An attempt to understand preferred extremals of Kähler action

M. Pitkänen
Email: matpitka@luukku.com.
http://tgd.wippiespace.com/public_html/.

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1 Introduction

There are pressing motivations for understanding the preferred extremals of Kähler action [2]. For instance, the conformal invariance of string models naturally generalizes to 4-D invariance defined by quantum Yangian of quantum affine algebra (Kac-Moody type algebra) characterized by two complex coordinates and therefore explaining naturally the effective 2-dimensionality [13]. One problem is how to assign a complex coordinate with the string world sheet having Minkowskian signature of metric. One can hope that the understanding of preferred extremals could allow to identify two preferred complex coordinates whose existence is also suggested by number theoretical vision giving preferred role for the rational points of partonic 2-surfaces in preferred coordinates. The best one could hope is a general solution of field equations in accordance with the hints that TGD is integrable quantum theory.

A lot is is known about properties of preferred extremals and just by trying to integrate all this understanding, one might gain new visions. The problem is that all these arguments are heuristic and rely heavily on physical intuition. The following considerations relate to the space-time regions having Minkowskian signature of the induced metric. The attempt to generalize the construction also to Euclidian regions could be very rewarding. Only a humble attempt to combine various ideas to a more coherent picture is in question.

The core observations and visions are following.

1. Hamilton-Jacobi coordinates for M^4 [2] define natural preferred coordinates for Minkowskian space-time sheet and might allow to identify string world sheets for X^4 as those for M^4. Hamilton-Jacobi coordinates consist of light-like coordinate m and its dual defining local 2-plane $M^2 \subset M^4$ and complex transversal complex coordinates $(w, \overline{w})$ for a plane E^2_x orthogonal to M^2_x at each point of M^4. Clearly, hyper-complex analyticity and complex analyticity are in question.

2. Space-time sheets allow a slicing by string world sheets (partonic 2-surfaces) labelled by partonic 2-surfaces (string world sheets).

3. The quaternionic planes of octonion space (for quaternions containing preferred hyper-complex plane are labelled by CP_2, which might be called CP_2^{mod} [11]. The identification $CP_2 = CP_2^{mod}$ motivates the notion of $M^8 - -M^4 \times CP_2$ duality [3]. It also inspires a concrete solution ansatz assuming the equivalence of two different identifications of the quaternionic tangent space of the space-time sheet and implying that string world sheets can be regarded as strings in the 6-D coset space $G_2/SU(3)$. The group G_2 of octonion automorphisms has already earlier appeared in TGD framework.

4. The duality between partonic 2-surfaces and string world sheets in turn suggests that the $CP_2 = CP_2^{mod}$ conditions reduce to string model for partonic 2-surfaces in $CP_2 = SU(3)/U(2)$. String model in both cases could mean just hypercomplex/complex analyticity for the coordinates of the coset space as functions of hyper-complex/complex coordinate of string world sheet/partonic 2-surface.

The considerations of this section lead to a revival of an old very ambitious and very romantic number theoretic idea.

1. To begin with express octonions in the form $o = q_1 + Iq_2$, where q_i is quaternion and I is an octonionic imaginary unit in the complement of fixed a quaternionic sub-space of octonions. Map preferred coordinates of $H = M^4 \times CP_2$ to octonionic coordinate, form an arbitrary octonion analytic function having expansion with real Taylor or Laurent coefficients to avoid problems due to non-commutativity and non-associativity. Map the outcome to a point of H to get a map $H \to H$. This procedure is nothing but a generalization of Wick rotation to get an 8-D generalization of analytic map.

2. Identify the preferred extremals of Kähler action as surfaces obtained by requiring the vanishing of the imaginary part of an octonion analytic function. Partonic 2-surfaces and string world sheets would correspond to commutative sub-manifolds of the space-time surface and of imbedding space and would emerge naturally. The ends of braid strands at partonic 2-surface would naturally correspond to the poles of the octonion analytic functions. This would mean a huge generalization of conformal invariance of string models to octonionic conformal invariance and an exact solution of the field equations of TGD and presumably of quantum TGD itself.

2 Basic ideas about preferred extremals

2.1 The slicing of the space-time sheet by partonic 2-surfaces and string world sheets

The basic vision is that space-time sheets are sliced by partonic 2-surfaces and string world sheets. The challenge is to formulate this more precisely at the level of the preferred extremals of Kähler action.

1. Almost topological QFT property means that the Kähler action reduces to Chern-Simons terms assignable to 3-surfaces [6]. This is guaranteed by the vanishing of the Coulomb term in the action density implied automatically if conserved Kähler current is proportional to the instanton current with proportionality coefficient some scalar function. An essential role is played by the weak form of electric magnetic duality [6] in transforming the boundary term to Chern-Simons term.

2. The field equations reduce to the conservation of isometry currents. An attractive ansatz is that the flow lines of these currents define global coordinates. This means that these currents are Beltrami flows [6] so that corresponding 1-forms J satisfy the condition $J \wedge dJ = 0$. These conditions are satisfied if

$$J = \Phi \nabla \Psi$$

hold true for conserved currents. From this one obtains that Ψ defines global coordinate varying along flow lines of J.

3. A possible interpretation is in terms of local polarization and momentum directions defined by the scalar functions involved and natural additional conditions are that the gradients of Ψ and Φ are orthogonal:

$$\nabla \Phi \cdot \nabla \Psi = 0 \ ,$$

and that the Ψ satisfies massless d'Alembert equation

$$\nabla^2 \Psi = 0$$

as a consequence of current conservation. If Ψ defines a light-like vector field - in other words

$$\nabla \Psi \cdot \nabla \Psi = 0 \ ,$$

the light-like dual of Φ -call it Φ_c- defines a light-like like coordinate and Φ and Φ_c defines a light-like plane at each point of space-time sheet.

If also Φ satisfies d'Alembert equation

$$\nabla^2 \Phi = 0 \ ,$$

also the current

$$K = \Psi \nabla \Phi$$

is conserved and its flow lines define a global coordinate in the polarization plane orthogonal to time-lik plane defined by local light-like momentum direction.

If Φ allows a contination to an analytic function of the transversal complex coordinate, one obtains a coordinatization of spacetime surface by Ψ and its dual (defining hyper-complex coordinate) and $w, \overline{w}$. Complex analyticity and its hyper-complex variant would allow to provide space-time surface with four coordinates very much analogous with Hamilton-Jacobi coordinates of M^4.

This would mean a decomposition of the tangent space of space-time surface to orthogonal planes defined by light-like momentum and plane orthogonal to it. If the flow lines of J defined Beltrami flow it seems that the distribution of momentum planes is integrable.

4. General arguments suggest that the space-time sheets allow a slicing by string world sheets parametrized by partonic 2-surfaces or vice versa. This would mean a intimate connection with the mathematics of string models. The two complex coordinates assignable to the Yangian of affine algebra would naturally relate to string world sheets and partonic 2-surfaces and the highly non-trivial challenge is to identify them appropriately.

2.2 Hamilton-Jacobi coordinates for M^4

The earlier attempts to construct preferred extremals led to the realization that so called Hamilton-Jacobi coordinates (m, w) for M^4 define its slicing by string world sheets parametrized by partonic 2-surfaces. m would be pair of light-like conjugate coordinates associated with an integrable distribution of planes M^2 and w would define a complex coordinate for the integrable distribution of 2-planes E^2 orthogonal to M^2. There is a great temptation to assume that these coordinates define prefered coorinates for M^4.

1. The slicing is very much analogous to that for space-time sheets and the natural question is how these slicings relate. What is of special interest is that the momentum plane M^2 can be defined by massless momentum. The scaling of this vector does not matter so that these planes are labelled by points z of sphere S^2 telling the direction of the line $M^2 \cap E^3$, when one assigns rest frame and therefore S^2 with the preferred time coordinate defined by the line connecting the tips of CD. This direction vector can be mapped to a twistor consisting of a spinor and its conjugate. The complex scalings of the twistor $(u, \overline{u}) \to \lambda u, \overline{u}/\lambda)$ define the same plane. Projective twistor like entities defining CP_1 having only one complex component instead of three are in question. This complex number defines with certain prerequisites a local coordinate for space-time sheet and together with the complex coordinate of E^2 could serve as a pair of complex coordinates (z, w) for space-time sheet. This brings strongly in mind the two complex coordinates appearing in the expansion of the generators of quantum Yangian of quantum affine algebra [13].

2. The coordinate Ψ appearing in Beltrami flow defines the light-like vector field defining M^2 distribution. Its hyper-complex conjugate would define Ψ_c and conjugate light-like direction. An attractive possibility is that Φ allows analytic continuation to a holomorphic function of w. In this manner one would have four coordinates for M^4 also for space-time sheet.

3. The general vision is that at each point of space-time surface one can decompose the tangent space to $M^2(x) \subset M^4 = M^2_x \times E^2_x$ representing momentum plane and polarization plane $E^2 \subset E^2_x \times T(CP_2)$. The moduli space of planes $E^2 \subset E^6$ is 8-dimensional and parametrized by $SO(6)/SO(2) \times SO(4)$ for a given E^2_x. How can one achieve this selection and what conditions it must satisfy? Certainly the choice must be integrable but this is not the only condition.

2.3 Space-time surfaces as quaternionic surfaces

The idea that number theory determines classical dynamics in terms of associativity condition means that space-time surfaces are in some sense quaternionic surfaces of an octonionic space-time [11]. It took several trials before the recent form of this hypothesis was achieved.

1. Octonionic structure is defined in terms of the octonionic representaton of gamma matrices of the imbedding space existing only in dimension $D = 8$ since octonion units are in one-one correspondence with tangent vectors of the tangent space. Octonionic real unit corresponds to a preferred time axes (and rest frame) identified naturally as that connecting the tips of CD. What modified gamma matrices mean depends on variational principle for space-time surface. For volume action one would obtain induced gamma matrices. For Kähler action one obtains something different. In particular, the modified gamma matrices do not define vector basis identical with tangent vector basis of space-time surface.

2. Quaternionicity means that the modified gamma matrices defined as contractions of gamma matrices of H with canonical momentum densities for Kähler action span quaternionic sub-space of the octonionic tangent space [6]. A further condition is that each quaternionic space defined in this manner contains a preferred hyper-complex subspace of octonions.

3. The sub-space defined by the modified gamma matrices does not co-incide with the tangent space of space-time surface in general so that the interpretation of this condition is far from obvious. The canonical momentum densities need not define four independent vectors at given point. For instance, for massless extremals these densities are proportional to light-like vector so that the situation is degenerate and the space in question reduces to 2-D hyper-complex sub-space since light-like vector defines plane M^2.

The obvious questions are following.

1. Does the analog of tangent space defined by the octonionic modified gammas contain the local tangent space $M^2 \subset M^4$ for preferred extremals? For massless extremals this condition would be true. The orthogonal decomposition $T(X^4) = M^2 \oplus_\perp E^2$ can be defined at each point if this is true. For massless extremals also the functions Ψ and Φ can be identified.

2. One should answer also the following delicate question. Can M^2 really depend on point x of space-time? CP_2 as a moduli space of quaternionic planes emerges naturally if M^2 is *same* everywhere. It however seems that one should allow an integrable distribution of M^2_x such that M^2_x is same for all points of a given partonic 2-surface.

   How could one speak about fixed CP_2 (the imbedding space) at the entire space-time sheet even when M^2_x varies?

   (a) Note first that G_2 [3] defines the Lie group of octonionic automorphisms and G_2 action is needed to change the preferred hyper-octonionic sub-space. Various $SU(3)$ subgroups of G_2 are related by G_2 automorphism. Clearly, one must assign to each point of a string world sheet in the slicing parameterizing the partonic 2-surfaces an element of G_2. One would have Minkowskian string model with G_2 as a target space. As a matter fact, this string model is defined in the target space $G_2/SU(3)$ having dimension $D = 6$ since $SU(3)$ automorphisms leave given $SU(3)$ invariant.

   (b) This would allow to identify at each point of the string world sheet standard quaternionic basis - say in terms of complexified basis vectors consisting of two hyper-complex units and octonionic unit q_1 with "color isospin" $I_3 = 1/2$ and "color hypercharge" $Y = -1/3$ and its conjugate $\overline{q}_1$ with opposite color isospin and hypercharge.

   (c) The CP_2 point assigned with the quaternionic basis would correspond to the $SU(3)$ rotation needed to rotate the standard basis to this basis and would actually correspond to the first row of $SU(3)$ rotation matrix. Hyper-complex analycity is the basic property of the solutions of the field equations representing Minkowskian string world sheets. Also now the same assumption is highly natural. In the case of string models in Minkowski space, the reduction of the induced metric to standard form implies Virasoro conditions and similar conditions are expected also now. There is no need to introduce action principle -just the hyper-complex analyticitity is enough-since Kähler action already defines it.

3. The WZW model inspired approach to the situation would be following. The parametrization corresponds to a map $g : X^2 \rightarrow G_2$ for which g defines a flat G_2 connection at string world sheet. WZW type action would give rise to this kind of situation. The transition $G_2 \rightarrow G_2/SU(3)$ would require that one gauges $SU(3)$ degrees of freedom by bringing in $SU(3)$ connection. Similar procedure for $CP_2 = SU(3)/U(2)$ would bring in $SU(3)$ valued chiral field and $U(2)$ gauge field. Instead of introducing these connections one can simply introduce $G_2/SU(3)$ and $SU(3)/U(2)$ valued chiral fields. What this observation suggests that this ansatz indeed predicts gluons and electroweak gauge bosons assignable to string like objects so that the mathematical picture would be consistent with physical intuition.

2.4 The two interpretations of CP_2

An old observation very relevant for what I have called M^8-H duality [3] is that the moduli space of quaternionic sub-spaces of octonionic space (identifiable as M^8) containing preferred hyper-complex plane is CP_2. Or equivalently, the space of two planes whose addition extends hyper-complex plane to some quaternionic subspace can be parametrized by CP_2. This CP_2 can be called it CP_2^{mod} to avoid confusion. In the recent case this would mean that the space $E^2(x) \subset E^2_x \times T(CP_2)$ is represented by a point of CP_2^{mod}. On the other hand, the imbedding of space-time surface to H defines a point of "real" CP_2. This gives two different CP_2s.

1. The highly suggestive idea is that the identification $CP_2^{mod} = CP_2$ (apart from isometry) is crucial for the construction of preferred extremals. Indeed, the projection of the space-time point to CP_2 would fix the local polarization plane completely. This condition for $E^2(x)$ would be purely local and depend on the values of CP_2 coordinates only. Second condition for $E^2(x)$ would involve the gradients of imbedding space coordinates including those of CP_2 coordinates.

2. The conditions that the planes M^2_x form an integrable distribution at space-like level and that M^2_x is determined by the modified gamma matrices. The integrability of this distribution for M^4 could imply the integrability for X^2. X^4 would differ from M^4 only by a deformation in degrees of freedom transversal to the string world sheets defined by the distribution of M^2s.

   Does this mean that one can begin from vacuum extremal with constant values of CP_2 coordinates and makes them non-constant but allows to depend only on transversal degrees of freedom? This condition is too strong even for simplest massless extremals for which CP_2 coordinates depend on transversal coorinates defined by $\epsilon \cdot m$ and $\epsilon \cdot k$. One could however allow dependence of CP_2 coordinates on light-like M^4 coordinate since the modification of the induced metric is light-like so that light-like coordinate remains light-like coordinate in this modification of the metric.

   Therefore, if one generalizes directly what is known about massless extremals, the most general dependence of CP_2 points on the light-like coordinates assignable to the distribution of M^2_x would be dependence on either of the light-like coordinates of Hamilton-Jacobi coordinates but not both.

3 What could be the construction recipe for the preferred extremals assuming $CP_2 = CP_2^{mod}$ identification?

The crucial condition is that the planes $E^2(x)$ determined by the point of $CP_2 = CP_2^{mod}$ identification and by the tangent space of $E^2_x \times CP_2$ are same. The challenge is to transform this condition to an explicit form. $CP_2 = CP_2^{mod}$ identification should be general coordinate invariant. This requires that also the representation of E^2 as (e^2, e^3) plane is general coordinate invariant suggesting that the use of preferred CP_2 coordinates -presumably complex Eguchi-Hanson coordinates- could make life easy. Preferred coordinates are also suggested by number theoretical vision. A careful consideration of the situation would be required.

The modified gamma matrices define a quaternionic sub-space analogous to tangent space of X^4 but not in general identical with the tangent space: this would be the case only if the action were 4-volume. I will use the notation $T^m_x(X^4)$ about the modified tangent space and call the vectors of $T^m_x(X^4)$ modified tangent vectors. I hope that this would not cause confusion.

3.1 $CP_2 = CP_2^{mod}$ condition

Quaternionic property of the counterpart of $T^m_x(X^4)$ allows an explicit formulation using the tangent vectors of $T^m_x(X^4)$.

1. The unit vector pair (e_2, e_3) should correspond to a unique tangent vector of H defined by the coordinate differentials dh^k in some natural coordinates used. Complex Eguchi-Hanson coordinates [1] are a natural candidate for CP_2 and require complexified octonionic imaginary units. If octonionic units correspond to the tangent vector basis of H uniquely, this is possible.

2. The pair (e_2, e_3) as also its complexification $(q_1 = e_2 + ie_3, \overline{q}_1 = e_2 - ie_3)$ is expressible as a linear combination of octonionic units $I_2, ...I_7$ should be mapped to a point of $CP_2^{mod} = CP_2$ in canonical manner. This mapping is what should be expressed explicitly. One should express given (e_2, e_3) in terms of $SU(3)$ rotation applied to a standard vector. After that one should define the corresponding CP_2 point by the bundle projection $SU(3) \rightarrow CP_2$.

3. The tangent vector pair

$$(\partial_w h^k, \partial_{\overline{w}} h^k)$$

defines second representation of the tangent space of $E^2(x)$. This pair should be equivalent with the pair $(q_1, \overline{q}_1)$. Here one must be however very cautious with the choice of coordinates. If the choice of w is unique apart from constant the gradients should be unique. One can use also real coordinates (x, y) instead of $(w = x + iy, \overline{w} = x - iy)$ and the pair (e_2, e_3). One can project the tangent vector pair to the standard vielbein basis which must correspond to the octonioni basis

$$(\partial_x h^k, \partial_y h^k) \rightarrow (\partial_x h^k e_k^A e_A, \partial_y h^k e_k^A) e_A) \leftrightarrow (e_2, e_3) \ ,$$

where the e_A denote the octonion units in 1-1 correspondence with vielbein vectors. This expression can be compared to the expression of (e_2, e_3) derived from the knowledge of CP_2 projection.

3.2 Formulation of quaternionicity condition in terms of octonionic structure constants

One can consider also a formulation of the quaternionic tangent planes in terms of (e_2, e_3) expressed in terms of octonionic units deducible from the condition that unit vectors obey quaternionic algebra. The expressions for octonionic *resp.* quaternionic structure constants can be found at [5] *resp.* [6].

1. The ansatz is

$$\begin{aligned} \{E_k\} &= \{1, I_1, E_2, E_3\} \ , \\ E_2 &= E_{2k}e^k \equiv \sum_{k=2}^{7} E_{2k}e^k \ \ , \ \ E_3 = E_{3k}e^k \equiv \sum_{k=2}^{7} E_{3k}e^k \ \ , \\ |E_2| &= 1 \ \ , \ \ |E_3| = 1 \ \ . \end{aligned} \tag{3.1}$$

2. The multiplication table for quaternionic units gives

$$f^{1kl} E_{2k} = E_{3l} \ \ , \ \ f^{1kl} E_{3k} = -E_{2l} \ \ , \ \ f^{klr} E_{2k} E_{3l} = \delta_1^r \ \ . \tag{3.2}$$

Here the indices are raised by unit metric so that there is no difference between lower and upper indices. Summation convention is assumed. Also the contribution of the real unit is present in the structure constants of third equation but this contribution must vanish.

3. The conditions are linear and quadratic in the coefficients E_{2k} and E_{3k} and are expected to allow an explicit solution. The first two conditions define homogenous equations which must allow solution. The coefficient matrix acting on (E_2, E_3) is of the form

$$\begin{pmatrix} f_1 & 1 \\ -1 & f_1 \end{pmatrix} \ ,$$

where 1 denotes unit matrix. The vanishing of the determinant of this matrix should be due to the highly symmetric properties of the structure constants. In fact the equations can be written as eigen conditions

$$f_1 \circ (E_2 \pm iE_3) = \mp i(E_2 \pm iE_3) \ ,$$

and one can say that the structure constants are eigenstates of the hermitian operator defined by I_1 analogous to color hyper charge. Both values of color hyper charged are obtained.

3.3 Explicit expression for the $CP_2 = CP_2^{mod}$ conditions

The symmetry under SU(3) allows to construct the solutions of the above equations directly.

1. One can introduce complexified basis of octonion units transforming like $(1, 1, 3, \overline{3})$ under SU(3). Note the analogy of triplet with color triplet of quarks. One can write complexified basis as $(1, e_1, (q_1, q_2, q_3), (\overline{q}_1\overline{q}_2, \overline{q}_3))$. The expressions for complexified basis elements are

$$(q_1, q_2, q_3) = \frac{1}{\sqrt{2}}(e_2 + ie_3, e_4 + ie_5, e_6 + ie_7) \ .$$

   These options can be seen to be possible by studying octonionic triangle in which all lines containing 3 units defined associative triple: any pair of octonion units at this kind o fline can be used to form pair of complexified unit and its conjugate. In the tangent space of $M^4 \times CP_2$ the basis vectors q_1, and q_2 are mixtures of E_x^2 and CP_2 tangent vectors. q_3 involves only CP_2 tangent vectors and there is a temptation to interpret it as the analog of the quark having no color isospin.

2. The quaternionic basis is real and must transform like $(1, 1, q_1, \overline{q}_1)$, where q_1 is any quark in the triplet and $\overline{q}_1$ its conjugate in antitriplet. Having fixed some basis one can perform SU(3) rotations to get a new basis. The action of the rotation is by 3×3 special unitary matrix. The over all phases of its rows do not matter since they induce only a rotation in (e_2, e_3) plane not affecting the plane itself. The action of $SU(3)$ on q_1 is simply the action of its first row on (q_1, q_2, q_3) triplet:

$$\begin{aligned} q_1 &\rightarrow (Uq)_1 = U_{11}q_1 + U_{12}q_2 + U_{13}q_3 \equiv z_1q_1 + z_2q_2 + z_3q_3 \\ &= z_1(e_2 + ie_3) + z_2(e_4 + ie_5) + z_3(e_6 + ie_7) \ . \end{aligned} \tag{3.3}$$

   The triplets (z_1, z_2, z_3) defining a complex unit vector and point of S^5. Since overall phase does not matter a point of CP_2 is in question. The new real octonion units are given by the formulas

$$\begin{aligned} e_2 &\rightarrow Re(z_1)e_2 + Re(z_2)e_4 + Re(z_3)e_6 - Im(z_1)e_3 - Im(z_2)e_5 - Im(z_3)e_7 \ , \\ e_3 &\rightarrow Im(z_1)e_2 + Im(z_2)e_4 + Im(z_3)e_6 + Re(z_1)e_3 + Re(z_2)e_5 + Re(z_3)e_7 \ . \end{aligned} \tag{3.4}$$

   For instance the CP_2 coordinates corresponding to the coordinate patch (z_1, z_2, z_3) with $z_3 \neq 0$ are obtained as $(\xi_1, \xi_2) = (z_1/z_3, z_2/z_3)$.

Using these expressions the equations expressing the conjecture $CP_2 = CP_2^{mod}$ equivalence can be expressed explicitly as first order differential equations. The conditions state the equivalence

$$(e_2, e_3) \ \leftrightarrow \ (\partial_x h^k e_k^A e_A, \partial_y h^k e_k^A e_A) \ , \tag{3.5}$$

where e_A denote octonion units. The comparison of two pairs of vectors requires normalization of the tangent vectors on the right hand side to unit vectors so that one takes unit vector in the direction of the tangent vector. After this the vectors can be equated. This allows to expresses the contractions of the partial derivatives with vielbein vectors with the 6 components of e_2 and e_3. Each condition gives 6+6 first order partial differential equations which are non-linear by the presence of the overal normalization factor for the right hand side. The equations are invariant under scalings of (x, y). The very special form of these equations suggests that some symmetry is involved.

It must be emphasized that these equations make sense only in preferred coordinates: ordinary Minkowski coordinates and Hamiltonin-Jacobi coordinates for M^4 and Eguchi-Hanson complex coordinates in which $SU(2) \times U(1)$ is represented linearly for CP_2. These coordinates are preferred because they carry deep physical meaning.

3.4 Does TGD boil down to two string models?

It is good to look what have we obtained. Besides Hamilton-Jacobi conditions, and $CP_2 = CP_2^{mod}$ conditions one has what one might call string model with 6-dimensional $G_2/SU(3)$ as targent space. The orbit of string in $G_2/SU(3)$ allows to deduce the G_2 rotation identifiable as a point of $G_2/SU(3)$ defining what one means with standard quaternionic plane at given point of string world sheet. The hypothesis is that hyper-complex analyticity solves these equations.

The conjectured electric-magnetic duality implies duality between string world sheet and partonic 2-surfaces central for the proposed mathematical applications of TGD [7, 8, 10, 14]. This duality suggests that the solutions to the $CP_2 = CP_2^{mod}$ conditions could reduce to holomorphy with respect to the coordinate w for partonic 2-surface plus the analogs of Virasoro conditions. The dependence on light-like coordinate would appear as a parametric dependence.

If this were the case, TGD would reduce at least partially to what might be regaded as dual string models in $G_2/SU(3)$ and $SU(3)/U(2)$ and also to string model in M^4 and X^4! In the previous arguments one ends up to string models in moduli spaces of string world sheets and partonic 2-surfaces. TGD seems to yield an inflation of string models! This not actually surprising since the slicing of space-time sheets by string world sheets and partonic 2-surfaces implies automatically various kinds of maps having interpretation in terms of string orbits.

3.5 Could octonion analyticity solve the field equations?

The interesting question is what happens in the space-time regions with Euclidian signature of induced metric. In this case it is not possible to introduce light-like plane at each point of the space-time sheet. Nothing however prevents from applying the above described procedure to construct conserved currents whose flow lines define global coordinates. In both cases analytic continuation allows to extend the coordinates to complex coordinates. Therefore one would have two complex functions satisfying Laplace equation and having orthogonal gradients.

1. When CP_2 projection is 4-dimensional, there is strong temptation to assume that these functions could be reduced to complex CP_2 coordinates analogous to the Hamilton-Jacobi coordinates for M^4. Complex Eguchi-Hanson coordinates transforming linearly under $U(2) \subset SU(3)$ define the simplest candidates in this respect. Laplace-equations are satisfied utomatically since holomorphic functions are in question. The gradients are also orthogonal automatically since the metric is Kähler metric. Note however that one could argue that in innner product the conjugate of the function appears. Any holomorphic map defines new coordinates of this kind. Note that the maps need not be globally holomorphic since CP_2 projection of space-time sheet need not cover the entire CP_2.

2. For string like objects $X^4 = X^2 \times Y^2 \subset M^4 \times CP_2$ with Minkowskian signature of the metric the coordinate pair would be hyper-complex coordinate in M^4 and complex coordinate in CP_2. If X^2 has Euclidian signature of induced metric the coordinate in question would be complex coordinate. The proposal in the case of CP_2 allows all holomorphic functions of the complex coordinates.

There is an objection against this construction. There should be a symmetry between M^4 and CP_2 but this is not the case. Therefore this picture cannot be quite correct.

Could the construction of new preferred coordinates by holomorphic maps generalize as electic-magnetic duality suggests? One can imagine several options, which bring in mind old ideas that what I have christened as "romantic stuff" [11].

1. Should one generalize the holomorphic map to a quaternion analytic map with real Taylor coefficients so that non-commutativity would not produce problems. One would map first M^4 coordinates to quaternions, map these coordinates to new ones by quaternion analytic map defined by a Taylor or even Laurnte expansion with real coefficients, and then map the resulting quaternion valued coordinate back to hyper-quaternion defining four coordinates as fuctions in M^4. This procedure would be very much analogous to Wick rotation used in quantum field theories. Similar quaternion analytic map be applied also in CP_2 degrees of freedom followed by the map of the quaternion to two complex numbers. This would give additional constraints on the map. This option could be seen as a quaternionic generalization of conformal invariance.

   The problem is that one decouples M^4 and CP_2 degrees of freedom completely. These degrees are however coupled in the proposed construction since the $E^2(x)$ corresponds to subspace of $E^2_x \times T(CP_2)$. Something goes still wrong.

2. This motivates to imagine even more ambitious and even more romantic option realizing the original idea about octonionic generalization of conformal invariance. Assume linear $M^4 \times CP_2$ coordinates (Eguchi-Hanson coordinates transforming linearly under $U(2)$ in the case of CP_2). Map these to octonionic coordinate h. Map the octonionic coordinate to itself by an octonionic analytic map defined by Taylor or even Laurent series with real coefficients so that non-commutativity and non-associativity do not cause troubles. Map the resulting octonion valued coordinates back to ordinary H-coordinates and expressible as functions of original coordinates.

   It must be emphasized that this would be nothing but a generalization of Wick rotation and its inverse used routinely in quantum field theories in order to define loop integrals.

3.5.1 Could octonion real-analyticity make sense?

Suppose that one -for a fleeting moment- takes octonionic analyticity seriously. For space-time surfaces themselves one should have in some sense quaternionic variant of conformal invariance. What does this mean?

1. Could one regard space-time surfaces analogous to the curves at which the imaginary part of analytic function of complex argument vanishes so that complex analyticity reduces to real analyticity. One can indeed divide octonion to quaternion and its imaginary part to give $o = q_1 + Iq_2$: q_1 and q_2 are quaternionis and I is octonionic imaginary unit in the complement of the quaternionic sub-space. This decomposition actually appears in the standard construction of octonions. Therefore 4-dimensional surfaces at which the imaginary part of octonion valued function vanishes make sense and defined in well-defined sense quaternionic 4-surfaces.

   This kind of definition would be in nice accord with the vision about physics as algebraic geometry. Now the algebraic geometry would be extended from complex realm to the octonionic realm since quaternionic surfaces/string world sheets could be regarded as associative/commutative sub-algebras of the algebra of the octionic real-analytic functions.

2. Could these surfaces correspond to quaternionic 4-surfaces defined in terms of the modified gamma matrices or induced gamma matrices? Contrary to the original expectations it will be found that only induced gamma matrices is a plausible option. This would be an enormous simplification and would mean that the theory is exactly solvable in the same sense as string models are: complex analyticity would be replaced with octonion analyticity. I have considered this option in several variants using the notion of real octonion analyticity [11] but have not managed to build any satisfactory scenario.

3. Hyper-complex and complex conformal symmetries would result by a restriction to hyper-complex *resp.* complex sub-manifods of the imbedding space defined by string world sheets *resp.* partonic 2-surfaces. The principle forcing this restriction would be commutativity. Yangian of an affine algebra would unify these views to single coherent view [13].

   4-D n-point functions of the theory should result from the restriction on partonic 2-surfaces or string world sheets with arguments of n-point functions identified as the ends of braid strands so that a kind of analytic continuation from 2-D to the 4-D case would be in question. The octonionic conformal invariance would be induced by the ordinary conformal invariance in accordance with strong form of General Coordinate Invariance.

4. This algebraic continuation of the ordinary conformal invariance could help to construct also the representations of Yangians of affine Kac-Moody type algebras. For the Yangian symmetry of 1+1 D integrable QFTs the charges are multilocal involving multiple integrals over ordered multiple points of 1-D space. I

   In the recent case multiple 1-D space is replaced with a space-like 3-surface at the light-like end of CD. The point of the 1-D space appearing in the multiple integral are replaced by a partonic 2-surface represented by a collection of punctures. There is a strong temptation to assume that the intermediate points on the line correspond to genuine physical particles and therefore to partonic 2-surfaces at which the signature of the induced metric changes. If so, the 1-D space would correspond to a closed curve connecting punctures of different partonic 2-surfaces representing physical particles and ordered along a loop. The integral over multiple points would correspond to an integral over WCW rather than over fixed back-ground space-time.

   1-D space would be replaced with a closed curve going through punctures of a subset of partonic 2-surfaces associated with a space-like 3-surface. If a given partonic surface or a given puncture can contribute only once to the multiple integral the multi-locality is bounded from above and only a finite number of Yangian generators are obtained in this manner unless one allows the number of partonic 2-surfaces and of punctures for them to vary. This variation is physically natural and would correspond to generation of particle pairs by vacuum polarization. Although only punctures would contribute, the Yangian charges would be defined in WCW rather than in fixed space-time. Integral over positions of punctures and possible numbers of them would be actually an integral over WCW. 2-D modular invariance of Yangian charges for the partonic 2-surfaces is a natural constraint.

   The question is whether some conformal fields at the punctures of the partonic 2-surfaces appearing in the multiple integral define the basic building bricks of the conserved quantum charges representing the multilocal generators of the Yangian algebra? Note that Wick rotation would be involved.

3.5.2 What the non-triviality of the moduli space of the octonionic structures means?

The moduli space G_2 of the octonionic structures is essentially the Galois group defined as maps of octonions to itself respecting octonionic sum and multiplication. This raises the question whether octonion analyticity should be generalized in such a manner that the global choice of the octonionic imaginary units - in particular that of preferred commuting complex sub-space- would become local. Physically this would correspond to the choice of momentum plane M^2_x for a position dependent light-iike momentum defining the plane of non-physical polarizations.

This question is inspired by the general solution ansatz based on the slicing of space-time sheets which involves the dependence of the choice of the momentum plane M^2_x on the point of string world sheet. This dependence is parameterized by a point of $G_2/SU(3)$ and assumed to be constant along partonic 2-surfaces. These slicings would be naturally associated with the two complex parts c_i of the quaternionic coordinate $q_1 = c_1 + Ic_2$ of the space-time sheet.

This dependence is well-defined only for the quaternionic 4-surface defining the space-time surface and can be seen as a local choice of a preferred complex imaginary unit along string world sheets. CP_2 would parametrize the remaining geometric degrees of freedom. Should/could one extend this dependence to entire 8-D imbedding space? This is possible if the 8-D imbedding space allows a slicing by the string world sheets. If the string world sheets correspond to the string world sheets appearing in the slicing of M^4 defined by Hamiton-Jacobi coordinates [2], this slicing indeed exists.

3.5.3 Zero energy ontology and octonion analyticity

How does this picture relate to zero energy ontology and how partonic 2-surfaces and string world sheets could be identified in this framework?

1. The intersection of the quaternionic four-surfaces with the 7-D light-like boundaries of CDs is 3-D space-like surface. String world sheets are obtained as 2-D complex surfaces by putting $c_2 = 0$, where c_2 is the imaginary part of the quaternion coordinate $q = c_1 + Ic_2$. Their intersections with CD boundaries are generally 1-dimensional and represent space-like strings.

2. Partonic 2-surfaces could correspond to the intersections of $Re(c_1) = constant$ 3-surfaces with the boundaries of CD. The variation of $Re(c_1)$ would give a family of (possibly light-like) 3-surfaces whose intersection with the boundaries of CD would be 2-dimensional. The interpretation $Re(c_1) = constant$ surfaces as (possibly light-like) orbits of partonic 2-surfaces would be natural. Wormhole throats at which the signature of the induced metric changes (by definition) would correspond to some special value of $Re(c_1)$, naturally $Re(c_1) = 0$.

   What comes first in mind is that partonic 2-surfaces assignable to wormhole throats correspond to co-complex 2-surfaces obtained by putting $c_1 = 0$ (or $c_1 = constant$) in the decomposition $q = c_1 + ic_2$. This option is consistent with the above assumption if $Im(c_1) = 0$ holds true at the boundaries of CD. Note that also co-quaternionic surfaces make sense and would have Euclidian signature of the induced metric: the interpretation as counterparts of lines of generalized Feynman graphs might make sense.

3. One can of course wonder whether also the poles of c_1 might be relevant. The most natural idea is that the value of $Re(c_1)$ varies between 0 and ∞ between the ends of the orbit of partonic 2-surface. This would mean that c_1 has a pole at the other end of CD (or light-like orbit of partonic 2-surface). In light of this the earlier proposal [10] that zero energy states might correspond to rational functions assignable to infinite primes and that the zeros/poles of these functions correspond to the positive/negative energy part of the state is interesting.

   The intersections of string world sheets and partonic 2-surfaces identifiable as the common ends of space-like and time like brand strands would correspond to the points $q = c_1 + Ic_2 = 0$ and $q = \infty + Ic_2$, where ∞ means real infinity. In other words, to the zeros and real poles of quaternion analytic function with real coefficients. In the number theoretic vision especially interesting situations correspond to polynomials with rational number valued coefficients and rational functions formed from these. In this kind of situations the number of zeros and therefore of braid strands is always finite.

3.5.4 Do induced or modified gamma matrices define quaternionicity?

The are two options to be considered: either induced or modified gamma matrices define quaternionicity.

1. There are several arguments supporting this view that induced gamma matrices define quaternionicity and that quaternionic planes are therefore tangent planes for space-time sheet.

   (a) $H - M^8$ correspondence is based on the observation that quaternionic sub-spaces of octonions containing preferred complex sub-space are labelled by points of CP_2. The integrability of the distribution of quaternionic spaces could follow from the parametrization by points of CP_2 ($CP_2 = CP_{mod}$ condition). Quaternionic planes would be necessarily tangent planes of space-time surface. Induced gamma matrices correspond naturally to the tangent space vectors of the space-time surface.

   Here one should however understand the role of the M^4 coordinates. What is the functional form of M^4 coordinates as functions of space-time coordinates or does this matter at all (general coordinate invariance): could one choose the space-time coordinates as M^4 coordinates for surfaces representable as graphs for maps $M^4 \to CP_2$? What about other cases such as cosmic strings [4]?

   (b) Could one do entirely without gamma matrices and speak only about induced octonion structure in 8-D tangent space (raising also dimension $D = 8$ to preferred role) with reduces to quaternionic structure for quaternionic 4-surfaces. The interpretation of quaternionic plane as tangent space would be unavoidable also now. In this approach there would be no question about whether one should identify octonionic gamma matrices as induced gamma matrices or as modified octonionic gamma matrices.

   (c) If quaternion analyticity is defined in terms of modified gamma matrices defined by the volume action why it would solve the field equations for Kähler action rather than for minimal surfaces? Is the reason that quaternionic and octonionic analyticities defined as generalized differentiability are not possible. The real and imaginary parts of quaternionic real-analytic function with quaternion interpreted as bi-complex number are not analytic functions of two complex variables of either complex variable. In 4-D situation minimal surface property would be too strong a condition whereas Kähler action poses much weaker conditions. Octonionic real-analyticity however poses strong symmetries and suggests effective 2-dimensionality.

2. The following argument suggest that modified gamma matrices cannot define the notion of quaternionic plane.

   (a) Modified gamma matrices can define sub-spaces of lower dimensionality so that they do not defined a 4-plane. In this case they cannot define CP_2 point so that $CP_2 = CP_2^{mod}$ identity fails. Massless extremals represents the basic example about this. Hydrodynamic solutions defined in terms of Beltrami flows could represent a more general phase of this kind.

   (b) Modified gamma matrices are not in general parallel to the space-time surface. The CP_2 part of field equations coming from the variation of Kähler form gives the non-tangential contribution. If the distribution of the quaternionic planes is integrable it defines another space-time surface and this looks rather strange.

   (c) Integrable quaternionicity can mean only tangent space quaternionicity. For modified gamma matrices this cannot be the case. One cannot assign to the octonion analytic map modified gamma matrices in any natural manner.

The conclusion seems to be that induced gamma matrices or induced octonion structure must define quaternionicity and quaternionic planes are tangent planes of space-time surface and therefore define an integrable distribution. An open question is whether $CP_2 = CP_2^{mod}$ condition implies the integrability automatically.

3.5.5 Volume action or Kähler action?

What seems clear is that quaternionicity must be defined by the induced gamma matrices obtained as contractions of canonical momentum densities associated with volume action with imbedding space gamma matrices. Probably equivalent definition is in terms of induced octonion structure. For the believer in strings this would suggest that the volume action is the correct choice. There are however strong objections against this choice.

1. In 2-dimensional case the minimal surfaces allow conformal invariance and one can speak of complex structure in their tangent space. In particular, string world sheets can be regarded as complex 2-surfaces of quaternionic space-time surfaces. In 4-dimensional case the situation is different since quaternionic differentiability fails by non-commutativity. It is quite possible that only very few minimal surfaces (volume action) are quaternionic.

2. The possibility of Beltrami flows is a rather plausible property of quite many preferred extremals of Kähler action. Beltrami flows are also possible for a 4-D minimal surface action. In particular, M^4 translations would define Beltrami flows for which the 1-forms would be gradients of linear M^4 coordinates. If M^4 coordinate can be used on obtains flows in directions of all coordinate axes. Hydrodynamical picture in the strong form therefore fails whereas for Kähler action various isometry currents could be parallel (as they are for massless extremals).

3. For volume action topological QFT property fails as also fails the decomposition of solutions to massless quanta in Minkowskian regions. The same applies to criticality. The crucial vacuum degeneracy responsible for most nice features of Kähler action is absent and also the effective 2-dimensionality and almost topological QFT property are lost since the action does not reduce to 3-D term.

One can however keep Kähler action and define quaternionicity in terms of induced gamma matrices or induced octonion structure. Preferred extremals could be identified as extremals of Kähler action which are also quaternionic 4-surfaces.

1. Preferred extremal property for Kähler action could be much weaker condition than minimal surface property so that much larger set of quaternionic space-time surfaces would be extremals of the Kähler action than of volume action. The reason would be that the rank of energy momentum tensor for Maxwell action tends to be smaller than maximal. This expectation is supported by the vacuum degeneracy, the properties of massless extremals and of CP_2 type vacuum extremals, and by the general hydrodynamical picture.

2. There is also a long list of beautiful properties supporting Kähler action which should be also familiar: effective 2-dimensionality and slicing of space-time surface by string world sheets and partonic 2-surfaces, reduction to almost topological QFT and to abelian Chern-Simons term, weak form of electric-magnetic duality, quantum criticality, spin glass degeneracy, etc...

3.5.6 Are quaternionicities defined in terms of induced gamma matrices *resp.* octonion real-analytic maps equivalent?

Quaternionicity could be defined by induced gamma matrices or in terms of octonion real-analytic maps. Are these two definitions equivalent and how could one test the equivalence?

1. The calculation technical problem is that space-time surfaces are not defined in terms of imbedding map involving some coordinate choice but in terms of four vanishing conditions for the imaginary part of the octonion real-analytic function expressible as biquaternion valued functions.

2. Integrability to 4-D surface is achieved if there exists a 4-D closed Lie algebra defined by vector fields identifiable as tangent vector fields. This Lie algebra can be generalized to a local 4-D Lie algebra. One cannot however represent octonionic units in terms of 8-D vector fields since the commutators of the latter do not form an associative algebra. Also the representation of 7 octonionic imaginary units as 8-D vector fields is impossible since the algebra in question is non-assciative Malcev algebra [4] which can be seen as a Lie algebra over non-associative number field (one speaks of 7-dimensional cross product [8]). One must use instead of vector fields either octonionic units as such or octonionic gamma "matrices" to represent tangent vectors. The use of octonionic units as such would mean the introduction of the notion of octonionic tangent space structure. That the subalgebra generated by any two octonionic units is associative brings strongly in mind effective 2-dimensionality.

3. The tangent vector fields of space-time surface in the representation using octonionic units can be identified in the following manner. Map can be defined using 8-D octonionic coordinates defined by standard M^4 coordinates or possibly Hamilton-Jacobi coordinates and CP_2 complex coordinates for which U(2) is represented linearly. Gamma "matrices" for H using octonionic representation are known in these coordinates. One can introduce the 8 components of the image of a given point under the octonion real-analytic map as new imbedding space coordinates. One can calculate the covariant gamma matrices of H in these coordinates.

   What should check whether the octonionic gamma matrices associated with the four non-vanishing coordinates define quaternionic (and thus associative) algebra in the octonionic basis for the gamma matrices. Also the interpretation as a associative subspace of local Malcev algebra elements is possible and one should check whether if the algebra reduces to a quaternionic Lie-algebra. Local $SO(2) \times U(1)$ algebra should emerge in this manner.

4. Can one identify quaternionic imaginary units with vector fields generating SO(3) Lie algebra or its local variant? The Lie algebra of rotation generators defines algebra equivalent with that based on commutars of quaternionic units. Could the slicing of space-time sheet by time axis define local $SO(3)$ algebra? Light-like momentum direction and momentum direction and its dual define as their sum space-like vector field and together with vector fields defining transversal momentum directions they might generate a local $SO(3)$ algebra.

3.5.7 Questions related to quaternion real-analyticity

There are many poorly understood issues and and the following questions represent only some of very many such questions picked up rather randomly.

1. The above considerations are restricted to Minkowskian regions of space-time sheets. What happens in the Euclidian regions? Does the existence of light-like Beltrami field and its dual generalize to the existence of complex vector field and its dual?

2. It would be nice to find a justification for the notion of CD from basic principles. The condition $q\overline{q} = 0$ implies $q = 0$ for quaternions. For hyper-quaternionic subspace of complexified quaternions obtained by Wick rotation it implies $q\overline{q} = 0$ corresponds the entire light-cone boundary. If n-point functions can be identified identified as products of quaternion valued n-point functions and their quaternionic conjugates, the outcome could be proportional to $1/q\overline{q}$ having poles at light-cone boundaries or CD boundaries rather than at single point as in Euclidian realm.

3. This correspondence of points and light-cone boundaries would effectively identify the points at future and past light-like boundaries of CD along light rays. Could one think that only the 2-sphere at which the upper and lower light-like boundaries of CD meet remains after this identification. The structure would be homologically very much like CP_2 which is obtained by compactifying E^4 by adding a 2-sphere at infinity. Could this $CD - CP_2$ correspondence have some deep physical meaning? Do the boundaries of CD somehow correspond to zeros and/or poles of quaternionic analytic functions in the Minkowskian realm? Could the light-like orbits of partonic 2-surfaces at which the signature of the induced metric changes correspond to similar counterparts of zeros or poles when the quaternion analytic variables is obtained as quaternion real analytic function of H coordinates regarded as bi-quaternions?

4. Could braids correspond to zeros and poles of an octonion real-analytic function? Consider the partonic 2-surfaces at which the signature of the induced metric changes. The intersections of these surfaces with string world sheets at the ends of CDs. contain only complex and thus commutative points meaning that the imaginary part of bi-complex number representing quaternionic value of octonion real-analytic function vanishes. Braid ends would thus correspond to the origins of local complex coordinate patches. Finite measurement resolution would be forced by commutativity condition and correlate directly with the complexity of the partonic 2-surface measured by the minimal number of coordinate patches. Its realization would be as an upper bound on the number of braid strands. A natural expectation would be that only the values of n-point functions at these points contribute to scattering amplitudes. Number theoretic braids would be realized but in a manner different from the original guess.

3.5.8 How complex analysis could generalize?

One can make several questions related to the possible generalization of complex analysis to the quaternionic and octonionic situation.

1. Does the notion of analyticity in the sense that derivatives df/dq and df/do make sense hold true? The answer is "No": non-commutativity destroys all hopes about this kind of generalization. Octonion and quaternion real-analyticity has however a well-defined meaning.

2. Could the generalization of residue calculus by keeping interaction contours as 1-D curves make sense? Since residue formulas is the outcome of the fact that any analytic function g can be written as $g = df/dz$ locally, the answer is "No".

3. Could one generalize of the residue calculus by replacing 1-dimensional curves with 4-D surfaces -possibly quaternionic 4-surfaces? Could one reduce the 4-D integral of quaternion analytic function to a double residue integral? This would be the case if the quaternion real-analytic function of $q = c_1 + Ic_2$ could be regarded as an analytic function of complex arguments c_1 and c_2. This is not the case. The product of two octonions decomposed to two quaternions as $o_i = q_{i1} + Iq_{i2}$, $i = a, b$ reads as

$$o_a o_b = q_{a1}q_{b1} - \overline{q}_{a2}q_{b2} + I(\overline{q}_{a1}q_{b2} - q_{a2}q_{b1}) \ . \tag{3.6}$$

The conjugations result from the anticommutativity of imaginary parts and I. This formula gives similar formula for quaternions by restriction. As a special cas $o_a = o_b = q_1 + Iq_2$ one has

$$o^2 = q_1^2 - \overline{q}_2 q_2 + I(\overline{q}_1 q_2 - q_2 q_1)$$

From this it is clear that the real part of an octonion real-analytic function cannot be regarded as quaternion-analytic function unless one assumes that the imaginary part q_2 vanishes. By similar argument real part of quaternion real-analytic function $q = c_1 + Ic_2$ fails to be analytic unless one restricts the consideration to a surface at which one has $c_2 = 0$. These negative results are obviously consistent with the effective 2-dimensionality.

4. One must however notice that physicists use often what might be called analytization trick [1] working if the non-analytic function $f(x, y) = f(z, \overline{z})$ is differentiable. The trick is to interpret z and $\overline{z}$ as independent variables. In the recent case this is rather natural. Wick rotation could be used to transform the integral over the space-time sheet to integral in quaternionic domain. For 4-dimensional integrals of quaternion real-analytic function with integration measure proportional to $dc_1 d\overline{c}_1 dc_2 d\overline{c}_2$ one could formally define the integral using multiple residue integration with four complex variables. The constraint is that the poles associated with c_i and $\overline{c}_i$ are conjugates of each other. Quaternion real-analyticity should guarantee this. This would of course be a *definition* of four-dimensional integral and might work for the 4-D generalization of conformal field theory.

Mandelbrot and Julia sets are fascinating fractals and already now more or less a standard piece of complex analysis. The fact that the iteration of octonion real-analytic map produces a sequence of space-time surfaces and partonic 2-surfaces encourages to ask whether these notions -and more generally, the dynamics based on iteration of analytic functions - might have a higher-dimensional generalization in the proposed framework.

1. The canonical Mandelbrot set corresponds to the set of the complex parameters c in $f(z) = z^2 + c$ for which iterates of $z = 0$ remain finite. In octonionic and quaternionic real-analytic case c would be real so that one would obtain only the intersection of the Mandelbrot set with real axes and the outcome would be rather uninteresting. This is true quite generally.

2. Julia set corresponds to the boundary of the Fatou set in which the dynamics defined by the iteration of $f(z)$ by definition behaves in a regular manner. In Julia set the behavior is chaotic. Julia set can be defined as a set of complex plane resulting by taking inverse images of a generic point belonging to the Julia set. For polynomials Julia set is the boundary of the region in which iterates remain finite. In Julia set the dynamics defined by the iteration is chaotic.

   Julia set could be interesting also in the recent case since it could make sense for real analytic functions of both quaternions and octonions, and one might hope that the dynamics determined by the iterations of octonion real-analytic function could have a physical meaning as a space-time correlate for quantal self-organization by quantum jump in TGD framework. Single step in iteration would be indeed a very natural space-time correlate for quantum jump. The restriction of octonion analytic functions to string world sheets should produce the counterparts of the ordinary Julia sets since these surfaces are mapped to themselves under iteration and octonion real-analytic functions reduces to ordinary complex real-analytic functions at them. Therefore one might obtain the counterparts of Julia sets in 4-D sense as extensions of ordinary Julia sets. These extensions would be 3-D sets obtained as piles of ordinary Julia sets labelled by partonic 2-surfaces.

4 In what sense TGD could be an integrable theory?

During years evidence supporting the idea that TGD could be an integrable theory in some sense has accumulated. The challenge is to show that various ideas about what integrability means form pieces of a bigger coherent picture. Of course, some of the ideas are doomed to be only partially correct or simply wrong. Since it is not possible to know beforehand what ideas are wrong and what are right the situation is very much like in experimental physics and it is easy to claim (and has been and will be claimed) that all this argumentation is useless speculation. This is the price that must be paid from real thinking.

Integrable theories allow to solve nonlinear classical dynamics in terms of scattering data for a linear system. In TGD framework this translates to quantum classical correspondence: the solutions of modified Dirac equation define the scattering data. This data should define a real analytic function whose octonionic extension defines the space-time surface as a surface for which its imaginary part in the representation as bi-quaternion vanishes. There are excellent hopes about this thanks to the reduction of the modified Dirac equation to geometric optics.

In the following I will first discuss briefly what integrability means in (quantum) field theories, list some bits of evidence for integrability in TGD framework, discuss once again the question whether the different pieces of evidence are consistent with other and what one really means with various notions. An an outcome I represent what I regard as a more coherent view about integrability of TGD. The notion of octonion analyticity developed in the previous section is essential for the for what follows.

4.1 What integrable theories are?

The following is an attempt to get some bird's eye of view about the landscape of integrable theories.

4.1.1 Examples of integrable theories

Integrable theories are typically non-linear 1+1-dimensional (quantum) field theories. Solitons and various other particle like structures are the characteristic phenomenon in these theories. Scattering matrix is trivial in the sense that the particles go through each other in the scattering and suffer only a phase change. In particular, momenta are conserved. Korteveg-de Vries equation [1]was motivated by the attempt to explain the experimentally discovered shallow water wave preserving its shape and moving with a constant velocity. Sine-Gordon equation [4] describes geometrically constant curvature surfaces and defines a Lorentz invariant non-linear field theory in 1+1-dimensional space-time, which can be applied to Josephson junctions (in TGD inspired quantum biology it is encountered in the model of nerve pulse [9]). Non-linear Schrödinger equation [3] having applications to optics and water waves represents a further example. All these equations have various variants.

From TGD point of view conformal field theories represent an especially interesting example of integrable theories. (Super-)conformal invariance is the basic underlying symmetry and by its infinite-dimensional character implies infinite number of conserved quantities. The construction of the theory reduces to the construction of the representations of (super-)conformal algebra. One can solve 2-point functions exactly and characterize them in terms of (possibly anomalous) scaling dimensions of conformal fields involved and the coefficients appearing in 3-point functions can be solved in terms of fusion rules leading to an associative algebra for conformal fields. The basic applications are to 2-dimensional critical thermodynamical systems whose scaling invariance generalizes to conformal invariance. String models represent second application in which a collection of super-conformal field theories associated with various genera of 2-surface is needed to describe loop corrections to the scattering amplitudes. Also moduli spaces of conformal equivalence classes become important.

Topological quantum field theories are also examples of integrable theories. Because of its independence on the metric Chern-Simons action is in 3-D case the unique action defining a topological quantum field theory. The calculations of knot invariants (for TGD approach see [7]), topological invariants of 3-manifolds and 4-manifolds, and topological quantum computation (for a model of DNA as topological quantum computer see [5]) represent applications of this approach. TGD as almost topological QFT means that the Kähler action for preferred extremals reduces to a surface term by the vanishing of Coulomb term in action and by the weak form of electric-magnetic duality reduces to Chern-Simons action. Both Euclidian and Minkowskian regions give this kind of contribution.

$\mathcal{N} = 4$ SYM is the a four-dimensional and very nearly realistic candidate for an integral quantum field theory. The observation that twistor amplitudes allow also a dual of the 4-D conformal symmetry motivates the extension of this symmetry to its infinite-dimensional Yangian variant [9]. Also the enormous progress in the construction of scattering amplitudes suggests integrability. In TGD framework Yangian symmetry would emerge naturally by extending the symplectic variant of Kac-Moody algebra from light-cone boundary to the interior of causal diamond and the Kac-Moody algebra from light-like 3-surface representing wormhole throats at which the signature of the induced metric changes to the space-time interior [13].

4.1.2 About mathematical methods

The mathematical methods used in integrable theories are rather refined and have contributed to the development of the modern mathematical physics. Mention only quantum groups, conformal algebras, and Yangian algebras.

The basic element of integrability is the possibility to transform the non-linear classical problem for which the interaction is characterized by a potential function or its analog to a linear scattering problem depending on time. For instance, for the ordinary Schrödinger function one can solve potential once single solution of the equation is known. This does not work in practice. One can however gather information about the asymptotic states in scattering to deduce the potential. One cannot do without information about bound state energies too.

In TGD framework asymptotic states correspond to partonic 2-surfaces at the two light-like boundaries of CD (more precisely: the largest CD involved and defining the IR resolution for momenta). From the scattering data coding information about scattering for various values of energy of the incoming particle one deduced the potential function or its analog.

1. The basic tool is inverse scattering transform known as Gelfand-Marchenko-Levitan (GML) transform described in simple terms in [5].

   (a) In 1+1 dimensional case the S-matrix characterizing scattering is very simple since the only thing that can take place in scattering is reflection or transmission. Therefore the S-matrix elements describe either of these processes and by unitarity the sum of corresponding probabilities equals to 1. The particle can arrive to the potential either from left or right and is characterized by a momentum. The transmission coefficient can have a pole meaning complex (imaginary in the simplest case) wave vector serving as a signal for the formation of a bound state or resonance. The scattering data are represented by the reflection and transmission coefficients as function of time.

   (b) One can deduce an integral equation for a propagator like function $K(t, x)$ describing how delta pulse moving with light velocity is scattered from the potential and is expressible in terms of time integral over scattering data with contributions from both scattering states and bound states. The derivation of GML transform [5] uses time reversal and time translational invariance and causality defined in terms of light velocity. After some tricks one obtains the integral equation as well as an expression for the time independent potential as $V(x) = K(x, x)$. The argument can be generalized to more complex problems to deduce the GML transform.

2. The so called Lax pair is one manner to describe integrable systems [2]. Lax pair consists of two operators L and M. One studies what might be identified as "energy" eigenstates satisfying $L(x, t)\Psi = \lambda\Psi$. λ does not depend on time and one can say that the dynamics is associated with x coordinate whereas as t is time coordinate parametrizing different variants of eigenvalue problem with the same spectrum for L. The operator $M(t)$ does not depend on x at all and the independence of λ on time implies the condition

   $$\partial_t L = [L, M] \ .$$

   This equation is analogous to a quantum mechanical evolution equation for an operator induced by time dependent "Hamiltonian" M and gives the non-linear classical evolution equation when the commutator on the right hand side is a multiplicative operator (so that it does not involve differential operators acting on the coordinate x). Non-linear classical dynamics for the time dependent potential emerges as an integrability condition.

   One could say that $M(t)$ introduces the time evolution of $L(t, x)$ as an automorphism which depends on time and therefore does not affect the spectrum. One has $L(t, x) = U(t)L(0, x)U^{-1}(t)$ with $dU(t)/dt = M(t)U(t)$. The time evolution of the analog of the quantum state is given by a similar equation.

3. A more refined view about Lax pair is based on the observation that the above equation can be generalized so that M depends also on x. The generalization of the basic equation for $M(x, t)$ reads as

$$\partial_t L - \partial_x M - [L, M] = 0 \ .$$

The condition has interpretation as a vanishing of the curvature of a gauge potential having components $A_x = L, A_t = M$. This generalization allows a beautiful geometric formulation of the integrability conditions and extends the applicability of the inverse scattering transform. The monodromy of the flat connection becomes important in this approach. Flat connections in moduli spaces are indeed important in topological quantum field theories and in conformal field theories.

4. There is also a connection with the so called Riemann-Hilbert problem [7]. The monodromies of the flat connection define monodromy group and Riemann-Hilbert problem concerns the existence of linear differential equations having a given monodromy group. Monodromy group emerges in the analytic continuation of an analytic function and the action of the element of the monodromy group tells what happens for the resulting many-valued analytic function as one turns around a singularity once ('mono-'). The linear equations obviously relate to the linear scattering problem. The flat connection (M, L) in turn defines the monodromy group. What is needed is that the functions involved are analytic functions of (t, x) replaced with a complex or hyper-complex variable. Again Wick rotation is involved. Similar approach generalizes also to higher dimensional moduli spaces with complex structures.

   In TGD framework the effective 2-dimensionality raises the hope that this kind of mathematical apparatus could be used. An interesting possibility is that finite measurement resolution could be realized in terms of a gauge group or Kac-Moody type group represented by trivial gauge potential defining a monodromy group for n-point functions. Monodromy invariance would hold for the full n-point functions constructed in terms of analytic n-point functions and their conjugates. The ends of braid strands are natural candidates for the singularities around which monodromies are defined.

4.2 Why TGD could be integrable theory in some sense?

There are many indications that TGD could be an integrable theory in some sense. The challenge is to see which ideas are consistent with each other and to build a coherent picture where everything finds its own place.

1. 2-dimensionality or at least effective 2-dimensionality seems to be a prerequisite for integrability. Effective 2-dimensionality is suggested by the strong form of General Coordinate Invariance implying also holography and generalized conformal invariance predicting infinite number of conservation laws. The dual roles of partonic 2-surfaces and string world sheets supports a four-dimensional generalization of conformal invariance. Twistor considerations [12] indeed suggest that Yangian invariance and Kac-Moody invariances combine to a 4-D analog of conformal invariance induced by 2-dimensional one by algebraic continuation.

2. Octonionic representation of imbedding space Clifford algebra and the identification of the space-time surfaces as quaternionic space-time surfaces would define a number theoretically natural generalization of conformal invariance. The reason for using gamma matrix representation is that vector field representation for octonionic units does not exist. The problem concerns the precise meaning of the octonionic representation of gamma matrices.

   Space-time surfaces could be quaternionic also in the sense that conformal invariance is analytically continued from string curve to 8-D space by octonion real-analyticity. The question is whether the Clifford algebra based notion of tangent space quaternionicity is equivalent with octonionic real-analyticity based notion of quaternionicity.

   The notions of co-associativity and co-quaternionicity make also sense and one must consider seriously the possibility that associativity-co-associativity dichotomy corresponds to Minkowskian-Euclidian dichotomy.

3. Field equations define hydrodynamic Beltrami flows satisfying integrability conditions of form $J \wedge dJ = 0$.

(a) One can assign local momentum and polarization directions to the preferred extremals and this gives a decomposition of Minkowskian space-time regions to massless quanta analogous to the 1+1-dimensional decomposition to solitons. The linear superposition of modes with 4-momenta with different directions possible for free Maxwell action does not look plausible for the preferred extremals of Kähler action. This rather quantal and solitonic character is in accordance with the quantum classical correspondence giving very concrete connection between quantal and classical particle pictures. For 4-D volume action one does not obtain this kind of decomposition. In 2-D case volume action gives superposition of solutions with different polarization directions so that the situation is nearer to that for free Maxwell action and is not like soliton decomposition.

(b) Beltrami property in strong sense allows to identify 4 preferred coordinates for the space-time surface in terms of corresponding Beltrami flows. This is possible also in Euclidian regions using two complex coordinates instead of hyper-complex coordinate and complex coordinate. The assumption that isometry currents are parallel to the same light-like Beltrami flow implies hydrodynamic character of the field equations in the sense that one can say that each flow line is analogous to particle carrying some quantum numbers. This property is not true for all extremals (say cosmic strings).

(c) The tangent bundle theoretic view about integrability is that one can find a Lie algebra of vector fields in some manifold spanning the tangent space of a lower-dimensional manifolds and is expressed in terms of Frobenius theorem [2]). The gradients of scalar functions defining Beltrami flows appearing in the ansatz for preferred exremals would define these vector fields and the slicing. Partonic 2-surfaces would correpond to two complex conjugate vector fields (local polarization direction) and string world sheets to light-like vector field and its dual (light-like momentum directions). This slicing generalizes to the Euclidian regions.

4. Infinite number of conservation laws is the signature of integrability. Classical field equations follow from the condition that the vector field defined by modified gamma matrices has vanishing divergence and can be identified an integrability condition for the modified Dirac equation guaranteing also the conservation of super currents so that one obtains an infinite number of conserved charges.

5. Quantum criticality is a further signal of integrability. 2-D conformal field theories describe critical systems so that the natural guess is that quantum criticality in TGD framework relates to the generalization of conformal invariance and to integrability. Quantum criticality implies that Kähler coupling strength is analogous to critical temperature. This condition does affects classical field equations only via boundary conditions expressed as weak form of electric magnetic duality at the wormhole throats at which the signature of the metric changes.

   For finite-dimensional systems the vanishing of the determinant of the matrix defined by the second derivatives of potential is similar signature and applies in catastrophe theory. Therefore the existence of vanishing second variations of Kähler action should characterize criticality and define a property of preferred extremals. The vanishing of second variations indeed leads to an infinite number of conserved currents [6, 2].

4.3 Questions

There are several questions which are not completely settled yet. Even the question what preferred extremals are is still partially open. In the following I try to de-learn what I have possibly learned during these years and start from scratch to see which assumptions might be un-necessarily strong or even wrong.

4.4 Could TGD be an integrable theory?

Consider first the abstraction of integrability in TGD framework. Quantum classical correspondence could be seen as a correspondence between linear quantum dynamics and non-linear classical dynamics. Integrability would realize this correspondence. In integrable models such as Sine-Gordon equation particle interactions are described by potential in 1+1 dimensions. This too primitive for the purposes of TGD. The vertices of generalized Feynman diagrams take care of this. At lines one has free particle dynamics so that the situation could be much simpler than in integrable models if one restricts the considerations to the lines or Minkowskian space-time regions surrounding them.

The non-linear dynamics for the space-time sheets representing incoming lines of generalized Feynman diagram should be obtainable from the linear dynamics for the induced spinor fields defined by modified Dirac operator. There are two options.

1. Strong form of the quantum classical correspondence states that each solution for the linear dynamics of spinor fields corresponds to space-time sheet. This is analogous to solving the potential function in terms of a single solution of Schrödinger equation. Coupling of space-time geometry to quantum numbers via measurement interaction term is a proposal for realizing this option. It is howwever the quantum numbers of positive/negative energy parts of zero energy state which would be visible in the classical dynamics rather than those of induced spinor field modes.

2. Only overall dynamics characterized by scattering data- the counterpart of S-matrix for the modified Dirac operator- is mapped to the geometry of the space-time sheet. This is much more abstract realization of quantum classical correspondence.

3. Can these two approaches be equivalent? This might be the case since quantum numbers of the state are not those of the modes of induced spinor fields.

What the scattering data could be for the induced spinor field satisfying modified Dirac equation?

1. If the solution of field equation has hydrodynamic character, the solutions of the modified Dirac equation can be localized to light-like Beltrami flow lines of hydrodynamic flow. These correspond to basic solutions and the general solution is a superposition of these. There is no dispersion and the dynamics is that of geometric optics at the basic level. This means geometric optics like character of the spinor dynamics.

   Solutions of the modified Dirac equation are completely analogous to the pulse solutions defining the fundamental solution for the wave equation in the argument leading from wave equation with external time independent potential to Marchenko-Gelfand-Levitan equation allowing to identify potential in terms of scattering data. There is however no potential present now since the interactions are described by the vertices of Feynman diagram where the particle lines meet. Note that particle like regions are Euclidian and that this picture applies only to the Minkowskian exteriors of particles.

2. Partonic 2-surfaces at the ends of the line of generalized Feynman diagram are connected by flow lines. Partonic 2-surfaces at which the signature of the induced metric changes are in a special position. Only the imaginary part of the bi-quaternionic value of the octonion valued map is non-vanishing at these surfaces which can be said to be co-complex 2-surfaces. By geometric optics behavior the scattering data correspond to a diffeomorphism mapping initial partonic 2-surface to the final one in some preferred complex coordinates common to both ends of the line.

3. What could be these preferred coordinates? Complex coordinates for S^2 at light-cone boundary define natural complex coordinates for the partonic 2-surface. With these coordinates the diffeomorphism defining scattering data is diffeomorphism of S^2. Suppose that this map is real analytic so that maps "real axis" of S^2 to itself. This map would be same as the map defining the octonionic real analyticity as algebraic extension of the complex real analytic map. By octonionic analyticity one can make large number of alternative choices for the coordinates of partonic 2-surface.

4. There can be non-uniqueness due to the possibility of $G_2/SU(3)$ valued map characterizing the local octonionic units. The proposal is that the choice of octonionic imaginary units can depend on the point of string like orbit: this would give string model in $G_2/SU(3)$. Conformal invariance for this string model would imply analyticity and helps considerably but would not probably fix the situation completely since the element of the coset space would constant at the partonic 2-surfaces at the ends of CD. One can of course ask whether the $G_2/SU(3)$ element could be constant for each propagator line and would change only at the 2-D vertices?

This would be the inverse scattering problem formulated in the spirit of TGD. There could be also dependence of space-time surface on quantum numbers of quantum states but not on indididual solution for the induced spinor field since the scattering data of this solution would be purely geometric.

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