the end of the paragraph.

Definition 2.1:

Since this is a complex definition, we will first try to give an informal idea of the entities we'll define in it. The definition is by induction on positive integers, now we list the sets and concepts we'll define for a generic positive integer n.

K(n) is the set of 'contexts' at step n. A context k is a soop, we can represent a (non empty) context k with a notation like this: $k = (x_1,\varphi_1) \parallel \dots \parallel (x_m,\varphi_m)$ where for each i x_i is a variable and φ_i is an expression.

For each $k \in K(n)$ $\Xi(k)$ is the set of 'states' bound to context k. If n>1 and $k \in K(n-1)$ then $\Xi(k)$ has already been defined at step n-1 or formerly, otherwise it will be defined at step n.

If $k = (x_1,\varphi_1) \parallel \dots \parallel (x_m,\varphi_m)$ is a context, a state on k is a soop $\sigma = (x_1,s_1) \parallel \dots \parallel (x_m,s_m)$, where (roughly speaking) s_i are members of the meaning of the corresponding espression φ_i .

For each $k \in K(n)$ E(n,k) is the set of expressions bound to step n and context k.

E(n) is the union of E(n,k) for $k \in K(n)$ (this will not be explicitly recalled on each iteration in the definition).

For each $k \in K(n)$, $t \in E(n,k)$, $\sigma \in \Xi(k)$ we'll define #(k,t,σ) which stays for 'the meaning of t bound to k and σ'. If n>1, $k \in K(n-1)$ and $t \in E(n-1,k)$ then #(k,t,σ) has already been defined at step n-1 or formerly, otherwise it will be defined at step n.

For each $k \in K(n)$, $t \in E(n,k)$
$V_b(t)$ is the set of the variables that occur within t, bound to a quantifier ;
$V_f(t)$ is the set of the variables that occur within t, not bound to a quantifier ;
V(t) is the set of the variables that occur within t (of course $V(t) = V_b(t) \cup V_f(t)$, so V(t) will not be explicitly defined each time).
If n>1, $k \in K(n-1)$ and $t \in E(n-1,k)$ then $V_b(t)$ and $V_f(t)$ have already been defined at step n-1 or formerly, otherwise they will be defined at step n.

We will also use some sets that will be defined in the same way at each step, so we will not define them each time.
For each $k \in K(n)$, we define $E_s(n,k) = \{t \mid t \in E(n,k), \forall \sigma \in \Xi(k)\ \#(k,t,\sigma) \text{ is a set}\}$.
For each $k \in K(n)$, $t \in E_s(n,k)$ we define $M(k,t) = \bigcup_{\sigma \in \Xi(k)} \#(k,t,\sigma)$.

For each $k \in K(n)$ we define $M(n,k) = \bigcup_{t \in E(s)(n,k)} M(k,t)$.

We finally define $M(n) = \bigcup_{k \in K(n)} M(n,k)$.

We have seen that some entities may have been defined before step n and in this case we are not to define them at step n, however at step n we need to check the definition that has been given is coherent with what we would expect.

Now we are ready to perform the simple initial step of our inductive process.

We set $K(1) = \{\varepsilon\}$, $\Xi(\varepsilon) = \{\varepsilon\}$, $E(1,\varepsilon) = \mathcal{C}$.
For each $t \in E(1,\varepsilon)$ ($=\mathcal{C}$) we define $\#(\varepsilon,t,\varepsilon) = \#(t)$, $V_b(t) = \emptyset$, $V_f(t) = \emptyset$.

The inductive step is much more complex. Suppose all our definitions have been given at step n and let's proceed with step n+1. In this inductive step we will need several assumptions which will be identified with a title like 'Assumption 2.1.x'. Each assumption is a statement that must be valid at step 1, we suppose is valid at step n and needs to be proved true at step n+1 at the end of our definition process.

The first assumption we need is the following.

Assumption 2.1.1: For each $k \in K(n)$: $k \neq \varepsilon$, $\sigma \in \Xi(k)$ there exist a positive integer m and $x_1,..x_m \in \mathcal{V}$, $\varphi_1,..,\varphi_m \in E(n)$, $s_1,..,s_m \in M(n)$ such that

- $\forall i,j=1..m\ i \neq j \rightarrow x_i \neq x_j$
- $k = (x_1,\varphi_1) \parallel \ldots \parallel (x_m,\varphi_m)$
- $\sigma = (x_1,s_1) \parallel \ldots \parallel (x_m,s_m)$. ⊣

Thanks to this assumption for each $k \in K(n)$: $k \neq \varepsilon$, $\sigma \in \Xi(k)$ the objects m, $x_1,..x_m \in \mathcal{V}$, $\varphi_1,..,\varphi_m \in E(n)$, $s_1,..,s_m \in M(n)$ are uniquely determined and for each i=1..m we can use the notation $k(x_i)$ to identify φ_i, and use the notation $\sigma(x_i)$ to identify s_i. Furthermore for each $k \in K(n)$ we can define $\mathcal{R}(k)$ and dom(k) etc. Similarly for each $\sigma \in \Xi(k)$ we can define $\mathcal{R}(\sigma)$ and dom(σ) etc.

We can proceed with the inductive step and set

$K(n)^+ = \{ h \parallel (y,\varphi) \mid h \in K(n) , \varphi \in E(n,h), y \in (\mathcal{V} - \text{dom}(h)), \forall \rho \in \Xi(h)\ \#(h,\varphi,\rho) \text{ is a set} \}$,

$K(n+1) = K(n) \cup K(n)^+$.

For each k in $K(n)^+$ there exist $h \in K(n)$, $y \in (\mathcal{V} - \text{dom}(h))$, $\varphi \in E(n,h)$ such that $k = h \parallel (y,\varphi)$, and it is clear that h, y, φ are unique.

So if $k \notin K(n)$ we can define

$\Xi(k) = \{ \sigma \parallel (y,s) \mid \sigma \in \Xi(h), s \in \#(h,\varphi,\sigma) \}$.

If we accept this definition this implies that the same definition of $\Xi(k)$ is true also for k in $K(n)^+ \cap K(n)$.

To prove this we need a second assumption.

Assumption 2.1.2: for each κ in K(n)

$(\kappa = \varepsilon) \vee$

$(n>1 \wedge$
$\exists g\in K(n\text{-}1),\ z\in \mathcal{V}\text{-dom}(g),\ \psi\in E(n\text{-}1,g){:}\ \kappa= g \parallel (z,\psi) \wedge \forall\sigma\in \Xi(g)\ \#(g,\psi,\sigma)$ is a set $\wedge$
$\Xi(\kappa) = \{\ \sigma \parallel (z,s) \mid \sigma\in \Xi(g),\ s\in \#(g,\psi,\sigma)\ \}\ )$

⊣

So let us consider k in $K(n)^+ \cap K(n)$, we have $k\neq\varepsilon$, so

$n>1 \wedge \exists g\in K(n\text{-}1),\ z\in \mathcal{V}\text{-dom}(g),\ \psi\in E(n\text{-}1,g){:}\ k= g \parallel (z,\psi) \wedge \forall\sigma\in \Xi(g)\ \#(g,\psi,\sigma)$ is a set $\wedge$
$\Xi(k) = \{\ \sigma \parallel (z,s) \mid \sigma\in \Xi(g),\ s\in \#(g,\psi,\sigma)\ \}$.

But we also have $k = h \parallel (y,\varphi)$, so $g=h$, $z=y$, $\psi=\varphi$,
$\Xi(k) = \{\ \sigma \parallel (y,s) \mid \sigma\in \Xi(h),\ s\in \#(h,\varphi,\sigma)\ \}$.

Therefore we have proved the following result, which is a consequence of the previous assumption.

Consequence 2.1.3:

for each $k\in K(n)^+$, $h\in K(n)$, $y\in (\mathcal{V}\text{–dom}(h))$, $\varphi\in E(n,h)$ such that $\forall\rho\in \Xi(h)\ \#(h,\varphi,\rho)$ is a set and $k = h \parallel (y,\varphi)$ we have

$\Xi(k) = \{\ \sigma \parallel (y,s) \mid \sigma\in \Xi(h),\ s\in \#(h,\varphi,\sigma)\ \}$.

⊣

To ensure the unique readability of our expressions we need the following assumption (which is clearly satisfied for n=1).

Assumption 2.1.4: For each $t\in E(n)$

- $t[\ell(t)] \neq$ '('
- if $t[\ell(t)] =$ ')' then $d(t,\ell(t)) = 1$, else $d(t,\ell(t)) = 0$.
- given an integer α in $\{1, .. , \ell(t)\}$ if ($t[\alpha]$=':' or $t[\alpha]$=',' or $t[\alpha]$=')') then $d(t,\alpha)\geq 1$.

⊣

It is time to define E(n+1,k), for each k in K(n+1), and for each t in E(n+1,k) and σ in Ξ(k) we need to define #(k,t,σ), and also we need to define $V_b(t)$ and $V_f(t)$.

We have to warn that the definition of #(k,t,σ), $V_b(t)$ and $V_f(t)$ is not an easy matter.

In fact, E(n+1,k) will be defined as the union of different sets. Consider for instance the situation where $k\in K(n)$. One of these sets is E(n,k), the old set of expressions bound to context k. But of course there also are new sets. If an expression belongs just to E(n,k) and not to the new sets, then we don't need to reason about #(k,t,σ), because simply it has already been defined.
However, if an expression belongs both to E(n,k) and to one or more of the new sets, we will have a proposed definition of #(k,t,σ) for each new set, and we have to check that this proposed definition is equal to the real definition.

If an expression belongs to just one new set and not to E(n,k) then we will simply define #(k,t,σ) with the proposed definition of #(k,t,σ) for the new set.
If an expression belongs to more than one new set, and not to E(n,k), we will need to check that the proposed definitions of #(k,t,σ) for each new set are equivalent, and then we will be authorized to set #(k,t,σ) with one of these proposed definitions.

When $k \notin K(n)$ the discussion is simpler: it cannot be $t \in E(n,k)$, so we just have to consider the other situations. For the definition of $V_b(t)$ and $V_f(t)$ the reasoning is similar but slightly different.

To have a formal approach to the subject we define the new sets of expressions bound to context k, and for expressions in each of them we define the proposed values of #(k,t,σ) and $V_b(t)$, $V_f(t)$.

For each $k = h \parallel (y,\varphi)$ in $K(n)^+$ we define

$E_a(n+1,k) = \{t \mid t \in E(n,h) \wedge y \notin V_b(t) \}$.

For each $t \in E_a(n+1,k)$, $\sigma = \rho \parallel (y,s) \in \Xi(k)$ we define the proposed values of #(k,t,σ) and $V_b(t)$, $V_f(t)$:

$\#(k,t,\sigma)_{(n+1,k,a)} = \#(h,t,\rho)$,
$V_f(t)_{(n+1,k,a)} = V_f(t)$, $V_b(t)_{(n+1,k,a)} = V_b(t)$.

For each $k = h \parallel (y,\varphi)$ in $K(n)^+$ we define

$E_b(n+1,k) = \{y\}$.

For each $t \in E_b(n+1,k)$, $\sigma = \rho \parallel (y,s) \in \Xi(k)$ we define

$\#(k,t,\sigma)_{(n+1,k,b)} = \sigma(y)$,
$V_f(t)_{(n+1,k,b)} = \{y\}$, $V_b(t)_{(n+1,k,b)} = \varnothing$.

As a premise to the following definition we specify that, given a positive integer m and a set D, we call D^m the set $D \times .. \times D$ where D appears m times (when m=1 of course D^1=D), and a function whose domain is a subset of D^m is called a 'function with m arguments'.

For each k in K(n) we define $E_c(n+1,k)$ as the set of the strings $(\varphi)(\varphi_1, \dots, \varphi_m)$ such that

- m is a positive integer
- $\varphi, \varphi_1, .., \varphi_m \in E(n,k)$;
- for each $\sigma \in \Xi(k)$ #(k,φ,σ) is a function with m arguments and
  $(\#(k, \varphi_1, \sigma), ..., \#(k, \varphi_m, \sigma))$ is a member of its domain.

This means that for each $t \in E_c(n+1,k)$ there are a positive integer m and $\varphi, \varphi_1, .., \varphi_m \in E(n)$ such that $t = (\varphi)(\varphi_1, \dots, \varphi_m)$. We will now show that m, $\varphi, \varphi_1, \dots, \varphi_m$ are uniquely determined. Within this complex definition this proof of unique readability may be considered as a technical detail, and can be skipped at first reading.

Suppose there are also a positive integer p and $\psi, \psi_1, \dots, \psi_p$ such that $t = (\psi)(\psi_1, \dots, \psi_p)$. We want to show that p=m, ψ=φ and for each i=1..m $\psi_i=\varphi_i$.

If we assign m we can give an ‘explicit representation’ of t. In fact if m=2 $t = (\varphi)(\varphi_1,\varphi_2)$, if m=3 $t = (\varphi)(\varphi_1,\varphi_2,\varphi_3)$, and so on. In this explicit representation of t we can see explicit occurrences of symbols ‘,’ and ‘)’. There are explicit occurrences of ‘,’ only when m>1. We indicate with q the position of the first explicit occurrence of ‘)’ and the second explicit occurrence of ‘)’ is clearly in position $\ell(t)$. If m>1 we indicate with $q_1, .. , q_{m-1}$ the positions of explicit occurrences of ‘,’.

In the same way if we assign p we can give another ‘explicit representation’ of t. In fact if p=2 $t = (\psi)(\psi_1,\psi_2)$, if p=3 $t = (\psi)(\psi_1,\psi_2,\psi_3)$, and so on. In this explicit representation of t we can see explicit occurrences of symbols ‘,’ and ‘)’. There are explicit occurrences of ‘,’ only when p>1. We indicate with r the position of the first explicit occurrence of ‘)’ and the second explicit occurrence of ‘)’ is clearly in position $\ell(t)$. If p>1 we indicate with $r_1, .. , r_{p-1}$ the positions of explicit occurrences of ‘,’.

We have $d(t,q-1) = d(t, 1 + \ell(\varphi)) = d(t,1 + 1) + d(\varphi, \ell(\varphi)) = 1 + d(\varphi, \ell(\varphi))$.

If $t[q-1] = \varphi[\ell(\varphi)] =$ ‘)’ then $d(t,q) = d(t,q-1) - 1 = d(\varphi, \ell(\varphi)) = 1$.

Else $t[q-1] = \varphi[\ell(\varphi)] \notin$ {‘(‘,’)’} so $d(t,q) = d(t,q-1) = 1 + d(\varphi, \ell(\varphi)) = 1$.

Suppose q<r. Obviously q>1, $q-1\geq 1$, $q-1\leq r-2 = \ell(\psi)$; $\psi[q-1] = t[q] =$ ‘)’. So
$d(t,q) = d(t, 1 + (q - 1)) = d(t,2) + d(\psi, q-1) = 1 + d(\psi, q-1) \geq 2$. This is a contradiction because we have proved $d(t,q) = 1$. So $q\geq r$.
In the same way we can prove that $r\geq q$, so it follows that r=q .

Clearly $\ell(\psi) = r - 2 = q - 2 = \ell(\varphi)$, and for each $\alpha=1..\ell(\varphi)$ $\varphi[\alpha] = t[\alpha+1] = \psi[\alpha]$. In other words $\psi=\varphi$.

Of course we have also

$d(t,r) = d(t,q) = 1$, $d(t, r+2) = d(t,r) - 1 + 1 = 1$, $d(t, q+2) = d(t,q) - 1 + 1 = 1$.

First we examine the case where m=1. First of all we want to show that p=1. Suppose p>1.

In this situation we have

$d(t,r_1 - 1) = d(t, r + 1 + (r_1 - 1 - (r + 1))) = d(t, r+1 + \ell(\psi_1)) = d(t,r+2) + d(\psi_1, \ell(\psi_1)) =$

$= 1 + d(\psi_1, \ell(\psi_1))$.

If $t[r_1 - 1] = \psi_1[\ell(\psi_1)] =$ ‘)’ then $d(t,r_1) = d(t,r_1 - 1) - 1 = d(\psi_1, \ell(\psi_1)) = 1$.

Else $t[r_1 - 1] = \psi_1[\ell(\psi_1)] \notin$ {‘(‘,’)’} so $d(t,r_1) = d(t,r_1 - 1) = 1 + d(\psi_1, \ell(\psi_1)) = 1$.

On the other side we have to consider that

$\ell(\varphi_1) = \ell(t) - 1 - (q + 1) = \ell(t) - q - 2$,

$r_1 \leq \ell(t) - 1$,

$r_1 - (q + 1) \le \ell(t) - 1 - (q + 1) = \ell(t) - q - 2 = \ell(\varphi_1)$,
$r_1 \ge q + 2$, $r_1 - (q + 1) \ge 1$,
$\varphi_1[r_1 - (q+1)] = t[r_1] =$ ‘,’ ,
$1 = d(t,r_1) = d(t, q+2) + d(\varphi_1, r_1 - (q+1)) = 1 + d(\varphi_1, r_1 - (q+1))$.

This causes $d(\varphi_1, r_1 - (q+1)) = 0$, but for assumption 2.1.4 we must have $d(\varphi_1, r_1 - (q+1)) \ge 1$.

So it must be p=1.

Of course $\ell(\psi_1) = \ell(t) - 1 - (r + 1) = \ell(t) - r - 2 = \ell(t) - q - 2 = \ell(\varphi_1)$.

For each $\alpha=1..\ell(\varphi_1)$ $\varphi_1[\alpha] = t[q + 1 + \alpha] = t[r + 1 + \alpha] = \psi_1[\alpha]$. Therefore $\psi_1 = \varphi_1$.

Now let’s discuss the case where m>1.

First we want to prove that for each i=1..m-1 p>i, $d(t,q_i)=1$, $r_i=q_i$, $\psi_i = \varphi_i$.

Let’s show that p>1, $d(t,q_1)=1$, $r_1=q_1$, $\psi_1=\varphi_1$.

If p=1 of course m=1, so p>1 holds. Suppose $q_1<r_1$.

We have that $d(t, q_1 - 1) = d(t, q + 1 + \ell(\varphi_1)) = d(t, q + 1 + 1) + d(\varphi_1, \ell(\varphi_1)) = 1 + d(\varphi_1, \ell(\varphi_1))$.

If $t[q_1 - 1] = \varphi_1[\ell(\varphi_1)] =$ ‘)’ then $d(t,q_1) = d(t,q_1 - 1) - 1 = d(\varphi_1, \ell(\varphi_1)) = 1$.
Else $t[q_1 - 1] = \varphi_1[\ell(\varphi_1)] \notin$ {‘(‘,’)’} so $d(t,q_1) = d(t,q_1 - 1) = 1 + d(\varphi_1, \ell(\varphi_1)) = 1$.

And we have also

$\ell(\psi_1) = r_1 - 1 - (r + 1) = r_1 - r - 2$,
$q_1 - r - 1 < r_1 - r - 1$, $q_1 - r - 1 \le \ell(\psi_1)$,
$q_1 > q+1$, $q_1 > r+1$, $q_1 - r - 1 \ge 1$,

$1 = d(t,q_1) = d(t, r + 1 + (q_1 - r - 1)) = d(t, r+2) + d(\psi_1, q_1 - r - 1) = 1 + d(\psi_1, q_1 - r - 1)$.

So $d(\psi_1, q_1 - r - 1) = 0$. But since $\psi_1[q_1 - r - 1] = t[q_1] =$ ‘,’ by assumption 2.1.4 we must have

$d(\psi_1, q_1 - r - 1) \ge 1$, so we have a contradiction .

Hence $q_1 \ge r_1$ and in the same way we can show that $r_1 \ge q_1$, therefore $r_1 = q_1$.

At this point we observe that $\ell(\varphi_1) = q_1 - 1 - (q + 1) = q_1 - q - 2 = r_1 - r - 2 = \ell(\psi_1)$ and for each $\alpha=1..\ell(\varphi_1)$ $\varphi_1[\alpha] = t[q + 1 + \alpha] = t[r + 1 + \alpha] = \psi_1[\alpha]$, hence $\psi_1 = \varphi_1$.

We have proved that p>1, $d(t,q_1)=1$, $r_1=q_1$, $\psi_1=\varphi_1$, and if m=2 we have also shown that for each i=1..m-1 p>i, $d(t,q_i)=1$, $r_i=q_i$, $\psi_i = \varphi_i$.

Now suppose m>2, let i=1..m-2, suppose we have proved $p>i$, $d(t,q_i)=1$, $r_i=q_i$, $\psi_i = \varphi_i$, we want to show that $p>i+1$, $d(t,q_{i+1})=1$, $r_{i+1}=q_{i+1}$, $\psi_{i+1}=\varphi_{i+1}$.

First of all $d(t, q_{i+1} - 1) = d(t, q_i + \ell(\varphi_{i+1})) = d(t, q_i + 1) + d(\varphi_{i+1}, \ell(\varphi_{i+1})) = 1 + d(\varphi_{i+1}, \ell(\varphi_{i+1}))$.

If $t[q_{i+1} - 1] = \varphi_{i+1}[\ell(\varphi_{i+1})] =$ ')' then $d(t,q_{i+1}) = d(t,q_{i+1} - 1) - 1 = d(\varphi_{i+1}, \ell(\varphi_{i+1})) = 1$.

Else $t[q_{i+1} - 1] = \varphi_{i+1}[\ell(\varphi_{i+1})] \notin$ {'(',')'} so $d(t,q_{i+1}) = d(t,q_{i+1} - 1) = 1 + d(\varphi_{i+1}, \ell(\varphi_{i+1})) = 1$.

Suppose p=i+1. We have $i \le m - 2$, $i + 2 \le m$, $t[q_{i+1}] =$ ','. And we have also

$\ell(\psi_p) = \ell(t) - 1 - r_i$,

$q_{i+1} \le \ell(t) - 1$, $q_{i+1} - r_i \le \ell(t) - 1 - r_i = \ell(\psi_p)$,
$q_{i+1} - r_i = q_{i+1} - q_i \ge 1$,
$\psi_p[q_{i+1} - r_i] = t[q_{i+1}] =$ ',' ,

$1 = d(t,q_{i+1}) = d(t, r_i + (q_{i+1} - r_i)) = d(t, r_i + 1) + d(\psi_p, q_{i+1} - r_i) = 1 + d(\psi_p, q_{i+1} - r_i)$.

So $d(\psi_p, q_{i+1} - r_i) = 0$, and this contradicts assumption 2.1.4. Therefore $p > i+1$.

Now suppose $q_{i+1} < r_{i+1}$. In this case

$\ell(\psi_{i+1}) = r_{i+1} - 1 - r_i$,

$q_{i+1} \le r_{i+1} - 1$, $q_{i+1} - r_i \le r_{i+1} - 1 - r_i = \ell(\psi_{i+1})$,
$q_{i+1} - r_i = q_{i+1} - q_i \ge 1$,
$\psi_{i+1}[q_{i+1} - r_i] = t[q_{i+1}] =$ ',' ,

$1 = d(t,q_{i+1}) = d(t, r_i + (q_{i+1} - r_i)) = d(t, r_i + 1) + d(\psi_{i+1}, q_{i+1} - r_i) = 1 + d(\psi_{i+1}, q_{i+1} - r_i)$.

So $d(\psi_{i+1}, q_{i+1} - r_i) = 0$, and this contradicts assumption 2.1.4. Therefore $q_{i+1} \ge r_{i+1}$.

In the same way we can prove that $q_{i+1} \le r_{i+1}$, hence $r_{i+1} = q_{i+1}$ is proved .

Moreover $\ell(\varphi_{i+1}) = q_{i+1} - 1 - q_i = r_{i+1} - 1 - r_i = \ell(\psi_{i+1})$, for each $\alpha = 1 .. \ell(\psi_{i+1})$
$\psi_{i+1}[\alpha] = t[r_i + \alpha] = t[q_i + \alpha] = \varphi_{i+1}[\alpha]$. And so $\psi_{i+1}=\varphi_{i+1}$.

We have proved that for each i=1..m-1 $p>i$, $d(t,q_i)=1$, $r_i=q_i$, $\psi_i = \varphi_i$.

So $p\ge m$, and in the same way we could prove $m\ge p$, therefore p=m.

We have seen that $r_{m-1} = q_{m-1}$, it follows
$\ell(\varphi_m) = \ell(t) - 1 - q_{m-1} = \ell(t) - 1 - r_{m-1} = \ell(\psi_m)$, and for each $\alpha = 1..\ell(\varphi_m)$
$\varphi_m[\alpha] = t[q_{m-1}+\alpha] = t[r_{m-1}+\alpha] = \psi_m[\alpha]$, therefore $\psi_m = \varphi_m$.

So also in the case m>1 it is shown that p=m and for each i=1..m $\psi_i=\varphi_i$

$\dashv$

For each $t = (\varphi)(\varphi_1, \dots, \varphi_m) \in E_c(n+1,k)$ we can define

$\#(k,t,\sigma)_{(n+1,k,c)} = \#(k,\varphi,\sigma)\ (\ \#(k,\varphi_1,\sigma), \dots, \#(k,\varphi_m,\sigma)\ )$,
$V_f(t)_{(n+1,k,c)} = V_f(\varphi) \cup V_f(\varphi_1) \cup \dots \cup V_f(\varphi_m)$,
$V_b(t)_{(n+1,k,c)} = V_b(\varphi) \cup V_b(\varphi_1) \cup \dots \cup V_b(\varphi_m)$.

For each k in K(n) we define $E_d(n+1,k)$ as the set of the strings $(f)(\varphi_1, \dots, \varphi_m)$ such that

- f belongs to F
- m is a positive integer
- $\varphi_1, .., \varphi_m \in E(n,k)$;
- for each $\sigma \in \Xi(k)$ $A_f(\ \#(k, \varphi_1, \sigma), \dots, \#(k, \varphi_m, \sigma)\ )$ is true.

For instance, this means that if the logical conjunction symbol $\wedge$ belong to F, φ_1, φ_2 belong to E(n,k) and for each $\sigma \in \Xi(k)$ both $\#(k, \varphi_1, \sigma)$ and $\#(k, \varphi_2, \sigma)$ are true or false, then $(\wedge)(\varphi_1,\varphi_2)$ belongs to $E_d(n+1,k)$.

This implies that for each $t \in E_d(n+1,k)$ there are f in F, a positive integer m and $\varphi_1, .., \varphi_m \in E(n)$ such that $t = (f)(\varphi_1, \dots, \varphi_m)$. We will now show that f, m, $\varphi_1, \dots, \varphi_m$ are uniquely determined. Within this complex definition this proof of unique readability may be considered as a technical detail, and can be skipped at first reading.

Suppose there are also $g \in F$, a positive integer p and $\psi_1, \dots, \psi_p$ such that $t = (g)(\psi_1, \dots, \psi_p)$. We want to show that g=f, p=m and for each i=1..m $\psi_i = \varphi_i$.

If we assign m we can give an 'explicit representation' of t. In fact if m=2 $t = (f)(\varphi_1,\varphi_2)$, if m=3 $t = (f)(\varphi_1,\varphi_2,\varphi_3)$, and so on. In this explicit representation of t we can see explicit occurrences of symbols ',' and ')'. There are explicit occurrences of ',' only when m>1. The explicit occurrences of ')' are clearly in positions 3 and $\ell(t)$. If m>1 we indicate with $q_1, .., q_{m-1}$ the positions of explicit occurrences of ','.

In the same way if we assign p we can give another 'explicit representation' of t. In fact if p=2 $t = (g)(\psi_1,\psi_2)$, if p=3 $t = (g)(\psi_1,\psi_2,\psi_3)$, and so on. In this explicit representation of t we can see explicit occurrences of symbols ',' and ')'. There are explicit occurrences of ',' only when p>1. The explicit occurrences of ')' are clearly in positions 3 and $\ell(t)$. If p>1 we indicate with $r_1, .., r_{p-1}$ the positions of explicit occurrences of ','.

It is immediate to see that g = t[2] = f.

We first consider the case where m=1. Here we have to show that p=1, $\psi_1 = \varphi_1$.

Suppose p>1. In this situation we have

$$d(t,r_1 - 1) = d(t, 4 + (r_1 - 1 - 4)) = d(t, 4 + \ell(\psi_1)) = d(t,4+1) + d(\psi_1, \ell(\psi_1)) =$$
$$= 1 + d(\psi_1, \ell(\psi_1)).$$

If $t[r_1 - 1] = \psi_1[\ell(\psi_1)] =$ ')' then $d(t,r_1) = d(t,r_1 - 1) - 1 = d(\psi_1, \ell(\psi_1)) = 1$.

Else $t[r_1 - 1] = \psi_1[\ell(\psi_1)] \notin$ {'(',')'} so $d(t,r_1) = d(t,r_1 - 1) = 1 + d(\psi_1, \ell(\psi_1)) = 1$.

On the other side we have to consider that

$\ell(\varphi_1) = \ell(t) - 1 - 4 = \ell(t) - 5$,

$r_1 - 4 \le \ell(t) - 1 - 4 = \ell(t) - 5 = \ell(\varphi_1)$,
$r_1 \ge 4 + 1$, $r_1 - 4 \ge 1$,
$\varphi_1[r_1 - 4] = t[r_1] =$ ',' ,

$1 = d(t,r_1) = d(t, 4 + (r_1 - 4)) = d(t, 4+1) + d(\varphi_1, r_1 - 4) = 1 + d(\varphi_1, r_1 - 4)$.

This causes $d(\varphi_1, r_1 - 4) = 0$, but for assumption 2.1.4 we must have $d(\varphi_1, r_1 - 4) \ge 1$.

So it must be p=1.

Of course $\ell(\psi_1) = \ell(t) - 1 - 4 = \ell(\varphi_1)$.

For each $\alpha=1..\ell(\varphi_1)$ $\varphi_1[\alpha] = t[4 + \alpha] = \psi_1[\alpha]$. Therefore $\psi_1 = \varphi_1$.

Now let's discuss the case where m>1.

First we want to prove that for each i=1..m-1 p>i, $d(t,q_i)=1$, $r_i=q_i$, $\psi_i = \varphi_i$.

Let's show that p>1, $d(t,q_1)=1$, $r_1=q_1$, $\psi_1=\varphi_1$.

If p=1 of course m=1, so p>1 holds. Suppose $q_1<r_1$.

We have that $d(t, q_1 - 1) = d(t, 4 + \ell(\varphi_1)) = d(t, 4 + 1) + d(\varphi_1, \ell(\varphi_1)) = 1 + d(\varphi_1, \ell(\varphi_1))$.

If $t[q_1 - 1] = \varphi_1[\ell(\varphi_1)] =$ ')' then $d(t,q_1) = d(t,q_1 - 1) - 1 = d(\varphi_1, \ell(\varphi_1)) = 1$.
Else $t[q_1 - 1] = \varphi_1[\ell(\varphi_1)] \notin$ {'(',')'} so $d(t,q_1) = d(t,q_1 - 1) = 1 + d(\varphi_1, \ell(\varphi_1)) = 1$.

And we have also

$\ell(\psi_1) = r_1 - 1 - 4$,

$q_1 - 4 \le r_1 - 1 - 4 = \ell(\psi_1)$,
$q_1 \ge 4 + 1$, $q_1 - 4 \ge 1$,

$1 = d(t,q_1) = d(t, 4 + (q_1 - 4)) = d(t, 4+1) + d(\psi_1, q_1 - 4) = 1 + d(\psi_1, q_1 - 4)$.

So $d(\psi_1, q_1 - 4) = 0$. But since $\psi_1[q_1 - 4] = t[q_1] =$ ',' by assumption 2.1.4 we must have

$d(\psi_1, q_1 - 4) \ge 1$, so we have a contradiction .

Hence $q_1 \ge r_1$ and in the same way we can show that $r_1 \ge q_1$, therefore $r_1 = q_1$.

At this point we observe that $\ell(\varphi_1) = q_1 - 1 - 4 = r_1 - 1 - 4 = \ell(\psi_1)$ and for each $\alpha=1..\ell(\varphi_1)$ $\varphi_1[\alpha] = t[4 + \alpha] = \psi_1[\alpha]$, hence $\psi_1 = \varphi_1$.

We have proved that $p>1$, $d(t,q_1)=1$, $r_1=q_1$, $\psi_1=\varphi_1$, and if $m=2$ we have also shown that for each $i=1..m-1$ $p>i$, $d(t,q_i)=1$, $r_i=q_i$, $\psi_i = \varphi_i$.

Now suppose $m>2$, let $i=1..m-2$, suppose we have proved $p>i$, $d(t,q_i)=1$, $r_i=q_i$, $\psi_i = \varphi_i$, we want to show that $p>i+1$, $d(t,q_{i+1})=1$, $r_{i+1}=q_{i+1}$, $\psi_{i+1}=\varphi_{i+1}$.

First of all $d(t, q_{i+1} - 1) = d(t, q_i + \ell(\varphi_{i+1})) = d(t, q_i + 1) + d(\varphi_{i+1}, \ell(\varphi_{i+1})) = 1 + d(\varphi_{i+1}, \ell(\varphi_{i+1}))$.

If $t[q_{i+1} - 1] = \varphi_{i+1}[\ell(\varphi_{i+1})] =$ ')' then $d(t,q_{i+1}) = d(t,q_{i+1} - 1) - 1 = d(\varphi_{i+1}, \ell(\varphi_{i+1})) = 1$.

Else $t[q_{i+1} - 1] = \varphi_{i+1}[\ell(\varphi_{i+1})] \notin$ { '(',')' } so $d(t,q_{i+1}) = d(t,q_{i+1} - 1) = 1 + d(\varphi_{i+1}, \ell(\varphi_{i+1})) = 1$.

Suppose $p=i+1$. We have $i \le m - 2$, $i + 2 \le m$, $t[q_{i+1}] =$ ','. And we have also

$\ell(\psi_p) = \ell(t) - 1 - r_i$,

$q_{i+1} \le \ell(t) - 1$, $q_{i+1} - r_i \le \ell(t) - 1 - r_i = \ell(\psi_p)$,
$q_{i+1} - r_i = q_{i+1} - q_i \ge 1$,
$\psi_p[q_{i+1} - r_i] = t[q_{i+1}] =$ ',' ,

$1 = d(t,q_{i+1}) = d(t, r_i + (q_{i+1} - r_i)) = d(t, r_i + 1) + d(\psi_p, q_{i+1} - r_i) = 1 + d(\psi_p, q_{i+1} - r_i)$.

So $d(\psi_p, q_{i+1} - r_i) = 0$, and this contradicts assumption 2.1.4. Therefore $p > i+1$.

Now suppose $q_{i+1} < r_{i+1}$. In this case

$\ell(\psi_{i+1}) = r_{i+1} - 1 - r_i$,

$q_{i+1} \le r_{i+1} - 1$, $q_{i+1} - r_i \le r_{i+1} - 1 - r_i = \ell(\psi_{i+1})$,
$q_{i+1} - r_i = q_{i+1} - q_i \ge 1$,
$\psi_{i+1}[q_{i+1} - r_i] = t[q_{i+1}] =$ ',' ,

$1 = d(t,q_{i+1}) = d(t, r_i + (q_{i+1} - r_i)) = d(t, r_i + 1) + d(\psi_{i+1}, q_{i+1} - r_i) = 1 + d(\psi_{i+1}, q_{i+1} - r_i)$.

So $d(\psi_{i+1}, q_{i+1} - r_i) = 0$, and this contradicts assumption 2.1.4. Therefore $q_{i+1} \ge r_{i+1}$.

In the same way we can prove that $q_{i+1} \le r_{i+1}$, hence $r_{i+1} = q_{i+1}$ is proved .

Moreover $\ell(\varphi_{i+1}) = q_{i+1} - 1 - q_i = r_{i+1} - 1 - r_i = \ell(\psi_{i+1})$, for each $\alpha = 1 .. \ell(\psi_{i+1})$ $\psi_{i+1}[\alpha] = t[r_i + \alpha] = t[q_i + \alpha] = \varphi_{i+1}[\alpha]$. And so $\psi_{i+1}=\varphi_{i+1}$.

We have proved that for each $i=1..m-1$ $p>i$, $d(t,q_i)=1$, $r_i=q_i$, $\psi_i = \varphi_i$.

So $p \ge m$, and in the same way we could prove $m \ge p$, therefore $p=m$.

We have seen that $r_{m-1} = q_{m-1}$, it follows

$\ell(\varphi_m) = \ell(t) - 1 - q_{m-1} = \ell(t) - 1 - r_{m-1} = \ell(\psi_m)$, and for each $\alpha = 1..\ell(\varphi_m)$
$\varphi_m[\alpha] = t[q_{m-1}+\alpha] = t[r_{m-1}+\alpha] = \psi_m[\alpha]$, therefore $\psi_m = \varphi_m$.

So also in the case m>1 it is shown that p=m and for each i=1..m $\psi_i=\varphi_i$.

$\dashv$

For each $t = (f)(\varphi_1, \dots , \varphi_m) \in E_d(n+1,k)$ we can define

$\#(k,t,\sigma)_{(n+1,k,d)} = P_f (\#(k,\varphi_1,\sigma), ... , \#(k,\varphi_m,\sigma))$,
$V_f(t)_{(n+1,k,d)} = V_f(\varphi_1) \cup \dots \cup V_f(\varphi_m)$,
$V_b(t)_{(n+1,k,d)} = V_b(\varphi_1) \cup \dots \cup V_b(\varphi_m)$.

For each k in K(n) we define $E_e(n+1,k)$ as the set of strings $\{\}(x_1{:}\varphi_1, \dots , x_m{:}\varphi_m, \varphi)$ such that

- m is a positive integer
- $x_1, \dots , x_m$ distinct $\in \mathcal{V}$-dom(k) ;
- $\varphi_1 \in E(n,k)$, for each $\sigma\in \Xi(k)$ $\#(k,\varphi_1,\sigma)$ is a set ;
- if m>1, for each i=1..m-1 if we define $k'_i = k||(x_1,\varphi_1)|| .. ||(x_i,\varphi_i)$ it follows
  $k'_i \in K(n) \wedge \varphi_{i+1}\in E(n, k'_i) \wedge$ for each $\sigma'_i\in \Xi(k'_i)$ $\#(k'_i, \varphi_{i+1}, \sigma'_i)$ is a set ;
- if we define $k'_m = k||(x_1,\varphi_1)|| .. ||(x_m,\varphi_m)$ it follows $k'_m\in K(n) \wedge \varphi\in E(n,k'_m)$.

This means that for each $t\in E_e(n+1,k)$ there are a positive integer m, $x_1, \dots , x_m \in \mathcal{V}$ and $\varphi, \varphi_1, .. , \varphi_m \in E(n)$ such that $t = \{\}(x_1{:}\varphi_1, \dots , x_m{:}\varphi_m, \varphi)$. We'll now show that m, $x_1, \dots , x_m$, $\varphi, \varphi_1, .. , \varphi_m$ are uniquely determined. Within this complex definition this proof of unique readability may be considered as a technical detail, and can be skipped at first reading.

Suppose there are also a positive integer p, $y_1, \dots , y_p \in \mathcal{V}$, $\psi, \psi_1, \dots , \psi_p \in E(n)$ such that $t = \{\}(y_1{:}\psi_1, \dots , y_m{:}\psi_m, \psi)$. We want to show that p=m, for each i=1..m $y_i = x_i$, $\psi_i = \varphi_i$, $\psi=\varphi$.

If we assign m we can give an 'explicit representation' of t. In fact if m=2 $t = \{\}(x_1{:}\varphi_1, x_2{:}\varphi_2, \varphi)$, if m=3 $t = \{\}(x_1{:}\varphi_1, x_2{:}\varphi_2, x_3{:}\varphi_3, \varphi)$, and so on. In this explicit representation of t we can see explicit occurrences of symbols ',' and ':'. We indicate with $q_1, .., q_m$ the positions of explicit occurrences of ':' and with $r_1, .., r_m$ the the positions of explicit occurrences of ','.

In the same way if we assign p we can give another 'explicit representation' of t. In fact if m=2 $t = \{\}(y_1{:}\psi_1, y_2{:}\psi_2, \psi)$, if p=3 $t = \{\}(y_1{:}\psi_1, y_2{:}\psi_2, y_3{:}\psi_3, \psi)$, and so on. In this explicit representation of t we can see explicit occurrences of symbols ',' and ':'. We indicate with $q'_1, .., q'_p$ the positions of explicit occurrences of ':' and with $r'_1, .., r'_p$ the the positions of explicit occurrences of ','.

We want to show that for each i=1..m $p\geq i$ $y_i=x_i$, $q'_i=q_i$, $d(t,r_i)=1$, $r'_i=r_i$, $\psi_i=\varphi_i$.

The first step is to show that $y_1=x_1$, $q'_1=q_1$, $d(t,r_1)=1$, $r'_1=r_1$, $\psi_1=\varphi_1$.

Of course $y_1 = t[3] = x_1$, $q'_1 = 4 = q_1$. Moreover

$d(t,r_1 - 1) = d(t, q_1 + (r_1 - 1 - q_1)) = d(t, q_1 + \ell(\varphi_1)) = d(t, q_1 + 1) + d(\varphi_1, \ell(\varphi_1)) = 1 + d(\varphi_1, \ell(\varphi_1))$.

If $t[r_1 - 1] = \varphi_1[\ell(\varphi_1)] =$ ')' then $d(t,r_1) = d(t,r_1 - 1) - 1 = d(\varphi_1, \ell(\varphi_1)) = 1$.

Else $t[r_1 - 1] = \varphi_1[\ell(\varphi_1)] \notin$ {'(',')'} so $d(t,r_1) = d(t,r_1 - 1) = 1 + d(\varphi_1, \ell(\varphi_1)) = 1$.

Now suppose $r_1 < r'_1$. This would mean that

$\ell(\psi_1) = r'_1 - 1 - q'_1$,

$r_1 - q'_1 \leq r'_1 - 1 - q'_1 = \ell(\psi_1)$,
$r_1 - q'_1 = r_1 - q_1 \geq 1$,
$\psi_1[r_1 - q'_1] = t[r_1] =$ ',' ,

$1 = d(t,r_1) = d(t, q'_1 + (r_1 - q'_1)) = d(t, q'_1 + 1) + d(\psi_1, r_1 - q'_1) = 1 + d(\psi_1, r_1 - q'_1)$.

So $d(\psi_1, r_1 - q'_1) = 0$, and this contradicts assumption 2.1.4.

Hence $r_1 \geq r'_1$, in the same way we can show that $r_1 \leq r'_1$, therefore $r_1 = r'_1$.

At this point we observe that

$\ell(\varphi_1) = r_1 - 1 - q_1 = \ell(\psi_1)$ and for each $\alpha = 1 .. \ell(\psi_1)$

$\psi_1[\alpha] = t[q'_1 + \alpha] = t[q_1 + \alpha] = \varphi_1[\alpha]$, hence $\psi_1 = \varphi_1$.

If m=1 we have proved that for each i=1..m $p \geq i$ $y_i=x_i$, $q'_i=q_i$, $d(t,r_i)=1$, $r'_i=r_i$, $\psi_i=\varphi_i$.

Consider the case where m>1. Let i=1..m-1, we suppose $p \geq i$ $y_i=x_i$, $q'_i=q_i$, $d(t,r_i)=1$, $r'_i=r_i$, $\psi_i=\varphi_i$ and want to show $p \geq i+1$ $y_{i+1}=x_{i+1}$, $q'_{i+1}=q_{i+1}$, $d(t,r_{i+1})=1$, $r'_{i+1}=r_{i+1}$, $\psi_{i+1}=\varphi_{i+1}$.

We can immediately show that $d(t,r_{i+1})=1$. In fact

$d(t, q_{i+1} + 1) = d(t, r_i) = 1$,

$d(t,r_{i+1} - 1) = d(t, q_{i+1} + (r_{i+1} - 1 - q_{i+1})) = d(t, q_{i+1} + \ell(\varphi_{i+1})) = d(t, q_{i+1} + 1) + d(\varphi_{i+1}, \ell(\varphi_{i+1})) =$
$= 1 + d(\varphi_{i+1}, \ell(\varphi_{i+1}))$.

If $t[r_{i+1} - 1] = \varphi_{i+1}[\ell(\varphi_{i+1})] =$ ')' then $d(t,r_{i+1}) = d(t,r_{i+1} - 1) - 1 = d(\varphi_{i+1}, \ell(\varphi_{i+1})) = 1$.
Else $t[r_{i+1} - 1] = \varphi_{i+1}[\ell(\varphi_{i+1})] \notin$ {'(',')'} so $d(t,r_{i+1}) = d(t,r_{i+1} - 1) = 1 + d(\varphi_{i+1}, \ell(\varphi_{i+1})) = 1$.

Suppose p=i . In this case

$\ell(\psi) = \ell(t) - 1 - r'_i$,

$r_{i+1} - r'_i \leq \ell(t) - 1 - r'_i = \ell(\psi)$,
$r_{i+1} - r'_i = r_{i+1} - r_i \geq 1$,
$\psi[r_{i+1} - r'_i] = t[r_{i+1}] =$ ',' ,

$1 = d(t,r_{i+1}) = d(t, r'_i + (r_{i+1} - r'_i)) = d(t, r'_i + 1) + d(\psi, r_{i+1} - r'_i) = 1 + d(\psi, r_{i+1} - r'_i)$.

So $d(\psi, r_{i+1} - r'_i) = 0$, and this contradicts assumption 2.1.4. Therefore $p \geq i+1$. It follows immediately that $y_{i+1} = t[r'_i+1] = t[r_i+1] = x_{i+1}$ and $q'_{i+1}=q_{i+1}$.

Now we suppose $r_{i+1} < r'_{i+1}$. This would mean that

$\ell(\psi_{i+1}) = r'_{i+1} - 1 - q'_{i+1}$,

$r_{i+1} - q'_{i+1} \leq r'_{i+1} - 1 - q'_{i+1} = \ell(\psi_1)$,

$r_{i+1} - q'_{i+1} = r_{i+1} - q_{i+1} \geq 1$,

$\psi_{i+1}[r_{i+1} - q'_{i+1}] = t[r_{i+1}] =$ ',' ,

$1 = d(t,r_{i+1}) = d(t, q'_{i+1} + (r_{i+1} - q'_{i+1})) = d(t, q'_{i+1} + 1) + d(\psi_{i+1}, r_{i+1} - q'_{i+1}) = 1 + d(\psi_{i+1}, r_{i+1} - q'_{i+1})$.

So $d(\psi_{i+1}, r_{i+1} - q'_{i+1}) = 0$, and this contradicts assumption 2.1.4. Hence $r_{i+1} \geq r'_{i+1}$, in the same way we can show that $r_{i+1} \leq r'_{i+1}$, therefore $r_{i+1} = r'_{i+1}$.

At this point we observe that

$\ell(\varphi_{i+1}) = r_{i+1} - 1 - q_{i+1} = \ell(\psi_{i+1})$ and for each $\alpha = 1 .. \ell(\psi_{i+1})$

$\psi_{i+1}[\alpha] = t[q'_{i+1} + \alpha] = t[q_{i+1} + \alpha] = \varphi_{i+1}[\alpha]$, hence $\psi_{i+1} = \varphi_{i+1}$.

It is shown that for each i=1..m $p \geq i$ $y_i=x_i$, $q'_i=q_i$, $d(t,r_i)=1$, $r'_i=r_i$, $\psi_i=\varphi_i$.

So $p \geq m$. In the same way we could prove that $m \geq p$, so p=m. In our proof we just need a final step, which is proving that $\psi=\varphi$. This clearly holds because of

$\ell(\psi) = \ell(t) - 1 - r'_p = \ell(t) - 1 - r_m = \ell(\varphi)$, for each $\alpha = 1..\ell(\psi)$

$\psi[\alpha] = t[r'_p + \alpha] = t[r_m + \alpha] = \varphi[\alpha]$.

$\dashv$

For every $t = \{\}(x_1:\varphi_1, \dots, x_m:\varphi_m, \varphi) \in E_e(n+1,k)$ we can define

$\#(k,t,\sigma)_{(n+1,k,e)} = \{\#(k'_m,\varphi,\sigma'_m) \mid \sigma'_m \in \Xi(k'_m), \sigma \sqsubseteq \sigma'_m \}$.

If we use a notation closer to the one of our formulas, we can write

$\#(k,t,\sigma)_{(n+1,k,e)} = \{\} (\sigma'_m \in \Xi(k'_m): \sigma \sqsubseteq \sigma'_m , \#(k'_m,\varphi,\sigma'_m))$.

In the paper we will often use a notation like $\{\} (\sigma'_m \in \Xi(k'_m): \sigma \sqsubseteq \sigma'_m , \#(k'_m,\varphi,\sigma'_m))$ to define our sets, in this example the meaning of this notation is clearly the same meaning of $\{\#(k'_m,\varphi,\sigma'_m) \mid \sigma'_m \in \Xi(k'_m), \sigma \sqsubseteq \sigma'_m \}$.

Moreover, if m=1

$V_f(t)_{(n+1,k,e)} = V_f(\varphi_1) \cup (V_f(\varphi)-\{x_1\})$;

$V_b(t)_{(n+1,k,e)} = \{x_1\} \cup V_b(\varphi_1) \cup V_b(\varphi)$;

If m>1

$V_f(t)_{(n+1,k,e)} = V_f(\varphi_1) \cup (V_f(\varphi_2)\text{-}\{x_1\}) \cup \ldots \cup (V_f(\varphi_m)\text{-}\{x_1,..,x_{m-1}\}) \cup (V_f(\varphi)\text{-}\{x_1,\ldots,x_m\})$;
$V_b(t)_{(n+1,k,e)} = \{x_{1,..,}x_m\} \cup V_b(\varphi_1) \cup .. \cup V_b(\varphi_m) \cup V_b(\varphi)$.

We have terminated the definition of the 'new sets' (of expressions bound to context k) and the related work, we are now ready to define E(n+1,k).

We establish that

- for each $k\in K(n)^+ - K(n)$ $E(n+1,k) = E_a(n+1,k) \cup E_b(n+1,k)$
- for each $k\in K(n) - K(n)^+$ $E(n+1,k) = E(n,k) \cup E_c(n+1,k) \cup E_d(n+1,k) \cup E_e(n+1,k)$
- for each $k\in (K(n)^+ \cap K(n))$
  $E(n+1,k) = E(n,k) \cup E_a(n+1,k) \cup E_b(n+1,k) \cup E_c(n+1,k) \cup E_d(n+1,k) \cup E_e(n+1,k)$

We can also have an unified definition by setting, for each $k\in K(n+1)$:

- if $k\in K(n)^+$
  $E'_a(n+1,k) = E_a(n+1,k)$, $E'_b(n+1,k) = E_b(n+1,k)$

  else
  $E'_a(n+1,k) = \varnothing$, $E'_b(n+1,k) = \varnothing$;

- if $k\in K(n)$
  $E'(n,k) = E(n,k)$, $E'_c(n+1,k) = E_c(n+1,k)$, $E'_d(n+1,k) = E_d(n+1,k)$, $E'_e(n+1,k) = E_e(n+1,k)$

  else
  $E'(n,k) = \varnothing$, $E'_c(n+1,k) = \varnothing$, $E'_d(n+1,k) = \varnothing$, $E'_e(n+1,k) = \varnothing$.

and finally setting

$$E(n+1,k) = E'(n,k) \cup E'_a(n+1,k) \cup E'_b(n+1,k) \cup E'_c(n+1,k) \cup E'_d(n+1,k) \cup E'_e(n+1,k) .$$

For every t in E(n+1,k), with respect to the definition of $\#(k,t,\sigma)$, there are three possibilities:

1) t is in E'(n,k): then $\#(k,t,\sigma)$ is already defined; if t is in one of the sets $E'_w(n+1,k)$ we need to verify that $\#(k,t,\sigma) = \#(k,t,\sigma)_{(n+1,k,w)}$
2) t is not in E'(n,k) and t is in just one of the sets $E'_w(n+1,k)$: then we just define $\#(k,t,\sigma) = \#(k,t,\sigma)_{(n+1,k,w)}$
3) t is not in E'(n,k) and t is more than one of the sets $E'_w(n+1,k)$: in this case we need to verify that for each distinct w_1, w_2 such that $t\in E'_{w1}(n+1,k) \cap E'_{w2}(n+1,k)$
   $\#(k,t,\sigma)_{(n+1,k,w1)} = \#(k,t,\sigma)_{(n+1,k,w2)}$.
   Then $\#(k,t,\sigma)$ will be defined equal to $\#(k,t,\sigma)_{(n+1,k,w)}$ for whatever w such that $t\in E'_w(n+1,k)$.

By point 1) we are required to verify that for each $k\in K(n+1)$, $w\in\{a,b,c,d,e\}$, t in $E'(n,k)\cap E'_w(n+1,k)$ and $\sigma\in \Xi(k)$ $\#(k,t,\sigma) = \#(k,t,\sigma)_{(n+1,k,w)}$.

By point 3) we are required to verify that for each $k\in K(n+1)$, $w_1,w_2 \in\{a,b,c,d,e\}$: $w_1\neq w_2$, t in $E'_{w1}(n+1,k) \cap E'_{w2}(n+1,k)$ and $\sigma\in \Xi(k)$ $\#(k,t,\sigma)_{(n+1,k,w1)} = \#(k,t,\sigma)_{(n+1,k,w2)}$.

Those verifications ensure us that $\#(k,t,\sigma)$ is defined for every t in E(n+1,k), and also we are enabled to state that

for each $k\in K(n+1)$, $w\in\{a,b,c,d,e\}$, t in $E'_w(n+1,k)$ and $\sigma\in\Xi(k)$ $\#(k,t,\sigma)_{(n+1,k,w)} = \#(k,t,\sigma)$.

As regards the definition of $V_b(t)$ and $V_f(t)$ we can make a similar argument. For each $t\in E(n+1)$ there are three possibilities:

1) t is in E(n): then $V_b(t)$ and $V_f(t)$ are already defined; if t is in one of the sets $E'_w(n+1,k)$ we need to verify that $V_b(t) = V_b(t)_{(n+1,k,w)}$ (and the same for $V_f(t)$)
2) t is not in E(n) and there are just one $k\in K(n+1)$ and $w\in\{a,b,c,d,e\}$ such that t is in $E'_w(n+1,k)$: then we just define $V_b(t) = V_b(t)_{(n+1,k,w)}$ (and the same for $V_f(t)$).
3) t is not in E(n) and there are more than one $k\in K(n+1)$ and $w\in\{a,b,c,d,e\}$ such that t is in $E'_w(n+1,k)$: in this case we need to verify that for each $k_1,k_2 \in K(n+1)$ and w_1,w_2 with $t\in E'_{w1}(n+1,k_1)\cap E'_{w2}(n+1,k_2)$ we have $V_b(t)_{(n+1,k1,w1)} = V_b(t)_{(n+1,k2,w2)}$ (and the same for $V_f(t)$). Then $V_b(t)$ will be defined equal to $V_b(t)_{(n+1,k,w)}$ for whatever k,w such that t is in $E'_w(n+1,k)$.

By point 1) we are required to verify that for each $k\in K(n+1)$, $w\in\{a,b,c,d,e\}$ and for each t in $E(n)\cap E'_w(n+1,k)$ $V_b(t) = V_b(t)_{(n+1,k,w)}$ (and the same for $V_f(t)$) .

By point 3) we are required to verify that for each $k_1,k_2 \in K(n+1)$, $w_1,w_2\in\{a,b,c,d,e\}$, $t\in E'_{w1}(n+1,k_1)\cap E'_{w2}(n+1,k_2)$ (such that $t\notin E(n)$) we have $V_b(t)_{(n+1,k1,w1)} = V_b(t)_{(n+1,k2,w2)}$ (and the same for $V_f(t)$).

Those verifications ensure us $V_b(t)$ and $V_f(t)$ are defined for every t in E(n+1,k), and also we are enabled to state that

for each $k\in K(n+1)$, $w\in\{a,b,c,d,e\}$ and t in $E'_w(n+1,k)$ $V_b(t)_{(n+1,k,w)} = V_b(t)$ (and the same for $V_f(t)$).

We now have to perform the required verifications. These verifications require a further set of assumptions. We will immediately list those assumptions, and also significant consequences to them that will in turn be used in our verification process.

<u>Assumption 2.1.5:</u> if n>1 then $K(n-1)\subseteq K(n)$. $\dashv$

<u>Assumption 2.1.6:</u> Let κ,k in K(n) such that for each x in $dom(\kappa)\cap dom(k)$ $\kappa(x)=k(x)$. Let $t\in E(n,\kappa)\cap E(n,k)$. Let $\sigma_\kappa\in\Xi(\kappa)$, $\rho_k\in\Xi(k)$ such that $\forall x\in(dom(\kappa)\cap dom(k))$ $\sigma_\kappa(x) = \rho_k(x)$.
Then $\#(\kappa,t,\sigma_\kappa) = \#(k,t,\rho_k)$. $\dashv$

The next assumption has a central role in our verification process.

<u>Assumption 2.1.7:</u> For each $\kappa\in K(n)$, $t\in E(n,\kappa)$ one and only one of these 5 alternative situations is verified:

a. $t\in \mathcal{C}$, $\forall\sigma\in\Xi(\kappa)$ $\#(\kappa,t,\sigma) = \#(t)$, $V_f(t)=\emptyset$, $V_b(t)=\emptyset$

b. $n>1$, $t \in dom(\kappa)$, $\forall \sigma \in \Xi(\kappa)$ $\#(\kappa,t,\sigma) = \sigma(t)$, $V_f(t)=\{t\}$, $V_b(t)=\varnothing$

c. $n>1$, $\exists h \in K(n\text{-}1)$: $h \sqsubseteq \kappa$, $\exists\ \varphi, \varphi_1, .. , \varphi_m \in E(n\text{-}1,h)$:
$t = (\varphi)(\varphi_1, \dots , \varphi_m)$, $t \in E(n,h)$,
for each $\rho \in \Xi(h)$ $\#(h,\varphi,\rho)$ is a function with m arguments,
$(\#(h, \varphi_1, \rho), ... , \#(h, \varphi_m, \rho))$ is a member of its domain,
$\#(h, t, \rho) = \#(h,\varphi,\rho) (\#(h,\varphi_1,\rho), ... , \#(h,\varphi_m,\rho))$
$V_f(t) = V_f(\varphi) \cup V_f(\varphi_1) \cup \dots \cup V_f(\varphi_m)$,
$V_b(t) = V_b(\varphi) \cup V_b(\varphi_1) \cup \dots \cup V_b(\varphi_m)$,
for each $\sigma \in \Xi(\kappa)$, $\rho \in \Xi(h)$: $\rho \sqsubseteq \sigma$ it results
$\#(\kappa, t, \sigma) = \#(h, t, \rho)$

d. $n>1$, $\exists h \in K(n\text{-}1)$: $h \sqsubseteq \kappa$, $\exists\ f \in F$, $\varphi_1, .. , \varphi_m \in E(n\text{-}1,h)$:
$t = (f)(\varphi_1, \dots , \varphi_m)$, $t \in E(n,h)$,
for each $\rho \in \Xi(h)$ $A_f(\#(h, \varphi_1, \rho), ... , \#(h, \varphi_m, \rho))$,
$\#(h, t, \rho) = P_f (\#(h,\varphi_1,\rho), ... , \#(h,\varphi_m,\rho))$
$V_f(t) = V_f(\varphi_1) \cup \dots \cup V_f(\varphi_m)$,
$V_b(t) = V_b(\varphi_1) \cup \dots \cup V_b(\varphi_m)$,
for each $\sigma \in \Xi(\kappa)$, $\rho \in \Xi(h)$: $\rho \sqsubseteq \sigma$ it results
$\#(\kappa, t, \sigma) = \#(h, t, \rho)$

e. $n>1$, $\exists h \in K(n\text{-}1)$: $h \sqsubseteq \kappa$, $\exists\ \varphi, \varphi_1, .. , \varphi_m \in E(n\text{-}1)$,
$\exists x_1, \dots , x_m$ distinct $\in V\text{-}dom(h)$:
$t = \{\}(x_1{:}\varphi_1, \dots , x_m{:}\varphi_m, \varphi)$, $t \in E(n,h)$,

$\varphi_1 \in E(n\text{-}1,h)$, for each $\rho \in \Xi(h)$ $\#(h,\varphi_1,\rho)$ is a set ;
if $m>1$ for each $i=1..m\text{-}1$ if we define $h'_i = h||(x_1,\varphi_1)|| .. ||(x_i,\varphi_i)$ it follows
$h'_i \in K(n\text{-}1) \wedge \varphi_{i+1} \in E(n\text{-}1, h'_i) \wedge$ for each $\rho'_i \in \Xi(h'_i)$ $\#(h'_i, \varphi_{i+1}, \rho'_i)$ is a set ;
if we define $h'_m = h||(x_1,\varphi_1)|| .. ||(x_m,\varphi_m)$ it follows $h'_m \in K(n\text{-}1) \wedge \varphi \in E(n\text{-}1,h'_m)$;

for each $\rho \in \Xi(h)$
$\#(h,t,\rho) = \{\}(\rho'_m \in \Xi(h'_m){:}\ \rho \sqsubseteq \rho'_m , \#(h'_m,\varphi,\rho'_m))$;

if $m=1$ $V_f(t) = V_f(\varphi_1) \cup (V_f(\varphi)\text{-}\{x_1\})$;
$V_b(t) = \{x_1\} \cup V_b(\varphi_1) \cup V_b(\varphi)$.

if $m>1$
$V_f(t) = V_f(\varphi_1) \cup (V_f(\varphi_2)\text{-}\{x_1\}) \cup \dots \cup (V_f(\varphi_m)\text{-}\{x_1,..,x_{m-1}\}) \cup (V_f(\varphi)\text{-}\{x_1,\dots,x_m\})$;
$V_b(t) = \{x_{1,..,}x_m\} \cup V_b(\varphi_1) \cup .. \cup V_b(\varphi_m) \cup V_b(\varphi)$,

for each $\sigma \in \Xi(\kappa)$ and for each $\rho \in \Xi(h)$: $\rho \sqsubseteq \sigma$ it results
$\#(\kappa, t, \sigma) = \#(h, t, \rho)$.

$\dashv$

Assumption 2.1.8: Let $n>1$, $k \in K(n)$, $h \in R(k)$: $h \neq k$. Then $h \in K(n\text{-}1)$, for each $\sigma \in \Xi(k)$ if we define $\rho = \sigma/dom(h)$ then $\rho \in \Xi(h)$.

$\dashv$

Assumption 2.1.9: if $n>1$ then for each $g \in K(n\text{-}1)$ $E(n\text{-}1,g) \subseteq E(n,g)$. $\dashv$

Consequence 2.1.10:
Suppose k, k'$\in$K(n), y$\in$$V$-dom(k), φ$\in$E(n,k): k' = k ǁ (y,φ). Moreover let σ$\in$Ξ(k) and σ'$\in$Ξ(k') such that σ $\sqsubseteq$ σ'. Then there is s in #(k,φ,σ) such that σ' = σ ǁ (y,s).

Proof:

By our assumption 2.1.2

(n>1 $\wedge$
$\exists$g$\in$K(n-1), z$\in$$V$-dom(g), ψ$\in$E(n-1,g): k'= g ǁ (z,ψ) $\wedge$ $\forall$ρ$\in$Ξ(g) #(g,ψ,ρ) is a set $\wedge$
Ξ(k') = { ρ ǁ (z,s) | ρ$\in$Ξ(g), s$\in$#(g,ψ,ρ) }).

So k ǁ (y,φ) = k'= g ǁ (z,ψ). Clearly this means that y=z, φ=ψ, k=g, and

Ξ(k') = { ρ ǁ (y,s) | ρ$\in$Ξ(k), s$\in$#(k,φ,ρ) } .

Hence there exist ρ$\in$Ξ(k), s$\in$#(k,φ,ρ) such that σ' = ρ ǁ (y,s).

Now dom(ρ) = dom(k) = dom(σ), and ρ = σ'/dom(ρ) = σ'/dom(σ) = σ .

The obvious conclusion is that σ' = σ ǁ (y,s) and s$\in$#(k,φ,σ) .

$\dashv$

Consequence 2.1.11:
Suppose k, k'$\in$K(n), y$\in$$V$-dom(k), φ$\in$E(n,k): k' = k ǁ (y,φ). Moreover let σ$\in$Ξ(k) and σ' = σ ǁ (y,s), with s$\in$#(k,φ,σ). Then σ'$\in$Ξ(k'), and clearly σ$\sqsubseteq$σ' .

Proof:

By our assumption 2.1.2

(n>1 $\wedge$
$\exists$g$\in$K(n-1), z$\in$$V$-dom(g), ψ$\in$E(n-1,g): k'= g ǁ (z,ψ) $\wedge$ $\forall$ρ$\in$Ξ(g) #(g,ψ,ρ) is a set $\wedge$
Ξ(k') = { ρ ǁ (z,s) | ρ$\in$Ξ(g), s$\in$#(g,ψ,ρ) }).

So k ǁ (y,φ) = k'= g ǁ (z,ψ). Clearly this means that y=z, φ=ψ, k=g, and

Ξ(k') = { ρ ǁ (y,s) | ρ$\in$Ξ(k), s$\in$#(k,φ,ρ) } .

It follows immediately that σ'$\in$Ξ(k'), and clearly σ$\sqsubseteq$σ' .

$\dashv$

Consequence 2.1.12:

Let g,h$\in$K(n), φ, $φ_1$, .. , $φ_m$ $\in$ E(n), x_1, … , x_m distinct $\in$ (V-dom(g)) $\cap$ (V-dom(h)) :

$t = \{\}(x_1{:}\varphi_1, \dots , x_m{:}\varphi_m, \varphi)$;

$\varphi_1 \in E(n,g)$, for each $\rho\in \Xi(g)$ $\#(g,\varphi_1,\rho)$ is a set ;
if $m>1$ then for each $i=1..m-1$ if we define $g'_i = g\|(x_1,\varphi_1)\| .. \|(x_i,\varphi_i)$ it follows
$g'_i \in K(n) \wedge \varphi_{i+1}\in E(n, g'_i) \wedge$ for each $\rho'_i\in \Xi(g'_i)$ $\#(g'_i, \varphi_{i+1}, \rho'_i)$ is a set ;
if we define $g'_m = g\|(x_1,\varphi_1)\| .. \|(x_m,\varphi_m)$ it follows $g'_m\in K(n) \wedge \varphi\in E(n,g'_m)$

$\varphi_1 \in E(n,h)$, for each $\sigma\in \Xi(h)$ $\#(h,\varphi_1,\sigma)$ is a set ;
if $m>1$ then for each $i=1..m-1$ if we define $h'_i = h\|(x_1,\varphi_1)\| .. \|(x_i,\varphi_i)$ it follows
$h'_i \in K(n) \wedge \varphi_{i+1}\in E(n, h'_i) \wedge$ for each $\sigma'_i\in \Xi(h'_i)$ $\#(h'_i, \varphi_{i+1}, \sigma'_i)$ is a set ;
if we define $h'_m = h\|(x_1,\varphi_1)\| .. \|(x_m,\varphi_m)$ it follows $h'_m\in K(n) \wedge \varphi\in E(n,h'_m)$

Moreover we suppose that $\forall x\in dom(g)\cap dom(h)$ $h(x)=g(x)$, and let $\rho\in \Xi(g)$, $\sigma\in \Xi(h)$ such that $\forall x\in dom(g)\cap dom(h)$ $\rho(x)=\sigma(x)$. Then

$\{\} (\sigma'_m \in\Xi(h'_m): \sigma \sqsubseteq \sigma'_m , \#(h'_m,\varphi,\sigma'_m)) = \{\}(\rho'_m \in\Xi(g'_m): \rho \sqsubseteq \rho'_m , \#(g'_m,\varphi,\rho'_m))$.

Proof:

Let $u \in \{\} (\sigma'_m \in\Xi(h'_m): \sigma \sqsubseteq \sigma'_m , \#(h'_m,\varphi,\sigma'_m))$, we want to show that
$u \in \{\}(\rho'_m \in\Xi(g'_m): \rho \sqsubseteq \rho'_m , \#(g'_m,\varphi,\rho'_m))$.

There exists $\sigma'_m \in\Xi(h'_m)$ such that $\sigma \sqsubseteq \sigma'_m$, $u = \#(h'_m,\varphi,\sigma'_m)$.

First of all we may observe that $h'_m\in K(n)$, $h'_m\neq\varepsilon$, so $n>1$.

We also observe that $h\in K(n)$ and so h can be expressed in the form $(z_1,\psi_1) \| .. \| (z_p,\psi_p)$ (if $h=\varepsilon$ we assume $p=0$ and this expression reduces to ε), and $\sigma\in \Xi(h)$ can be expressed in the form
$(z_1,r_1) \| .. \| (z_p,r_p)$.

So we have $h'_m = (z_1,\psi_1) \| .. \| (z_p,\psi_p) \| (x_1,\varphi_1) \| .. \| (x_m,\varphi_m)$.

Since $\sigma'_m \in\Xi(h'_m)$ σ'_m can be expressed as $(z_1,v_1) \| .. \| (z_p,v_p) \| (x_1,w_1) \| .. \| (x_m,w_m)$.

Because of $\sigma\sqsubseteq\sigma'_m$ it follows that $\sigma'_m = (z_1,r_1) \| .. \| (z_p,r_p) \| (x_1,w_1) \| .. \| (x_m,w_m)$.

For each $i=1..m-1$ we have $h'_i = (z_1,\psi_1) \| .. \| (z_p,\psi_p) \| (x_1,\varphi_1) \| .. \| (x_i,\varphi_i)$, and

$dom(h'_i) = \{z_1,..,z_p,x_1,..,x_i\}$ so we can define $\sigma'_i = \sigma'_m/dom(h'_i)$ and we have

$\sigma'_i = (z_1,r_1) \| .. \| (z_p,r_p) \| (x_1,w_1) \| .. \| (x_i,w_i)$.

We also define $h'_0=h$, $\sigma'_0 = \sigma$. We can immediately observe that for each $i=1..m-1$ $\sigma'_i\sqsubseteq\sigma'_{i+1}$.

We can prove that for each $i=1..m$ $\sigma'_i\in \Xi(h'_i)$ and there is $s_i\in \#(h'_{i-1},\varphi_i,\sigma'_{i-1})$ such that
$\sigma'_i = \sigma'_{i-1} \| (x_i,s_i)$.

We will prove this by induction on i. Let us perform the initial step of our induction process.

We have $\sigma'_1=\sigma'_m/dom(h'_1)$. If m=1 then $\sigma'_1=\sigma'_m\in\Xi(h'_1)$, otherwise $h'_1\in \mathcal{R}(h'_m)$ $h'_1\neq h'_m$, and this means (by 2.1.8) that $\sigma'_1\in\Xi(h'_1)$.

We have h'_0, $h'_1 \in K(n)$, $x_1\in \mathcal{V}\text{-}dom(h'_0)$, $\varphi_1 \in E(n,h'_0)$, $h'_1 = h'_0 \| (x_1,\varphi_1)$, $\sigma'_0\in\Xi(h'_0)$, $\sigma'_1\in\Xi(h'_1)$, $\sigma'_0\sqsubseteq\sigma'_1$.

We can apply consequence 2.1.10 and state there is $s_1\in\#(h'_0,\varphi_1,\sigma'_0)$ such that $\sigma'_1 = \sigma'_0 \| (x_1,s_1)$.

We now perform the inductive step. This is needed only if m>1, let i=1..m-1. We suppose $\sigma'_i\in\Xi(h'_i)$ and there is $s_i\in\#(h'_{i-1},\varphi_i,\sigma'_{i-1})$ such that $\sigma'_i = \sigma'_{i-1} \| (x_i,s_i)$.

We have $\sigma'_{i+1}=\sigma'_m/dom(h'_{i+1})$. If i+1=m then $\sigma'_{i+1}=\sigma'_m\in\Xi(h'_{i+1})$, otherwise $h'_{i+1}\in \mathcal{R}(h'_m)$ $h'_{i+1}\neq h'_m$, and this means (by 2.1.8) that $\sigma'_{i+1}\in\Xi(h'_{i+1})$.

We have h'_i, $h'_{i+1}\in K(n)$, $x_{i+1}\in \mathcal{V}\text{-}dom(h'_i)$, $\varphi_{i+1}\in E(n,h'_i)$, $h'_{i+1}=h'_i\|(x_{i+1},\varphi_{i+1})$, $\sigma'_i\in\Xi(h'_i)$, $\sigma'_{i+1}\in\Xi(h'_{i+1})$, $\sigma'_i\sqsubseteq\sigma'_{i+1}$.

We can apply consequence 2.1.10 and state there is $s_{i+1}\in\#(h'_i,\varphi_{i+1},\sigma'_i)$ such that
$\sigma'_{i+1} = \sigma'_i \| (x_{i+1},s_{i+1})$.

We now define $\rho'_1 = \rho \| (x_1,s_1)$, and, if m>1, for each i = 1 .. m-1 $\rho'_{i+1} = \rho'_i \| (x_{i+1}, s_{i+1})$.

We will show that for each i=1..m $\rho'_i\in\Xi(g'_i)$.

We begin by showing that $\rho'_1\in\Xi(g'_1)$. We intend to use assumption 2.1.6 to show that $s_1\in\#(g,\varphi_1,\rho)$.

We consider that $g,h \in K(n)$, $\forall x\in dom(g)\cap dom(h)$ $h(x)=g(x)$, $\rho\in\Xi(g)$, $\sigma\in\Xi(h)$, $\forall x\in dom(g)\cap dom(h)$ $\rho(x)=\sigma(x)$. Then by assumption 2.1.6 $\#(g,\varphi_1,\rho) = \#(h,\varphi_1,\sigma)$, so $s_1\in\#(g,\varphi_1,\rho)$.

We can now use consequence 2.1.11 to show that $\rho'_1\in\Xi(g'_1)$. In fact g, $g'_1\in K(n)$, $x_1\in \mathcal{V}\text{-}dom(g)$, $\varphi_1\in E(n,g)$, $g'_1 = g\|(x_1,\varphi_1)$, $\rho\in\Xi(g)$, $\rho'_1 = \rho \| (x_1,s_1)$, $s_1\in\#(g,\varphi_1,\rho)$. So by 2.1.11 we get $\rho'_1\in\Xi(g'_1)$.

If m>1 we need to perform an inductive step. Let i=1..m-1, we suppose that $\rho'_i\in\Xi(g'_i)$ and want to show that $\rho'_{i+1}\in\Xi(g'_{i+1})$. First we intend to use assumption 2.1.6 to show that $s_{i+1}\in\#(g'_i,\varphi_{i+1},\rho'_i)$.

We consider that $g'_i,h'_i \in K(n)$, $\forall x\in dom(g'_i)\cap dom(h'_i)$ $h'_i(x)=g'_i(x)$. Furthermore $\varphi_{i+1}\in E(n,h'_i) \cap E(n,g'_i)$, $\rho'_i\in\Xi(g'_i)$, $\sigma'_i\in\Xi(h'_i)$, $\forall x\in dom(\rho'_i)\cap dom(\sigma'_i)$ $\rho'_i(x)=\sigma'_i(x)$. Then by assumption 2.1.6 $\#(g'_i,\varphi_{i+1},\rho'_i) = \#(h'_i,\varphi_{i+1},\sigma'_i)$, so $s_{i+1}\in\#(g'_i,\varphi_{i+1},\rho'_i)$.

We can now use consequence 2.1.11 to show that $\rho'_{i+1}\in\Xi(g'_{i+1})$. In fact $g'_i,g'_{i+1} \in K(n)$, $x_{i+1}\in \mathcal{V}\text{-}dom(g'_i)$, $\varphi_{i+1} \in E(n,g'_i)$, $g'_{i+1} = g'_i\|(x_{i+1},\varphi_{i+1})$, $\rho'_i\in\Xi(g'_i)$, $s_{i+1}\in\#(g'_i,\varphi_{i+1},\rho'_i)$, $\rho'_{i+1} = \rho'_i \| (x_{i+1},s_{i+1})$. So by 2.1.11 we get $\rho'_{i+1}\in\Xi(g'_{i+1})$.

We can conclude that $\rho'_m\in\Xi(g'_m)$. By 2.1.6 we can derive that $\#(h'_m,\varphi,\sigma'_m) = \#(g'_m,\varphi,\rho'_m)$. In fact $g'_m,h'_m \in K(n)$, $\forall x\in dom(g'_m)\cap dom(h'_m)$ $h'_m(x)=g'_m(x)$, $\varphi\in E(n,h'_m) \cap E(n,g'_m)$, $\rho'_m\in\Xi(g'_m)$, $\sigma'_m\in\Xi(h'_m)$, $\forall x\in dom(\rho'_m)\cap dom(\sigma'_m)$ $\rho'_m(x)=\sigma'_m(x)$. Therefore $\#(h'_m,\varphi,\sigma'_m) = \#(g'_m,\varphi,\rho'_m)$.

Of course we have also $\rho\sqsubseteq\rho'_m$, so $\rho'_m\in\Xi(g'_m)$, $\rho\sqsubseteq\rho'_m$, $u = \#(g'_m,\varphi,\rho'_m)$. In other words

$u \in \{\}(\rho'_m \in \Xi(g'_m): \rho \sqsubseteq \rho'_m , \#(g'_m,\varphi,\rho'_m))$.

The proof of the converse implication (if $u \in \{\}(\rho'_m \in \Xi(g'_m): \rho \sqsubseteq \rho'_m , \#(g'_m,\varphi,\rho'_m))$ then $u \in \{\} (\sigma'_m \in \Xi(h'_m): \sigma \sqsubseteq \sigma'_m , \#(h'_m,\varphi,\sigma'_m)))$ is perfectly analogous.

$\dashv$

We now start with the verifications required to define $\#(k,t,\sigma)$. There we have to verify that

- for each $w \in \{a,b,c,d,e\}$, t in $E'(n,k) \cap E'_w(n+1,k)$ and $\sigma \in \Xi(k)$ $\#(k,t,\sigma) = \#(k,t,\sigma)_{(n+1,k,w)}$;
- for each $w_1,w_2 \in \{a,b,c,d,e\}$: $w_1 \neq w_2$, t in $E'_{w1}(n+1,k) \cap E'_{w2}(n+1,k)$ and $\sigma \in \Xi(k)$ $\#(k,t,\sigma)_{(n+1,k,w1)} = \#(k,t,\sigma)_{(n+1,k,w2)}$.

Suppose t in $\mathbf{E'(n,k) \cap E'_a(n+1,k)}$, and so $t \in E(n,k) \cap E_a(n+1,k)$. As a consequence of $t \in E_a(n+1,k)$ we have that $k \in K(n)^+$, so $k = h \parallel (y,\varphi)$ where $h \in K(n)$, $\varphi \in E(n,h)$, $y \in (V\text{–dom}(h))$, and we also have $t \in E(n,h)$; $\sigma = \rho \parallel (y,s)$ with $\rho \in \Xi(h)$, $s \in \#(h,\varphi,\rho)$; $\#(k,t,\sigma)_{(n+1,k,a)} = \#(h,t,\rho)$.

By assumption 2.1.6, since $t \in E(n,k) \cap E(n,h)$, we get $\#(h,t,\rho) = \#(k,t,\sigma)$, and then

$\#(k,t,\sigma)_{(n+1,k,a)} = \#(k,t,\sigma)$.

Now we consider the situation in which t is in $\mathbf{E'(n,k) \cap E'_b(n+1,k)}$ and then t belongs to $E(n,k) \cap E_b(n+1,k)$. As a consequence of $t \in E_b(n+1,k)$ we have that $k \in K(n)^+$, so $k = h \parallel (y,\varphi)$ where $h \in K(n)$, $\varphi \in E(n,h)$, $y \in (V\text{–dom}(h))$, and we also have $t=y$; $\sigma = \rho \parallel (y,s)$ with $\rho \in \Xi(h)$, $s \in \#(h,\varphi,\rho)$; $\#(k,t,\sigma)_{(n+1,k,b)} = \sigma(y)$.

By assumption 2.1.7, which applies because of $t \in E(n,k)$, we must have $t \in dom(k)$, $\#(k,t,\sigma) = \sigma(t) = \sigma(y) = \#(k,t,\sigma)_{(n+1,k,b)}$.

Let's examine the situation in which t is in $\mathbf{E'(n,k) \cap E'_c(n+1,k)}$ and then t belongs to $E(n,k) \cap E_c(n+1,k)$. As a consequence of $t \in E_c(n+1,k)$ there exist $\varphi, \varphi_1, .. , \varphi_m \in E(n,k)$ such that $t = (\varphi)(\varphi_1, \dots , \varphi_m)$, $\#(k,t,\sigma)_{(n+1,k,c)} = \#(k,\varphi,\sigma) (\#(k,\varphi_1,\sigma), ... , \#(k,\varphi_m,\sigma))$.

Since $t \in E(n,k)$ we can apply assumption 2.1.7 and obtain that $n>1$, there exists $h \in K(n-1)$: $h \sqsubseteq \kappa$, $t \in E(n,h)$, for each $\rho \in \Xi(h)$ $\#(h,\varphi,\rho)$ is a function with m arguments, $(\#(h, \varphi_1, \rho), ... , \#(h, \varphi_m, \rho))$ is a member of its domain, $\#(h, t, \rho) = \#(h,\varphi,\rho) (\#(h,\varphi_1,\rho), ... , \#(h,\varphi_m,\rho))$.

We define $\rho = \sigma/dom(h)$. If $h=k$ then $\rho = \sigma \in \Xi(h)$. Otherwise by assumption 2.1.8 we still get $\rho \in \Xi(h)$.

By assumption 2.1.7 we have

$\#(k, t, \sigma) = \#(h, t, \rho) = \#(h,\varphi,\rho) (\#(h,\varphi_1,\rho), ... , \#(h,\varphi_m,\rho))$.

Now we can consider that $k,h \in K(n)$, $\varphi,\varphi_i \in E(n,k) \cap E(n,h)$, $\sigma \in \Xi(k)$, $\rho \in \Xi(h)$, we can apply assumption 2.1.6 to obtain that

$\#(h,\varphi,\rho) (\#(h,\varphi_1,\rho), ... , \#(h,\varphi_m,\rho)) = \#(k,\varphi,\sigma) (\#(k,\varphi_1,\sigma), ... , \#(k,\varphi_m,\sigma)) = \#(k,t,\sigma)_{(n+1,k,c)}$, so we have proved

$\#(k, t, \sigma) = \#(k,t,\sigma)_{(n+1,k,c)}$.

Next we consider the case in which t is in **E'(n,k) ∩ E'$_d$(n+1,k)** and then t belongs to $E(n,k) \cap E_d(n+1,k)$. As a consequence of $t \in E_d(n+1,k)$ there exist $f \in F$, $\varphi_1, .. , \varphi_m \in E(n,k)$ such that $t = (f)(\varphi_1, \dots , \varphi_m)$, $\#(k,t,\sigma)_{(n+1,k,d)} = P_f (\#(k,\varphi_1,\sigma), ... , \#(k,\varphi_m,\sigma))$.

Since $t \in E(n,k)$ we can apply assumption 2.1.7 and obtain that n>1, there exists $h \in K(n-1)$: $h \sqsubseteq \kappa$, $t \in E(n,h)$, for each $\rho \in \Xi(h)$ $A_f(\#(h, \varphi_1, \rho), ... , \#(h, \varphi_m, \rho))$, $\#(h, t, \rho) = P_f (\#(h,\varphi_1,\rho), ... , \#(h,\varphi_m,\rho))$.

We define $\rho = \sigma/dom(h)$. If h=k then $\rho = \sigma \in \Xi(h)$. Otherwise by assumption 2.1.8 we still get $\rho \in \Xi(h)$.

By assumption 2.1.7 we have

$\#(k, t, \sigma) = \#(h, t, \rho) = P_f (\#(h,\varphi_1,\rho), ... , \#(h,\varphi_m,\rho))$.

Now we can consider that $k,h \in K(n)$, $\varphi_i \in E(n,k) \cap E(n,h)$, $\sigma \in \Xi(k)$, $\rho \in \Xi(h)$, we can apply assumption 2.1.6 to obtain that

$P_f (\#(h,\varphi_1,\rho), ... , \#(h,\varphi_m,\rho)) = P_f (\#(k,\varphi_1,\sigma), ... , \#(k,\varphi_m,\sigma)) = \#(k,t,\sigma)_{(n+1,k,d)}$, so we have proved

$\#(k, t, \sigma) = \#(k,t,\sigma)_{(n+1,k,d)}$.

In this part of our verification we just need to examine the case in which t is in **E'(n,k) ∩ E'$_e$(n+1,k)** and so t belongs to $E(n,k) \cap E_e(n+1,k)$. As a consequence to $t \in E_e(n+1,k)$ there exist a positive integer m, $x_1, .., x_m$ distinct $\in V$-dom(k), $\varphi, \varphi_1, .. , \varphi_m \in E(n)$ such that $t = \{\}(x_1{:}\varphi_1, \dots , x_m{:}\varphi_m, \varphi)$. Moreover we have

- $\varphi_1 \in E(n,k)$, for each $\sigma \in \Xi(k)$ $\#(k,\varphi_1,\sigma)$ is a set ;
- if m>1, for each i=1..m-1 if we define $k'_i = k||(x_1,\varphi_1)|| .. ||(x_i,\varphi_i)$ it follows
  $k'_i \in K(n) \wedge \varphi_{i+1} \in E(n, k'_i) \wedge$ for each $\sigma'_i \in \Xi(k'_i)$ $\#(k'_i, \varphi_{i+1}, \sigma'_i)$ is a set ;
- if we define $k'_m = k||(x_1,\varphi_1)|| .. ||(x_m,\varphi_m)$ it follows $k'_m \in K(n) \wedge \varphi \in E(n,k'_m)$.

For a fixed $\sigma \in \Xi(k)$

$\#(k,t,\sigma)_{(n+1,k,e)} = \{\} (\sigma'_m \in \Xi(k'_m): \sigma \sqsubseteq \sigma'_m , \#(k'_m,\varphi,\sigma'_m))$.

Since $t \in E(n,k)$ we can apply assumption 2.1.7 and obtain that n>1, there exists $h \in K(n-1)$: $h \sqsubseteq k$, $t \in E(n,h)$, and also

$\varphi_1 \in E(n-1,h)$, for each $\rho \in \Xi(h)$ $\#(h,\varphi_1,\rho)$ is a set ;
if m>1 for each i=1..m-1 if we define $h'_i = h||(x_1,\varphi_1)|| .. ||(x_i,\varphi_i)$ it follows
$h'_i \in K(n-1) \wedge \varphi_{i+1} \in E(n-1, h'_i) \wedge$ for each $\rho'_i \in \Xi(h'_i)$ $\#(h'_i, \varphi_{i+1}, \rho'_i)$ is a set ;
if we define $h'_m = h||(x_1,\varphi_1)|| .. ||(x_m,\varphi_m)$ it follows $h'_m \in K(n-1) \wedge \varphi \in E(n-1,h'_m)$;

for each $\rho \in \Xi(h)$ $\#(h,t,\rho) = \{\}(\rho'_m \in \Xi(h'_m): \rho \sqsubseteq \rho'_m , \#(h'_m,\varphi,\rho'_m))$.

We define $\rho = \sigma/\mathrm{dom}(h)$. If $h=k$ then $\rho = \sigma \in \Xi(h)$. Otherwise by assumption 2.1.8 we still get $\rho \in \Xi(h)$.

By assumption 2.1.7

$\#(k, t, \sigma) = \#(h, t, \rho) = \{\}(\rho'_m \in \Xi(h'_m){:}\ \rho \sqsubseteq \rho'_m\ , \#(h'_m,\varphi,\rho'_m)\ )$.

To complete our proof, we need to show that

$\{\}\ (\sigma'_m \in \Xi(k'_m){:}\ \sigma \sqsubseteq \sigma'_m\ \ , \#(k'_m,\varphi,\sigma'_m)\ ) = \{\}(\rho'_m \in \Xi(h'_m){:}\ \rho \sqsubseteq \rho'_m\ , \#(h'_m,\varphi,\rho'_m)\ )$.

This follows by consequence 2.1.12 , that can be applied because of:

$k,h \in K(n)$, $\varphi, \varphi_1, .., \varphi_m \in E(n)$, $x_1, .., x_m$ distinct $\in (V\text{-dom}(k)) \cap (V\text{-dom}(h))$,
$t = \{\}(x_1{:}\varphi_1, \dots , x_m{:}\varphi_m, \varphi)$,

> $\varphi_1 \in E(n,k)$, for each $\sigma \in \Xi(k)$ $\#(k,\varphi_1,\sigma)$ is a set ;
> if $m>1$, for each $i=1..m-1$ if we define $k'_i = k||(x_1,\varphi_1)|| .. ||(x_i,\varphi_i)$ it follows
> $k'_i \in K(n) \wedge \varphi_{i+1} \in E(n, k'_i) \wedge$ for each $\sigma'_i \in \Xi(k'_i)$ $\#(k'_i, \varphi_{i+1}, \sigma'_i)$ is a set ;
> if we define $k'_m = k||(x_1,\varphi_1)|| .. ||(x_m,\varphi_m)$ it follows $k'_m \in K(n) \wedge \varphi \in E(n,k'_m)$.
>
> $\varphi_1 \in E(n,h)$, for each $\rho \in \Xi(h)$ $\#(h,\varphi_1,\rho)$ is a set ;
> if $m>1$ for each $i=1..m-1$ if we define $h'_i = h||(x_1,\varphi_1)|| .. ||(x_i,\varphi_i)$ it follows
> $h'_i \in K(n) \wedge \varphi_{i+1} \in E(n, h'_i) \wedge$ for each $\rho'_i \in \Xi(h'_i)$ $\#(h'_i, \varphi_{i+1}, \rho'_i)$ is a set ;
> if we define $h'_m = h||(x_1,\varphi_1)|| .. ||(x_m,\varphi_m)$ it follows $h'_m \in K(n) \wedge \varphi \in E(n,h'_m)$.

For each $x \in \mathrm{dom}(k) \cap \mathrm{dom}(h)$ $k(x)=h(x)$, $\sigma \in \Xi(k)$, $\rho \in \Xi(h)$, for each $x \in \mathrm{dom}(k) \cap \mathrm{dom}(h)$ $\sigma(x)=\rho(x)$.

$\dashv$

We now need to verify

- for each $w_1,w_2 \in \{a,b,c,d,e\}$: $w_1 \neq w_2$, t in $E'_{w1}(n+1,k) \cap E'_{w2}(n+1,k)$ and $\sigma \in \Xi(k)$
  $\#(k,t,\sigma)_{(n+1,k,w1)} = \#(k,t,\sigma)_{(n+1,k,w2)}$.

Fortunately for us, for many values of w_1,w_2 it is easy to see that $E'_{w1}(n+1,k) \cap E'_{w2}(n+1,k) = \varnothing$.

We use a table to list all cases where this happens (of course in the table we have barred the cells which are duplicates or not of interest).

	$E'_a(n+1,k)$	$E'_b(n+1,k)$	$E'_c(n+1,k)$	$E'_d(n+1,k)$	$E'_e(n+1,k)$
$E'_a(n+1,k)$		$\varnothing$			
$E'_b(n+1,k)$			$\varnothing$	$\varnothing$	$\varnothing$
$E'_c(n+1,k)$				$\varnothing$	$\varnothing$
$E'_d(n+1,k)$					$\varnothing$
$E'_e(n+1,k)$					

It is immediate to see that when $w_1,w_2 \in \{b,c,d,e\}$ and $w_1 \neq w_2$ we have

$E'_{w1}(n+1,k) \cap E'_{w2}(n+1,k) = \varnothing$.

We can also easily prove that $E'_a(n+1,k) \cap E'_b(n+1,k) = \varnothing$.

Suppose t is in $E'_a(n+1,k) \cap E'_b(n+1,k)$. This means that $t \in E_a(n+1,k)$ and $k \in K(n)^+$, so we can write $k = h \parallel (y,\varphi)$, with $h \in K(n)$, $\varphi \in E(n,h)$, $y \in (V - dom(h))$. We have $t \in E(n,h)$, and since $t \in E_b(n+1,k)$ we have t=y. We can apply assumption 2.1.7 to $t \in E(n,h)$, situations a,c,d,e can not occur, so situation b must occur, but this means that $y \in dom(h)$, against our hypothesis.

Therefore we just need to examine three cases: t in $E'_a(n+1,k) \cap E'_c(n+1,k)$, t in $E'_a(n+1,k) \cap E'_d(n+1,k)$, t in $E'_a(n+1,k) \cap E'_e(n+1,k)$.

Consider the case where t in $\mathbf{E'_a(n+1,k) \cap E'_c(n+1,k)}$, and so $t \in E_a(n+1,k) \cap E_c(n+1,k)$.

As a consequence of $t \in E_c(n+1,k)$ there exist $\varphi, \varphi_1, .., \varphi_m \in E(n,k)$ such that $t = (\varphi)(\varphi_1, \dots, \varphi_m)$,
$\#(k,t,\sigma)_{(n+1,k,c)} = \#(k,\varphi,\sigma) (\#(k,\varphi_1,\sigma), ... , \#(k,\varphi_m,\sigma))$.

As a consequence of $t \in E_a(n+1,k)$ we have that $k \in K(n)^+$, so $k = h \parallel (y,\varphi)$ where $h \in K(n)$, $\varphi \in E(n,h)$, $y \in (V - dom(h))$, and we also have $t \in E(n,h)$; $\sigma = \rho \parallel (y,s)$ with $\rho \in \Xi(h)$, $s \in \#(h,\varphi,\rho)$;
$\#(k,t,\sigma)_{(n+1,k,a)} = \#(h,t,\rho)$.

Since $t \in E(n,h)$ we can apply assumption 2.1.7 and obtain that n>1, there exist $g \in K(n-1)$: $g \sqsubseteq h$, $t \in E(n,g)$, for each $\delta \in \Xi(g)$ $\#(g,\varphi,\delta)$ is a function with m arguments, $(\#(g, \varphi_1, \delta), ... , \#(g, \varphi_m, \delta))$ is a member of its domain, $\#(g, t, \delta) = \#(g,\varphi,\delta) (\#(g,\varphi_1,\delta), ... , \#(g,\varphi_m,\delta))$.

Let $\delta = \rho/dom(g)$. If g=h then $\delta = \rho \in \Xi(g)$, otherwise by assumption 2.1.8 we still get $\delta \in \Xi(g)$.

By assumption 2.1.7 we have

$\#(k,t,\sigma)_{(n+1,k,a)} = \#(h, t, \rho) = \#(g, t, \delta) = \#(g,\varphi,\delta) (\#(g,\varphi_1,\delta), ... , \#(g,\varphi_m,\delta))$.

Since $g, k \in K(n)$, $\varphi, \varphi_i \in E(n,g) \cap E(n,k)$, $\sigma \in \Xi(k)$, $\delta \in \Xi(g)$, etc., we can apply assumption 2.1.6 and obtain that

$\#(k,t,\sigma)_{(n+1,k,a)} = \#(g,\varphi,\delta) (\#(g,\varphi_1,\delta), ... , \#(g,\varphi_m,\delta)) =$
$= \#(k,\varphi,\sigma) (\#(k,\varphi_1,\sigma), ... , \#(k,\varphi_m,\sigma)) = \#(k,t,\sigma)_{(n+1,k,c)}$.

Consider now the case where t in $\mathbf{E'_a(n+1,k) \cap E'_d(n+1,k)}$, and so $t \in E_a(n+1,k) \cap E_d(n+1,k)$.

As a consequence of $t \in E_d(n+1,k)$ there exist $f \in F$, $\varphi_1, .., \varphi_m \in E(n,k)$ such that $t = (f)(\varphi_1, \dots, \varphi_m)$,
$\#(k,t,\sigma)_{(n+1,k,d)} = P_f (\#(k,\varphi_1,\sigma), ... , \#(k,\varphi_m,\sigma))$.

As a consequence of $t \in E_a(n+1,k)$ we have that $k \in K(n)^+$, so $k = h \parallel (y,\varphi)$ where $h \in K(n)$, $\varphi \in E(n,h)$, $y \in (V - dom(h))$, and we also have $t \in E(n,h)$; $\sigma = \rho \parallel (y,s)$ with $\rho \in \Xi(h)$, $s \in \#(h,\varphi,\rho)$;
$\#(k,t,\sigma)_{(n+1,k,a)} = \#(h,t,\rho)$.

Since $t \in E(n,h)$ we can apply assumption 2.1.7 and obtain that n>1, there exist $g \in K(n-1)$: $g \sqsubseteq h$, $t \in E(n,g)$, for each $\delta \in \Xi(g)$ $A_f(\#(g, \varphi_1, \delta), ... , \#(g, \varphi_m, \delta))$, $\#(g, t, \delta) = P_f (\#(g,\varphi_1,\delta), ... , \#(g,\varphi_m,\delta))$.

Let $\delta=\rho/dom(g)$. If g=h then $\delta=\rho\in\Xi(g)$, otherwise by assumption 2.1.8 we still get $\delta\in\Xi(g)$.

By assumption 2.1.7 we have

$\#(k,t,\sigma)_{(n+1,k,a)} = \#(h, t, \rho) = \#(g, t, \delta) = P_f (\#(g,\varphi_1,\delta), ... , \#(g,\varphi_m,\delta))$.

Since g, k $\in$ K(n), $\varphi_i \in E(n,g)\cap E(n,k)$, $\sigma\in\Xi(k)$, $\delta\in\Xi(g)$, etc., we can apply assumption 2.1.6 and obtain that

$\#(k,t,\sigma)_{(n+1,k,a)} = P_f (\#(g,\varphi_1,\delta), ... , \#(g,\varphi_m,\delta)) = P_f (\#(k,\varphi_1,\sigma), ... , \#(k,\varphi_m,\sigma)) = \#(k,t,\sigma)_{(n+1,k,d)}$.

Finally we examine the case where t in $\mathbf{E'_a(n+1,k) \cap E'_e(n+1,k)}$, and so $t \in E_a(n+1,k) \cap E_e(n+1,k)$.

As a consequence to $t\in E_e(n+1,k)$ there exist a positive integer m, $x_1, .., x_m$ distinct $\in$ V-dom(k), φ, $\varphi_1, .. , \varphi_m \in E(n)$ such that $t = \{\}(x_1:\varphi_1, \dots , x_m:\varphi_m, \varphi)$. Moreover we have

- $\varphi_1 \in E(n,k)$, for each $\sigma\in\Xi(k)$ $\#(k,\varphi_1,\sigma)$ is a set ;
- if m>1, for each i=1..m-1 if we define $k'_i = k||(x_1,\varphi_1)|| .. ||(x_i,\varphi_i)$ it follows
  $k'_i \in K(n) \wedge \varphi_{i+1}\in E(n, k'_i) \wedge$ for each $\sigma'_i\in\Xi(k'_i)$ $\#(k'_i, \varphi_{i+1}, \sigma'_i)$ is a set ;
- if we define $k'_m = k||(x_1,\varphi_1)|| .. ||(x_m,\varphi_m)$ it follows $k'_m\in K(n) \wedge \varphi\in E(n,k'_m)$.

For a fixed $\sigma\in\Xi(k)$

$\#(k,t,\sigma)_{(n+1,k,e)} = \{\} (\sigma'_m \in\Xi(k'_m): \sigma \sqsubseteq \sigma'_m , \#(k'_m,\varphi,\sigma'_m))$.

As a consequence of $t\in E_a(n+1,k)$ we have that $k\in K(n)^+$, so $k = h \parallel (y,\varphi)$ where $h\in K(n)$, $\varphi\in E(n,h)$, $y\in (V-dom(h))$, and we also have $t\in E(n,h)$; $\sigma = \rho \parallel (y,s)$ with $\rho \in \Xi(h)$, $s\in \#(h,\varphi,\rho)$;
$\#(k,t,\sigma)_{(n+1,k,a)} = \#(h,t,\rho)$.

Since $t\in E(n,h)$ we can apply assumption 2.1.7 and obtain that n>1, there exists $g\in K(n-1)$: $g \sqsubseteq h$, $t\in E(n,g)$, and also

$\varphi_1 \in E(n-1,g)$, for each $\delta\in\Xi(g)$ $\#(g,\varphi_1,\delta)$ is a set ;
if m>1 for each i=1..m-1 if we define $g'_i = g||(x_1,\varphi_1)|| .. ||(x_i,\varphi_i)$ it follows
$g'_i \in K(n-1) \wedge \varphi_{i+1}\in E(n-1, g'_i) \wedge$ for each $\delta'_i\in\Xi(g'_i)$ $\#(g'_i, \varphi_{i+1}, \delta'_i)$ is a set ;
if we define $g'_m = g||(x_1,\varphi_1)|| .. ||(x_m,\varphi_m)$ it follows $g'_m\in K(n-1) \wedge \varphi\in E(n-1,g'_m)$;

for each $\delta\in\Xi(g)$ $\#(g,t,\delta) = \{\}(\delta'_m \in\Xi(g'_m): \delta \sqsubseteq \delta'_m , \#(g'_m,\varphi,\delta'_m))$.

Let $\delta=\rho/dom(g)$. If g=h then $\delta=\rho\in\Xi(g)$, otherwise by assumption 2.1.8 we still get $\delta\in\Xi(g)$.

By assumption 2.1.7

$\#(k,t,\sigma)_{(n+1,k,a)} = \#(h, t, \rho) = \#(g,t,\delta) = \{\}(\delta'_m \in\Xi(g'_m): \delta \sqsubseteq \delta'_m , \#(g'_m,\varphi,\delta'_m))$.

To complete our proof, we need to show that

$\{\} (\sigma'_m \in\Xi(k'_m): \sigma \sqsubseteq \sigma'_m , \#(k'_m,\varphi,\sigma'_m)) = \{\}(\delta'_m \in\Xi(g'_m): \delta \sqsubseteq \delta'_m , \#(g'_m,\varphi,\delta'_m))$.

This follows by consequence 2.1.12 , that can be applied because of:

$k,g \in K(n)$, $\varphi, \varphi_1, .., \varphi_m \in E(n)$, $x_1, .., x_m$ distinct $\in (V\text{-dom}(k)) \cap (V\text{-dom}(g))$,
$t = \{\}(x_1{:}\varphi_1, \ldots, x_m{:}\varphi_m, \varphi)$,

$\varphi_1 \in E(n,k)$, for each $\sigma \in \Xi(k)$ $\#(k,\varphi_1,\sigma)$ is a set ;
if $m>1$, for each $i=1..m-1$ if we define $k'_i = k \| (x_1,\varphi_1) \| .. \| (x_i,\varphi_i)$ it follows
$k'_i \in K(n) \wedge \varphi_{i+1} \in E(n, k'_i) \wedge$ for each $\sigma'_i \in \Xi(k'_i)$ $\#(k'_i, \varphi_{i+1}, \sigma'_i)$ is a set ;
if we define $k'_m = k \| (x_1,\varphi_1) \| .. \| (x_m,\varphi_m)$ it follows $k'_m \in K(n) \wedge \varphi \in E(n,k'_m)$.

$\varphi_1 \in E(n,g)$, for each $\delta \in \Xi(g)$ $\#(g,\varphi_1,\delta)$ is a set ;
if $m>1$ for each $i=1..m-1$ if we define $g'_i = g \| (x_1,\varphi_1) \| .. \| (x_i,\varphi_i)$ it follows
$g'_i \in K(n) \wedge \varphi_{i+1} \in E(n, g'_i) \wedge$ for each $\delta'_i \in \Xi(g'_i)$ $\#(g'_i, \varphi_{i+1}, \delta'_i)$ is a set ;
if we define $g'_m = g \| (x_1,\varphi_1) \| .. \| (x_m,\varphi_m)$ it follows $g'_m \in K(n) \wedge \varphi \in E(n,g'_m)$.

For each $x \in \text{dom}(k) \cap \text{dom}(g)$ $k(x)=g(x)$, $\sigma \in \Xi(k)$, $\delta \in \Xi(g)$, for each $x \in \text{dom}(k) \cap \text{dom}(g)$ $\sigma(x)=\delta(x)$.

$\dashv$

Let's now perform the verifications required to define $V_b(t)$ and $V_f(t)$. We have to verify that

- for each $k \in K(n+1)$, $w \in \{a,b,c,d,e\}$ and for each t in $E(n) \cap E'_w(n+1,k)$ $V_b(t) = V_b(t)_{(n+1,k,w)}$ (and the same for $V_f(t)$);
- for each $k_1,k_2 \in K(n+1)$, $w_1,w_2 \in \{a,b,c,d,e\}$, $t \in E'_{w1}(n+1,k_1) \cap E'_{w2}(n+1,k_2)$ (such that $t \notin E(n)$) we have $V_b(t)_{(n+1,k1,w1)} = V_b(t)_{(n+1,k2,w2)}$ (and the same for $V_f(t)$).

Suppose t is in $\mathbf{E(n) \cap E'_a(n+1,k)}$. As a consequence of $t \in E_a(n+1,k)$ we have that $k \in K(n)^+$, so $k = h \| (y,\varphi)$ where $h \in K(n)$, $\varphi \in E(n,h)$, $y \in (V\text{–dom}(h))$, and we also have $t \in E(n,h)$,
$V_f(t)_{(n+1,k,a)} = V_f(t)$, $V_b(t)_{(n+1,k,a)} = V_b(t)$.

Suppose t is in $\mathbf{E(n) \cap E'_b(n+1,k)}$. As a consequence of $t \in E_b(n+1,k)$ we have that $k \in K(n)^+$, so $k = h \| (y,\varphi)$ where $h \in K(n)$, $\varphi \in E(n,h)$, $y \in (V\text{–dom}(h))$, and we also have $t=y$, $V_f(t)_{(n+1,k,b)} = \{y\}$, $V_b(t)_{(n+1,k,b)} = \varnothing$.
There exists $g \in K(n)$ such that $t \in E(n,g)$. By assumption 2.1.7 we get $t \in \text{dom}(g)$, $V_f(t)=\{t\}=\{y\}=V_f(t)_{(n+1,k,b)}$, $V_b(t)=\varnothing= V_b(t)_{(n+1,k,b)}$.

Suppose t is in $\mathbf{E(n) \cap E'_c(n+1,k)}$. As a consequence of $t \in E_c(n+1,k)$ there exist $\varphi, \varphi_1, .., \varphi_m \in E(n,k)$ such that $t = (\varphi)(\varphi_1, \ldots, \varphi_m)$, $V_f(t)_{(n+1,k,c)} = V_f(\varphi) \cup V_f(\varphi_1) \cup \ldots \cup V_f(\varphi_m)$, $V_b(t)_{(n+1,k,c)} = V_b(\varphi) \cup V_b(\varphi_1) \cup \ldots \cup V_b(\varphi_m)$.

There exists $\kappa \in K(n)$ such that $t \in E(n,\kappa)$. By assumption 2.1.7 we get $n>1$, $\exists h \in K(n-1)$: $h \sqsubseteq \kappa$, $t \in E(n,h)$, $V_f(t) = V_f(\varphi) \cup V_f(\varphi_1) \cup \ldots \cup V_f(\varphi_m) = V_f(t)_{(n+1,k,c)}$,
$V_b(t) = V_b(\varphi) \cup V_b(\varphi_1) \cup \ldots \cup V_b(\varphi_m) = V_b(t)_{(n+1,k,c)}$.

Suppose t is in $\mathbf{E(n) \cap E'_d(n+1,k)}$. As a consequence of $t \in E_d(n+1,k)$ there exist $f \in F$, $\varphi_1, .. ,\varphi_m \in E(n,k)$ such that $t = (f)(\varphi_1, \ldots, \varphi_m)$, $V_f(t)_{(n+1,k,d)} = V_f(\varphi_1) \cup \ldots \cup V_f(\varphi_m)$,
$V_b(t)_{(n+1,k,d)} = V_b(\varphi_1) \cup \ldots \cup V_b(\varphi_m)$.

There exists $\kappa \in K(n)$ such that $t \in E(n,\kappa)$. By assumption 2.1.7 we get $n>1$, $\exists h \in K(n-1)$: $h \sqsubseteq \kappa$, $t \in E(n,h)$, $V_f(t) = V_f(\varphi_1) \cup \ldots \cup V_f(\varphi_m) = V_f(t)_{(n+1,k,d)}$, $V_b(t) = V_b(\varphi_1) \cup \ldots \cup V_b(\varphi_m) = V_b(t)_{(n+1,k,d)}$.

Suppose t is in **$E(n) \cap E'_e(n+1,k)$**. As a consequence of $t \in E_e(n+1,k)$ there exist a positive integer m, x_1, .., x_m distinct $\in$ V-dom(k), φ, φ_1, .. , $\varphi_m \in E(n)$ such that $t = \{\}(x_1:\varphi_1, \ldots , x_m:\varphi_m, \varphi)$.

Moreover, if m=1

$V_f(t)_{(n+1,k,e)} = V_f(\varphi_1) \cup (V_f(\varphi)-\{x_1\})$;
$V_b(t)_{(n+1,k,e)} = \{x_1\} \cup V_b(\varphi_1) \cup V_b(\varphi)$;

If m>1

$V_f(t)_{(n+1,k,e)} = V_f(\varphi_1) \cup (V_f(\varphi_2)-\{x_1\}) \cup \ldots \cup (V_f(\varphi_m)-\{x_1,..,x_{m-1}\}) \cup (V_f(\varphi)-\{x_1,\ldots,x_m\})$;
$V_b(t)_{(n+1,k,e)} = \{x_{1,..,}x_m\} \cup V_b(\varphi_1) \cup .. \cup V_b(\varphi_m) \cup V_b(\varphi)$.

There exists $\kappa \in K(n)$ such that $t \in E(n,\kappa)$. By assumption 2.1.7 we get $n>1$, $\exists h \in K(n-1)$: $h \sqsubseteq \kappa$, $t \in E(n,h)$,

if m=1

$V_f(t) = V_f(\varphi_1) \cup (V_f(\varphi)-\{x_1\}) = V_f(t)_{(n+1,k,e)}$,
$V_b(t) = \{x_1\} \cup V_b(\varphi_1) \cup V_b(\varphi) = V_b(t)_{(n+1,k,e)}$,

if m>1

$V_f(t) = V_f(\varphi_1) \cup (V_f(\varphi_2)-\{x_1\}) \cup \ldots \cup (V_f(\varphi_m)-\{x_1,..,x_{m-1}\}) \cup (V_f(\varphi)-\{x_1,\ldots,x_m\}) = V_f(t)_{(n+1,k,e)}$,
$V_b(t) = \{x_{1,..,}x_m\} \cup V_b(\varphi_1) \cup .. \cup V_b(\varphi_m) \cup V_b(\varphi) = V_b(t)_{(n+1,k,e)}$.

We now need to verify

- for each $k_1,k_2 \in K(n+1)$, $w_1,w_2 \in \{a,b,c,d,e\}$, $t \in E'_{w1}(n+1,k_1) \cap E'_{w2}(n+1,k_2)$ (such that $t \notin E(n)$) we have $V_b(t)_{(n+1,k1,w1)} = V_b(t)_{(n+1,k2,w2)}$ (and the same for $V_f(t)$).

First of all we observe that for each $k \in K(n+1)$, $t \in E'_a(n+1,k)$ we have that $k \in K(n)^+$, so $k = h \parallel (y,\varphi)$ where $h \in K(n)$, $\varphi \in E(n,h)$, $y \in (V\text{–dom}(h))$, and we also have $t \in E(n,h)$, this means that $t \in E(n)$. This implies that we just need to verify

- for each $k_1,k_2 \in K(n+1)$, $w_1,w_2 \in \{b,c,d,e\}$, $t \in E'_{w1}(n+1,k_1) \cap E'_{w2}(n+1,k_2)$ (such that $t \notin E(n)$) we have $V_b(t)_{(n+1,k1,w1)} = V_b(t)_{(n+1,k2,w2)}$ (and the same for $V_f(t)$).

For each $k_1,k_2 \in K(n+1)$, $w_1,w_2 \in \{b,c,d,e\}$, if $w_1 \neq w_2$ then $E'_{w1}(n+1,k_1) \cap E'_{w2}(n+1,k_2) = \varnothing$.

So we just need to verify

- for each $k_1,k_2 \in K(n+1)$, $w \in \{b,c,d,e\}$, $t \in E'_w(n+1,k_1) \cap E'_w(n+1,k_2)$ (such that $t \notin E(n)$) we have $V_b(t)_{(n+1,k1,w)} = V_b(t)_{(n+1,k2,w)}$ (and the same for $V_f(t)$).

Suppose t is in **$E'_b(n+1,k_1) \cap E'_b(n+1,k_2)$** .

From $t\in E_b(n+1,k_1)$ we obtain that $k_1\in K(n)^+$, so $k_1 = h_1 \parallel (y_1,\varphi_1)$ where $h_1\in K(n)$, $\varphi_1\in E(n,h_1)$, $y_1\in (V\text{–dom}(h_1))$, and we also have $t=y_1$, $V_f(t)_{(n+1,k(1),b)} = \{y_1\}$, $V_b(t)_{(n+1,k(1),b)} = \varnothing$.
From $t\in E_b(n+1,k_2)$ we obtain that $k_2\in K(n)^+$, so $k_2 = h_2 \parallel (y_2,\varphi_2)$ where $h_2\in K(n)$, $\varphi_2\in E(n,h_2)$, $y_2\in (V\text{–dom}(h_2))$, and we also have $t=y_2$, $V_f(t)_{(n+1,k(2),b)} = \{y_2\}$, $V_b(t)_{(n+1,k(2),b)} = \varnothing$.

Hence $V_f(t)_{(n+1,k(1),b)} = \{t\} = V_f(t)_{(n+1,k(2),b)}$; $V_b(t)_{(n+1,k(1),b)} = \varnothing = V_b(t)_{(n+1,k(2),b)}$.

Suppose t is in $\mathbf{E'_c(n+1,k_1) \cap E'_c(n+1,k_2)}$.
As a consequence of $t\in E_c(n+1,k_1)$ there exist $\varphi, \varphi_1, .., \varphi_m\in E(n,k_1)$ such that $t = (\varphi)(\varphi_1, \dots , \varphi_m)$, $V_f(t)_{(n+1,k(1),c)} = V_f(\varphi) \cup V_f(\varphi_1) \cup \dots \cup V_f(\varphi_m)$, $V_b(t)_{(n+1,k(1),c)} = V_b(\varphi) \cup V_b(\varphi_1) \cup \dots \cup V_b(\varphi_m)$.
As a consequence of $t\in E_c(n+1,k_2)$ there exist $\psi, \psi_1, .., \psi_p\in E(n,k_1)$ such that $t = (\psi)(\psi_1, \dots , \psi_p)$, $V_f(t)_{(n+1,k(2),c)} = V_f(\psi) \cup V_f(\psi_1) \cup \dots \cup V_f(\psi_p)$, $V_b(t)_{(n+1,k(2),c)} = V_b(\psi) \cup V_b(\psi_1) \cup \dots \cup V_b(\psi_p)$.

So $(\varphi)(\varphi_1, \dots , \varphi_m) = t = (\psi)(\psi_1, \dots , \psi_p)$, it follows $p=m$, $\psi=\varphi$, $\psi_i=\varphi_i$, hence

$V_f(t)_{(n+1,k(1),c)} = V_f(\varphi) \cup V_f(\varphi_1) \cup \dots \cup V_f(\varphi_m) = V_f(t)_{(n+1,k(2),c)}$;
$V_b(t)_{(n+1,k(1),c)} = V_b(\varphi) \cup V_b(\varphi_1) \cup \dots \cup V_b(\varphi_m) = V_b(t)_{(n+1,k(2),c)}$.

Suppose t is in $\mathbf{E'_d(n+1,k_1) \cap E'_d(n+1,k_2)}$.
As a consequence of $t\in E_d(n+1,k_1)$ there exist $f\in F$, $\varphi_1, .. ,\varphi_m \in E(n,k_1)$ such that $t = (f)(\varphi_1, \dots , \varphi_m)$, $V_f(t)_{(n+1,k(1),d)} = V_f(\varphi_1) \cup \dots \cup V_f(\varphi_m)$, $V_b(t)_{(n+1,k(1),d)} = V_b(\varphi_1) \cup \dots \cup V_b(\varphi_m)$.
As a consequence of $t\in E_d(n+1,k_2)$ there exist $g\in F$, $\psi_1, .. ,\psi_p \in E(n,k_2)$ such that $t = (g)(\psi_1, \dots , \psi_p)$, $V_f(t)_{(n+1,k(2),d)} = V_f(\psi_1) \cup \dots \cup V_f(\psi_p)$, $V_b(t)_{(n+1,k(2),d)} = V_b(\psi_1) \cup \dots \cup V_b(\psi_p)$.

So $(f)(\varphi_1, \dots , \varphi_m) = t = (g)(\psi_1, \dots , \psi_p)$, it follows $f=g$, $p=m$, $\psi_i=\varphi_i$, hence

$V_f(t)_{(n+1,k(1),d)} = V_f(\varphi_1) \cup \dots \cup V_f(\varphi_m) = V_f(t)_{(n+1,k(2),d)}$.
$V_b(t)_{(n+1,k(1),d)} = V_b(\varphi_1) \cup \dots \cup V_b(\varphi_m) = V_b(t)_{(n+1,k(2),d)}$.

Suppose t is in $\mathbf{E'_e(n+1,k_1) \cap E'_e(n+1,k_2)}$.

As a consequence of $t\in E_e(n+1,k_1)$ there exist a positive integer m, $x_1, .., x_m$ distinct $\in V\text{-dom}(k_1)$, $\varphi, \varphi_1, .. , \varphi_m \in E(n)$ such that $t = \{\}(x_1{:}\varphi_1, \dots , x_m{:}\varphi_m, \varphi)$.

Moreover, if m=1

$V_f(t)_{(n+1,k(1),e)} = V_f(\varphi_1) \cup (V_f(\varphi)-\{x_1\})$;
$V_b(t)_{(n+1,k(1),e)} = \{x_1\} \cup V_b(\varphi_1) \cup V_b(\varphi)$;

If m>1

$V_f(t)_{(n+1,k(1),e)} = V_f(\varphi_1) \cup (V_f(\varphi_2)-\{x_1\}) \cup \dots \cup (V_f(\varphi_m)-\{x_1,..,x_{m-1}\}) \cup (V_f(\varphi)-\{x_1,\dots,x_m\})$;
$V_b(t)_{(n+1,k(1),e)} = \{x_{1,..,}x_m\} \cup V_b(\varphi_1) \cup .. \cup V_b(\varphi_m) \cup V_b(\varphi)$.

As a consequence of $t\in E_e(n+1,k_2)$ there exist a positive integer p, $y_1, .., y_p$ distinct $\in V\text{-dom}(k_2)$, ψ, $\psi_1, .. , \psi_p \in E(n)$ such that $t = \{\}(y_1{:}\psi_1, \dots , y_p{:}\psi_p, \psi)$.

Moreover, if p=1

$V_f(t)_{(n+1,k(2),e)} = V_f(\psi_1) \cup (V_f(\psi)\text{-}\{y_1\})$;
$V_b(t)_{(n+1,k(2),e)} = \{y_1\} \cup V_b(\psi_1) \cup V_b(\psi)$;

If p>1

$V_f(t)_{(n+1,k(2),e)} = V_f(\psi_1) \cup (V_f(\psi_2)\text{-}\{y_1\}) \cup \ldots \cup (V_f(\psi_p)\text{-}\{y_1,..,y_{p-1}\}) \cup (V_f(\psi)\text{-}\{y_1,\ldots,y_p\})$;
$V_b(t)_{(n+1,k(2),e)} = \{y_{1,..,}y_p\} \cup V_b(\psi_1) \cup .. \cup V_b(\psi_p) \cup V_b(\psi)$.

So $\{\}(x_1:\varphi_1, \ldots , x_m:\varphi_m, \varphi) = t = \{\}(y_1:\psi_1, \ldots , y_p:\psi_p, \psi)$, it follows p=m, $y_i=x_i$, $\psi_i=\varphi_i$, $\psi=\varphi$, hence if m=1

$V_f(t)_{(n+1,k(1),e)} = V_f(\varphi_1) \cup (V_f(\varphi)\text{-}\{x_1\}) = V_f(t)_{(n+1,k(2),e)}$;
$V_b(t)_{(n+1,k(1),e)} = \{x_1\} \cup V_b(\varphi_1) \cup V_b(\varphi) = V_b(t)_{(n+1,k(2),e)}$.

If m>1

$V_f(t)_{(n+1,k(1),e)} = V_f(\varphi_1) \cup (V_f(\varphi_2)\text{-}\{x_1\}) \cup \ldots \cup (V_f(\varphi_m)\text{-}\{x_1,..,x_{m-1}\}) \cup (V_f(\varphi)\{x_1,\ldots,x_m\}) =$
$= V_f(t)_{(n+1,k(2),e)}$;
$V_b(t)_{(n+1,k(1),e)} = \{x_{1,..,}x_m\} \cup V_b(\varphi_1) \cup .. \cup V_b(\varphi_m) \cup V_b(\varphi) = V_b(t)_{(n+1,k(2),e)}$.

⊣

In the last part of our definition we need to prove all assumptions we have made at step n are true at step n+1. The order in which we will provide these proofs is not the same in which we have listed the assumptions, but this of course is not a problem.

<u>Proof of (assumption) 2.1.5 (at level n+1) :</u>

We need to prove that $K(n) \subseteq K(n+1)$, this is obvious by definition. ⊣

<u>Proof of 2.1.9 :</u>

We need to prove that for each $k\in K(n)$ $E(n,k) \subseteq E(n+1,k)$.

For each $k\in K(n)$ we have $k\in K(n+1)$,

$E(n+1,k) = E'(n,k) \cup E'_a(n+1,k) \cup E'_b(n+1,k) \cup E'_c(n+1,k) \cup E'_d(n+1,k) \cup E'_e(n+1,k) =$
$= E(n,k) \cup E'_a(n+1,k) \cup E'_b(n+1,k) \cup E'_c(n+1,k) \cup E'_d(n+1,k) \cup E'_e(n+1,k)$. ⊣

<u>Proof of 2.1.4 :</u>

We need to prove that for each $k\in K(n+1)$, $t \in E(n+1.k)$

- $t[\ell(t)] \neq$ '('
- if $t[\ell(t)]$ = ')' then $d(t,\ell(t)) = 1$, else $d(t,\ell(t)) = 0$.
- given an integer α in $\{1, .. , \ell(t)\}$ if (t[α]=':' or t[α]=',' or t[α]=')') then $d(t,\alpha)\geq 1$.

We recall that

$E(n+1,k) = E'(n,k) \cup E'_a(n+1,k) \cup E'_b(n+1,k) \cup E'_c(n+1,k) \cup E'_d(n+1,k) \cup E'_e(n+1,k)$.

Let **t∈E'(n,k)**, this means that $t\in E(n,k)$ and $k\in K(n)$. We just need to apply assumption 2.1.4.

Let **t∈E'$_a$(n+1,k)**, this means that $t\in E_a(n+1,k)$ and $k\in K(n)^+$. We can write $k = h \parallel (y,\varphi)$, with $h\in K(n)$, $\varphi\in E(n,h)$, $y\in(V\text{–dom}(h))$. We have $t\in E(n,h)$, so we apply assumption 2.1.4 and the proof is finished.

Let **t∈E'$_b$(n+1,k)**, this means that $t\in E_b(n+1,k)$ and $k\in K(n)^+$. We can write $k = h \parallel (y,\varphi)$, with $h\in K(n)$, $\varphi\in E(n,h)$, $y\in(V\text{–dom}(h))$. We have t=y, so t has just one character and t[1] differs from '(', ')', ':', ','. Therefore the proof is finished.

Let **t∈E'$_c$(n+1,k)**, this means that $t\in E_c(n+1,k)$ and $k\in K(n)$. As a consequence of $t\in E_c(n+1,k)$ there exist $\varphi, \varphi_1, .. , \varphi_m\in E(n,k)$ such that $t = (\varphi)(\varphi_1, \dots , \varphi_m)$.

If we assign m we can give an 'explicit representation' of t. In fact if m=2 $t = (\varphi)(\varphi_1,\varphi_2)$, if m=3 $t = (\varphi)(\varphi_1,\varphi_2,\varphi_3)$, and so on. In this explicit representation of t we can see explicit occurrences of symbols ',' and ')'. There are explicit occurrences of ',' only when m>1. We indicate with q the position of the first explicit occurrence of ')' and the second explicit occurrence of ')' is clearly in position $\ell(t)$. If m>1 we indicate with $q_1, .. , q_{m-1}$ the positions of explicit occurrences of ','.

We have $d(t,q-1) = d(t, 1 + \ell(\varphi)) = d(t,1 + 1) + d(\varphi, \ell(\varphi)) = 1 + d(\varphi, \ell(\varphi))$.

If $t[q-1] = \varphi[\ell(\varphi)] =$ ')' then $d(t,q) = d(t,q-1) - 1 = d(\varphi, \ell(\varphi)) = 1$.

Else $t[q-1] = \varphi[\ell(\varphi)] \notin$ {'(',')'} so $d(t,q) = d(t,q-1) = 1 + d(\varphi, \ell(\varphi)) = 1$.

If m>1 we can prove for each i in 1 .. m-1 $d(t,q_i) = 1$.

First of all we agree that $d(t,q+2) = d(t,q) - 1 + 1 = 1$.

And we agree that $d(t, q_1 - 1) = d(t, q + 1 + \ell(\varphi_1)) = d(t, q + 1 + 1) + d(\varphi_1, \ell(\varphi_1)) = 1 + d(\varphi_1, \ell(\varphi_1))$.

If $t[q_1 - 1] = \varphi_1[\ell(\varphi_1)] =$ ')' then $d(t,q_1) = d(t,q_1 - 1) - 1 = d(\varphi_1, \ell(\varphi_1)) = 1$.

Else $t[q_1 - 1] = \varphi_1[\ell(\varphi_1)] \notin$ {'(',')'} so $d(t,q_1) = d(t,q_1 - 1) = 1 + d(\varphi_1, \ell(\varphi_1)) = 1$.

If m=2 we have finished this step. Suppose now m>2. Let i = 1 .. m – 2 and suppose $d(t,q_i) = 1$. We will show that also $d(t,q_{i+1}) = 1$ holds.

In fact $d(t, q_{i+1} - 1) = d(t, q_i + \ell(\varphi_{i+1})) = d(t, q_i + 1) + d(\varphi_{i+1}, \ell(\varphi_{i+1})) = 1 + d(\varphi_{i+1}, \ell(\varphi_{i+1}))$.

If $t[q_{i+1} - 1] = \varphi_{i+1}[\ell(\varphi_{i+1})] =$ ')' then $d(t,q_{i+1}) = d(t,q_{i+1} - 1) - 1 = d(\varphi_{i+1}, \ell(\varphi_{i+1})) = 1$.

Else $t[q_{i+1} - 1] = \varphi_{i+1}[\ell(\varphi_{i+1})] \notin$ {'(',')'} so $d(t,q_{i+1}) = d(t,q_{i+1} - 1) = 1 + d(\varphi_{i+1}, \ell(\varphi_{i+1})) = 1$.

So it is shown that for each i in 1 .. m-1 $d(t,q_i) = 1$.

We now want to show that $d(t,\ell(t)) = 1$.

If m=1 then

$d(t, \ell(t) - 1) = d(t, q + 1 + \ell(\varphi_1)) = d(t, q + 2) + d(\varphi_1, \ell(\varphi_1)) = d(t,q) + d(\varphi_1, \ell(\varphi_1)) = 1 + d(\varphi_m, \ell(\varphi_m))$.

If m>1 then

$d(t, \ell(t) - 1) = d(t, q_{m-1} + \ell(\varphi_m)) = d(t, q_{m-1} + 1) + d(\varphi_m, \ell(\varphi_m)) = 1 + d(\varphi_m, \ell(\varphi_m))$.

If $t[\ell(t) - 1] = \varphi_m[\ell(\varphi_m)] =$ ')' then $d(t,\ell(t)) = d(t,\ell(t) - 1) - 1 = d(\varphi_m, \ell(\varphi_m)) = 1$.
Else $t[\ell(t) - 1] = \varphi_m[\ell(\varphi_m)] \notin$ {'(',')'} so $d(t,\ell(t)) = d(t,\ell(t) - 1) = 1 + d(\varphi_m, \ell(\varphi_m)) = 1$.

Let's now examine the facts we have to prove. It is true that $t[\ell(t)] \neq$ '(' . It's also true that $t[\ell(t)] =$ ')', $d(t,\ell(t)) = 1$.
Moreover suppose α is in $\{1, .., \ell(t)\}$ and ($t[\alpha]$=':' or $t[\alpha]$=',' or $t[\alpha]$=')').
If α is in $\{q, q_1, .., q_{m-1}, \ell(t)\}$ we have already shown that $d(t,\alpha) = 1$.
Otherwise there are these alternative possibilities:

a) $\alpha>1 \wedge \alpha<q$,
b) $m=1 \wedge \alpha>q+1 \wedge \alpha<\ell(t)$,
c) $m>1 \wedge \alpha>q+1 \wedge \alpha<q_1$,
d) $m>2 \wedge \exists i=1..m-2: \alpha>q_i \wedge \alpha<q_{i+1}$,
e) $m>1 \wedge \alpha>q_{m-1} \wedge \alpha<\ell(t)$.

In situation a) $t[\alpha] = \varphi[\alpha - 1]$; $d(t,\alpha) = d(t,1 + (\alpha - 1)) = d(t,2) + d(\varphi,\alpha-1) = 1 + d(\varphi,\alpha-1) \geq 2$.

In situations b) and c) $t[\alpha] = \varphi_1[\alpha - (q + 1)]$;
$d(t,\alpha) = d(t,q + 1 + (\alpha - (q+1))) = d(t,q + 2) + d(\varphi_1,\alpha - (q + 1)) = 1 + d(\varphi_1,\alpha - (q + 1)) \geq 2$.

In situation d) $t[\alpha] = \varphi_{i+1}[\alpha - q_i]$;
$d(t,\alpha) = d(t,q_i + (\alpha - q_i)) = d(t,q_i + 1) + d(\varphi_{i+1},\alpha - q_i) = 1 + d(\varphi_{i+1},\alpha - q_i) \geq 2$.

In situation e) $t[\alpha] = \varphi_m[\alpha - q_{m-1}]$;
$d(t,\alpha) = d(t,q_{m-1} + (\alpha - q_{m-1})) = d(t,q_{m-1} + 1) + d(\varphi_m,\alpha - q_{m-1}) = 1 + d(\varphi_m,\alpha - q_{m-1}) \geq 2$.

Let **$t\in E'_d(n+1,k)$**, this means that $t\in E_d(n+1,k)$ and $k\in K(n)$. As a consequence of $t\in E_d(n+1,k)$ there exist $f\in F$, $\varphi_1, .. , \varphi_m \in E(n,k)$ such that $t = (f)(\varphi_1, \dots , \varphi_m)$.

If we assign m we can give an 'explicit representation' of t. In fact if m=2 $t = (f)(\varphi_1,\varphi_2)$, if m=3 $t = (f)(\varphi_1,\varphi_2,\varphi_3)$, and so on. In this explicit representation of t we can see explicit occurrences of symbols ',' and ')'. There are explicit occurrences of ',' only when m>1. The occurrences of ')' are clearly in positions 3 and $\ell(t)$. If m>1 we indicate with $q_1, .., q_{m-1}$ the positions of explicit occurrences of ‘,’.

It is immediate to see that d(t,3)=1.

If m>1 we can prove for each i in 1 .. m-1 $d(t,q_i) = 1$.

We have $d(t, q_1 - 1) = d(t, 4 + \ell(\varphi_1)) = d(t, 4 + 1) + d(\varphi_1, \ell(\varphi_1)) = 1 + d(\varphi_1, \ell(\varphi_1))$.

If $t[q_1 - 1] = \varphi_1[\ell(\varphi_1)] =$ ‘)’ then $d(t,q_1) = d(t,q_1 - 1) - 1 = d(\varphi_1, \ell(\varphi_1)) = 1$.

Else $t[q_1 - 1] = \varphi_1[\ell(\varphi_1)] \notin$ {‘(‘,’)’} so $d(t,q_1) = d(t,q_1 - 1) = 1 + d(\varphi_1, \ell(\varphi_1)) = 1$.

If m=2 we have finished this step. Suppose now m>2. Let i = 1 .. m – 2 and suppose $d(t,q_i) = 1$. We will show that also $d(t,q_{i+1}) = 1$ holds.

In fact $d(t, q_{i+1} - 1) = d(t, q_i + \ell(\varphi_{i+1})) = d(t, q_i + 1) + d(\varphi_{i+1}, \ell(\varphi_{i+1})) = 1 + d(\varphi_{i+1}, \ell(\varphi_{i+1}))$.

If $t[q_{i+1} - 1] = \varphi_{i+1}[\ell(\varphi_{i+1})] =$ ‘)’ then $d(t,q_{i+1}) = d(t,q_{i+1} - 1) - 1 = d(\varphi_{i+1}, \ell(\varphi_{i+1})) = 1$.

Else $t[q_{i+1} - 1] = \varphi_{i+1}[\ell(\varphi_{i+1})] \notin$ {‘(‘,’)’} so $d(t,q_{i+1}) = d(t,q_{i+1} - 1) = 1 + d(\varphi_{i+1}, \ell(\varphi_{i+1})) = 1$.

So it is shown that for each i in 1 .. m-1 $d(t,q_i) = 1$.

We now want to show that $d(t,\ell(t)) = 1$.

If m=1 then

$d(t, \ell(t) - 1) = d(t, 4 + \ell(\varphi_1)) = d(t, 4+1) + d(\varphi_1, \ell(\varphi_1)) = 1 + d(\varphi_1, \ell(\varphi_1)) = 1 + d(\varphi_m, \ell(\varphi_m))$.

If m>1 then

$d(t, \ell(t) - 1) = d(t, q_{m-1} + \ell(\varphi_m)) = d(t, q_{m-1} + 1) + d(\varphi_m, \ell(\varphi_m)) = 1 + d(\varphi_m, \ell(\varphi_m))$.

If $t[\ell(t) - 1] = \varphi_m[\ell(\varphi_m)] =$ ‘)’ then $d(t,\ell(t)) = d(t,\ell(t) - 1) - 1 = d(\varphi_m, \ell(\varphi_m)) = 1$.

Else $t[\ell(t) - 1] = \varphi_m[\ell(\varphi_m)] \notin$ {‘(‘,’)’} so $d(t,\ell(t)) = d(t,\ell(t) - 1) = 1 + d(\varphi_m, \ell(\varphi_m)) = 1$.

Let’s now examine the facts we have to prove. It is true that $t[\ell(t)] \neq$ ‘(‘ . It’s also true that $t[\ell(t)] =$ ‘)’, $d(t,\ell(t)) = 1$.

Moreover suppose α is in $\{1, .. , \ell(t)\}$ and ($t[\alpha]$=’:’ or $t[\alpha]$=’,’ or $t[\alpha]$=’)’).

If α is in $\{3, q_1, .., q_{m-1}, \ell(t)\}$ we have already shown that $d(t,\alpha) = 1$.

Otherwise there are these alternative possibilities:

a) $m=1 \wedge \alpha>4 \wedge \alpha<\ell(t)$,

b) $m>1 \wedge \alpha>4 \wedge \alpha<q_1$,

c) $m>2 \wedge \exists i=1..m\text{-}2: \alpha>q_i \wedge \alpha<q_{i+1}$,

d) $m>1 \wedge \alpha>q_{m-1} \wedge \alpha<\ell(t)$.

In situations a) and b) $t[\alpha] = \varphi_1[\alpha - 4]$;
$d(t,\alpha) = d(t,4 + (\alpha - 4)) = d(t,4 + 1) + d(\varphi_1,\alpha - 4) = 1 + d(\varphi_1,\alpha - 4) \geq 2$.

In situation c) $t[\alpha] = \varphi_{i+1}[\alpha - q_i]$;
$d(t,\alpha) = d(t,q_i + (\alpha - q_i)) = d(t,q_i + 1) + d(\varphi_{i+1},\alpha - q_i) = 1 + d(\varphi_{i+1},\alpha - q_i) \geq 2$.

In situation d) $t[\alpha] = \varphi_m[\alpha - q_{m-1}]$;
$d(t,\alpha) = d(t,q_{m-1} + (\alpha - q_{m-1})) = d(t,q_{m-1} + 1) + d(\varphi_m,\alpha - q_{m-1}) = 1 + d(\varphi_m,\alpha - q_{m-1}) \geq 2$.

Let $\mathbf{t \in E'_e(n+1,k)}$, this means that $t \in E_e(n+1,k)$ and $k \in K(n)$. As a consequence of $t \in E_e(n+1,k)$ there exist a positive integer m, x_1, .., x_m distinct $\in$ V-dom(k), φ, φ_1, .. , $\varphi_m \in$ E(n) such that
$t = \{\}(x_1{:}\varphi_1, \ldots , x_m{:}\varphi_m, \varphi)$.

If we assign m we can give an 'explicit representation' of t. In fact if m=2 $t = \{\}(x_1{:}\varphi_1, x_2{:}\varphi_2, \varphi)$, if m=3 $t = \{\}(x_1{:}\varphi_1, x_2{:}\varphi_2, x_3{:}\varphi_3, \varphi)$, and so on. In this explicit representation of t we can see explicit occurrences of symbols ',', ':' and ')'. We indicate with q_1, .., q_m the positions of explicit occurrences of ':' and with r_1, .., r_m the positions of explicit occurrences of ','. The only explicit occurrence of ')' has the position $\ell(t)$. We want to show that for each i=1..m $d(t,q_i)=1$, $d(t,r_i)=1$, and that $d(t,\ell(t))=1$.

It is obvious that $d(t,q_1)=1$. Moreover

$d(t,r_1 - 1) = d(t, q_1 + (r_1 - 1 - q_1)) = d(t, q_1 + \ell(\varphi_1)) = d(t, q_1 + 1) + d(\varphi_1, \ell(\varphi_1)) = 1 + d(\varphi_1, \ell(\varphi_1))$.

If $t[r_1 - 1] = \varphi_1[\ell(\varphi_1)] =$ ')' then $d(t,r_1) = d(t,r_1 - 1) - 1 = d(\varphi_1, \ell(\varphi_1)) = 1$.
Else $t[r_1 - 1] = \varphi_1[\ell(\varphi_1)] \notin$ {'(',')'} so $d(t,r_1) = d(t,r_1 - 1) = 1 + d(\varphi_1, \ell(\varphi_1)) = 1$.

If m=1 we have shown that for each i=1..m $d(t,q_i)=1$, $d(t,r_i)=1$. Now suppose m>1, let i=1..m-1 and suppose $d(t,q_i)=1$, $d(t,r_i)=1$. We show that $d(t,q_{i+1})=1$, $d(t,r_{i+1})=1$.

We have $q_{i+1} = r_i + 2$ and it is immediate that $d(t,q_{i+1})=1$. Moreover

$d(t,r_{i+1} - 1) = d(t, q_{i+1} + (r_{i+1} - 1 - q_{i+1})) = d(t, q_{i+1} + \ell(\varphi_{i+1})) = d(t, q_{i+1} + 1) + d(\varphi_{i+1}, \ell(\varphi_{i+1})) =$
$= 1 + d(\varphi_{i+1}, \ell(\varphi_{i+1}))$.

If $t[r_{i+1} - 1] = \varphi_{i+1}[\ell(\varphi_{i+1})] =$ ')' then $d(t,r_{i+1}) = d(t,r_{i+1} - 1) - 1 = d(\varphi_{i+1}, \ell(\varphi_{i+1})) = 1$.
Else $t[r_{i+1} - 1] = \varphi_{i+1}[\ell(\varphi_{i+1})] \notin$ {'(',')'} so $d(t,r_{i+1}) = d(t,r_{i+1} - 1) = 1 + d(\varphi_{i+1}, \ell(\varphi_{i+1})) = 1$.

Furthermore

$d(t, \ell(t) - 1) = d(t, r_m + (\ell(t) - 1 - r_m)) = d(t, r_m + \ell(\varphi)) = d(t, r_m + 1) + d(\varphi, \ell(\varphi)) = 1 + d(\varphi, \ell(\varphi))$.

If $t[\ell(t) - 1] = \varphi[\ell(\varphi)] =$ ')' then $d(t,\ell(t)) = d(t,\ell(t) - 1) - 1 = d(\varphi, \ell(\varphi)) = 1$.

Else $t[\ell(t) - 1] = \varphi[\ell(\varphi)] \notin$ {'(',')'} so $d(t,\ell(t)) = d(t,\ell(t) - 1) = 1 + d(\varphi, \ell(\varphi)) = 1$.

Let's now examine the facts we have to prove. It is true that $t[\ell(t)] \neq$ '(' . It's also true that $t[\ell(t)] =$ ')', $d(t,\ell(t)) = 1$.

Moreover suppose α is in $\{1, .. , \ell(t)\}$ and ($t[\alpha]$=':' or $t[\alpha]$=',' or $t[\alpha]$=')').

If α is in $\{q_1, .., q_m, r_1, \ldots, r_m, \ell(t)\}$ we have already shown that $d(t,\alpha) = 1$.

Otherwise there are these alternative possibilities:

a) $\exists i=1..m$ such that $q_i<\alpha<r_i$,
b) $r_m < \alpha < \ell(t)$.

In situation a) $t[\alpha] = \varphi_i[\alpha - q_i]$; $d(t,\alpha) = d(t, q_i + (\alpha - q_i)) = d(t, q_i + 1) + d(\varphi_i, \alpha - q_i) =$
$= 1 + d(\varphi_i, \alpha - q_i) \geq 2$.

In situation b) $t[\alpha] = \varphi[\alpha - r_m]$; $d(t,\alpha) = d(t, r_m + (\alpha - r_m)) = d(t, r_m + 1) + d(\varphi, \alpha - r_m) =$
$= 1 + d(\varphi, \alpha - r_m) \geq 2$.

$\dashv$

Proof of 2.1.1 :

We need to prove that for each $k\in K(n+1)$: $k\neq\varepsilon$, $\sigma\in\Xi(k)$ there exist a positive integer m and $x_1,..x_m\in V$, $\varphi_1,..,\varphi_m\in E(n+1)$, $s_1,..,s_m \in M(n+1)$ such that

- $\forall i,j=1..m\ i\neq j \rightarrow x_i\neq x_j$
- $k = (x_1,\varphi_1) \parallel \ldots \parallel (x_m,\varphi_m)$
- $\sigma = (x_1,s_1) \parallel \ldots \parallel (x_m,s_m)$.

We can notice that $E(n) = \bigcup_{k\in K(n)} E(n,k) \subseteq \bigcup_{k\in K(n)} E(n+1,k) \subseteq E(n+1)$.

We can also notice that for each $k\in K(n)$

$E_s(n,k) = \{t \mid t\in E(n,k), \forall\sigma\in\Xi(k)\ \#(k,t,\sigma) \text{ is a set}\} \subseteq$
$\subseteq \{t \mid t\in E(n+1,k), \forall\sigma\in\Xi(k)\ \#(k,t,\sigma) \text{ is a set}\} = E_s(n+1,k)$

$M(n,k) = \bigcup_{t\in E(s)(n,k)} M(k,t) \subseteq \bigcup_{t\in E(s)(n+1,k)} M(k,t) = M(n+1,k)$;

$M(n) = \bigcup_{k\in K(n)} M(n,k) \subseteq \bigcup_{k\in K(n)} M(n+1,k) \subseteq M(n+1)$.

Now let $k\in K(n+1)$ such that $k\neq\varepsilon$, $\sigma\in\Xi(k)$.

If $k \in K(n)$ we can apply our assumption and infer that there exist a positive integer m and $x_1,..x_m \in V$, $\varphi_1,..,\varphi_m \in E(n)$, $s_1,..,s_m \in M(n)$ such that

- $\forall i,j=1..m\ i \neq j \rightarrow x_i \neq x_j$
- $k = (x_1,\varphi_1) \parallel \ldots \parallel (x_m,\varphi_m)$
- $\sigma = (x_1,s_1) \parallel \ldots \parallel (x_m,s_m)$.

So if $k \in K(n)$ our proof is complete.

Now suppose $k \notin K(n)$, i.e. $k \in K(n)^+$.
There exist $h \in K(n)$, $y \in (V\text{–dom}(h))$, $\varphi \in E_s(n,h)$ such that $k = h \parallel (y,\varphi)$.

Also, by consequence 2.1.3, there exist $\rho \in \Xi(h)$, $s \in \#(h,\varphi,\rho)$ such that $\sigma = \rho \parallel (y,s)$.

We can observe that $\varphi \in E(n) \subseteq E(n+1)$, $s \in M(h,\varphi) \subseteq M(n,h) \subseteq M(n) \subseteq M(n+1)$.

If $h=\varepsilon$ then $k = (y,\varphi)$ and $\sigma = (y,s)$, with $y \in V$, $\varphi \in E(n+1)$, $s \in M(n+1)$.

If $h \neq \varepsilon$ we can apply our assumption 2.1.1 to h and ρ, so there exist a positive integer m and $x_1,..x_m \in V$, $\varphi_1,..,\varphi_m \in E(n)$, $s_1,..,s_m \in M(n)$ such that

- $\forall i,j=1..m\ i \neq j \rightarrow x_i \neq x_j$
- $h = (x_1,\varphi_1) \parallel \ldots \parallel (x_m,\varphi_m)$
- $\rho = (x_1,s_1) \parallel \ldots \parallel (x_m,s_m)$.

Therefore

$k = h \parallel (y,\varphi) = (x_1,\varphi_1) \parallel \ldots \parallel (x_m,\varphi_m) \parallel (y,\varphi)$;
$\sigma = \rho \parallel (y,s) = (x_1,s_1) \parallel \ldots \parallel (x_m,s_m) \parallel (y,s)$,

and $x_1,..x_m,y \in V$, $\varphi_1,..,\varphi_m,\varphi \in E(n+1)$, $s_1,..,s_m,s \in M(n+1)$, $\forall i=1..m\ x_i \neq y$ etc. .

$\dashv$

<u>Proof of 2.1.2 :</u>

We need to prove that for each κ in $K(n+1)$

$(\kappa = \varepsilon) \vee$

$(\exists g \in K(n),\ z \in V\text{-dom}(g),\ \psi \in E(n,g){:}\ \kappa = g \parallel (z,\psi) \wedge \forall \sigma \in \Xi(g)\ \#(g,\psi,\sigma) \text{ is a set} \wedge$
$\Xi(\kappa) = \{\ \sigma \parallel (z,s) \mid \sigma \in \Xi(g),\ s \in \#(g,\psi,\sigma)\ \}\ )$.

If $\kappa \in K(n)$ we can apply assumption 2.1.2 and get

$(\kappa = \varepsilon) \vee$

$(n>1 \wedge$
$\exists g \in K(n-1),\ z \in V\text{-dom}(g),\ \psi \in E(n-1,g){:}\ \kappa = g \parallel (z,\psi) \wedge \forall \sigma \in \Xi(g)\ \#(g,\psi,\sigma) \text{ is a set} \wedge$
$\Xi(\kappa) = \{\ \sigma \parallel (z,s) \mid \sigma \in \Xi(g),\ s \in \#(g,\psi,\sigma)\ \}\ )$.

If we consider that $g\in K(n)$, $\psi\in E(n,g)$ the proof is complete, in this case.

Now suppose $\kappa\notin K(n)$, i.e. $\kappa\in K(n)^+$.
There exist $h\in K(n)$, $y\in(V-dom(h))$, $\varphi\in E_s(n,h)$ such that $\kappa = h \,||\, (y,\varphi)$. By consequence 2.1.3 we also get $\Xi(\kappa) = \{ \sigma \,||\, (y,s) \mid \sigma\in\Xi(h), s\in\#(h,\varphi,\sigma) \}$.

$\dashv$

Proof of 2.1.8 :

We need to prove that for each $k\in K(n+1)$, $h\in R(k)$: $h\neq k$ we have $h\in K(n)$ and for each $\sigma\in\Xi(k)$ if we define $\rho = \sigma/dom(h)$ then $\rho\in\Xi(h)$.

If $k\in K(n)$, since $k\neq\varepsilon$ we can exploit assumption 2.1.8 and say that $h\in K(n-1)\subseteq K(n)$ and for each $\sigma\in\Xi(k)$ if we define $\rho = \sigma/dom(h)$ then $\rho\in\Xi(h)$.

Now suppose $k\notin K(n)$, i.e. $k\in K(n)^+$.
There exist $g\in K(n)$, $y\in(V-dom(h))$, $\varphi\in E_s(n,h)$ such that $k = g \,||\, (y,\varphi)$. By consequence 2.1.3 we also get $\Xi(k) = \{ \delta \,||\, (y,s) \mid \delta\in\Xi(g), s\in\#(g,\varphi,\delta) \}$.

Of course we have $h\in R(g)$ and we can distinguish two cases: $h=g$ and $h\neq g$.

If $h=g$ then $h\in K(n)$ and for each $\sigma\in\Xi(k)$ if we define $\rho= \sigma/dom(h)$ then we have to consider there exist $\delta\in\Xi(g)$, $s\in\#(g,\varphi,\delta)$ such that $\sigma = \delta \,||\, (y,s)$, so $\delta = \sigma/dom(\delta) = \sigma/dom(g) = \sigma/dom(h) = \rho$, and $\rho\in\Xi(h)$.

If $h\neq g$ then we can apply assumption 2.1.8 to g and h and obtain that $h\in K(n-1)$, for each $\delta\in\Xi(g)$ if we define $\rho = \delta/dom(h)$ then $\rho\in\Xi(h)$. So $h\in K(n)$. Let $\sigma\in\Xi(k)$ and define $\rho = \sigma/dom(h)$. There exist $\delta\in\Xi(g)$, $s\in\#(g,\varphi,\delta)$ such that $\sigma = \delta \,||\, (y,s)$, so $\rho = \sigma/dom(h) = \delta/dom(h) \in \Xi(h)$.

$\dashv$

Proof of 2.1.7 :

We need to prove that for each $\kappa\in K(n+1)$, $t\in E(n+1,\kappa)$ one and only one of these 5 alternative situations is verified:

a. $t\in C$, $\forall\sigma\in\Xi(\kappa)\ \#(\kappa,t,\sigma) = \#(t)$, $V_f(t)=\varnothing$, $V_b(t)=\varnothing$

b. $t\in dom(\kappa)$, $\forall\sigma\in\Xi(\kappa)\ \#(\kappa,t,\sigma) = \sigma(t)$, $V_f(t)=\{t\}$, $V_b(t)=\varnothing$

c. $\exists h\in K(n)$: $h \sqsubseteq \kappa$, $\exists\ \varphi, \varphi_1, .., \varphi_m \in E(n,h)$:
$t = (\varphi)(\varphi_1, \dots , \varphi_m)$, $t\in E(n+1,h)$,
for each $\rho\in\Xi(h)$ $\#(h,\varphi,\rho)$ is a function with m arguments,
$(\#(h, \varphi_1, \rho), ... , \#(h, \varphi_m, \rho))$ is a member of its domain,
$\#(h, t, \rho) = \#(h,\varphi,\rho) (\#(h,\varphi_1,\rho), ... , \#(h,\varphi_m,\rho))$
$V_f(t) = V_f(\varphi) \cup V_f(\varphi_1) \cup \dots \cup V_f(\varphi_m)$,
$V_b(t) = V_b(\varphi) \cup V_b(\varphi_1) \cup \dots \cup V_b(\varphi_m)$,
for each $\sigma\in\Xi(\kappa)$, $\rho\in\Xi(h)$: $\rho \sqsubseteq \sigma$ it results

$\#(\kappa, t, \sigma) = \#(h, t, \rho)$

d. $\exists h \in K(n)$: $h \sqsubseteq \kappa$, $\exists\, f \in F$, $\varphi_1, .., \varphi_m \in E(n,h)$:
$t = (f)(\varphi_1, \dots, \varphi_m)$, $t \in E(n+1,h)$,
for each $\rho \in \Xi(h)$ $A_f(\#(h, \varphi_1, \rho), ..., \#(h, \varphi_m, \rho))$,
$\#(h, t, \rho) = P_f (\#(h,\varphi_1,\rho), ..., \#(h,\varphi_m,\rho))$
$V_f(t) = V_f(\varphi_1) \cup \dots \cup V_f(\varphi_m)$,
$V_b(t) = V_b(\varphi_1) \cup \dots \cup V_b(\varphi_m)$,
for each $\sigma \in \Xi(\kappa)$, $\rho \in \Xi(h)$: $\rho \sqsubseteq \sigma$ it results
$\#(\kappa, t, \sigma) = \#(h, t, \rho)$

e. $\exists h \in K(n)$: $h \sqsubseteq \kappa$, $\exists\, \varphi, \varphi_1, .., \varphi_m \in E(n)$,
$\exists x_1, \dots, x_m$ distinct $\in V$-dom(h) :
$t = \{\}(x_1{:}\varphi_1, \dots, x_m{:}\varphi_m, \varphi)$, $t \in E(n+1,h)$,

$\varphi_1 \in E(n,h)$, for each $\rho \in \Xi(h)$ $\#(h,\varphi_1,\rho)$ is a set ;
if $m>1$ for each $i=1..m-1$ if we define $h'_i = h\|(x_1,\varphi_1)\| .. \|(x_i,\varphi_i)$ it follows
$h'_i \in K(n) \wedge \varphi_{i+1} \in E(n, h'_i) \wedge$ for each $\rho'_i \in \Xi(h'_i)$ $\#(h'_i, \varphi_{i+1}, \rho'_i)$ is a set ;
if we define $h'_m = h\|(x_1,\varphi_1)\| .. \|(x_m,\varphi_m)$ it follows $h'_m \in K(n) \wedge \varphi \in E(n,h'_m)$;

for each $\rho \in \Xi(h)$
$\#(h,t,\rho) = \{\}(\rho'_m \in \Xi(h'_m): \rho \sqsubseteq \rho'_m , \#(h'_m,\varphi,\rho'_m))$;

if $m=1$ $V_f(t) = V_f(\varphi_1) \cup (V_f(\varphi)-\{x_1\})$;
$V_b(t) = \{x_1\} \cup V_b(\varphi_1) \cup V_b(\varphi)$.

if $m>1$
$V_f(t) = V_f(\varphi_1) \cup (V_f(\varphi_2)-\{x_1\}) \cup \dots \cup (V_f(\varphi_m)-\{x_1,..,x_{m-1}\}) \cup (V_f(\varphi)-\{x_1,\dots,x_m\})$;
$V_b(t) = \{x_{1,..,}x_m\} \cup V_b(\varphi_1) \cup .. \cup V_b(\varphi_m) \cup V_b(\varphi)$,

for each $\sigma \in \Xi(\kappa)$ and for each $\rho \in \Xi(h)$: $\rho \sqsubseteq \sigma$ it results
$\#(\kappa, t, \sigma) = \#(h, t, \rho)$.

We recall that

$E(n+1,\kappa) = E'(n,\kappa) \cup E'_a(n+1,\kappa) \cup E'_b(n+1,\kappa) \cup E'_c(n+1,\kappa) \cup E'_d(n+1,\kappa) \cup E'_e(n+1,\kappa)$.

So we need to prove that
- for each $t \in E'(n,\kappa)$ one of the five alternative situations is verified;
- for each $w \in \{a,b,c,d,e\}$ and $t \in E'_w(n+1,\kappa)$ one of the five alternative situations is verified .

Let **$t \in E'(n,\kappa)$**, this means that $t \in E(n,\kappa)$ and $\kappa \in K(n)$. This case is easily solved, in fact we apply assumption 2.1.7 and obtain that one of the five situations is verified at level n, but this means the situation is also verified at n+1.

Let **$t \in E'_a(n+1,\kappa)$**, this means that $t \in E_a(n+1,\kappa)$ and $\kappa \in K(n)^+$. We can write $\kappa = h \| (y,\varphi)$, with $h \in K(n)$, $\varphi \in E(n,h)$, $y \in (V-\text{dom}(h))$. We have $t \in E(n,h)$, so we can apply assumption 2.1.7 to h and t. Assumption 2.1.7 says that one of five alternative situations (referred to h,n) is true; we need to show that the corresponding situation, referred to κ,n+1, is also true. If unclear, this statement will be immediately clarified.

Let's consider the situation in which

$t \in \mathcal{C}$, $\forall \rho \in \Xi(h)$ $\#(h,t,\rho) = \#(t)$, $V_f(t)=\varnothing$, $V_b(t)=\varnothing$.

In this case for each $\sigma \in \Xi(\kappa)$ there exist $\rho \in \Xi(h)$, $s \in \#(h,\varphi,\rho)$ such that $\sigma = \rho \parallel (y,s)$ and
$\#(\kappa,t,\sigma) = \#(\kappa,t,\sigma)_{(n+1,\kappa,a)} = \#(h,t,\rho) = \#(t)$.

So one of the 5 alternative situations at level n+1 is verified, and there is nothing else we need to show.

Consider the situation where $n>1$, $t \in dom(h)$, $\forall \rho \in \Xi(h)$ $\#(h,t,\rho) = \rho(t)$, $V_f(t)=\{t\}$, $V_b(t)=\varnothing$.
In this case $t \in dom(\kappa)$ and for each $\sigma = \rho \parallel (y,s) \in \Xi(\kappa)$
$\#(\kappa,t,\sigma) = \#(\kappa,t,\sigma)_{(n+1,\kappa,a)} = \#(h,t,\rho) = \rho(t) = \sigma(t)$.

Consider the situation where

$n>1$, $\exists g \in K(n-1)$: $g \sqsubseteq h$, $\exists$ ψ, ψ_1, .. , $\psi_m \in$ E(n-1,g) :
$t = (\psi)(\psi_1, \dots , \psi_m)$, $t \in E(n,g)$,
for each $\delta \in \Xi(g)$ $\#(g,\psi,\delta)$ is a function with m arguments,
($\#(g, \psi_1, \delta)$, ... , $\#(g, \psi_m, \delta)$) is a member of its domain,
$\#(g, t, \delta) = \#(g,\psi,\delta)$ ($\#(g,\psi_1,\delta)$, ... , $\#(g,\psi_m,\delta)$)
$V_f(t) = V_f(\psi) \cup V_f(\psi_1) \cup \dots \cup V_f(\psi_m)$,
$V_b(t) = V_b(\psi) \cup V_b(\psi_1) \cup \dots \cup V_b(\psi_m)$,
for each $\rho \in \Xi(h)$, $\delta \in \Xi(g)$: $\delta \sqsubseteq \rho$ it results
$\#(h, t, \rho) = \#(g, t, \delta)$.

We have

$g \in K(n)$, $g \sqsubseteq \kappa$, ψ, ψ_1, .. , $\psi_m \in$ E(n,g),
$t = (\psi)(\psi_1, \dots , \psi_m)$, $t \in E(n+1,g)$,
for each $\delta \in \Xi(g)$ $\#(g,\psi,\delta)$ is a function with m arguments,
($\#(g, \psi_1, \delta)$, ... , $\#(g, \psi_m, \delta)$) is a member of its domain,
$\#(g, t, \delta) = \#(g,\psi,\delta)$ ($\#(g,\psi_1,\delta)$, ... , $\#(g,\psi_m,\delta)$)
$V_f(t) = V_f(\psi) \cup V_f(\psi_1) \cup \dots \cup V_f(\psi_m)$,
$V_b(t) = V_b(\psi) \cup V_b(\psi_1) \cup \dots \cup V_b(\psi_m)$,

for each $\sigma = \rho \parallel (y,s) \in \Xi(\kappa)$, $\delta \in \Xi(g)$: $\delta \sqsubseteq \sigma$, since $\delta \sqsubseteq \rho$ it results
$\#(\kappa,t,\sigma) = \#(\kappa,t,\sigma)_{(n+1,\kappa,a)} = \#(h,t,\rho) = \#(g, t, \delta)$.

Consider the situation where

$n>1$, $\exists g \in K(n-1)$: $g \sqsubseteq h$, $\exists$ $f \in \mathcal{F}$, ψ_1, .. , $\psi_m \in$ E(n-1,g) :
$t = (f)(\psi_1, \dots , \psi_m)$, $t \in E(n,g)$,
for each $\delta \in \Xi(g)$ A_f($\#(g, \psi_1, \delta)$, ... , $\#(g, \psi_m, \delta)$) ,

$\#(g, t, \delta) = P_f (\#(g,\psi_1,\delta), \ldots , \#(g,\psi_m,\delta))$
$V_f(t) = V_f(\psi_1) \cup \ldots \cup V_f(\psi_m)$,
$V_b(t) = V_b(\psi_1) \cup \ldots \cup V_b(\psi_m)$,
for each $\rho \in \Xi(h)$, $\delta \in \Xi(g)$: $\delta \sqsubseteq \rho$ it results
$\#(h, t, \rho) = \#(g, t, \delta)$.

We have

$g \in K(n)$, $g \sqsubseteq \kappa$, $f \in F$, $\psi_1, .. , \psi_m \in E(n,g)$:
$t = (f)(\psi_1, \ldots , \psi_m)$, $t \in E(n+1,g)$,
for each $\delta \in \Xi(g)$ $A_f(\#(g, \psi_1, \delta), \ldots , \#(g, \psi_m, \delta))$,
$\#(g, t, \delta) = P_f (\#(g,\psi_1,\delta), \ldots , \#(g,\psi_m,\delta))$
$V_f(t) = V_f(\psi_1) \cup \ldots \cup V_f(\psi_m)$,
$V_b(t) = V_b(\psi_1) \cup \ldots \cup V_b(\psi_m)$,

for each $\sigma = \rho \,||\, (y,s) \in \Xi(\kappa)$, $\delta \in \Xi(g)$: $\delta \sqsubseteq \sigma$, since $\delta \sqsubseteq \rho$ it results
$\#(\kappa,t,\sigma) = \#(\kappa,t,\sigma)_{(n+1,\kappa,a)} = \#(h,t,\rho) = \#(g, t, \delta)$.

Finally consider the situation where

$n>1$, $\exists g \in K(n-1)$: $g \sqsubseteq h$, $\exists\ \psi, \psi_1, .. , \psi_m \in E(n-1)$,
$\exists x_1, \ldots , x_m$ distinct $\in V$-dom(g) :
$t = \{\}(x_1{:}\psi_1, \ldots , x_m{:}\psi_m, \psi)$, $t \in E(n,g)$,

$\psi_1 \in E(n-1,g)$, for each $\delta \in \Xi(g)$ $\#(g,\psi_1,\delta)$ is a set ;
if $m>1$ for each $i=1..m-1$ if we define $g'_i = g||(x_1,\psi_1)|| .. ||(x_i,\psi_i)$ it follows
$g'_i \in K(n-1) \wedge \psi_{i+1} \in E(n-1, g'_i) \wedge$ for each $\delta'_i \in \Xi(g'_i)$ $\#(g'_i, \psi_{i+1}, \delta'_i)$ is a set ;
if we define $g'_m = g||(x_1,\psi_1)|| .. ||(x_m,\psi_m)$ it follows $g'_m \in K(n-1) \wedge \psi \in E(n-1,g'_m)$;

for each $\delta \in \Xi(g)$
$\#(g,t,\delta) = \{\}(\delta'_m \in \Xi(g'_m): \delta \sqsubseteq \delta'_m , \#(g'_m,\psi,\delta'_m))$;

if $m=1$ $V_f(t) = V_f(\psi_1) \cup (V_f(\psi)-\{x_1\})$;
$V_b(t) = \{x_1\} \cup V_b(\psi_1) \cup V_b(\psi)$.

if $m>1$
$V_f(t) = V_f(\psi_1) \cup (V_f(\psi_2)-\{x_1\}) \cup \ldots \cup (V_f(\psi_m)-\{x_1,..,x_{m-1}\}) \cup (V_f(\psi)-\{x_1,\ldots,x_m\})$;
$V_b(t) = \{x_{1,..,}x_m\} \cup V_b(\psi_1) \cup .. \cup V_b(\psi_m) \cup V_b(\psi)$,

for each $\rho \in \Xi(h)$ and for each $\delta \in \Xi(g)$: $\delta \sqsubseteq \rho$ it results
$\#(h, t, \rho) = \#(g, t, \delta)$.

Here we have

$g \in K(n)$: $g \sqsubseteq \kappa$, $\psi, \psi_1, .. , \psi_m \in E(n)$,
$x_1, \ldots , x_m$ distinct $\in V$-dom(g) :
$t = \{\}(x_1{:}\psi_1, \ldots , x_m{:}\psi_m, \psi)$, $t \in E(n+1,g)$,

$\psi_1 \in E(n,g)$, for each $\delta \in \Xi(g)$ $\#(g,\psi_1,\delta)$ is a set ;
if m>1 for each i=1..m-1 if we define $g'_i = g \| (x_1,\psi_1) \| .. \| (x_i,\psi_i)$ it follows
$g'_i \in K(n) \wedge \psi_{i+1} \in E(n, g'_i) \wedge$ for each $\delta'_i \in \Xi(g'_i)$ $\#(g'_i, \psi_{i+1}, \delta'_i)$ is a set ;
if we define $g'_m = g \| (x_1,\psi_1) \| .. \| (x_m,\psi_m)$ it follows $g'_m \in K(n) \wedge \psi \in E(n,g'_m)$;

for each $\delta \in \Xi(g)$
$\#(g,t,\delta) = \{\}(\delta'_m \in \Xi(g'_m): \delta \sqsubseteq \delta'_m , \#(g'_m,\psi,\delta'_m))$;

if m=1 $V_f(t) = V_f(\psi_1) \cup (V_f(\psi)-\{x_1\})$;
$V_b(t) = \{x_1\} \cup V_b(\psi_1) \cup V_b(\psi)$.

if m>1
$V_f(t) = V_f(\psi_1) \cup (V_f(\psi_2)-\{x_1\}) \cup \ldots \cup (V_f(\psi_m)-\{x_1,..,x_{m-1}\}) \cup (V_f(\psi)-\{x_1,\ldots,x_m\})$;
$V_b(t) = \{x_{1,..,}x_m\} \cup V_b(\psi_1) \cup .. \cup V_b(\psi_m) \cup V_b(\psi)$,

for each $\sigma = \rho \| (y,s) \in \Xi(\kappa)$, $\delta \in \Xi(g)$: $\delta \sqsubseteq \sigma$, since $\delta \sqsubseteq \rho$ it results
$\#(\kappa,t,\sigma) = \#(\kappa,t,\sigma)_{(n+1,\kappa,a)} = \#(h,t,\rho) = \#(g, t, \delta)$.

Let **$t \in E'_b(n+1,\kappa)$**, this means that $t \in E_b(n+1,\kappa)$ and $\kappa \in K(n)^+$. We can write $\kappa = h \| (y,\varphi)$, with $h \in K(n)$, $\varphi \in E(n,h)$, $y \in (V\text{–dom}(h))$. We have t=y, so $t \in dom(\kappa)$, for each $\sigma = \rho \| (y,s) \in \Xi(\kappa)$ $\#(\kappa,t,\sigma) = \#(\kappa,t,\sigma)_{(n+1,k,b)} = \sigma(y) = \sigma(t)$, $V_f(t) = V_f(t)_{(n+1,k,b)} = \{t\}$, $V_b(t) = V_b(t)_{(n+1,k,b)} = \varnothing$.

Let **$t \in E'_c(n+1,\kappa)$**, this means that $t \in E_c(n+1,\kappa)$ and $\kappa \in K(n)$. As a consequence of $t \in E_c(n+1,\kappa)$ there exist $\varphi, \varphi_1, .., \varphi_m \in E(n,\kappa)$ such that $t = (\varphi)(\varphi_1, \ldots , \varphi_m)$, $t \in E(n+1,\kappa)$,
for each $\sigma \in \Xi(\kappa)$ $\#(\kappa,\varphi,\sigma)$ is a function with m arguments,
$(\#(\kappa, \varphi_1, \sigma), ... , \#(\kappa, \varphi_m, \sigma))$ is a member of its domain,
$\#(\kappa, t, \sigma) = \#(\kappa,t,\sigma)_{(n+1,\kappa,c)} = \#(\kappa,\varphi,\sigma) (\#(\kappa,\varphi_1,\sigma), ... , \#(\kappa,\varphi_m,\sigma))$,
$V_f(t) = V_f(t)_{(n+1,\kappa,c)} = V_f(\varphi) \cup V_f(\varphi_1) \cup \ldots \cup V_f(\varphi_m)$,
$V_b(t) = V_b(t)_{(n+1,\kappa,c)} = V_b(\varphi) \cup V_b(\varphi_1) \cup \ldots \cup V_b(\varphi_m)$,
for each $\sigma \in \Xi(\kappa)$, $\rho \in \Xi(\kappa)$: $\rho \sqsubseteq \sigma$ it results $\rho=\sigma$ and obviously $\#(\kappa, t, \sigma) = \#(\kappa, t, \rho)$.

Since $\kappa \sqsubseteq \kappa$ there is nothing else to prove.

Let **$t \in E'_d(n+1,\kappa)$**, this means that $t \in E_d(n+1,\kappa)$ and $\kappa \in K(n)$. As a consequence of $t \in E_d(n+1,\kappa)$ there exist $f \in F$, $\varphi_1, .. , \varphi_m \in E(n,\kappa)$ such that $t = (f)(\varphi_1, \ldots , \varphi_m)$, $t \in E(n+1,\kappa)$,
for each $\sigma \in \Xi(\kappa)$ $A_f(\#(\kappa, \varphi_1, \sigma), ... , \#(\kappa, \varphi_m, \sigma))$ is true ,
$\#(\kappa,t,\sigma) = \#(\kappa,t,\sigma)_{(n+1,\kappa,d)} = P_f (\#(\kappa,\varphi_1,\sigma), ... , \#(\kappa,\varphi_m,\sigma))$,
$V_f(t) = V_f(t)_{(n+1,\kappa,d)} = V_f(\varphi_1) \cup \ldots \cup V_f(\varphi_m)$,
$V_b(t) = V_b(t)_{(n+1,\kappa,d)} = V_b(\varphi_1) \cup \ldots \cup V_b(\varphi_m)$,
for each $\sigma \in \Xi(\kappa)$, $\rho \in \Xi(\kappa)$: $\rho \sqsubseteq \sigma$ it results $\rho=\sigma$ and obviously $\#(\kappa, t, \sigma) = \#(\kappa, t, \rho)$.

Since $\kappa \sqsubseteq \kappa$ there is nothing else to prove.

Let **$t \in E'_e(n+1,\kappa)$**, this means that $t \in E_e(n+1,\kappa)$ and $\kappa \in K(n)$. As a consequence of $t \in E_e(n+1,\kappa)$ there exist a positive integer m, $x_1, .., x_m$ distinct $\in V\text{-dom}(\kappa)$, $\varphi, \varphi_1, .. , \varphi_m \in E(n)$ such that

$t = \{\}(x_1{:}\varphi_1, \dots , x_m{:}\varphi_m, \varphi)$. Moreover we have $t \in E(n+1,\kappa)$,

- $\varphi_1 \in E(n,\kappa)$, for each $\sigma \in \Xi(\kappa)$ $\#(\kappa,\varphi_1,\sigma)$ is a set ;
- if m>1, for each i=1..m-1 if we define $\kappa'_i = \kappa || (x_1,\varphi_1) || \, .. \, || (x_i,\varphi_i)$ it follows
  $\kappa'_i \in K(n) \wedge \varphi_{i+1} \in E(n, \kappa'_i) \wedge$ for each $\sigma'_i \in \Xi(\kappa'_i)$ $\#(\kappa'_i, \varphi_{i+1}, \sigma'_i)$ is a set ;
- if we define $\kappa'_m = \kappa || (x_1,\varphi_1) || \, .. \, || (x_m,\varphi_m)$ it follows $\kappa'_m \in K(n) \wedge \varphi \in E(n,\kappa'_m)$.

For a fixed $\sigma \in \Xi(\kappa)$

$$\#(\kappa,t,\sigma) = \#(\kappa,t,\sigma)_{(n+1,\kappa,e)} = \{\} \, (\sigma'_m \in \Xi(\kappa'_m): \sigma \sqsubseteq \sigma'_m \, , \#(\kappa'_m,\varphi,\sigma'_m)) \, .$$

If m=1

$V_f(t) = V_f(t)_{(n+1,\kappa,e)} = V_f(\varphi_1) \cup (V_f(\varphi)-\{x_1\})$;
$V_b(t) = V_b(t)_{(n+1,\kappa,e)} = \{x_1\} \cup V_b(\varphi_1) \cup V_b(\varphi)$.

If m>1

$V_f(t) = V_f(t)_{(n+1,\kappa,e)} = V_f(\varphi_1) \cup (V_f(\varphi_2)-\{x_1\}) \cup \dots \cup (V_f(\varphi_m)-\{x_1,..,x_{m-1}\}) \cup (V_f(\varphi)-\{x_1,\dots,x_m\})$;
$V_b(t) = V_b(t)_{(n+1,\kappa,e)} = \{x_{1,..,}x_m\} \cup V_b(\varphi_1) \cup .. \cup V_b(\varphi_m) \cup V_b(\varphi)$.

Finally, for each $\sigma \in \Xi(\kappa)$, $\rho \in \Xi(\kappa)$: $\rho \sqsubseteq \sigma$ it results $\rho=\sigma$ and obviously $\#(\kappa, t, \sigma) = \#(\kappa, t, \rho)$.

Since $\kappa \sqsubseteq \kappa$ there is nothing else to prove.

$\dashv$

Proof of 2.1.6 :

Let κ,k in K(n+1) such that for each x in $dom(\kappa) \cap dom(k)$ $\kappa(x)=k(x)$. Let $t \in E(n+1,\kappa) \cap E(n+1,k)$. Let $\sigma_\kappa \in \Xi(\kappa)$, $\rho_k \in \Xi(k)$ such that $\forall x \in (dom(\kappa) \cap dom(k))$ $\sigma_\kappa(x) = \rho_k(x)$.
We need to show that $\#(\kappa,t,\sigma_\kappa) = \#(k,t,\rho_k)$.

We have proved 2.1.7 is true at level n+1, so
- $t \in E(n+1,\kappa)$ implies one of five alternative situations is verified ,
- $t \in E(n+1,k)$ implies one of five alternative situations is verified .

Suppose situation a is the true situation caused by $t \in E(n+1,\kappa)$. We have

$t \in \mathcal{C}$, $\#(\kappa,t,\sigma_\kappa) = \#(t)$.

This entails situation a is also the true situation caused by $t \in E(n+1,k)$. So

$t \in \mathcal{C}$, $\#(k,t,\rho_k) = \#(t) = \#(\kappa,t,\sigma_\kappa)$.

The same kind of reasoning applies for other situations. We now analyze the case where situation b is the true situation caused by $t \in E(n+1,\kappa)$. We have

$t \in dom(\kappa)$, $\#(\kappa,t,\sigma_\kappa) = \sigma_\kappa(t)$;
$t \in dom(k)$, $\#(k,t,\rho_k) = \rho_k(t)$;

Since $t \in (\mathrm{dom}(\kappa) \cap \mathrm{dom}(k))$ $\sigma_\kappa(t) = \rho_k(t)$, and $\#(\kappa,t,\sigma_\kappa) = \#(k,t,\rho_k)$.

We turn to examine the case where situation c is the true situation caused by $t \in E(n+1,\kappa)$. We have

$\exists h \in K(n)$: $h \sqsubseteq \kappa$, $\exists\ \varphi, \varphi_1, .., \varphi_m \in E(n,h)$:
$t = (\varphi)(\varphi_1, \dots, \varphi_m)$, $t \in E(n+1,h)$,
for each $\sigma \in \Xi(h)$ $\#(h,\varphi,\sigma)$ is a function with m arguments,
$(\#(h, \varphi_1, \sigma), ..., \#(h, \varphi_m, \sigma))$ is a member of its domain,
$\#(h, t, \sigma) = \#(h,\varphi,\sigma) (\#(h,\varphi_1,\sigma), ..., \#(h,\varphi_m,\sigma))$,
for each $\sigma \in \Xi(h)$: $\sigma \sqsubseteq \sigma_\kappa$ it results
$\#(\kappa, t, \sigma_\kappa) = \#(h, t, \sigma)$;

$\exists g \in K(n)$: $g \sqsubseteq k$, $\exists\ \psi, \psi_1, .., \psi_p \in E(n,g)$:
$t = (\psi)(\psi_1, \dots, \psi_p)$, $t \in E(n+1,g)$,
for each $\rho \in \Xi(g)$ $\#(g,\psi,\rho)$ is a function with p arguments,
$(\#(g, \psi_1, \rho), ..., \#(g, \psi_p, \rho))$ is a member of its domain,
$\#(g, t, \rho) = \#(g,\psi,\rho) (\#(g,\psi_1,\rho), ..., \#(g,\psi_p,\rho))$,
for each $\rho \in \Xi(g)$: $\rho \sqsubseteq \rho_k$ it results
$\#(k, t, \rho_k) = \#(g, t, \rho)$.

Of course p=m, $\psi=\varphi$ and $\psi_i=\varphi_i$ for i=1..m .

Let $\sigma = \sigma_\kappa/\mathrm{dom}(h)$ and $\rho = \rho_k/\mathrm{dom}(g)$. We have proved 2.1.8 is true at level n+1, so $\sigma \in \Xi(h)$ and $\rho \in \Xi(g)$. Therefore

$\#(\kappa, t, \sigma_\kappa) = \#(h, t, \sigma) = \#(h,\varphi,\sigma) (\#(h,\varphi_1,\sigma), ..., \#(h,\varphi_m,\sigma))$;
$\#(k, t, \rho_k) = \#(g, t, \rho) = \#(g,\varphi,\rho) (\#(g,\varphi_1,\rho), ..., \#(g,\varphi_m,\rho))$.

Now $h,g \in K(n)$, for each $x \in \mathrm{dom}(h) \cap \mathrm{dom}(g)$ $x \in \mathrm{dom}(\kappa) \cap \mathrm{dom}(k)$ so $h(x)=\kappa(x)=k(x)=g(x)$ and $\sigma(x)=\sigma_\kappa(x)=\rho_k(x)=\rho(x)$. By assumption 2.1.6 $\#(h,\varphi,\sigma) = \#(g,\varphi,\rho)$ and for each i=1..m $\#(h,\varphi_i,\sigma) = \#(g,\varphi_i,\rho)$ hence

$\#(\kappa, t, \sigma_\kappa) = \#(h, t, \sigma) = \#(g, t, \rho) = \#(k, t, \rho_k)$.

Next we examine the case where situation d is the true situation caused by $t \in E(n+1,\kappa)$. We have

$\exists h \in K(n)$: $h \sqsubseteq \kappa$, $\exists\ f \in F$, $\varphi_1, .., \varphi_m \in E(n,h)$:
$t = (f)(\varphi_1, \dots, \varphi_m)$, $t \in E(n+1,h)$,
for each $\sigma \in \Xi(h)$ $A_f(\#(h, \varphi_1, \sigma), ..., \#(h, \varphi_m, \sigma))$,
$\#(h, t, \sigma) = P_f (\#(h,\varphi_1,\sigma), ..., \#(h,\varphi_m,\sigma))$
for each $\sigma \in \Xi(h)$: $\sigma \sqsubseteq \sigma_\kappa$ it results
$\#(\kappa, t, \sigma_\kappa) = \#(h, t, \sigma)$;

$\exists g \in K(n)$: $g \sqsubseteq k$, $\exists\ f \in F$, $\psi_1, .., \psi_p \in E(n,g)$:
$t = (f)(\psi_1, \dots, \psi_p)$, $t \in E(n+1,g)$,

for each $\rho \in \Xi(g)$ $A_f(\#(g, \psi_1, \rho), ... , \#(g, \psi_p, \rho))$,
$\#(g, t, \rho) = P_f (\#(g,\psi_1,\rho), ... , \#(g,\psi_p,\rho))$
for each $\rho \in \Xi(g)$: $\rho \sqsubseteq \rho_k$ it results
$\#(k, t, \rho_k) = \#(g, t, \rho)$.

Of course p=m and $\psi_i=\varphi_i$ for i=1..m .

Let $\sigma = \sigma_\kappa/dom(h)$ and $\rho = \rho_k/dom(g)$. We have proved 2.1.8 is true at level n+1, so $\sigma \in \Xi(h)$ and $\rho \in \Xi(g)$. Therefore

$\#(\kappa, t, \sigma_\kappa) = \#(h, t, \sigma) = P_f (\#(h,\varphi_1,\sigma), ... , \#(h,\varphi_m,\sigma))$;
$\#(k, t, \rho_k) = \#(g, t, \rho) = P_f (\#(g,\varphi_1,\rho), ... , \#(g,\varphi_m,\rho))$.

Now $h,g \in K(n)$, for each $x \in dom(h) \cap dom(g)$ $x \in dom(\kappa) \cap dom(k)$ so $h(x)=\kappa(x)=k(x)=g(x)$ and $\sigma(x)=\sigma_\kappa(x)=\rho_k(x)=\rho(x)$. By assumption 2.1.6, for each i=1..m $\#(h,\varphi_i,\sigma) = \#(g,\varphi_i,\rho)$ hence

$\#(\kappa, t, \sigma_\kappa) = \#(h, t, \sigma) = \#(g, t, \rho) = \#(k, t, \rho_k)$.

We still need to examine the case where situation e is the true situation caused by $t \in E(n+1,\kappa)$. We have

$\exists h \in K(n)$: $h \sqsubseteq \kappa$, $\exists\ \varphi, \varphi_1, .. , \varphi_m \in E(n)$,
$\exists x_1, \dots , x_m$ distinct $\in \mathcal{V}$-dom(h) :
$t = \{\}(x_1:\varphi_1, \dots , x_m:\varphi_m, \varphi)$, $t \in E(n+1,h)$,

$\varphi_1 \in E_s(n,h)$;
if m>1 for each i=1..m-1 if we define $h'_i = h||(x_1,\varphi_1)|| .. ||(x_i,\varphi_i)$ it follows
$h'_i \in K(n) \wedge \varphi_{i+1} \in E_s(n, h'_i)$;
if we define $h'_m = h||(x_1,\varphi_1)|| .. ||(x_m,\varphi_m)$ it follows $h'_m \in K(n) \wedge \varphi \in E(n,h'_m)$;

for each $\sigma \in \Xi(h)$
$\#(h,t,\sigma) = \{\}(\sigma'_m \in \Xi(h'_m): \sigma \sqsubseteq \sigma'_m , \#(h'_m,\varphi,\sigma'_m))$;

for each $\sigma \in \Xi(h)$: $\sigma \sqsubseteq \sigma_\kappa$ it results $\#(\kappa, t, \sigma_\kappa) = \#(h, t, \sigma)$;

$\exists g \in K(n)$: $g \sqsubseteq k$, $\exists\ \psi, \psi_1, .. , \psi_p \in E(n)$,
$\exists y_1, \dots , y_p$ distinct $\in \mathcal{V}$-dom(g) :
$t = \{\}(y_1:\psi_1, \dots , y_p:\psi_p, \psi)$, $t \in E(n+1,g)$,

$\varphi_1 \in E_s(n,g)$;
if m>1 for each i=1..m-1 if we define $g'_i = g||(y_1,\psi_1)|| .. ||(y_i,\psi_i)$ it follows
$g'_i \in K(n) \wedge \psi_{i+1} \in E_s(n, g'_i)$;
if we define $g'_p = g||(y_1,\psi_1)|| .. ||(y_p,\psi_p)$ it follows $g'_p \in K(n) \wedge \psi \in E(n,g'_p)$;

for each $\rho \in \Xi(g)$
$\#(g,t,\rho) = \{\}(\rho'_m \in \Xi(g'_m): \rho \sqsubseteq \rho'_m , \#(g'_m,\psi,\rho'_m))$;

for each $\rho \in \Xi(g)$: $\rho \sqsubseteq \rho_k$ it results $\#(k, t, \rho_k) = \#(g, t, \rho)$.

Of course p=m, $\psi=\varphi$ and $y_i=x_i$, $\psi_i=\varphi_i$ for i=1..m .

Let $\sigma = \sigma_\kappa/dom(h)$ and $\rho = \rho_k/dom(g)$. We have proved 2.1.8 is true at level n+1, so $\sigma \in \Xi(h)$ and $\rho \in \Xi(g)$. Therefore

$\#(\kappa, t, \sigma_\kappa) = \#(h, t, \sigma) = \{\}(\sigma'_m \in \Xi(h'_m): \sigma \sqsubseteq \sigma'_m , \#(h'_m,\varphi,\sigma'_m))$;
$\#(k, t, \rho_k) = \#(g, t, \rho) = \{\}(\rho'_m \in \Xi(g'_m): \rho \sqsubseteq \rho'_m , \#(g'_m,\varphi,\rho'_m))$.

Now $h,g \in K(n)$, for each $x \in dom(h) \cap dom(g)$ $x \in dom(\kappa) \cap dom(k)$ so $h(x)=\kappa(x)=k(x)=g(x)$ and $\sigma(x)=\sigma_\kappa(x)=\rho_k(x)=\rho(x)$. So we can apply consequence 2.1.12 and get

$\{\} (\sigma'_m \in \Xi(h'_m): \sigma \sqsubseteq \sigma'_m , \#(h'_m,\varphi,\sigma'_m)) = \{\}(\rho'_m \in \Xi(g'_m): \rho \sqsubseteq \rho'_m , \#(g'_m,\varphi,\rho'_m))$.

In other words $\#(\kappa, t, \sigma_\kappa) = \#(k, t, \rho_k)$.

□

We have finished with definition 2.1. We now prove a result that is closely related to the definition.

Lemma 2.2: for each positive integer n and $t \in E(n,k)$ $(V_f(t) \subseteq dom(k) \wedge V_b(t) \subseteq V\text{-}dom(k))$.

Proof:

We use induction on n.

Initial step:
For each $t \in C$ $V_f(t) = \emptyset \subseteq \emptyset = dom(\varepsilon)$; $V_b(t) = \emptyset \subseteq V = V - dom(\varepsilon)$.

Inductive step:

Let $k \in K(n+1)$, $t \in E(n+1,k)$. We have seen that

$E(n+1,k) = E'(n,k) \cup E'_a(n+1,k) \cup E'_b(n+1,k) \cup E'_c(n+1,k) \cup E'_d(n+1,k) \cup E'_e(n+1,k)$.

If $t \in E'(n,k)$ then $t \in E(n,k)$ and by induction our statement holds.

If $t \in E'_a(n+1,k)$ then we have $k \in K(n)^+$, so $k = h \parallel (y,\varphi)$ where $h \in K(n)$, $\varphi \in E(n,h)$, $y \in (V\text{-}dom(h))$, and we also have $t \in E(n,h)$, $y \notin V_b(t)$,
$V_f(t) \subseteq dom(h) \subseteq dom(k)$; $V_b(t) \subseteq V - dom(h)$, $y \notin V_b(t)$ so $V_b(t) \subseteq V - dom(k)$.

If $t \in E'_b(n+1,k)$ then we have $k \in K(n)^+$, so $k = h \parallel (y,\varphi)$ where $h \in K(n)$, $\varphi \in E(n,h)$, $y \in (V\text{-}dom(h))$, and we also have t=y, $V_f(t) = V_f(t)_{(n+1,k,b)} = \{y\} \subseteq dom(k)$, $V_b(t) = V_b(t)_{(n+1,k,b)} = \emptyset \subseteq V - dom(k)$.

If $t \in E'_c(n+1,k)$ then there exist $\varphi, \varphi_1, .. , \varphi_m \in E(n,k)$ such that $t = (\varphi)(\varphi_1, \dots , \varphi_m)$,
$V_f(t) = V_f(t)_{(n+1,k,c)} = V_f(\varphi) \cup V_f(\varphi_1) \cup \dots \cup V_f(\varphi_m) \subseteq dom(k)$
$V_b(t) = V_b(t)_{(n+1,k,c)} = V_b(\varphi) \cup V_b(\varphi_1) \cup \dots \cup V_b(\varphi_m) \subseteq V - dom(k)$.

If $t \in E'_d(n+1,k)$ then there exist $f \in F$, $\varphi_1, .., \varphi_m \in E(n,k)$ such that $t = (f)(\varphi_1, \dots, \varphi_m)$,
$V_f(t) = V_f(t)_{(n+1,k,d)} = V_f(\varphi_1) \cup \dots \cup V_f(\varphi_m) \subseteq dom(k)$
$V_b(t) = V_b(t)_{(n+1,k,d)} = V_b(\varphi_1) \cup \dots \cup V_b(\varphi_m) \subseteq V - dom(k)$.

If $t \in E'_e(n+1,k)$ then there are a positive integer m, $x_1, \dots, x_m \in V$ and $\varphi, \varphi_1, .., \varphi_m \in E(n)$ such that $t = \{\}(x_1:\varphi_1, \dots, x_m:\varphi_m, \varphi)$. Furthermore

- $x_1, \dots, x_m$ distinct $\in V$-dom(k) ;
- $\varphi_1 \in E(n,k)$, for each $\sigma \in \Xi(k)$ $\#(k,\varphi_1,\sigma)$ is a set ;
- if m>1, for each i=1..m-1 if we define $k'_i = k||(x_1,\varphi_1)|| .. ||(x_i,\varphi_i)$ it follows
  $k'_i \in K(n) \wedge \varphi_{i+1} \in E(n, k'_i) \wedge$ for each $\sigma'_i \in \Xi(k'_i)$ $\#(k'_i, \varphi_{i+1}, \sigma'_i)$ is a set ;
- if we define $k'_m = k||(x_1,\varphi_1)|| .. ||(x_m,\varphi_m)$ it follows $k'_m \in K(n) \wedge \varphi \in E(n,k'_m)$.

If m=1

$V_f(t) = V_f(t)_{(n+1,k,e)} = V_f(\varphi_1) \cup (V_f(\varphi)-\{x_1\})$;
$V_b(t) = V_b(t)_{(n+1,k,e)} = \{x_1\} \cup V_b(\varphi_1) \cup V_b(\varphi)$.

If m>1

$V_f(t) = V_f(t)_{(n+1,k,e)} = V_f(\varphi_1) \cup (V_f(\varphi_2)-\{x_1\}) \cup \dots \cup (V_f(\varphi_m)-\{x_1,..,x_{m-1}\}) \cup (V_f(\varphi)-\{x_1,\dots,x_m\})$;
$V_b(t) = V_b(t)_{(n+1,k,e)} = \{x_1,..,x_m\} \cup V_b(\varphi_1) \cup .. \cup V_b(\varphi_m) \cup V_b(\varphi)$.

Let's consider the case where m=1.

By the inductive hypothesis

$V_f(\varphi_1) \subseteq dom(k)$; $V_f(\varphi) \subseteq dom(k'_m) = dom(k) \cup \{x_1\}$; so $V_f(t) \subseteq dom(k)$;

$V_b(\varphi_1) \subseteq V$-dom(k); $V_b(\varphi) \subseteq V$-dom(k'_m) = V- (dom(k) $\cup \{x_1\}) \subseteq V$ - dom(k) ; therefore
$V_b(t) = \{x_1\} \cup V_b(\varphi_1) \cup V_b(\varphi) \subseteq V$ - dom(k) .

We now turn to examine the case where m>1 .

By the inductive hypothesis

$V_f(\varphi_1) \subseteq dom(k)$;
for each i=1..m-1 $V_f(\varphi_{i+1}) \subseteq dom(k'_i) = dom(k) \cup \{x_1, .., x_i\}$,
so $V_f(\varphi_{i+1}) - \{x_1, .., x_i\} \subseteq dom(k)$;
$V_f(\varphi) \subseteq dom(k'_m) = dom(k) \cup \{x_1, .., x_m\}$,
so $V_f(\varphi) - \{x_1, .., x_m\} \subseteq dom(k)$.

It follows

$V_f(t) = V_f(\varphi_1) \cup (V_f(\varphi_2)-\{x_1\}) \cup \dots \cup (V_f(\varphi_m)-\{x_1,..,x_{m-1}\}) \cup (V_f(\varphi)-\{x_1,\dots,x_m\}) \subseteq dom(k)$.

And also by the inductive hypothesis

$V_b(\varphi_1) \subseteq V$-dom(k) ;
for each i=1..m-1 $V_b(\varphi_{i+1}) \subseteq V$-dom(k'$_i$) $\subseteq V$-dom(k) ;
$V_b(\varphi) \subseteq V$-dom(k'$_m$) $\subseteq V$-dom(k) .

Therefore

$V_b(t) = \{x_{1,..,}x_m\} \cup V_b(\varphi_1) \cup .. \cup V_b(\varphi_m) \cup V_b(\varphi) \subseteq V$-dom(k) .

□

This result ensures that $V_b(t)$ and $V_f(t)$ are always disjoint, so a variabile cannot have both bound and free occurrences in the same expression.

3. Introduction to the deductive methodology

In this section we will cover some fundamental principles that underlie our inferences. An important target will be achieved with the proof of theorem 3.5, which is a simple but significant step to justify our deductive methodology.

Some preliminary definitions. Let $K = \bigcup_{n\geq 1} K(n)$, for each $k\in K$ let

$E(k) = \bigcup_{n\geq 1:\ k\in K(n)} E(n,k)$.

Let $E = \bigcup_{k\in K} E(k)$; E is the set of all expressions in our language.
One expression $t\in E(k)$ is a 'sentence with respect to k' when
for each $\sigma\in \Xi(k)$ #(k,t,σ) is true or #(k,t,σ) is false.

We define $S(k) = \{t|\ t\in E(k)$, t is a sentence with respect to k$\}$.

At the beginning of section 2 we have introduced the logical connectives. In our deductions, expressions will make an extensive use of the logical connectives, so we assume that all of these symbols: $\wedge, \vee, \rightarrow, \neg, \forall, \exists$ are in our set F. For each of these operators f $A_f(x_1,\ldots,x_n)$ and $P_f(x_1,..,x_n)$ are defined as specified at the beginning of section 2.

For each $t\in E(\varepsilon)$ we define $\#(t) = \#(\varepsilon, t, \varepsilon)$.

On the way to theorem 3.5 we need some other preliminary work, beginning with the following lemma 3.1.

Lemma 3.1:
Let $h\in K$, $\varphi\in E(h)$, $y\in(V\text{-dom}(h))$ such that $\forall\rho\in\Xi(h)$ #(h,φ,ρ) is a set. If k = h ‖ (y,φ) then we have

- $k \in K$;
- If $\gamma\in S(k)$ then
  - $\{\}(y{:}\varphi, \gamma) \in E(h)$
  - $(\forall)\ (\{\}(y{:}\varphi, \gamma)), (\exists)\ (\{\}(y{:}\varphi, \gamma)) \in S(h)$
  - $\forall\rho\in\Xi(h)\ \#(h, (\forall)\ (\{\}(y{:}\varphi, \gamma)), \rho) = P_\forall\ (\{\}(\sigma\in\Xi(k){:}\ \rho\sqsubseteq\sigma, \#(k, \gamma, \sigma))\ )$
  - $\forall\rho\in\Xi(h)\ \#(h, (\exists)\ (\{\}(y{:}\varphi, \gamma)), \rho) = P_\exists\ (\{\}(\sigma\in\Xi(k){:}\ \rho\sqsubseteq\sigma, \#(k, \gamma, \sigma))\ )$

Proof:

Since $\varphi\in E(h)$ there is a positive integer n such that $h\in K(n)$, $\varphi\in E(n,h)$. This implies that
$h \parallel (y,\varphi) \in K(n)^+ \subseteq K(n+1) \subseteq K$.

Let $\gamma\in S(k)$. There is a positive integer m such that $\gamma\in E(m,k)$. We define p=max{n+1,m}, then we have

- $h\in K(p)$
- $y\in(V\text{-dom}(h))$;
- $\varphi\in E(p,h)$, $\forall\rho\in\Xi(h)$ #(h,φ,ρ) is a set;
- $k\in K(p)$, $\gamma\in E(p,k)$.

This implies that $\{\}(y{:}\varphi, \gamma) \in E_e(p+1, h) \subseteq E(p+1,h) \subseteq E(h)$.

Moreover for each $\rho\in\Xi(h)$ $\#(h, \{\}(y{:}\varphi, \gamma), \rho) = \#(h, \{\}(y{:}\varphi, \gamma), \rho)_{(p+1,h,e)} =$
$= \{\} (\sigma \in\Xi(k){:}\ \rho \sqsubseteq \sigma\ , \#(k, \gamma, \sigma))$.

For each $\rho\in\Xi(h)$ $A_\forall(\#(h, \{\}(y{:}\varphi, \gamma), \rho)) =$
$= \#(h, \{\}(y{:}\varphi, \gamma), \rho)$ is a set, for each u in $\#(h, \{\}(y{:}\varphi, \gamma), \rho)$ (u is true or u is false) .

Clearly $\#(h, \{\}(y{:}\varphi, \gamma), \rho)$ is a set, furthermore for each u in $\#(h, \{\}(y{:}\varphi, \gamma), \rho)$ there is $\sigma\in\Xi(k)$: $\rho\sqsubseteq\sigma$, $u = \#(k, \gamma, \sigma)$, and since $\gamma\in S(k)$ u must be true or false.
Therefore for each $\rho\in\Xi(h)$ $A_\forall(\#(h, \{\}(y{:}\varphi, \gamma), \rho))$ holds, so $(\forall) (\{\}(y{:}\varphi, \gamma)) \in E(p+2, h)$.
And for each $\rho\in\Xi(h)$ $A_\exists(\#(h, \{\}(y{:}\varphi, \gamma), \rho)) = A_\forall(\#(h, \{\}(y{:}\varphi, \gamma), \rho))$ holds,
so $(\exists) (\{\}(y{:}\varphi, \gamma)) \in E(p+2, h)$.

For each $\rho\in\Xi(h)$

$\#(h, (\forall) (\{\}(y{:}\varphi, \gamma)), \rho) = \#(h, (\forall) (\{\}(y{:}\varphi, \gamma)), \rho)_{(p+2,h,e)} = P_\forall (\#(h, \{\}(y{:}\varphi, \gamma), \rho)) =$
$= P_\forall (\{\}(\sigma\in\Xi(k){:}\ \rho\sqsubseteq\sigma, \#(k, \gamma, \sigma)))$;

$\#(h, (\exists) (\{\}(y{:}\varphi, \gamma)), \rho) = \#(h, (\exists) (\{\}(y{:}\varphi, \gamma)), \rho)_{(p+2,h,e)} = P_\exists (\#(h, \{\}(y{:}\varphi, \gamma), \rho)) =$
$= P_\exists (\{\}(\sigma\in\Xi(k){:}\ \rho\sqsubseteq\sigma, \#(k, \gamma, \sigma)))$.

Finally, for each $\rho\in\Xi(h)$ $\#(h, (\forall) (\{\}(y{:}\varphi, \gamma)), \rho) = P_\forall (\{\}(\sigma\in\Xi(k){:}\ \rho\sqsubseteq\sigma, \#(k, \gamma, \sigma)))$, and $P_\forall (\{\}(\sigma\in\Xi(k){:}\ \rho\sqsubseteq\sigma, \#(k, \gamma, \sigma)))$ is clearly true or false, hence $(\forall) (\{\}(y{:}\varphi, \gamma)) \in S(h)$. And similarly we obtain that $(\exists) (\{\}(y{:}\varphi, \gamma)) \in S(h)$.

□

Definition 3.2:

Let m be a positive integer. Let $x_1, \dots , x_m \in V$, with $x_i\neq x_j$ for $i\neq j$. Let $\varphi_1, .. , \varphi_m\in E$.

Let $k_0 = \varepsilon \in K$. Let $\varphi_1\in E(\varepsilon)$ such that $\#(\varphi_1)$ is a set. We define $k_1 = (x_1,\varphi_1)$, so $k_1\in K$.
If m>1 for each i=1..m-1 suppose we have defined $k_i = (x_1,\varphi_1) \parallel .. \parallel (x_i,\varphi_i) \in K$. Let $\varphi_{i+1}\in E(k_i)$ such that $\forall\rho\in\Xi(k_i)$ $\#(k_i,\varphi_{i+1},\rho)$ is a set. We define $k_{i+1} = k_i \parallel (x_{i+1}, \varphi_{i+1})$, so $k_{i+1}\in K$.

We indicate this situation with $H[x_1{:}\varphi_1, \dots , x_m{:}\varphi_m]$ and in this case we indicate k_m with $k[x_1{:}\varphi_1, .. , x_m{:}\varphi_m]$.

□

Definition 3.3:

Let m be a positive integer. Let $x_1, \dots , x_m \in V$, with $x_i\neq x_j$ for $i\neq j$. Let $\varphi_1, .. , \varphi_m\in E$ and assume $H[x_1{:}\varphi_1, \dots , x_m{:}\varphi_m]$. Let $\varphi \in S(k[x_1{:}\varphi_1, .. , x_m{:}\varphi_m])$.

Define

$\gamma[x_m{:}\varphi_m, \varphi] = (\forall) (\{\}(x_m{:}\varphi_m, \varphi))$. By 3.1 we have $\gamma[x_m{:}\varphi_m, \varphi] \in S(k_{m-1})$.

If m>1 for each i=2..m suppose we have defined $\gamma[x_i{:}\varphi_i, \dots, x_m{:}\varphi_m, \varphi] \in S(k_{i-1})$, and define

$\gamma[x_{i-1}{:}\varphi_{i-1}, \dots, x_m{:}\varphi_m, \varphi] = (\forall)\ (\{\}(x_{i-1}{:}\varphi_{i-1}, \gamma[x_i{:}\varphi_i, \dots, x_m{:}\varphi_m, \varphi]))$

By lemma 3.1 $\gamma[x_{i-1}{:}\varphi_{i-1}, \dots, x_m{:}\varphi_m, \varphi] \in S(k_{i-2})$.

□

Lemma 3.4:

Let m be a positive integer. Let $x_1, \dots, x_m \in V$, with $x_i \neq x_j$ for $i \neq j$. Let $\varphi_1, .., \varphi_m \in E$ and assume $H[x_1{:}\varphi_1, \dots, x_m{:}\varphi_m]$. Let $\varphi \in S(k[x_1{:}\varphi_1, .., x_m{:}\varphi_m])$, m>1, j=2..m .

We have $\gamma[x_j{:}\varphi_j, \dots, x_m{:}\varphi_m, \varphi] \in S(k_{j-1})$. We can show that for each i=1..j-1

$\gamma[x_i{:}\varphi_i, \dots, x_m{:}\varphi_m, \varphi] = \gamma[x_i{:}\varphi_i, \dots, x_{j-1}{:}\varphi_{j-1}, \gamma[x_j{:}\varphi_j, \dots, x_m{:}\varphi_m, \varphi]\ ]$.

Proof:

We show this by induction on i. First we show the property for i = j-1.

$\gamma[\ x_{j-1}{:}\varphi_{j-1}, \dots, x_m{:}\varphi_m, \varphi] = (\forall)\ (\{\}(x_{j-1}{:}\varphi_{j-1}, \gamma[x_j{:}\varphi_j, \dots, x_m{:}\varphi_m, \varphi])) =$
$= \gamma[\ x_{j-1}{:}\varphi_{j-1}, \gamma[x_j{:}\varphi_j, \dots, x_m{:}\varphi_m, \varphi]\ ]$.

Now we assume $j-1 \geq 2$ and i between 2 and j-1. We assume the property is true for i and want to show it holds also for i-1. We have

$\gamma[x_{i-1}{:}\varphi_{i-1}, \dots, x_m{:}\varphi_m, \varphi] = (\forall)\ (\{\}(x_{i-1}{:}\varphi_{i-1}, \gamma[x_i{:}\varphi_i, \dots, x_m{:}\varphi_m, \varphi])) =$
$= (\forall)\ (\{\}(x_{i-1}{:}\varphi_{i-1}, \gamma[x_i{:}\varphi_i, \dots, x_{j-1}{:}\varphi_{j-1}, \gamma[x_j{:}\varphi_j, \dots, x_m{:}\varphi_m, \varphi]\ ]\ )) =$
$= \gamma[x_{i-1}{:}\varphi_{i-1}, \dots, x_{j-1}{:}\varphi_{j-1}, \gamma[x_j{:}\varphi_j, \dots, x_m{:}\varphi_m, \varphi]\ ]$.

□

Theorem 3.5:

Let m be a positive integer. Let $x_1, \dots, x_m \in V$, with $x_i \neq x_j$ for $i \neq j$. Let $\varphi_1, .., \varphi_m \in E$ and assume $H[x_1{:}\varphi_1, \dots, x_m{:}\varphi_m]$. Let $\varphi \in S(k[x_1{:}\varphi_1, .., x_m{:}\varphi_m])$. Then

$\#(\ \gamma[x_1{:}\varphi_1, \dots, x_m{:}\varphi_m, \varphi]) \leftrightarrow$
$P_\forall\ (\ \{\}\ (\sigma' \in \Xi(k[x_1{:}\varphi_1, .., x_m{:}\varphi_m])\ , \#(\ k[x_1{:}\varphi_1, .., x_m{:}\varphi_m], \varphi, \sigma')\ )\ )$.

Proof:

We'll use the symbols $k_0, .., k_m$ with the same meaning they have in the former definitions 3.2 and 3.3.

So we need to show that

$\#(\ \gamma[x_1{:}\varphi_1, \dots, x_m{:}\varphi_m, \varphi]) \leftrightarrow P_\forall\ (\ \{\}\ (\sigma' \in \Xi(k_m)\ , \#(\ k_m, \varphi, \sigma')\ )\ )$.

To this end we need to show that for each i = m .. 1 and for each $\sigma \in \Xi(k_{i-1})$

$\#(k_{i-1}, \gamma[x_i{:}\varphi_i, \ldots , x_m{:}\varphi_m, \varphi] , \sigma) \leftrightarrow P_\forall (\{\} (\sigma' \in \Xi(k_m){:}\ \sigma \sqsubseteq \sigma' , \#(k_m, \varphi, \sigma')))$.

We prove this by induction on i, starting with the case where i=m. Here we need to show that for each $\sigma \in \Xi(k_{m-1})$

$\#(k_{m-1}, \gamma[x_m{:}\varphi_m, \varphi] , \sigma) \leftrightarrow P_\forall (\{\} (\sigma' \in \Xi(k_m){:}\ \sigma \sqsubseteq \sigma' , \#(k_m, \varphi, \sigma')))$.

Actually

$\#(k_{m-1}, \gamma[x_m{:}\varphi_m, \varphi] , \sigma) = \#(k_{m-1}, (\forall) (\{\}(x_m{:}\varphi_m, \varphi)) , \sigma) =$
$= P_\forall (\{\} (\sigma' \in \Xi(k_m){:}\ \sigma \sqsubseteq \sigma' , \#(k_m, \varphi, \sigma')))$.

Now suppose m>1, let i=2..m and suppose the property holds for i, we show it also holds for i-1. We have

$\#(k_{i-2}, \gamma[x_{i-1}{:}\varphi_{i-1}, \ldots , x_m{:}\varphi_m, \varphi] , \sigma) = \#(k_{i-2}, (\forall) (\{\}(x_{i-1}{:}\varphi_{i-1}, \gamma[x_i{:}\varphi_i, \ldots , x_m{:}\varphi_m, \varphi])) , \sigma) =$

$= P_\forall (\{\}(\sigma' \in \Xi(k_{i-1}){:}\ \sigma \sqsubseteq \sigma', \#(k_{i-1}, \gamma[x_i{:}\varphi_i, \ldots , x_m{:}\varphi_m, \varphi] , \sigma'))) \leftrightarrow$

$= P_\forall (\{\}(\sigma' \in \Xi(k_{i-1}){:}\ \sigma \sqsubseteq \sigma', P_\forall (\{\} (\sigma'' \in \Xi(k_m){:}\ \sigma' \sqsubseteq \sigma'' , \#(k_m, \varphi, \sigma'')))))$.

So it comes to showing that

$P_\forall (\{\}(\sigma' \in \Xi(k_{i-1}), \sigma \sqsubseteq \sigma', P_\forall (\{\} (\sigma'' \in \Xi(k_m){:}\ \sigma' \sqsubseteq \sigma'' , \#(k_m, \varphi, \sigma''))))) \leftrightarrow$

$P_\forall (\{\} (\sigma'' \in \Xi(k_m){:}\ \sigma \sqsubseteq \sigma'' , \#(k_m, \varphi, \sigma'')))$.

Suppose $P_\forall (\{\}(\sigma' \in \Xi(k_{i-1}){:}\ \sigma \sqsubseteq \sigma', P_\forall (\{\} (\sigma'' \in \Xi(k_m){:}\ \sigma' \sqsubseteq \sigma'' , \#(k_m, \varphi, \sigma'')))))$.
This means that for each $\sigma' \in \Xi(k_{i-1}){:}\ \sigma \sqsubseteq \sigma'$ and $\sigma'' \in \Xi(k_m){:}\ \sigma' \sqsubseteq \sigma''$ $\#(k_m, \varphi, \sigma'')$ holds.

Let $\sigma'' \in \Xi(k_m){:}\ \sigma \sqsubseteq \sigma''$, we need to prove $\#(k_m, \varphi, \sigma'')$. Let $\sigma' = \sigma''/dom(k_{i-1})$. We have $\sigma' \in \Xi(k_{i-1})$, since $\sigma = \sigma''/dom(\sigma) = \sigma''/dom(k_{i-2})$ then $\sigma \sqsubseteq \sigma'$, moreover it's clear that $\sigma' \sqsubseteq \sigma''$, therefore $\#(k_m,\varphi,\sigma'')$ holds.

Conversely suppose $P_\forall (\{\} (\sigma'' \in \Xi(k_m){:}\ \sigma \sqsubseteq \sigma'' , \#(k_m, \varphi, \sigma'')))$ holds, and so that for each $\sigma'' \in \Xi(k_m){:}\ \sigma \sqsubseteq \sigma''$ $\#(k_m, \varphi, \sigma'')$ is true. Let $\sigma' \in \Xi(k_{i-1}){:}\ \sigma \sqsubseteq \sigma'$ and $\sigma'' \in \Xi(k_m){:}\ \sigma' \sqsubseteq \sigma''$, we wonder if $\#(k_m, \varphi, \sigma'')$ holds. The answer is yes, because of $\sigma'' \in \Xi(k_m)$ and $\sigma \sqsubseteq \sigma''$.

We conclude that

$\#(k_{i-2}, \gamma[x_{i-1}{:}\varphi_{i-1}, \ldots , x_m{:}\varphi_m, \varphi] , \sigma) \leftrightarrow$

$P_\forall (\{\}(\sigma' \in \Xi(k_{i-1}), \sigma \sqsubseteq \sigma', P_\forall (\{\} (\sigma'' \in \Xi(k_m){:}\ \sigma' \sqsubseteq \sigma'' , \#(k_m, \varphi, \sigma''))))) \leftrightarrow$

$P_\forall (\{\} (\sigma'' \in \Xi(k_m){:}\ \sigma \sqsubseteq \sigma'' , \#(k_m, \varphi, \sigma'')))$.

And clearly the proof is finished, since

$\#(\gamma[x_1{:}\varphi_1, \ldots, x_m{:}\varphi_m, \varphi]) \leftrightarrow \#(k_0, \gamma[x_1{:}\varphi_1, \ldots, x_m{:}\varphi_m, \varphi], \varepsilon) \leftrightarrow$
$P_\forall(\{\}(\sigma' \in \Xi(k_m), \#(k_m, \varphi, \sigma')))$.

□

We will soon apply theorem 3.5 to show its importance. First of all we need to prove lemma 3.6, which in some way is similar to 3.1 but involves other logical connectives.

Lemma 3.6

Let $h \in K$, $\varphi_1, \varphi_2 \in S(h)$. Then

- $(\wedge)(\varphi_1,\varphi_2) \in S(h)$
- $\forall \rho \in \Xi(h)\ \#(h, (\wedge)(\varphi_1, \varphi_2), \rho) = P_\wedge(\#(h, \varphi_1, \rho), \#(h, \varphi_2, \rho))$

- $(\vee)(\varphi_1,\varphi_2) \in S(h)$
- $\forall \rho \in \Xi(h)\ \#(h, (\vee)(\varphi_1, \varphi_2), \rho) = P_\vee(\#(h, \varphi_1, \rho), \#(h, \varphi_2, \rho))$

- $(\rightarrow)(\varphi_1,\varphi_2) \in S(h)$
- $\forall \rho \in \Xi(h)\ \#(h, (\rightarrow)(\varphi_1, \varphi_2), \rho) = P_\rightarrow(\#(h, \varphi_1, \rho), \#(h, \varphi_2, \rho))$

- $(\neg)(\varphi_1) \in S(h)$
- $\forall \rho \in \Xi(h)\ \#(h, (\neg)(\varphi_1), \rho) = P_\neg(\#(h, \varphi_1, \rho))$

Proof:

For each $\rho \in \Xi(h)$ $\#(h, \varphi_1, \rho)$ is true or $\#(h, \varphi_1, \rho)$ is false; $\#(h, \varphi_2, \rho)$ is true or $\#(h, \varphi_2, \rho)$ is false.

We recall that

$A_\wedge(\#(h,\varphi_1,\rho), \#(h,\varphi_2,\rho)) = A_\vee(\#(h,\varphi_1,\rho), \#(h,\varphi_2,\rho)) = A_\rightarrow(\#(h,\varphi_1,\rho), \#(h,\varphi_2,\rho)) =$
$=$ ($\#(h,\varphi_1,\rho)$ is true or $\#(h,\varphi_1,\rho)$ is false) and ($\#(h,\varphi_2,\rho)$ is true or $\#(h,\varphi_2,\rho)$ is false) ,

and $A_\neg(\#(h,\varphi_1,\rho)) =$ ($\#(h,\varphi_1,\rho)$ is true or $\#(h,\varphi_1,\rho)$ is false) .

So $A_\wedge(\#(h,\varphi_1,\rho), \#(h,\varphi_2,\rho))$, $A_\vee(\#(h,\varphi_1,\rho), \#(h,\varphi_2,\rho))$, $A_\rightarrow(\#(h,\varphi_1,\rho), \#(h,\varphi_2,\rho))$ and $A_\neg(\#(h,\varphi_1,\rho))$ all hold true.

There exists a positive integer n such that $\varphi_1, \varphi_2 \in E(n,h)$, so

$(\wedge)(\varphi_1,\varphi_2), (\vee)(\varphi_1,\varphi_2), (\rightarrow)(\varphi_1,\varphi_2), (\neg)(\varphi_1) \in E(h)$.

Moreover for each $\rho \in \Xi(h)$

$\#(h, (\wedge)(\varphi_1, \varphi_2), \rho) = P_\wedge(\#(h, \varphi_1, \rho), \#(h, \varphi_2, \rho))$;
$\#(h, (\vee)(\varphi_1, \varphi_2), \rho) = P_\vee(\#(h, \varphi_1, \rho), \#(h, \varphi_2, \rho))$;
$\#(h, (\rightarrow)(\varphi_1, \varphi_2), \rho) = P_\rightarrow(\#(h, \varphi_1, \rho), \#(h, \varphi_2, \rho))$;
$\#(h, (\neg)(\varphi_1), \rho) = P_\neg(\#(h, \varphi_1, \rho))$;

so

#(h, $(\wedge)$ (φ_1, φ_2), ρ) is true or false ;
#(h, $(\vee)$ (φ_1, φ_2), ρ) is true or false ;
#(h, $(\rightarrow)$ (φ_1, φ_2), ρ) is true or false ;
#(h, $(\neg)$ (φ_1), ρ) is true or false .

Therefore $(\wedge)(\varphi_1,\varphi_2)$, $(\vee)(\varphi_1,\varphi_2)$, $(\rightarrow)(\varphi_1,\varphi_2)$, $(\neg)(\varphi_1) \in$ S(h).

□

The following lemmas, 3.7 and 3.8, are examples of how theorem 3.5 is applied.

Lemma 3.7

Let m be a positive integer. Let $x_1, \dots, x_m \in V$, with $x_i \neq x_j$ for $i \neq j$. Let $\varphi_1, .., \varphi_m \in E$ and assume $H[x_1{:}\varphi_1, \dots, x_m{:}\varphi_m]$. Define $k = k[x_1{:}\varphi_1, \dots, x_m{:}\varphi_m]$ and let $\varphi, \psi_1, \psi_2 \in S(k)$.

Under these assumptions we have $(\rightarrow)(\varphi, \psi_1)$, $(\rightarrow)(\varphi, \psi_2)$, $(\rightarrow)(\varphi, (\wedge)(\psi_1, \psi_2)) \in S(k)$.

Moreover, if
$\#(\gamma[x_1{:}\varphi_1, \dots, x_m{:}\varphi_m, (\rightarrow)(\varphi, \psi_1)])$, $\#(\gamma[x_1{:}\varphi_1, \dots, x_m{:}\varphi_m, (\rightarrow)(\varphi, \psi_2)])$
then
$\#(\gamma[x_1{:}\varphi_1, \dots, x_m{:}\varphi_m, (\rightarrow)(\varphi, (\wedge)(\psi_1, \psi_2))])$.

Proof:

We need to show

$$\#(\gamma[x_1, .., x_m ; \varphi_1, .. \varphi_m; (\rightarrow)(\varphi, (\wedge)(\psi_1, \psi_2))]) ,$$

that is

$$P_\forall\Big(\{\}\big(\sigma \in \Xi(k), \#\big(k, (\rightarrow)\big(\varphi, (\wedge)(\psi_1, \psi_2)\big), \sigma\big)\big)\Big) ,$$

$$P_\forall\Big(\{\}\big(\sigma \in \Xi(k), P_\rightarrow\big(\#(k, \varphi, \sigma), \#\big(k, (\wedge)(\psi_1, \psi_2), \sigma\big)\big)\big)\Big),$$

$$(1) \quad P_\forall\Big(\{\}\big(\sigma \in \Xi(k), P_\rightarrow\big(\#(k, \varphi, \sigma), P_\wedge\big(\#(k, \psi_1, \sigma), \#(k, \psi_2, \sigma)\big)\big)\big)\Big) .$$

But we have

$\#(\gamma[x_1{:}\varphi_1, \dots, x_m{:}\varphi_m, (\rightarrow)(\varphi, \psi_1)])$, from which we get

$$P_\forall\Big(\{\}\big(\sigma \in \Xi(k), \#\big(k, (\rightarrow)(\varphi, \psi_1), \sigma\big)\big)\Big) ,$$

$$P_\forall\Big(\{\}\big(\sigma \in \Xi(k), P_\rightarrow\big(\#(k, \varphi, \sigma), \#(k, \psi_1, \sigma)\big)\big)\Big) .$$

And we have

$\#(\gamma[x_1{:}\varphi_1, \ldots, x_m{:}\varphi_m, (\to)(\varphi, \psi_2)])$, from which we get

$$P_\forall\Big(\{\}\big(\sigma\in \Xi(k), \#\big(k, (\to)(\varphi, \psi_2), \sigma\big)\big)\Big) ,$$
$$P_\forall\Big(\{\}\big(\sigma\in \Xi(k), P_\to\big(\#(k, \varphi, \sigma), \#(k, \psi_2, \sigma)\big)\big)\Big) .$$

So for each $\sigma\in\Xi(k)$ if $\#(k,\varphi,\sigma)$ holds true then both $\#(k,\psi_1,\sigma)$ and $\#(k,\psi_2,\sigma)$ hold. This implies (1) holds true in turn.

□

Lemma 3.8

Let m be a positive integer. Let $x_1, \ldots, x_{m+1} \in V$, with $x_i \neq x_j$ for $i\neq j$. Let $\varphi_1, .., \varphi_{m+1}\in E$ and assume $H[x_1{:}\varphi_1, \ldots, x_m{:}\varphi_{m+1}]$. Define $k = k[x_1{:}\varphi_1, \ldots, x_{m+1}{:}\varphi_{m+1}]$.
Of course $H[x_1{:}\varphi_1, \ldots, x_m{:}\varphi_m]$ holds and we can define $h = k[x_1{:}\varphi_1, \ldots, x_m{:}\varphi_m]$.
Let $\psi\in S(h)$: $x_{m+1}\notin V_b(\psi)$, $\varphi\in S(k)$.

Under these assumptions we have $\psi\in S(k)$ and $(\to)(\psi,\varphi)\in S(k)$,
moreover $(\forall)(\{\}(x_{m+1}{:}\varphi_{m+1}, \varphi)) \in S(h)$, $(\to)(\psi, (\forall)(\{\}(x_{m+1}{:}\varphi_{m+1}, \varphi))) \in S(h)$.

Finally,

$$\#(\gamma[x_1{:}\varphi_1, \ldots, x_m{:}\varphi_m, (\to)(\psi, (\forall)(\{\}(x_{m+1}{:}\varphi_{m+1}, \varphi)))]) \leftrightarrow$$
$$\#(\gamma[x_1{:}\varphi_1, \ldots, x_{m+1}{:}\varphi_{m+1}, (\to)(\psi, \varphi)])$$

Proof:

There exists a positive integer n such that $\psi,\varphi_{m+1} \in E(n,h)$. This implies that $k\in K(n)^+$. Since $x_{m+1}\notin V_b(\psi)$ we have $\psi\in E(n+1,k)$, moreover for each $\sigma\in\Xi(k)$ we have $\sigma = \rho \parallel (x_{m+1}, s)$, with $\rho\in\Xi(h)$, $s\in\#(h,\varphi_{m+1},\rho)$ and $\#(k,\psi,\sigma) = \#(h, \psi, \rho)$; $\#(h, \psi, \rho)$ is true or false, so $\#(k,\psi,\sigma)$ also is true or false. Therefore $\psi\in S(k)$ and $(\to)(\psi,\varphi)\in S(k)$.

By lemma 3.1, since $\varphi\in S(k)$, we derive that $(\forall)(\{\}(x_{m+1}{:}\varphi_{m+1}, \varphi)) \in S(h)$, and it immediately follows that $(\to)(\psi, (\forall)(\{\}(x_{m+1}{:}\varphi_{m+1}, \varphi))) \in S(h)$.

We can rewrite

$$\#(\gamma[x_1{:}\varphi_1, \ldots, x_m{:}\varphi_m, (\to)(\psi, (\forall)(\{\}(x_{m+1}{:}\varphi_{m+1}, \varphi)))])$$

in these ways

$$P_\forall\Big(\{\}\Big(\rho\in \Xi(h), \#\Big(h, (\to)\big(\psi, (\forall)\big(\{\}(x_{m+1}:\varphi_{m+1}, \varphi)\big)\big), \rho\Big)\Big)\Big) ,$$
$$P_\forall\Big(\{\}\Big(\rho\in \Xi(h), P_\to\Big(\#(h, \psi, \rho), \#\big(h, (\forall)\big(\{\}(x_{m+1}:\varphi_{m+1}, \varphi)\big), \rho\big)\Big)\Big)\Big) ,$$

$$P\forall\Big(\{\}\big(\rho\in\Xi(h), P\rightarrow\big(\#(h,\psi,\rho), P\forall\big(\{\}\big(\sigma\in\Xi(k):\rho\sqsubseteq\sigma, \#(k,\varphi,\sigma)\big)\big)\big)\big)\Big) .$$

We can rewrite #($\gamma[x_1{:}\varphi_1, \ldots, x_{m+1}{:}\varphi_{m+1}, (\rightarrow)(\psi, \varphi)]$) in these ways

$$P\forall\big(\{\}\big(\sigma\in\Xi(k), \#(k,(\rightarrow)(\psi,\varphi),\sigma)\big)\big) ,$$

$$P\forall\big(\{\}\big(\sigma\in\Xi(k), P\rightarrow\big(\#(k,\psi,\sigma), \#(k,\varphi,\sigma)\big)\big)\big) .$$

So it comes to proving that

$$P\forall\Big(\{\}\big(\rho\in\Xi(h), P\rightarrow\big(\#(h,\psi,\rho), P\forall\big(\{\}\big(\sigma\in\Xi(k):\rho\sqsubseteq\sigma, \#(k,\varphi,\sigma)\big)\big)\big)\big)\Big) \text{ and}$$

$$P\forall\big(\{\}\big(\sigma\in\Xi(k), P\rightarrow\big(\#(k,\psi,\sigma), \#(k,\varphi,\sigma)\big)\big)\big)$$

are equivalent .

Assume

$$P\forall\Big(\{\}\big(\rho\in\Xi(h), P\rightarrow\big(\#(h,\psi,\rho), P\forall\big(\{\}\big(\sigma\in\Xi(k):\rho\sqsubseteq\sigma, \#(k,\varphi,\sigma)\big)\big)\big)\big)\Big) ,$$

let $\sigma\in\Xi(k)$ such that $\#(k,\psi,\sigma)$, we ask whether $\#(k,\varphi,\sigma)$ holds true.
There exist $\rho\in\Xi(h)$, $s\in\#(h,\varphi_{m+1},\rho)$ such that $\sigma = \rho \parallel (x_{m+1},s)$; $\#(k,\psi,\sigma)$ implies (h,ψ,ρ), and since $\rho\sqsubseteq\sigma$ we have $\#(k,\varphi,\sigma)$.

Conversely assume $P\forall\big(\{\}\big(\sigma\in\Xi(k), P\rightarrow\big(\#(k,\psi,\sigma), \#(k,\varphi,\sigma)\big)\big)\big)$.
Let $\rho'\in\Xi(h)$ such that $\#(h,\psi,\rho')$, let $\sigma\in\Xi(k)$: $\rho'\sqsubseteq\sigma$, we wish to prove $\#(k, \varphi, \sigma)$.
We've seen there exist $\rho\in\Xi(h)$, $s\in\#(h,\varphi_{m+1},\rho)$ such that $\sigma = \rho \parallel (x_{m+1},s)$ and $\#(k,\psi,\sigma) = \#(h, \psi, \rho)$.
Clearly $\rho' = \sigma/dom(\rho') = \sigma/dom(h) = \sigma/dom(\rho) = \rho$, so both $\#(h, \psi, \rho)$ and $\#(k,\psi,\sigma)$ hold, and therefore $\#(k, \varphi, \sigma)$ also is true.

□

Next we list some other general results, similar to the ones we have just seen in this section.

Lemma 3.9

Let $k\in K$, m positive integer, $\varphi, \varphi_1, \ldots, \varphi_m \in E(k)$. Suppose for each $\sigma\in\Xi(k)$ $\#(k,\varphi,\sigma)$ is a function with m arguments and ($\#(k, \varphi_1, \sigma), \ldots, \#(k, \varphi_m, \sigma)$) is a member of its domain. Then

- $(\varphi)(\varphi_1, \ldots, \varphi_m) \in E(k)$
- for each $\sigma\in\Xi(k)$ $\#(k, (\varphi)(\varphi_1, \ldots, \varphi_m), \sigma) = \#(k, \varphi, \sigma)$ ($\#(k, \varphi_1, \sigma), \ldots, \#(k, \varphi_m, \sigma)$)
- $V_b((\varphi)(\varphi_1, \ldots, \varphi_m)) = V_b(\varphi) \cup V_b(\varphi_1) \cup \ldots \cup V_b(\varphi_m)$.
- $V_f((\varphi)(\varphi_1, \ldots, \varphi_m)) = V_f(\varphi) \cup V_f(\varphi_1) \cup \ldots \cup V_f(\varphi_m)$.

Proof:

There exists a positive integer n such that $\varphi, \varphi_1, \ldots, \varphi_m \in E(n,k)$. This implies that

$(\varphi)(\varphi_1, \ldots, \varphi_m) \in E(n+1,k)$, and for each $\sigma\in\Xi(k)$

$\#(k, (\varphi)(\varphi_1, \dots, \varphi_m), \sigma) = \#(k, \varphi, \sigma) (\#(k, \varphi_1, \sigma), \dots, \#(k, \varphi_m, \sigma))$.

Clearly the following also hold:

$V_f((\varphi)(\varphi_1, \dots, \varphi_m)) = V_f(\varphi) \cup V_f(\varphi_1) \cup \dots \cup V_f(\varphi_m)$,
$V_b((\varphi)(\varphi_1, \dots, \varphi_m)) = V_b(\varphi) \cup V_b(\varphi_1) \cup \dots \cup V_b(\varphi_m)$.

□

Lemma 3.10

Let $k \in K$, $f \in F$, m positive integer, $\varphi_1, \dots, \varphi_m \in E(k)$.
Suppose for each $\sigma \in \Xi(k)$ $A_f(\#(k, \varphi_1, \sigma), ..., \#(k, \varphi_m, \sigma))$ is true . Then

- $(f)(\varphi_1, \dots, \varphi_m) \in E(k)$
- for each $\sigma \in \Xi(k)$ $\#(k, (f)(\varphi_1, \dots, \varphi_m), \sigma) = P_f (\#(k, \varphi_1, \sigma), \dots, \#(k, \varphi_m, \sigma))$
- $V_b((f)(\varphi_1, \dots, \varphi_m)) = V_b(\varphi_1) \cup \dots \cup V_b(\varphi_m)$.
- $V_f((f)(\varphi_1, \dots, \varphi_m)) = V_f(\varphi_1) \cup \dots \cup V_f(\varphi_m)$.

Proof:

There exists a positive integer n such that $\varphi_1, \dots, \varphi_m \in E(n,k)$. This implies that

$(f)(\varphi_1, \dots, \varphi_m) \in E(n+1,k)$, and for each $\sigma \in \Xi(k)$

$\#(k, (f)(\varphi_1, \dots, \varphi_m), \sigma) = P_f (\#(k, \varphi_1, \sigma), \dots, \#(k, \varphi_m, \sigma))$.

Clearly the following also hold:

$V_b((f)(\varphi_1, \dots, \varphi_m)) = V_b(\varphi_1) \cup \dots \cup V_b(\varphi_m)$.
$V_f((f)(\varphi_1, \dots, \varphi_m)) = V_f(\varphi_1) \cup \dots \cup V_f(\varphi_m)$.

□

Lemma 3.11

Let $k \in K$, $\varphi, \varphi_1, .., \varphi_m \in E$, $x_1,\dots,x_m$ distinct $\in V$-dom(k)

- $\varphi_1 \in E(k)$, for each $\sigma \in \Xi(k)$ $\#(k,\varphi_1,\sigma)$ is a set;
- if m>1 then for each j=1..m-1 if we define $k'_j = k||(x_1,\varphi_1)||\dots||(x_j,\varphi_j)$ then
  $k'_j \in K \wedge \varphi_{j+1} \in E(k'_j) \wedge$ for each $\sigma'_j \in \Xi(k'_j)$ $\#(k'_j, \varphi_{j+1}, \sigma'_j)$ is a set;
- if we define $k'_m = k||(x_1,\varphi_1)||\dots||(x_m,\varphi_m)$ then $k'_m \in K \wedge \varphi \in E(k'_m)$

Define $\psi = \{\}(x_1{:}\varphi_1, .., x_m{:}\varphi_m, \varphi)$, then

- $\psi \in E(k)$,
- for each $\sigma \in \Xi(k)$ $\#(k, \psi, \sigma) = \{\} (\sigma'_m \in \Xi(k'_m){:}\ \sigma \sqsubseteq \sigma'_m , \#(k'_m,\varphi,\sigma'_m))$,
- $V_b(\psi) = \{x_{1,..,}x_m\} \cup V_b(\varphi_1) \cup .. \cup V_b(\varphi_m) \cup V_b(\varphi)$.

Proof:

There exists a positive integer n_1 such that $\varphi_1 \in E(n_1,k)$; if m>1 then for each j=1..m-1 there exists a positive integer n_{j+1} such that $\varphi_{j+1} \in E(n_{j+1}, k'_j)$; there exists a positive integer n_{m+1} such that $\varphi \in E(n_{m+1},k'_m)$.

If we define $n = \max\{n_1, .., n_{m+1}\}$ then $k \in K(n)$ and

- $\varphi_1 \in E(n,k)$, for each $\sigma \in \Xi(k)$ $\#(k,\varphi_1,\sigma)$ is a set;
- if m>1 then for each j=1..m-1 if we define $k'_j = k \| (x_1,\varphi_1) \| \ldots \| (x_j,\varphi_j)$ then
  $k'_j \in K(n) \wedge \varphi_{j+1} \in E(n,k'_j) \wedge$ for each $\sigma'_j \in \Xi(k'_j)$ $\#(k'_j, \varphi_{j+1}, \sigma'_j)$ is a set;
- if we define $k'_m = k \| (x_1,\varphi_1) \| \ldots \| (x_m,\varphi_m)$ then $k'_m \in K(n) \wedge \varphi \in E(n,k'_m)$.

Of course this implies that $\{\}(x_1:\varphi_1, .., x_m:\varphi_m, \varphi) \in E(n+1,k)$, for each $\sigma \in \Xi(k)$

$\#(k, \psi, \sigma) = \{\} (\sigma'_m \in \Xi(k'_m): \sigma \sqsubseteq \sigma'_m , \#(k'_m,\varphi,\sigma'_m))$, and finally

$V_b(\psi) = \{x_{1,..,}x_m\} \cup V_b(\varphi_1) \cup .. \cup V_b(\varphi_m) \cup V_b(\varphi)$.

□

Lemma 3.12

Let $c \in C$. For each positive integer n and $k \in K(n)$ $c \in E(n,k)$ and for each $\sigma \in \Xi(k)$ $\#(k,c,\sigma) = \#(c)$.

Proof:

The proof is by induction on n.

For n=1 we have $c \in E(1,\varepsilon)$ and by definition $\#(\varepsilon,c,\varepsilon) = \#(c)$.

Let n be a positive integer and $k \in K(n+1) = K(n) \cup K(n)^+$.

If $k \in K(n)$ then $c \in E(n,k) \subseteq E(n+1,k)$ and for each $\sigma \in \Xi(k)$ $\#(k,c,\sigma) = \#(c)$.

Otherwise $k \in K(n)^+$ so there exist $h \in K(n)$, $\varphi \in E(n,h)$, $y \in (V - \mathrm{dom}(h))$ such that $k = h \| (y,\varphi)$.
We have $c \in E(n,h)$ and for each $\rho \in \Xi(h)$ $\#(h,c,\rho) = \#(c)$.
It follows that $c \in E(n+1,k)$ and for each $\sigma = \rho \| (y,s) \in \Xi(k)$ we have $\#(k,c,\sigma) = \#(h,c,\rho) = \#(c)$.

□

Lemma 3.13

Suppose the equality predicate symbol = we defined at the beginning of section 2 belongs to F. Suppose $\varphi_1, \varphi_2 \in E(k)$. Then $(=)(\varphi_1,\varphi_2) \in S(k)$.

Proof:

For each $\sigma \in \Xi(k)$ $A_=(\#(k, \varphi_1, \sigma), \#(k, \varphi_2, \sigma))$ is true, so $(=)(\varphi_1,\varphi_2) \in E(k)$.
Moreover for each $\sigma \in \Xi(k)$ $\#(k, (=)(\varphi_1,\varphi_2), \sigma) = P_= (\#(k, \varphi_1, \sigma), \#(k, \varphi_2, \sigma)) =$

= #(k, φ_1, σ) is equal to #(k, φ_2, σ) , so #(k, (=)(φ_1,φ_2), σ) is true or false.

Therefore (=)(φ_1,φ_2) ∈ S(k).

□

Lemma 3.14

Suppose the membership predicate symbol ∈ we defined at the beginning of section 2 belongs to F. Suppose t, φ ∈ E(k) and for each σ∈Ξ(k) #(k,φ,σ) is a set. Then (∈)(t,φ) ∈ S(k).

Proof:

For each σ∈Ξ(k) $A_{\in}$(#(k,t,σ), #(k,φ,σ)) = (#(k,φ,σ) is a set) is true.

So by lemma 3.10 (∈)(t,φ) ∈ E(k),

for each σ∈Ξ(k) #(k, (∈)(t,φ), σ) = $P_{\in}$(#(k,t,σ), #(k,φ,σ)) = (#(k,t,σ) is a member of #(k,φ,σ)),

so #(k, (∈)(t,φ), σ) is true or false and (∈)(t,φ) ∈ S(k).

□

Lemma 3.15

Let m be a positive integer. Let x_1, … , x_m ∈ V, with $x_i \neq x_j$ for i≠j. Let φ_1, .. , φ_m∈E and assume H[x_1:φ_1, ... , x_m:φ_m]. Define k = k[x_1:φ_1, ... , x_m:φ_m]. Let i=0..m-1 and let ψ∈E(k_i) such that for each j=i+1 .. m $x_j \notin V_b(\psi)$ .Then ψ∈E(k) and for each σ∈Ξ(k) there exists ρ∈Ξ(k_i) such that ρ⊑σ and #(k, ψ, σ) = #(k_i, ψ, ρ) .

Proof:

We prove by induction on j that for each j=i..m ψ∈E(k_j) and for each σ∈Ξ(k_j) there exists ρ∈Ξ(k_i) such that ρ⊑σ and #(k_j, ψ, σ) = #(k_i, ψ, ρ) .

The initial step of the proof is obvious, so let j=i..m-1, and assume ψ∈E(k_j) and for each σ∈Ξ(k_j) there exists ρ∈Ξ(k_i) such that ρ⊑σ and #(k_j, ψ, σ) = #(k_i, ψ, ρ) .

Since $x_{j+1} \notin V_b(\psi)$ by lemma 4.2 we have ψ∈E(k_{j+1}) and for each σ = η ‖ (y,s) ∈ Ξ(k_{j+1}) #(k_{j+1},ψ,σ) = #(k_j,ψ,η). Since η∈Ξ(k_j) there exists ρ∈Ξ(k_i) such that ρ⊑η⊑σ and #(k_i, ψ, ρ) = #(k_j, ψ, η) = #(k_{j+1},ψ,σ).

□

4. Substitution

First-order logic features the notion of 'substitution' (see e.g. Enderton's book [1]). Under appropriate assumptions, we can apply substitution to a formula φ and obtain a new formula φ^x_t by replacing the free occurrences of the variable x by the term t. In our approach we'll define a similar notion, with the difference that for us t is a generic expression.

We begin with some preliminary definitions and results, and then substitution will be defined through the complex definition process 4.6.

Lemma 4.1:

Let $h\in K$, $y\in \mathcal{V}$-dom(h), $\varphi\in E(h)$: $\forall\rho\in\Xi(h)$ #(h,φ,ρ) is a set, k = h ǁ (y,φ). Then $k\in K$.

Proof:

There exists a positive integer n such that $\varphi\in E(n,h)$. Of course $h\in K(n)$, so $k\in K(n)^+ \subseteq K(n+1)$.

□

Lemma 4.2:

Let $h\in K$, $y\in \mathcal{V}$-dom(h), $\varphi\in E(h)$: $\forall\rho\in\Xi(h)$ #(h,φ,ρ) is a set, k = h ǁ (y,φ). Let $\psi\in E(h)$ such that $y\notin V_b(\psi)$. Then $\psi\in E(k)$ and for each σ = ρ ǁ (y, s) ∈ Ξ(k) #(k,ψ,σ) = #(h,ψ,ρ) .

Proof:

There exists a positive integer n such that $\varphi\in E(n,h)$, $\psi\in E(n,h)$. Of course $h\in K(n)$, so $k\in K(n)^+$, and $\psi\in E(n+1,k)$. Let σ = ρ ǁ (y, s) ∈ Ξ(k), we have $\#(k,\psi,\sigma) = \#(k,\psi,\sigma)_{(n+1,k,a)} = \#(h,\psi,\rho)$.

□

Lemma 4.3:

Let $h\in K$, $y\in \mathcal{V}$-dom(h), $\varphi\in E(h)$: $\forall\rho\in\Xi(h)$ #(h,φ,ρ) is a set, k = h ǁ (y,φ). Then $y\in E(k)$ and and for each σ ∈ Ξ(k) #(k,y,σ) = σ(y) .

Proof:

There exists a positive integer n such that $\varphi\in E(n,h)$. Of course $h\in K(n)$, so $k\in K(n)^+$, and $y\in E(n+1,k)$. For each $\sigma\in\Xi(k)$ we have #(k,y,σ) = σ(y) .

□

Definition 4.4

Let n be a positive integer, $k\in K(n)$, $k\neq\varepsilon$, n>1.

Let p be a positive integer with p < n, $x_1, .. , x_p \in \mathcal{V}$: $x_i\neq x_j$ for $i\neq j$, $\varphi_1, .. , \varphi_p \in E$.

We define $k_0=\varepsilon$ and (when p>1) for each i=1..p-1 $k_i = (x_1, \varphi_1) \parallel .. \parallel (x_i,\varphi_i)$.

Suppose

- for each i=1..p
  - $k_{i-1} \in K(n-1)$
  - $\varphi_i \in E(n-1, k_{i-1})$
  - for each $\rho_{i-1}\in\Xi(k_{i-1})$ $\#(k_{i-1}, \varphi_i, \rho_{i-1})$ is a set

- $k = (x_1, \varphi_1) \| .. \| (x_p,\varphi_p)$
- for each $\sigma\in\Xi(k)$ if we define $\sigma_0=\varepsilon$ and (when $p>1$) for each $i=1..p-1$ $\sigma_i = \sigma/dom(k_i)$ then there exist $s_1\in\#(k_0,\varphi_1,\sigma_0)$, … , $s_p\in\#(k_{p-1},\varphi_p,\sigma_{p-1})$ such that $\sigma = (x_1, s_1) \| .. \| (x_p,s_p)$.

We'll indicate this situation with the expression $K(n; k; x_1, .. , x_p; \varphi_1, .. , \varphi_p)$.

□

Lemma 4.5

For each positive integer n and k in K(n) we have
$k = \varepsilon$ or
($n>1$ and there exist

- a positive integer p such that $p<n$,
- $x_1, .. , x_p \in V$ such that $x_i\neq x_j$ for $i\neq j$,
- $\varphi_1, .. , \varphi_p \in E$

such that $K(n; k; x_1, .. , x_p; \varphi_1, .. , \varphi_p)$) .

Proof:

We prove this by induction on n.

The initial step is clearly satisfied because if $k\in K(1)$ then $k=\varepsilon$.

Then suppose the statement holds for n and let's see it holds also for n+1.

So let $k\in K(n+1)$ and $k\neq\varepsilon$. By assumption 2.1.2:

there exist $g\in K(n)$, $z\in V$-dom(g), $\psi\in E(n,g)$: $k= g \| (z,\psi) \wedge \forall\sigma\in\Xi(g)\ \#(g,\psi,\sigma)$ is a set $\wedge$
$\Xi(k) = \{ \sigma \| (z,s) \mid \sigma\in\Xi(g), s\in\#(g,\psi,\sigma) \}$.

By the inductive hypothesis

$g = \varepsilon$ or
($n>1$ and there exist

- a positive integer p such that $p<n$,
- $x_1, .. , x_p \in V$ such that $x_i\neq x_j$ for $i\neq j$,
- $\varphi_1, .. , \varphi_p \in E$

such that $K(n; g; x_1, .. , x_p; \varphi_1, .. , \varphi_p)$) .

We first consider the case where $g=\varepsilon$.

Here we define $p=1 < n+1$, $x_1= z\in V$, $\varphi_1=\psi\in E$.

We have $k_0 = \varepsilon \in K(n)$, $\varphi_1=\psi\in E(n,g)=E(n,\varepsilon)=E(n,k_0)$,
for each $\rho_0\in\Xi(k_0)$ $\#(k_0, \varphi_1, \rho_0) = \#(g, \psi, \rho_0)$ is a set .

Moreover $k=(x_1,\varphi_1)$.

Let $\sigma \in \Xi(k)$ and define $\sigma_0 = \varepsilon$. We need to show there exists $s_1 \in \#(k_0, \varphi_1, \sigma_0)$ such that $\sigma = (x_1, s_1)$.

For each $\sigma \in \Xi(k)$ there exist $\rho \in \Xi(g)$, $s \in \#(g, \psi, \rho)$ such that $\sigma = \rho \parallel (z,s)$.
Since $g = \varepsilon$ then $\rho = \varepsilon$ and $\sigma = (z,s) = (x_1,s)$ and $s \in \#(\varepsilon, \varphi_1, \varepsilon) = \#(k_0, \varphi_1, \sigma_0)$.

Therefore we have shown $\mathcal{K}(n+1; k; x_1; \varphi_1)$.

We now turn to consider the case where

n>1 and there exist

- a positive integer p such that $p<n$,
- $x_1, .. , x_p \in V$ such that $x_i \neq x_j$ for $i \neq j$,
- $\varphi_1, .. , \varphi_p \in E$

such that $\mathcal{K}(n; g; x_1, .. , x_p; \varphi_1, .. , \varphi_p)$.

In this case p+1 is a positive integer and $p+1 < n+1$. We define $x_{p+1} = z \in V$, $\varphi_{p+1} = \psi \in E$ and need to show that $\mathcal{K}(n+1; k; x_1, .. , x_{p+1}; \varphi_1, .. , \varphi_{p+1})$ holds.

We define $k_0 = \varepsilon$ and for each i=1..p $k_i = (x_1, \varphi_1) \parallel .. \parallel (x_i, \varphi_i)$.

Since $\mathcal{K}(n; g; x_1, .. , x_p; \varphi_1, .. , \varphi_p)$ holds, for each i=1..p

- $k_{i-1} \in K(n-1) \subseteq K(n)$,
- $\varphi_i \in E(n-1, k_{i-1}) \subseteq E(n, k_{i-1})$,
- for each $\rho_{i-1} \in \Xi(k_{i-1})$ $\#(k_{i-1}, \varphi_i, \rho_{i-1})$ is a set .

Moreover

- $k_p = g \in K(n)$,
- $\varphi_{p+1} = \psi \in E(n,g) = E(n,k_p)$,
- for each $\rho_p \in \Xi(k_p)$ we have $\rho_p \in \Xi(g)$ and $\#(k_p, \varphi_{p+1}, \rho_p) = \#(g, \psi, \rho_p)$ is a set.

Of course it also holds

$k = g \parallel (z,\psi) = (x_1, \varphi_1) \parallel .. \parallel (x_p,\varphi_p) \parallel (x_{p+1},\varphi_{p+1})$.

Let $\sigma \in \Xi(k)$. We define $\sigma_0 = \varepsilon$ and for each i=1..p $\sigma_i = \sigma/\text{dom}(k_i)$, and we need to show there exist $s_1 \in \#(k_0,\varphi_1,\sigma_0), \dots , s_{p+1} \in \#(k_p,\varphi_{p+1},\sigma_p)$ such that $\sigma = (x_1, s_1) \parallel .. \parallel (x_{p+1},s_{p+1})$.

There exist $\rho \in \Xi(g)$, $s \in \#(g,\psi,\rho)$ such that $\sigma = \rho \parallel (z,s) = \rho \parallel (x_{p+1},s)$.

Clearly $\sigma_p = \sigma/\text{dom}(k_p) = \sigma/\text{dom}(g) = \sigma/\text{dom}(\rho) = \rho$, so $s \in \#(g,\psi,\rho) = \#(k_p,\varphi_{p+1},\sigma_p)$.

Since $\mathcal{K}(n; g; x_1, .. , x_p; \varphi_1, .. , \varphi_p)$ we also know that if we define $\rho_0 = \varepsilon$ and (if p>1) for each i=1..p-1 $\rho_i = \rho/\text{dom}(k_i)$ then there exist $s_1 \in \#(k_0,\varphi_1,\rho_0), \dots , s_p \in \#(k_{p-1},\varphi_p,\rho_{p-1})$ such that

$\rho = (x_1, s_1) \parallel .. \parallel (x_p,s_p)$.

Of course $\rho_0 = \varepsilon = \sigma_0$ and if p>1 for each i=1..p-1 $\rho_i = \rho/\text{dom}(k_i) = \sigma/\text{dom}(k_i) = \sigma_i$. Therefore

$s_1 \in \#(k_0,\varphi_1,\sigma_0), \ldots, s_p \in \#(k_{p-1},\varphi_p,\sigma_{p-1}), s \in \#(k_p,\varphi_{p+1},\sigma_p)$,

$\sigma = \rho \,||\, (x_{p+1},s) = (x_1, s_1) \,||\, .. \,||\, (x_p,s_p) \,||\, (x_{p+1},s)$.

□

The former result is useful to the next definition. In fact suppose n is a positive integer, with $n \geq 2$, suppose $k \in K(n)$ and $k \neq \varepsilon$. In this situation there exist a positive integer p such that $p<n$, $x_1, .., x_p \in V$: $x_i \neq x_j$ for $i \neq j$, $\varphi_1, .., \varphi_p \in$ E such that $K(n; k; x_1, .., x_p; \varphi_1, .., \varphi_p)$.

And of course p, $x_1, .., x_p; \varphi_1, .., \varphi_p$ are univocally determined. In fact suppose $K(n; k; y_1, .., y_q; \psi_1, .., \psi_q)$ also holds. In this case clearly

$(x_1, \varphi_1) \,||\, .. \,||\, (x_p,\varphi_p) = k = (y_1, \psi_1) \,||\, .. \,||\, (y_q,\psi_q)$, therefore

$q=p$, for each $i=1..p$ $y_i = x_i$, $\psi_i = \varphi_i$.

Definition 4.6

For each positive integer $n \geq 2$:
for each $k \in K(n)$, if $k \neq \varepsilon$ then there exist a positive integer p such that $p<n$, $x_1, .., x_p \in V$: $x_i \neq x_j$ for $i \neq j$, $\varphi_1, .., \varphi_p \in$ E such that $K(n; k; x_1, .., x_p; \varphi_1, .., \varphi_p)$, and p, $x_1, .., x_p; \varphi_1, .., \varphi_p$ are univocally determined.
Given $i=1..p$, $t \in E(k_{i-1})$ such that

- $\forall \rho_{i-1} \in \Xi(k_{i-1})$ $\#(k_{i-1},t,\rho_{i-1}) \in \#(k_{i-1},\varphi_i,\rho_{i-1})$,
- $\forall j=1..p$: $j \neq i$ $x_j \notin V_b(t)$
- $\forall j=i+1..p$ $V_b(t) \cap V_b(\varphi_j) = \varnothing$

what we want to do is:

- If $i=p$ we want to define $k\{x_i/t\}$ if not already defined.
  If $k\{x_i/t\}$ is already defined we'll verify it is $k\{x_i/t\} = k_{p-1}$, otherwise we'll explicitly define $k\{x_i/t\} = k_{p-1}$.
- If $i<p$ we want to verify the following
  - $k_{p-1}\{x_i/t\}$ is defined and belongs to K;
  - $x_p \in V\text{-dom}(k_{p-1}\{x_i/t\})$;
  - $(\varphi_p)_{k(p-1)}\{x_i/t\}$ is defined and belongs to $E(k_{p-1}\{x_i/t\})$;
  - for each $\rho \in \Xi(k_{p-1}\{x_i/t\})$ $\#(k_{p-1}\{x_i/t\}, (\varphi_p)_{k(p-1)}\{x_i/t\}, \rho)$ is a set ;

  Then if $k\{x_i/t\}$ is already defined we'll verify it is
  - $k\{x_i/t\} = k_{p-1}\{x_i/t\} \,||\, (x_p, (\varphi_p)_{k(p-1)}\{x_i/t\})$.

  Otherwise we'll explicitly define
  - $k\{x_i/t\} = k_{p-1}\{x_i/t\} \,||\, (x_p, (\varphi_p)_{k(p-1)}\{x_i/t\})$.
- In both cases $i=p$ and $i<p$ we'll verify
  - $\text{dom}(k\{x_i/t\}) = \text{dom}(k) - \{x_i\}$;
  - $k\{x_i/t\} \in K$;
  - for each $\rho \in \Xi(k\{x_i/t\})$, if we define $\rho_{i-1} = \rho/\text{dom}(k_{i-1})$, and define σ as the soop $(x_1,r_1) \,||\, .. \,||\, (x_p,r_p)$ where $\forall j \neq i$ $r_j=\rho(x_j)$ and $r_i = \#(k_{i-1},t,\rho_{i-1})$ then $\sigma \in \Xi(k)$;
- for each $\varphi \in E(n,k)$, with $V_b(t) \cap V_b(\varphi) = \varnothing$
  - we wish to define $\varphi_k\{x_i/t\}$
  - we wish to show that $\varphi_k\{x_i/t\} \in E(k\{x_i/t\})$

- we wish to prove that for each $\rho \in \Xi(k\{x_i/t\})$, if we define $\rho_{i-1} = \rho/dom(k_{i-1})$, and define σ as the soop $(x_1,r_1) \parallel .. \parallel (x_p,r_p)$ where $\forall j \neq i\ r_j=\rho(x_j)$ and $r_i = \#(k_{i-1},t,\rho_{i-1})$ then
  $\#(k, \varphi, \sigma) = \#(k\{x_i/t\}, \varphi_k\{x_i/t\}, \rho)$.
- we wish to prove that $V_b(\varphi_k\{x_i/t\}) \subseteq V_b(\varphi) \cup V_b(t)$.

- we wish to verify one of the following five conditions holds
  - $\varphi \in C \wedge \varphi_k\{x_i/t\} = \varphi$,

  - $\varphi \in dom(k) \wedge (\varphi=x_i \rightarrow \varphi_k\{x_i/t\} = t) \wedge (\varphi \neq x_i \rightarrow \varphi_k\{x_i/t\} = \varphi)$

  - n>1, there exist $\kappa \in K(n-1)$ such that $\kappa \sqsubseteq k$ and $\psi, \psi_1, \dots, \psi_m \in E(n-1,\kappa)$ such that $\varphi = (\psi)(\psi_1, \dots, \psi_m)$, $\varphi \in E(n,\kappa)$,
    for each $\rho \in \Xi(\kappa)$ $\#(\kappa,\psi,\rho)$ is a function with m arguments and
    $(\#(\kappa, \psi_1, \rho), ... , \#(\kappa, \psi_m, \rho))$ is a member of its domain,
    if $x_i \in dom(\kappa)$ then $\varphi_k\{x_i/t\} = (\psi_\kappa\{x_i/t\}) ((\psi_1)_\kappa\{x_i/t\}, \dots, (\psi_m)_\kappa\{x_i/t\})$,
    else $\varphi_k\{x_i/t\} = \varphi$.

  - n>1, there exist $\kappa \in K(n-1)$ such that $\kappa \sqsubseteq k$ and $f \in F$, $\psi_1, \dots, \psi_m \in E(n-1,\kappa)$ such that $\varphi = (f)(\psi_1, \dots, \psi_m)$, $\varphi \in E(n,\kappa)$,
    for each $\rho \in \Xi(\kappa)$ $A_f(\#(\kappa, \psi_1, \rho), ... , \#(\kappa, \psi_m, \rho))$ is true,
    if $x_i \in dom(\kappa)$ then $\varphi_k\{x_i/t\} = (f) ((\psi_1)_\kappa\{x_i/t\}, \dots, (\psi_m)_\kappa\{x_i/t\})$,
    else $\varphi_k\{x_i/t\} = \varphi$.

  - n>1, there exist $\kappa \in K(n-1)$ such that $\kappa \sqsubseteq k$ and $\psi, \psi_1, .. , \psi_m \in E(n-1)$,
    $y_1, \dots, y_m$ distinct $\in V$-dom(κ) such that
    $\varphi = \{\}(y_1{:}\psi_1, \dots, y_m{:}\psi_m, \psi)$, $\varphi \in E(n,\kappa)$,

    $\psi_1 \in E(n-1,\kappa)$, for each $\sigma \in \Xi(\kappa)$ $\#(\kappa,\psi_1,\sigma)$ is a set ;
    if m>1 then for each j=1..m-1 we define $\kappa'_j = \kappa \parallel (y_1,\psi_1) \parallel .. \parallel (y_j,\psi_j)$ and we have $\kappa'_j \in K(n-1)$, $\psi_{j+1} \in E(n-1, \kappa'_j)$, for each $\sigma'_j \in \Xi(\kappa'_j)$ $\#(\kappa'_j, \psi_{j+1}, \sigma'_j)$ is a set;
    if we define $\kappa'_m = \kappa \parallel (x_1,\psi_1) \parallel .. \parallel (x_m,\psi_m)$ then $\kappa'_m \in K(n-1) \wedge \psi \in E(n-1,\kappa'_m)$;

    if $x_i \in dom(\kappa)$ then we can observe that

    $\psi_1 \in E(n,\kappa)$, $V_b(t) \cap V_b(\psi_1) \subseteq V_b(t) \cap V_b(\varphi) = \varnothing$, therefore $(\psi_1)_\kappa\{x_i/t\}$ is defined;

    for each j=1..m-1 $\psi_{j+1} \in E(n, \kappa'_j)$, for each α=1..j $y_\alpha \in V_b(\varphi)$ so $y_\alpha \notin V_b(t)$,
    $V_b(t) \cap V_b(\psi_\alpha) \subseteq V_b(t) \cap V_b(\varphi) = \varnothing$, $V_b(t) \cap V_b(\psi_{j+1}) \subseteq V_b(t) \cap V_b(\varphi) = \varnothing$,
    therefore $(\psi_{j+1})_{\kappa'(j)}\{x_i/t\}$ is defined ;

    $\psi \in E(n, \kappa'_m)$, for each α=1..m $y_\alpha \in V_b(\varphi)$ so $y_\alpha \notin V_b(t)$,
    $V_b(t) \cap V_b(\psi_\alpha) \subseteq V_b(t) \cap V_b(\varphi) = \varnothing$, $V_b(t) \cap V_b(\psi) \subseteq V_b(t) \cap V_b(\varphi) = \varnothing$,
    therefore $\psi_{\kappa'(m)}\{x_i/t\}$ is defined ;

    it results $\varphi_k\{x_i/t\} =$
    $= \{\}(y_1{:}\ (\psi_1)_\kappa\{x_i/t\}, y_2{:}\ (\psi_2)_{\kappa'(1)}\{x_i/t\} , \dots, y_m{:}\ (\psi_m)_{\kappa'(m-1)}\{x_i/t\}, \psi_{\kappa'(m)}\{x_i/t\})$;

if $x_i \notin dom(\kappa)$ then $\varphi_k\{x_i/t\} = \varphi$.

- we wish to verify the following: let $h \in K(n)$ such that $k_i \sqsubseteq h$.
  There exists a positive integer $q<n$, $y_1, \ldots, y_q \in V$: $y_\alpha \neq y_\beta$ for $\alpha \neq \beta$, $\psi_1, .., \psi_q \in E$ such that $K(n; h; y_1, .., y_q; \psi_1, .., \psi_q)$.
  Therefore $h = (y_1, \psi_1) \parallel .. \parallel (y_q,\psi_q)$, $i \leq q$, for each $j=1..i$ $y_j = x_j$, $\psi_j = \varphi_j$.
  If $i<q$ then assume for each $j = i+1 .. q$
    - $y_j \notin V_b(t)$
    - $V_b(t) \cap V_b(\psi_j) = \varnothing$.

  Also assume $\varphi \in E(n,h)$.
  Then $\varphi_k\{x_i/t\} = \varphi_h\{x_i/t\}$.

- we wish to verify the following: let $h \in K(n)$ such that $\varphi \in E(n,h)$, $x_i \notin dom(h)$.
  Then $\varphi_k\{x_i/t\} = \varphi$.

Our definition process uses induction on $n \geq 2$, therefore in the initial step we have $n=2$.
If $k \in K(2)$ and $k \neq \varepsilon$ then there exist $x_1 \in V$, $\varphi_1 \in E$ such that $K(1; k; x_1, \varphi_1)$.
It results $k = (x_1, \varphi_1)$ and $\varphi_1 \in E(1,\varepsilon)$.
Let $t \in E(\varepsilon)$ such that $\#(t) \in \#(\varphi_1)$.

We can define $k\{x_1/t\} = \varepsilon \in K(1)$; we have $dom(\varepsilon) = \varnothing = \{x_1\} - \{x_1\}$, $k\{x_1/t\} = \varepsilon = k_{p-1}$.

For each $\rho \in \Xi(\varepsilon)$, suppose we define $\rho_0 = \rho/dom(k_0) = \varepsilon$, and define $\sigma = (x_1,\#(t))$. We have $k = \varepsilon \parallel (x_1, \varphi_1)$, where $\varepsilon \in K(1)$, $x_1 \in V\text{-}dom(\varepsilon)$, $\varphi_1 \in E(1,\varepsilon)$, for each $\sigma_0 \in \Xi(\varepsilon)$ $\#(\varepsilon, \varphi_1, \sigma_0)$ is a set. So $k \in K(1)^+$ and since $\varepsilon \in \Xi(\varepsilon)$, $\#(t) \in \#(\varepsilon,\varphi_1,\varepsilon)$, then $\sigma = \varepsilon \parallel (x_1,\#(t)) \in \Xi(k)$.

Let $\varphi \in E(2,k)$ such that $V_b(t) \cap V_b(\varphi) = \varnothing$.

Of course

$$E(2,k) = E'(1,k) \cup E'_a(2,k) \cup E'_b(2,k) \cup E'_c(2,k) \cup E'_d(2,k) \cup E'_e(2,k) .$$

Suppose **$\varphi \in E'(1,k)$**, so $\varphi \in E(1,k)$ and $k \in K(1)$, $k=\varepsilon$. This is against our assumption that $k \neq \varepsilon$, so we must exclude the case where $\varphi \in E'(1,k)$.

Now suppose **$\varphi \in E'_a(2,k)$**. This means $\varphi \in E_a(2,k)$, $k \in K(1)^+$. So there exist $h \in K(1)$, $y_1 \in (V\text{-}dom(h))$, $\psi_1 \in E_s(1,h)$ such that $(x_1, \varphi_1) = k = h \parallel (y_1,\psi_1)$. This implies $h = \varepsilon$, $y_1 = x_1$, $\psi_1 = \varphi_1$.
We have also $\varphi \in E(1,\varepsilon)$ (which implies $V_b(\varphi)=\varnothing$ and $x_1 \notin V_b(\varphi)$).

We define $\varphi_k\{x_1/t\} = \varphi \in E(\varepsilon) = E(k\{x_1/t\})$.

For each $\rho \in \Xi(\varepsilon)$, if we define $\rho_0 = \rho/dom(k_0) = \varepsilon$, and define $\sigma = (x_1,\#(t))$ then we have seen that $\sigma \in \Xi(k)$, and furthermore $\sigma = \varepsilon \parallel (x_1,\#(t))$, where $k = \varepsilon \parallel (x_1, \varphi_1)$, $\varepsilon \in \Xi(\varepsilon)$, $\#(t) \in \#(\varepsilon,\varphi_1,\varepsilon)$. So it must be

$$\#(k, \varphi, \sigma) = \#(\varepsilon,\varphi,\varepsilon) = \#(k\{x_1/t\}, \varphi_k\{x_1/t\}, \rho) .$$

Of course $V_b(\varphi_k\{x_1/t\}) = V_b(\varphi) = \varnothing \subseteq V_b(\varphi) \cup V_b(t)$.

Let's go on with the other verifications.

Suppose $h \in K(2)$ such that $k_i \sqsubseteq h$. This implies $h=k$ and so it's obvious that $\varphi_k\{x_i/t\} = \varphi_h\{x_i/t\}$.

Now let $h \in K(2)$ such that $\varphi \in E(2,h)$, $x_i \notin dom(h)$. Actually these hypothesis are not needed to verify that $\varphi_k\{x_1/t\} = \varphi$, since this is the definition of $\varphi_k\{x_1/t\}$.

Finally, the condition $\varphi \in \mathcal{C} \wedge \varphi_k\{x_i/t\} = \varphi$ is clearly satisfied.

Let's examine the case where **$\varphi \in E'_b(2,k)$**. This means $\varphi \in E_b(2,k)$, $k \in K(1)^+$. So there exist $h \in K(1)$, $y_1 \in (\mathcal{V}\text{-dom}(h))$, $\psi_1 \in E_s(1,h)$ such that $(x_1, \varphi_1) = k = h \parallel (y_1, \psi_1)$. This implies $h = \varepsilon$, $y_1 = x_1$, $\psi_1 = \varphi_1$. We have $\varphi = x_1$ and we define $\varphi_k\{x_1/t\} = t \in E(\varepsilon) = E(k\{x_1/t\})$.

For each $\rho \in \Xi(\varepsilon)$, if we define $\rho_0 = \rho/dom(k_0) = \varepsilon$, and define $\sigma = (x_1, \#(t))$ then we have seen that $\sigma \in \Xi(k)$, and furthermore $\#(k, \varphi, \sigma) = \sigma(x_1) = \#(t) = \#(\varepsilon,t,\varepsilon) = \#(k\{x_1/t\}, \varphi_k\{x_1/t\}, \rho)$.

Of course $V_b(\varphi_k\{x_1/t\}) = V_b(t) \subseteq V_b(\varphi) \cup V_b(t)$.

Let's see the other points.

Suppose $h \in K(2)$ such that $k_i \sqsubseteq h$. This implies $h=k$ and so it's obvious that $\varphi_k\{x_i/t\} = \varphi_h\{x_i/t\}$.

Now let $h \in K(2)$ such that $\varphi \in E(2,h)$, $x_i \notin dom(h)$. We have $h = (y_1, \psi_1)$ where $y_1 \in \mathcal{V}$, $y_1 \neq x_1$ and $\psi_1 \in E(1,\varepsilon)$. This implies $\varphi = y_1$ or $\varphi \in E(1,\varepsilon) = \mathcal{C}$ (we'll see very soon there aren't other possibilities). In both cases $\varphi \neq x_1$, but this contradicts our hypothesis. So it is never the case there exist $h \in K(2)$ such that $\varphi \in E(2,h)$, $x_i \notin dom(h)$.

Finally, the following condition holds

$\varphi \in dom(k) \wedge \varphi = x_i \wedge \varphi_k\{x_i/t\} = t$, and therefore the following is satisfied:

$\varphi \in dom(k) \wedge (\ \varphi = x_i \rightarrow \varphi_k\{x_i/t\} = t\ ) \wedge (\ \varphi \neq x_i \rightarrow \varphi_k\{x_i/t\} = \varphi\ )$.

Now suppose **$\varphi \in E'_c(2,k)$**. This implies $\varphi \in E_c(2,k) \neq \emptyset$, so $k \in K(1)$, $k=\varepsilon$. This is against our assumption that $k \neq \varepsilon$, so we must exclude the case where $\varphi \in E'_c(2,k)$, and the same way we have to exclude the cases where $\varphi \in E'_d(2,k)$, and $\varphi \in E'_e(2,k)$.

We've seen the only two 'real' cases are $\varphi \in E'_a(2,k)$ and $\varphi \in E'_b(2,k)$, and the definition of $\varphi_k\{x_1/t\}$ depends on which case is verified. Clearly $E'_a(2,k)$ and $E'_b(2,k)$ are disjoint sets so the definition we have set out is correct.

This wraps up the initial step of our definition process. To deal with the inductive step let $n \geq 2$, suppose we have given our definitions and verified the results at step n, and let's go on with step n+1.

Let $k \in K(n+1)$ such that $k \neq \varepsilon$. Let p be a positive integer such that $p<n+1$, $x_1, .., x_p \in \mathcal{V}$: $x_i \neq x_j$ for $i \neq j$, $\varphi_1, .., \varphi_p \in E$ such that $\mathcal{K}(n+1; k; x_1, .., x_p; \varphi_1, .., \varphi_p)$.

Let $i=1..p$, $t\in$ $t\in E(k_{i-1})$ such that

- $\forall\rho_{i-1}\in\Xi(k_{i-1})\ \#(k_{i-1},t,\rho_{i-1}) \in \#(k_{i-1},\varphi_i,\rho_{i-1})$,
- $\forall j=1..p$: $j\neq i$ $x_j\notin V_b(t)$
- $\forall j=i+1..p$ $V_b(t) \cap V_b(\varphi_j) = \varnothing$

Consider the case where **i=p**.

If **k∈K(n)** there exist a positive integer $q<n$, $y_1, .. ,y_q \in V$: $y_i\neq y_j$ for $i\neq j$, $\psi_1, .. , \psi_q \in E$ such that $K(n; k; y_1, .. , y_q; \psi_1, .. , \psi_q)$. Therefore

$(y_1,\psi_1) \parallel .. \parallel (y_q,\psi_q) = k = (x_1,\varphi_1) \parallel .. \parallel (x_p,\varphi_p)$, so $q=p$, $y_i=x_i$, $\psi_i=\varphi_i$ and $K(n; k; x_1, .. , x_p; \varphi_1, .. , \varphi_p)$.

For this reason, by the inductive hypothesis, $k\{x_i/t\}$ is already defined and $k\{x_i/t\} = k_{p-1}$; $dom(k\{x_i/t\}) = dom(k) - \{x_i\}$; $k\{x_i/t\}\in K$; for each $\rho\in\Xi(k\{x_i/t\})$, if we define $\rho_{i-1} = \rho/dom(k_{i-1})$, and define σ as the soop $(x_1,r_1) \parallel .. \parallel (x_p,r_p)$ where $\forall j\neq i$ $r_j=\rho(x_j)$ and $r_i = \#(k_{i-1},t,\rho_{i-1})$ then $\sigma\in\Xi(k)$.

If on the contrary **k∉K(n)** then we define $k\{x_i/t\} = k_{p-1} \in K(n)$, of course
$dom(k\{x_i/t\}) = dom(k_{p-1}) = dom(k) - \{x_i\}$.

Let $\rho\in\Xi(k\{x_i/t\})$, define $\rho_{i-1} = \rho/dom(k_{i-1})$, define σ as the soop $(x_1,r_1) \parallel .. \parallel (x_p,r_p)$ where $\forall j\neq i$ $r_j=\rho(x_j)$ and $r_i = \#(k_{i-1},t,\rho_{i-1})$. We have $k = k_{p-1} \parallel (x_p, \varphi_p) \in K(n)^+$,
$\sigma = \rho \parallel (x_p, \#(k_{p-1},t,\rho_{i-1})) = \rho \parallel (x_p, \#(k_{p-1},t,\rho))$, and also $\rho\in\Xi(k_{p-1})$, $\#(k_{p-1},t,\rho) \in \#(k_{p-1},\varphi_p,\rho)$. This implies $\sigma\in\Xi(k)$.

Now we turn to examine the case where **i<p**.

If **k∈K(n)** then $K(n; k; x_1, .. , x_p; \varphi_1, .. , \varphi_p)$. By the inductive hypothesis,

- $k_{p-1}\{x_i/t\}$ is defined and belongs to K;
- $x_p\in V\text{-}dom(k_{p-1}\{x_i/t\})$;
- $(\varphi_p)_{k(p-1)}\{x_i/t\}$ is defined and belongs to $E(k_{p-1}\{x_i/t\})$;
- for each $\rho\in\Xi(k_{p-1}\{x_i/t\})$ $\#(k_{p-1}\{x_i/t\}, (\varphi_p)_{k(p-1)}\{x_i/t\}, \rho)$ is a set ;

$k\{x_i/t\}$ is already defined and $k\{x_i/t\} = k_{p-1}\{x_i/t\} \parallel (x_p, (\varphi_p)_{k(p-1)}\{x_i/t\})$;
$dom(k\{x_i/t\}) = dom(k) - \{x_i\}$; $k\{x_i/t\}\in K$; for each $\rho\in\Xi(k\{x_i/t\})$, if we define $\rho_{i-1} = \rho/dom(k_{i-1})$, and define σ as the soop $(x_1,r_1) \parallel .. \parallel (x_p,r_p)$ where $\forall j\neq i$ $r_j=\rho(x_j)$ and $r_i = \#(k_{i-1},t,\rho_{i-1})$ then $\sigma\in\Xi(k)$.

If on the contrary **k∉K(n)** then $k_{p-1} \in K(n)$ and there exist $q<n$, $y_1, .. ,y_q \in V$: $y_i\neq y_j$ for $i\neq j$, $\psi_1, .. , \psi_q \in E$ such that $K(n; k_{p-1}; y_1, .. , y_q; \psi_1, .. , \psi_q)$. Therefore

$(y_1,\psi_1) \parallel .. \parallel (y_q,\psi_q) = k_{p-1} = (x_1,\varphi_1) \parallel .. \parallel (x_{p-1},\varphi_{p-1})$, so $q=p-1$, $y_i=x_i$, $\psi_i=\varphi_i$, and
$K(n; k_{p-1}; x_1, .. , x_{p-1}; \varphi_1, .. , \varphi_{p-1})$.

By the inductive hypothesis $k_{p-1}\{x_i/t\}$ is defined and $k_{p-1}\{x_i/t\}\in K$,
$dom(k_{p-1}\{x_i/t\}) = dom(k_{p-1}) - \{x_i\}$.

We also consider that $\varphi_p \in E(n, k_{p-1})$, $V_b(t) \cap V_b(\varphi_p) = \varnothing$, so $(\varphi_p)_{k(p-1)}\{x_i/t\}$ is also defined and belongs to $E(k_{p-1}\{x_i/t\})$.

We have $k_{p-1}\{x_i/t\}\in K$, $x_p\in V\text{-dom}(k_{p-1}\{x_i/t\})$, $(\varphi_p)_{k(p-1)}\{x_i/t\}\in E(k_{p-1}\{x_i/t\})$.

We still need to prove that for each $\rho\in\Xi(k_{p-1}\{x_i/t\})$ $\#(k_{p-1}\{x_i/t\}, (\varphi_p)_{k(p-1)}\{x_i/t\}, \rho)$ is a set.
Let $\rho_{i-1} = \rho/\text{dom}(k_{i-1})$ and define σ as the soop $(x_1,r_1) \parallel .. \parallel (x_{p-1},r_{p-1})$ where $\forall j\neq i$ $r_j=\rho(x_j)$ and $r_i = \#(k_{i-1},t,\rho_{i-1})$. Since $\varphi_p \in E(n, k_{p-1})$, $V_b(t) \cap V_b(\varphi_p) = \emptyset$, we have $\sigma\in\Xi(k_{p-1})$, $\#(k_{p-1}\{x_i/t\}, (\varphi_p)_{k(p-1)}\{x_i/t\}, \rho) = \#(k_{p-1}, \varphi_p, \sigma)$. And since $\#(k_{p-1}, \varphi_p, \sigma)$ is a set then also $\#(k_{p-1}\{x_i/t\}, (\varphi_p)_{k(p-1)}\{x_i/t\}, \rho)$ is a set.

So we can define $k\{x_i/t\} = k_{p-1}\{x_i/t\} \parallel (x_p, (\varphi_p)_{k(p-1)}\{x_i/t\})$, and $k\{x_i/t\} \in K$.

Moreover $\text{dom}(k\{x_i/t\}) = \text{dom}(k_{p-1}\{x_i/t\}) \cup \{x_p\} = (\text{dom}(k_{p-1})-\{x_i\}) \cup \{x_p\} =$
$= (\text{dom}(k_{p-1}) \cup \{x_p\}) - \{x_i\} = \text{dom}(k) - \{x_i\}$.

Let $\rho\in\Xi(k\{x_i/t\})$, define $\rho_{i-1} = \rho/\text{dom}(k_{i-1})$, define σ as the soop $(x_1,r_1) \parallel .. \parallel (x_p,r_p)$ where $\forall j\neq i$ $r_j=\rho(x_j)$ and $r_i = \#(k_{i-1},t,\rho_{i-1})$. We need to show that $\sigma\in\Xi(k)$.

Of course, there must exist $\rho_{p-1}\in\Xi(k_{p-1}\{x_i/t\})$ and $s\in\#(k_{p-1}\{x_i/t\},(\varphi_p)_{k(p-1)}\{x_i/t\}, \rho_{p-1})$ such that
$\rho = \rho_{p-1} \parallel (x_p, s)$. We define $\sigma_{p-1} = \sigma/\text{dom}(k_{p-1})$.

It's pretty obvious that $\rho_{i-1} = \rho_{p-1}/\text{dom}(k_{i-1})$.

To show this holds consider that $\rho_{p-1}\in\Xi(k_{p-1}\{x_i/t\})$, $\text{dom}(\rho_{p-1}) = \text{dom}(k_{p-1}\{x_i/t\}) = \text{dom}(k_{p-1}) - \{x_i\}$; $\text{dom}(\rho_{i-1}) = \text{dom}(k_{i-1}) \subseteq \text{dom}(\rho_{p-1})$. Since $\rho_{p-1} \sqsubseteq \rho$, $\rho_{i-1} \sqsubseteq \rho$, $\text{dom}(\rho_{i-1}) \subseteq \text{dom}(\rho_{p-1})$ we have $\rho_{i-1} \sqsubseteq \rho_{p-1}$ and so $\rho_{i-1} = \rho_{p-1}/\text{dom}(\rho_{i-1}) = \rho_{p-1}/\text{dom}(k_{i-1})$.

It's also obvious that $\sigma_{p-1} = (x_1,r_1) \parallel .. \parallel (x_{p-1},r_{p-1})$, for each $j=1..p-1$ if $j\neq i$ then $r_j=\rho(x_j)=\rho_{p-1}(x_j)$, $r_i = \#(k_{i-1},t,\rho_{i-1})$.

So we can apply the inductive hypothesis to obtain that $\sigma_{p-1}\in\Xi(k_{p-1})$.

To show that $\sigma\in\Xi(k)$ we consider that $k = k_{p-1} \parallel (x_p, \varphi_p) \in K(n)^+$, $\sigma = \sigma_{p-1} \parallel (x_p, \rho(x_p))$. So, to say that $\sigma\in\Xi(k)$ holds, we just need to show that $\rho(x_p) \in \#(k_{p-1}, \varphi_p, \sigma_{p-1})$.

We have $\rho = \rho_{p-1} \parallel (x_p, s) \in\Xi(k\{x_i/t\})$ and $\rho(x_p) = s \in \#(k_{p-1}\{x_i/t\}, (\varphi_p)_{k(p-1)}\{x_i/t\}, \rho_{p-1})$, if we apply the inductive hypotesis to $k_{p-1}\in K(n)$, $\varphi_p\in E(n, k_{p-1})$ (this is possible since $V_b(t) \cap V_b(\varphi_p) = \emptyset$) and consider that $\rho_{p-1}\in\Xi(k_{p-1}\{x_i/t\})$, $\rho_{i-1} = \rho_{p-1}/\text{dom}(k_{i-1})$, $\sigma_{p-1} = (x_1,r_1) \parallel .. \parallel (x_{p-1},r_{p-1})$, for each $j=1..p-1$ if $j\neq i$ then $r_j=\rho(x_j)=\rho_{p-1}(x_j)$, $r_i = \#(k_{i-1},t,\rho_{i-1})$ it comes out that

$\#(k_{p-1}, \varphi_p, \sigma_{p-1}) = \#(k_{p-1}\{x_i/t\}, (\varphi_p)_{k(p-1)}\{x_i/t\}, \rho_{p-1})$,

and therefore $\rho(x_p) \in \#(k_{p-1}, \varphi_p, \sigma_{p-1})$.

Let $\varphi\in E(n+1,k)$ such that $V_b(t) \cap V_b(\varphi) = \emptyset$. Remember that

$E(n+1,k) = E'(n,k) \cup E'_a(n+1,k) \cup E'_b(n+1,k) \cup E'_c(n+1,k) \cup E'_d(n+1,k) \cup E'_e(n+1,k)$.

The definition of $\varphi_k\{x_i/t\}$ depends on the set to which φ belongs to, actually φ may belong to more than one of these sets, but this problem will be addressed later when we'll show that the definitions match each other.

Suppose **φ is in E'(n,k)**. This means $\varphi \in E(n,k)$, $k \in K(n)$. In this case, by the inductive hypothesis, $\varphi_k\{x_i/t\}$ is already defined and has all the requested properties.

Now suppose **$\varphi \in E'_a(n+1,k)$**. This implies $\varphi \in E_a(n+1,k)$, $k \in K(n)^+$.

We have $k = k_{p-1} \parallel (x_p, \varphi_p)$, and there exist $h \in K(n)$, $y \in \mathcal{V}$-dom(h), ψ such that $k = h \parallel (y,\psi)$, $\varphi \in E(n,h)$, $y \notin V_b(\varphi)$. Of course $h = k_{p-1}$, $y = x_p$, $\psi = \varphi_p$ so $\varphi \in E(n,k_{p-1})$, $x_p \notin V_b(\varphi)$.

If **i=p** we define $\varphi_k\{x_i/t\} = \varphi \in E(k_{p-1}) = E(k\{x_i/t\})$.

Let $\rho \in \Xi(k\{x_i/t\})$, define $\rho_{i-1} = \rho/\text{dom}(k_{i-1})$, define σ as the soop $(x_1,r_1) \parallel .. \parallel (x_p,r_p)$ where $\forall j \neq i$ $r_j=\rho(x_j)$ and $r_i = \#(k_{i-1},t,\rho_{i-1})$. We need to show that $\sigma \in \Xi(k)$, $\#(k, \varphi, \sigma) = \#(k\{x_i/t\}, \varphi_k\{x_i/t\}, \rho)$.

We have $\sigma = \rho \parallel (x_p, \#(k_{p-1},t,\rho_{i-1})) = \rho \parallel (x_p, \#(k_{p-1},t,\rho))$, and also
$\rho \in \Xi(k_{p-1})$, $\#(k_{p-1},t,\rho) \in \#(k_{p-1},\varphi_p,\rho)$. This implies $\sigma \in \Xi(k)$. By lemma 4.2 we have also

$\#(k,\varphi,\sigma) = \#(k_{p-1},\varphi,\rho) = \#(k\{x_i/t\}, \varphi_k\{x_i/t\}, \rho)$.

Moreover $V_b(\varphi_k\{x_i/t\}) = V_b(\varphi) \subseteq V_b(\varphi) \cup V_b(t)$.

If **i<p** we consider that $k_{p-1} \in K(n)$ and therefore $\mathcal{K}(n;\, k_{p-1};\, x_1, .. , x_{p-1};\, \varphi_1, .. , \varphi_{p-1})$, and also $\varphi \in E(n,k_{p-1})$. This implies $\varphi_{k(p-1)}\{x_i/t\}$ is defined and belongs to $E(k_{p-1}\{x_i/t\})$.

So we can define $\varphi_k\{x_i/t\} = \varphi_{k(p-1)}\{x_i/t\} \in E(k_{p-1}\{x_i/t\})$.

We need to show that $\varphi_k\{x_i/t\} \in E(k\{x_i/t\})$. We consider that

- $k_{p-1}\{x_i/t\} \in K$;
- $x_p \in \mathcal{V}$-dom$(k_{p-1}\{x_i/t\})$;
- $(\varphi_p)_{k(p-1)}\{x_i/t\} \in E(k_{p-1}\{x_i/t\})$;
- for each $\rho \in \Xi(k_{p-1}\{x_i/t\})$ $\#(k_{p-1}\{x_i/t\}, (\varphi_p)_{k(p-1)}\{x_i/t\}, \rho)$ is a set ;
- $k\{x_i/t\} = k_{p-1}\{x_i/t\} \parallel (x_p, (\varphi_p)_{k(p-1)}\{x_i/t\})$.

Moreover we can show that $x_p \notin V_b(\varphi_{k(p-1)}\{x_i/t\})$.
In fact, by the inductive hypothesis, $V_b(\varphi_{k(p-1)}\{x_i/t\}) \subseteq V_b(\varphi) \cup V_b(t)$. We know that $V_b(\varphi) \subseteq \mathcal{V}$-dom(k), so $x_p \notin V_b(\varphi)$; and we know also $x_p \notin V_b(t)$, hence $x_p \notin V_b(\varphi_{k(p-1)}\{x_i/t\})$.

Therefore by lemma 4.2 we obtain that $\varphi_k\{x_i/t\} = \varphi_{k(p-1)}\{x_i/t\} \in E(k\{x_i/t\})$.

Let $\rho \in \Xi(k\{x_i/t\})$, define $\rho_{i-1} = \rho/\text{dom}(k_{i-1})$, define σ as the soop $(x_1,r_1) \parallel .. \parallel (x_p,r_p)$ where $\forall j \neq i$ $r_j=\rho(x_j)$ and $r_i = \#(k_{i-1},t,\rho_{i-1})$. We have shown that $\sigma \in \Xi(k)$ and we need to show that
$\#(k, \varphi, \sigma) = \#(k\{x_i/t\}, \varphi_k\{x_i/t\}, \rho)$.

We can use the proof of '$\sigma \in \Xi(k)$' as a reference. In that proof we have seen there must exist $\rho_{p-1} \in \Xi(k_{p-1}\{x_i/t\})$ and $s \in \#(k_{p-1}\{x_i/t\}, (\varphi_p)_{k(p-1)}\{x_i/t\}, \rho_{p-1})$ such that $\rho = \rho_{p-1} \parallel (x_p, s)$. We have also defined $\sigma_{p-1} = \sigma/\text{dom}(k_{p-1})$. We have seen that $\rho_{i-1} = \rho_{p-1}/\text{dom}(k_{i-1})$ and that
$\sigma_{p-1} = (x_1,r_1) \parallel .. \parallel (x_{p-1},r_{p-1})$, for each j=1..p-1 if $j \neq i$ then $r_j=\rho(x_j)=\rho_{p-1}(x_j)$, $r_i = \#(k_{i-1},t,\rho_{i-1})$.

By lemma 4.2 we have $\#(k\{x_i/t\}, \varphi_k\{x_i/t\}, \rho) = \#(k_{p-1}\{x_i/t\}, \varphi_{k(p-1)}\{x_i/t\}, \rho_{p-1})$.

We can apply the inductive hypothesis to obtain that $\sigma_{p-1} \in \Xi(k_{p-1})$ and

$\#(k_{p-1}\{x_i/t\}, \varphi_{k(p-1)}\{x_i/t\}, \rho_{p-1}) = \#(k_{p-1}, \varphi, \sigma_{p-1})$.

It remains to show that $\#(k_{p-1}, \varphi, \sigma_{p-1}) = \#(k, \varphi, \sigma)$.

This holds because of $k = k_{p-1} \parallel (x_p, \varphi_p)$, $\varphi \in E(k_{p-1})$, $x_p \notin V_b(\varphi)$, $\sigma = \sigma_{p-1} \parallel (x_p,\rho(x_p))$ and lemma 4.2 .

Finally we have $V_b(\varphi_k\{x_i/t\}) = V_b(\varphi_{k(p-1)}\{x_i/t\}) \subseteq V_b(\varphi) \cup V_b(t)$.

Consider the case where $\boldsymbol{\varphi \in E'_b(n+1,k)}$. This implies $\varphi \in E_b(n+1,k)$, $k \in K(n)^+$.

We have $k = k_{p-1} \parallel (x_p, \varphi_p)$, and there exist $h \in K(n)$, $y \in \mathcal{V}\text{-dom}(h)$, ψ such that $k = h \parallel (y,\psi)$, $\varphi = y$, for each $\sigma \in \Xi(k)$ $\#(k,\varphi,\sigma) = \sigma(y)$.
Of course $h = k_{p-1}$, $y = x_p$, $\psi = \varphi_p$ so $\varphi = x_p$, $\#(k,\varphi,\sigma) = \sigma(x_p)$.

If **i=p** we define $\varphi_k\{x_i/t\} = t \in E(k_{p-1}) = E(k\{x_i/t\})$.

Let $\rho \in \Xi(k\{x_i/t\})$, define $\rho_{i-1} = \rho/\text{dom}(k_{i-1})$, define σ as the soop $(x_1,r_1) \parallel .. \parallel (x_p,r_p)$ where $\forall j \neq i$ $r_j=\rho(x_j)$ and $r_i = \#(k_{i-1},t,\rho_{i-1})$. We need to show that $\sigma \in \Xi(k)$, $\#(k, \varphi, \sigma) = \#(k\{x_i/t\}, \varphi_k\{x_i/t\}, \rho)$.

We have $\sigma = \rho \parallel (x_p, \#(k_{p-1},t,\rho_{i-1})) = \rho \parallel (x_p, \#(k_{p-1},t,\rho))$, and also
$\rho \in \Xi(k_{p-1})$, $\#(k_{p-1},t,\rho) \in \#(k_{p-1},\varphi_p,\rho)$. This implies $\sigma \in \Xi(k)$.

We have also $\#(k, \varphi, \sigma) = \sigma(x_p) = \#(k_{p-1},t,\rho_{i-1}) = \#(k_{p-1}, t, \rho) = \#(k\{x_i/t\}, \varphi_k\{x_i/t\}, \rho)$.

Moreover $V_b(\varphi_k\{x_i/t\}) = V_b(t) \subseteq V_b(\varphi) \cup V_b(t)$.

If **i<p** we define $\varphi_k\{x_i/t\} = \varphi = x_p$.

We consider that

- $k_{p-1}\{x_i/t\} \in K$;
- $x_p \in \mathcal{V}\text{-dom}(k_{p-1}\{x_i/t\})$;
- $(\varphi_p)_{k(p-1)}\{x_i/t\} \in E(k_{p-1}\{x_i/t\})$;
- for each $\rho \in \Xi(k_{p-1}\{x_i/t\})$ $\#(k_{p-1}\{x_i/t\}, (\varphi_p)_{k(p-1)}\{x_i/t\}, \rho)$ is a set ;
- $k\{x_i/t\} = k_{p-1}\{x_i/t\} \parallel (x_p, (\varphi_p)_{k(p-1)}\{x_i/t\})$.

Therefore $\varphi_k\{x_i/t\} = x_p \in E(k\{x_i/t\})$.

Let $\rho \in \Xi(k\{x_i/t\})$, define $\rho_{i-1} = \rho/dom(k_{i-1})$, define σ as the soop $(x_1,r_1) \parallel .. \parallel (x_p,r_p)$ where $\forall j \neq i$ $r_j = \rho(x_j)$ and $r_i = \#(k_{i-1},t,\rho_{i-1})$. We have shown that $\sigma \in \Xi(k)$, we also need to show that $\#(k, \varphi, \sigma) = \#(k\{x_i/t\}, \varphi_k\{x_i/t\}, \rho)$.

By lemma 4.3 we have
$\#(k, \varphi, \sigma) = \#(k, x_p, \sigma) = \sigma(x_p) = \rho(x_p) = \#(k\{x_i/t\}, x_p, \rho) = \#(k\{x_i/t\}, \varphi_k\{x_i/t\}, \rho)$.

Finally $V_b(\varphi_k\{x_i/t\}) = V_b(\varphi) \subseteq V_b(\varphi) \cup V_b(t)$.

We turn to the case where **$\varphi \in E'_c(n+1,k)$**. This implies $\varphi \in E_c(n+1,k)$, $k \in K(n)$.

There exist a positive integer m and $\psi, \psi_1, .. , \psi_m \in E(n,k)$ such that

- $\varphi = (\psi)\,(\psi_1, .. , \psi_m)$,
- for each $\sigma \in \Xi(k)$ $\#(k,\psi,\sigma)$ is a function with m arguments and $(\ \#(k, \psi_1, \sigma), ... , \#(k, \psi_m, \sigma)\ )$ is a member of its domain.

Since $k \in K(n)$ we have $K(n; k; x_1, .. , x_p; \varphi_1, .. , \varphi_p)$. We have

$V_b(\varphi) = V_b(\psi) \cup V_b(\psi_1) \cup \ldots \cup V_b(\psi_m)$,

and since $V_b(t) \cap V_b(\varphi) = \emptyset$ we have

- $V_b(t) \cap V_b(\psi) = \emptyset$,
- for each j=1.m $V_b(t) \cap V_b(\psi_j) = \emptyset$.

By the inductive hypothesis $\psi_k\{x_i/t\}$ is defined and belongs to $E(k\{x_i/t\})$, and for each j=1..m $(\psi_j)_k\{x_i/t\}$ is defined and belongs to $E(k\{x_i/t\})$, so we can define

$\varphi_k\{x_i/t\} = (\psi_k\{x_i/t\})\,((\psi_1)_k\{x_i/t\}, .. , (\psi_m)_k\{x_i/t\})$.

We need to show that $\varphi_k\{x_i/t\} \in E(k\{x_i/t\})$.

Let $\rho \in \Xi(k\{x_i/t\})$, we want to show that $\#(\ k\{x_i/t\}, \psi_k\{x_i/t\}, \rho)$ is a function with m arguments and $(\#(\ k\{x_i/t\}, (\psi_1)_k\{x_i/t\}, \rho), .. , \#(\ k\{x_i/t\}, (\psi_m)_k\{x_i/t\}, \rho))$ is a member of its domain.

We define $\rho_{i-1} = \rho/dom(k_{i-1})$, and define σ as the soop $(x_1,r_1) \parallel .. \parallel (x_p,r_p)$ where $\forall j \neq i$ $r_j = \rho(x_j)$ and $r_i = \#(k_{i-1},t,\rho_{i-1})$. By the inductive hypothesis $\sigma \in \Xi(k)$ and

$\#(k, \psi, \sigma) = \#(k\{x_i/t\}, \psi_k\{x_i/t\}, \rho)$, and for each j=1..m $\#(k, \psi_j, \sigma) = \#(k\{x_i/t\}, (\psi_j)_k\{x_i/t\}, \rho)$.

So $\#(k\{x_i/t\}, \psi_k\{x_i/t\}, \rho) = \#(k, \psi, \sigma)$ is a function with m arguments and

$(\#(\ k\{x_i/t\}, (\psi_1)_k\{x_i/t\}, \rho), .. , \#(\ k\{x_i/t\}, (\psi_m)_k\{x_i/t\}, \rho)) = (\ \#(k, \psi_1, \sigma), ... , \#(k, \psi_m, \sigma)\ )$ is a member of the domain of $\#(k\{x_i/t\}, \psi_k\{x_i/t\}, \rho)$.

Therefore, by lemma 3.9, $\varphi_k\{x_i/t\} \in E(k\{x_i/t\})$.

Moreover if, as defined above, $\rho \in \Xi(k\{x_i/t\})$, $\rho_{i-1} = \rho/dom(k_{i-1})$, and σ is the soop $(x_1,r_1) \parallel .. \parallel (x_p,r_p)$ where $\forall j \neq i$ $r_j = \rho(x_j)$ and $r_i = \#(k_{i-1},t,\rho_{i-1})$ we have

$\#(k, (\psi)(\psi_1, .., \psi_m), \sigma) = \#(k, \psi, \sigma)(\#(k, \psi_1, \sigma), \dots, \#(k, \psi_m, \sigma)) =$

$= \#(k\{x_i/t\}, \psi_k\{x_i/t\}, \rho)(\#(k\{x_i/t\}, (\psi_1)_k\{x_i/t\}, \rho), .., \#(k\{x_i/t\}, (\psi_m)_k\{x_i/t\}, \rho)) =$

$= \#(k\{x_i/t\}, (\psi_k\{x_i/t\})((\psi_1)_k\{x_i/t\}, .., (\psi_m)_k\{x_i/t\}), \rho) = \#(k\{x_i/t\}, \varphi_k\{x_i/t\}, \rho)$.

Finally

$V_b(\varphi_k\{x_i/t\}) = V_b(\psi_k\{x_i/t\}) \cup V_b((\psi_1)_k\{x_i/t\}) \cup \dots \cup V_b((\psi_m)_k\{x_i/t\}) \subseteq$

$\subseteq V_b(\psi) \cup V_b(t) \cup V_b(\psi_1) \cup V_b(t) \cup \dots \cup V_b(\psi_m) \cup V_b(t) =$

$= V_b(\psi) \cup V_b(\psi_1) \cup \dots \cup V_b(\psi_m) \cup V_b(t) = V_b(\varphi) \cup V_b(t)$.

We examine the case where $\boldsymbol{\varphi \in E'_d(n+1,k)}$. This implies $\varphi \in E_d(n+1,k)$, $k \in K(n)$.

There exist f in F, a positive integer m and $\psi_1, .., \psi_m \in E(n,k)$ such that
- $\varphi = (f)(\psi_1, \dots, \psi_m)$
- for each $\sigma \in \Xi(k)$ $A_f(\#(k, \psi_1, \sigma), ..., \#(k, \psi_m, \sigma))$ is true.

Since $k \in K(n)$ we have $K(n; k; x_1, .., x_p; \varphi_1, .., \varphi_p)$. We have

$V_b(\varphi) = V_b(\psi_1) \cup \dots \cup V_b(\psi_m)$,

and since $V_b(t) \cap V_b(\varphi) = \varnothing$ we have
- for each j=1.m $V_b(t) \cap V_b(\psi_j) = \varnothing$.

By the inductive hypothesis for each j=1..m $(\psi_j)_k\{x_i/t\}$ is defined and belongs to $E(k\{x_i/t\})$, so we can define

$\varphi_k\{x_i/t\} = (f)((\psi_1)_k\{x_i/t\}, .., (\psi_m)_k\{x_i/t\})$.

We need to show that $\varphi_k\{x_i/t\} \in E(k\{x_i/t\})$.

We have $k\{x_i/t\} \in K$, $f \in F$, for each j=1..m $(\psi_j)_k\{x_i/t\} \in E(k\{x_i/t\})$.

Let $\rho \in \Xi(k\{x_i/t\})$, we want to show that $A_f(\#(k\{x_i/t\}, (\psi_1)_k\{x_i/t\}, \rho), ..., \#(k\{x_i/t\}, (\psi_m)_k\{x_i/t\}, \rho))$ is true.

We define $\rho_{i-1} = \rho/\mathrm{dom}(k_{i-1})$, and define σ as the soop $(x_1,r_1) \parallel .. \parallel (x_p,r_p)$ where $\forall j \neq i$ $r_j = \rho(x_j)$ and $r_i = \#(k_{i-1},t,\rho_{i-1})$. We have already shown that $\sigma \in K$, moreover, by the inductive hypothesis, for each j=1..m $\#(k, \psi_j, \sigma) = \#(k\{x_i/t\}, (\psi_j)_k\{x_i/t\}, \rho)$.

So $A_f(\#(k, \psi_1, \sigma), ..., \#(k, \psi_m, \sigma))$ is true, and consequently

$A_f(\#(k\{x_i/t\}, (\psi_1)_k\{x_i/t\}, \rho), ..., \#(k\{x_i/t\}, (\psi_m)_k\{x_i/t\}, \rho))$ is true.

So by lemma 3.10 it is proved that $\varphi_k\{x_i/t\} \in E(k\{x_i/t\})$.

Moreover if, as defined above, $\rho \in \Xi(k\{x_i/t\})$, $\rho_{i-1} = \rho/dom(k_{i-1})$, and σ is the soop $(x_1,r_1) \| .. \| (x_p,r_p)$ where $\forall j \neq i \; r_j = \rho(x_j)$ and $r_i = \#(k_{i-1},t,\rho_{i-1})$ we have

$$\#(k, (f)(\psi_1, .. , \psi_m), \sigma) = P_f(\#(k, \psi_1, \sigma), \dots , \#(k, \psi_m, \sigma)) =$$

$$= P_f(\#(k\{x_i/t\}, (\psi_1)_k\{x_i/t\}, \rho), .. , \#(k\{x_i/t\}, (\psi_m)_k\{x_i/t\}, \rho)) =$$

$$= \#(k\{x_i/t\}, (f)((\psi_1)_k\{x_i/t\}, .. , (\psi_m)_k\{x_i/t\}), \rho) = \#(k\{x_i/t\}, \varphi_k\{x_i/t\} , \rho) .$$

Finally

$$V_b(\varphi_k\{x_i/t\}) = V_b((\psi_1)_k\{x_i/t\}) \cup \dots \cup V_b((\psi_m)_k\{x_i/t\}) \subseteq$$

$$\subseteq V_b(\psi_1) \cup V_b(t) \cup \dots \cup V_b(\psi_m) \cup V_b(t) =$$

$$= V_b(\psi_1) \cup \dots \cup V_b(\psi_m) \cup V_b(t) = V_b(\varphi) \cup V_b(t) .$$

Finally let's consider the case where $\boldsymbol{\varphi \in E'_e(n+1,k)}$. This implies $\varphi \in E_e(n+1,k)$, $k \in K(n)$.

There exist a positive integer m, y_1, .., y_m distinct $\in$ V-dom(k), ψ, ψ_1, .. , $\psi_m \in E(n)$ such that $\varphi = \{\}(y_1{:}\psi_1, \dots , y_m{:}\psi_m, \psi)$. Moreover we have

- $\psi_1 \in E(n,k)$, for each $\sigma \in \Xi(k)$ $\#(k,\psi_1,\sigma)$ is a set ;
- if m>1, for each i=j..m-1 if we define $k'_j = k \| (y_1,\psi_1) \| .. \| (y_j,\psi_j)$ it follows
  $k'_j \in K(n) \wedge \psi_{j+1} \in E(n, k'_j) \wedge$ for each $\sigma'_j \in \Xi(k'_j)$ $\#(k'_j, \psi_{j+1}, \sigma'_j)$ is a set ;
- if we define $k'_m = k \| (y_1,\psi_1) \| .. \| (y_m,\psi_m)$ it follows $k'_m \in K(n) \wedge \psi \in E(n,k'_m)$.

If m=1 we define

$$\varphi_k\{x_i/t\} = \{\}(y_1{:}\ (\psi_1)_k\{x_i/t\}, \psi_{k'(1)}\{x_i/t\}) ;$$

if m>1 we define

$$\varphi_k\{x_i/t\} = \{\}(y_1{:}\ (\psi_1)_k\{x_i/t\}, y_2{:}\ (\psi_2)_{k'(1)}\{x_i/t\} , \dots , y_m{:}\ (\psi_m)_{k'(m-1)}\{x_i/t\}, \psi_{k'(m)}\{x_i/t\}) .$$

We need to verify these definitions are correct, in the sense that they rely on well defined concepts.

So we need to verify that $(\psi_1)_k\{x_i/t\}$ is defined, if m>1 then for each j=2..m $(\psi_j)_{k'(j-1)}\{x_i/t\}$ is defined, and finally that $\psi_{k'(m)}\{x_i/t\}$ is defined.

Since $k \in K(n)$ we have $K(n; k; x_1, .. , x_p; \varphi_1, .. , \varphi_p)$.

It results $\psi_1 \in E(n,k)$, and since $V_b(\psi_1) \subseteq V_b(\varphi)$ we have $V_b(\psi_1) \cap V_b(t) = \varnothing$. This ensures $(\psi_1)_k\{x_i/t\}$ is defined, and belongs to $E(k\{x_i/t\})$.

Suppose m>1 and let j=2..m, we want to verify that $(\psi_j)_{k'(j-1)}\{x_i/t\}$ is defined. We have $k'_{j-1} \in K(n)$, and $K(n; k'_{j-1}; x_1, .. , x_p, y_1, .. , y_{j-1}; \varphi_1, .. , \varphi_p, \psi_1, .. , \psi_{j-1})$. For each u=1..j-1 $y_u \in V_b(\varphi)$, so $y_u \notin V_b(t)$. Moreover for each u=1..j-1 $V_b(\psi_u) \subseteq V_b(\varphi)$, so $V_b(t) \cap V_b(\psi_u) = \varnothing$. We have $\psi_j \in E(n,k'_{j-1})$, and also $V_b(\psi_j) \subseteq V_b(\varphi)$, so $V_b(t) \cap V_b(\psi_j) = \varnothing$. Therefore $(\psi_j)_{k'(j-1)}\{x_i/t\}$ is defined, and belongs to $E(k'_{j-1}\{x_i/t\})$.

To verify that $\psi_{k'(m)}\{x_i/t\}$ is defined we consider that $k'_m \in K(n)$ and
$\mathcal{K}(n; k'_m; x_1, .. , x_p, y_1, .. , y_m; \varphi_1, .. , \varphi_p, \psi_1, .. , \psi_m)$. For each u=1..m $y_u \in V_b(\varphi)$, so $y_u \notin V_b(t)$. Moreover for each u=1..m $V_b(\psi_u) \subseteq V_b(\varphi)$, so $V_b(t) \cap V_b(\psi_u) = \varnothing$. We have $\psi \in E(n,k'_m)$, and also $V_b(\psi) \subseteq V_b(\varphi)$, so $V_b(t) \cap V_b(\psi) = \varnothing$. Therefore $(\psi)_{k'(m)}\{x_i/t\}$ is defined, and belongs to $E(k'_m\{x_i/t\})$.

At this point we have verified the definition of $\varphi_k\{x_i/t\}$ is an acceptable definition, but we need to prove $\varphi_k\{x_i/t\} \in E(k\{x_i/t\})$. We try to apply lemma 3.11 to show this.

We see that $k\{x_i/t\} \in K$, for each j=1..m $y_j \in \mathcal{V}$ - dom(k) $\subseteq \mathcal{V}$ - dom($k\{x_i/t\}$).

We also see $(\psi_1)_k\{x_i/t\} \in E(k\{x_i/t\})$. Let $\rho \in \Xi(k\{x_i/t\})$, by the inductive hypothesis we know that if we define $\rho_{i-1} = \rho/\mathrm{dom}(k_{i-1})$, and define σ as the soop $(x_1,r_1) \parallel .. \parallel (x_p,r_p)$ where $\forall j\neq i\ r_j=\rho(x_j)$ and $r_i = \#(k_{i-1},t,\rho_{i-1})$ then $\sigma \in \Xi(k)$ and $\#(k\{x_i/t\}, (\psi_1)_k\{x_i/t\}, \rho) = \#(k, \psi_1, \sigma)$ is a set .

If m>1 we know that for each j=2..m $(\psi_j)_{k'(j-1)}\{x_i/t\} \in E(k'_{j-1}\{x_i/t\})$. Let $\rho'_{j-1} \in \Xi(k'_{j-1}\{x_i/t\})$, by the inductive hypothesis we know that if we define $\rho_{i-1} = \rho'_{j-1}/\mathrm{dom}(k_{i-1})$, and define σ'_{j-1} as the soop
$(x_1,r_1) \parallel .. \parallel (x_p,r_p) \parallel (y_1,q_1) \parallel .. \parallel (y_{j-1},q_{j-1})$ where $\forall u=1..j-1\ q_u=\rho'_{j-1}(y_u)$, $\forall j\neq i\ r_j=\rho'_{j-1}(x_j)$ and $r_i = \#(k_{i-1},t,\rho_{i-1})$ then $\sigma'_{j-1} \in \Xi(k'_{j-1})$ and $\#(k'_{j-1}\{x_i/t\}, (\psi_j)_{k'(j-1)}\{x_i/t\}, \rho'_{j-1}) = \#(k'_{j-1}, \psi_j, \sigma'_{j-1})$ is a set.

We also know that $(\psi)_{k'(m)}\{x_i/t\} \in E(k'_m\{x_i/t\})$.

In order to apply lemma 3.11 we still need to know that for each j = 1..m
$k'_j\{x_i/t\} = k\{x_i/t\} \parallel (y_1, (\psi_1)_{k'(0)}\{x_i/t\}) \parallel \ldots \parallel (y_j, (\psi_j)_{k'(j-1)}\{x_i/t\})$

(where we adopt by convention that $k'_0 = k$)

In fact $k'_1\{x_i/t\} = k\{x_i/t\} \parallel (y_1, (\psi_1)_k\{x_i/t\})$.

If m>1, for each j=1..m-1 if we assume
$k'_j\{x_i/t\} = k\{x_i/t\} \parallel (y_1, (\psi_1)_{k'(0)}\{x_i/t\}) \parallel \ldots \parallel (y_j, (\psi_j)_{k'(j-1)}\{x_i/t\})$ then

$k'_{j+1}\{x_i/t\} = k'_j\{x_i/t\} \parallel (y_{j+1}, (\psi_{j+1})_{k'(j)}\{x_i/t\}) =$
$= k\{x_i/t\} \parallel (y_1, (\psi_1)_{k'(0)}\{x_i/t\}) \parallel \ldots \parallel (y_{j+1}, (\psi_{j+1})_{k'(j)}\{x_i/t\})$.

By lemma 3.11 we conclude that $\varphi_k\{x_i/t\} \in E(k\{x_i/t\})$.

Another point we have to verify is the following. Let $\rho \in \Xi(k\{x_i/t\})$, define $\rho_{i-1} = \rho/\mathrm{dom}(k_{i-1})$, and define σ as the soop $(x_1,r_1) \parallel .. \parallel (x_p,r_p)$ where $\forall j\neq i\ r_j=\rho(x_j)$ and $r_i = \#(k_{i-1},t,\rho_{i-1})$. It has been shown that $\sigma \in \Xi(k)$, and we need to prove $\#(k, \varphi, \sigma) = \#(k\{x_i/t\}, \varphi_k\{x_i/t\}, \rho)$.

Of course we have

$\#(k, \varphi, \sigma) = \{\}\ (\sigma'_m \in \Xi(k'_m): \sigma \sqsubseteq \sigma'_m\ , \#(k'_m,\psi,\sigma'_m)\ )$, and, since

$\varphi_k\{x_i/t\} = \{\}(y_1: (\psi_1)_k\{x_i/t\}, y_2: (\psi_2)_{k'(1)}\{x_i/t\}\ , \ldots , y_m: (\psi_m)_{k'(m-1)}\{x_i/t\}, \psi_{k'(m)}\{x_i/t\})$, by lemma 3.11

$\#(k\{x_i/t\}, \varphi_k\{x_i/t\}, \rho) =$
$= \{\}\ (\rho'_m \in \Xi(k'_m\{x_i/t\}): \rho \sqsubseteq \rho'_m\ , \#(k'_m\{x_i/t\},\psi_{k'(m)}\{x_i/t\},\rho'_m)\ )$.

So we need to show

$\{\}\ (\sigma'_m \in \Xi(k'_m): \sigma \sqsubseteq \sigma'_m\ , \#(k'_m,\psi,\sigma'_m)\ ) =$
$= \{\}\ (\rho'_m \in \Xi(k'_m\{x_i/t\}): \rho \sqsubseteq \rho'_m\ , \#(k'_m\{x_i/t\},\psi_{k'(m)}\{x_i/t\},\rho'_m)\ )$.

Suppose u is in $\{\}\ (\sigma'_m \in \Xi(k'_m): \sigma \sqsubseteq \sigma'_m\ , \#(k'_m,\psi,\sigma'_m)\ )$, there exist

$\sigma'_m \in \Xi(k'_m)$ such that $\sigma \sqsubseteq \sigma'_m$ and $u = \#(k'_m,\psi,\sigma'_m)$.

We define ρ'_m as the soop $\rho \parallel (y_1, \sigma'_m(y_1)) \parallel .. \parallel (y_m, \sigma'_m(y_m))$ and we'll show that $\rho'_m \in \Xi(k'_m\{x_i/t\})$,

$\#(k'_m\{x_i/t\},\psi_{k'(m)}\{x_i/t\},\rho'_m) = \#(k'_m,\psi,\sigma'_m)$.

To show that $\rho'_m \in \Xi(k'_m\{x_i/t\})$ we need to define for each j=1..m-1

$\sigma'_j = \sigma \parallel (y_1, \sigma'_m(y_1)) \parallel .. \parallel (y_j, \sigma'_m(y_j))$ and

$\rho'_j = \rho \parallel (y_1, \sigma'_m(y_1)) \parallel .. \parallel (y_j, \sigma'_m(y_j))$,

and then we need to show by induction on j that for each j=1..m $\rho'_j \in \Xi(k'_j\{x_i/t\})$.

Since $\sigma'_m \in \Xi(k'_m)$ we have $\sigma'_1 \in \Xi(k'_1)$. Given that k, $k'_1 \in K(n)$, $k'_1 = k \parallel (y_1,\psi_1)$, $\sigma \in \Xi(k)$, $\sigma'_1 \in \Xi(k'_1)$ we can apply consequence 2.1.10 to obtain that $\sigma'_m(y_1) \in \#(k,\psi_1,\sigma)$.

We have applied the inductive hypothesis to show $(\psi_1)_k\{x_i/t\}$ is defined, of course we have also

$\#(k\{x_i/t\}, (\psi_1)_k\{x_i/t\}, \rho) = \#(k, \psi_1, \sigma)$, so $\sigma'_m(y_1) \in \#(k\{x_i/t\}, (\psi_1)_k\{x_i/t\}, \rho)$.

Given that $k\{x_i/t\} \in K$, $k'_1\{x_i/t\} \in K$, $k'_1\{x_i/t\} = k\{x_i/t\} \parallel (y_1, (\psi_1)_k\{x_i/t\})$, $\rho \in \Xi(k\{x_i/t\})$, $\rho'_1 = \rho \parallel (y_1, \sigma'_m(y_1))$, $\sigma'_m(y_1) \in \#(k\{x_i/t\}, (\psi_1)_k\{x_i/t\}, \rho)$, we can apply consequence 2.1.11 to obtain that $\rho'_1 \in \Xi(k'_1\{x_i/t\})$.

If m>1 we need an inductive step in which j=1..m-1, we assume $\rho'_j \in \Xi(k'_j\{x_i/t\})$ and show $\rho'_{j+1} \in \Xi(k'_{j+1}\{x_i/t\})$.

We have $\sigma'_{j+1} \in \Xi(k'_{j+1})$. Given that k'_j, $k'_{j+1} \in K(n)$, $k'_{j+1} = k'_j \parallel (y_{j+1},\psi_{j+1})$, $\sigma'_j \in \Xi(k'_j)$ $\sigma'_{j+1} \in \Xi(k'_{j+1})$, $\sigma'_{j+1} = \sigma'_j \parallel (y_{j+1}, \sigma'_m(y_{j+1}))$ we apply consequence 2.1.10 to obtain that $\sigma'_m(y_{j+1}) \in \#(k'_j,\psi_{j+1},\sigma'_j)$.

We have applied the inductive hypothesis to show that $(\psi_{j+1})_{k'(j)}\{x_i/t\}$ is defined, of course we have also

$\#(k'_j\{x_i/t\}, (\psi_{j+1})_{k'(j)}\{x_i/t\}, \rho'_j) = \#(k'_j, \psi_{j+1}, \sigma'_j)$, so $\sigma'_m(y_{j+1}) \in \#(k'_j\{x_i/t\}, (\psi_{j+1})_{k'(j)}\{x_i/t\}, \rho'_j)$.

Given that $k'_j\{x_i/t\} \in K$, $k'_{j+1}\{x_i/t\} \in K$, $k'_{j+1}\{x_i/t\} = k'_j\{x_i/t\} \parallel (y_{j+1}, (\psi_{j+1})_{k'(j)}\{x_i/t\})$, $\rho'_j \in \Xi(k'_j\{x_i/t\})$, $\rho'_{j+1} = \rho'_j \parallel (y_{j+1}, \sigma'_m(y_{j+1}))$, $\sigma'_m(y_{j+1}) \in \#(k'_j\{x_i/t\}, (\psi_{j+1})_{k'(j)}\{x_i/t\}, \rho'_j)$, we can apply consequence 2.1.11 to obtain that $\rho'_{j+1} \in \Xi(k'_{j+1}\{x_i/t\})$.

So it is proved that $\rho'_m \in \Xi(k'_m\{x_i/t\})$.

We have applied the inductive hypothesis to show that $\psi_{k'(m)}\{x_i/t\}$ is defined, of course we have also

$\#(k'_m\{x_i/t\},\psi_{k'(m)}\{x_i/t\},\rho'_m) = \#(k'_m,\psi,\sigma'_m)$.

So $u = \#(k'_m\{x_i/t\},\psi_{k'(m)}\{x_i/t\},\rho'_m)$, and therefore

u belongs to $\{\}\ (\rho'_m \in \Xi(k'_m\{x_i/t\}): \rho \sqsubseteq \rho'_m\ , \#(k'_m\{x_i/t\},\psi_{k'(m)}\{x_i/t\},\rho'_m)\ )$.

For the converse implication suppose
u is in $\{\}\ (\rho'_m \in \Xi(k'_m\{x_i/t\}): \rho \sqsubseteq \rho'_m\ , \#(k'_m\{x_i/t\},\psi_{k'(m)}\{x_i/t\},\rho'_m)\ )$.

There exist $\rho'_m \in \Xi(k'_m\{x_i/t\})$ such that $\rho \sqsubseteq \rho'_m$ and $u = \#(k'_m\{x_i/t\},\psi_{k'(m)}\{x_i/t\},\rho'_m)\ )$.

We define σ'_m as the soop $\sigma \parallel (y_1, \rho'_m(y_1)) \parallel .. \parallel (y_m, \rho'_m(y_m))$ and we'll show that

$\sigma'_m \in \Xi(k'_m)$ and $\#(k'_m,\psi,\sigma'_m) = \#(k'_m\{x_i/t\},\psi_{k'(m)}\{x_i/t\},\rho'_m)$.

We've already seen that our assumptions ensure that $(\psi)_{k'(m)}\{x_i/t\}$ is defined, and belongs to $E(k'_m\{x_i/t\})$. The same assumptions, together with the definitions of ρ'_m and σ'_m, ensure that

$\sigma'_m \in \Xi(k'_m)$ and $\#(k'_m,\psi,\sigma'_m) = \#(k'_m\{x_i/t\},\psi_{k'(m)}\{x_i/t\},\rho'_m)$.

Therefore $u = \#(k'_m,\psi,\sigma'_m)$, and

u belongs to $\{\}\ (\sigma'_m \in \Xi(k'_m): \sigma \sqsubseteq \sigma'_m\ , \#(k'_m,\psi,\sigma'_m)\ )$.

This ends the proof that $\#(k, \varphi, \sigma) = \#(k\{x_i/t\}, \varphi_k\{x_i/t\}, \rho)$.

To finish with the case where $\varphi \in E'_e(n+1,k)$ we just need to show that

$V_b(\varphi_k\{x_i/t\}) \subseteq V_b(\varphi) \cup V_b(t)$.

By lemma 3.11 and the inductive hypotesis

$V_b(\varphi_k\{x_i/t\}) = \{y_1, \dots, y_m\} \cup V_b((\psi_1)_k\{x_i/t\}) \cup .. \cup V_b((\psi_m)_{k'(m-1)}\{x_i/t\}) \cup V_b(\psi_{k'(m)}\{x_i/t\}) \subseteq$

$\subseteq \{y_1, \dots, y_m\} \cup (\ V_b(\psi_1) \cup V_b(t)\ ) \cup \dots \cup (\ V_b(\psi_m) \cup V_b(t)\ ) \cup (\ V_b(\psi) \cup V_b(t)\ ) =$

$= \{y_1, \dots, y_m\} \cup V_b(\psi_1) \cup \dots \cup V_b(\psi_m) \cup V_b(\psi) \cup V_b(t) = V_b(\varphi) \cup V_b(t)$.

We have defined $\varphi_k\{x_i/t\}$ for each $\varphi\in E(n+1,k)$ such that $V_b(t) \cap V_b(\varphi) = \emptyset$. Recall that

$E(n+1,k) = E'(n,k) \cup E'_a(n+1,k) \cup E'_b(n+1,k) \cup E'_c(n+1,k) \cup E'_d(n+1,k) \cup E'_e(n+1,k)$,

and recall the definition of $\varphi_k\{x_i/t\}$ depends on the set to which φ belongs to. Actually φ may belong to more than one of these sets. We need to check that, in every case in which φ belongs to two of the six sets, the two definitions of $\varphi_k\{x_i/t\}$ match each other.

We split the task in two steps. The first step requires to verify that

- for each $w\in\{a,b,c,d,e\}$, φ in $E'(n,k) \cap E'_w(n+1,k)$ $(\varphi_k\{x_i/t\})_w = \varphi_k\{x_i/t\}$.

The second step requires to verify that

- for each w_1, $w_2 \in \{a,b,c,d,e\}$: $w_1\neq w_2$, φ in $E'_{w1}(n+1,k) \cap E'_{w2}(n+1,k)$ $(\varphi_k\{x_i/t\})_{w1} = (\varphi_k\{x_i/t\})_{w2}$.

We begin with the first step and examine the case where φ is in **E'(n,k) ∩ E'$_a$(n+1,k)**.

Of course $\varphi \in E(n,k) \cap E_a(n+1,k)$.

We have $k = k_{p-1} \parallel (x_p, \varphi_p)$, and there exist $h\in K(n)$, $y\in V\text{-dom}(h)$, ψ such that $k = h \parallel (y,\psi)$, $\varphi\in E(n,h)$, $y\notin V_b(\varphi)$. Of course $h = k_{p-1}$, $y = x_p$, $\psi = \varphi_p$ so $\varphi\in E(n,k_{p-1})$, $x_p\notin V_b(\varphi)$.

Consider the case where i=p. Here we have $(\varphi_k\{x_i/t\})_a = \varphi$.

We also see that $k_{p-1} \in K(n)$, $\varphi\in E(n,k_{p-1})$, $x_i \notin \text{dom}(k_{p-1})$. Therefore $\varphi_k\{x_i/t\} = \varphi = (\varphi_k\{x_i/t\})_a$.

We now examine the case where i<p. Here we defined $(\varphi_k\{x_i/t\})_a = \varphi_{k(p-1)}\{x_i/t\}$.

It also holds true that $k_{p-1} \in K(n)$, $k_i \sqsubseteq k_{p-1}$, $\varphi\in E(n,k_{p-1})$, $K(n;\ k_{p-1};\ x_1, .. , x_{p-1};\ \varphi_1, .. , \varphi_{p-1})$. Therefore $\varphi_k\{x_i/t\} = \varphi_{k(p-1)}\{x_i/t\} = (\varphi_k\{x_i/t\})_a$.

Let's turn to examine the case where φ is in **E'(n,k) ∩ E'$_b$(n+1,k)**.

Of course $\varphi \in E(n,k) \cap E_b(n+1,k)$.

We have $k = k_{p-1} \parallel (x_p, \varphi_p)$, and there exist $h\in K(n)$, $y\in V\text{-dom}(h)$, ψ such that $k = h \parallel (y,\psi)$, $\varphi = y$, for each $\sigma\in\Xi(k)$ $\#(k,\varphi,\sigma) = \sigma(y)$.
Of course $h = k_{p-1}$, $y = x_p$, $\psi = \varphi_p$ so $\varphi = x_p$, $\#(k,\varphi,\sigma) = \sigma(x_p)$.

Since $\varphi\in E(n,k)$ the following condition holds:

$\varphi\in \text{dom}(k) \wedge (\ \varphi=x_i \rightarrow \varphi_k\{x_i/t\} = t\ ) \wedge (\ \varphi\neq x_i \rightarrow \varphi_k\{x_i/t\} = \varphi\ )$.

Consider the case where i=p. Here we defined $(\varphi_k\{x_i/t\})_b = t$, since $\varphi = x_p = x_i$ we have $\varphi_k\{x_i/t\} = t = (\varphi_k\{x_i/t\})_b$.

Turn to the case where i<p. Here we defined $(\varphi_k\{x_i/t\})_b = \varphi$, and since $\varphi = x_p \neq x_i$ we have $\varphi_k\{x_i/t\} = \varphi = (\varphi_k\{x_i/t\})_b$.

Let's examine the case where φ is in **E'(n,k) ∩ E'$_c$(n+1,k)**.

Of course $\varphi \in E(n,k) \cap E_c(n+1,k)$.

Since $\varphi \in E(n,k)$ the following condition holds:

n>1, there exist $\kappa \in K(n-1)$ such that $\kappa \sqsubseteq k$, and $\psi, \psi_1, \dots, \psi_m \in E(n-1,\kappa)$ such that
$\varphi = (\psi)(\psi_1, \dots, \psi_m)$, $\varphi \in E(n,\kappa)$,
for each $\rho \in \Xi(\kappa)$ $\#(\kappa,\psi,\rho)$ is a function with m arguments and $(\#(\kappa, \psi_1, \rho), \dots, \#(\kappa, \psi_m, \rho))$ is a member of its domain,
if $x_i \in dom(\kappa)$ then $\varphi_k\{x_i/t\} = (\psi_\kappa\{x_i/t\}) ((\psi_1)_\kappa\{x_i/t\}, \dots, (\psi_m)_\kappa\{x_i/t\})$, else $\varphi_k\{x_i/t\} = \varphi$.

Since $\varphi \in E_c(n+1,k)$ the following condition holds:

there exist a positive integer q and $\chi, \chi_1, .., \chi_q \in E(n,k)$ such that

- $\varphi = (\chi) (\chi_1, .., \chi_q)$,
- $(\varphi_k\{x_i/t\})_c = (\chi_k\{x_i/t\}) ((\chi_1)_k\{x_i/t\}, .., (\chi_q)_k\{x_i/t\})$.

Clearly $\chi = \psi$, q=m, $\chi_1 = \psi_1, \dots, \chi_m = \psi_m$ (this has been shown within definition 2.1), therefore

$(\varphi_k\{x_i/t\})_c = (\psi_k\{x_i/t\}) ((\psi_1)_k\{x_i/t\}, .., (\psi_m)_k\{x_i/t\})$.

Suppose $x_i \in dom(\kappa)$, in this case

$\varphi_k\{x_i/t\} = (\psi_\kappa\{x_i/t\}) ((\psi_1)_\kappa\{x_i/t\}, \dots, (\psi_m)_\kappa\{x_i/t\})$.

By one of our assumptions, since $k_i \sqsubseteq \kappa$, it follows

$\psi_\kappa\{x_i/t\} = \psi_k\{x_i/t\}$, $(\psi_1)_\kappa\{x_i/t\} = (\psi_1)_k\{x_i/t\}, \dots, (\psi_m)_\kappa\{x_i/t\} = (\psi_m)_k\{x_i/t\}$. Therefore

$\varphi_k\{x_i/t\} = (\psi_\kappa\{x_i/t\}) ((\psi_1)_\kappa\{x_i/t\}, \dots, (\psi_m)_\kappa\{x_i/t\}) = (\psi_k\{x_i/t\}) ((\psi_1)_k\{x_i/t\}, .., (\psi_m)_k\{x_i/t\}) =$

$= (\varphi_k\{x_i/t\})_c$.

Now suppose $x_i \notin dom(\kappa)$. In this case $\varphi_k\{x_i/t\} = \varphi$.

Since $\kappa \in K(n)$, $\psi \in E(n,\kappa)$, $\psi_j \in E(n,\kappa)$, $x_i \notin dom(\kappa)$ we can apply one of our assumptions and get that

$\psi_k\{x_i/t\} = \psi$, $(\psi_j)_k\{x_i/t\} = \psi_j$, so

$(\varphi_k\{x_i/t\})_c = (\psi_k\{x_i/t\}) ((\psi_1)_k\{x_i/t\}, .., (\psi_m)_k\{x_i/t\}) = (\psi)(\psi_1, \dots, \psi_m) = \varphi = \varphi_k\{x_i/t\}$.

Let's examine the case where φ is in **E'(n,k) ∩ E'$_d$(n+1,k)**.

Of course $\varphi \in E(n,k) \cap E_d(n+1,k)$.

Since $\varphi \in E(n,k)$ the following condition holds:

n>1, there exist $\kappa \in K(n-1)$ such that $\kappa \sqsubseteq k$, and $f \in F$, $\psi_1, \dots, \psi_m \in E(n-1,\kappa)$ such that
$\varphi = (f)(\psi_1, \dots, \psi_m)$, $\varphi \in E(n,\kappa)$,
for each $\rho \in \Xi(\kappa)$ $A_f(\#(\kappa, \psi_1, \rho), \dots, \#(\kappa, \psi_m, \rho))$ is true,
if $x_i \in dom(\kappa)$ then $\varphi_k\{x_i/t\} = (f)((\psi_1)_\kappa\{x_i/t\}, \dots, (\psi_m)_\kappa\{x_i/t\})$, else $\varphi_k\{x_i/t\} = \varphi$.

Since $\varphi \in E_d(n+1,k)$ the following condition holds:

there exist g in F, a positive integer q and $\chi_1, .., \chi_q \in E(n,k)$ such that

- $\varphi = (g)(\chi_1, \dots, \chi_q)$,
- $(\varphi_k\{x_i/t\})_d = (g)((\chi_1)_k\{x_i/t\}, .., (\chi_q)_k\{x_i/t\})$.

Clearly g=f, q=m, $\chi_1 = \psi_1, \dots, \chi_m = \psi_m$ (this has been shown within definition 2.1), therefore

$(\varphi_k\{x_i/t\})_d = (f)((\psi_1)_k\{x_i/t\}, .., (\psi_m)_k\{x_i/t\})$.

Suppose $x_i \in dom(\kappa)$, in this case

$\varphi_k\{x_i/t\} = (f)((\psi_1)_\kappa\{x_i/t\}, \dots, (\psi_m)_\kappa\{x_i/t\})$.

By one of our assumptions, since $k_i \sqsubseteq \kappa$, it follows

$(\psi_1)_\kappa\{x_i/t\} = (\psi_1)_k\{x_i/t\}, \dots, (\psi_m)_\kappa\{x_i/t\} = (\psi_m)_k\{x_i/t\}$. Therefore

$\varphi_k\{x_i/t\} = (f)((\psi_1)_\kappa\{x_i/t\}, \dots, (\psi_m)_\kappa\{x_i/t\}) = (f)((\psi_1)_k\{x_i/t\}, \dots, (\psi_m)_k\{x_i/t\}) = (\varphi_k\{x_i/t\})_d$.

Now suppose $x_i \notin dom(\kappa)$. In this case $\varphi_k\{x_i/t\} = \varphi$.

Since $\kappa \in K(n)$, $\psi_j \in E(n,\kappa)$, $x_i \notin dom(\kappa)$ we can apply one of our assumptions and get that

$(\psi_j)_k\{x_i/t\} = \psi_j$, so

$(\varphi_k\{x_i/t\})_d = (f)((\psi_1)_k\{x_i/t\}, .., (\psi_m)_k\{x_i/t\}) = (f)(\psi_1, \dots, \psi_m) = \varphi = \varphi_k\{x_i/t\}$.

We turn to the case where φ is in **E'(n,k) $\cap$ E'$_e$(n+1,k)**.

Of course $\varphi \in E(n,k) \cap E_e(n+1,k)$.

Since $\varphi \in E(n,k)$ the following condition holds:

n>1, there exist $\kappa \in K(n-1)$ such that $\kappa \sqsubseteq k$ and $\psi, \psi_1, .., \psi_m \in E(n-1)$, $y_1, \dots, y_m$ distinct $\in V$-$dom(\kappa)$ such that $\varphi = \{\}(y_1:\psi_1, \dots, y_m:\psi_m, \psi)$, $\varphi \in E(n,\kappa)$,

$\psi_1 \in E(n-1,\kappa)$, for each $\sigma \in \Xi(\kappa)$ $\#(\kappa,\psi_1,\sigma)$ is a set ;
if m>1 then for each j=1..m-1 we define $\kappa'_j = \kappa \| (y_1,\psi_1) \| \, .. \, \| (y_j,\psi_j)$ and we have $\kappa'_j \in K(n-1)$, $\psi_{j+1} \in E(n-1, \kappa'_j)$, for each $\sigma'_j \in \Xi(\kappa'_j)$ $\#(\kappa'_j, \psi_{j+1}, \sigma'_j)$ is a set;
if we define $\kappa'_m = \kappa \| (x_1,\psi_1) \| \, .. \, \| (x_m,\psi_m)$ then $\kappa'_m \in K(n-1) \wedge \psi \in E(n-1,\kappa'_m)$;

if $x_i \in dom(\kappa)$ then

$\varphi_k\{x_i/t\} = \{\}(y_1{:}\ (\psi_1)_\kappa\{x_i/t\},\ y_2{:}\ (\psi_2)_{\kappa'(1)}\{x_i/t\}\ ,\ \ldots\ ,\ y_m{:}\ (\psi_m)_{\kappa'(m-1)}\{x_i/t\},\ \psi_{\kappa'(m)}\{x_i/t\});$

if $x_i \notin dom(\kappa)$ then $\varphi_k\{x_i/t\} = \varphi$.

Since $\varphi \in E_e(n+1,k)$ the following condition holds:

there exist a positive integer q, $z_1, .., z_q$ distinct $\in$ V-dom(k), $\chi, \chi_1, .. , \chi_q \in E(n)$ such that $\varphi = \{\}(z_1{:}\chi_1, \ldots , z_q{:}\chi_q, \chi)$. Moreover we have

- $\chi_1 \in E(n,k)$, for each $\sigma \in \Xi(k)$ $\#(k,\chi_1,\sigma)$ is a set ;
- if q>1, for each i=j..q-1 if we define $k'_j = k||(z_1,\chi_1)|| .. ||(z_j,\chi_j)$ it follows
  $k'_j \in K(n) \wedge \chi_{j+1} \in E(n, k'_j) \wedge$ for each $\sigma'_j \in \Xi(k'_j)$ $\#(k'_j, \chi_{j+1}, \sigma'_j)$ is a set ;
- if we define $k'_q = k||(z_1,\chi_1)|| .. ||(z_q,\chi_q)$ it follows $k'_q \in K(n) \wedge \psi \in E(n,k'_q)$.

and we define

$(\varphi_k\{x_i/t\})_e = \{\}(z_1{:}\ (\chi_1)_k\{x_i/t\},\ z_2{:}\ (\chi_2)_{k'(1)}\{x_i/t\}\ ,\ \ldots\ ,\ z_m{:}\ (\chi_q)_{k'(q-1)}\{x_i/t\},\ \chi_{k'(q)}\{x_i/t\})$.

Clearly q=m, $z_j = y_j$, $\chi_j = \psi_j$, $\chi = \psi$ (this has been shown within definition 2.1), therefore we have

$(\varphi_k\{x_i/t\})_e = \{\}(y_1{:}\ (\psi_1)_k\{x_i/t\},\ y_2{:}\ (\psi_2)_{k'(1)}\{x_i/t\}\ ,\ \ldots\ ,\ y_m{:}\ (\psi_m)_{k'(m-1)}\{x_i/t\},\ \psi_{k'(m)}\{x_i/t\})$.

Suppose $x_i \in dom(\kappa)$, then

$\varphi_k\{x_i/t\} = \{\}(y_1{:}\ (\psi_1)_\kappa\{x_i/t\},\ y_2{:}\ (\psi_2)_{\kappa'(1)}\{x_i/t\}\ ,\ \ldots\ ,\ y_m{:}\ (\psi_m)_{\kappa'(m-1)}\{x_i/t\},\ \psi_{\kappa'(m)}\{x_i/t\})$.

Since $\kappa \in K(n)$, $k_i \sqsubseteq \kappa$, $\psi_1 \in E(n,\kappa)$, we can apply one of our assumptions and obtain that $(\psi_1)_k\{x_i/t\} = (\psi_1)_\kappa\{x_i/t\}$.

If m>1 suppose j=1..m-1, we want to show that $(\psi_{j+1})_{k'(j)}\{x_i/t\} = (\psi_{j+1})_{\kappa'(j)}\{x_i/t\}$.

We have $\psi_{j+1} \in E(n,k'_j)$, $\kappa'_j \in K(n)$, $k_i \sqsubseteq \kappa \sqsubseteq \kappa'_j$, $\psi_{j+1} \in E(n, \kappa'_j)$, for each α=1..j $y_\alpha \in V_b(\varphi)$ so $y_\alpha \notin V_b(t)$, $V_b(t) \cap V_b(\psi_\alpha) \subseteq V_b(t) \cap V_b(\varphi) = \varnothing$. Therefore we can apply one of our assumptions (the same we used for ψ_1) to obtain that

$(\psi_{j+1})_{k'(j)}\{x_i/t\} = (\psi_{j+1})_{\kappa'(j)}\{x_i/t\}$.

We have also $\psi \in E(n,k'_m)$, $\kappa'_m \in K(n)$, $k_i \sqsubseteq \kappa \sqsubseteq \kappa'_m$, $\psi \in E(n, \kappa'_m)$, for each α=1..m $y_\alpha \in V_b(\varphi)$ so $y_\alpha \notin V_b(t)$, $V_b(t) \cap V_b(\psi_\alpha) \subseteq V_b(t) \cap V_b(\varphi) = \varnothing$. Therefore we can apply one of our assumptions (the same we used for ψ_1) to obtain that

$(\psi)_{k'(m)}\{x_i/t\} = (\psi)_{\kappa'(m)}\{x_i/t\}$.

Hence

$\varphi_k\{x_i/t\} = \{\}(y_1{:}\ (\psi_1)_\kappa\{x_i/t\},\ y_2{:}\ (\psi_2)_{\kappa'(1)}\{x_i/t\}\ ,\ \ldots\ ,\ y_m{:}\ (\psi_m)_{\kappa'(m-1)}\{x_i/t\},\ \psi_{\kappa'(m)}\{x_i/t\}) =$

$= \{\}(y_1{:}\ (\psi_1)_k\{x_i/t\},\ y_2{:}\ (\psi_2)_{k'(1)}\{x_i/t\}\ ,\ \ldots\ ,\ y_m{:}\ (\psi_m)_{k'(m-1)}\{x_i/t\},\ \psi_{k'(m)}\{x_i/t\}) = (\varphi_k\{x_i/t\})_e$.

Now suppose $x_i \notin dom(\kappa)$. In this case $\varphi_k\{x_i/t\} = \varphi = \{\}(y_1{:}\psi_1, \dots, y_m{:}\psi_m, \psi)$.

Since $\kappa \in K(n)$, $x_i \notin dom(\kappa)$, $\psi_1 \in E(n,\kappa)$ we can apply one of our assumptions and conclude that $(\psi_1)_k\{x_i/t\} = \psi_1$.

If m>1 suppose j=1..m-1, we want to show that $(\psi_{j+1})_{k'(j)}\{x_i/t\} = \psi_{j+1}$. This holds because the following conditions hold: $\psi_{j+1} \in E(n,k'_j)$, $\kappa'_j \in K(n)$, $x_i \notin dom(\kappa'_j)$, $\psi_{j+1} \in E(n, \kappa'_j)$, and we can apply the same assumption we used for ψ_1.

Moreover we need to show $\psi_{k'(m)}\{x_i/t\} = \psi$. This holds because the following conditions hold: $\psi \in E(n,k'_m)$, $\kappa'_m \in K(n)$, $x_i \notin dom(\kappa'_m)$, $\psi \in E(n, \kappa'_m)$, and we can apply the same assumption we used for ψ_1.

Therefore

$\varphi_k\{x_i/t\} = \varphi = \{\}(y_1{:}\psi_1, \dots, y_m{:}\psi_m, \psi) =$

$= \{\}(y_1{:}\ (\psi_1)_k\{x_i/t\}, y_2{:}\ (\psi_2)_{k'(1)}\{x_i/t\}, \dots, y_m{:}\ (\psi_m)_{k'(m-1)}\{x_i/t\}, \psi_{k'(m)}\{x_i/t\}) = (\varphi_k\{x_i/t\})_e$.

We now turn to the second step of our task. This requires to verify that

- for each $w_1, w_2 \in \{a,b,c,d,e\}$: $w_1 \neq w_2$, φ in $E'_{w1}(n+1,k) \cap E'_{w2}(n+1,k)$
  $(\varphi_k\{x_i/t\})_{w1} = (\varphi_k\{x_i/t\})_{w2}$.

In section two we have seen that for many values of w_1, w_2 it results $E'_{w1}(n+1,k) \cap E'_{w2}(n+1,k) = \varnothing$. We used a table to list all cases where this happens (in the table we have barred the cells which are duplicates or not of interest). Below we reproduce the table.

	$E'_a(n+1,k)$	$E'_b(n+1,k)$	$E'_c(n+1,k)$	$E'_d(n+1,k)$	$E'_e(n+1,k)$
$E'_a(n+1,k)$		$\varnothing$			
$E'_b(n+1,k)$			$\varnothing$	$\varnothing$	$\varnothing$
$E'_c(n+1,k)$				$\varnothing$	$\varnothing$
$E'_d(n+1,k)$					$\varnothing$
$E'_e(n+1,k)$					

The results $E'_{w1}(n+1,k) \cap E'_{w2}(n+1,k) = \varnothing$ were discussed and proved in section 2. Therefore we just need to examine three cases: φ in $E'_a(n+1,k) \cap E'_c(n+1,k)$, φ in $E'_a(n+1,k) \cap E'_d(n+1,k)$, φ in $E'_a(n+1,k) \cap E'_e(n+1,k)$.

We start with the case where φ belongs to $\mathbf{E'_a(n+1,k) \cap E'_c(n+1,k)}$. Clearly φ belongs to $E_a(n+1,k) \cap E_c(n+1,k)$.

Since $\varphi \in E_a(n+1,k)$ we have $\varphi \in E(n,k_{p-1})$, $x_p \notin V_b(\varphi)$. We have to distinguish the case where i=p and $(\varphi_k\{x_i/t\})_a = \varphi$ from the case where i<p and $(\varphi_k\{x_i/t\})_a = \varphi_{k(p-1)}\{x_i/t\}$.

Since $\varphi \in E_c(n+1,k)$ the following condition holds:

there exist a positive integer m and $\psi, \psi_1, .., \psi_m \in E(n,k)$ such that

- $\varphi = (\psi)(\psi_1, .., \psi_m)$,
- $(\varphi_k\{x_i/t\})_c = (\psi_k\{x_i/t\})((\psi_1)_k\{x_i/t\}, .., (\psi_m)_k\{x_i/t\})$.

Suppose i=p. In this case, given $\varphi \in E(n,k_{p-1})$, we can apply assumption 2.1.7 to obtain that

n>1, there exist $h \in K(n-1)$ such that $h \sqsubseteq k_{p-1}$, $\chi, \chi_1, .., \chi_q \in E(n-1,h)$ such that $\varphi = (\chi)(\chi_1, .., \chi_q)$.

Clearly q=m, $\chi=\psi$, $\chi_j = \psi_j$ therefore $\psi, \psi_1, .., \psi_m \in E(n-1,h)$.

Since $x_i \notin dom(h)$ we can apply one of our assumptions and conclude that

$\psi_k\{x_i/t\} = \psi$, $(\psi_j)_k\{x_i/t\} = \psi_j$. Hence

$(\varphi_k\{x_i/t\})_c = (\psi_k\{x_i/t\})((\psi_1)_k\{x_i/t\}, .., (\psi_m)_k\{x_i/t\}) = (\psi)(\psi_1, .., \psi_m) = \varphi = (\varphi_k\{x_i/t\})_a$.

Now suppose i<p. Since $\varphi \in E(n,k_{p-1})$ we can apply one of our inductive assumptions and obtain the following:

n>1, there exist $\kappa \in K(n-1)$ such that $\kappa \sqsubseteq k_{p-1}$ and $\chi, \chi_1, \dots, \chi_q \in E(n-1,\kappa)$ such that $\varphi = (\chi)(\chi_1, \dots, \chi_q)$, $\varphi \in E(n,\kappa)$, if $x_i \in dom(\kappa)$ then $\varphi_{k(p-1)}\{x_i/t\} = (\chi_\kappa\{x_i/t\})((\chi_1)_\kappa\{x_i/t\}, \dots, (\chi_m)_\kappa\{x_i/t\})$, else $\varphi_{k(p-1)}\{x_i/t\} = \varphi$.

Clearly q=m, $\chi=\psi$, $\chi_j = \psi_j$, so
if $x_i \in dom(\kappa)$ then $\varphi_{k(p-1)}\{x_i/t\} = (\psi_\kappa\{x_i/t\})((\psi_1)_\kappa\{x_i/t\}, \dots, (\psi_m)_\kappa\{x_i/t\})$, else $\varphi_{k(p-1)}\{x_i/t\} = \varphi$.

Therefore
if $x_i \in dom(\kappa)$ then $(\varphi_k\{x_i/t\})_a = (\psi_\kappa\{x_i/t\})((\psi_1)_\kappa\{x_i/t\}, \dots, (\psi_m)_\kappa\{x_i/t\})$, else $(\varphi_k\{x_i/t\})_a = \varphi$.

Suppose $x_i \in dom(\kappa)$. By one of our inductive assumptions $\psi_k\{x_i/t\} = \psi_\kappa\{x_i/t\}$, $(\psi_j)_k\{x_i/t\} = (\psi_j)_\kappa\{x_i/t\}$, therefore

$(\varphi_k\{x_i/t\})_a = (\psi_\kappa\{x_i/t\})((\psi_1)_\kappa\{x_i/t\}, \dots, (\psi_m)_\kappa\{x_i/t\}) = (\psi_k\{x_i/t\})((\psi_1)_k\{x_i/t\}, \dots, (\psi_m)_k\{x_i/t\}) =$
$= (\varphi_k\{x_i/t\})_c$.

Suppose instead $x_i \notin dom(\kappa)$. In this case $(\varphi_k\{x_i/t\})_a = \varphi$. Moreover $\psi, \psi_1, \dots, \psi_m \in E(n,\kappa)$, so by one of our inductive assumptions $\psi_k\{x_i/t\} = \psi$, $(\psi_j)_k\{x_i/t\} = \psi_j$. Therefore

$(\varphi_k\{x_i/t\})_c = (\psi_k\{x_i/t\})((\psi_1)_k\{x_i/t\}, .., (\psi_m)_k\{x_i/t\}) = (\psi)(\psi_1, .., \psi_m) = \varphi = (\varphi_k\{x_i/t\})_a$.

We now examine the case where φ belongs to $\mathbf{E'_a(n+1,k) \cap E'_d(n+1,k)}$. Clearly φ belongs to $E_a(n+1,k) \cap E_d(n+1,k)$.

Since $\varphi \in E_a(n+1,k)$ we have $\varphi \in E(n,k_{p-1})$, $x_p \notin V_b(\varphi)$. We have to distinguish the case where i=p and $(\varphi_k\{x_i/t\})_a = \varphi$ from the case where i<p and $(\varphi_k\{x_i/t\})_a = \varphi_{k(p-1)}\{x_i/t\}$.

Since $\varphi \in E_d(n+1,k)$ the following condition holds:

there exist f in F, a positive integer m and $\psi_1, .., \psi_m \in E(n,k)$ such that

- $\varphi = (f)(\psi_1, \dots, \psi_m)$,
- $(\varphi_k\{x_i/t\})_d = (f)\ ((\psi_1)_k\{x_i/t\}, .., (\psi_m)_k\{x_i/t\})$.

Suppose i=p. In this case, given $\varphi \in E(n,k_{p-1})$, we can apply assumption 2.1.7 to obtain that

n>1, there exist $h \in K(n-1)$ such that $h \sqsubseteq k_{p-1}$, $g \in F$, $\chi_1, .., \chi_q \in E(n-1,h)$ such that $\varphi = (g)(\chi_1, .., \chi_q)$.

Clearly q=m, g=f, $\chi_j = \psi_j$ therefore $\psi_1, .., \psi_m \in E(n-1,h)$.

Since $x_i \notin dom(h)$ we can apply one of our assumptions and conclude that $(\psi_j)_k\{x_i/t\} = \psi_j$. Hence

$(\varphi_k\{x_i/t\})_d = (f)\ ((\psi_1)_k\{x_i/t\}, .., (\psi_m)_k\{x_i/t\}) = (f)\ (\psi_1, .., \psi_m) = \varphi = (\varphi_k\{x_i/t\})_a$.

Now suppose i<p. Since $\varphi \in E(n,k_{p-1})$ we can apply one of our inductive assumptions and obtain the following:

n>1, there exist $\kappa \in K(n-1)$ such that $\kappa \sqsubseteq k_{p-1}$, $g \in F$, $\chi_1, \dots, \chi_q \in E(n-1,\kappa)$ such that $\varphi = (g)(\chi_1, \dots, \chi_q)$, $\varphi \in E(n,\kappa)$, if $x_i \in dom(\kappa)$ then $\varphi_{k(p-1)}\{x_i/t\} = (g)\ ((\chi_1)_\kappa\{x_i/t\}, \dots, (\chi_m)_\kappa\{x_i/t\})$, else $\varphi_{k(p-1)}\{x_i/t\} = \varphi$.

Clearly q=m, g=f, $\chi_j = \psi_j$, so
if $x_i \in dom(\kappa)$ then $\varphi_{k(p-1)}\{x_i/t\} = (f)\ ((\psi_1)_\kappa\{x_i/t\}, \dots, (\psi_m)_\kappa\{x_i/t\})$, else $\varphi_{k(p-1)}\{x_i/t\} = \varphi$.

Therefore
if $x_i \in dom(\kappa)$ then $(\varphi_k\{x_i/t\})_a = (f)\ ((\psi_1)_\kappa\{x_i/t\}, \dots, (\psi_m)_\kappa\{x_i/t\})$, else $(\varphi_k\{x_i/t\})_a = \varphi$.

Suppose $x_i \in dom(\kappa)$. By one of our inductive assumptions $(\psi_j)_k\{x_i/t\} = (\psi_j)_\kappa\{x_i/t\}$, therefore

$(\varphi_k\{x_i/t\})_a = (f)\ ((\psi_1)_\kappa\{x_i/t\}, \dots, (\psi_m)_\kappa\{x_i/t\}) = (f)\ ((\psi_1)_k\{x_i/t\}, .., (\psi_m)_k\{x_i/t\}) = (\varphi_k\{x_i/t\})_d$.

Suppose instead $x_i \notin dom(\kappa)$. In this case $(\varphi_k\{x_i/t\})_a = \varphi$. Moreover $\psi_1, \dots, \psi_m \in E(n,\kappa)$, so by one of our inductive assumptions $(\psi_j)_k\{x_i/t\} = \psi_j$. Therefore

$(\varphi_k\{x_i/t\})_d = (f)\ ((\psi_1)_k\{x_i/t\}, .., (\psi_m)_k\{x_i/t\}) = (f)\ (\psi_1, .., \psi_m) = \varphi = (\varphi_k\{x_i/t\})_a$.

Finally we turn to the case where φ belongs to **$E'_a(n+1,k) \cap E'_e(n+1,k)$** . Clearly φ belongs to $E_a(n+1,k) \cap E_e(n+1,k)$.

Since $\varphi \in E_a(n+1,k)$ we have $\varphi \in E(n,k_{p-1})$, $x_p \notin V_b(\varphi)$. We have to distinguish the case where i=p and $(\varphi_k\{x_i/t\})_a = \varphi$ from the case where i<p and $(\varphi_k\{x_i/t\})_a = \varphi_{k(p-1)}\{x_i/t\}$.

Since $\varphi \in E_e(n+1,k)$ the following condition holds:

there exist a positive integer m, $y_1, .., y_m$ distinct $\in V$-dom(k), $\psi, \psi_1, .., \psi_m \in E(n)$ such that $\varphi = \{\}(y_1{:}\psi_1, \dots, y_m{:}\psi_m, \psi)$. Moreover we have

- $\psi_1 \in E(n,k)$, for each $\sigma \in \Xi(k)$ $\#(k,\psi_1,\sigma)$ is a set ;
- if m>1, for each i=j..m-1 if we define $k'_j = k \| (y_1,\psi_1) \| .. \| (y_j,\psi_j)$ it follows

$k'_j \in K(n) \wedge \psi_{j+1} \in E(n, k'_j) \wedge$ for each $\sigma'_j \in \Xi(k'_j)$ $\#(k'_j, \psi_{j+1}, \sigma'_j)$ is a set ;

- if we define $k'_m = k||(y_1,\psi_1)|| .. ||(y_m,\psi_m)$ it follows $k'_m \in K(n) \wedge \psi \in E(n,k'_m)$.

and we define

$(\varphi_k\{x_i/t\})_e = \{\}(y_1: (\psi_1)_k\{x_i/t\}, y_2: (\psi_2)_{k'(1)}\{x_i/t\} , \dots , y_m: (\psi_m)_{k'(m-1)}\{x_i/t\}, \psi_{k'(m)}\{x_i/t\})$.

Suppose i=p. In this case, given $\varphi \in E(n,k_{p-1})$, we can apply assumption 2.1.7 to obtain that

n>1, there exists $h \in K(n-1)$: $h \sqsubseteq k_{p-1}$, $\chi, \chi_1, .. , \chi_q \in E(n-1)$, $z_1, \dots , z_q$ distinct $\in V$-dom(h) :
$\varphi = \{\}(z_1:\chi_1, \dots , z_q:\chi_q, \chi)$, $\varphi \in E(n,h)$. Moreover

- $\chi_1 \in E(n-1,h)$, for each $\rho \in \Xi(h)$ $\#(h,\chi_1,\rho)$ is a set ;
- if q>1 for each j=1..q-1 if we define $h'_j = h||(z_1,\chi_1)|| .. ||(z_j,\chi_j)$ it follows
  $h'_j \in K(n-1) \wedge \chi_{j+1} \in E(n-1, h'_j) \wedge$ for each $\rho'_j \in \Xi(h'_j)$ $\#(h'_j, \chi_{j+1}, \rho'_j)$ is a set ;
- if we define $h'_q = h||(y_1,\chi_1)|| .. ||(y_q,\chi_q)$ it follows $h'_q \in K(n-1) \wedge \chi \in E(n-1,h'_q)$.

Clearly q=m, $\chi=\psi$, $\chi_j=\psi_j$, $z_j=y_j$.

We can see that $h \in K(n-1) \subseteq K(n)$, $\psi_1 \in E(n,h)$, $h \sqsubseteq k_{p-1}$ and so $x_i \notin$ dom(h). We can apply one of our assumptions and determine $(\psi_1)_k\{x_i/t\} = \psi$.

Suppose m>1, and let j=1..m-1. We can see $h'_j \in K(n)$, $\psi_{j+1} \in E(n, h'_j)$, $x_i \notin$ dom(h'_j). We can apply one of our assumptions and determine $(\psi_{j+1})_{k'(j)}\{x_i/t\} = \psi_{j+1}$.

Finally we can see $h'_m \in K(n)$, $\psi \in E(n,h'_m)$, $x_i \notin$ dom(h'_m). We can apply one of our assumptions and determine $\psi_{k'(m)}\{x_i/t\} = \psi$.

Therefore

$(\varphi_k\{x_i/t\})_e = \{\}(y_1: (\psi_1)_k\{x_i/t\}, y_2: (\psi_2)_{k'(1)}\{x_i/t\} , \dots , y_m: (\psi_m)_{k'(m-1)}\{x_i/t\}, \psi_{k'(m)}\{x_i/t\})$ =

$= \{\}(y_1:\psi_1, \dots , y_m:\psi_m, \psi) = \varphi = (\varphi_k\{x_i/t\})_a$.

Now suppose i<p. Since $\varphi \in E(n,k_{p-1})$ we can apply one of our inductive assumptions and obtain the following:

n>1, there exist $\kappa \in K(n-1)$ such that $\kappa \sqsubseteq k_{p-1}$ and $\chi, \chi_1, .. , \chi_q \in E(n-1)$, $z_1, \dots , z_q$ distinct $\in V$-dom(κ) such that $\varphi = \{\}(z_1:\chi_1, \dots , z_q:\chi_q, \chi)$, $\varphi \in E(n,\kappa)$,

$\chi_1 \in E(n-1,\kappa)$, for each $\sigma \in \Xi(\kappa)$ $\#(\kappa,\chi_1,\sigma)$ is a set ;
if q>1 then for each j=1..q-1 we define $\kappa'_j = \kappa||(z_1,\chi_1)|| .. ||(z_j,\chi_j)$ and we have $\kappa'_j \in K(n-1)$, $\chi_{j+1} \in E(n-1, \kappa'_j)$, for each $\sigma'_j \in \Xi(\kappa'_j)$ $\#(\kappa'_j, \chi_{j+1}, \sigma'_j)$ is a set;
if we define $\kappa'_q = \kappa||(z_1,\chi_1)|| .. ||(z_q,\chi_q)$ then $\kappa'_q \in K(n-1) \wedge \chi \in E(n-1,\kappa'_q)$;

if $x_i \in$ dom(κ) then
$\varphi_{k(p-1)}\{x_i/t\} = \{\}(z_1: (\chi_1)_\kappa\{x_i/t\}, z_2: (\chi_2)_{\kappa'(1)}\{x_i/t\} , \dots , z_q: (\chi_q)_{\kappa'(q-1)}\{x_i/t\}, \chi_{\kappa'(q)}\{x_i/t\})$;

if $x_i \notin$ dom(κ) then $\varphi_{k(p-1)}\{x_i/t\} = \varphi$.

Clearly q=m, $\chi=\psi$, $\chi_j=\psi_j$, $z_j=y_j$, so

if $x_i \in dom(\kappa)$ then
$(\varphi_k\{x_i/t\})_a = \varphi_{k(p-1)}\{x_i/t\} = \{\}(y_1: (\psi_1)_\kappa\{x_i/t\}, y_2: (\psi_2)_{\kappa'(1)}\{x_i/t\}, \dots, y_m: (\psi_m)_{\kappa'(m-1)}\{x_i/t\}, \psi_{\kappa'(m)}\{x_i/t\})$;

if $x_i \notin dom(\kappa)$ then $(\varphi_k\{x_i/t\})_a = \varphi_{k(p-1)}\{x_i/t\} = \varphi$.

Suppose $x_i \in dom(\kappa)$.

Since $\kappa \in K(n)$, $k_i \sqsubseteq \kappa$, $\psi_1 \in E(n,\kappa)$, we can apply one of our assumptions and obtain that $(\psi_1)_k\{x_i/t\} = (\psi_1)_\kappa\{x_i/t\}$.

If m>1 suppose j=1..m-1, we want to show that $(\psi_{j+1})_{k'(j)}\{x_i/t\} = (\psi_{j+1})_{\kappa'(j)}\{x_i/t\}$.

We have $\psi_{j+1} \in E(n,k'_j)$, $\kappa'_j \in K(n)$, $k_i \sqsubseteq \kappa \sqsubseteq \kappa'_j$, $\psi_{j+1} \in E(n, \kappa'_j)$, for each α=1..j $y_\alpha \in V_b(\varphi)$ so $y_\alpha \notin V_b(t)$, $V_b(t) \cap V_b(\psi_\alpha) \subseteq V_b(t) \cap V_b(\varphi) = \emptyset$. Therefore we can apply one of our assumptions (the same we used for ψ_1) to obtain that

$(\psi_{j+1})_{k'(j)}\{x_i/t\} = (\psi_{j+1})_{\kappa'(j)}\{x_i/t\}$.

We have also $\psi \in E(n,k'_m)$, $\kappa'_m \in K(n)$, $k_i \sqsubseteq \kappa \sqsubseteq \kappa'_m$, $\psi \in E(n, \kappa'_m)$, for each α=1..m $y_\alpha \in V_b(\varphi)$ so $y_\alpha \notin V_b(t)$, $V_b(t) \cap V_b(\psi_\alpha) \subseteq V_b(t) \cap V_b(\varphi) = \emptyset$. Therefore we can apply one of our assumptions (the same we used for ψ_1) to obtain that

$(\psi)_{k'(m)}\{x_i/t\} = (\psi)_{\kappa'(m)}\{x_i/t\}$.

Hence

$(\varphi_k\{x_i/t\})_a = \{\}(y_1: (\psi_1)_\kappa\{x_i/t\}, y_2: (\psi_2)_{\kappa'(1)}\{x_i/t\}, \dots, y_m: (\psi_m)_{\kappa'(m-1)}\{x_i/t\}, \psi_{\kappa'(m)}\{x_i/t\}) =$

$= \{\}(y_1: (\psi_1)_k\{x_i/t\}, y_2: (\psi_2)_{k'(1)}\{x_i/t\}, \dots, y_m: (\psi_m)_{k'(m-1)}\{x_i/t\}, \psi_{k'(m)}\{x_i/t\}) = (\varphi_k\{x_i/t\})_e$.

Now let $x_i \notin dom(\kappa)$. In this case $(\varphi_k\{x_i/t\})_a = \varphi = \{\}(y_1:\psi_1, \dots, y_m:\psi_m, \psi)$.

Since $\kappa \in K(n)$, $x_i \notin dom(\kappa)$, $\psi_1 \in E(n,\kappa)$ we can apply one of our assumptions and conclude that $(\psi_1)_k\{x_i/t\} = \psi_1$.

If m>1 suppose j=1..m-1, we want to show that $(\psi_{j+1})_{k'(j)}\{x_i/t\} = \psi_{j+1}$. This holds because the following conditions hold: $\psi_{j+1} \in E(n,k'_j)$, $\kappa'_j \in K(n)$, $x_i \notin dom(\kappa'_j)$, $\psi_{j+1} \in E(n, \kappa'_j)$, and we can apply the same assumption we used for ψ_1.

Moreover we need to show $\psi_{k'(m)}\{x_i/t\} = \psi$. This holds because the following conditions hold: $\psi \in E(n,k'_m)$, $\kappa'_m \in K(n)$, $x_i \notin dom(\kappa'_m)$, $\psi \in E(n, \kappa'_m)$, and we can apply the same assumption we used for ψ_1.

Therefore

$(\varphi_k\{x_i/t\})_a = \{\}(y_1:\psi_1, \dots, y_m:\psi_m, \psi) =$

$= \{\}(y_1: (\psi_1)_k\{x_i/t\}, y_2: (\psi_2)_{k'(1)}\{x_i/t\}, \dots, y_m: (\psi_m)_{k'(m-1)}\{x_i/t\}, \psi_{k'(m)}\{x_i/t\}) = (\varphi_k\{x_i/t\})_e$.

At this point we have completed the proof that $\varphi_k\{x_i/t\}$ is defined unambiguously. Our definition process requires now to verify that (for $\varphi\in E(n+1,k)$ such that $V_b(t) \cap V_b(\varphi) = \emptyset$) one of the following five conditions holds

a1) $\varphi\in C \wedge \varphi_k\{x_i/t\} = \varphi$,

a2) $\varphi\in dom(k) \wedge (\varphi=x_i \rightarrow \varphi_k\{x_i/t\} = t) \wedge (\varphi\neq x_i \rightarrow \varphi_k\{x_i/t\} = \varphi)$

a3) there exist $\kappa\in K(n)$ such that $\kappa\sqsubseteq k$ and $\psi, \psi_1, \dots , \psi_m \in E(n,\kappa)$ such that
$\varphi = (\psi)(\psi_1, \dots , \psi_m)$, $\varphi\in E(n+1,\kappa)$,
for each $\rho\in\Xi(\kappa)$ $\#(\kappa,\psi,\rho)$ is a function with m arguments and
$(\#(\kappa, \psi_1, \rho), ... , \#(\kappa, \psi_m, \rho))$ is a member of its domain,
if $x_i\in dom(\kappa)$ then $\varphi_k\{x_i/t\} = (\psi_\kappa\{x_i/t\}) ((\psi_1)_\kappa\{x_i/t\}, \dots , (\psi_m)_\kappa\{x_i/t\})$,
else $\varphi_k\{x_i/t\} = \varphi$.

a4) there exist $\kappa\in K(n)$ such that $\kappa\sqsubseteq k$ and $f\in F$, $\psi_1, \dots , \psi_m \in E(n,\kappa)$ such that
$\varphi = (f)(\psi_1, \dots , \psi_m)$, $\varphi\in E(n+1,\kappa)$,
for each $\rho\in\Xi(\kappa)$ $A_f(\#(\kappa, \psi_1, \rho), ... , \#(\kappa, \psi_m, \rho))$ is true,
if $x_i\in dom(\kappa)$ then $\varphi_k\{x_i/t\} = (f) ((\psi_1)_\kappa\{x_i/t\}, \dots , (\psi_m)_\kappa\{x_i/t\})$,
else $\varphi_k\{x_i/t\} = \varphi$.

a5) there exist $\kappa\in K(n)$ such that $\kappa\sqsubseteq k$ and $\psi, \psi_1, .. , \psi_m \in E(n)$,
$y_1, \dots , y_m$ distinct $\in V\text{-}dom(\kappa)$ such that
$\varphi = \{\}(y_1{:}\psi_1, \dots , y_m{:}\psi_m, \psi)$, $\varphi\in E(n+1,\kappa)$,

$\psi_1 \in E(n,\kappa)$, for each $\sigma\in\Xi(\kappa)$ $\#(\kappa,\psi_1,\sigma)$ is a set ;
if $m>1$ then for each $j=1..m-1$ we define $\kappa'_j = \kappa\|(y_1,\psi_1)\| .. \|(y_j,\psi_j)$ and we have $\kappa'_j \in K(n)$, $\psi_{j+1}\in E(n, \kappa'_j)$, for each $\sigma'_j\in\Xi(\kappa'_j)$ $\#(\kappa'_j, \psi_{j+1}, \sigma'_j)$ is a set;
if we define $\kappa'_m = \kappa\|(x_1,\psi_1)\| .. \|(x_m,\psi_m)$ then $\kappa'_m\in K(n) \wedge \psi\in E(n,\kappa'_m)$;

if $x_i\in dom(\kappa)$ then we can observe that

$\psi_1 \in E(n+1,\kappa)$, $V_b(t) \cap V_b(\psi_1) \subseteq V_b(t) \cap V_b(\varphi) = \emptyset$, therefore $(\psi_1)_\kappa\{x_i/t\}$ is defined;

for each $j=1..m-1$ $\psi_{j+1}\in E(n+1, \kappa'_j)$, for each $\alpha=1..j$ $y_\alpha\in V_b(\varphi)$ so $y_\alpha\notin V_b(t)$,
$V_b(t) \cap V_b(\psi_\alpha) \subseteq V_b(t) \cap V_b(\varphi) = \emptyset$, $V_b(t) \cap V_b(\psi_{j+1}) \subseteq V_b(t) \cap V_b(\varphi) = \emptyset$,
therefore $(\psi_{j+1})_{\kappa'(j)}\{x_i/t\}$ is defined ;

$\psi\in E(n+1, \kappa'_m)$, for each $\alpha=1..m$ $y_\alpha\in V_b(\varphi)$ so $y_\alpha\notin V_b(t)$,
$V_b(t) \cap V_b(\psi_\alpha) \subseteq V_b(t) \cap V_b(\varphi) = \emptyset$, $V_b(t) \cap V_b(\psi) \subseteq V_b(t) \cap V_b(\varphi) = \emptyset$,
therefore $\psi_{\kappa'(m)}\{x_i/t\}$ is defined ;

it results $\varphi_k\{x_i/t\} =$
$= \{\}(y_1{:}\ (\psi_1)_\kappa\{x_i/t\}, y_2{:}\ (\psi_2)_{\kappa'(1)}\{x_i/t\} , \dots , y_m{:}\ (\psi_m)_{\kappa'(m-1)}\{x_i/t\}, \psi_{\kappa'(m)}\{x_i/t\})$;

if $x_i\notin dom(\kappa)$ then $\varphi_k\{x_i/t\} = \varphi$.

In this case too we need to remember that

$E(n+1,k) = E'(n,k) \cup E'_a(n+1,k) \cup E'_b(n+1,k) \cup E'_c(n+1,k) \cup E'_d(n+1,k) \cup E'_e(n+1,k)$.

Suppose **φ is in E'(n,k)** .

By the inductive hypothesis one of the following five conditions holds

b1) $\varphi \in C \wedge \varphi_k\{x_i/t\} = \varphi$,

b2) $\varphi \in dom(k) \wedge (\varphi = x_i \rightarrow \varphi_k\{x_i/t\} = t) \wedge (\varphi \neq x_i \rightarrow \varphi_k\{x_i/t\} = \varphi)$

b3) n>1, there exist $\kappa \in K(n-1)$ such that $\kappa \sqsubseteq k$ and $\psi, \psi_1, \ldots, \psi_m \in E(n-1,\kappa)$ such that
$\varphi = (\psi)(\psi_1, \ldots, \psi_m)$, $\varphi \in E(n,\kappa)$,
for each $\rho \in \Xi(\kappa)$ $\#(\kappa,\psi,\rho)$ is a function with m arguments and
$(\#(\kappa, \psi_1, \rho), \ldots, \#(\kappa, \psi_m, \rho))$ is a member of its domain,
if $x_i \in dom(\kappa)$ then $\varphi_k\{x_i/t\} = (\psi_\kappa\{x_i/t\}) ((\psi_1)_\kappa\{x_i/t\}, \ldots, (\psi_m)_\kappa\{x_i/t\})$,
else $\varphi_k\{x_i/t\} = \varphi$.

b4) n>1, there exist $\kappa \in K(n-1)$ such that $\kappa \sqsubseteq k$ and $f \in F$, $\psi_1, \ldots, \psi_m \in E(n-1,\kappa)$ such that $\varphi = (f)(\psi_1, \ldots, \psi_m)$, $\varphi \in E(n,\kappa)$,
for each $\rho \in \Xi(\kappa)$ $A_f(\#(\kappa, \psi_1, \rho), \ldots, \#(\kappa, \psi_m, \rho))$ is true,
if $x_i \in dom(\kappa)$ then $\varphi_k\{x_i/t\} = (f) ((\psi_1)_\kappa\{x_i/t\}, \ldots, (\psi_m)_\kappa\{x_i/t\})$,
else $\varphi_k\{x_i/t\} = \varphi$.

b5) n>1, there exist $\kappa \in K(n-1)$ such that $\kappa \sqsubseteq k$ and $\psi, \psi_1, .., \psi_m \in E(n-1)$,
$y_1, \ldots, y_m$ distinct $\in V$-dom(κ) such that
$\varphi = \{\}(y_1{:}\psi_1, \ldots, y_m{:}\psi_m, \psi)$, $\varphi \in E(n,\kappa)$,

$\psi_1 \in E(n-1,\kappa)$, for each $\sigma \in \Xi(\kappa)$ $\#(\kappa,\psi_1,\sigma)$ is a set ;
if m>1 then for each j=1..m-1 we define $\kappa'_j = \kappa \| (y_1,\psi_1) \| .. \| (y_j,\psi_j)$ and we have $\kappa'_j \in K(n-1)$, $\psi_{j+1} \in E(n-1, \kappa'_j)$, for each $\sigma'_j \in \Xi(\kappa'_j)$ $\#(\kappa'_j, \psi_{j+1}, \sigma'_j)$ is a set;
if we define $\kappa'_m = \kappa \| (x_1,\psi_1) \| .. \| (x_m,\psi_m)$ then $\kappa'_m \in K(n-1) \wedge \psi \in E(n-1,\kappa'_m)$;

if $x_i \in dom(\kappa)$ then we can observe that

$\psi_1 \in E(n,\kappa)$, $V_b(t) \cap V_b(\psi_1) \subseteq V_b(t) \cap V_b(\varphi) = \emptyset$, therefore $(\psi_1)_\kappa\{x_i/t\}$ is defined;

for each j=1..m-1 $\psi_{j+1} \in E(n, \kappa'_j)$, for each α=1..j $y_\alpha \in V_b(\varphi)$ so $y_\alpha \notin V_b(t)$,
$V_b(t) \cap V_b(\psi_\alpha) \subseteq V_b(t) \cap V_b(\varphi) = \emptyset$, $V_b(t) \cap V_b(\psi_{j+1}) \subseteq V_b(t) \cap V_b(\varphi) = \emptyset$,
therefore $(\psi_{j+1})_{\kappa'(j)}\{x_i/t\}$ is defined ;

$\psi \in E(n, \kappa'_m)$, for each α=1..m $y_\alpha \in V_b(\varphi)$ so $y_\alpha \notin V_b(t)$,
$V_b(t) \cap V_b(\psi_\alpha) \subseteq V_b(t) \cap V_b(\varphi) = \emptyset$, $V_b(t) \cap V_b(\psi) \subseteq V_b(t) \cap V_b(\varphi) = \emptyset$,
therefore $\psi_{\kappa'(m)}\{x_i/t\}$ is defined ;

it results $\varphi_k\{x_i/t\}$ =
= $\{\}(y_1: (\psi_1)_\kappa\{x_i/t\}, y_2: (\psi_2)_{\kappa'(1)}\{x_i/t\}, \dots, y_m: (\psi_m)_{\kappa'(m-1)}\{x_i/t\}, \psi_{\kappa'(m)}\{x_i/t\})$;

if $x_i \notin dom(\kappa)$ then $\varphi_k\{x_i/t\} = \varphi$.

Clearly if b1) holds then a1) holds too, if b2) holds then a2) holds too. If b3) holds then a3) holds too, if b4) holds then a4) holds too. Finally if b5) holds then a5) holds too.

We turn to the case where **φ is in $E'_a(n+1,k)$** .

In this case we have $\varphi \in E(n,k_{p-1})$, $x_p \notin V_b(\varphi)$. We have to distinguish the case where $i<p$ from the one where $i=p$.

First we suppose $i<p$. We can apply the inductive hypothesis to φ and obtain that one of the following five conditions holds

- $\varphi \in \mathcal{C} \wedge \varphi_{k(p-1)}\{x_i/t\} = \varphi$,
- $\varphi \in dom(k_{p-1}) \wedge (\varphi = x_i \rightarrow \varphi_{k(p-1)}\{x_i/t\} = t) \wedge (\varphi \neq x_i \rightarrow \varphi_{k(p-1)}\{x_i/t\} = \varphi)$
- $n>1$, there exist $\kappa \in K(n-1)$ such that $\kappa \sqsubseteq k_{p-1}$ and $\psi, \psi_1, \dots, \psi_m \in E(n-1,\kappa)$ such that $\varphi = (\psi)(\psi_1, \dots, \psi_m)$, $\varphi \in E(n,\kappa)$,
  for each $\rho \in \Xi(\kappa)$ $\#(\kappa,\psi,\rho)$ is a function with m arguments and
  $(\#(\kappa, \psi_1, \rho), \dots, \#(\kappa, \psi_m, \rho))$ is a member of its domain,
  if $x_i \in dom(\kappa)$ then $\varphi_{k(p-1)}\{x_i/t\} = (\psi_\kappa\{x_i/t\}) ((\psi_1)_\kappa\{x_i/t\}, \dots, (\psi_m)_\kappa\{x_i/t\})$,
  else $\varphi_{k(p-1)}\{x_i/t\} = \varphi$.
- $n>1$, there exist $\kappa \in K(n-1)$ such that $\kappa \sqsubseteq k_{p-1}$ and $f \in \mathcal{F}$, $\psi_1, \dots, \psi_m \in E(n-1,\kappa)$ such that $\varphi = (f)(\psi_1, \dots, \psi_m)$, $\varphi \in E(n,\kappa)$,
  for each $\rho \in \Xi(\kappa)$ $A_f(\#(\kappa, \psi_1, \rho), \dots, \#(\kappa, \psi_m, \rho))$ is true,
  if $x_i \in dom(\kappa)$ then $\varphi_{k(p-1)}\{x_i/t\} = (f) ((\psi_1)_\kappa\{x_i/t\}, \dots, (\psi_m)_\kappa\{x_i/t\})$,
  else $\varphi_{k(p-1)}\{x_i/t\} = \varphi$.
- $n>1$, there exist $\kappa \in K(n-1)$ such that $\kappa \sqsubseteq k_{p-1}$ and $\psi, \psi_1, .., \psi_m \in E(n-1)$,
  $y_1, \dots, y_m$ distinct $\in \mathcal{V}$-dom(κ) such that
  $\varphi = \{\}(y_1:\psi_1, \dots, y_m:\psi_m, \psi)$, $\varphi \in E(n,\kappa)$,

  $\psi_1 \in E(n-1,\kappa)$, for each $\sigma \in \Xi(\kappa)$ $\#(\kappa,\psi_1,\sigma)$ is a set ;
  if $m>1$ then for each $j=1..m-1$ we define $\kappa'_j = \kappa || (y_1,\psi_1) || .. || (y_j,\psi_j)$ and we have $\kappa'_j \in K(n-1)$, $\psi_{j+1} \in E(n-1, \kappa'_j)$, for each $\sigma'_j \in \Xi(\kappa'_j)$ $\#(\kappa'_j, \psi_{j+1}, \sigma'_j)$ is a set;
  if we define $\kappa'_m = \kappa || (x_1,\psi_1) || .. || (x_m,\psi_m)$ then $\kappa'_m \in K(n-1) \wedge \psi \in E(n-1,\kappa'_m)$;

  if $x_i \in dom(\kappa)$ then we can observe that

  $\psi_1 \in E(n,\kappa)$, $V_b(t) \cap V_b(\psi_1) \subseteq V_b(t) \cap V_b(\varphi) = \varnothing$, therefore $(\psi_1)_\kappa\{x_i/t\}$ is defined;

for each j=1..m-1 $\psi_{j+1} \in E(n, \kappa'_j)$, for each α=1..j $y_\alpha \in V_b(\varphi)$ so $y_\alpha \notin V_b(t)$, $V_b(t) \cap V_b(\psi_\alpha) \subseteq V_b(t) \cap V_b(\varphi) = \emptyset$, $V_b(t) \cap V_b(\psi_{j+1}) \subseteq V_b(t) \cap V_b(\varphi) = \emptyset$, therefore $(\psi_{j+1})_{\kappa'(j)}\{x_i/t\}$ is defined ;

$\psi \in E(n, \kappa'_m)$, for each α=1..m $y_\alpha \in V_b(\varphi)$ so $y_\alpha \notin V_b(t)$, $V_b(t) \cap V_b(\psi_\alpha) \subseteq V_b(t) \cap V_b(\varphi) = \emptyset$, $V_b(t) \cap V_b(\psi) \subseteq V_b(t) \cap V_b(\varphi) = \emptyset$, therefore $\psi_{\kappa'(m)}\{x_i/t\}$ is defined ;

it results $\varphi_{k(p-1)}\{x_i/t\} =$
$= \{\}(y_1\text{: } (\psi_1)_\kappa\{x_i/t\}, y_2\text{: } (\psi_2)_{\kappa'(1)}\{x_i/t\} , \dots , y_m\text{: } (\psi_m)_{\kappa'(m-1)}\{x_i/t\}, \psi_{\kappa'(m)}\{x_i/t\})$;

if $x_i \notin dom(\kappa)$ then $\varphi_{k(p-1)}\{x_i/t\} = \varphi$.

When i<p we defined $\varphi_k\{x_i/t\} = \varphi_{k(p-1)}\{x_i/t\}$, therefore one of the following five conditions holds

- $\varphi \in C \wedge \varphi_k\{x_i/t\} = \varphi$,
- $\varphi \in dom(k) \wedge (\varphi=x_i \rightarrow \varphi_k\{x_i/t\} = t) \wedge (\varphi \neq x_i \rightarrow \varphi\{x_i/t\} = \varphi)$
- there exist $\kappa \in K(n)$ such that $\kappa \sqsubseteq k$ and $\psi, \psi_1, \dots , \psi_m \in E(n,\kappa)$ such that $\varphi = (\psi)(\psi_1, \dots , \psi_m)$, $\varphi \in E(n+1,\kappa)$,
for each $\rho \in \Xi(\kappa)$ $\#(\kappa,\psi,\rho)$ is a function with m arguments and
$(\#(\kappa, \psi_1, \rho), ... , \#(\kappa, \psi_m, \rho))$ is a member of its domain,
if $x_i \in dom(\kappa)$ then $\varphi_k\{x_i/t\} = (\psi_\kappa\{x_i/t\}) ((\psi_1)_\kappa\{x_i/t\}, \dots , (\psi_m)_\kappa\{x_i/t\})$,
else $\varphi_k\{x_i/t\} = \varphi$.
- there exist $\kappa \in K(n)$ such that $\kappa \sqsubseteq k$ and $f \in F$, $\psi_1, \dots , \psi_m \in E(n,\kappa)$ such that $\varphi = (f)(\psi_1, \dots , \psi_m)$, $\varphi \in E(n+1,\kappa)$,
for each $\rho \in \Xi(\kappa)$ $A_f(\#(\kappa, \psi_1, \rho), ... , \#(\kappa, \psi_m, \rho))$ is true,
if $x_i \in dom(\kappa)$ then $\varphi_k\{x_i/t\} = (f) ((\psi_1)_\kappa\{x_i/t\}, \dots , (\psi_m)_\kappa\{x_i/t\})$,
else $\varphi_k\{x_i/t\} = \varphi$.
- there exist $\kappa \in K(n)$ such that $\kappa \sqsubseteq k$ and $\psi, \psi_1, .. , \psi_m \in E(n)$,
$y_1, \dots , y_m$ distinct $\in V$-dom(κ) such that
$\varphi = \{\}(y_1{:}\psi_1, \dots , y_m{:}\psi_m, \psi)$, $\varphi \in E(n+1,\kappa)$,

  $\psi_1 \in E(n,\kappa)$, for each $\sigma \in \Xi(\kappa)$ $\#(\kappa,\psi_1,\sigma)$ is a set ;
  if m>1 then for each j=1..m-1 we define $\kappa'_j = \kappa \| (y_1,\psi_1) \| .. \| (y_j,\psi_j)$ and we have $\kappa'_j \in K(n)$, $\psi_{j+1} \in E(n, \kappa'_j)$, for each $\sigma'_j \in \Xi(\kappa'_j)$ $\#(\kappa'_j, \psi_{j+1}, \sigma'_j)$ is a set;
  if we define $\kappa'_m = \kappa \| (x_1,\psi_1) \| .. \| (x_m,\psi_m)$ then $\kappa'_m \in K(n) \wedge \psi \in E(n,\kappa'_m)$;

  if $x_i \in dom(\kappa)$ then we can observe that

  $\psi_1 \in E(n+1,\kappa)$, $V_b(t) \cap V_b(\psi_1) \subseteq V_b(t) \cap V_b(\varphi) = \emptyset$, therefore $(\psi_1)_\kappa\{x_i/t\}$ is defined;

for each j=1..m-1 $\psi_{j+1} \in E(n+1, \kappa'_j)$, for each α=1..j $y_\alpha \in V_b(\varphi)$ so $y_\alpha \notin V_b(t)$, $V_b(t) \cap V_b(\psi_\alpha) \subseteq V_b(t) \cap V_b(\varphi) = \varnothing$, $V_b(t) \cap V_b(\psi_{j+1}) \subseteq V_b(t) \cap V_b(\varphi) = \varnothing$, therefore $(\psi_{j+1})_{\kappa'(j)}\{x_i/t\}$ is defined ;

$\psi \in E(n+1, \kappa'_m)$, for each α=1..m $y_\alpha \in V_b(\varphi)$ so $y_\alpha \notin V_b(t)$, $V_b(t) \cap V_b(\psi_\alpha) \subseteq V_b(t) \cap V_b(\varphi) = \varnothing$, $V_b(t) \cap V_b(\psi) \subseteq V_b(t) \cap V_b(\varphi) = \varnothing$, therefore $\psi_{\kappa'(m)}\{x_i/t\}$ is defined ;

it results $\varphi_k\{x_i/t\}$ =
$= \{\}(y_1: (\psi_1)_\kappa\{x_i/t\}, y_2: (\psi_2)_{\kappa'(1)}\{x_i/t\}, \ldots, y_m: (\psi_m)_{\kappa'(m-1)}\{x_i/t\}, \psi_{\kappa'(m)}\{x_i/t\})$;

if $x_i \notin dom(\kappa)$ then $\varphi_k\{x_i/t\} = \varphi$.

We now consider the case where i=p, in which we defined $\varphi_k\{x_i/t\} = \varphi$.

Since $\varphi \in E(n,k_{p-1})$, we can apply assumption 2.1.7 to establish that one of the following five conditions hold

c1) $\varphi \in \mathcal{C}$;

c2) n>1, $\varphi \in dom(k_{p-1})$;

c3) n>1, $\exists h \in K(n-1): h \sqsubseteq k_{p-1}, \exists\ \psi, \psi_1, .., \psi_m \in E(n-1,h)$:
$\varphi = (\psi)(\psi_1, \ldots, \psi_m)$, $\varphi \in E(n,h)$,
for each $\rho \in \Xi(h)$ $\#(h,\psi,\rho)$ is a function with m arguments,
$(\#(h, \psi_1, \rho), \ldots, \#(h, \psi_m, \rho))$ is a member of its domain;

c4) n>1, $\exists h \in K(n-1): h \sqsubseteq k_{p-1}, \exists\ f \in \mathcal{F}, \psi_1, .., \psi_m \in E(n-1,h)$:
$\varphi = (f)(\psi_1, \ldots, \psi_m)$, $\varphi \in E(n,h)$,
for each $\rho \in \Xi(h)$ $A_f(\#(h, \psi_1, \rho), \ldots, \#(h, \psi_m, \rho))$;

c5) n>1, $\exists h \in K(n-1): h \sqsubseteq k_{p-1}, \exists\ \psi, \psi_1, .., \psi_m \in E(n-1)$,
$\exists y_1, \ldots, y_m$ distinct $\in \mathcal{V}$-dom(h) :
$\varphi = \{\}(y_1{:}\psi_1, \ldots, y_m{:}\psi_m, \psi)$, $\varphi \in E(n,h)$,

$\psi_1 \in E(n-1,h)$, for each $\rho \in \Xi(h)$ $\#(h,\psi_1,\rho)$ is a set ;
if m>1 for each j=1..m-1 if we define $h'_j = h \| (y_1,\psi_1) \| .. \| (y_j,\psi_j)$ it follows
$h'_j \in K(n-1) \wedge \psi_{j+1} \in E(n-1, h'_j) \wedge$ for each $\rho'_j \in \Xi(h'_j)$ $\#(h'_j, \psi_{j+1}, \rho'_j)$ is a set ;
if we define $h'_m = h \| (y_1,\psi_1) \| .. \| (y_m,\psi_m)$ it follows $h'_m \in K(n-1) \wedge \psi \in E(n-1,h'_m)$.

If c1) holds then $\varphi \in \mathcal{C} \wedge \varphi_k\{x_i/t\} = \varphi$, so a1) holds.

If c2) holds then $\varphi \in dom(k)$, $\varphi \neq x_i$, $\varphi_k\{x_i/t\} = \varphi$, so a2) holds.

If c3) holds then $h \in K(n)$, $h \sqsubseteq k$, $\psi, \psi_1, .., \psi_m \in E(n,h)$, $\varphi = (\psi)(\psi_1, \ldots, \psi_m)$, $\varphi \in E(n+1,h)$, for each $\rho \in \Xi(h)$ $\#(h,\psi,\rho)$ is a function with m arguments, $(\#(h, \psi_1, \rho), \ldots, \#(h, \psi_m, \rho))$ is a member of its domain, $x_i \notin dom(h)$, $\varphi_k\{x_i/t\} = \varphi$. Therefore a3) holds.

If c4) holds then $h \in K(n)$, $h \sqsubseteq k$, $f \in \mathcal{F}$, $\psi_1, .. , \psi_m \in E(n,h)$, $\varphi = (f)(\psi_1, \dots , \psi_m)$, $\varphi \in E(n+1,h)$, for each $\rho \in \Xi(h)$ $A_f(\#(h, \psi_1, \rho), ... , \#(h, \psi_m, \rho))$, $x_i \notin dom(h)$, $\varphi_k\{x_i/t\} = \varphi$. Therefore a4) holds.

If c5) holds then $h \in K(n)$, $h \sqsubseteq k$, $\psi, \psi_1, .. , \psi_m \in E(n)$, $y_1, \dots , y_m$ distinct $\in \mathcal{V}$-dom(h), $\varphi = \{\}(y_1{:}\psi_1, \dots , y_m{:}\psi_m, \psi)$, $\varphi \in E(n+1,h)$,

$\psi_1 \in E(n,h)$, for each $\rho \in \Xi(h)$ $\#(h,\psi_1,\rho)$ is a set ;
if m>1 for each j=1..m-1 if we define $h'_j = h||(y_1,\psi_1)|| .. ||(y_j,\psi_j)$ it follows
$h'_j \in K(n) \wedge \psi_{j+1} \in E(n, h'_j) \wedge$ for each $\rho'_j \in \Xi(h'_j)$ $\#(h'_j, \psi_{j+1}, \rho'_j)$ is a set ;
if we define $h'_m = h||(y_1,\psi_1)|| .. ||(y_m,\psi_m)$ it follows $h'_m \in K(n) \wedge \psi \in E(n,h'_m)$.

Moreover $x_i \notin dom(h)$, $\varphi_k\{x_i/t\} = \varphi$. Therefore a5) holds.

Let's examine the case where **φ is in $E'_b(n+1,k)$** .

We have $\varphi = x_p \in dom(k)$. If $\varphi = x_i$ then i=p and (as we defined) $\varphi_k\{x_i/t\} = t$.
If $\varphi \neq x_i$ then $i \neq p$ and (as we defined) $\varphi_k\{x_i/t\} = \varphi$.

We now consider the case where **φ is in $E'_c(n+1,k)$** .

In this case there exist a positive integer m and $\psi, \psi_1, .. , \psi_m \in E(n,k)$ such that
- $\varphi = (\psi)\ (\psi_1, .. , \psi_m)$,
- for each $\sigma \in \Xi(k)$ $\#(k,\psi,\sigma)$ is a function with m arguments and ($\#(k, \psi_1, \sigma), ... , \#(k, \psi_m, \sigma)$) is a member of its domain.

Moreover $x_i \in dom(k) \wedge \varphi_k\{x_i/t\} = (\psi_k\{x_i/t\})\ ((\psi_1)_k\{x_i/t\}, .. , (\psi_m)_k\{x_i/t\})$.

The case where **φ is in $E'_d(n+1,k)$** is similar. In fact

there exist f in $\mathcal{F}$, a positive integer m and $\psi_1, .. , \psi_m \in E(n,k)$ such that
- $\varphi = (f)(\psi_1, \dots , \psi_m)$
- for each $\sigma \in \Xi(k)$ $A_f(\#(k, \psi_1, \sigma), ... , \#(k, \psi_m, \sigma))$ is true.

Moreover $x_i \in dom(k) \wedge \varphi_k\{x_i/t\} = (f)\ ((\psi_1)_k\{x_i/t\}, .. , (\psi_m)_k\{x_i/t\})$.

Finally we examine the case where **φ is in $E'_e(n+1,k)$**.

In this case $k \in K(n)$ and there exist a positive integer m, $y_1, ..., y_m$ distinct $\in \mathcal{V}$-dom(k), $\psi, \psi_1, .. , \psi_m \in E(n)$ such that $\varphi = \{\}(y_1{:}\psi_1, \dots , y_m{:}\psi_m, \psi)$. Moreover we have

- $\psi_1 \in E(n,k)$, for each $\sigma \in \Xi(k)$ $\#(k,\psi_1,\sigma)$ is a set ;
- if m>1, for each i=j..m-1 if we define $k'_j = k||(y_1,\psi_1)|| .. ||(y_j,\psi_j)$ it follows $k'_j \in K(n) \wedge \psi_{j+1} \in E(n, k'_j) \wedge$ for each $\sigma'_j \in \Xi(k'_j)$ $\#(k'_j, \psi_{j+1}, \sigma'_j)$ is a set ;
- if we define $k'_m = k||(y_1,\psi_1)|| .. ||(y_m,\psi_m)$ it follows $k'_m \in K(n) \wedge \psi \in E(n,k'_m)$.

Furthermore $x_i \in dom(k)$ and we can observe that

$\psi_1 \in E(n+1,k)$, $V_b(t) \cap V_b(\psi_1) \subseteq V_b(t) \cap V_b(\varphi) = \emptyset$, therefore $(\psi_1)_k\{x_i/t\}$ is defined;

for each j=1..m-1 $\psi_{j+1} \in E(n+1, k'_j)$, for each α=1..j $y_\alpha \in V_b(\varphi)$ so $y_\alpha \notin V_b(t)$,
$V_b(t) \cap V_b(\psi_\alpha) \subseteq V_b(t) \cap V_b(\varphi) = \emptyset$, $V_b(t) \cap V_b(\psi_{j+1}) \subseteq V_b(t) \cap V_b(\varphi) = \emptyset$,
therefore $(\psi_{j+1})_{k'(j)}\{x_i/t\}$ is defined ;

$\psi \in E(n+1, k'_m)$, for each α=1..m $y_\alpha \in V_b(\varphi)$ so $y_\alpha \notin V_b(t)$, $V_b(t) \cap V_b(\psi_\alpha) \subseteq V_b(t) \cap V_b(\varphi) = \emptyset$,
$V_b(t) \cap V_b(\psi) \subseteq V_b(t) \cap V_b(\varphi) = \emptyset$, therefore $\psi_{k'(m)}\{x_i/t\}$ is defined ;

$\varphi_k\{x_i/t\} = \{\}(y_1: (\psi_1)_k\{x_i/t\}, y_2: (\psi_2)_{k'(1)}\{x_i/t\} , \dots , y_m: (\psi_m)_{k'(m-1)}\{x_i/t\}, \psi_{k'(m)}\{x_i/t\})$.

Another step has been completed. It's always assumed that $\varphi \in E(n+1,k)$ is such that $V_b(t) \cap V_b(\varphi) = \emptyset$. To perform the next step we assume $h \in K(n+1)$ is such that $k_i \sqsubseteq h$.

There exists a positive integer r<n+1, $w_1, \dots , w_r \in V$: $w_\alpha \neq w_\beta$ for $\alpha \neq \beta$, $\vartheta_1, .. , \vartheta_r \in E$ such that
$K(n+1; h; w_1, .. , w_r; \vartheta_1, .. , \vartheta_r)$. Therefore $h = (w_1, \vartheta_1) \parallel .. \parallel (w_r,\vartheta_r)$, $i \le r$, for each j=1..i $w_j=x_j$, $\vartheta_j=\varphi_j$.

If i<r then we assume for each j = i+1 .. r

- $w_j \notin V_b(t)$
- $V_b(t) \cap V_b(\vartheta_j) = \emptyset$.

We also assume $\varphi \in E(n+1,h)$.

We need to show that $\varphi_k\{x_i/t\} = \varphi_h\{x_i/t\}$. We have just seen that one of the following five conditions holds:

a1) $\varphi \in C \wedge \varphi_k\{x_i/t\} = \varphi$,

a2) $\varphi \in dom(k) \wedge (\varphi=x_i \rightarrow \varphi_k\{x_i/t\} = t) \wedge (\varphi \neq x_i \rightarrow \varphi_k\{x_i/t\} = \varphi)$

a3) there exist $\kappa \in K(n)$ such that $\kappa \sqsubseteq k$ and $\psi, \psi_1, \dots , \psi_m \in E(n,\kappa)$ such that
$\varphi = (\psi)(\psi_1, \dots , \psi_m)$, $\varphi \in E(n+1,\kappa)$,
for each $\rho \in \Xi(\kappa)$ $\#(\kappa,\psi,\rho)$ is a function with m arguments and
$(\#(\kappa, \psi_1, \rho), ... , \#(\kappa, \psi_m, \rho))$ is a member of its domain,
if $x_i \in dom(\kappa)$ then $\varphi_k\{x_i/t\} = (\psi_\kappa\{x_i/t\}) ((\psi_1)_\kappa\{x_i/t\}, \dots , (\psi_m)_\kappa\{x_i/t\})$,
else $\varphi_k\{x_i/t\} = \varphi$.

a4) there exist $\kappa \in K(n)$ such that $\kappa \sqsubseteq k$ and $f \in F$, $\psi_1, \dots , \psi_m \in E(n,\kappa)$ such that
$\varphi = (f)(\psi_1, \dots , \psi_m)$, $\varphi \in E(n+1,\kappa)$,
for each $\rho \in \Xi(\kappa)$ $A_f(\#(\kappa, \psi_1, \rho), ... , \#(\kappa, \psi_m, \rho))$ is true,
if $x_i \in dom(\kappa)$ then $\varphi_k\{x_i/t\} = (f) ((\psi_1)_\kappa\{x_i/t\}, \dots , (\psi_m)_\kappa\{x_i/t\})$,
else $\varphi_k\{x_i/t\} = \varphi$.

a5) there exist $\kappa \in K(n)$ such that $\kappa \sqsubseteq k$ and $\psi, \psi_1, .. , \psi_m \in E(n)$,
$y_1, \dots , y_m$ distinct $\in V\text{-}dom(\kappa)$ such that
$\varphi = \{\}(y_1:\psi_1, \dots , y_m:\psi_m, \psi)$, $\varphi \in E(n+1,\kappa)$,

$\psi_1 \in E(n,\kappa)$, for each $\sigma \in \Xi(\kappa)$ $\#(\kappa,\psi_1,\sigma)$ is a set ;
if m>1 then for each j=1..m-1 we define $\kappa'_j = \kappa \| (y_1,\psi_1) \| \dots \| (y_j,\psi_j)$ and we have $\kappa'_j \in K(n)$, $\psi_{j+1} \in E(n, \kappa'_j)$, for each $\sigma'_j \in \Xi(\kappa'_j)$ $\#(\kappa'_j, \psi_{j+1}, \sigma'_j)$ is a set;
if we define $\kappa'_m = \kappa \| (y_1,\psi_1) \| \dots \| (y_m,\psi_m)$ then $\kappa'_m \in K(n) \wedge \psi \in E(n,\kappa'_m)$;

if $x_i \in dom(\kappa)$ then we can observe that

$\psi_1 \in E(n+1,\kappa)$, $V_b(t) \cap V_b(\psi_1) \subseteq V_b(t) \cap V_b(\varphi) = \emptyset$, therefore $(\psi_1)_\kappa\{x_i/t\}$ is defined;

for each j=1..m-1 $\psi_{j+1} \in E(n+1, \kappa'_j)$, for each α=1..j $y_\alpha \in V_b(\varphi)$ so $y_\alpha \notin V_b(t)$,
$V_b(t) \cap V_b(\psi_\alpha) \subseteq V_b(t) \cap V_b(\varphi) = \emptyset$, $V_b(t) \cap V_b(\psi_{j+1}) \subseteq V_b(t) \cap V_b(\varphi) = \emptyset$, therefore $(\psi_{j+1})_{\kappa'(j)}\{x_i/t\}$ is defined ;

$\psi \in E(n+1, \kappa'_m)$, for each α=1..m $y_\alpha \in V_b(\varphi)$ so $y_\alpha \notin V_b(t)$,
$V_b(t) \cap V_b(\psi_\alpha) \subseteq V_b(t) \cap V_b(\varphi) = \emptyset$, $V_b(t) \cap V_b(\psi) \subseteq V_b(t) \cap V_b(\varphi) = \emptyset$, therefore $\psi_{\kappa'(m)}\{x_i/t\}$ is defined ;

it results $\varphi_k\{x_i/t\}$ =
$= \{\}(y_1: (\psi_1)_\kappa\{x_i/t\}, y_2: (\psi_2)_{\kappa'(1)}\{x_i/t\} , \dots , y_m: (\psi_m)_{\kappa'(m-1)}\{x_i/t\}, \psi_{\kappa'(m)}\{x_i/t\})$;

if $x_i \notin dom(\kappa)$ then $\varphi_k\{x_i/t\} = \varphi$.

Given that $\varphi \in E(n+1,h)$ we also have to accept one of the following five conditions holds

e1) $\varphi \in C \wedge \varphi_h\{x_i/t\} = \varphi$,

e2) $\varphi \in dom(h) \wedge (\varphi = x_i \rightarrow \varphi_h\{x_i/t\} = t) \wedge (\varphi \neq x_i \rightarrow \varphi_h\{x_i/t\} = \varphi)$

e3) there exist $\eta \in K(n)$ such that $\eta \sqsubseteq h$ and $\chi, \chi_1, \dots , \chi_q \in E(n,\eta)$ such that
$\varphi = (\chi)(\chi_1, \dots , \chi_q)$, $\varphi \in E(n+1,\eta)$,
for each $\rho \in \Xi(\eta)$ $\#(\eta,\chi,\rho)$ is a function with m arguments and
$(\#(\eta, \chi_1, \rho), \dots , \#(\eta, \chi_q, \rho))$ is a member of its domain,
if $x_i \in dom(\eta)$ then $\varphi_h\{x_i/t\} = (\chi_\eta\{x_i/t\}) ((\chi_1)_\eta\{x_i/t\}, \dots , (\chi_q)_\eta\{x_i/t\})$,
else $\varphi_h\{x_i/t\} = \varphi$.

e4) there exist $\eta \in K(n)$ such that $\eta \sqsubseteq h$ and $g \in F$, $\chi_1, \dots , \chi_q \in E(n,\eta)$ such that
$\varphi = (g)(\chi_1, \dots , \chi_q)$, $\varphi \in E(n+1,\eta)$,
for each $\rho \in \Xi(\eta)$ $A_g(\#(\eta, \chi_1, \rho), \dots , \#(\eta, \chi_q, \rho))$ is true,
if $x_i \in dom(\eta)$ then $\varphi_h\{x_i/t\} = (g) ((\chi_1)_\eta\{x_i/t\}, \dots , (\chi_q)_\eta\{x_i/t\})$,
else $\varphi_h\{x_i/t\} = \varphi$.

e5) there exist $\eta \in K(n)$ such that $\eta \sqsubseteq h$ and $\chi, \chi_1, \dots , \chi_q \in E(n)$,
$z_1, \dots , z_q$ distinct $\in V$-dom(η) such that
$\varphi = \{\}(z_1:\chi_1, \dots , z_q:\chi_q, \chi)$, $\varphi \in E(n+1,\eta)$,

$\chi_1 \in E(n,\eta)$, for each $\sigma \in \Xi(\eta)$ $\#(\eta,\chi_1,\sigma)$ is a set ;

if $q>1$ then for each $j=1..q-1$ we define $\eta'_j = \eta \| (z_1,\chi_1) \| .. \| (z_j,\chi_j)$ and we have $\eta'_j \in K(n)$, $\chi_{j+1} \in E(n, \eta'_j)$, for each $\sigma'_j \in \Xi(\eta'_j)$ $\#(\eta'_j, \chi_{j+1}, \sigma'_j)$ is a set;
if we define $\eta'_q = \eta \| (z_1,\chi_1) \| .. \| (z_q,\chi_q)$ then $\eta'_q \in K(n) \wedge \chi \in E(n,\eta'_q)$;

if $x_i \in dom(\eta)$ then we can observe that

$\chi_1 \in E(n+1,\eta)$, $V_b(t) \cap V_b(\chi_1) \subseteq V_b(t) \cap V_b(\varphi) = \varnothing$, therefore $(\psi_1)_\eta\{x_i/t\}$ is defined;

for each $j=1..q-1$ $\chi_{j+1} \in E(n+1, \eta'_j)$, for each $\alpha=1..j$ $z_\alpha \in V_b(\varphi)$ so $z_\alpha \notin V_b(t)$, $V_b(t) \cap V_b(\chi_\alpha) \subseteq V_b(t) \cap V_b(\varphi) = \varnothing$, $V_b(t) \cap V_b(\chi_{j+1}) \subseteq V_b(t) \cap V_b(\varphi) = \varnothing$, therefore $(\chi_{j+1})_{\eta'(j)}\{x_i/t\}$ is defined ;

$\chi \in E(n+1, \eta'_q)$, for each $\alpha=1..q$ $z_\alpha \in V_b(\varphi)$ so $z_\alpha \notin V_b(t)$, $V_b(t) \cap V_b(\chi_\alpha) \subseteq V_b(t) \cap V_b(\varphi) = \varnothing$, $V_b(t) \cap V_b(\chi) \subseteq V_b(t) \cap V_b(\varphi) = \varnothing$, therefore $\chi_{\eta'(q)}\{x_i/t\}$ is defined ;

it results $\varphi_h\{x_i/t\} =$
$= \{\}(z_1: (\chi_1)_\eta\{x_i/t\}, z_2: (\chi_2)_{\eta'(1)}\{x_i/t\} , \dots , z_q: (\chi_q)_{\eta'(q-1)}\{x_i/t\}, \chi_{\eta'(q)}\{x_i/t\})$;

if $x_i \notin dom(\eta)$ then $\varphi_h\{x_i/t\} = \varphi$.

If **a1)** occurs then e1) also occurs. So $\varphi_k\{x_i/t\} = \varphi = \varphi_h\{x_i/t\}$.

If **a2)** occurs then e2) also occurs. So $\varphi=x_i \rightarrow \varphi_k\{x_i/t\} = t = \varphi_h\{x_i/t\}$; $\varphi \neq x_i \rightarrow \varphi_k\{x_i/t\} = \varphi = \varphi_h\{x_i/t\}$.

We now consider the case where **a3)** occurs. In this case e3) also occurs.

There exists $u=1..p$ such that $\kappa = (x_1,\varphi_1) \| .. \| (x_u,\varphi_u)$,
and there exists $v=1..r$ such that $\eta = (w_1, \vartheta_1) \| .. \| (w_v,\vartheta_v)$.

We have

$(\chi)(\chi_1, \dots , \chi_q) = \varphi = (\psi)(\psi_1, \dots , \psi_m)$,

and therefore $\chi = \psi$, $q = m$, for each $j=1..m$ $\chi_j = \psi_j$.

We have to distinguish the following cases:

- $x_i \in dom(\kappa) \wedge x_i \in dom(\eta)$
- $x_i \in dom(\kappa) \wedge x_i \notin dom(\eta)$
- $x_i \notin dom(\kappa) \wedge x_i \in dom(\eta)$
- $x_i \notin dom(\kappa) \wedge x_i \notin dom(\eta)$.

Suppose $x_i \in dom(\kappa) \wedge x_i \in dom(\eta)$. In this case:

$\varphi_k\{x_i/t\} = (\psi_\kappa\{x_i/t\}) ((\psi_1)_\kappa\{x_i/t\}, \dots , (\psi_m)_\kappa\{x_i/t\})$,

$\varphi_h\{x_i/t\} = (\psi_\eta\{x_i/t\})\ ((\psi_1)_\eta\{x_i/t\}, \dots, (\psi_m)_\eta\{x_i/t\})$.

Clearly $\kappa\in K(n)$ and $u\geq i$, $\psi\in E(n,\kappa)$, $V_b(t) \cap V_b(\psi) = \emptyset$, $\eta\in K(n)$, $\eta\sqsubseteq h$, $k_i\sqsubseteq h$, $x_i\in dom(\eta)$, therefore $\kappa_i = k_i \sqsubseteq \eta$. It results $v\geq i$ and for each $j = i+1 .. v$

- $w_j\notin V_b(t)$
- $V_b(t) \cap V_b(\vartheta_j) = \emptyset$.

Moreover $\psi\in E(n,\eta)$ and therefore, by the inductive hypothesis, $\psi_\kappa\{x_i/t\} = \psi_\eta\{x_i/t\}$.

Given $\alpha=1..m$ we can add the observations that $\psi_\alpha\in E(n,\kappa)$, $V_b(t) \cap V_b(\psi_\alpha) = \emptyset$, $\psi_\alpha\in E(n,\eta)$.

Therefore $(\psi_\alpha)_\kappa\{x_i/t\} = (\psi_\alpha)_\eta\{x_i/t\}$, and

$\varphi_k\{x_i/t\} = (\psi_\kappa\{x_i/t\})\ ((\psi_1)_\kappa\{x_i/t\}, \dots, (\psi_m)_\kappa\{x_i/t\}) = (\psi_\eta\{x_i/t\})\ ((\psi_1)_\eta\{x_i/t\}, \dots, (\psi_m)_\eta\{x_i/t\}) =$

$= \varphi_h\{x_i/t\}$.

Now consider the case where $x_i\in dom(\kappa) \wedge x_i\notin dom(\eta)$. In this case:

$\varphi_k\{x_i/t\} = (\psi_\kappa\{x_i/t\})\ ((\psi_1)_\kappa\{x_i/t\}, \dots, (\psi_m)_\kappa\{x_i/t\})$,

$\varphi_h\{x_i/t\} = \varphi$.

We have $\kappa\in K(n)$ and $u\geq i$, $\psi\in E(n,\kappa)$, $V_b(t) \cap V_b(\psi) = \emptyset$, $\eta\in K(n)$, $\psi\in E(n,\eta)$, $x_i\notin dom(\eta)$. Therefore by the inductive hypothesis $\psi_\kappa\{x_i/t\} = \psi$.

Given $\alpha=1..m$ we can add the observations that $\psi_\alpha\in E(n,\kappa)$, $V_b(t) \cap V_b(\psi_\alpha) = \emptyset$, $\psi_\alpha\in E(n,\eta)$. Therefore $(\psi_\alpha)_\kappa\{x_i/t\} = \psi_\alpha$.

So the conclusion is

$\varphi_k\{x_i/t\} = (\psi_\kappa\{x_i/t\})\ ((\psi_1)_\kappa\{x_i/t\}, \dots, (\psi_m)_\kappa\{x_i/t\}) = (\psi)(\psi_1, \dots, \psi_m) = \varphi = \varphi_h\{x_i/t\}$.

We turn to the case where $x_i\notin dom(\kappa) \wedge x_i\in dom(\eta)$. This is similar to the former one, in fact

$\varphi_k\{x_i/t\} = \varphi$,

$\varphi_h\{x_i/t\} = (\psi_\eta\{x_i/t\})\ ((\psi_1)_\eta\{x_i/t\}, \dots, (\psi_m)_\eta\{x_i/t\})$.

We have $\eta\in K(n)$. Since $\eta\sqsubseteq h$, $k_i\sqsubseteq h$, $x_i\in dom(\eta)$ it results $k_i \sqsubseteq \eta$ and $v\geq i$, for each $j=1..i$ $w_j=x_j$, $\vartheta_j=\varphi_j$, and for each $j=i+1 .. r$ $w_j\notin V_b(t)$, $V_b(t) \cap V_b(\vartheta_j) = \emptyset$.

We have $\eta_{i-1} = k_{i-1}$, so

- $\forall\rho_{i-1}\in \Xi(\eta_{i-1})\ \#(\eta_{i-1},t,\rho_{i-1}) \in \#(\eta_{i-1},\vartheta_i,\rho_{i-1})$,
- $\forall j=1..v$: $j\neq i$ $w_j\notin V_b(t)$
- $\forall j=i+1..v$ $V_b(t) \cap V_b(\vartheta_j) = \emptyset$.

We have $\psi\in E(n,\eta)$, $V_b(t) \cap V_b(\psi) = \emptyset$, $\psi\in E(n,\kappa)$, $x_i\notin dom(\kappa)$. So $\psi_\eta\{x_i/t\} = \psi$.

Given $\alpha=1..m$ we can add the observations that $\psi_\alpha \in E(n,\eta)$, $V_b(t) \cap V_b(\psi_\alpha) = \varnothing$, $\psi_\alpha \in E(n,\kappa)$. Therefore $(\psi_\alpha)_\eta\{x_i/t\} = \psi_\alpha$.

So the conclusion is

$\varphi_h\{x_i/t\} = (\psi_\eta\{x_i/t\})\ ((\psi_1)_\eta\{x_i/t\}, \ldots , (\psi_m)_\eta\{x_i/t\}) = (\psi)(\psi_1, \ldots , \psi_m) = \varphi = \varphi_k\{x_i/t\}$.

Finally let's see what happens when $x_i \notin dom(\kappa) \wedge x_i \notin dom(\eta)$.

It's the simpler case, in fact $\varphi_k\{x_i/t\} = \varphi = \varphi_h\{x_i/t\}$.

We turn to the case where **a4)** holds, and accordingly e4) holds too.

There exists $u=1..p$ such that $\kappa = (x_1,\varphi_1) \parallel .. \parallel (x_u,\varphi_u)$,
and there exists $v=1..r$ such that $\eta = (w_1, \vartheta_1) \parallel .. \parallel (w_v,\vartheta_v)$.

We have

$(g)(\chi_1, \ldots , \chi_q) = \varphi = (f)(\psi_1, \ldots , \psi_m)$,

and therefore $g = f$, $q = m$, for each $j=1..m$ $\chi_j = \psi_j$.

We have to distinguish the following cases:

- $x_i \in dom(\kappa) \wedge x_i \in dom(\eta)$
- $x_i \in dom(\kappa) \wedge x_i \notin dom(\eta)$
- $x_i \notin dom(\kappa) \wedge x_i \in dom(\eta)$
- $x_i \notin dom(\kappa) \wedge x_i \notin dom(\eta)$.

Suppose $x_i \in dom(\kappa) \wedge x_i \in dom(\eta)$. In this case:

$\varphi_k\{x_i/t\} = (f)\ ((\psi_1)_\kappa\{x_i/t\}, \ldots , (\psi_m)_\kappa\{x_i/t\})$,

$\varphi_h\{x_i/t\} = (f)\ ((\psi_1)_\eta\{x_i/t\}, \ldots , (\psi_m)_\eta\{x_i/t\})$.

Let $\alpha=1..m$, in order to show that $(\psi_\alpha)_\kappa\{x_i/t\} = (\psi_\alpha)_\eta\{x_i/t\}$ we consider that $\kappa \in K(n)$ and $u \geq i$, $\psi_\alpha \in E(n,\kappa)$, $V_b(t) \cap V_b(\psi_\alpha) = \varnothing$, moreover $\eta \in K(n)$, $\eta \sqsubseteq h$, $k_i \sqsubseteq h$, $x_i \in dom(\eta)$, therefore $\kappa_i = k_i \sqsubseteq \eta$. It results $v \geq i$ and for each $j = i+1\ ..\ v$

- $w_j \notin V_b(t)$
- $V_b(t) \cap V_b(\vartheta_j) = \varnothing$;

and we have also $\psi_\alpha \in E(n,\eta)$. Therefore $(\psi_\alpha)_\kappa\{x_i/t\} = (\psi_\alpha)_\eta\{x_i/t\}$ and $\varphi_k\{x_i/t\} = \varphi_h\{x_i/t\}$.

Now consider the case where $x_i \in dom(\kappa) \wedge x_i \notin dom(\eta)$. In this situation:

$\varphi_k\{x_i/t\} = (f)\ ((\psi_1)_\kappa\{x_i/t\}, \ldots , (\psi_m)_\kappa\{x_i/t\})$,

$\varphi_h\{x_i/t\} = \varphi = (f)(\psi_1, \dots, \psi_m)$.

Let α=1..m, in order to show that $(\psi_\alpha)_\kappa\{x_i/t\} = \psi_\alpha$ we consider that $\kappa \in K(n)$ and $u \geq i$, $\psi_\alpha \in E(n,\kappa)$, $V_b(t) \cap V_b(\psi_\alpha) = \emptyset$, and moreover $\eta \in K(n)$, $x_i \notin dom(\eta)$, $\psi_\alpha \in E(n,\eta)$. This clearly implies that $(\psi_\alpha)_\kappa\{x_i/t\} = \psi_\alpha$, and therefore

$\varphi_k\{x_i/t\} = (f)\,((\psi_1)_\kappa\{x_i/t\}, \dots, (\psi_m)_\kappa\{x_i/t\}) = (f)(\psi_1, \dots, \psi_m) = \varphi_h\{x_i/t\}$.

We turn to the case where $x_i \notin dom(\kappa) \wedge x_i \in dom(\eta)$. This is similar to the former one, in fact

$\varphi_k\{x_i/t\} = \varphi = (f)(\psi_1, \dots, \psi_m)$,

$\varphi_h\{x_i/t\} = (f)\,((\psi_1)_\eta\{x_i/t\}, \dots, (\psi_m)_\eta\{x_i/t\})$.

Let α=1..m, in order to show that $(\psi_\alpha)_\eta\{x_i/t\} = \psi_\alpha$ we consider the following facts.

First of all we have $\eta \in K(n)$. Since $\eta \sqsubseteq h$, $k_i \sqsubseteq h$, $x_i \in dom(\eta)$ it results $k_i \sqsubseteq \eta$ and $v \geq i$, for each j=1..i $w_j = x_j$, $\vartheta_j = \varphi_j$, and for each j=i+1 .. r $w_j \notin V_b(t)$, $V_b(t) \cap V_b(\vartheta_j) = \emptyset$.

We have $\eta_{i-1} = k_{i-1}$, so

- $\forall \rho_{i-1} \in \Xi(\eta_{i-1})\ \#(\eta_{i-1},t,\rho_{i-1}) \in \#(\eta_{i-1},\vartheta_i,\rho_{i-1})$,
- $\forall j$=1..v: $j \neq i$ $w_j \notin V_b(t)$
- $\forall j$=i+1..v $V_b(t) \cap V_b(\vartheta_j) = \emptyset$.

We have $\psi_\alpha \in E(n,\eta)$, $V_b(t) \cap V_b(\psi_\alpha) = \emptyset$, $\psi_\alpha \in E(n,\kappa)$, $x_i \notin dom(\kappa)$. So $(\psi_\alpha)_\eta\{x_i/t\} = \psi_\alpha$.

Therefore we conclude

$\varphi_k\{x_i/t\} = (f)(\psi_1, \dots, \psi_m) = (f)\,((\psi_1)_\eta\{x_i/t\}, \dots, (\psi_m)_\eta\{x_i/t\}) = \varphi_h\{x_i/t\}$.

Finally let's see what happens when $x_i \notin dom(\kappa) \wedge x_i \notin dom(\eta)$.

It's the simpler case, in fact $\varphi_k\{x_i/t\} = \varphi = \varphi_h\{x_i/t\}$.

Finally we examine the case where **a5)** holds, and accordingly e5) also occurs.

There exists u=1..p such that $\kappa = (x_1,\varphi_1) \parallel .. \parallel (x_u,\varphi_u)$,
and there exists v=1..r such that $\eta = (w_1, \vartheta_1) \parallel .. \parallel (w_v,\vartheta_v)$.

We have

$\{\}(z_1{:}\chi_1, \dots, z_q{:}\chi_q, \chi) = \varphi = \{\}(y_1{:}\psi_1, \dots, y_m{:}\psi_m, \psi)$,

therefore

$q = m$; for each j=1..m $\chi_j = \psi_j$, $z_j = y_j$; $\chi = \psi$.

We have to distinguish the following cases:

- $x_i \in dom(\kappa) \wedge x_i \in dom(\eta)$
- $x_i \in dom(\kappa) \wedge x_i \notin dom(\eta)$
- $x_i \notin dom(\kappa) \wedge x_i \in dom(\eta)$
- $x_i \notin dom(\kappa) \wedge x_i \notin dom(\eta)$.

Suppose $x_i \in dom(\kappa) \wedge x_i \in dom(\eta)$. In this case:

$\varphi_k\{x_i/t\} = \{\}(y_1: (\psi_1)_\kappa\{x_i/t\}, y_2: (\psi_2)_{\kappa'(1)}\{x_i/t\} , \ldots , y_m: (\psi_m)_{\kappa'(m-1)}\{x_i/t\}, \psi_{\kappa'(m)}\{x_i/t\})$;

$\varphi_h\{x_i/t\} = \{\}(y_1: (\psi_1)_\eta\{x_i/t\}, y_2: (\psi_2)_{\eta'(1)}\{x_i/t\} , \ldots , y_m: (\psi_m)_{\eta'(m-1)}\{x_i/t\}, \psi_{\eta'(m)}\{x_i/t\})$.

Clearly $\kappa \in K(n)$ and $u \geq i$, $\psi_1 \in E(n,\kappa)$, $V_b(t) \cap V_b(\psi_1) = \varnothing$, $\eta \in K(n)$, $\eta \sqsubseteq h$, $k_i \sqsubseteq h$, $x_i \in dom(\eta)$, therefore $\kappa_i = k_i \sqsubseteq \eta$. It results $v \geq i$ and for each $j = i+1 .. v$

- $w_j \notin V_b(t)$
- $V_b(t) \cap V_b(\vartheta_j) = \varnothing$.

Moreover $\psi_1 \in E(n,\eta)$ and therefore, by the inductive hypothesis, $(\psi_1)_\kappa\{x_i/t\} = (\psi_1)_\eta\{x_i/t\}$.

Suppose $m>1$ and let $j=1..m-1$. It results $\kappa'_j \in K(n)$, for each $\alpha=1..j$ $y_\alpha \in V_b(\varphi)$ so $y_\alpha \notin V_b(t)$, $V_b(t) \cap V_b(\psi_\alpha) \subseteq V_b(t) \cap V_b(\varphi) = \varnothing$, $\psi_{j+1} \in E(n, \kappa'_j)$, $V_b(t) \cap V_b(\psi_{j+1}) = \varnothing$.
Moreover $\eta'_j \in K(n)$, $(\kappa'_j)_i = k_i \sqsubseteq \eta \sqsubseteq \eta'_j$. As we've just seen for each $\alpha=1..j$ $y_\alpha \notin V_b(t)$ and $V_b(t) \cap V_b(\psi_\alpha) = \varnothing$, and $\psi_{j+1} \in E(n, \eta'_j)$ also holds. Therefore $(\psi_{j+1})_{\kappa'(j)}\{x_i/t\} = (\psi_{j+1})_{\eta'(j)}\{x_i/t\}$.

We still need to show that $\psi_{\kappa'(m)}\{x_i/t\} = \psi_{\eta'(m)}\{x_i/t\}$.

To this end we see that $\kappa'_m \in K(n)$, for each $\alpha=1..m$ $y_\alpha \in V_b(\varphi)$ so $y_\alpha \notin V_b(t)$,
$V_b(t) \cap V_b(\psi_\alpha) \subseteq V_b(t) \cap V_b(\varphi) = \varnothing$, $\psi \in E(n, \kappa'_m)$, $V_b(t) \cap V_b(\psi) = \varnothing$.

Moreover $\eta'_m \in K(n)$, $(\kappa'_m)_i = k_i \sqsubseteq \eta \sqsubseteq \eta'_m$. As we've just seen for each $\alpha=1..m$ $y_\alpha \notin V_b(t)$ and $V_b(t) \cap V_b(\psi_\alpha) = \varnothing$, and $\psi \in E(n, \eta'_m)$ also holds. Therefore $\psi_{\kappa'(m)}\{x_i/t\} = \psi_{\eta'(m)}\{x_i/t\}$.

Finally we can establish

$\varphi_k\{x_i/t\} = \{\}(y_1: (\psi_1)_\kappa\{x_i/t\}, y_2: (\psi_2)_{\kappa'(1)}\{x_i/t\} , \ldots , y_m: (\psi_m)_{\kappa'(m-1)}\{x_i/t\}, \psi_{\kappa'(m)}\{x_i/t\}) =$
$= \{\}(y_1: (\psi_1)_\eta\{x_i/t\}, y_2: (\psi_2)_{\eta'(1)}\{x_i/t\} , \ldots , y_m: (\psi_m)_{\eta'(m-1)}\{x_i/t\}, \psi_{\eta'(m)}\{x_i/t\}) = \varphi_h\{x_i/t\}$.

Now consider the case where $x_i \in dom(\kappa) \wedge x_i \notin dom(\eta)$. In this case:

$\varphi_k\{x_i/t\} = \{\}(y_1: (\psi_1)_\kappa\{x_i/t\}, y_2: (\psi_2)_{\kappa'(1)}\{x_i/t\} , \ldots , y_m: (\psi_m)_{\kappa'(m-1)}\{x_i/t\}, \psi_{\kappa'(m)}\{x_i/t\})$;

$\varphi_h\{x_i/t\} = \varphi = \{\}(y_1:\psi_1, \ldots , y_m:\psi_m, \psi)$.

Clearly $\kappa \in K(n)$ and $u \geq i$, $\psi_1 \in E(n,\kappa)$, $V_b(t) \cap V_b(\psi_1) = \varnothing$. Moreover $\eta \in K(n)$, $\psi_1 \in E(n,\eta)$, $x_i \notin dom(\eta)$. This implies $(\psi_1)_\kappa\{x_i/t\} = \psi_1$.

Suppose $m>1$ and let $j=1..m-1$. It results $\kappa'_j \in K(n)$, for each $\alpha=1..j$ $y_\alpha \in V_b(\varphi)$ so $y_\alpha \notin V_b(t)$, $V_b(t) \cap V_b(\psi_\alpha) \subseteq V_b(t) \cap V_b(\varphi) = \varnothing$, $\psi_{j+1} \in E(n, \kappa'_j)$, $V_b(t) \cap V_b(\psi_{j+1}) = \varnothing$.
Moreover $\eta'_j \in K(n)$, $\psi_{j+1} \in E(n, \eta'_j)$, $x_i \notin dom(\eta'_j)$. This implies $(\psi_{j+1})_{\kappa'(j)}\{x_i/t\} = \psi_{j+1}$.

It also results $\kappa'_m \in K(n)$, for each α=1..m $y_\alpha \in V_b(\varphi)$ so $y_\alpha \notin V_b(t)$,
$V_b(t) \cap V_b(\psi_\alpha) \subseteq V_b(t) \cap V_b(\varphi) = \varnothing$, $\psi \in E(n, \kappa'_m)$, $V_b(t) \cap V_b(\psi) = \varnothing$.
Moreover $\eta'_m \in K(n)$, $\psi \in E(n, \eta'_m)$, $x_i \notin dom(\eta'_m)$. This implies $\psi_{\kappa'(m)}\{x_i/t\} = \psi$.

Therefore

$\varphi_k\{x_i/t\} = \{\}(y_1: (\psi_1)_\kappa\{x_i/t\}, y_2: (\psi_2)_{\kappa'(1)}\{x_i/t\} , \ldots , y_m: (\psi_m)_{\kappa'(m-1)}\{x_i/t\}, \psi_{\kappa'(m)}\{x_i/t\}) =$

$= \{\}(y_1:\psi_1, \ldots , y_m:\psi_m, \psi) = \varphi = \varphi_h\{x_i/t\}$.

We turn to the case where $x_i \notin dom(\kappa) \wedge x_i \in dom(\eta)$. This is similar to the former one, in fact

$\varphi_k\{x_i/t\} = \varphi = \{\}(y_1:\psi_1, \ldots , y_m:\psi_m, \psi)$;

$\varphi_h\{x_i/t\} = \{\}(y_1: (\psi_1)_\eta\{x_i/t\}, y_2: (\psi_2)_{\eta'(1)}\{x_i/t\} , \ldots , y_m: (\psi_m)_{\eta'(m-1)}\{x_i/t\}, \psi_{\eta'(m)}\{x_i/t\})$.

Clearly $\eta \in K(n)$ and $v \geq i$, $\psi_1 \in E(n,\eta)$, $V_b(t) \cap V_b(\psi_1) = \varnothing$. Moreover $\kappa \in K(n)$, $\psi_1 \in E(n,\kappa)$, $x_i \notin dom(\kappa)$. This implies $(\psi_1)_\eta\{x_i/t\} = \psi_1$.

Suppose m>1 and let j=1..m-1. It results $\eta'_j \in K(n)$, for each α=1..j $y_\alpha \in V_b(\varphi)$ so $y_\alpha \notin V_b(t)$, $V_b(t) \cap V_b(\psi_\alpha) \subseteq V_b(t) \cap V_b(\varphi) = \varnothing$, $\psi_{j+1} \in E(n, \eta'_j)$, $V_b(t) \cap V_b(\psi_{j+1}) = \varnothing$.
Moreover $\kappa'_j \in K(n)$, $\psi_{j+1} \in E(n, \kappa'_j)$, $x_i \notin dom(\kappa'_j)$. This implies $(\psi_{j+1})_{\eta'(j)}\{x_i/t\} = \psi_{j+1}$.

It also results $\eta'_m \in K(n)$, for each α=1..m $y_\alpha \in V_b(\varphi)$ so $y_\alpha \notin V_b(t)$,
$V_b(t) \cap V_b(\psi_\alpha) \subseteq V_b(t) \cap V_b(\varphi) = \varnothing$, $\psi \in E(n, \eta'_m)$, $V_b(t) \cap V_b(\psi) = \varnothing$.
Moreover $\kappa'_m \in K(n)$, $\psi \in E(n, \kappa'_m)$, $x_i \notin dom(\kappa'_m)$. This implies $\psi_{\eta'(m)}\{x_i/t\} = \psi$.

Therefore

$\varphi_k\{x_i/t\} = \{\}(y_1:\psi_1, \ldots , y_m:\psi_m, \psi) =$

$= \{\}(y_1: (\psi_1)_\eta\{x_i/t\}, y_2: (\psi_2)_{\eta'(1)}\{x_i/t\} , \ldots , y_m: (\psi_m)_{\eta'(m-1)}\{x_i/t\}, \psi_{\eta'(m)}\{x_i/t\}) = \varphi_h\{x_i/t\}$.

Finally let's see what happens when $x_i \notin dom(\kappa) \wedge x_i \notin dom(\eta)$.

It's the simpler case, in fact $\varphi_k\{x_i/t\} = \varphi = \varphi_h\{x_i/t\}$.

Our definition process requires just a final step. This consists in proving that if there exists $h \in K(n+1)$ such that $\varphi \in E(n+1,h)$, $x_i \notin dom(h)$ then it results $\varphi_k\{x_i/t\} = \varphi$.

Because of $\varphi \in E(n+1,k)$, and φ is such that $V_b(t) \cap V_b(\varphi) = \varnothing$, one of the following five conditions holds:

a1) $\varphi \in \mathcal{C} \wedge \varphi_k\{x_i/t\} = \varphi$,

a2) $\varphi \in dom(k) \wedge (\varphi = x_i \rightarrow \varphi_k\{x_i/t\} = t) \wedge (\varphi \neq x_i \rightarrow \varphi_k\{x_i/t\} = \varphi)$

a3) there exist $\kappa \in K(n)$ such that $\kappa \sqsubseteq k$ and $\psi, \psi_1, \dots, \psi_m \in E(n,\kappa)$ such that

$\varphi = (\psi)(\psi_1, \dots, \psi_m)$, $\varphi \in E(n+1,\kappa)$,

for each $\rho \in \Xi(\kappa)$ $\#(\kappa,\psi,\rho)$ is a function with m arguments and

$(\#(\kappa, \psi_1, \rho), \dots, \#(\kappa, \psi_m, \rho))$ is a member of its domain,

if $x_i \in dom(\kappa)$ then $\varphi_k\{x_i/t\} = (\psi_\kappa\{x_i/t\}) ((\psi_1)_\kappa\{x_i/t\}, \dots, (\psi_m)_\kappa\{x_i/t\})$,

else $\varphi_k\{x_i/t\} = \varphi$.

a4) there exist $\kappa \in K(n)$ such that $\kappa \sqsubseteq k$ and $f \in F$, $\psi_1, \dots, \psi_m \in E(n,\kappa)$ such that

$\varphi = (f)(\psi_1, \dots, \psi_m)$, $\varphi \in E(n+1,\kappa)$,

for each $\rho \in \Xi(\kappa)$ $A_f(\#(\kappa, \psi_1, \rho), \dots, \#(\kappa, \psi_m, \rho))$ is true,

if $x_i \in dom(\kappa)$ then $\varphi_k\{x_i/t\} = (f) ((\psi_1)_\kappa\{x_i/t\}, \dots, (\psi_m)_\kappa\{x_i/t\})$

else $\varphi_k\{x_i/t\} = \varphi$.

a5) there exist $\kappa \in K(n)$ such that $\kappa \sqsubseteq k$ and $\psi, \psi_1, .., \psi_m \in E(n)$,

$y_1, \dots, y_m$ distinct $\in V$-dom(κ) such that

$\varphi = \{\}(y_1{:}\psi_1, \dots, y_m{:}\psi_m, \psi)$, $\varphi \in E(n+1,\kappa)$,

$\psi_1 \in E(n,\kappa)$, for each $\sigma \in \Xi(\kappa)$ $\#(\kappa,\psi_1,\sigma)$ is a set ;

if m>1 then for each j=1..m-1 we define $\kappa'_j = \kappa \| (y_1,\psi_1) \| .. \| (y_j,\psi_j)$ and we have $\kappa'_j \in K(n)$, $\psi_{j+1} \in E(n, \kappa'_j)$, for each $\sigma'_j \in \Xi(\kappa'_j)$ $\#(\kappa'_j, \psi_{j+1}, \sigma'_j)$ is a set;

if we define $\kappa'_m = \kappa \| (y_1,\psi_1) \| .. \| (y_m,\psi_m)$ then $\kappa'_m \in K(n) \wedge \psi \in E(n,\kappa'_m)$;

if $x_i \in dom(\kappa)$ then we can observe that

$\psi_1 \in E(n+1,\kappa)$, $V_b(t) \cap V_b(\psi_1) \subseteq V_b(t) \cap V_b(\varphi) = \varnothing$, therefore $(\psi_1)_\kappa\{x_i/t\}$ is defined;

for each j=1..m-1 $\psi_{j+1} \in E(n+1, \kappa'_j)$, for each α=1..j $y_\alpha \in V_b(\varphi)$ so $y_\alpha \notin V_b(t)$, $V_b(t) \cap V_b(\psi_\alpha) \subseteq V_b(t) \cap V_b(\varphi) = \varnothing$, $V_b(t) \cap V_b(\psi_{j+1}) \subseteq V_b(t) \cap V_b(\varphi) = \varnothing$, therefore $(\psi_{j+1})_{\kappa'(j)}\{x_i/t\}$ is defined ;

$\psi \in E(n+1, \kappa'_m)$, for each α=1..m $y_\alpha \in V_b(\varphi)$ so $y_\alpha \notin V_b(t)$,

$V_b(t) \cap V_b(\psi_\alpha) \subseteq V_b(t) \cap V_b(\varphi) = \varnothing$, $V_b(t) \cap V_b(\psi) \subseteq V_b(t) \cap V_b(\varphi) = \varnothing$, therefore $\psi_{\kappa'(m)}\{x_i/t\}$ is defined ;

it results

$\varphi_k\{x_i/t\} = \{\}(y_1{:}\ (\psi_1)_\kappa\{x_i/t\}, y_2{:}\ (\psi_2)_{\kappa'(1)}\{x_i/t\} , \dots, y_m{:}\ (\psi_m)_{\kappa'(m-1)}\{x_i/t\}, \psi_{\kappa'(m)}\{x_i/t\})$;

if $x_i \notin dom(\kappa)$ then $\varphi_k\{x_i/t\} = \varphi$.

Since $\varphi \in E(n+1,h)$, we can apply assumption 2.1.7 to establish that one of the following five conditions holds:

d1) $\varphi \in C$;

d2) $\varphi \in dom(h)$;

d3) $\exists \eta \in K(n)$: $\eta \sqsubseteq h$, $\exists\ \chi, \chi_1, .., \chi_q \in E(n,\eta)$:

$\varphi = (\chi)(\chi_1, \dots, \chi_q)$, $\varphi \in E(n+1,\eta)$,

for each $\rho\in\Xi(\eta)$ $\#(\eta,\chi,\rho)$ is a function with m arguments,
$(\#(\eta, \chi_1, \rho), \ldots, \#(\eta, \chi_q, \rho))$ is a member of its domain;

d4) $\exists\eta\in K(n)$: $\eta \sqsubseteq h$, $\exists\, g\in F$, $\chi_1, .., \chi_q \in E(n,\eta)$:
$\varphi = (g)(\chi_1, \ldots, \chi_q)$, $\varphi\in E(n+1,\eta)$,
for each $\rho\in\Xi(\eta)$ $A_f(\#(\eta, \chi_1, \rho), \ldots, \#(\eta, \chi_q, \rho))$;

d5) $n>1$, $\exists\eta\in K(n)$: $\eta \sqsubseteq h$, $\exists\, \chi, \chi_1, .., \chi_q \in E(n)$,
$\exists z_1, \ldots, z_q$ distinct $\in V\text{-dom}(\eta)$:
$\varphi = \{\}(z_1:\chi_1, \ldots, z_q:\chi_q, \chi)$, $\varphi\in E(n+1,\eta)$,

$\chi_1 \in E(n,\eta)$, for each $\rho\in\Xi(\eta)$ $\#(\eta,\chi_1,\rho)$ is a set ;
if $q>1$ for each $j=1..q-1$ if we define $\eta'_j = \eta\|(z_1,\chi_1)\| .. \|(z_j,\chi_j)$ it follows
$\eta'_j \in K(n) \wedge \chi_{j+1}\in E(n, \eta'_j) \wedge$ for each $\rho'_j\in\Xi(\eta'_j)$ $\#(\eta'_j, \chi_{j+1}, \rho'_j)$ is a set ;
if we define $\eta'_q = \eta\|(z_1,\chi_1)\| .. \|(z_q,\chi_q)$ it follows $\eta'_q\in K(n) \wedge \chi\in E(n,\eta'_q)$.

If **a1)** occurs then clearly $\varphi_k\{x_i/t\} = \varphi$.

If **a2)** occurs then d2) also occurs. If $\varphi=x_i$ then $\varphi\notin \text{dom}(h)$, so we have $\varphi\neq x_i$ and $\varphi_k\{x_i/t\} = \varphi$.

If **a3)** occurs then d3) also occurs. We have

$(\psi)(\psi_1, \ldots, \psi_m) = \varphi = (\chi)(\chi_1, \ldots, \chi_q)$, therefore

$\chi = \psi$, $q=m$, for each $j=1..m$ $\chi_j=\psi_j$.

We have to distinguish the following cases:

- $x_i \in \text{dom}(\kappa)$
- $x_i \notin \text{dom}(\kappa)$

Suppose $x_i \in \text{dom}(\kappa)$. We have

$\varphi_k\{x_i/t\} = (\psi_\kappa\{x_i/t\})\,((\psi_1)_\kappa\{x_i/t\}, \ldots, (\psi_m)_\kappa\{x_i/t\})$,

$\varphi = (\psi)(\psi_1, \ldots, \psi_m)$.

Clearly there exists $u=1..p$ such that $\kappa = (x_1,\varphi_1) \| .. \| (x_u,\varphi_u)$, $u\geq i$, $\psi\in E(n,\kappa)$, $V_b(t) \cap V_b(\psi) = \varnothing$, $\eta\in K(n)$, $\psi\in E(n,\eta)$, $x_i\notin \text{dom}(\eta)$, therefore $\psi_\kappa\{x_i/t\} = \psi$.

Given $\alpha=1..m$ $\psi_\alpha\in E(n,\kappa)$, $V_b(t) \cap V_b(\psi_\alpha) = \varnothing$, $\psi_\alpha\in E(n,\eta)$, therefore $(\psi_\alpha)_\kappa\{x_i/t\} = \psi_\alpha$.

So we conclude

$\varphi_k\{x_i/t\} = (\psi_\kappa\{x_i/t\})\,((\psi_1)_\kappa\{x_i/t\}, \ldots, (\psi_m)_\kappa\{x_i/t\}) = (\psi)(\psi_1, \ldots, \psi_m) = \varphi$.

Now suppose $x_i \notin \text{dom}(\kappa)$. Here it's easier, as we immediately see that $\varphi_k\{x_i/t\} = \varphi$.

If **a4)** occurs then d4) also occurs. We have

$(f)(\psi_1, \dots, \psi_m) = \varphi = (g)(\chi_1, \dots, \chi_q)$, therefore

$g = f$, $q=m$, for each $j=1..m$ $\chi_j=\psi_j$.

We have to distinguish the following cases:

- $x_i \in dom(\kappa)$
- $x_i \notin dom(\kappa)$

Suppose $x_i \in dom(\kappa)$. We have

$\varphi_k\{x_i/t\} = (f)\ ((\psi_1)_\kappa\{x_i/t\}, \dots, (\psi_m)_\kappa\{x_i/t\})$,

$\varphi = (f)(\psi_1, \dots, \psi_m)$.

Let $\alpha = 1..m$. There exists $u=1..p$ such that $\kappa = (x_1,\varphi_1) \parallel .. \parallel (x_u,\varphi_u)$, $u\geq i$, $\psi_\alpha\in E(n,\kappa)$, $V_b(t) \cap V_b(\psi_\alpha) = \varnothing$, $\eta\in K(n)$, $\psi_\alpha\in E(n,\eta)$, $x_i\notin dom(\eta)$, therefore $(\psi_\alpha)_\kappa\{x_i/t\} = \psi_\alpha$.

So we conclude

$\varphi_k\{x_i/t\} = (f)\ ((\psi_1)_\kappa\{x_i/t\}, \dots, (\psi_m)_\kappa\{x_i/t\}) = (f)(\psi_1, \dots, \psi_m) = \varphi$.

Now suppose $x_i \notin dom(\kappa)$. Here it's easier, as we immediately see that $\varphi_k\{x_i/t\} = \varphi$.

If **a5)** occurs then d5) also occurs. We have

$\{\}(y_1{:}\psi_1, \dots, y_m{:}\psi_m, \psi) = \varphi = \{\}(z_1{:}\chi_1, \dots, z_q{:}\chi_q, \chi)$, therefore

$q=m$, for each $j=1..m$ $\chi_j=\psi_j$, $\chi = \psi$.

We have to distinguish the following cases:

- $x_i \in dom(\kappa)$,
- $x_i \notin dom(\kappa)$.

Suppose $x_i \in dom(\kappa)$. We have

$\varphi_k\{x_i/t\} = \{\}(y_1{:}\ (\psi_1)_\kappa\{x_i/t\}, y_2{:}\ (\psi_2)_{\kappa'(1)}\{x_i/t\}\ , \dots, y_m{:}\ (\psi_m)_{\kappa'(m-1)}\{x_i/t\}, \psi_{\kappa'(m)}\{x_i/t\})$;

$\varphi = \{\}(y_1{:}\psi_1, \dots, y_m{:}\psi_m, \psi)$.

There exists $u=1..p$ such that $\kappa = (x_1,\varphi_1) \parallel .. \parallel (x_u,\varphi_u)$, $u\geq i$.

Clearly $\psi_1 \in E(n,\kappa)$, $V_b(t) \cap V_b(\psi_1) = \varnothing$, $\psi_1 \in E(n,\eta)$, $x_i\notin dom(\eta)$, and therefore $(\psi_1)_\kappa\{x_i/t\} = \psi_1$.

Suppose m>1 and let j=1..m-1. We have $\kappa'_j = \kappa \| (y_1,\psi_1) \| \ldots \| (y_j,\psi_j)$, for each α=1..j $y_\alpha \in V_b(\varphi)$ so $y_\alpha \notin V_b(t)$, $V_b(t) \cap V_b(\psi_\alpha) \subseteq V_b(t) \cap V_b(\varphi) = \emptyset$, $\psi_{j+1} \in E(n,\kappa'_j)$, $V_b(t) \cap V_b(\psi_{j+1}) = \emptyset$, $\eta'_j = \eta \| (y_1,\psi_1) \| \ldots \| (y_j,\psi_j) \in K(n)$, $\psi_{j+1} \in E(n, \eta'_j)$, $x_i \notin dom(\eta'_j)$. Therefore $(\psi_{j+1})_{\kappa'(j)}\{x_i/t\} = \psi_{j+1}$.
We have also $\kappa'_m = \kappa \| (y_1,\psi_1) \| \ldots \| (y_m,\psi_m)$, for each α=1..m $y_\alpha \in V_b(\varphi)$ so $y_\alpha \notin V_b(t)$, $V_b(t) \cap V_b(\psi_\alpha) \subseteq V_b(t) \cap V_b(\varphi) = \emptyset$, $\psi \in E(n,\kappa'_m)$, $V_b(t) \cap V_b(\psi) = \emptyset$, $\eta'_m = \eta \| (y_1,\psi_1) \| \ldots \| (y_m,\psi_m)$, $\eta'_m \in K(n)$, $\psi \in E(n, \eta'_m)$, $x_i \notin dom(\eta'_m)$. Therefore $\psi_{\kappa'(m)}\{x_i/t\} = \psi$.

We conclude

$$\varphi_k\{x_i/t\} = \{\}(y_1\text{: } (\psi_1)_\kappa\{x_i/t\},\ y_2\text{: } (\psi_2)_{\kappa'(1)}\{x_i/t\}\ , \ldots ,\ y_m\text{: } (\psi_m)_{\kappa'(m-1)}\{x_i/t\},\ \psi_{\kappa'(m)}\{x_i/t\}) =$$

$$= \{\}(y_1\text{:}\psi_1, \ldots , y_m\text{:}\psi_m, \psi) = \varphi .$$

Now suppose $x_i \notin dom(\kappa)$. Here we immediately see that $\varphi_k\{x_i/t\} = \varphi$.

The final step of our definition process has been completed.

□

5. Proofs and deductive methodology

In section 2 we have seen that our language is identified by a 4-tuple (V, F, C, #). In section 3 we have given some definitions which are important with respect to the deductive metodology. For instance we have defined the set S(k) of sentences with respect to a context k. A sentence with respect to ε can be simply called a 'sentence'.

At this point we need to define what is a proof in our language. To define this we need to define the notions of axiom and rule.

An axiom is a set A such that
- $A \subseteq S(\varepsilon)$
- for each $\varphi \in A$ #(φ) holds .

We'll say that the property 'for each $\varphi \in A$ #(φ) holds' states the 'soundness' of A.

Given a positive integer n we indicate with $S(\varepsilon)^n$ the set of all n-tuples (φ_1, .. , φ_n) for $\varphi_1, .. , \varphi_n \in S(\varepsilon)$. An n-ary rule is a set $R \subseteq S(\varepsilon)^{n+1}$ such that
- for each $(\varphi_1, .. , \varphi_n, \varphi) \in R$ if #(φ_1), …, #(φ_n) hold then #(φ) holds.

We'll say that the property 'for each $(\varphi_1, .. , \varphi_n, \varphi) \in R$ if #(φ_1), …, #(φ_n) hold then #(φ) holds' states the 'soundness' of R.

Both in the definition of axiom and rule we have included a requirement of soundness.

A deductive system is built on top of a language L = (V, F, C, #), and so it is identified by a 3-tuple (L, A, R) where L of course is the language, A is a set of axioms in L, R is a set of rules in L.

Given a language L, a deductive system D = (L, A, R) and a sentence φ in L, a proof of φ in D is a sequence of sentences (ψ_1, .. , ψ_m) such that

- there exists $A \in \mathcal{A}$ such that $\psi_1 \in A$;
- if m>1 then for each j=2..m one of the following hold
  - there exists $A \in \mathcal{A}$ such that $\psi_j \in A$
  - there exist an n-ary rule R in $\mathcal{R}$ and $i_1, .., i_n < j$ such that $(\psi_{i(1)}, .. , \psi_{i(n)}, \psi_j) \in R$
- $\psi_m = \varphi$.

Suppose there exists a proof (ψ_1, .. , ψ_m) of φ in our system D. We can easily show that for each i=1..m #(ψ_i) holds.

In fact #(ψ_1) holds. If m>1 suppose j=2..m. If there exists $A \in \mathcal{A}$ such that $\psi_j \in A$ then #(ψ_j) holds, otherwise there exist an n-ary rule R in $\mathcal{R}$ and $i_1, .., i_n < j$ such that $(\psi_{i(1)}, .. , \psi_{i(n)}, \psi_j) \in R$. Since #($\psi_{i(1)}$), … , #($\psi_{i(n)}$) all hold then #($\psi_j$) also holds.

Therefore #(φ) holds. This proves what is called the 'soundness' of our system: if we can derive φ in our system then #(φ) holds.

In section 3 we have assumed all the logical connectives $\wedge$, $\vee$, $\rightarrow$, $\neg$, $\forall$, $\exists$ are in our set F. This assumption is still valid, and here we also add to F the membership predicate $\in$ and the equality predicate = (which have been explained at the beginning of section 2).

We also add to F a new predicate $\leftrightarrow$ to represent double logical implication which is described as follows:

> For $n \neq 2$ $A_{\leftrightarrow}(x_1,\dots,x_n)$ = false
> $A_{\leftrightarrow}(x_1,x_2)$ = (x_1 true or x_1 false) and (x_2 true or x_2 false)
> $P_{\leftrightarrow}(x_1,x_2) = P_{\rightarrow}(x_1,x_2)$ and $P_{\rightarrow}(x_2,x_1)$.

We can apply the results of lemma 3.6 to this new predicate. In other words, if $h \in K$ and $\varphi_1, \varphi_2 \in S(h)$ we have

- $(\leftrightarrow)(\varphi_1,\varphi_2) \in S(h)$
- $\forall \rho \in \Xi(h)$ $\#(h, (\leftrightarrow)(\varphi_1, \varphi_2), \rho) = P_{\leftrightarrow}(\#(h, \varphi_1, \rho), \#(h, \varphi_2, \rho))$.

In fact for each $\rho \in \Xi(h)$ $\#(h, \varphi_1, \rho)$ is true or $\#(h, \varphi_1, \rho)$ is false; $\#(h, \varphi_2, \rho)$ is true or $\#(h, \varphi_2, \rho)$ is false. We also consider that

$A_{\leftrightarrow}(\ \#(h,\varphi_1,\rho), \#(h,\varphi_2,\rho)\ ) =$
= ($\#(h,\varphi_1,\rho)$ is true or $\#(h,\varphi_1,\rho)$ is false) and ($\#(h,\varphi_2,\rho)$ is true or $\#(h,\varphi_2,\rho)$ is false) ,

so $A_{\leftrightarrow}(\ \#(h,\varphi_1,\rho), \#(h,\varphi_2,\rho)\ )$ holds true.

There exists a positive integer n such that $\varphi_1, \varphi_2 \in E(n,h)$, so $(\leftrightarrow)(\varphi_1,\varphi_2) \in E(h)$.

Moreover for each $\rho \in \Xi(h)$

$\#(h, (\leftrightarrow)(\varphi_1, \varphi_2), \rho) = P_{\leftrightarrow}(\#(h, \varphi_1, \rho), \#(h, \varphi_2, \rho))$, so

$\#(h, (\leftrightarrow)(\varphi_1, \varphi_2), \rho)$ is true or false .

Therefore $(\leftrightarrow)(\varphi_1,\varphi_2) \in S(h)$.

We now need to list a set of axioms and rules which can be used in every language with the aforementioned symbols within the set F. For every axiom/rule we first prove a result which ensures the soundness of the axiom/rule and then define properly the axiom/rule.

We begin with a simple rule.

<u>Lemma 5.1</u>

Let m be a positive integer. Let $x_1, \dots, x_m \in V$, with $x_i \neq x_j$ for $i \neq j$. Let $\varphi_1, .., \varphi_m \in E$ and assume $H[x_1{:}\varphi_1, \dots, x_m{:}\varphi_m]$. Define $k = k[x_1{:}\varphi_1, \dots, x_m{:}\varphi_m]$ and let $\varphi, \psi \in S(k)$.

Under these assumptions we have $(\leftrightarrow)(\varphi,\psi)$ and $(\rightarrow)(\varphi,\psi) \in S(k)$,

$\gamma[x_1:\varphi_1, \ldots, x_m:\varphi_m, (\leftrightarrow)(\varphi, \psi)]$, $\gamma[x_1:\varphi_1, \ldots, x_m:\varphi_m, (\rightarrow)(\varphi, \psi)]$,
$\gamma[x_1:\varphi_1, \ldots, x_m:\varphi_m, (\rightarrow)(\psi, \varphi)] \in S(\varepsilon)$.

Moreover if $\#(\gamma[x_1:\varphi_1, \ldots, x_m:\varphi_m, (\leftrightarrow)(\varphi, \psi)])$ then
$\#(\gamma[x_1:\varphi_1, \ldots, x_m:\varphi_m, (\rightarrow)(\varphi, \psi)])$ and $\#(\gamma[x_1:\varphi_1, \ldots, x_m:\varphi_m, (\rightarrow)(\psi, \varphi)])$

Proof:

We suppose $\#(\gamma[x_1:\varphi_1, \ldots, x_m:\varphi_m, (\leftrightarrow)(\varphi, \psi)])$ and therefore

$$P_\forall\Big(\{\}\big(\sigma\in \Xi(k), \#\big(k,(\leftrightarrow)(\varphi,\psi),\sigma\big)\big)\Big)$$
$$P_\forall\Big(\{\}\big(\sigma\in \Xi(k), P_\leftrightarrow\big(\#(k,\varphi,\sigma), \#(k,\psi,\sigma)\big)\big)\Big)$$
$$P_\forall\Big(\{\}\big(\sigma\in \Xi(k), P_\rightarrow\big(\#(k,\varphi,\sigma), \#(k,\psi,\sigma)\big)\big)\Big)$$
$$P_\forall\Big(\{\}\big(\sigma\in \Xi(k), \#\big(k,(\rightarrow)(\varphi,\psi),\sigma\big)\big)\Big)$$

and $\#(\gamma[x_1:\varphi_1, \ldots, x_m:\varphi_m, (\rightarrow)(\varphi, \psi)])$.

We have also

$$P_\forall\Big(\{\}\big(\sigma\in \Xi(k), P_\rightarrow\big(\#(k,\psi,\sigma), \#(k,\varphi,\sigma)\big)\big)\Big)$$
$$P_\forall\Big(\{\}\big(\sigma\in \Xi(k), \#\big(k,(\rightarrow)(\psi,\varphi),\sigma\big)\big)\Big)$$

and $\#(\gamma[x_1:\varphi_1, \ldots, x_m:\varphi_m, (\rightarrow)(\psi, \varphi)])$.

□

This lemma allows us to create a unary rule $R_{5.1}$ which is the union of two sets of couples:

the set of all couples

$(\gamma[x_1:\varphi_1, \ldots, x_m:\varphi_m, (\leftrightarrow)(\varphi, \psi)], \gamma[x_1:\varphi_1, \ldots, x_m:\varphi_m, (\rightarrow)(\varphi, \psi)])$

such that

- m is a positive integer, $x_1, \ldots, x_m \in V$, $x_i \neq x_j$ for $i \neq j$, $\varphi_1, \ldots, \varphi_m \in E$, $H[x_1:\varphi_1, \ldots, x_m:\varphi_m]$
- if we define $k = k[x_1:\varphi_1, \ldots, x_m:\varphi_m]$ then $\varphi, \psi \in S(k)$;

and the set of all couples

$(\gamma[x_1:\varphi_1, \ldots, x_m:\varphi_m, (\leftrightarrow)(\varphi, \psi)], \gamma[x_1:\varphi_1, \ldots, x_m:\varphi_m, (\rightarrow)(\psi, \varphi)])$

with the same requirements for their components.

Lemma 5.2

Let m be a positive integer. Let $x_1, \dots, x_m \in V$, with $x_i \neq x_j$ for $i \neq j$. Let $\varphi_1, .., \varphi_m \in E$ and assume $H[x_1:\varphi_1, ..., x_m:\varphi_m]$. Define $k = k[x_1:\varphi_1, ..., x_m:\varphi_m]$ and let $\varphi, \psi \in S(k)$.

Under these assumptions we have $(\wedge)(\varphi,\psi) \in S(k)$, $(\rightarrow)((\wedge)(\varphi,\psi), \varphi) \in S(k)$, and

$\gamma[x_1:\varphi_1, ..., x_m:\varphi_m, (\rightarrow)((\wedge)(\varphi,\psi), \varphi)]$, $\gamma[x_1:\varphi_1, ..., x_m:\varphi_m, (\rightarrow)((\wedge)(\varphi,\psi), \psi)] \in S(\varepsilon)$.

Moreover $\#(\gamma[x_1:\varphi_1, ..., x_m:\varphi_m, (\rightarrow)((\wedge)(\varphi,\psi), \varphi)])$ and $\#(\gamma[x_1:\varphi_1, ..., x_m:\varphi_m, (\rightarrow)((\wedge)(\varphi,\psi), \psi)])$ are both true .

Proof:

$\#(\gamma[x_1:\varphi_1, ..., x_m:\varphi_m, (\rightarrow)((\wedge)(\varphi,\psi), \varphi)])$ can be rewritten as

$$P_\forall\Big(\{\}\big(\sigma\in\Xi(k),\#\big(k,(\rightarrow)\big((\wedge)(\varphi,\psi),\varphi\big),\sigma\big)\big)\Big)$$
$$P_\forall\Big(\{\}\big(\sigma\in\Xi(k),P_\rightarrow\big(\#\big(k,(\wedge)(\varphi,\psi),\sigma\big),\#(k,\varphi,\sigma)\big)\big)\Big)$$
$$P_\forall\Big(\{\}\big(\sigma\in\Xi(k),P_\rightarrow\big(P_\wedge\big(\#(k,\varphi,\sigma),\#(k,\psi,\sigma)\big),\#(k,\varphi,\sigma)\big)\big)\Big)\ .$$

This can be expressed as 'for each $\sigma\in\Xi(k)$ if $\#(k,\varphi,\sigma)$ and $\#(k,\psi,\sigma)$ then $\#(k,\varphi,\sigma)$'.

This condition is clearly true, and in the same way we can show that

$\#(\gamma[x_1:\varphi_1, ..., x_m:\varphi_m, (\rightarrow)((\wedge)(\varphi,\psi), \psi)])$ is true .

□

Lemma 5.2 allows us to create an axiom $A_{5.2}$ which is the union of two sets of sentences:

the set of all sentences $\gamma[x_1:\varphi_1, ..., x_m:\varphi_m, (\rightarrow)((\wedge)(\varphi,\psi), \varphi)]$ such that

- m is a positive integer, $x_1, \dots, x_m \in V$, $x_i \neq x_j$ for $i \neq j$, $\varphi_1, .., \varphi_m \in E$, $H[x_1:\varphi_1, ..., x_m:\varphi_m]$
- if we define $k = k[x_1:\varphi_1, ..., x_m:\varphi_m]$ then $\varphi, \psi \in S(k)$;

and the set of all sentences $\gamma[x_1:\varphi_1, ..., x_m:\varphi_m, (\rightarrow)((\wedge)(\varphi,\psi), \psi)]$ with the same requirements for their components.

Lemma 5.3

Let m be a positive integer. Let $x_1, \dots, x_m \in V$, with $x_i \neq x_j$ for $i \neq j$. Let $\varphi_1, .., \varphi_m \in E$ and assume $H[x_1:\varphi_1, ..., x_m:\varphi_m]$. Define $k = k[x_1:\varphi_1, ..., x_m:\varphi_m]$ and let $\varphi, \psi, \chi \in S(k)$.

Under these assumptions we have $(\rightarrow)(\varphi,\psi)$, $(\rightarrow)(\psi,\chi)$, $(\rightarrow)(\varphi,\chi) \in S(k)$, and
$\gamma[x_1:\varphi_1, ..., x_m:\varphi_m, (\rightarrow)(\varphi,\psi)]$, $\gamma[x_1:\varphi_1, ..., x_m:\varphi_m, (\rightarrow)(\psi,\chi)]$, $\gamma[x_1:\varphi_1, ..., x_m:\varphi_m, (\rightarrow)(\varphi,\chi)] \in S(\varepsilon)$.

Moreover if $\#(\gamma[x_1:\varphi_1, \dots, x_m:\varphi_m, (\rightarrow)(\varphi,\psi)])$ and $\#(\gamma[x_1:\varphi_1, \dots, x_m:\varphi_m, (\rightarrow)(\psi,\chi)])$ then $\#(\gamma[x_1:\varphi_1, \dots, x_m:\varphi_m, (\rightarrow)(\varphi,\chi)])$.

Proof:

We can rewrite $\#(\gamma[x_1:\varphi_1, \dots, x_m:\varphi_m, (\rightarrow)(\varphi,\psi)])$ as follows:

$$P\forall\Big(\{\}\big(\sigma\in\Xi(k),\#\big(k,(\rightarrow)(\varphi,\psi),\sigma\big)\big)\Big)$$

$$P\forall\Big(\{\}\big(\sigma\in\Xi(k),P_\rightarrow\big(\#(k,\varphi,\sigma),\#(k,\psi,\sigma)\big)\big)\Big) .$$

And we can rewrite $\#(\gamma[x_1:\varphi_1, \dots, x_m:\varphi_m, (\rightarrow)(\psi,\chi)])$ as follows

$$P\forall\Big(\{\}\big(\sigma\in\Xi(k),\#\big(k,(\rightarrow)(\psi,\chi),\sigma\big)\big)\Big)$$

$$P\forall\Big(\{\}\big(\sigma\in\Xi(k),P_\rightarrow\big(\#(k,\psi,\sigma),\#(k,\chi,\sigma)\big)\big)\Big) .$$

In other words for each $\sigma\in\Xi(k)$ if $\#(k,\varphi,\sigma)$ then $\#(k,\psi,\sigma)$ and if $\#(k,\psi,\sigma)$ then $\#(k,\chi,\sigma)$, so if $\#(k,\varphi,\sigma)$ then $\#(k,\chi,\sigma)$. This can be written like this

$$P\forall\Big(\{\}\big(\sigma\in\Xi(k),P_\rightarrow\big(\#(k,\varphi,\sigma),\#(k,\chi,\sigma)\big)\big)\Big)$$

$$P\forall\Big(\{\}\big(\sigma\in\Xi(k),\#\big(k,(\rightarrow)(\varphi,\chi),\sigma\big)\big)\Big)$$

And so we have $\#(\gamma[x_1:\varphi_1, \dots, x_m:\varphi_m, (\rightarrow)(\varphi,\chi)])$.

□

Lemma 5.3 allows us to create a rule $R_{5.3}$ which is the set of all 3-tuples

$(\gamma[x_1:\varphi_1, \dots, x_m:\varphi_m, (\rightarrow)(\varphi,\psi)], \gamma[x_1:\varphi_1, \dots, x_m:\varphi_m, (\rightarrow)(\psi,\chi)], \gamma[x_1:\varphi_1, \dots, x_m:\varphi_m, (\rightarrow)(\varphi,\chi)])$

such that

- m is a positive integer, $x_1, \dots, x_m \in V$, $x_i\neq x_j$ for $i\neq j$, $\varphi_1, .., \varphi_m\in E$, $H[x_1:\varphi_1, \dots, x_m:\varphi_m]$
- if we define $k = k[x_1:\varphi_1, \dots, x_m:\varphi_m]$ then $\varphi, \psi, \chi \in S(k)$.

Lemma 5.4

Let m be a positive integer. Let $x_1, \dots, x_{m+1} \in V$, with $x_i\neq x_j$ for $i\neq j$. Let $\varphi_1, .., \varphi_{m+1}\in E$ and assume $H[x_1:\varphi_1, \dots, x_{m+1}:\varphi_{m+1}]$. Define $k = k[x_1:\varphi_1, \dots, x_{m+1}:\varphi_{m+1}]$. Of course $H[x_1:\varphi_1, \dots, x_m:\varphi_m]$ also holds and we define $h = k[x_1:\varphi_1, \dots, x_m:\varphi_m]$. Let $\chi\in S(h)$.

Let $t\in E(h)$ such that $\forall\rho\in\Xi(h)\ \#(h,t,\rho) \in \#(h,\varphi_{m+1},\rho)$.
Let $t'\in E(h)$ such that $\forall\rho\in\Xi(h)\ \#(h,t',\rho) \in \#(h,\varphi_{m+1},\rho)$.
Let $\varphi\in S(k)$ such that $V_b(t) \cap V_b(\varphi) = \emptyset$, $V_b(t') \cap V_b(\varphi) = \emptyset$.

Then we can define $\varphi_k\{x_{m+1}/t'\}\in S(h)$ and therefore

$\gamma[x_1{:}\varphi_1, \dots, x_m{:}\varphi_m, (\rightarrow)(\chi,\varphi_k\{x_{m+1}/t\})]$, $\gamma[x_1{:}\varphi_1, \dots, x_m{:}\varphi_m, (\rightarrow)(\chi,(=)(t,t'))]$, $\gamma[x_1{:}\varphi_1, \dots, x_m{:}\varphi_m, (\rightarrow)(\chi,\varphi_k\{x_{m+1}/t'\})]$ all belong to S(ε), and moreover

if $\#(\gamma[x_1{:}\varphi_1, \dots, x_m{:}\varphi_m, (\rightarrow)(\chi,\varphi_k\{x_{m+1}/t\})])$ and $\#(\gamma[x_1{:}\varphi_1, \dots, x_m{:}\varphi_m, (\rightarrow)(\chi,(=)(t,t'))])$ then $\#(\gamma[x_1{:}\varphi_1, \dots, x_m{:}\varphi_m, (\rightarrow)(\chi,\varphi_k\{x_{m+1}/t'\})])$.

Proof:

There exist a positive integer n such that $\varphi \in E(n,k)$.
By lemma 4.5 there exist a positive integer p such that p<n, $y_1, .. , y_p \in V$ such that $y_i \neq y_j$ for $i \neq j$, $\psi_1, .. , \psi_p \in$ E such that $K(n; k; y_1, .. , y_p; \psi_1, .. , \psi_p)$.

Since $(y_1,\psi_1) \parallel \dots \parallel (y_p,\psi_p) = k = (x_1,\varphi_1) \parallel \dots \parallel (x_{m+1},\varphi_{m+1})$ it follows

$p=m+1$, $y_j=x_j$, $\psi_j=\varphi_j$, and $K(n; k; x_1, .. , x_{m+1}; \varphi_1, .. , \varphi_{m+1})$.

Since $t \in E(h)$ we have $V_b(t) \subseteq V\text{-dom}(h)$, so for each j=1..m $x_j \notin V_b(t)$, and similarly for each j=1..m $x_j \notin V_b(t')$.

Therefore we can define $\varphi_k\{x_{m+1}/t\} \in E(h)$, $\varphi_k\{x_{m+1}/t'\} \in E(h)$.

By definition 4.6 for each $\rho \in \Xi(h)$ we can define $\sigma \in \Xi(k)$ such that

$\#(h, \varphi_k\{x_{m+1}/t\}, \rho) = \#(k, \varphi, \sigma)$, and since $\#(k, \varphi, \sigma)$ is true or false we have $\varphi_k\{x_{m+1}/t\} \in S(h)$.

Similarly $\varphi_k\{x_{m+1}/t'\} \in S(h)$.

Suppose the following both hold:

a) $\#(\gamma[x_1{:}\varphi_1, \dots, x_m{:}\varphi_m, (\rightarrow)(\chi,\varphi_k\{x_{m+1}/t\})])$
b) $\#(\gamma[x_1{:}\varphi_1, \dots, x_m{:}\varphi_m, (\rightarrow)(\chi,(=)(t,t'))])$

We can rewrite a) like this:

$$P_\forall\Big(\{\}\Big(\rho \in \Xi(h), \#\big(h, (\rightarrow)\big(\chi, \varphi_k\{x_{m+1}/t\}\big), \rho\big)\Big)\Big)$$

$$P_\forall\Big(\{\}\Big(\rho \in \Xi(h), P_\rightarrow\Big(\#\big(h,\chi,\rho\big), \#\big(h, \varphi_k\{x_{m+1}/t\}, \rho\big)\Big)\Big)\Big)$$

And we can rewrite b) like this:

$$P_\forall\Big(\{\}\Big(\rho \in \Xi(h), \#\big(h, (\rightarrow)\big(\chi, (=)(t,t')\big), \rho\big)\Big)\Big)$$

$$P_\forall\Big(\{\}\Big(\rho \in \Xi(h), P_\rightarrow\Big(\#\big(h,\chi,\rho\big), \#\big(h, (=)(t,t'), \rho\big)\Big)\Big)\Big)$$

$$P_\forall\Big(\{\}\Big(\rho \in \Xi(h), P_\rightarrow\Big(\#\big(h,\chi,\rho\big), P_=\big(\#(h,t,\rho), \#(h,t',\rho)\big)\Big)\Big)\Big)$$

We need to show $\#(\gamma[x_1{:}\varphi_1, \dots, x_m{:}\varphi_m, (\rightarrow)(\chi,\varphi_k\{x_{m+1}/t'\})])$, which can be rewritten

$$P_\forall\Big(\{\}\Big(\rho\in\Xi(h), P_\rightarrow\Big(\#\big(h,\chi,\rho\big),\#\big(h,\varphi_k\{x_{m+1}/t'\},\rho\big)\Big)\Big)\Big)\ .$$

In other words we need to show that for each $\rho\in\Xi(h)$ if $\#(h,\chi,\rho)$ then $\#(h, \varphi_k\{x_{m+1}/t'\},\rho)$.

Let $\rho\in\Xi(h)$ and assume $\#(h,\chi,\rho)$.

Define σ to be the soop $(x_1,\rho(x_1)) \parallel \dots \parallel (x_m,\rho(x_m)) \parallel (x_{m+1}, \#(h,t',\rho))$.

From definition 4.6 follows that $\sigma\in\Xi(k)$ and $\#(h, \varphi_k\{x_{m+1}/t'\},\rho) = \#(k, \varphi, \sigma)$.

So we need to show that $\#(k, \varphi, \sigma)$ holds.

Because of b) we have $\#(h,t,\rho) = \#(h,t',\rho)$, so $\sigma = (x_1,\rho(x_1)) \parallel \dots \parallel (x_m,\rho(x_m)) \parallel (x_{m+1}, \#(h,t,\rho))$.

From definition 4.6 follows that $\#(h, \varphi_k\{x_{m+1}/t\},\rho) = \#(k, \varphi, \sigma)$.

Because of a) we have $\#(h, \varphi_k\{x_{m+1}/t\},\rho)$, so $\#(k, \varphi, \sigma)$ holds as we needed to show.

□

Lemma 5.4 allows us to create a rule $R_{5.4}$ which is the set of all 3-tuples

$(\gamma[x_1:\varphi_1, \dots , x_m:\varphi_m, (\rightarrow)(\chi,\varphi_k\{x_{m+1}/t\})]$,
$\gamma[x_1:\varphi_1, \dots , x_m:\varphi_m, (\rightarrow)(\chi,(=)(t,t'))]$,
$\gamma[x_1:\varphi_1, \dots , x_m:\varphi_m, (\rightarrow)(\chi,\varphi_k\{x_{m+1}/t'\})]$)

such that
- m is a positive integer, $x_1, \dots , x_{m+1} \in V$, with $x_i\neq x_j$ for $i\neq j$, $\varphi_1, .. , \varphi_{m+1}\in E$, $H[x_1:\varphi_1, \dots ,x_{m+1}:\varphi_{m+1}]$.
- if we define $k = k[x_1:\varphi_1, \dots , x_{m+1}:\varphi_{m+1}]$ and $h = k[x_1:\varphi_1, \dots , x_m:\varphi_m]$ then
  - $\chi\in S(h)$
  - $t\in E(h)$, $\forall\rho\in\Xi(h)\ \#(h,t,\rho) \in \#(h,\varphi_{m+1},\rho)$;
  - $t'\in E(h)$, $\forall\rho\in\Xi(h)\ \#(h,t',\rho) \in \#(h,\varphi_{m+1},\rho)$;
  - $\varphi\in S(k)$, $V_b(t) \cap V_b(\varphi) = \varnothing$, $V_b(t') \cap V_b(\varphi) = \varnothing$.

Lemma 5.5

Let m be a positive integer. Let $x_1, \dots , x_m \in V$, with $x_i\neq x_j$ for $i\neq j$. Let $\varphi_1, .. , \varphi_m\in E$ and assume $H[x_1:\varphi_1, \dots , x_m:\varphi_m]$. Define $k = k[x_1:\varphi_1, \dots , x_m:\varphi_m]$ and let $\varphi, \psi \in S(k)$.

Under these assumptions we have $(\rightarrow)(\psi,\varphi) \in S(k)$ and

$\gamma[x_1:\varphi_1, \dots , x_m:\varphi_m, \varphi], \gamma[x_1:\varphi_1, \dots , x_m:\varphi_m, (\rightarrow) (\psi, \varphi)] \in S(\varepsilon)$.

Moreover if $\#(\gamma[x_1:\varphi_1, \dots , x_m:\varphi_m, \varphi])$ then $\#(\gamma[x_1:\varphi_1, \dots , x_m:\varphi_m, (\rightarrow) (\psi, \varphi)])$ also holds.

Proof:

$\#(\gamma[x_1:\varphi_1, \dots , x_m:\varphi_m, (\rightarrow) (\psi, \varphi)])$ can be rewritten as

$$P_\forall\Big(\{\}\big(\sigma\in\Xi(k),\#\big(k,(\rightarrow)(\psi,\varphi),\sigma\big)\big)\Big)$$

$$P_\forall\Big(\{\}\big(\sigma\in\Xi(k),P_\rightarrow\big(\#(k,\psi,\sigma),\#(k,\varphi,\sigma)\big)\big)\Big)\ .$$

$\gamma[x_1{:}\varphi_1, \ldots, x_m{:}\varphi_m, \varphi]$ can be rewritten as

$$P_\forall\Big(\{\}\big(\sigma\in\Xi(k),\#(k,\varphi,\sigma)\big)\Big)$$

So if $\#(\gamma[x_1{:}\varphi_1, \ldots, x_m{:}\varphi_m, \varphi])$ then for each $\sigma\in\Xi(k)$ $\#(k,\varphi,\sigma)$ holds, and therefore $\#(\gamma[x_1{:}\varphi_1, \ldots, x_m{:}\varphi_m, (\rightarrow)(\psi, \varphi)])$ also holds.

□

Lemma 5.5 allows us to create a rule $R_{5.5}$ which is the set of all couples

$(\gamma[x_1{:}\varphi_1, \ldots, x_m{:}\varphi_m, \varphi], \gamma[x_1{:}\varphi_1, \ldots, x_m{:}\varphi_m, (\rightarrow)(\psi, \varphi)])$

such that

- m is a positive integer, $x_1, \ldots, x_m \in V$, $x_i \neq x_j$ for $i\neq j$, $\varphi_1, \ldots, \varphi_m\in E$, $H[x_1{:}\varphi_1, \ldots, x_m{:}\varphi_m]$
- if we define $k = k[x_1{:}\varphi_1, \ldots, x_m{:}\varphi_m]$ then $\varphi, \psi \in S(k)$.

Lemma 5.6

Let m be a positive integer. Let $x_1, \ldots, x_m \in V$, with $x_i\neq x_j$ for $i\neq j$. Let $\varphi_1, \ldots, \varphi_m\in E$ and assume $H[x_1{:}\varphi_1, \ldots, x_m{:}\varphi_m]$. Define $k = k[x_1{:}\varphi_1, \ldots, x_m{:}\varphi_m]$ and let $\varphi, \psi, \chi \in E(k)$, $\vartheta\in S(k)$.

Under these assumptions we have $(\rightarrow)(\vartheta, (=)(\varphi,\psi))$, $(\rightarrow)(\vartheta, (=)(\psi,\chi))$, $(\rightarrow)(\vartheta, (=)(\varphi,\chi)) \in S(k)$ and

$\gamma[x_1{:}\varphi_1, \ldots, x_m{:}\varphi_m, (\rightarrow)(\vartheta, (=)(\varphi,\psi))] \in S(\varepsilon)$,
$\gamma[x_1{:}\varphi_1, \ldots, x_m{:}\varphi_m, (\rightarrow)(\vartheta, (=)(\psi,\chi))] \in S(\varepsilon)$,
$\gamma[x_1{:}\varphi_1, \ldots, x_m{:}\varphi_m, (\rightarrow)(\vartheta, (=)(\varphi,\chi))] \in S(\varepsilon)$.

Moreover if $\#(\gamma[x_1{:}\varphi_1, \ldots, x_m{:}\varphi_m, (\rightarrow)(\vartheta, (=)(\varphi,\psi))])$ and $\#(\gamma[x_1{:}\varphi_1, \ldots, x_m{:}\varphi_m, (\rightarrow)(\vartheta, (=)(\psi,\chi))])$ hold then $\#(\gamma[x_1{:}\varphi_1, \ldots, x_m{:}\varphi_m, (\rightarrow)(\vartheta, (=)(\varphi,\chi))])$ also holds.

Proof:

We rewrite $\#(\gamma[x_1{:}\varphi_1, \ldots, x_m{:}\varphi_m, (\rightarrow)(\vartheta, (=)(\varphi,\psi))])$ as

$$P_\forall\Big(\{\}\big(\sigma\in\Xi(k),\#\big(k,(\rightarrow)(\vartheta,(=)(\varphi,\psi)),\sigma\big)\big)\Big)$$

$$P_\forall\Big(\{\}\big(\sigma\in\Xi(k),P_\rightarrow\big(\#(k,\vartheta,\sigma),\#\big(k,(=)(\varphi,\psi),\sigma\big)\big)\big)\Big)$$

$$P_\forall\Big(\{\}\big(\sigma\in\Xi(k),P_\rightarrow\big(\#(k,\vartheta,\sigma),P_=\big(\#(k,\varphi,\sigma),\#(k,\psi,\sigma)\big)\big)\big)\Big)$$

Similarly we rewrite $\#(\gamma[x_1{:}\varphi_1, \ldots, x_m{:}\varphi_m, (\rightarrow)(\vartheta, (=)(\psi,\chi))])$ as

$$P\forall\Big(\{\}\Big(\sigma\in\Xi(k),P\rightarrow\big(\#(k,\vartheta,\sigma),P=\big(\#(k,\psi,\sigma),\#(k,\chi,\sigma)\big)\big)\Big)\Big)$$

Therefore for each $\sigma\in\Xi(k)$ if $\#(k,\vartheta,\sigma)$ then $\#(k,\varphi,\sigma)$ equals $\#(k,\psi,\sigma)$, and $\#(k,\psi,\sigma)$ equals $\#(k,\chi,\sigma)$, so $\#(k,\varphi,\sigma)$ equals $\#(k,\chi,\sigma)$.

Since $\#(\ \gamma[x_1:\varphi_1, \ldots , x_m:\varphi_m, (\rightarrow)(\vartheta, (=)(\varphi,\chi))\ ]\ )$ can be rewritten as

$$P\forall\Big(\{\}\Big(\sigma\in\Xi(k),P\rightarrow\big(\#(k,\vartheta,\sigma),P=\big(\#(k,\varphi,\sigma),\#(k,\chi,\sigma)\big)\big)\Big)\Big)$$

it clearly holds true.

□

Lemma 5.6 allows us to create a rule $R_{5.6}$ which is the set of all 3-tuples

$(\gamma[x_1:\varphi_1, \ldots , x_m:\varphi_m, (\rightarrow)(\vartheta, (=)(\varphi,\psi))\ ],$
$\gamma[x_1:\varphi_1, \ldots , x_m:\varphi_m, (\rightarrow)(\vartheta, (=)(\psi,\chi))\ ],$
$\gamma[x_1:\varphi_1, \ldots , x_m:\varphi_m, (\rightarrow)(\vartheta, (=)(\varphi,\chi))\ ])$

such that

- m is a positive integer, $x_1, \ldots , x_m \in V$, $x_i\neq x_j$ for $i\neq j$, $\varphi_1, .. , \varphi_m\in E$, $H[x_1:\varphi_1, \ldots , x_m:\varphi_m]$
- if we define $k = k[x_1:\varphi_1, \ldots , x_m:\varphi_m]$ then $\varphi, \psi, \chi \in E(k)$, $\vartheta\in S(k)$.

Lemma 5.7

Let m be a positive integer. Let $x_1, \ldots , x_{m+1} \in V$, with $x_i\neq x_j$ for $i\neq j$. Let $\varphi_1, .. , \varphi_{m+1}\in E$ and assume $H[x_1:\varphi_1, \ldots , x_{m+1}:\varphi_{m+1}]$. Define $k = k[x_1:\varphi_1, \ldots , x_{m+1}:\varphi_{m+1}]$. Of course $H[x_1:\varphi_1, \ldots , x_m:\varphi_m]$ also holds and we define $h = k[x_1:\varphi_1, \ldots , x_m:\varphi_m]$. Let $\chi\in S(h)$.

Let $t\in E(h)$ such that $\forall\rho\in\Xi(h)\ \#(h,t,\rho) \in \#(h,\varphi_{m+1},\rho)$.
Let $\varphi\in S(k)$ such that $V_b(t) \cap V_b(\varphi) = \varnothing$.

We can define $\varphi_k\{x_{m+1}/t\}\in S(h)$ and $(\exists)(\{\}(x_{m+1}:\varphi_{m+1}, \varphi)\ ) \in S(h)$.

Therefore $\gamma[x_1:\varphi_1, \ldots , x_m:\varphi_m, (\rightarrow)(\chi,\varphi_k\{x_{m+1}/t\})]$ and
$\gamma[x_1:\varphi_1, \ldots , x_m:\varphi_m, (\rightarrow)(\chi, (\exists)(\{\}(x_{m+1}:\varphi_{m+1}, \varphi)\ )\ )]$ both belong to $S(\varepsilon)$ and

if $\#(\ \gamma[x_1:\varphi_1, \ldots , x_m:\varphi_m, (\rightarrow)(\chi,\varphi_k\{x_{m+1}/t\})]\ )$ then
$\#(\ \gamma[x_1:\varphi_1, \ldots , x_m:\varphi_m, (\rightarrow)(\chi, (\exists)(\{\}(x_{m+1}:\varphi_{m+1}, \varphi)\ )\ )]\ )$.

Proof:

It's easy to see that $\varphi_k\{x_{m+1}/t\}\in S(h)$ (this has been shown in lemma 5.4).

By lemma 3.1 we get $(\exists)(\{\}(x_{m+1}:\varphi_{m+1}, \varphi)\ ) \in S(h)$.

Suppose $\#(\ \gamma[x_1:\varphi_1, \ldots , x_m:\varphi_m, (\rightarrow)(\chi,\varphi_k\{x_{m+1}/t\})]\ )$ holds, it can be rewritten as

$$P_\forall\Big(\{\}\Big(\rho\in\Xi(h), P_\to\Big(\#\big(h,\chi,\rho\big), \#\big(h,\varphi_k\{x_{m+1}/t\},\rho\big)\Big)\Big)\Big) .$$

We need to prove #($\gamma[x_1:\varphi_1, \dots , x_m:\varphi_m, (\to)(\chi, (\exists)(\{\}(x_{m+1}:\varphi_{m+1}, \varphi)))]$), and this can be rewritten as

$$P_\forall\Big(\{\}\Big(\rho\in\Xi(h), P_\to\Big(\#\big(h,\chi,\rho\big), \#\big(h,(\exists)\big(\{\}\big(x_{m+1}:\varphi_{m+1},\varphi\big)\big),\rho\big)\Big)\Big)\Big)$$

$$P_\forall\Big(\{\}\Big(\rho\in\Xi(h), P_\to\Big(\#\big(h,\chi,\rho\big), P_\exists\big(\{\}\big(\sigma\in\Xi(k):\rho\sqsubseteq\sigma,\#(k,\varphi,\sigma)\big)\big)\Big)\Big)\Big) .$$

Let $\rho\in\Xi(h)$ and suppose $\#(h,\chi,\rho)$. We need to show there exists $\sigma\in\Xi(k)$ such that $\rho\sqsubseteq\sigma$ and $\#(k,\varphi,\sigma)$.

We have $\#(h,\varphi_k\{x_{m+1}/t\},\rho)$ and by definition 4.6 we can define $\sigma\in\Xi(k)$ such that $\rho \sqsubseteq \sigma$ and

$\#(h, \varphi_k\{x_{m+1}/t\}, \rho) = \#(k, \varphi, \sigma)$. This completes the proof.

□

Lemma 5.7 allows us to create a rule $R_{5.7}$ which is the set of all couples

($\gamma[x_1:\varphi_1, \dots , x_m:\varphi_m, (\to)(\chi,\varphi_k\{x_{m+1}/t\})]$, $\gamma[x_1:\varphi_1, \dots , x_m:\varphi_m, (\to)(\chi, (\exists)(\{\}(x_{m+1}:\varphi_{m+1}, \varphi)))]$)

such that
- m is a positive integer, $x_1, \dots , x_{m+1} \in V$, with $x_i\neq x_j$ for $i\neq j$, $\varphi_1, .. , \varphi_{m+1}\in E$, $H[x_1:\varphi_1, \dots ,x_{m+1}:\varphi_{m+1}]$.
- if we define $k = k[x_1:\varphi_1, \dots , x_{m+1}:\varphi_{m+1}]$ and $h = k[x_1:\varphi_1, \dots , x_m:\varphi_m]$ then
  - $\chi\in S(h)$
  - $t\in E(h)$, $\forall\rho\in\Xi(h)\ \#(h,t,\rho) \in \#(h,\varphi_{m+1},\rho)$;
  - $\varphi\in S(k)$, $V_b(t) \cap V_b(\varphi) = \varnothing$.

Lemma 5.8

Let m be a positive integer. Let $x_1, \dots , x_m \in V$, with $x_i\neq x_j$ for $i\neq j$. Let $\varphi_1, .. , \varphi_m\in E$ and assume $H[x_1:\varphi_1, \dots , x_m:\varphi_m]$. Define $k = k[x_1:\varphi_1, \dots , x_m:\varphi_m]$ and let $\varphi, \psi, \chi \in S(k)$.

Under these assumptions we have $(\to)(\varphi,\psi), (\to)(\psi,\chi), (\to)(\varphi,\chi) \in S(k)$ and

$\gamma[x_1:\varphi_1, \dots , x_m:\varphi_m, (\to)(\varphi,\psi)] \in S(\varepsilon)$,
$\gamma[x_1:\varphi_1, \dots , x_m:\varphi_m, (\to)(\psi,\chi)] \in S(\varepsilon)$,
$\gamma[x_1:\varphi_1, \dots , x_m:\varphi_m, (\to)(\varphi,\chi)] \in S(\varepsilon)$.

Moreover if #($\gamma[x_1:\varphi_1, \dots , x_m:\varphi_m, (\to)(\varphi,\psi)]$) and #($\gamma[x_1:\varphi_1, \dots , x_m:\varphi_m, (\to)(\psi,\chi)]$) then
#($\gamma[x_1:\varphi_1, \dots , x_m:\varphi_m, (\to)(\varphi,\chi)]$) .

Proof:

We rewrite $\gamma[x_1:\varphi_1, \dots , x_m:\varphi_m, (\to)(\varphi,\psi)]$ as

$P\forall\Big(\{\}\big(\sigma\in\Xi(k),P\rightarrow\big(\#(k,\varphi,\sigma),\#(k,\psi,\sigma)\big)\big)\Big)$,

$\gamma[x_1{:}\varphi_1, \ldots , x_m{:}\varphi_m, (\rightarrow)(\psi,\chi)]$ as

$P\forall\Big(\{\}\big(\sigma\in\Xi(k),P\rightarrow\big(\#(k,\psi,\sigma),\#(k,\chi,\sigma)\big)\big)\Big)$,

$\gamma[x_1{:}\varphi_1, \ldots , x_m{:}\varphi_m, (\rightarrow)(\varphi,\chi)]$ as

$P\forall\Big(\{\}\big(\sigma\in\Xi(k),P\rightarrow\big(\#(k,\varphi,\sigma),\#(k,\chi,\sigma)\big)\big)\Big)$.

If $\#(\gamma[x_1{:}\varphi_1, \ldots , x_m{:}\varphi_m, (\rightarrow)(\varphi,\psi)])$ and $\#(\gamma[x_1{:}\varphi_1, \ldots , x_m{:}\varphi_m, (\rightarrow)(\psi,\chi)])$ then

for each $\sigma\in\Xi(k)$ if $\#(k,\varphi,\sigma)$ then $\#(k,\psi,\sigma)$ and so $\#(k,\chi,\sigma)$, in other words we have

$\#(\gamma[x_1{:}\varphi_1, \ldots , x_m{:}\varphi_m, (\rightarrow)(\varphi,\chi)])$.

□

Lemma 5.8 allows us to create a rule $R_{5.8}$ which is the set of all 3-tuples

($\gamma[x_1{:}\varphi_1, \ldots , x_m{:}\varphi_m, (\rightarrow)(\varphi,\psi)]$,
$\gamma[x_1{:}\varphi_1, \ldots , x_m{:}\varphi_m, (\rightarrow)(\psi,\chi)]$,
$\gamma[x_1{:}\varphi_1, \ldots , x_m{:}\varphi_m, (\rightarrow)(\varphi,\chi)]$)

such that

- m is a positive integer, $x_1, \ldots , x_m \in V$, $x_i\neq x_j$ for $i\neq j$, $\varphi_1, .. , \varphi_m\in E$, $H[x_1{:}\varphi_1, \ldots , x_m{:}\varphi_m]$
- if we define $k = k[x_1{:}\varphi_1, \ldots , x_m{:}\varphi_m]$ then $\varphi, \psi, \chi \in S(k)$.

Lemma 5.9

Let m be a positive integer. Let $x_1, \ldots , x_m \in V$, with $x_i\neq x_j$ for $i\neq j$. Let $\varphi_1, .. , \varphi_m\in E$ and assume $H[x_1{:}\varphi_1, \ldots , x_m{:}\varphi_m]$. Define $k = k[x_1{:}\varphi_1, \ldots , x_m{:}\varphi_m]$ and let $\varphi, \psi, \chi \in S(k)$.

Under these assumptions we have $(\rightarrow)((\wedge)(\varphi,\psi),\chi)$, $(\rightarrow)(\varphi,(\rightarrow)(\psi,\chi)) \in S(k)$ and

$\gamma[x_1{:}\varphi_1, \ldots , x_m{:}\varphi_m, (\rightarrow)((\wedge)(\varphi,\psi),\chi)] \in S(\varepsilon)$,
$\gamma[x_1{:}\varphi_1, \ldots , x_m{:}\varphi_m, (\rightarrow)(\varphi,(\rightarrow)(\psi,\chi))] \in S(\varepsilon)$.

Moreover if $\#(\gamma[x_1{:}\varphi_1, \ldots , x_m{:}\varphi_m, (\rightarrow)((\wedge)(\varphi,\psi),\chi)])$ then
$\#(\gamma[x_1{:}\varphi_1, \ldots , x_m{:}\varphi_m, (\rightarrow)(\varphi,(\rightarrow)(\psi,\chi))])$.

Proof:

We assume $\#(\gamma[x_1{:}\varphi_1, \ldots , x_m{:}\varphi_m, (\rightarrow)((\wedge)(\varphi,\psi),\chi)])$, which can be rewritten as

$P\forall\Big(\{\}\big(\sigma\in\Xi(k),\#\big(k,(\rightarrow)\big((\wedge)(\varphi,\psi),\chi\big),\sigma\big)\big)\Big)$

$$P_{\forall}\Big(\{\}\big(\sigma\in\Xi(k),P_{\rightarrow}\big(\#(k,(\wedge)(\varphi,\psi),\sigma),\#(k,\chi,\sigma)\big)\big)\Big)$$

$$P_{\forall}\Big(\{\}\big(\sigma\in\Xi(k),P_{\rightarrow}\big(P_{\wedge}\big(\#(k,\varphi,\sigma),\#(k,\psi,\sigma)\big),\#(k,\chi,\sigma)\big)\big)\Big)\ .$$

We try to show #($\gamma[x_1:\varphi_1, \ldots, x_m:\varphi_m, (\rightarrow)(\varphi,(\rightarrow)(\psi,\chi))]$) which in turn can be rewritten

$$P_{\forall}\Big(\{\}\big(\sigma\in\Xi(k),\#\big(k,(\rightarrow)(\varphi,(\rightarrow)(\psi,\chi)),\sigma\big)\big)\Big)$$

$$P_{\forall}\Big(\{\}\big(\sigma\in\Xi(k),P_{\rightarrow}\big(\#(k,\varphi,\sigma),\#(k,(\rightarrow)(\psi,\chi),\sigma)\big)\big)\Big)$$

$$P_{\forall}\Big(\{\}\big(\sigma\in\Xi(k),P_{\rightarrow}\big(\#(k,\varphi,\sigma),P_{\rightarrow}\big(\#(k,\psi,\sigma),\#(k,\chi,\sigma)\big)\big)\big)\Big)\ .$$

Let $\sigma\in\Xi(k)$, suppose $\#(k,\varphi,\sigma)$ and $\#(k,\psi,\sigma)$, then we have $\#(k,\chi,\sigma)$ and this completes the proof.

□

Lemma 5.9 allows us to create a rule $R_{5.9}$ which is the set of all couples

($\gamma[x_1:\varphi_1, \ldots, x_m:\varphi_m, (\rightarrow)((\wedge)(\varphi,\psi),\chi)]$, $\gamma[x_1:\varphi_1, \ldots, x_m:\varphi_m, (\rightarrow)(\varphi,(\rightarrow)(\psi,\chi))]$)

such that

- m is a positive integer, $x_1, \ldots, x_m \in V$, $x_i\neq x_j$ for $i\neq j$, $\varphi_1, \ldots, \varphi_m\in E$, $H[x_1:\varphi_1, \ldots, x_m:\varphi_m]$
- if we define $k = k[x_1:\varphi_1, \ldots, x_m:\varphi_m]$ then $\varphi, \psi, \chi \in S(k)$.

Lemma 5.10

Let m be a positive integer. Let $x_1, \ldots, x_{m+1} \in V$, with $x_i\neq x_j$ for $i\neq j$. Let $\varphi_1, \ldots, \varphi_{m+1}\in E$ and assume $H[x_1:\varphi_1, \ldots, x_{m+1}:\varphi_{m+1}]$. Define $k = k[x_1:\varphi_1, \ldots, x_{m+1}:\varphi_{m+1}]$. Of course $H[x_1:\varphi_1, \ldots, x_m:\varphi_m]$ also holds and we define $h = k[x_1:\varphi_1, \ldots, x_m:\varphi_m]$. Let $\psi \in S(h)\cap S(k)$ and $\varphi\in S(k)$.

Then $(\rightarrow)(\psi,\varphi) \in S(k)$ and $\gamma[x_{m+1}:\varphi_{m+1}, (\rightarrow)(\psi,\varphi)] \in S(h)$, and $(\rightarrow)(\psi, \gamma[x_{m+1}:\varphi_{m+1}, \varphi]) \in S(h)$.

Therefore

$\gamma[x_1:\varphi_1, \ldots, x_m:\varphi_m, \gamma[x_{m+1}:\varphi_{m+1}, (\rightarrow)(\psi,\varphi)]] \in S(\varepsilon)$,
$\gamma[x_1:\varphi_1, \ldots, x_m:\varphi_m, (\rightarrow)(\psi, \gamma[x_{m+1}:\varphi_{m+1}, \varphi])] \in S(\varepsilon)$, and

if #($\gamma[x_1:\varphi_1, \ldots, x_m:\varphi_m, \gamma[x_{m+1}:\varphi_{m+1}, (\rightarrow)(\psi,\varphi)]]$)
then #($\gamma[x_1:\varphi_1, \ldots, x_m:\varphi_m, (\rightarrow)(\psi, \gamma[x_{m+1}:\varphi_{m+1}, \varphi])]$) .

Proof:

We can rewrite #($\gamma[x_1:\varphi_1, \ldots, x_m:\varphi_m, \gamma[x_{m+1}:\varphi_{m+1}, (\rightarrow)(\psi,\varphi)]]$) as

$$P_{\forall}\Big(\{\}\big(\rho\in\Xi(h),\#\big(h,\gamma\big[x_{m+1}:\varphi_{m+1},(\rightarrow)(\psi,\varphi)\big],\rho\big)\big)\Big)$$

$$P_{\forall}\Big(\{\}\big(\rho\in\Xi(h),\#\big(h,(\forall)\big(\{\}\big(x_{m+1}:\varphi_{m+1},(\rightarrow)(\psi,\varphi)\big)\big),\rho\big)\big)\Big)$$

$$P_\forall\Big(\{\}\Big(\rho\in\Xi(h),P_\forall\Big(\{\}\Big(\sigma\in\Xi(k):\rho\sqsubseteq\sigma,\#\big(k,(\to)(\psi,\varphi),\sigma\big)\Big)\Big)\Big)\Big)$$
$$P_\forall\Big(\{\}\Big(\rho\in\Xi(h),P_\forall\Big(\{\}\Big(\sigma\in\Xi(k):\rho\sqsubseteq\sigma,P_\to\big(\#(k,\psi,\sigma),\#(k,\varphi,\sigma)\big)\Big)\Big)\Big)\Big)$$

$\#(\ \gamma[x_1{:}\varphi_1, \dots , x_m{:}\varphi_m, (\to)(\psi, \gamma[x_{m+1}{:}\varphi_{m+1}, \varphi])]\ )$ in turn can be rewritten

$$P_\forall\Big(\{\}\Big(\rho\in\Xi(h),\#\big(h,(\to)\big(\psi,\gamma[x_{m+1}:\varphi_{m+1},\varphi]\big),\rho\big)\Big)\Big)$$
$$P_\forall\Big(\{\}\Big(\rho\in\Xi(h),P_\to\big(\#(h,\psi,\rho),\#\big(h,\gamma[x_{m+1}:\varphi_{m+1},\varphi],\rho\big)\big)\Big)\Big)$$
$$P_\forall\Big(\{\}\Big(\rho\in\Xi(h),P_\to\Big(\#(h,\psi,\rho),P_\forall\big(\{\}\big(\sigma\in\Xi(k):\rho\sqsubseteq\sigma,\#(k,\varphi,\sigma)\big)\big)\Big)\Big)\Big)\ .$$

We suppose $\#(\ \gamma[x_1{:}\varphi_1, \dots , x_m{:}\varphi_m, \gamma[x_{m+1}{:}\varphi_{m+1}, (\to)(\psi,\varphi)]]\ )$ holds and try to show $\#(\ \gamma[x_1{:}\varphi_1, \dots , x_m{:}\varphi_m, (\to)(\psi, \gamma[x_{m+1}{:}\varphi_{m+1}, \varphi])]\ )$ holds too.

To this purpose let $\rho\in\Xi(h)$ such that $\#(h,\psi,\rho)$, let $\sigma\in\Xi(k)$ such that $\rho\sqsubseteq\sigma$. We want to show that $\#(k,\varphi,\sigma)$ holds.

Since $\psi\in E(k)$ $V_b(\psi) \in V\text{-dom}(k)$ and so $x_{m+1}\notin V_b(\psi)$.

Given that $\psi\in E(h)$ there exists a positive integer n such that $\psi\in E(n,h)$ and therefore $\psi\in E(n+1,k)$,

and $\#(k,\psi,\sigma) = \#(h,\psi,\rho)$.

Since $\#(h,\psi,\rho)$ holds then $\#(k,\psi,\sigma)$ holds too, and this of course means that $\#(k,\varphi,\sigma)$ holds.

□

Lemma 5.10 allows us to create a rule $R_{5.10}$ which is the set of all couples

$(\ \gamma[x_1{:}\varphi_1, \dots , x_m{:}\varphi_m, \gamma[x_{m+1}{:}\varphi_{m+1}, (\to)(\psi,\varphi)]], \gamma[x_1{:}\varphi_1, \dots , x_m{:}\varphi_m, (\to)(\psi, \gamma[x_{m+1}{:}\varphi_{m+1}, \varphi])]\ )$

such that

- m is a positive integer, $x_1, \dots , x_{m+1} \in V$, $x_i\neq x_j$ for $i\neq j$, $\varphi_1, .. , \varphi_{m+1}\in E$, $H[x_1{:}\varphi_1, \dots , x_{m+1}{:}\varphi_{m+1}]$;
- if we define $k = k[x_1{:}\varphi_1, \dots , x_{m+1}{:}\varphi_{m+1}]$ and $h = k[x_1{:}\varphi_1, \dots , x_m{:}\varphi_m]$ then $\psi \in S(k)\cap S(h)$, $\varphi\in S(k)$.

Lemma 5.11

Let m be a positive integer. Let $x_1, \dots , x_{m+1} \in V$, with $x_i\neq x_j$ for $i\neq j$. Let $\varphi_1, .. , \varphi_{m+1}\in E$ and assume $H[x_1{:}\varphi_1, \dots , x_{m+1}{:}\varphi_{m+1}]$. Define $k = k[x_1{:}\varphi_1, \dots , x_{m+1}{:}\varphi_{m+1}]$. Of course $H[x_1{:}\varphi_1, \dots , x_m{:}\varphi_m]$ also holds and we define $h = k[x_1{:}\varphi_1, \dots , x_m{:}\varphi_m]$. Let $\chi\in S(h)$, $\psi\in S(k)$ and $\varphi \in S(k)\cap S(h)$.

Under these assumptions we have

$(\to)(\psi,\varphi) \in S(k)$, $\gamma[x_{m+1}{:}\varphi_{m+1}, (\to)(\psi,\varphi)] \in S(h)$, $(\to)(\chi, \gamma[x_{m+1}{:}\varphi_{m+1}, (\to)(\psi,\varphi)]) \in S(h)$;

$\gamma\left[x_1:\varphi_1,\ldots,x_m:\varphi_m,(\to)\left(\chi,\gamma\left[x_{m+1}:\varphi_{m+1},(\to)(\psi,\varphi)\right]\right)\right]\in S(\varepsilon)$;

$(\exists)\ (\{\}(x_{m+1}:\varphi_{m+1},\ \psi))\in S(h)$, $(\to)\left(\chi,(\to)\left((\exists)\left(\{\}\left(x_{m+1}:\varphi_{m+1},\psi\right)\right),\varphi\right)\right)\in S(h)$;

$\gamma\left[x_1:\varphi_1,\ldots,x_m:\varphi_m,(\to)\left(\chi,(\to)\left((\exists)\left(\{\}\left(x_{m+1}:\varphi_{m+1},\psi\right)\right),\varphi\right)\right)\right]\in S(\varepsilon)$.

Moreover if $\#\left(\gamma\left[x_1:\varphi_1,\ldots,x_m:\varphi_m,(\to)\left(\chi,\gamma\left[x_{m+1}:\varphi_{m+1},(\to)(\psi,\varphi)\right]\right)\right]\right)$
then $\#\left(\gamma\left[x_1:\varphi_1,\ldots,x_m:\varphi_m,(\to)\left(\chi,(\to)\left((\exists)\left(\{\}\left(x_{m+1}:\varphi_{m+1},\psi\right)\right),\varphi\right)\right)\right]\right)$.

Proof:

Suppose $\#\left(\gamma\left[x_1:\varphi_1,\ldots,x_m:\varphi_m,(\to)\left(\chi,\gamma\left[x_{m+1}:\varphi_{m+1},(\to)(\psi,\varphi)\right]\right)\right]\right)$.

This can be rewritten

$P_\forall\left(\{\}\left(\rho\in\Xi(h),\#\left(h,(\to)\left(\chi,\gamma\left[x_{m+1}:\varphi_{m+1},(\to)(\psi,\varphi)\right]\right),\rho\right)\right)\right)$,

$P_\forall\left(\{\}\left(\rho\in\Xi(h),P_\to\left(\#(h,\chi,\rho),\#\left(h,\gamma\left[x_{m+1}:\varphi_{m+1},(\to)(\psi,\varphi)\right],\rho\right)\right)\right)\right)$,

$P_\forall\left(\{\}\left(\rho\in\Xi(h),P_\to\left(\#(h,\chi,\rho),P_\forall\left(\{\}\left(\sigma\in\Xi(k):\rho\sqsubseteq\sigma,\#\left(k,(\to)(\psi,\varphi),\sigma\right)\right)\right)\right)\right)\right)$,

$P_\forall\left(\{\}\left(\rho\in\Xi(h),P_\to\left(\#(h,\chi,\rho),P_\forall\left(\{\}\left(\sigma\in\Xi(k):\rho\sqsubseteq\sigma,P_\to\left(\#(k,\psi,\sigma),\#(k,\varphi,\sigma)\right)\right)\right)\right)\right)\right)$.

In turn $\#\left(\gamma\left[x_1:\varphi_1,\ldots,x_m:\varphi_m,(\to)\left(\chi,(\to)\left((\exists)\left(\{\}\left(x_{m+1}:\varphi_{m+1},\psi\right)\right),\varphi\right)\right)\right]\right)$ can be rewritten as

$P_\forall\left(\{\}\left(\rho\in\Xi(h),\#\left(h,(\to)\left(\chi,(\to)\left((\exists)\left(\{\}\left(x_{m+1}:\varphi_{m+1},\psi\right)\right),\varphi\right)\right),\rho\right)\right)\right)$,

$P_\forall\left(\{\}\left(\rho\in\Xi(h),P_\to\left(\#(h,\chi,\rho),\#\left(h,(\to)\left((\exists)\left(\{\}\left(x_{m+1}:\varphi_{m+1},\psi\right)\right),\varphi\right),\rho\right)\right)\right)\right)$,

$P_\forall\left(\{\}\left(\rho\in\Xi(h),P_\to\left(\#(h,\chi,\rho),P_\to\left(\#\left(h,(\exists)\left(\{\}\left(x_{m+1}:\varphi_{m+1},\psi\right)\right),\rho\right),\#(h,\varphi,\rho)\right)\right)\right)\right)$,

$P_\forall\left(\{\}\left(\rho\in\Xi(h),P_\to\left(\#(h,\chi,\rho),P_\to\left(P_\exists\left(\{\}\left(\sigma\in\Xi(k):\rho\sqsubseteq\sigma,\#(k,\psi,\sigma)\right)\right),\#(h,\varphi,\rho)\right)\right)\right)\right)$.

To prove the last statement we suppose $\rho\in\Xi(h)$, $\#(h,\chi,\rho)$ and suppose there exists $\sigma\in\Xi(k)$ such that $\rho\sqsubseteq\sigma$ and $\#(k,\psi,\sigma)$.

We know that $\#(k,\varphi,\sigma)$ holds, but we need to show $\#(h,\varphi,\rho)$.

There exists a positive integer n such that $h,k \in K(n)$. By consequence 2.1.10 there exists $s_{m+1} \in \#(h,\varphi_{m+1},\rho)$ such that $\sigma = \rho \parallel (x_{m+1},s_{m+1})$.

Since $\varphi \in S(k)$ we have $V_b(\varphi) \subseteq V\text{-dom}(k)$, so $x_{m+1} \notin V_b(\varphi)$. We can apply lemma 4.2 to determine that $\#(k,\varphi,\sigma) = \#(h,\varphi,\rho)$. This completes the proof.

□

Lemma 5.11 allows us to create a rule $R_{5.11}$ which is the set of all couples

(
$\gamma\left[x_1:\varphi_1,\ldots,x_m:\varphi_m,(\rightarrow)\left(\chi,\gamma\left[x_{m+1}:\varphi_{m+1},(\rightarrow)(\psi,\varphi)\right]\right)\right]$,
$\gamma\left[x_1:\varphi_1,\ldots,x_m:\varphi_m,(\rightarrow)\left(\chi,(\rightarrow)\left((\exists)\left(\{\}\left(x_{m+1}:\varphi_{m+1},\psi\right)\right),\varphi\right)\right)\right]$
)
such that

- m is a positive integer, $x_1, \ldots , x_{m+1} \in V$, $x_i \neq x_j$ for $i \neq j$, $\varphi_1, .. , \varphi_{m+1} \in E$, $H[x_1:\varphi_1, \ldots , x_{m+1}:\varphi_{m+1}]$;
- if we define $k = k[x_1:\varphi_1, \ldots , x_{m+1}:\varphi_{m+1}]$ and $h = k[x_1:\varphi_1, \ldots , x_m:\varphi_m]$ then $\chi \in S(h)$, $\psi \in S(k)$ and $\varphi \in S(k) \cap S(h)$.

We can now state a rule which is very similar to the former one, but simpler. Of course we start with the related lemma.

Lemma 5.12

Let m be a positive integer. Let $x_1, \ldots , x_{m+1} \in V$, with $x_i \neq x_j$ for $i \neq j$. Let $\varphi_1, .. , \varphi_{m+1} \in E$ and assume $H[x_1:\varphi_1, \ldots , x_{m+1}:\varphi_{m+1}]$. Define $k = k[x_1:\varphi_1, \ldots , x_{m+1}:\varphi_{m+1}]$. Of course $H[x_1:\varphi_1, \ldots , x_m:\varphi_m]$ also holds and we define $h = k[x_1:\varphi_1, \ldots , x_m:\varphi_m]$. Let $\psi \in S(k)$ and $\varphi \in S(k) \cap S(h)$.

Under these assumptions we have

$(\rightarrow)(\psi,\varphi) \in S(k)$, $\gamma[x_{m+1}:\varphi_{m+1}, (\rightarrow)(\psi,\varphi)] \in S(h)$,

$\gamma\left[x_1:\varphi_1,\ldots,x_m:\varphi_m,\gamma\left[x_{m+1}:\varphi_{m+1},(\rightarrow)(\psi,\varphi)\right]\right] \in S(\varepsilon)$;

$(\exists)\left(\{\}(x_{m+1}:\varphi_{m+1}, \psi)\right) \in S(h)$, $(\rightarrow)\left((\exists)\left(\{\}\left(x_{m+1}:\varphi_{m+1},\psi\right)\right),\varphi\right) \in S(h)$;

$\gamma\left[x_1:\varphi_1,\ldots,x_m:\varphi_m,(\rightarrow)\left((\exists)\left(\{\}\left(x_{m+1}:\varphi_{m+1},\psi\right)\right),\varphi\right)\right] \in S(\varepsilon)$.

Moreover if $\#\left(\gamma\left[x_1:\varphi_1,\ldots,x_m:\varphi_m,\gamma\left[x_{m+1}:\varphi_{m+1},(\rightarrow)(\psi,\varphi)\right]\right]\right)$
then $\#\left(\gamma\left[x_1:\varphi_1,\ldots,x_m:\varphi_m,(\rightarrow)\left((\exists)\left(\{\}\left(x_{m+1}:\varphi_{m+1},\psi\right)\right),\varphi\right)\right]\right)$.

Proof:

Suppose $\#\left(\gamma\left[x_1:\varphi_1,\ldots,x_m:\varphi_m,\gamma\left[x_{m+1}:\varphi_{m+1},(\rightarrow)(\psi,\varphi)\right]\right]\right)$.

This can be rewritten

$$P_\forall\left(\{\}\left(\rho\in\Xi(h),\#\left(h,\gamma\left[x_{m+1}:\varphi_{m+1},(\rightarrow)(\psi,\varphi)\right],\rho\right)\right)\right) ,$$

$$P_\forall\left(\{\}\left(\rho\in\Xi(h),P_\forall\left(\{\}\left(\sigma\in\Xi(k):\rho\sqsubseteq\sigma,\#\left(k,(\rightarrow)(\psi,\varphi),\sigma\right)\right)\right)\right)\right) ,$$

$$P_\forall\left(\{\}\left(\rho\in\Xi(h),P_\forall\left(\{\}\left(\sigma\in\Xi(k):\rho\sqsubseteq\sigma,P_\rightarrow\left(\#(k,\psi,\sigma),\#(k,\varphi,\sigma)\right)\right)\right)\right)\right) .$$

In turn $\#\left(\gamma\left[x_1:\varphi_1,\ldots,x_m:\varphi_m,(\rightarrow)\left((\exists)\left(\{\}\left(x_{m+1}:\varphi_{m+1},\psi\right)\right),\varphi\right)\right]\right)$ can be rewritten as

$$P_\forall\left(\{\}\left(\rho\in\Xi(h),\#\left(h,(\rightarrow)\left((\exists)\left(\{\}\left(x_{m+1}:\varphi_{m+1},\psi\right)\right),\varphi\right),\rho\right)\right)\right) ,$$

$$P_\forall\left(\{\}\left(\rho\in\Xi(h),P_\rightarrow\left(\#\left(h,(\exists)\left(\{\}\left(x_{m+1}:\varphi_{m+1},\psi\right)\right),\rho\right),\#(h,\varphi,\rho)\right)\right)\right) ,$$

$$P_\forall\left(\{\}\left(\rho\in\Xi(h),P_\rightarrow\left(P_\exists\left(\{\}\left(\sigma\in\Xi(k):\rho\sqsubseteq\sigma,\#(k,\psi,\sigma)\right)\right),\#(h,\varphi,\rho)\right)\right)\right) .$$

To prove the last statement we suppose $\rho\in\Xi(h)$ and suppose there exists $\sigma\in\Xi(k)$ such that $\rho\sqsubseteq\sigma$ and $\#(k,\psi,\sigma)$.

We know that $\#(k,\varphi,\sigma)$ holds, but we need to show $\#(h,\varphi,\rho)$.

There exists a positive integer n such that $h,k \in K(n)$. By consequence 2.1.10 there exists $s_{m+1}\in\#(h,\varphi_{m+1},\rho)$ such that $\sigma = \rho \parallel (x_{m+1},s_{m+1})$.

Since $\varphi\in S(k)$ we have $V_b(\varphi) \subseteq V\text{-dom}(k)$, so $x_{m+1}\notin V_b(\varphi)$. We can apply lemma 4.2 to determine that $\#(k,\varphi,\sigma) = \#(h,\varphi,\rho)$. This completes the proof.

□

Lemma 5.12 allows us to create a rule $R_{5.12}$ which is the set of all couples

(
$\gamma\left[x_1:\varphi_1,\ldots,x_m:\varphi_m,\gamma\left[x_{m+1}:\varphi_{m+1},(\rightarrow)(\psi,\varphi)\right]\right]$,
$\gamma\left[x_1:\varphi_1,\ldots,x_m:\varphi_m,(\rightarrow)\left((\exists)\left(\{\}\left(x_{m+1}:\varphi_{m+1},\psi\right)\right),\varphi\right)\right]$
)
such that

- m is a positive integer, $x_1, \ldots , x_{m+1} \in V$, $x_i\neq x_j$ for $i\neq j$, $\varphi_1, .. , \varphi_{m+1}\in E$, $H[x_1:\varphi_1, \ldots , x_{m+1}:\varphi_{m+1}]$;

- if we define $k = k[x_1{:}\varphi_1, \dots , x_{m+1}{:}\varphi_{m+1}]$ and $h = k[x_1{:}\varphi_1, \dots , x_m{:}\varphi_m]$ then $\psi \in S(k)$ and $\varphi \in S(k) \cap S(h)$.

The next rule features the one commonly referred as 'modus ponens'.

Lemma 5.13

Let m be a positive integer. Let $x_1, \dots , x_m \in V$, with $x_i \neq x_j$ for $i \neq j$. Let $\varphi_1, .. , \varphi_m \in E$ and assume $H[x_1{:}\varphi_1, \dots , x_m{:}\varphi_m]$. Define $k = k[x_1{:}\varphi_1, \dots , x_m{:}\varphi_m]$ and let $\varphi, \psi, \chi \in S(k)$.

Under these assumptions we have $(\rightarrow)(\varphi,\psi)$, $(\rightarrow)(\varphi,(\rightarrow)(\psi,\chi))$, $(\rightarrow)(\varphi,\chi) \in S(k)$, and
$\gamma[x_1{:}\varphi_1, \dots , x_m{:}\varphi_m, (\rightarrow)(\varphi,\psi)] \in S(\varepsilon)$,
$\gamma[x_1{:}\varphi_1, \dots , x_m{:}\varphi_m, (\rightarrow)(\varphi,(\rightarrow)(\psi,\chi)] \in S(\varepsilon)$,
$\gamma[x_1{:}\varphi_1, \dots , x_m{:}\varphi_m, (\rightarrow)(\varphi,\chi)] \in S(\varepsilon)$.

Moreover if $\#(\ \gamma[x_1{:}\varphi_1, \dots , x_m{:}\varphi_m, (\rightarrow)(\varphi,\psi)]\ )$ and $\#(\ \gamma[x_1{:}\varphi_1, \dots , x_m{:}\varphi_m, (\rightarrow)(\varphi,(\rightarrow)(\psi,\chi)]\ )$ then $\#(\ \gamma[x_1{:}\varphi_1, \dots , x_m{:}\varphi_m, (\rightarrow)(\varphi,\chi)]\ )$.

Proof:

We can rewrite $\#(\gamma[x_1{:}\varphi_1, \dots , x_m{:}\varphi_m, (\rightarrow)(\varphi,\psi)])$ as follows:

$$P_\forall\Big(\{\}\big(\sigma\in \Xi(k), \#\big(k,(\rightarrow)(\varphi,\psi),\sigma\big)\big)\Big)$$

$$P_\forall\Big(\{\}\big(\sigma\in \Xi(k), P_\rightarrow\big(\#(k,\varphi,\sigma), \#(k,\psi,\sigma)\big)\big)\Big)\ .$$

And we can rewrite $\#(\ \gamma[x_1{:}\varphi_1, \dots , x_m{:}\varphi_m, (\rightarrow)(\varphi,(\rightarrow)(\psi,\chi)]\ )$ as follows

$$P_\forall\Big(\{\}\big(\sigma\in \Xi(k), \#\big(k,(\rightarrow)(\varphi,(\rightarrow)(\psi,\chi)),\sigma\big)\big)\Big)\ ,$$

$$P_\forall\Big(\{\}\big(\sigma\in \Xi(k), P_\rightarrow\big(\#(k,\varphi,\sigma), \#\big(k,(\rightarrow)(\psi,\chi),\sigma\big)\big)\big)\Big)\ ,$$

$$P_\forall\Big(\{\}\big(\sigma\in \Xi(k), P_\rightarrow\big(\#(k,\varphi,\sigma), P_\rightarrow\big(\#(k,\psi,\sigma), \#(k,\chi,\sigma)\big)\big)\big)\Big)\ .$$

Therefore, for each $\sigma\in\Xi(k)$ if $\#(k,\varphi,\sigma)$ then

- $\#(k,\psi,\sigma)$ holds, and
- if $\#(k,\psi,\sigma)$ then $\#(k,\chi,\sigma)$.
- hence $\#(k,\chi,\sigma)$ holds .

This can be formally rewritten as

$$P_\forall\Big(\{\}\big(\sigma\in \Xi(k), P_\rightarrow\big(\#(k,\varphi,\sigma), \#(k,\chi,\sigma)\big)\big)\Big)\ ,$$

$$P_\forall\Big(\{\}\big(\sigma\in \Xi(k), \#\big(k,(\rightarrow)(\varphi,\chi),\sigma\big)\big)\Big)\ ,$$

$\#(\ \gamma[x_1{:}\varphi_1, \dots , x_m{:}\varphi_m, (\rightarrow)(\varphi,\chi)]\ )$.

□

Lemma 5.13 allows us to create a rule $R_{5.13}$ which is the set of all 3-tuples

$(\gamma[x_1{:}\varphi_1, \dots, x_m{:}\varphi_m, (\rightarrow)(\varphi,\psi)], \gamma[x_1{:}\varphi_1, \dots, x_m{:}\varphi_m, (\rightarrow)(\varphi,(\rightarrow)(\psi,\chi)], \gamma[x_1{:}\varphi_1, \dots, x_m{:}\varphi_m, (\rightarrow)(\varphi,\chi)])$

such that

- m is a positive integer, $x_1, \dots, x_m \in V$, $x_i \neq x_j$ for $i \neq j$, $\varphi_1, .., \varphi_m \in E$, $H[x_1{:}\varphi_1, \dots, x_m{:}\varphi_m]$
- if we define $k = k[x_1{:}\varphi_1, \dots, x_m{:}\varphi_m]$ then $\varphi, \psi, \chi \in S(k)$.

Lemma 5.14

Let m be a positive integer. Let $x_1, \dots, x_{m+1} \in V$, with $x_i \neq x_j$ for $i \neq j$. Let $\varphi_1, .., \varphi_{m+1} \in E$ and assume $H[x_1{:}\varphi_1, \dots, x_{m+1}{:}\varphi_{m+1}]$. Define $k = k[x_1{:}\varphi_1, \dots, x_{m+1}{:}\varphi_{m+1}]$. Of course $H[x_1{:}\varphi_1, \dots, x_m{:}\varphi_m]$ also holds and we define $h = k[x_1{:}\varphi_1, \dots, x_m{:}\varphi_m]$. Let $\chi \in S(h)$.

Let $t \in E(h)$ such that $\forall \rho \in \Xi(h)\ \#(h,t,\rho) \in \#(h,\varphi_{m+1},\rho)$.
Let $\varphi \in S(k)$ such that $V_b(t) \cap V_b(\varphi) = \varnothing$.

We can define $\varphi_k\{x_{m+1}/t\} \in S(h)$ and $(\forall)(\{\}(x_{m+1}{:}\varphi_{m+1}, \varphi)) \in S(h)$.

Therefore $\gamma[x_1{:}\varphi_1, \dots, x_m{:}\varphi_m, (\rightarrow)(\chi,\varphi_k\{x_{m+1}/t\})]$ and
$\gamma[x_1{:}\varphi_1, \dots, x_m{:}\varphi_m, (\rightarrow)(\chi, (\forall)(\{\}(x_{m+1}{:}\varphi_{m+1}, \varphi)))]$ both belong to $S(\varepsilon)$ and

if $\#(\gamma[x_1{:}\varphi_1, \dots, x_m{:}\varphi_m, (\rightarrow)(\chi, (\forall)(\{\}(x_{m+1}{:}\varphi_{m+1}, \varphi)))])$ then
$\#(\gamma[x_1{:}\varphi_1, \dots, x_m{:}\varphi_m, (\rightarrow)(\chi,\varphi_k\{x_{m+1}/t\})])$.

Proof:

It's easy to see that $\varphi_k\{x_{m+1}/t\} \in S(h)$ (this has been shown in lemma 5.4).

By lemma 3.1 we get $(\forall)(\{\}(x_{m+1}{:}\varphi_{m+1}, \varphi)) \in S(h)$.

Suppose $\#(\gamma[x_1{:}\varphi_1, \dots, x_m{:}\varphi_m, (\rightarrow)(\chi, (\forall)(\{\}(x_{m+1}{:}\varphi_{m+1}, \varphi)))])$ holds, it can be rewritten as

$$P_\forall\Big(\{\}\Big(\rho \in \Xi(h), P_\rightarrow\Big(\#\big(h,\chi,\rho\big), \#\Big(h, (\forall)\big(\{\}\big(x_{m+1} : \varphi_{m+1}, \varphi\big)\big), \rho\Big)\Big)\Big)\Big) ,$$

$$P_\forall\Big(\{\}\Big(\rho \in \Xi(h), P_\rightarrow\Big(\#\big(h,\chi,\rho\big), P_\forall\Big(\{\}\big(\sigma \in \Xi(k) : \rho \sqsubseteq \sigma, \#\big(k,\varphi,\sigma\big)\big)\Big)\Big)\Big)\Big) .$$

We need to prove $\#(\gamma[x_1{:}\varphi_1, \dots, x_m{:}\varphi_m, (\rightarrow)(\chi,\varphi_k\{x_{m+1}/t\})])$, and this can be rewritten as

$$P_\forall\Big(\{\}\Big(\rho \in \Xi(h), P_\rightarrow\Big(\#\big(h,\chi,\rho\big), \#\big(h, \varphi_k\{x_{m+1}/t\}, \rho\big)\Big)\Big)\Big) .$$

Let $\rho \in \Xi(h)$ and suppose $\#(h,\chi,\rho)$. We need to show $\#(h,\varphi_k\{x_{m+1}/t\},\rho)$. By definition 4.6 we can define $\sigma \in \Xi(k)$ such that $\rho \sqsubseteq \sigma$ and $\#(h, \varphi_k\{x_{m+1}/t\}, \rho) = \#(k, \varphi, \sigma)$.
By our hypothesis we have $\#(k, \varphi, \sigma)$ and so $\#(h, \varphi_k\{x_{m+1}/t\}, \rho)$ holds. This completes the proof.

□

Lemma 5.14 allows us to create a rule $R_{5.14}$ which is the set of all couples

$(\gamma[x_1{:}\varphi_1, \dots , x_m{:}\varphi_m, (\rightarrow)(\chi, (\forall)(\{\}(x_{m+1}{:}\varphi_{m+1}, \varphi)))] , \gamma[x_1{:}\varphi_1, \dots , x_m{:}\varphi_m, (\rightarrow)(\chi,\varphi_k\{x_{m+1}/t\})])$

such that

- m is a positive integer, $x_1, \dots , x_{m+1} \in V$, with $x_i \neq x_j$ for $i \neq j$, $\varphi_1, .. , \varphi_{m+1} \in E$, $H[x_1{:}\varphi_1, \dots ,x_{m+1}{:}\varphi_{m+1}]$.
- if we define $k = k[x_1{:}\varphi_1, \dots , x_{m+1}{:}\varphi_{m+1}]$ and $h = k[x_1{:}\varphi_1, \dots , x_m{:}\varphi_m]$ then
  - $\chi \in S(h)$
  - $t \in E(h)$, $\forall \rho \in \Xi(h)\ \#(h,t,\rho) \in \#(h,\varphi_{m+1},\rho)$;
  - $\varphi \in S(k)$, $V_b(t) \cap V_b(\varphi) = \varnothing$.

Lemma 5.15

Let m be a positive integer. Let $x_1, \dots , x_{m+1} \in V$, with $x_i \neq x_j$ for $i \neq j$. Let $\varphi_1, .. , \varphi_{m+1} \in E$ and assume $H[x_1{:}\varphi_1, \dots , x_{m+1}{:}\varphi_{m+1}]$. Define $k = k[x_1{:}\varphi_1, \dots , x_{m+1}{:}\varphi_{m+1}]$. Of course $H[x_1{:}\varphi_1, \dots , x_m{:}\varphi_m]$ also holds and we define $h = k[x_1{:}\varphi_1, \dots , x_m{:}\varphi_m]$. Let $\chi \in S(h)$, $t \in E(h)$.
Let $\varphi \in E(h)$ be such that for each $\rho \in \Xi(h)$ $\#(h,\varphi,\rho)$ is a set and $x_{m+1} \notin V_b(\varphi)$.

Under these assumptions

$(\in)(x_{m+1},\varphi) \in S(k)$, $(\forall)(\{\}(x_{m+1}{:}\varphi_{m+1}, (\in)(x_{m+1},\varphi))) \in S(h)$,

$\gamma[x_1{:}\varphi_1, \dots , x_m{:}\varphi_m, (\rightarrow)(\chi, (\forall)(\{\}(x_{m+1}{:}\varphi_{m+1}, (\in)(x_{m+1},\varphi))))] \in S(\varepsilon)$,

$(\in)(t,\varphi_{m+1}) \in S(h)$,

$\gamma[x_1{:}\varphi_1, \dots , x_m{:}\varphi_m, (\rightarrow)(\chi, (\in)(t,\varphi_{m+1}))] \in S(\varepsilon)$,

$(\in)(t,\varphi) \in S(h)$

$\gamma[x_1{:}\varphi_1, \dots , x_m{:}\varphi_m, (\rightarrow)(\chi, (\in)(t,\varphi))] \in S(\varepsilon)$.

Moreover if

$\#(\gamma[x_1{:}\varphi_1, \dots , x_m{:}\varphi_m, (\rightarrow)(\chi, (\forall)(\{\}(x_{m+1}{:}\varphi_{m+1}, (\in)(x_{m+1},\varphi))))])$ and

$\#(\gamma[x_1{:}\varphi_1, \dots , x_m{:}\varphi_m, (\rightarrow)(\chi, (\in)(t,\varphi_{m+1}))])$ then

$\#(\gamma[x_1{:}\varphi_1, \dots , x_m{:}\varphi_m, (\rightarrow)(\chi, (\in)(t,\varphi))])$.

Proof:

Clearly $x_{m+1} \in E(k)$, and for each $\sigma \in \Xi(k)$ $\#(k, x_{m+1}, \sigma) = \sigma(x_{m+1})$.
There exist a positive integer n such that $\varphi \in E(n,h)$ and $x_{m+1} \notin V_b(\varphi)$, therefore $\varphi \in E(k)$ and for each $\sigma = \rho \parallel (x_{m+1},s) \in \Xi(k)$ $\#(k, \varphi, \sigma) = \#(h, \varphi, \rho)$ is a set.

This implies $(\in)(x_{m+1},\varphi) \in S(k)$, and by lemma 3.1 $(\forall)(\{\}(x_{m+1}{:}\varphi_{m+1}, (\in)(x_{m+1},\varphi))) \in S(h)$.

Clearly $\gamma[x_1{:}\varphi_1, \dots , x_m{:}\varphi_m, (\rightarrow)(\chi, (\forall)(\{\}(x_{m+1}{:}\varphi_{m+1}, (\in)(x_{m+1},\varphi))))]$ is in $S(\varepsilon)$.

Furthermore we have $\varphi_{m+1} \in E(h)$ and for each $\rho\in\Xi(h)$ $\#(h, \varphi_{m+1}, \rho)$ is a set, therefore $(\in)(t,\varphi_{m+1}) \in S(h)$, $\gamma[x_1{:}\varphi_1, \dots , x_m{:}\varphi_m, (\rightarrow)(\chi, (\in)(t,\varphi_{m+1}))] \in S(\varepsilon)$.

We have also $\varphi\in E(h)$ and for each $\rho\in\Xi(h)$ $\#(h,\varphi,\rho)$ is a set, therefore $(\in)(t,\varphi) \in S(h)$ and $\gamma[x_1{:}\varphi_1, \dots , x_m{:}\varphi_m, (\rightarrow)(\chi, (\in)(t,\varphi))] \in S(\varepsilon)$.

We now assume

$\#(\gamma[x_1{:}\varphi_1, \dots , x_m{:}\varphi_m, (\rightarrow)(\chi, (\forall)(\{\}(x_{m+1}{:}\varphi_{m+1}, (\in)(x_{m+1},\varphi))))])$ and

$\#(\gamma[x_1{:}\varphi_1, \dots , x_m{:}\varphi_m, (\rightarrow)(\chi, (\in)(t,\varphi_{m+1}))])$ both hold and we try to prove

$\#(\gamma[x_1{:}\varphi_1, \dots , x_m{:}\varphi_m, (\rightarrow)(\chi, (\in)(t,\varphi))])$.

We can rewrite $\#(\gamma[x_1{:}\varphi_1, \dots , x_m{:}\varphi_m, (\rightarrow)(\chi, (\forall)(\{\}(x_{m+1}{:}\varphi_{m+1}, (\in)(x_{m+1},\varphi))))])$ as

$$P_\forall\Big(\{\}\Big(\rho\in\Xi(h), P_\rightarrow\Big(\#\big(h,\chi,\rho\big), \#\Big(h,(\forall)\Big(\{\}\big(x_{m+1}:\varphi_{m+1},(\in)\big(x_{m+1},\varphi\big)\big)\Big),\rho\Big)\Big)\Big)\Big) ,$$

$$P_\forall\Big(\{\}\Big(\rho\in\Xi(h), P_\rightarrow\Big(\#\big(h,\chi,\rho\big), P_\forall\Big(\{\}\Big(\sigma\in\Xi(k):\rho\sqsubseteq\sigma, \#\big(k,(\in)\big(x_{m+1},\varphi\big),\sigma\big)\Big)\Big)\Big)\Big)\Big) ,$$

$$P_\forall\Big(\{\}\Big(\rho\in\Xi(h), P_\rightarrow\Big(\#\big(h,\chi,\rho\big), P_\forall\Big(\{\}\Big(\sigma\in\Xi(k):\rho\sqsubseteq\sigma, P_\in\big(\#\big(k,x_{m+1},\sigma\big),\#\big(k,\varphi,\sigma\big)\big)\Big)\Big)\Big)\Big)\Big) .$$

We can rewrite $\#(\gamma[x_1{:}\varphi_1, \dots , x_m{:}\varphi_m, (\rightarrow)(\chi, (\in)(t,\varphi_{m+1}))])$ as

$$P_\forall\Big(\{\}\Big(\rho\in\Xi(h), P_\rightarrow\Big(\#\big(h,\chi,\rho\big), \#\big(h,(\in)\big(t,\varphi_{m+1}\big),\rho\big)\Big)\Big)\Big) ,$$

$$P_\forall\Big(\{\}\Big(\rho\in\Xi(h), P_\rightarrow\Big(\#\big(h,\chi,\rho\big), P_\in\big(\#\big(h,t,\rho\big),\#\big(h,\varphi_{m+1},\rho\big)\big)\Big)\Big)\Big) .$$

We can rewrite $\#(\gamma[x_1{:}\varphi_1, \dots , x_m{:}\varphi_m, (\rightarrow)(\chi, (\in)(t,\varphi))])$ as

$$P_\forall\Big(\{\}\Big(\rho\in\Xi(h), P_\rightarrow\Big(\#\big(h,\chi,\rho\big), \#\big(h,(\in)\big(t,\varphi\big),\rho\big)\Big)\Big)\Big) ,$$

$$P_\forall\Big(\{\}\Big(\rho\in\Xi(h), P_\rightarrow\Big(\#\big(h,\chi,\rho\big), P_\in\big(\#\big(h,t,\rho\big),\#\big(h,\varphi,\rho\big)\big)\Big)\Big)\Big) .$$

Let $\rho\in\Xi(h)$ and assume $\#(h, \chi, \rho)$. We need to show that $\#(h,t,\rho)$ belongs to $\#(h,\varphi,\rho)$.

Let $\sigma = \rho \parallel (x_{m+1}, \#(h,t,\rho))$.

Since $k = h \parallel (x_{m+1}, \varphi_{m+1})$ and $\#(h,t,\rho) \in \#(h, \varphi_{m+1}, \rho)$ we have $\sigma\in\Xi(k)$. We have also $\#(k,x_{m+1},\sigma) \in \#(k,\varphi,\sigma)$, but $\#(k,x_{m+1},\sigma) = \sigma(x_{m+1}) = \#(h,t,\rho)$, so $\#(h,t,\rho) \in \#(k,\varphi,\sigma) = \#(h, \varphi, \rho)$.

□

Lemma 5.15 allows us to create a rule $R_{5.15}$ which is the set of all 3-tuples

(

$\gamma[x_1:\varphi_1, \ldots, x_m:\varphi_m, (\rightarrow)(\chi, (\forall)(\{\}(x_{m+1}:\varphi_{m+1}, (\in)(x_{m+1},\varphi))))]$,
$\gamma[x_1:\varphi_1, \ldots, x_m:\varphi_m, (\rightarrow)(\chi, (\in)(t,\varphi_{m+1}))]$,
$\gamma[x_1:\varphi_1, \ldots, x_m:\varphi_m, (\rightarrow)(\chi, (\in)(t,\varphi))]$
)

such that
- m is a positive integer, $x_1, \ldots, x_{m+1} \in V$, with $x_i \neq x_j$ for $i \neq j$, $\varphi_1, .., \varphi_{m+1} \in E$, $H[x_1:\varphi_1, \ldots ,x_{m+1}:\varphi_{m+1}]$.
- if we define $k = k[x_1:\varphi_1, \ldots, x_{m+1}:\varphi_{m+1}]$ and $h = k[x_1:\varphi_1, \ldots, x_m:\varphi_m]$ then
  - $\chi \in S(h)$
  - $t \in E(h)$,
  - $\varphi \in E(h)$, for each $\rho \in \Xi(h)$ $\#(h,\varphi,\rho)$ is a set and $x_{m+1} \notin V_b(\varphi)$.

Lemma 5.16

Let m be a positive integer. Let $x_1, \ldots, x_m \in V$, with $x_i \neq x_j$ for $i \neq j$. Let $\varphi_1, .., \varphi_m \in E$ and assume $H[x_1:\varphi_1, \ldots, x_m:\varphi_m]$. Define $k = k[x_1:\varphi_1, \ldots, x_m:\varphi_m]$.

Let i=1..m such that for each j=i..m $x_j \notin V_b(\varphi_i)$.

Then $(\in)(x_i,\varphi_i) \in S(k)$, $\gamma[x_1:\varphi_1, \ldots, x_m:\varphi_m, (\in)(x_i,\varphi_i)] \in S(\varepsilon)$ and

$\#(\gamma[x_1:\varphi_1, \ldots, x_m:\varphi_m, (\in)(x_i,\varphi_i)])$ holds.

Proof

By lemma 4.3 we have $x_i \in E(k_i)$ and for each $\rho \in \Xi(k_i)$ $\#(k_i, x_i, \rho) = \rho(x_i)$.

If i=m then $x_i \in E(k)$ and for each $\rho \in \Xi(k)$ $\#(k, x_i, \rho) = \rho(x_i)$.

If i<m, since for each j=i+1..m $x_j \notin V_b(x_i)$, by lemma 3.15 $x_i \in E(k)$ and for each $\sigma \in \Xi(k)$ there exists $\rho \in \Xi(k_i)$ such that $\rho \sqsubseteq \sigma$ and $\#(k, x_i, \sigma) = \#(k_i, x_i, \rho) = \rho(x_i)$.

It also results $\varphi_i \in E(k_{i-1})$ and for each $\rho \in \Xi(k_{i-1})$ $\#(k_{i-1}, \varphi_i, \rho)$ is a set, and for each j=i..m $x_j \notin V_b(\varphi_i)$.

We can apply lemma 3.15 and obtain that $\varphi_i \in E(k)$ and for each $\sigma \in \Xi(k)$ there exists $\rho \in \Xi(k_{i-1})$ such that $\rho \sqsubseteq \sigma$ and $\#(k, \varphi_i, \sigma) = \#(k_{i-1}, \varphi_i, \rho)$, so $\#(k, \varphi_i, \sigma)$ is a set.

By lemma 3.14 we derive that $(\in)(x_i,\varphi_i) \in S(k)$, and consequently

$\gamma[x_1:\varphi_1, \ldots, x_m:\varphi_m, (\in)(x_i,\varphi_i)] \in S(\varepsilon)$.

Moreover we can rewrite

$\#(\gamma[x_1:\varphi_1, \ldots, x_m:\varphi_m, (\in)(x_i,\varphi_i)])$ as follows

$$P_\forall\Big(\{\}\big(\sigma \in \Xi(k), \#\big(k,(\in)(x_i,\varphi_i),\sigma\big)\big)\Big) ,$$

$P_{\forall}\Big(\{\}\big(\sigma\in\Xi(k), P_{\in}\big(\#(k,x_i,\sigma),\#(k,\varphi_i,\sigma)\big)\big)\Big)$.

Given $\sigma\in\Xi(k)$ there exists $\rho\in\Xi(k_i)$ such that $\rho\sqsubseteq\sigma$ and $\#(k, x_i, \sigma) = \#(k_i, x_i, \rho) = \rho(x_i)$.

There also exists $\rho'\in\Xi(k_{i-1})$ such that $\rho'\sqsubseteq\sigma$ and $\#(k, \varphi_i, \sigma) = \#(k_{i-1}, \varphi_i, \rho')$.

There exists a positive integer n such that $k_{i-1} \in K(n)$, so $k_i\in K(n)^+$ and since $\rho\in\Xi(k_i)$ there exist $\eta\in\Xi(k_{i-1})$, $s\in\#(k_{i-1},\varphi_i,\eta)$ such that $\rho = \eta \parallel (x_i,s)$.

Clearly $\rho(x_i) = s \in \#(k_{i-1},\varphi_i,\eta)$, and $\rho' = \sigma/dom(\rho') = \sigma/dom(k_{i-1}) = \sigma/dom(\eta) = \eta$, thus $\rho(x_i) \in \#(k_{i-1},\varphi_i,\rho')$, in other words $\#(k, x_i, \sigma)$ belongs to $\#(k, \varphi_i, \sigma)$.

At this point we have shown that $\#(\ \gamma[x_1:\varphi_1, ... , x_m:\varphi_m, (\in)(x_i,\varphi_i)\ ]\ )$ holds .

□

Lemma 5.16 allows us to create an axiom $A_{5.16}$ which is the set of all sentences

$\gamma[x_1:\varphi_1, ... , x_m:\varphi_m, (\in)(x_i,\varphi_i)\ ]$

such that

- m is a positive integer, $x_1, \dots , x_m \in V$, $x_i\neq x_j$ for $i\neq j$, $\varphi_1, .. , \varphi_m\in E$, $H[x_1:\varphi_1, ... , x_m:\varphi_m]$
- i=1..m and for each j=i..m $x_j \notin V_b(\varphi_i)$.

We can also use lemma 3.7 (from section 3 of course) to create a rule which we call rule $R_{3,7}$. This is the set of all 3-tuples

(
$\gamma[x_1:\varphi_1, ... , x_m:\varphi_m, (\rightarrow) (\varphi, \psi_1)]$,
$\gamma[x_1:\varphi_1, ... , x_m:\varphi_m, (\rightarrow) (\varphi, \psi_2)]$,
$\gamma[x_1:\varphi_1, ... , x_m:\varphi_m, (\rightarrow) (\varphi, (\wedge) (\psi_1, \psi_2))]$
)

such that

- m is a positive integer, $x_1, \dots , x_m \in V$, $x_i\neq x_j$ for $i\neq j$, $\varphi_1, .. , \varphi_m\in E$, $H[x_1:\varphi_1, ... , x_m:\varphi_m]$
- if we define $k = k[x_1:\varphi_1, ... , x_m:\varphi_m]$ then $\varphi, \psi_1, \psi_2 \in S(k)$.

Lemma 5.17

Let m be a positive integer. Let $x_1, \dots , x_m \in V$, with $x_i\neq x_j$ for $i\neq j$. Let $\varphi_1, .. , \varphi_m\in E$ and assume $H[x_1:\varphi_1, ... , x_m:\varphi_m]$. Define $k = k[x_1:\varphi_1, ... , x_m:\varphi_m]$ and let $\varphi, \psi \in S(k)$.

Under these assumptions we have $(\rightarrow)(\ \varphi, (\wedge)(\psi, (\neg)(\psi))\ )$ and $(\neg)(\varphi) \in S(k)$,

$\gamma[x_1:\varphi_1, ... , x_m:\varphi_m, (\rightarrow)(\ \varphi, (\wedge)(\psi, (\neg)(\psi))\ )\ ]$, $\gamma[x_1:\varphi_1, ... , x_m:\varphi_m, (\neg)(\varphi)\ ] \in S(\varepsilon)$.

Moreover if

$\#(\ \gamma[x_1{:}\varphi_1, \ldots, x_m{:}\varphi_m, (\rightarrow)(\ \varphi, (\wedge)(\psi, (\neg)(\psi))\ )\ ]\ )$ then
$\#(\ \gamma[x_1{:}\varphi_1, \ldots, x_m{:}\varphi_m, (\neg)(\varphi)\ ]\ )$

<u>Proof:</u>

We can rewrite $\#(\ \gamma[x_1{:}\varphi_1, \ldots, x_m{:}\varphi_m, (\rightarrow)(\ \varphi, (\wedge)(\psi, (\neg)(\psi))\ )\ ]\ )$ as

$$P_\forall\Big(\{\}\Big(\sigma\in\Xi(k),\#\Big(k,(\rightarrow)\big(\varphi,(\wedge)\big(\psi,(\neg)(\psi)\big)\big),\sigma\Big)\Big)\Big)\ ,$$
$$P_\forall\Big(\{\}\Big(\sigma\in\Xi(k),P_\rightarrow\Big(\#(k,\varphi,\sigma),\#\Big(k,(\wedge)\big(\psi,(\neg)(\psi)\big),\sigma\Big)\Big)\Big)\Big)\ ,$$
$$P_\forall\Big(\{\}\Big(\sigma\in\Xi(k),P_\rightarrow\Big(\#(k,\varphi,\sigma),P_\wedge\Big(\#(k,\psi,\sigma),\#\big(k,(\neg)(\psi),\sigma\big)\Big)\Big)\Big)\Big)\ ,$$
$$P_\forall\Big(\{\}\Big(\sigma\in\Xi(k),P_\rightarrow\Big(\#(k,\varphi,\sigma),P_\wedge\Big(\#(k,\psi,\sigma),P_\neg\big(\#(k,\psi,\sigma)\big)\Big)\Big)\Big)\Big)\ .$$

This can be expressed as

'for each $\sigma\in\Xi(k)$ $\#(k,\varphi,\sigma)$ is false or both $\#(k,\psi,\sigma)$ and ($\#(k,\psi,\sigma)$ is false) are true'

Since $\#(k,\psi,\sigma)$ and ($\#(k,\psi,\sigma)$ is false) cannot be both true we have that

for each $\sigma\in\Xi(k)$ $\#(k,\varphi,\sigma)$ is false.

This can be formally expressed as

$$P_\forall\Big(\{\}\Big(\sigma\in\Xi(k),P_\neg\big(\#(k,\varphi,\sigma)\big)\Big)\Big)\ ,$$
$$P_\forall\Big(\{\}\Big(\sigma\in\Xi(k),\#\big(k,(\neg)(\varphi),\sigma\big)\Big)\Big)\ ,$$

which we can finally rewrite as

$\#(\ \gamma[x_1{:}\varphi_1, \ldots, x_m{:}\varphi_m, (\neg)(\varphi)\ ]\ )$.

□

Lemma 5.17 allows us to create a rule $R_{5.17}$ which is the set of all couples

$(\ \gamma[x_1{:}\varphi_1, \ldots, x_m{:}\varphi_m, (\rightarrow)(\ \varphi, (\wedge)(\psi, (\neg)(\psi))\ )\ ], \gamma[x_1{:}\varphi_1, \ldots, x_m{:}\varphi_m, (\neg)(\varphi)\ ]\ )$

such that

- m is a positive integer, $x_1, \ldots, x_m \in V$, $x_i \neq x_j$ for $i\neq j$, $\varphi_1, \ldots, \varphi_m \in E$, $H[x_1{:}\varphi_1, \ldots, x_m{:}\varphi_m]$
- if we define $k = k[x_1{:}\varphi_1, \ldots, x_m{:}\varphi_m]$ then $\varphi, \psi \in S(k)$.

Lemma 5.18

Let m be a positive integer. Let $x_1, \dots, x_m \in V$, with $x_i \neq x_j$ for $i \neq j$. Let $\varphi_1, .., \varphi_m \in E$ and assume $H[x_1{:}\varphi_1, \dots, x_m{:}\varphi_m]$. Define $k = k[x_1{:}\varphi_1, \dots, x_m{:}\varphi_m]$ and let $\varphi, \psi \in S(k)$.

Under these assumptions we have $(\neg)(\,(\wedge)(\varphi,\psi)\,), (\rightarrow)(\,\varphi, (\neg)(\psi)\,) \in S(k)$,

$\gamma[x_1{:}\varphi_1, \dots, x_m{:}\varphi_m, (\neg)(\,(\wedge)(\varphi,\psi)\,)\,], \gamma[x_1{:}\varphi_1, \dots, x_m{:}\varphi_m, (\rightarrow)(\,\varphi, (\neg)(\psi)\,)\,] \in S(\varepsilon)$.

Moreover if

$\#(\,\gamma[x_1{:}\varphi_1, \dots, x_m{:}\varphi_m, (\neg)(\,(\wedge)(\varphi,\psi)\,)\,]\,)$ then
$\#(\,\gamma[x_1{:}\varphi_1, \dots, x_m{:}\varphi_m, (\rightarrow)(\,\varphi, (\neg)(\psi)\,)\,]\,)$.

Proof:

We can rewrite $\#(\,\gamma[x_1{:}\varphi_1, \dots, x_m{:}\varphi_m, (\neg)(\,(\wedge)(\varphi,\psi)\,)\,]\,)$ as

$$P_\forall\Big(\{\}\big(\sigma\in\Xi(k), \#\big(k,(\neg)\big((\wedge)(\varphi,\psi)\big),\sigma\big)\big)\Big) ,$$

$$P_\forall\Big(\{\}\big(\sigma\in\Xi(k), P_\neg\big(\#\big(k,(\wedge)(\varphi,\psi),\sigma\big)\big)\big)\Big) ,$$

$$P_\forall\Big(\{\}\big(\sigma\in\Xi(k), P_\neg\big(P_\wedge\big(\#(k,\varphi,\sigma),\#(k,\psi,\sigma)\big)\big)\big)\Big) .$$

We can rewrite $\#(\,\gamma[x_1{:}\varphi_1, \dots, x_m{:}\varphi_m, (\rightarrow)(\,\varphi, (\neg)(\psi)\,)\,]\,)$ as

$$P_\forall\Big(\{\}\big(\sigma\in\Xi(k), \#\big(k,(\rightarrow)\big(\varphi,(\neg)(\psi)\big),\sigma\big)\big)\Big) ,$$

$$P_\forall\Big(\{\}\big(\sigma\in\Xi(k), P_\rightarrow\big(\#(k,\varphi,\sigma),P_\neg\big(\#(k,\psi,\sigma)\big)\big)\big)\Big) .$$

Thus if $\#(\,\gamma[x_1{:}\varphi_1, \dots, x_m{:}\varphi_m, (\neg)(\,(\wedge)(\varphi,\psi)\,)\,]\,)$ we have that

for each $\sigma\in\Xi(k)$ it is false that $\#(k,\varphi,\sigma)$ and $\#(k,\psi,\sigma)$ are both true.

In other words for each $\sigma\in\Xi(k)$ ($\#(k,\varphi,\sigma)$ is false) or ($\#(k,\psi,\sigma)$ is false) .

In other words for each $\sigma\in\Xi(k)$ $P_\rightarrow\big(\#(k,\varphi,\sigma),P_\neg\big(\#(k,\psi,\sigma)\big)\big)$,

And this condition clearly implies $\#(\,\gamma[x_1{:}\varphi_1, \dots, x_m{:}\varphi_m, (\rightarrow)(\,\varphi, (\neg)(\psi)\,)\,]\,)$.

□

Lemma 5.18 allows us to create a rule $R_{5.18}$ which is the set of all couples

$(\,\gamma[x_1{:}\varphi_1, \dots, x_m{:}\varphi_m, (\neg)(\,(\wedge)(\varphi,\psi)\,)\,], \gamma[x_1{:}\varphi_1, \dots, x_m{:}\varphi_m, (\rightarrow)(\,\varphi, (\neg)(\psi)\,)\,]\,)$

such that

- m is a positive integer, $x_1, \dots, x_m \in V$, $x_i \neq x_j$ for $i \neq j$, $\varphi_1, \dots, \varphi_m \in E$, $H[x_1{:}\varphi_1, \dots, x_m{:}\varphi_m]$
- if we define $k = k[x_1{:}\varphi_1, \dots, x_m{:}\varphi_m]$ then $\varphi, \psi \in S(k)$.

Lemma 5.19

Let m be a positive integer. Let $x_1, \dots, x_{m+1} \in V$, with $x_i \neq x_j$ for $i \neq j$. Let $\varphi_1, \dots, \varphi_{m+1} \in E$ and assume $H[x_1{:}\varphi_1, \dots, x_{m+1}{:}\varphi_{m+1}]$. Define $k = k[x_1{:}\varphi_1, \dots, x_{m+1}{:}\varphi_{m+1}]$. Of course $H[x_1{:}\varphi_1, \dots, x_m{:}\varphi_m]$ also holds and we define $h = k[x_1{:}\varphi_1, \dots, x_m{:}\varphi_m]$. Let $\varphi \in S(k)$.

Under these assumptions we have (by lemmas 3.1 and 3.6)

$(\forall)(\{\}(x_{m+1}{:}\varphi_{m+1}, \varphi)) \in S(h)$, $(\neg)((\forall)(\{\}(x_{m+1}{:}\varphi_{m+1}, \varphi))) \in S(h)$,
$\gamma[x_1{:}\varphi_1, \dots, x_m{:}\varphi_m, (\neg)((\forall)(\{\}(x_{m+1}{:}\varphi_{m+1}, \varphi)))] \in S(\varepsilon)$;

$(\neg)(\varphi) \in S(k)$, $(\exists)(\{\}(x_{m+1}{:}\varphi_{m+1}, (\neg)(\varphi))) \in S(h)$,
$\gamma[x_1{:}\varphi_1, \dots, x_m{:}\varphi_m, (\exists)(\{\}(x_{m+1}{:}\varphi_{m+1}, (\neg)(\varphi)))] \in S(\varepsilon)$.

Moreover if $\#(\gamma[x_1{:}\varphi_1, \dots, x_m{:}\varphi_m, (\neg)((\forall)(\{\}(x_{m+1}{:}\varphi_{m+1}, \varphi)))])$ then
$\#(\gamma[x_1{:}\varphi_1, \dots, x_m{:}\varphi_m, (\exists)(\{\}(x_{m+1}{:}\varphi_{m+1}, (\neg)(\varphi)))])$.

Proof:

We can rewrite $\#(\gamma[x_1{:}\varphi_1, \dots, x_m{:}\varphi_m, (\neg)((\forall)(\{\}(x_{m+1}{:}\varphi_{m+1}, \varphi)))])$ as follows:

$$P_\forall\Big(\{\}\Big(\rho\in \Xi(h), \#\Big(h,(\neg)\Big((\forall)\big(\{\}(x_{m+1}:\varphi_{m+1},\varphi)\big)\Big),\rho\Big)\Big)\Big) ,$$
$$P_\forall\Big(\{\}\Big(\rho\in \Xi(h), P_\neg\Big(\#\Big(h,(\forall)\big(\{\}(x_{m+1}:\varphi_{m+1},\varphi)\big),\rho\Big)\Big)\Big)\Big) ,$$
$$P_\forall\Big(\{\}\Big(\rho\in \Xi(h), P_\neg\Big(P_\forall\Big(\{\}\big(\sigma\in \Xi(k):\rho\sqsubseteq\sigma,\#(k,\varphi,\sigma)\big)\Big)\Big)\Big)\Big) .$$

In words this can be expressed as:

'for each $\rho \in \Xi(h)$ it is false that (for each $\sigma \in \Xi(k)$ such that $\rho \sqsubseteq \sigma$ $\#(k,\varphi,\sigma)$)', or also
'for each $\rho \in \Xi(h)$ (there exists $\sigma \in \Xi(k)$ such that $\rho \sqsubseteq \sigma$ and ($\#(k,\varphi,\sigma)$ is false))' .

We can rewrite $\#(\gamma[x_1{:}\varphi_1, \dots, x_m{:}\varphi_m, (\exists)(\{\}(x_{m+1}{:}\varphi_{m+1}, (\neg)(\varphi)))])$ as follows:

$$P_\forall\Big(\{\}\Big(\rho\in \Xi(h), \#\Big(h,(\exists)\big(\{\}(x_{m+1}:\varphi_{m+1},(\neg)(\varphi))\big),\rho\Big)\Big)\Big) ,$$
$$P_\forall\Big(\{\}\Big(\rho\in \Xi(h), P_\exists\Big(\{\}\big(\sigma\in \Xi(k):\rho\sqsubseteq\sigma,\#(h,(\neg)(\varphi),\sigma)\big)\Big)\Big)\Big) ,$$
$$P_\forall\Big(\{\}\Big(\rho\in \Xi(h), P_\exists\Big(\{\}\big(\sigma\in \Xi(k):\rho\sqsubseteq\sigma,P_\neg(\#(h,\varphi,\sigma))\big)\Big)\Big)\Big) .$$

Let $\rho \in \Xi(h)$, we need to show that there exists $\sigma \in \Xi(k)$ such that $\rho \sqsubseteq \sigma$ and $\#(h,\varphi,\sigma)$ is false.
We have seen this directly follows by our hypothesis.

□

Lemma 5.19 allows us to create a rule $R_{5.19}$ which is the set of all couples

$(\gamma[x_1{:}\varphi_1, \dots , x_m{:}\varphi_m,\ (\neg)((\forall) (\{\}(x_{m+1}{:}\varphi_{m+1}, \varphi)))],$
$\gamma[x_1{:}\varphi_1, \dots , x_m{:}\varphi_m,\ (\exists) (\{\}(x_{m+1}{:}\varphi_{m+1}, (\neg)(\varphi)))])$

such that
- m is a positive integer, $x_1, \dots , x_{m+1} \in V$, with $x_i \neq x_j$ for $i \neq j$, $\varphi_1, .. , \varphi_{m+1} \in E$, $H[x_1{:}\varphi_1, \dots ,x_{m+1}{:}\varphi_{m+1}]$.
- if we define $k = k[x_1{:}\varphi_1, \dots , x_{m+1}{:}\varphi_{m+1}]$ then $\varphi \in S(k)$.

The following rule is a degenerate case of rules $R_{5.11}$ and $R_{5.12}$.

Lemma 5.20

Let $x_1 \in V$, $\varphi_1 \in E$ and assume $H[x_1{:}\varphi_1]$. Define $k = k[x_1{:}\varphi_1]$. Let $\psi \in S(k)$ and $\varphi \in S(k) \cap S(\varepsilon)$.

Under these assumptions we have

$(\rightarrow)(\psi,\varphi) \in S(k)$, $\gamma[x_1{:}\varphi_1, (\rightarrow)(\psi,\varphi)] \in S(\varepsilon)$,

$(\exists) (\{\}(x_1{:}\varphi_1, \psi)) \in S(\varepsilon)$, $(\rightarrow)\big((\exists)(\{\}(x_1 : \varphi_1,\psi)),\varphi\big) \in S(\varepsilon)$;

Moreover if $\#\big(\gamma\big[x_1 : \varphi_1,(\rightarrow)(\psi,\varphi)\big]\big)$ then $\#\big((\rightarrow)\big((\exists)(\{\}(x_1 : \varphi_1,\psi)),\varphi\big)\big)$.

Proof:

Suppose $\#\big(\gamma\big[x_1 : \varphi_1,(\rightarrow)(\psi,\varphi)\big]\big)$. We can rewrite this as

$P_\forall\Big(\{\}\big(\rho \in \Xi(k), \#\big(k,(\rightarrow)(\psi,\varphi),\rho\big)\big)\Big)$,
$P_\forall\Big(\{\}\big(\rho \in \Xi(k), P_\rightarrow\big(\#(k,\psi,\rho),\#(k,\varphi,\rho)\big)\big)\Big)$.

In turn $\#\big((\rightarrow)\big((\exists)(\{\}(x_1 : \varphi_1,\psi)),\varphi\big)\big)$ can be rewritten as
$P_\rightarrow\Big(\#\big((\exists)(\{\}(x_1 : \varphi_1,\psi))\big),\#(\varphi)\Big)$,
$P_\rightarrow\Big(P_\exists\big(\{\}\big(\rho \in \Xi(k),\#(k,\psi,\rho)\big)\big),\#(\varphi)\Big)$.

To prove the last statement we suppose there exists $\rho \in \Xi(k)$ such that $\#(k,\psi,\rho)$. This implies $\#(k,\varphi,\rho)$ holds, but we need to show that $\#(\varphi)$ holds.

Since $\varphi \in S(k)$ $V_b(\varphi) \subseteq V$-dom(k), so $x_1 \notin V_b(\varphi)$. So by lemma 4.2 $\#(k,\varphi,\rho) = \#(\varepsilon,\varphi,\varepsilon) = \#(\varphi)$.
This completes the proof.

□

Lemma 5.20 allows us to create a rule $R_{5.20}$ which is the set of all couples

(
$\gamma\left[x_1:\varphi_1,(\rightarrow)(\psi,\varphi)\right]$,
$(\rightarrow)\left((\exists)(\{\}(x_1:\varphi_1,\psi)),\varphi\right)$
)

such that
- $x_1 \in V$, $\varphi_1 \in E$, $H[x_1:\varphi_1]$;
- if we define $k = k[x_1:\varphi_1]$ then $\psi \in S(k)$ and $\varphi \in S(k) \cap S(\varepsilon)$.

Lemma 5.21

Let $\varphi, \psi, \chi \in S(\varepsilon)$. We have $(\rightarrow)(\varphi, (\rightarrow)(\psi,\chi)) \in S(\varepsilon)$ and $(\rightarrow)((\wedge)(\varphi,\psi), \chi) \in S(\varepsilon)$.

Moreover if $\#((\rightarrow)(\varphi, (\rightarrow)(\psi,\chi)))$ then $\#((\rightarrow)((\wedge)(\varphi,\psi), \chi))$

Proof:

Suppose $\#((\rightarrow)(\varphi, (\rightarrow)(\psi,\chi)))$ holds. It can be rewritten

$$P_{\rightarrow}\left(\#(\varphi),\#\left((\rightarrow)(\psi,\chi)\right)\right) \quad ,$$
$$P_{\rightarrow}\left(\#(\varphi),P_{\rightarrow}\left(\#(\psi),\#(\chi)\right)\right) \quad .$$

In turn, $\#((\rightarrow)((\wedge)(\varphi,\psi), \chi))$ can be rewritten

$$P_{\rightarrow}\left(\#\left((\wedge)(\varphi,\psi)\right),\#(\chi)\right)$$
$$P_{\rightarrow}\left(P_{\wedge}\left(\#(\varphi),\#(\psi)\right),\#(\chi)\right) \; .$$

Suppose $\#(\varphi)$ and $\#(\psi)$ both hold, we need to show that $\#(\chi)$ holds. This is granted by $P_{\rightarrow}\left(\#(\varphi),P_{\rightarrow}\left(\#(\psi),\#(\chi)\right)\right)$.

□

Lemma 5.21 allows us to create a rule $R_{5.21}$ which is the set of all couples

(
$(\rightarrow)(\varphi, (\rightarrow)(\psi,\chi))$,
$(\rightarrow)((\wedge)(\varphi,\psi), \chi)$
)

such that $\varphi, \psi, \chi \in S(\varepsilon)$.

We have listed a set of axioms and rules that we need to complete the deduction examples in the next sections. Actually there is one rule, rule $R_{5.14}$, which will not be used in our examples, but it is listed since I perceive it as an important and useful rule.

In all our deductions we can assume our language extends the one we are using, so these ('general-purpose') axioms and rules can be applied in all our deductions. Often in a deduction we need also to apply rules which are more context-specific, in fact our first deduction example will require additional rules, and we'll see those rules in the next section.

For each axiom and rule we have checked its soundness. In fact, in our definition, axioms and rules are not such if they aren't sound. As we saw at the beginning of this section, thanks to the soundness of axioms and rules, the soundness of the deductive system is granted.

We can now ask ourselves if we defined all the 'general-purpose' (axioms and) rules we could need in a deduction. It seems an interesting question, but its meaning may sound unclear. It may be unclear what is the 'completeness' property we wish to verify with respect to our set of rules.

One first approach to the question is: have we listed all the axioms (and all the rules) we can create in our language? We know that for each $\varphi \in S(\varepsilon)$ if we can prove that $\#(\varphi)$ holds then we can use $\{\varphi\}$ as an axiom. Clearly it can be difficult to enumerate all the sentences φ such that we can prove $\#(\varphi)$. Moreover, if you look at the formal definition of axiom, $\{\varphi\}$ is an axiom whenever $\#(\varphi)$ holds (it doesn't matter whether we are able to prove $\#(\varphi)$). It seems trying to list all the axioms is not the correct approach to our question.

For another approach, let's call D_0 the deductive system we have listed in this section and consider the following property:

i) for each $\varphi \in S(\varepsilon)$ if $\#(\varphi)$ holds then we can derive φ in D_0.

This condition states the 'full completeness' of the system, but I'm not sure there exists a somehow enumerable set of (axioms and) rules which is able to ensure this kind of completeness. As we know Gödel has proved an incompleteness theorem that (under appropriate conditions) states more or less the negation of the completeness condition i). In Cutland's book ([3]) there is an interesting discussion about this, for accuracy we say that Cutland presents a 'simplified version of Gödel's incompleteness theorem' which, in the appropriate context, states the negation of the completeness condition i). In this paper I don't want to discuss if and how this theorem applies to our system, though (based on some past work) I suspect under certain conditions it could apply to our system. Moreover, though itself interesting and related to our question, the condition i) could slightly differ from the property we want to discuss.

The right approach seems to be like the following. We call $\Delta(L)$ the set of all deductive systems built on top of language L. We say that a sentence φ is 'provable' if there exists D in $\Delta(L)$ such that we can derive φ in D. Now consider the following two conditions:

ii) for each $\varphi \in S(\varepsilon)$ if $\#(\varphi)$ holds then φ is provable
iii) for each $\varphi \in S(\varepsilon)$ if φ is provable then we can derive φ in D_0 .

Clearly the condition i) holds when both ii) and iii) hold.

The property ii) seems to refer to the completeness of the deductive methodology, but we cannot fail to notice that, with our definition of axiom, if $\#(\varphi)$ holds then $\{\varphi\}$ is an axiom and so φ is provable. Therefore the condition ii) seems to be a tautology, and i) and iii) are thus equivalent. This may sound strange, because it may be the case that $\#(\varphi)$ holds and we cannot prove $\#(\varphi)$, so we'll not be able to use $\{\varphi\}$ as an axiom, but in our definition of axiom we could not require that

#(φ) is provable, since we had no formal notion of 'being provable' to use in that case. This might possibly be a kind of problem with the current state of our theory, but as far as we cannot see a solution to this, conditions i) and iii) are equivalent. The condition iii) can be seen as a formal definition of completeness for a specific deductive system, but we have no answer to the question whether iii) holds or not. Probably it doesn't hold for our D_0 and we don't know if and how we can improve D_0 to ensure it.

So we simply add rules because while performing a deduction we discover we need them, or because common sense tells us they can be useful.
With respect to the existential quantifier , we have introduced a rule $R_{5.7}$ which permits to introduce it and two similar rules $R_{5.11}$ and $R_{5.12}$ allowing to exploit it, and so to eliminate it. As for the universal quantifier, a rule permitting the introduction seems not make sense, while it seems appropriate to introduce a rule, in some way related to rule $R_{5,7}$, allowing the elimination, and this is rule $R_{5.14}$ we have introduced though it was not required in our examples.

6. A deduction example

For each x, y natural numbers we say that x divides y if there exists a natural number α such that $y = x * \alpha$.

In our example we want to show that for each x, y, z natural numbers if x divides y and y divides z then x divides z.

Of course, we first need to build an expression in our language to express this. To build that expression we must extend our language with two other constant symbols:

- a constant symbol ℕ to represent the set of natural numbers ℕ, so that we have #(ℕ) = ℕ ;
- a constant symbol | to represent the 'divides' relation, so that #(|) is a function defined on ℕ×ℕ and we have $\#(|)(\alpha,\beta) = (\exists\eta\in\mathbb{N}: \beta = \alpha\bullet\eta)$.

The statement we wish to prove is the following:

$$\gamma\Big[x:\mathbb{N}, y:\mathbb{N}, z:\mathbb{N}, (\rightarrow)\big((\wedge)\big((|)(x,y),(|)(y,z)\big),(|)(x,z)\big)\Big] \text{ (S1)} ,$$

where x,y,z of course are variables in our language.

First of all we need to know this is a sentence in our language and we need to see its meaning is as expected. To this purpose we'll use the following technical lemma.

Lemma 6.1

Let m be a positive integer, $x_1, .., x_m \in V$, with $x_i \neq x_j$ for $i \neq j$.
We have $H[x_1:\mathbb{N}, .. , x_m:\mathbb{N}]$ and we define $k = k[x_1:\mathbb{N}, .. , x_m:\mathbb{N}]$.

Then for each i=1..m $x_i \in E(k)$ and for each $\sigma\in\Xi(k)$ $\#(k, x_i, \sigma) = \sigma(x_i) \in \mathbb{N}$.

Moreover for each $\alpha_1, .., \alpha_m \in \mathbb{N}$ if we define $\sigma = (x_1,\alpha_1) \parallel \ldots \parallel (x_m,\alpha_m)$ then $\sigma\in\Xi(k)$.

Proof:

We first need to show $H[x_1:\mathbb{N}, .. , x_m:\mathbb{N}]$ holds.

First consider that $\mathbb{N}\in E(\varepsilon)$ and $\#(\mathbb{N})=\mathbb{N}$ is a set, so we can define $k_1 = (x_1,\mathbb{N})$.

If m>1 then for each i=1..m-1 we suppose to have defined $k_i = (x_1,\mathbb{N}) \parallel .. \parallel (x_i,\mathbb{N})$. By lemma 3.12 $\mathbb{N}\in E(k_i)$ and for each $\rho\in\Xi(k_i)$ $\#(k_i,\mathbb{N},\rho) = \#(\mathbb{N})$ is a set, so we can define $k_{i+1} = k_i \parallel (x_{i+1},\mathbb{N})$.

This proves that $H[x_1:\mathbb{N}, .. , x_m:\mathbb{N}]$ holds.

We have $k_i = k_{i-1} \parallel (x_i, \mathbb{N})$. There exists a positive integer n such that $k_{i-1} \in K(n)$, so $k_i \in K(n)^+$ and $x_i \in E(n+1,k_i) \subseteq E(k_i)$, for each $\sigma = \rho \parallel (x_i,s) \in \Xi(k_i)$ $\#(k_i, x_i, \sigma) = \sigma(x_i) = s \in \#(k_{i-1}, \mathbb{N}, \rho) = \#(\mathbb{N}) = \mathbb{N}$.

If i<m then for each j=i..m-1 we can assume $x_i \in E(k_j)$ and for each $\rho \in \Xi(k_j)$ $\#(k_j, x_i, \rho) = \rho(x_i) \in \mathbb{N}$. There exists a positive integer n such that $x_i \in E(n,k_j)$. Moreover $k_{j+1} \in K(n)^+$, $x_{j+1} \notin V_b(x_i)$, so $x_i \in E(n+1,k_{j+1}) \subseteq E(k_{j+1})$, and for each $\sigma = \rho \parallel (x_{j+1},s) \in \Xi(k_{j+1})$ $\#(k_{j+1},x_i,\sigma) = \#(k_j,x_i,\rho) = \rho(x_i) =$
$= \sigma(x_i)$, and $\sigma(x_i) = \rho(x_i) \in \mathbb{N}$.

Let $\alpha_1, .., \alpha_m \in \mathbb{N}$ and let $\sigma = (x_1,\alpha_1) \parallel \ldots \parallel (x_m,\alpha_m)$.

We define $\sigma_0 = \varepsilon$ and so we have $\sigma_0 \in \Xi(\varepsilon)=\Xi(k_0)$.

Given i=0..m-1 we assume we have defined $\sigma_i = (x_1,\alpha_1) \parallel \ldots \parallel (x_i,\alpha_i)$ and proved that $\sigma_i \in \Xi(k_i)$.
We define $\sigma_{i+1} = \sigma_i \parallel (x_{i+1},\alpha_{i+1})$. We know that $k_i \in K$, so there exists a positive integer n such that $k_i \in K(n)$, and $k_{i+1} = k_i \parallel (x_{i+1}, \mathbb{N}) \in K(n)^+$. Moreover $\alpha_{i+1} \in \mathbb{N}=\#(\mathbb{N})$, so $\sigma_{i+1} \in \Xi(k_{i+1})$.

□

To show that expression (S1) belongs to S(ε) we define $k = k[x:\mathbb{N}, y:\mathbb{N}, z:\mathbb{N}]$. By 6.1 we obtain that $x,y,z \in E(k)$, and for each $\sigma \in \Xi(k)$ $\#(k,x,\sigma) = \sigma(x) \in \mathbb{N}$, $\#(k,y,\sigma) = \sigma(y) \in \mathbb{N}$, $\#(k,z,\sigma) = \sigma(z) \in \mathbb{N}$.

Moreover $| \in E(k)$, for each $\sigma \in \Xi(k)$ $\#(k, |, \sigma) = \#(|)$ is a function with 2 arguments and
$(\#(k,x,\sigma), \#(k,y,\sigma))$, $(\#(k,y,\sigma), \#(k,z,\sigma))$, $(\#(k,x,\sigma), \#(k,z,\sigma))$ are members of its domain.

So, by lemma 3.9, $(|)(x,y)$, $(|)(y,z)$, $(|)(x,z)$ belong to E(k).

Moreover, for each $\sigma \in \Xi(k)$ $\#(k, (|)(x,y), \sigma) = \#(|) (\#(k,x,\sigma), \#(k,y,\sigma)) = \#(|) (\sigma(x), \sigma(y)) =$
$= (\exists \eta \in \mathbb{N}: \sigma(y) = \sigma(x) \bullet \eta)$, so $\#(k, (|)(x,y), \sigma)$ is true or false and $(|)(x,y) \in S(k)$. In the same way we can show that $(|)(y,z) \in S(k)$, $(|)(x,z) \in S(k)$.

By lemma 3.6 we have

$$(\wedge)\big((|)(x,y),(|)(y,z)\big) \in S(k),$$
$$(\rightarrow)\Big((\wedge)\big((|)(x,y),(|)(y,z)\big),(|)(x,z)\Big) \in S(k).$$

By definition 3.3

$$\gamma\Big[x:\mathbb{N}, y:\mathbb{N}, z:\mathbb{N}, (\rightarrow)\Big((\wedge)\big((|)(x,y),(|)(y,z)\big),(|)(x,z)\Big)\Big] \in S(\varepsilon) \ .$$

By theorem 3.5

$\#\Big(\gamma\Big[x:\mathbb{N}, y:\mathbb{N}, z:\mathbb{N}, (\rightarrow)\Big((\wedge)\big((|)(x,y),(|)(y,z)\big),(|)(x,z)\Big)\Big]\Big)$ is equivalent to

$P_\forall\Big(\{\}\Big(\sigma\in\Xi(k),\#\Big(k,(\to)\big((\wedge)\big((|)(x,y),(|)(y,z)\big),(|)(x,z)\big),\sigma\Big)\Big)\Big)$.

And then this can be rewritten in the following ways:

$P_\forall\Big(\{\}\Big(\sigma\in\Xi(k),P_\to\Big(\#\big(k,(\wedge)\big((|)(x,y),(|)(y,z)\big),\sigma\big),\#\big(k,(|)(x,z),\sigma\big)\Big)\Big)\Big)$,

$P_\forall\Big(\{\}\Big(\sigma\in\Xi(k),P_\to\Big(P_\wedge\Big(\#\big(k,(|)(x,y),\sigma\big),\#\big(k,(|)(y,z),\sigma\big)\Big),\#\big(k,(|)(x,z),\sigma\big)\Big)\Big)\Big)$,

$P_\forall\Big(\{\}\Big(\sigma\in\Xi(k),P_\to\Big(P_\wedge\Big(\#(|)\big(\#(k,x,\sigma),\#(k,y,\sigma)\big),\#(|)\big(\#(k,y,\sigma),\#(k,z,\sigma)\big)\Big),\#(|)\big(\#(k,y,\sigma),\#(k,z,\sigma)\big)\Big)\Big)\Big)$,

$P_\forall\Big(\{\}\Big(\sigma\in\Xi(k),P_\to\Big(P_\wedge\Big(\#(|)\big(\sigma(x),\sigma(y)\big),\#(|)\big(\sigma(y),\sigma(z)\big)\Big),\#(|)\big(\sigma(x),\sigma(z)\big)\Big)\Big)\Big)$.

The last statement can be rewritten

For each $\sigma\in\Xi(k)$ $P_\to\Big(P_\wedge\Big(\#(|)\big(\sigma(x),\sigma(y)\big),\#(|)\big(\sigma(y),\sigma(z)\big)\Big),\#(|)\big(\sigma(x),\sigma(z)\big)\Big)$.

Lemma 6.1 allows us to furtherly rewrite this:

for each $\alpha_1,\alpha_2,\alpha_3\in\mathbb{N}$ $P_\to\Big(P_\wedge\big(\#(|)(\alpha_1,\alpha_2),\#(|)(\alpha_2,\alpha_3)\big),\#(|)(\alpha_1,\alpha_3)\Big)$.

And finally this can be rewritten

for each $\alpha_1,\alpha_2,\alpha_3\in\mathbb{N}$ if $\#(|)(\alpha_1,\alpha_2)$ and $\#(|)(\alpha_2,\alpha_3)$ then $\#(|)(\alpha_1,\alpha_3)$.

This is the meaning of sentence S1 and that meaning is exactly as expected.

Our proof of statement S1 will begin by trying to exploit the definition of symbol |. To this end we need to add another constant symbol in our language. This is the symbol * that represents the product operation in the domain $\mathbb{N}$ of natural numbers. Therefore #(*) is a function defined on $\mathbb{N}\times\mathbb{N}$ and for each $\alpha,\beta\in\mathbb{N}$ $\#(*)(\alpha,\beta)$ is the product of α and β, in other words $\#(*)(\alpha,\beta)=\alpha\bullet\beta$. Given two variables x and y we'll abbreviate the expression (*)(x,y) with xy (as used in mathematics).

Consider the following lemma.

Lemma 6.2

Let m be a positive integer, $x_1, .., x_m \in V$, with $x_i\neq x_j$ for $i\neq j$.
We have $H[x_1:\mathbb{N}, .. , x_m:\mathbb{N}]$ and we define $k= k[x_1:\mathbb{N}, .. , x_m:\mathbb{N}]$.
Suppose i, j = 1..m, $i\neq j$, suppose $c\in V$-dom(k). Then

$$\gamma\Big[x_1:\mathbb{N},..,x_m:\mathbb{N},(\leftrightarrow)\Big((|)(x_i,x_j),(\exists)\Big(\{\}\big(c:\mathbb{N},(=)(x_j,x_ic)\big)\Big)\Big)\Big]\in S(\varepsilon)$$

$\#\left(\gamma\left[x_1:\mathbb{N},..,x_m:\mathbb{N},(\leftrightarrow)\left((|)(x_i,x_j),(\exists)\left(\{\}\left(c:\mathbb{N},(=)(x_j,x_ic)\right)\right)\right)\right]\right)$ is true .

Proof:

We have also $H[x_1:\mathbb{N}, .. , x_m:\mathbb{N}, c:\mathbb{N}]$ and we can define $k' = k[x_1:\mathbb{N}, .. , x_m:\mathbb{N}, c:\mathbb{N}]$.
By lemma 6.1 we obtain that $x_i, x_j, c \in E(k')$.
For each $\sigma' \in \Xi(k')$ $\#(k',*,\sigma') = \#(*)$ is a function with two arguments; $\#(k',x_i,\sigma') = \sigma'(x_i) \in \mathbb{N}$; $\#(k',c,\sigma') = \sigma'(c) \in \mathbb{N}$. Therefore $(*)(x_i,c) \in E(k')$, and $(=)(x_j, x_ic) \in E(k')$. By lemma 3.1

$\{\}(c:\mathbb{N}, (=)(x_j,x_ic)) \in E(k)$;
$(\exists)(\{\}(c:\mathbb{N}, (=)(x_j,x_ic))) \in S(k)$.

Lemma 6.1 also tells us that $x_i, x_j \in E(k)$ and for each $\sigma \in \Xi(k)$
$\#(k, x_i, \sigma) = \sigma(x_i) \in \mathbb{N}$; $\#(k, x_j, \sigma) = \sigma(x_j) \in \mathbb{N}$. For each $\sigma \in \Xi(k)$ $\#(k,|,\sigma) = \#(|)$ is a function with two arguments and $(\#(k, x_i, \sigma), \#(k, x_j, \sigma))$ is a member of its domain, therefore $(|)(x_i,x_j) \in E(k)$.

Moreover, for each $\sigma \in \Xi(k)$ $\#(k, (|)(x_i,x_j), \sigma) = \#(|) (\#(k,x_i,\sigma), \#(k,x_j,\sigma)) = \#(|) (\sigma(x_i), \sigma(x_j)) =$
$= (\exists \eta \in \mathbb{N}: \sigma(x_j) = \sigma(x_i)*\eta)$, so $\#(k, (|)(x_i,x_j), \sigma)$ is true or false and $(|)(x_i,x_j) \in S(k)$.

From there follows that $(\leftrightarrow)\left((|)(x_i,x_j),(\exists)\left(\{\}\left(c:\mathbb{N},(=)(x_j,x_ic)\right)\right)\right) \in S(k)$, and

$$\gamma\left[x_1:\mathbb{N},..,x_m:\mathbb{N},(\leftrightarrow)\left((|)(x_i,x_j),(\exists)\left(\{\}\left(c:\mathbb{N},(=)(x_j,x_ic)\right)\right)\right)\right] \in S(\varepsilon).$$

By theorem 3.5 we can rewrite

$$\#\left(\gamma\left[x_1:\mathbb{N},..,x_m:\mathbb{N},(\leftrightarrow)\left((|)(x_i,x_j),(\exists)\left(\{\}\left(c:\mathbb{N},(=)(x_j,x_ic)\right)\right)\right)\right]\right)$$

as follows

$$P_\forall\left(\{\}\left(\sigma \in \Xi(k),\#\left(k,(\leftrightarrow)\left((|)(x_i,x_j),(\exists)\left(\{\}\left(c:\mathbb{N},(=)(x_j,x_ic)\right)\right)\right),\sigma\right)\right)\right)$$

and this can be further rewritten

$$P_\forall\left(\{\}\left(\sigma \in \Xi(k),P_\leftrightarrow\left(\#\left(k,(|)(x_i,x_j),\sigma\right),\#\left(k,(\exists)\left(\{\}\left(c:\mathbb{N},(=)(x_j,x_ic)\right)\right),\sigma\right)\right)\right)\right)$$

$$P_\forall\left(\{\}\left(\sigma \in \Xi(k),P_\leftrightarrow\left(\#(|)\left(\sigma(x_i),\sigma(x_j)\right),P_\exists\left(\{\}\left(\rho \in \Xi(k'):\sigma \sqsubseteq \rho,\#\left(k',(=)(x_j,x_ic),\rho\right)\right)\right)\right)\right)\right)$$

$$P_{\forall}\left(\{\}\left(\sigma\in\Xi(k), P_{\leftrightarrow}\left(\#(|)(\sigma(x_i),\sigma(x_j)), P_{\exists}\left(\{\}\left(\rho\in\Xi(k'):\sigma\sqsubseteq\rho, P_{=}\left(\rho(x_j), \#(k', x_i c, \rho)\right)\right)\right)\right)\right)\right)$$

$$P_{\forall}\left(\{\}\left(\sigma\in\Xi(k), P_{\leftrightarrow}\left(\#(|)(\sigma(x_i),\sigma(x_j)), P_{\exists}\left(\{\}\left(\rho\in\Xi(k'):\sigma\sqsubseteq\rho, P_{=}\left(\rho(x_j), \rho(x_i)\bullet\rho(c)\right)\right)\right)\right)\right)\right)$$

The final statement can be written as follows:
For each $\sigma\in\Xi(k)$ $\#(|)$ $(\sigma(x_i), \sigma(x_j))$ if and only if
(there exists $\rho\in\Xi(k')$ such that $\sigma\sqsubseteq\rho$ and $\rho(x_j) = \rho(x_i) \bullet \rho(c)$) .

By definition we have $\#(|)$ $(\sigma(x_i), \sigma(x_j)) = (\exists\eta\in\mathbb{N}: \sigma(x_j) = \sigma(x_i)\bullet\eta)$.

Suppose $\#(|)$ $(\sigma(x_i), \sigma(x_j))$ holds. There exists $\eta\in\mathbb{N}$ such that $\sigma(x_j) = \sigma(x_i)\bullet\eta$.

By lemma 6.1 there exist $\alpha_1, .., \alpha_m \in\mathbb{N}$ such that $\sigma = (x_1,\alpha_1) \parallel \dots \parallel (x_m,\alpha_m)$.
We define $\rho = (x_1,\alpha_1) \parallel \dots \parallel (x_m,\alpha_m) \parallel (c,\eta)$, by 6.1 we have $\rho\in\Xi(k')$.

Moreover clearly $\sigma\sqsubseteq\rho$, and $\rho(x_j) = \sigma(x_j) = \sigma(x_i)\bullet\eta = \rho(x_i)\bullet\rho(c)$.

Conversely suppose there exists $\rho\in\Xi(k')$ such that $\sigma\sqsubseteq\rho$ and $\rho(x_j) = \rho(x_i) \bullet \rho(c)$.
By 6.1 $\rho(c)\in\mathbb{N}$ and $\sigma(x_j) = \rho(x_j) = \rho(x_i) \bullet \rho(c) = \sigma(x_i) \bullet \rho(c)$.

□

This lemma allows us to create an axiom which is the set $A_{6.2}$ of all expressions

$$\gamma\left[x_1:\mathbb{N},..,x_m:\mathbb{N},(\leftrightarrow)\left((|)(x_i,x_j),(\exists)\left(\{\}\left(c:\mathbb{N},(=)(x_j,x_i c)\right)\right)\right)\right]$$

such that m is a positive integer, $x_1, .., x_m \in V$, with $x_\alpha\neq x_\beta$ for $\alpha\neq\beta$, i, j = 1..m, $i\neq j$, $c\in V$-dom(k).

Lemma 6.3

Let m be a positive integer, $x_1, .., x_m \in V$, with $x_i\neq x_j$ for $i\neq j$.
We have $H[x_1:\mathbb{N}, .. , x_m:\mathbb{N}]$ and we define $k= k[x_1:\mathbb{N}, .. , x_m:\mathbb{N}]$.
Suppose i_1, i_2, i_3 distinct in {1, .. , m}. Then

$$\gamma\left[x_1:\mathbb{N},..,x_m:\mathbb{N},(=)\left((*)\left((*)(x_{i(1)},x_{i(2)}),x_{i(3)}\right),(*)\left(x_{i(1)},(*)(x_{i(2)},x_{i(3)})\right)\right)\right]\in S(\varepsilon) ,$$

$$\#\left(\gamma\left[x_1:\mathbb{N},..,x_m:\mathbb{N},(=)\left((*)\left((*)(x_{i(1)},x_{i(2)}),x_{i(3)}\right),(*)\left(x_{i(1)},(*)(x_{i(2)},x_{i(3)})\right)\right)\right]\right) \text{ is true.}$$

Proof:

By lemma 6.1 we obtain that for each j=1..3 $x_{i(j)} \in E(k)$,
for each $\sigma\in\Xi(k)$ $\#(k, x_{i(j)}, \sigma) = \sigma(x_{i(j)}) \in \mathbb{N}$.

For each $\sigma\in \Xi(k)$ $\#(k,*,\sigma) = \#(*)$ is a function with 2 arguments so $(*)(x_{i(1)}, x_{i(2)}) \in E(k)$ and for each $\sigma\in \Xi(k)$ $\#(k, (*)(x_{i(1)}, x_{i(2)}), \sigma) = \#(*) (\#(k, x_{i(1)}, \sigma), \#(k, x_{i(2)}, \sigma)) \in \mathbb{N}$, so $(*)\big((*)(x_{i(1)}, x_{i(2)}), x_{i(3)}\big) \in E(k)$.

Similarly $(*)\big(x_{i(1)},(*)(x_{i(2)}, x_{i(3)})\big) \in E(k)$, so by 3.13

$(=)\Big((*)\big((*)(x_{i(1)}, x_{i(2)}), x_{i(3)}\big),(*)\big(x_{i(1)},(*)(x_{i(2)}, x_{i(3)})\big)\Big)\in S(k)$ and

$$\gamma\Big[x_1:\mathbb{N},..,x_m:\mathbb{N},(=)\Big((*)\big((*)(x_{i(1)}, x_{i(2)}), x_{i(3)}\big),(*)\big(x_{i(1)},(*)(x_{i(2)}, x_{i(3)})\big)\Big)\Big]\in S(\varepsilon).$$

By theorem 3.5 we can rewrite

$$\#\Big(\gamma\Big[x_1:\mathbb{N},..,x_m:\mathbb{N},(=)\Big((*)\big((*)(x_{i(1)}, x_{i(2)}), x_{i(3)}\big),(*)\big(x_{i(1)},(*)(x_{i(2)}, x_{i(3)})\big)\Big)\Big]\Big)$$

as follows

$$P_\forall\Big(\{\}\Big(\sigma\in \Xi(k),\#\Big(k,(=)\Big((*)\big((*)(x_{i(1)}, x_{i(2)}), x_{i(3)}\big),(*)\big(x_{i(1)},(*)(x_{i(2)}, x_{i(3)})\big)\Big),\sigma\Big)\Big)\Big)$$

$$P_\forall\Big(\{\}\Big(\sigma\in \Xi(k),P_=\Big(\#\Big(k,(*)\big((*)(x_{i(1)}, x_{i(2)}), x_{i(3)}\big),\sigma\Big),\#\Big(k,(*)\big(x_{i(1)},(*)(x_{i(2)}, x_{i(3)})\big),\sigma\Big)\Big)\Big)\Big)$$

$$P_\forall\Big(\{\}\Big(\sigma\in \Xi(k),P_=\Big(\#(*)\Big(\#\big(k,(*)(x_{i(1)}, x_{i(2)}),\sigma\big),\sigma(x_{i(3)})\Big),\#(*)\Big(\sigma(x_{i(1)}),\#\big(k,(*)(x_{i(2)}, x_{i(3)}),\sigma\big)\Big)\Big)\Big)\Big)$$

$$P_\forall\Big(\{\}\Big(\sigma\in \Xi(k),P_=\Big(\#(*)\big(\sigma(x_{i(1)})\bullet\sigma(x_{i(2)}),\sigma(x_{i(3)})\big),\#(*)\big(\sigma(x_{i(1)}),\sigma(x_{i(2)})\bullet\sigma(x_{i(3)})\big)\Big)\Big)\Big)$$

$$P_\forall\Big(\{\}\Big(\sigma\in \Xi(k),P_=\Big(\big(\sigma(x_{i(1)})\bullet\sigma(x_{i(2)})\big)\bullet\sigma(x_{i(3)}),\sigma(x_{i(1)})\bullet\big(\sigma(x_{i(2)})\bullet\sigma(x_{i(3)})\big)\Big)\Big)\Big).$$

In words this can be expressed as:

for each $\sigma\in \Xi(k)$ $\big(\sigma(x_{i(1)})\bullet\sigma(x_{i(2)})\big)\bullet\sigma(x_{i(3)}) = \sigma(x_{i(1)})\bullet\big(\sigma(x_{i(2)})\bullet\sigma(x_{i(3)})\big)$

and this is clearly satisfied.

□

Lemma 6.3 allows us to create an axiom which is the set $A_{6.3}$ of all expressions

$$\gamma\Big[x_1:\mathbb{N},..,x_m:\mathbb{N},(=)\Big((*)\big((*)(x_{i(1)}, x_{i(2)}), x_{i(3)}\big),(*)\big(x_{i(1)},(*)(x_{i(2)}, x_{i(3)})\big)\Big)\Big]$$

such that m is a positive integer, $x_1, .., x_m \in V$, with $x_\alpha\neq x_\beta$ for $\alpha\neq\beta$, i_1, i_2, i_3 distinct in $\{1, .. , m\}$.

The proof

We can now proceed with the proof of statement S1. Let's recap how our language is structured.

$C = \{\mathbb{N}, |, *\}$
$F = \{\wedge, \vee, \rightarrow, \neg, \forall, \exists, \in, =, \leftrightarrow\}$
$V = \{x, y, z, c, d, e\}$.

The axioms and rules of our deductive system are the ones we've listed in sections 5 and 6.

The first step in our proof of statement S1 uses axiom $A_{6.2}$:

$$\text{(1)}\quad \gamma\Big[x:\mathbb{N}, y:\mathbb{N}, z:\mathbb{N}, (\leftrightarrow)\big((|)(x,y), (\exists)\big(\{\}(c:\mathbb{N}, (=)(y, xc))\big)\big)\Big]$$

Then we can use $R_{5.1}$ to derive a new statement from (1):

$$\text{(2)}\quad \gamma\Big[x:\mathbb{N}, y:\mathbb{N}, z:\mathbb{N}, (\rightarrow)\big((|)(x,y), (\exists)\big(\{\}(c:\mathbb{N}, (=)(y, xc))\big)\big)\Big]$$

In the next step we use axiom $A_{5.2}$:

$$\text{(3)}\quad \gamma\Big[x:\mathbb{N}, y:\mathbb{N}, z:\mathbb{N}, (\rightarrow)\big((\wedge)\big((|)(x,y), (|)(y,z)\big), (|)(x,y)\big)\Big]$$

At this point we can apply rule $R_{5.3}$ to (3) and (2) and obtain

$$\text{(4)}\quad \gamma\Big[x:\mathbb{N}, y:\mathbb{N}, z:\mathbb{N}, (\rightarrow)\big((\wedge)\big((|)(x,y), (|)(y,z)\big), (\exists)\big(\{\}(c:\mathbb{N}, (=)(y, xc))\big)\big)\Big]$$

In much the same way we can obtain

$$\text{(5)}\quad \gamma\Big[x:\mathbb{N}, y:\mathbb{N}, z:\mathbb{N}, (\rightarrow)\big((\wedge)\big((|)(x,y), (|)(y,z)\big), (\exists)\big(\{\}(d:\mathbb{N}, (=)(z, yd))\big)\big)\Big]$$

The next two statements are instances of $A_{6,2}$.

$$\text{(6)}\quad \gamma\Big[x:\mathbb{N}, y:\mathbb{N}, z:\mathbb{N}, c:\mathbb{N}, d:\mathbb{N}, (\rightarrow)\big((\wedge)\big((=)(y, xc), (=)(z, yd)\big), (=)\big(y, (*)(x,c)\big)\big)\Big]$$

$$\text{(7)}\quad \gamma\Big[x:\mathbb{N}, y:\mathbb{N}, z:\mathbb{N}, c:\mathbb{N}, d:\mathbb{N}, (\rightarrow)\big((\wedge)\big((=)(y, xc), (=)(z, yd)\big), (=)\big(z, (*)(y,d)\big)\big)\Big]$$

In fact if we define $h = k[x:\mathbb{N}, y:\mathbb{N}, z:\mathbb{N}, c:\mathbb{N}, d:\mathbb{N}]$ then $x,y,z,c,d \in E(h)$ and for each $\sigma \in \Xi(h)$

$\#(h, x, \sigma), \#(h, y, \sigma), \#(h, z, \sigma), \#(h, c, \sigma), \#(h, d, \sigma) \in \mathbb{N}$.

For each $\sigma \in \Xi(h)$ $\#(h, *, \sigma) = \#(*)$ is a function with two arguments and $(\#(h, x, \sigma), \#(h, c, \sigma))$ is a member of its domain, therefore $(*)(x,c) \in E(h)$, and similarly $(*)(y,d) \in E(h)$.
By lemma 3.13 it follows immediately that $(=)(y,xc) \in S(h)$, and similarly $(=)(z,yd) \in S(h)$.

To proceed with our proof, our idea is to leverage rule $R_{5.4.}$ We notice we have just shown that $(\wedge)((=)(y,xc),(=)(z,yd)) \in S(h)$.

If we define k = k[x:ℕ, y:ℕ, z:ℕ, c:ℕ, d:ℕ, e:ℕ] it is easy to see that $(=)(z,(*)(e,d)) \in S(k)$. In fact e,d∈E(k), for each σ∈Ξ(k) #(h, e, σ), #(h, d, σ) ∈ ℕ, therefore (*)(e,d) ∈ E(k), moreover z∈E(k), so by 3.13 $(=)(z,(*)(e,d)) \in S(k)$.

With respect to $R_{5.4}$ we want to use y as t and (*)(x,c) as t'. We have y∈E(h) and for each ρ∈Ξ(h) #(h,y,ρ) ∈ ℕ = #(h,ℕ,ρ). Moreover (*)(x,c)∈E(h) and for each ρ∈Ξ(h)

#(h, (*)(x,c), ρ) = #(*)(#(h,x,ρ), #(h,c,ρ)) = ρ(x) • ρ(c) ∈ ℕ = #(h,ℕ,ρ).

Furthermore to evaluate V_b(y) and V_b((*)(x,c)) we can apply assumption 2.1.7. That assumption tells us that $V_b(y)=\varnothing$ and $V_b((*)(x,c)) = V_b(*) \cup V_b(x) \cup V_b(c) = \varnothing$.
With respect to $R_{5.4}$ in the role of φ we have (=)(z, (*)(e,d)), of course
$V_b(y) \cap V_b((=)(z, (*)(e,d))) = \varnothing$; $V_b((*)(x,c)) \cap V_b((=)(z, (*)(e,d))) = \varnothing$.

In order to calculate $(=)(z, (*)(e,d))_k\{e/t\}$ and $(=)(z, (*)(e,d))_k\{e/t'\}$ we can exploit definition 4.6. In it we have established one of five condition is true and a consequent calculation of $\varphi_k\{x_i/t\}$. So

$(=)(z, (*)(e,d))_k\{e/y\} = (=)(z_k\{e/y\}, (*)(e,d)_k\{e/y\}) = (=)(z, (*)(y,d))$;

$(=)(z, (*)(e,d))_k\{e/(*)(x,c)\} = (=)(z_k\{e/(*)(x,c)\}, (*)(e,d)_k\{e/(*)(x,c)\}) =$

$= (=) (z, (*)((*)(x,c), d))$.

(8) $\gamma\Big[x:\mathbb{N},y:\mathbb{N},z:\mathbb{N},c:\mathbb{N},d:\mathbb{N},(\rightarrow)\Big((\wedge)\big((=)(y,xc),(=)(z,yd)\big),(=)\big(z,(*)\big((*)(x,c),d\big)\big)\Big)\Big]$

The next statement is an instance of axiom $A_{6.3}$:

(9) $\gamma\Big[x:\mathbb{N},y:\mathbb{N},z:\mathbb{N},c:\mathbb{N},d:\mathbb{N},(=)\Big((*)\big((*)(x,c),d\big),(*)\big(x,(*)(c,d)\big)\Big)\Big]$

By rule $R_{5.5}$ we obtain

(10) $\gamma\Big[x:\mathbb{N},y:\mathbb{N},z:\mathbb{N},c:\mathbb{N},d:\mathbb{N},(\rightarrow)\Big((\wedge)\big((=)(y,xc),(=)(z,yd)\big),(=)\Big((*)\big((*)(x,c),d\big),(*)\big(x,(*)(c,d)\big)\Big)\Big)\Big]$

We can apply rule $R_{5.6}$ to (8) and (10) to obtain

(11) $\gamma\Big[x:\mathbb{N},y:\mathbb{N},z:\mathbb{N},c:\mathbb{N},d:\mathbb{N},(\rightarrow)\Big((\wedge)\big((=)(y,xc),(=)(z,yd)\big),(=)\Big(z,(*)\big(x,(*)(c,d)\big)\Big)\Big)\Big]$

We want to apply rule $R_{5.7}$ to obtain

$$\gamma\Big[x:\mathbb{N},y:\mathbb{N},z:\mathbb{N},c:\mathbb{N},d:\mathbb{N},(\rightarrow)\Big((\wedge)\big((=)(y,xc),(=)(z,yd)\big),(\exists)\Big(\{\}\big(e:\mathbb{N},(=)(z,(*)(x,e))\big)\Big)\Big)\Big]$$

With respect to that rule, the idea is to have

- k = k[x:ℕ, y:ℕ, z:ℕ, c:ℕ, d:ℕ, e:ℕ]
- h = k[x:ℕ, y:ℕ, z:ℕ, c:ℕ, d:ℕ]
- $\chi = (\wedge)\big((=)(y,xc),(=)(z,yd)\big)$
- t = (*)(c,d)
- $\varphi = (=)\big(z,(*)(x,e)\big)$

It has been show above that $\chi \in S(h)$.

We have $c,d \in E(h)$, for each $\rho \in \Xi(h)$ #(h, c, ρ), #(h, d, ρ) ∈ ℕ, therefore (*)(c,d) ∈ E(h), for each $\rho \in \Xi(h)$ #(h, (*)(c,d), ρ) = #(*) (#(h, c, ρ), #(h, d, ρ)) = ρ(c) • ρ(d) ∈ ℕ = #(h, ℕ, ρ) .

Clearly (*)(x,e) ∈ E(k), $z \in E(k)$ and $(=)\big(z,(*)(x,e)\big) \in S(k)$.

Since $V_b((*)(c,d)) = \varnothing$ we also have $V_b(t) \cap V_b(\varphi) = \varnothing$.

Therefore we are able to obtain

$$(12)\ \gamma\Big[x:\mathbb{N},y:\mathbb{N},z:\mathbb{N},c:\mathbb{N},d:\mathbb{N},(\rightarrow)\Big((\wedge)\big((=)(y,xc),(=)(z,yd)\big),(\exists)\Big(\{\}\big(e:\mathbb{N},(=)(z,(*)(x,e))\big)\Big)\Big)\Big]$$

We can use the following instance of axiom 6.2:

$$(13)\ \gamma\Big[x:\mathbb{N},y:\mathbb{N},z:\mathbb{N},c:\mathbb{N},d:\mathbb{N},(\leftrightarrow)\Big((|)(x\,|\,z),(\exists)\Big(\{\}\big(e:\mathbb{N},(=)(z,(*)(x,e))\big)\Big)\Big)\Big]\ .$$

And we can use rule $R_{5.1}$ to derive

$$(14)\ \gamma\Big[x:\mathbb{N},y:\mathbb{N},z:\mathbb{N},c:\mathbb{N},d:\mathbb{N},(\rightarrow)\Big((\exists)\Big(\{\}\big(e:\mathbb{N},(=)(z,(*)(x,e))\big)\Big),(|)(x\,|\,z)\Big)\Big]\ .$$

We can apply rule $R_{5.8}$ to (12) and (14) and obtain

$$(15)\ \gamma\Big[x:\mathbb{N},y:\mathbb{N},z:\mathbb{N},c:\mathbb{N},d:\mathbb{N},(\rightarrow)\Big((\wedge)\big((=)(y,xc),(=)(z,yd)\big),(|)(x\,|\,z)\Big)\Big]\ .$$

We now apply rule $R_{5.9}$ using k = k[x:ℕ, y:ℕ, z:ℕ, c:ℕ, d:ℕ] and considering that (=)(y,xc) ∈ S(k), (=)(z,yd) ∈ S(k), (|)(x,z) ∈ S(k). We obtain

$$(16)\ \gamma\Big[x:\mathbb{N},y:\mathbb{N},z:\mathbb{N},c:\mathbb{N},d:\mathbb{N},(\rightarrow)\Big((=)(y,xc),(\rightarrow)\big((=)(z,yd),(|)(x,z)\big)\Big)\Big]\ .$$

This can be rewritten

(17) $\gamma\left[x:\mathbb{N},y:\mathbb{N},z:\mathbb{N},c:\mathbb{N},\gamma\left[d:\mathbb{N},(\to)\left((=)(y,xc),(\to)\left((=)(z,yd),(|)(x,z)\right)\right)\right]\right]$.

We can apply rule $R_{5.10}$ using k = k[x:ℕ, y:ℕ, z:ℕ, c:ℕ, d:ℕ] and h = k[x:ℕ, y:ℕ, z:ℕ, c:ℕ]. We consider that (=)(y,xc) ∈ S(k)∩S(h), $(\to)\left((=)(z,yd),(|)(x,z)\right)\in$ S(k) . We obtain

(18) $\gamma\left[x:\mathbb{N},y:\mathbb{N},z:\mathbb{N},c:\mathbb{N},(\to)\left((=)(y,xc),\gamma\left[d:\mathbb{N},(\to)\left((=)(z,yd),(|)(x,z)\right)\right]\right)\right]$.

This can be rewritten

(19) $\gamma\left[x:\mathbb{N},y:\mathbb{N},z:\mathbb{N},\gamma\left[c:\mathbb{N},(\to)\left((=)(y,xc),\gamma\left[d:\mathbb{N},(\to)\left((=)(z,yd),(|)(x,z)\right)\right]\right)\right]\right]$.

We intend to apply rule $R_{5.12}$ using

k = k[x:ℕ, y:ℕ, z:ℕ, c:ℕ],
h = k[x:ℕ, y:ℕ, z:ℕ],
ψ = (=)(y,xc) ∈ S(k) ,
φ = $\gamma\left[d:\mathbb{N},(\to)\left((=)(z,yd),(|)(x,z)\right)\right]$ ∈ S(k) .

To be able to apply that rule we need to show that φ∈S(h). Let κ = k[x:ℕ, y:ℕ, z:ℕ, d:ℕ] .
By lemma 6.1 x, y, z, d ∈ E(κ), for each σ∈Ξ(κ) #(k, x, σ) ∈ ℕ, and the same for y,z,d .

Therefore (*)(y,d) ∈ E(κ), (=)(z,yd) ∈ S(κ), (|)(x,z) ∈ S(κ), $(\to)\left((=)(z,yd),(|)(x,z)\right)\in$S(κ), and finally $\gamma\left[d:\mathbb{N},(\to)\left((=)(z,yd),(|)(x,z)\right)\right]$ ∈ S(h) .
So we obtain

(20) $\gamma\left[x:\mathbb{N},y:\mathbb{N},z:\mathbb{N},(\to)\left((\exists)\left(\{\}\left(c:\mathbb{N},(=)(y,xc)\right)\right),\gamma\left[d:\mathbb{N},(\to)\left((=)(z,yd),(|)(x,z)\right)\right]\right)\right]$.

Next we apply rule $R_{5.3}$ to (4) and (20). If h = k[x:ℕ, y:ℕ, z:ℕ] then

$(\wedge)\left((|)(x,y),(|)(y,z)\right)$ ∈ S(h), $(\exists)\left(\{\}\left(c:\mathbb{N},(=)(y,xc)\right)\right)$ ∈ S(h),
$\gamma\left[d:\mathbb{N},(\to)\left((=)(z,yd),(|)(x,z)\right)\right]$ ∈ S(h) .

So we obtain

(21) $\gamma\left[x:\mathbb{N},y:\mathbb{N},z:\mathbb{N},(\to)\left((\wedge)\left((|)(x,y),(|)(y,z)\right),\gamma\left[d:\mathbb{N},(\to)\left((=)(z,yd),(|)(x,z)\right)\right]\right)\right]$.

At this point we need to apply rule $R_{5.11}$ using

k = k[x:ℕ, y:ℕ, z:ℕ, d:ℕ],

$h = k[x:\mathbb{N},\ y:\mathbb{N},\ z:\mathbb{N}]$,
$\chi = (\wedge)\big((|)(x,y),(|)(y,z)\big) \in S(h)$,
$\psi = (=)(z,yd) \in S(k)$.
$\varphi = (|)(x,z) \in S(k)$.

Of course (|)(x,z) also belongs to S(h), therefore we obtain

$$(22)\ \gamma\Big[x:\mathbb{N}, y:\mathbb{N}, z:\mathbb{N}, (\rightarrow)\Big((\wedge)\big((|)(x,y),(|)(y,z)\big),(\rightarrow)\Big((\exists)\big(\{\}\big(d:\mathbb{N},(=)(z,yd)\big)\big),(|)(x,z)\Big)\Big)\Big]\ .$$

The final step in our proof consists in applying the 'modus ponens' rule 5.13 to (5) and (22). We obtain

$$(23)\ \gamma\Big[x:\mathbb{N}, y:\mathbb{N}, z:\mathbb{N}, (\rightarrow)\Big((\wedge)\big((|)(x,y),(|)(y,z)\big),(|)(x,z)\Big)\Big]\ .$$

7. Another example

We want to prove a form of the Bocardo syllogism. In Ferreiros' referenced paper ([4]), on paragraph 3.1, the syllogism is expressed as follows:

Some A are not B. All C are B. Therefore, some A are not C.

Suppose A, B, C represent sets, the statement we actually want to prove is the following:

If ((there exists x∈A such that x∉B) and (for each y∈C y∈B))
then (there exists z∈A such that z∉C) .

In order to formalize this in our approach, our language must be as assumed in section 5, and in addition we need
- 3 constants A, B and C, each representing a set
- 3 variables named x, y and z.
To recap, our language is as follows

$\mathcal{C}$ = {A,B,C}
$\mathcal{F}$ = {∧, ∨, →, ¬, ∀, ∃, ∈, =, ↔}
$\mathcal{V}$ = {x, y, z} .

At this point we suppose we can formalize the statement as

$$(\rightarrow)\left((\wedge)\begin{pmatrix}(\exists)\left(\{\}\left(x:A,(\neg)\left((\in)(x,B)\right)\right)\right),\\(\forall)\left(\{\}\left(y:C,(\in)(y,B)\right)\right)\end{pmatrix},(\exists)\left(\{\}\left(z:A,(\neg)\left((\in)(z,C)\right)\right)\right)\right)\quad \text{(S2)} .$$

We'll now see a proof of this statement (within the proof we'll also prove S2 is a sentence in our language).

First of all we need a lemma that can be applied to the general language of section 5.

Lemma 7.1

Let m be a positive integer, $x_1, .., x_m \in \mathcal{V}$, with $x_i \neq x_j$ for $i \neq j$. Let $A_1, .., A_m \in \mathcal{C}$ such that for each i=1..m $\#(A_i)$ is a set. We have $H[x_1:A_1, .. , x_m:A_m]$ and we define $k = k[x_1:A_1, .. , x_m:A_m]$. Let $D \in \mathcal{C}$ such that #(D) is a set. Then for each i=1..m $(\in)(x_i,D) \in S(k)$.

Proof:

First consider that $A_1 \in E(\varepsilon)$ and $\#(A_1)$ is a set, so we can define $k_1 = (x_1,A_1)$.
If m>1 then for each i=1..m-1 we suppose to have defined $k_i = (x_1,A_1) \parallel .. \parallel (x_i,A_i)$. By lemma 3.12 $A_{i+1} \in E(k_i)$ and for each $\rho \in \Xi(k_i)$ $\#(k_i,A_{i+1},\rho) = \#(A_{i+1})$ is a set, so we can define $k_{i+1} = k_i \parallel (x_{i+1},A_{i+1})$.

This proves that $H[x_1:A_1, .. , x_m:A_m]$ holds.

By lemma 4.3 we have $x_i \in E(k_i)$. If i=m this implies $x_i \in E(k)$, otherwise for each j=i+1..m $x_j \notin V_b(x_i)$. So by lemma 3.15 $x_i \in E(k)$.

Moreover $D \in E(k)$ and for each $\sigma \in \Xi(k)$ $\#(k,D,\sigma) = \#(D)$ is a set. By lemma 3.14 $(\in)(x_i,D) \in S(k)$.

□

So we have H[x:A] and we can define h = k[x:A]. Moreover $(\in)(x,B) \in S(h)$, so also $(\neg)((\in)(x,B)) \in S(h)$.

We also have H[x:A, y:C] and we define k_y = k[x:A, y:C]. We have $(\in)(y,B) \in S(k_y)$, and by lemma 3.1 $(\forall)(\{\}(y:C,(\in)(y,B))) \in S(h)$.

Thus $(\wedge)((\neg)((\in)(x,B)),(\forall)(\{\}(y:C,(\in)(y,B))))$ also belongs to S(h) .

We also have H[x:A, z:A] and we define k_z = k[x:A, z:A]. We have $(\in)(z,C) \in S(k_z)$, and by lemma 3.1 $(\forall)(\{\}(z:A,(\in)(z,C))) \in S(h)$.

We can apply axiom $A_{5.2}$ to obtain the first sentence in our proof

$$(1)\ \gamma\left[x:A,(\rightarrow)\left((\wedge)\left((\wedge)\begin{pmatrix}(\neg)((\in)(x,B)),\\(\forall)(\{\}(y:C,(\in)(y,B)))\end{pmatrix}, \atop (\forall)(\{\}(z:A,(\in)(z,C)))\right),(\wedge)\begin{pmatrix}(\neg)((\in)(x,B)),\\(\forall)(\{\}(y:C,(\in)(y,B)))\end{pmatrix}\right)\right]$$

By $A_{5.2}$ we also obtain

$$(2)\ \gamma\left[x:A,(\rightarrow)\left((\wedge)\begin{pmatrix}(\neg)((\in)(x,B)),\\(\forall)(\{\}(y:C,(\in)(y,B)))\end{pmatrix},(\neg)((\in)(x,B))\right)\right]$$

By (1), (2) and $R_{5,3}$

$$(3)\ \gamma\left[x:A,(\rightarrow)\left((\wedge)\left((\wedge)\begin{pmatrix}(\neg)((\in)(x,B)),\\(\forall)(\{\}(y:C,(\in)(y,B)))\end{pmatrix}, \atop (\forall)(\{\}(z:A,(\in)(z,C)))\right),(\neg)((\in)(x,B))\right)\right]$$

We apply $A_{5.2}$ again to obtain

$$(4)\quad \gamma\left[x:A,(\rightarrow)\left((\land)\begin{pmatrix}(\land)\begin{pmatrix}(\neg)((\in)(x,B)), \\ (\forall)(\{\}(y:C,(\in)(y,B)))\end{pmatrix}, \\ (\forall)(\{\}(z:A,(\in)(z,C)))\end{pmatrix},(\forall)(\{\}(z:A,(\in)(z,C)))\right)\right]$$

By $A_{5.16}$ we obtain

$$(5)\quad \gamma[x:A,(\in)(x,A)]$$

By (5) and $A_{5.5}$ we also get

$$(6)\quad \gamma\left[x:A,(\rightarrow)\left((\land)\begin{pmatrix}(\land)\begin{pmatrix}(\neg)((\in)(x,B)), \\ (\forall)(\{\}(y:C,(\in)(y,B)))\end{pmatrix}, \\ (\forall)(\{\}(z:A,(\in)(z,C)))\end{pmatrix},(\in)(x,A)\right)\right]$$

Since $x\in E(h)$, $C\in E(h)$, for each $\rho\in \Xi(h)$ $\#(h,C,\rho) = \#(C)$ is a set, $z\notin V_b(C)$ we can apply rule $R_{5,15}$ to (4) and (6) and obtain

$$(7)\quad \gamma\left[x:A,(\rightarrow)\left((\land)\begin{pmatrix}(\land)\begin{pmatrix}(\neg)((\in)(x,B)), \\ (\forall)(\{\}(y:C,(\in)(y,B)))\end{pmatrix}, \\ (\forall)(\{\}(z:A,(\in)(z,C)))\end{pmatrix},(\in)(x,C)\right)\right]$$

By $A_{5.2}$

$$(8)\quad \gamma\left[x:A,(\rightarrow)\left((\land)\begin{pmatrix}(\neg)((\in)(x,B)), \\ (\forall)(\{\}(y:C,(\in)(y,B)))\end{pmatrix},(\forall)(\{\}(y:C,(\in)(y,B)))\right)\right]$$

By (1), (8) and $R_{5,3}$

$$(9)\quad \gamma\left[x:A,(\rightarrow)\left((\land)\begin{pmatrix}(\land)\begin{pmatrix}(\neg)((\in)(x,B)), \\ (\forall)(\{\}(y:C,(\in)(y,B)))\end{pmatrix}, \\ (\forall)(\{\}(z:A,(\in)(z,C)))\end{pmatrix},(\forall)(\{\}(y:C,(\in)(y,B)))\right)\right]$$

Since $x\in E(h)$, $B\in E(h)$, for each $\rho\in \Xi(h)$ $\#(h,B,\rho) = \#(B)$ is a set, $y\notin V_b(B)$, we can apply $R_{5,15}$ to (9) and (7) to obtain

$$(10)\qquad \gamma\left[x:A,(\rightarrow)\left((\land)\begin{pmatrix}(\land)\begin{pmatrix}(\neg)((\in)(x,B)), \\ (\forall)(\{\}(y:C,(\in)(y,B)))\end{pmatrix}, \\ (\forall)(\{\}(z:A,(\in)(z,C)))\end{pmatrix},(\in)(x,B)\right)\right]$$

By (10), (3) and $R_{3,7}$

$$(11)\qquad \gamma\left[x:A,(\rightarrow)\left(\begin{array}{l} (\wedge)\left(\begin{array}{l} (\wedge)\left(\begin{array}{l} (\neg)\big((\in)(x,B)\big), \\ (\forall)\big(\{\}\big(y:C,(\in)(y,B)\big)\big) \end{array} \right), \\ (\forall)\big(\{\}\big(z:A,(\in)(z,C)\big)\big) \end{array} \right), (\wedge)\left(\begin{array}{l} (\in)(x,B), \\ (\neg)\big((\in)(x,B)\big) \end{array} \right) \end{array} \right) \right]$$

We apply rule $R_{5,17}$ and obtain

$$(12)\qquad \gamma\left[x:A,(\neg)\left((\wedge)\left(\begin{array}{l} (\wedge)\left(\begin{array}{l} (\neg)\big((\in)(x,B)\big), \\ (\forall)\big(\{\}\big(y:C,(\in)(y,B)\big)\big) \end{array} \right), \\ (\forall)\big(\{\}\big(z:A,(\in)(z,C)\big)\big) \end{array} \right) \right) \right]$$

By $R_{5,18}$ we obtain

$$(13)\qquad \gamma\left[x:A,(\rightarrow)\left((\wedge)\left(\begin{array}{l} (\neg)\big((\in)(x,B)\big), \\ (\forall)\big(\{\}\big(y:C,(\in)(y,B)\big)\big) \end{array} \right), (\neg)\Big((\forall)\big(\{\}\big(z:A,(\in)(z,C)\big)\big)\Big) \right) \right]$$

We have seen that H[x:A, z:A], we have defined k_z = k[x:A, z:A] and seen that $(\in)(z,C) \in S(k_z)$. So we can apply rule $R_{5.19}$ and obtain

$$(14)\qquad \gamma\left[x:A,(\rightarrow)\left((\wedge)\left(\begin{array}{l} (\neg)\big((\in)(x,B)\big), \\ (\forall)\big(\{\}\big(y:C,(\in)(y,B)\big)\big) \end{array} \right), (\exists)\Big(\{\}\big(z:A,(\neg)\big((\in)(z,C)\big)\big)\Big) \right) \right]$$

By (14) and $R_{5,9}$

$$(15)\qquad \gamma\left[x:A,(\rightarrow)\left((\neg)\big((\in)(x,B)\big),(\rightarrow)\left(\begin{array}{l} (\forall)\big(\{\}\big(y:C,(\in)(y,B)\big)\big), \\ (\exists)\Big(\{\}\big(z:A,(\neg)\big((\in)(z,C)\big)\big)\Big) \end{array} \right) \right) \right]$$

Remember we have seen that H[x:A] holds and defined h = k[x:A]. We have seen that $(\neg)((\in)(x,B)) \in S(h)$, $(\forall)\big(\{\}\big(y:C,(\in)(y,B)\big)\big) \in S(h)$.

Our assumptions clearly imply that $(\exists)\Big(\{\}\big(z:A,(\neg)\big((\in)(z,C)\big)\big)\Big) \in S(h)$.

It's also immediate that $(\in)(y,B) \in S(k[y:C])$ so by lemma 3.1 $(\forall)\big(\{\}\big(y:C,(\in)(y,B)\big)\big) \in S(\varepsilon)$.

Similarly $(\in)(z,C) \in S(k[z:A])$, $(\neg)\big((\in)(z,C)\big) \in S(k[z:A])$, $(\exists)\Big(\{\}\big(z:A,(\neg)\big((\in)(z,C)\big)\big)\Big) \in S(\varepsilon)$.

Therefore $(\rightarrow)\left(\begin{array}{l} (\forall)\big(\{\}\big(y:C,(\in)(y,B)\big)\big), \\ (\exists)\Big(\{\}\big(z:A,(\neg)\big((\in)(z,C)\big)\big)\Big) \end{array} \right) \in S(h)\cap S(\varepsilon)$, and we can apply rule $R_{5.20}$.

By (15) and $R_{5.20}$ we obtain

$$(16) \qquad (\to)\left((\exists)\big(\{\}\big(x:A,(\neg)\big((\in)(x,B)\big)\big)\big),(\to)\begin{pmatrix} (\forall)\big(\{\}\big(y:C,(\in)(y,B)\big)\big), \\ (\exists)\big(\{\}\big(z:A,(\neg)\big((\in)(z,C)\big)\big)\big) \end{pmatrix} \right)$$

By $R_{5.21}$ we finally get

$$(17) \qquad (\to)\left((\wedge)\begin{pmatrix} (\exists)\big(\{\}\big(x:A,(\neg)\big((\in)(x,B)\big)\big)\big), \\ (\forall)\big(\{\}\big(y:C,(\in)(y,B)\big)\big) \end{pmatrix},(\exists)\big(\{\}\big(z:A,(\neg)\big((\in)(z,C)\big)\big)\big) \right)$$

$\dashv$

So we have proved statement (S2), and this also means that (S2) is a sentence in our language. It seems quite evident that the statement's meaning is as expected, anyway to complete the argument we also want to prove this.

To this end we prove the following simple lemma.

Lemma 7.2

Let $x_1 \in V$, $A_1 \in C$ such that $\#(A_1)$ is a set. We have $H[x_1:A_1]$. Then

$\Xi(k[x_1:A_1]) = \{ (x_1,s_1) \mid s_1 \in \#(A_1) \}$.

Proof

We have $\varepsilon \in K(1)$, $A_1 \in E(1,\varepsilon)$, $x_1 \in V\text{-dom}(\varepsilon)$, $\#(\varepsilon,A_1,\varepsilon) = \#(A_1)$ is a set.

Therefore $k[x_1:A_1] = (x_1,A_1) \in K(1)^+$ and

$\Xi(k[x_1:A_1]) = \{ \varepsilon \parallel (x_1,s_1) \mid s_1 \in \#(\varepsilon,A_1,\varepsilon) \} = \{ (x_1,s_1) \mid s_1 \in \#(A_1) \}$.

□

We first examine the meaning of $(\exists)\big(\{\}\big(x:A,(\neg)\big((\in)(x,B)\big)\big)\big)$.

We can rewrite $\#\big((\exists)\big(\{\}\big(x:A,(\neg)\big((\in)(x,B)\big)\big)\big)\big)$ as

$P_\exists\big(\{\}\big(\sigma\in \Xi\big(k[x:A]\big),\#\big(k[x:A],(\neg)\big((\in)(x,B)\big),\sigma\big)\big)\big)$,

$P_\exists\big(\{\}\big(\sigma\in \Xi\big(k[x:A]\big),P_\neg\big(\#\big(k[x:A],(\in)(x,B),\sigma\big)\big)\big)\big)$,

$P_\exists\big(\{\}\big(\sigma\in \Xi\big(k[x:A]\big),P_\neg\big(P_\in\big(\#\big(k[x:A],x,\sigma\big),\#\big(k[x:A],B,\sigma\big)\big)\big)\big)\big)$,

$$P_\exists\Big(\{\}\big(\sigma\in\Xi\big(k[x:A]\big),P_\neg\big(P_\in\big(\sigma(x),\#(B)\big)\big)\big)\Big)\ ,$$

$$P_\exists\Big(\{\}\big(\alpha_x\in\#(A),P_\neg\big(P_\in\big(\alpha_x,\#(B)\big)\big)\big)\Big)\ .$$

Similarly we can rewrite $\#\big((\forall)\big(\{\}\big(y:C,(\in)(y,B)\big)\big)\big)$ as

$$P_\forall\Big(\{\}\big(\sigma\in\Xi\big(k[y:C]\big),\#\big(k[y:C],(\in)(y,B),\sigma\big)\big)\Big)\ ,$$

$$P_\forall\Big(\{\}\big(\sigma\in\Xi\big(k[y:C]\big),P_\in\big(\#\big(k[y:C],y,\sigma\big),\#\big(k[y:C],B,\sigma\big)\big)\big)\Big)\ ,$$

$$P_\forall\Big(\{\}\big(\sigma\in\Xi\big(k[y:C]\big),P_\in\big(\sigma(y),\#(B)\big)\big)\Big)\ ,$$

$$P_\forall\Big(\{\}\big(\alpha_y\in\#(C),P_\in\big(\alpha_y,\#(B)\big)\big)\Big)\ .$$

Similary we can rewrite $\#\big((\exists)\big(\{\}\big(z:A,(\neg)\big((\in)(z,C)\big)\big)\big)\big)$ as

$$P_\exists\Big(\{\}\big(\sigma\in\Xi\big(k[z:A]\big),\#\big(k[z:A],(\neg)\big((\in)(z,C)\big),\sigma\big)\big)\Big)\ ,$$

$$P_\exists\Big(\{\}\big(\sigma\in\Xi\big(k[z:A]\big),P_\neg\big(\#\big(k[z:A],(\in)(z,C),\sigma\big)\big)\big)\Big)\ ,$$

$$P_\exists\Big(\{\}\big(\sigma\in\Xi\big(k[z:A]\big),P_\neg\big(P_\in\big(\#\big(k[z:A],z,\sigma\big),\#\big(k[z:A],C,\sigma\big)\big)\big)\big)\Big)\ ,$$

$$P_\exists\Big(\{\}\big(\sigma\in\Xi\big(k[z:A]\big),P_\neg\big(P_\in\big(\sigma(z),\#(C)\big)\big)\big)\Big)\ ,$$

$$P_\exists\Big(\{\}\big(\alpha_z\in\#(A),P_\neg\big(P_\in\big(\alpha_z,\#(C)\big)\big)\big)\Big)\ .$$

At this point we can rewrite

$$\#\left((\rightarrow)\left((\wedge)\begin{pmatrix}(\exists)\big(\{\}\big(x:A,(\neg)\big((\in)(x,B)\big)\big)\big),\\(\forall)\big(\{\}\big(y:C,(\in)(y,B)\big)\big)\end{pmatrix},(\exists)\big(\{\}\big(z:A,(\neg)\big((\in)(z,C)\big)\big)\big)\right)\right)\quad\text{as}$$

$$P_\rightarrow\left(\#\left((\wedge)\begin{pmatrix}(\exists)\big(\{\}\big(x:A,(\neg)\big((\in)(x,B)\big)\big)\big),\\(\forall)\big(\{\}\big(y:C,(\in)(y,B)\big)\big)\end{pmatrix}\right),\#\big((\exists)\big(\{\}\big(z:A,(\neg)\big((\in)(z,C)\big)\big)\big)\big)\right)\ ,$$

$$P_\rightarrow\left(P_\wedge\begin{pmatrix}\#\big((\exists)\big(\{\}\big(x:A,(\neg)\big((\in)(x,B)\big)\big)\big)\big),\\\#\big((\forall)\big(\{\}\big(y:C,(\in)(y,B)\big)\big)\big)\end{pmatrix},\#\big((\exists)\big(\{\}\big(z:A,(\neg)\big((\in)(z,C)\big)\big)\big)\big)\right)\ ,$$

$$P_\rightarrow\left(P_\wedge\begin{pmatrix}P_\exists\big(\{\}\big(\alpha_x\in\#(A),P_\neg\big(P_\in\big(\alpha_x,\#(B)\big)\big)\big)\big),\\P_\forall\big(\{\}\big(\alpha_y\in\#(C),P_\in\big(\alpha_y,\#(B)\big)\big)\big)\end{pmatrix},P_\exists\big(\{\}\big(\alpha_z\in\#(A),P_\neg\big(P_\in\big(\alpha_z,\#(C)\big)\big)\big)\big)\right)\ .$$

Clearly the last statement can be expressed with the following words:

If ((there exists $\alpha_x \in \#(A)$ such that $\alpha_x \notin \#(B)$) and (for each $\alpha_y \in \#(C)$ $\alpha_y \in \#(B)$))
then (there exists $\alpha_z \in \#(A)$ such that $\alpha_z \notin \#(C)$) .

This confirms that the statement we have proved has the desired meaning.

8. Consistency and paradoxes

We have proved the soundness of our deductive system, i.e. if we can derive φ in our system then #(φ) holds. We now discuss the consistency of the system. We say our system is consistent if for each sentence φ we cannot derive both φ and ¬φ.

Suppose our system is not consistent. In this case there exists a sentence φ such that we can derive both φ and ¬φ. By the soundness property we have #(φ) and #(¬φ), and

$\#(\neg\varphi) = \#(\varepsilon,\neg\varphi,\varepsilon) = P_{\neg}(\#(\varphi)) = \#(\varphi)$ is false.

So #(φ) is true and false at the same time. This is a plain contradiction, so soundness implies consistency.

A paradox is usually a situation in which a contradiction or inconsistency occurs, in other words a paradox arises when we can build a sentence φ such that both φ and ¬φ can be derived. Since our system is consistent it shouldn't be possible to have real paradoxes in it, anyway it seems appropriate to discuss how our system relates with some of the most famous paradoxical arguments.

We begin with Russell's paradox. Assume we can build the set A of all those sets X such that X is not a member of X. Clearly, if A∈A then A∉A and conversely if A∉A then A∈A. We have proved both A∈A and its negation, and this is the Russell's paradox.
It seems in our system we cannot generate this paradox since building a set is permitted only if you rely on already defined sets. When trying to build set A in our language we could obtain something like this:

$$\{\}\big((\neg)\big((\in)(X,X)\big),X\big)\ .$$

But it is clear this isn't a legal expression in our language, since in our language if you want to build a context-independent expression using a variable X, then you have to assign a domain to X.

We now turn to Cantor's paradox. Often the wording of this paradox involves the theory of cardinal numbers (see Mendelson's book [2]), but here we give a simpler wording.

First of all we prove for each set A there doesn't exist a surjective function from domain A to codomain P(A) (where P(A) is the set of A's subsets).

Let f be a function from A to P(A). Let B = {x∈A| x∉f(x)} .
Suppose there exists y∈A such that B=f(y). If y∈B then y∉f(y)=B, and conversely if y∉B=f(y) then y∈B. So there isn't y∈A such that B=f(y), and therefore f is not surjective.

At this point, suppose there exists the set Ω of all sets. Clearly Ω and all of its subsets belong to Ω, so we can define a function f with domain Ω and codomain P(Ω) with this requirement: for each X⊆Ω f(X)=X. Obviously f is a surjective function, and this is a contradiction.

The contradiction is due to having assumed the existence of Ω. In this case too in our language we cannot build an expression with such meaning. One expression like the following:

$$\{\}(\mathrm{set}(X), X)$$

is not a valid expression in our language.

Finally we want to examine the liar paradox. Let's consider how the paradox is stated in Mendelson's book.

A man says, "I am lying". If he is lying, then what he says is true, so he is not lying. If he is not lying, then what he says is false, so he is lying. In any case, he is lying and he is not lying.

Mendelson classifies this paradox as a 'semantic paradox' because it makes use of concepts which need not occur within our standard mathematical language. I agree that, in his formulation, the paradox has some step which seems not mathematically rigorous.

Let's try to give a more rigorous wording of the paradox.

Let A be a set, and let δ be the condition 'for each x in A x is false'. Suppose δ is the only member of A. In this case if δ is true then it is false; if on the contrary δ is false then it is true.

The explanation of the paradox is the following: simply δ cannot be the only item in set A. In fact, suppose A has only one element, and let's call it φ. This implies δ is equivalent to '¬φ', so it seems acceptable that δ is not φ.

Another approach to the explanation is the following.

If δ is true then for each x in A x is false, so δ is not in A. By contraposition if δ is in A then δ is false.

Moreover if δ is false and the uniqueness condition 'for each x in A x=δ' is true then δ is true, thus if δ is false then 'for each x in A x=δ' is false too. By contraposition if 'for each x in A x=δ' then δ is true.

Therefore if δ is the only element in A then δ is true and false at the same time. This implies δ cannot be the only item in A.

On the basis of this argument I consider the liar paradox as an apparent paradox that actually has an explanation. What is the relation between our approach to logic and the liar paradox?

Standard logic isn't very suitable to express this paradox. In fact first-order logic is not designed to construct a condition like our condition δ (= 'for each x in A x is false'), and moreover, it is clearly not designed to say 'δ belongs to set A'. These conditions aren't plainly leading to inconsistency, so it is desirable they can be expressed in a general approach to logic. And our system permits to express them. The paradox isn't ought to simply using these conditions, it is due to an assumption that is clearly false, and the so-called paradox is simply the proof of its falseness.

9. References

[1] Enderton, H., *A Mathematical Introduction to Logic – Second Edition*, Academic Press, 2001 (first edition 1972).

[2] Mendelson, E., *Introduction to Mathematical Logic – Fourth Edition*, Chapman & Hall, 1997 (first edition 1964).

[3] Cutland, N.J., *Computability - An introduction to recursive function theory*, Cambridge University Press, 1980.

[4] Ferreirós, J., *The road to modern logic – an interpretation*, The Bulletin of Symbolic Logic, Volume 7, Number 4, Dec. 2001