Quantum Arithmetics and the Relationship between Real and p-Adic Physics

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Abstract

This chapter suggests answers to the basic questions of the p-adicization program, which are following.

1. Is there a duality between real and p-adic physics? What is its precice mathematic formulation? In particular, what is the concrete map p-adic physics in long scales (in real sense) to real physics in short scales? Can one find a rigorous mathematical formulationof canonical identification induced by the map $p \rightarrow 1/p$ in pinary expansion of p-adic number such that it is both continuous and respects symmetries.
2. What is the origin of the p-adic length scale hypothesis suggesting that primes near power of two are physically preferred? Why Mersenne primes are especially important?

The answer to these questions proposed in this chapter relies on the following ideas inspired by the model of Shnoll effect. The first piece of the puzzle is the notion of quantum arithmetics formulated in non-rigorous manner already in the model of Shnoll effect.

1. Quantum arithmetics is induced by the map of primes to quantum primes by the standard formula. Quantum integer is obtained by mapping the primes in the prime decomposition of integer to quantum primes. Quantum sum is induced by the ordinary sum by requiring that also sum commutes with the quantization.
2. The construction is especially interesting if the integer defining the quantum phase is prime. One can introduce the notion of quantum rational defined as series in powers of the preferred prime defining quantum phase. The coefficients of the series are quantum rationals for which neither numerator and denominator is divisible by the preferred prime.
3. p-Adic–real duality can be identified as the analog of canonical identification induced by the map $p \rightarrow 1/p$ in the pinary expansion of quantum rational. This maps maps p-adic and real physics to each other and real long distances to short ones and vice versa. This map is especially interesting as a map defining cognitive representations.

Quantum arithmetics inspires the notion of quantum matrix group as counterpart of quantum group for which matrix elements are ordinary numbers. Quantum classical correspondence and the notion of finite measurement resolution realized at classical level in terms of discretization suggest that these two views about quantum groups are closely related. The preferred prime p defining the quantum matrix group is identified as p-adic prime and canonical identification $p \rightarrow 1/p$ is group homomorphism so that symmetries are respected.

1. The quantum counterparts of special linear groups $SL(n, F)$ exists always. For the covering group $SL(2, C)$of $SO(3, 1)$ this is the case so that 4-dimensional Minkowski space is in a very special position. For orthogonal, unitary, and orthogonal groups the quantum counterpart exists only if quantum arithmetics is characterized by a prime rather than general integer and when the number of powers of p for the generating elements of the quantum matrix group satisfies an upper bound characterizing the matrix group.
2. For the quantum counterparts of $SO(3)$ ($SU(2)$/ $SU(3)$) the orthogonality conditions state that at least some multiples of the prime characterizing quantum arithmetics is sum of three (four/six) squares. For $SO(3)$ this condition is strongest and satisfied for all integers, which are not of form $n = 2^{2r}(8k + 7)$). The number $r_3(n)$ of representations as sum of squares is known and $r_3(n)$ is invariant under the scalings $n \rightarrow 2^{2r}n$. This means scaling by 2 for the integers appearing in the square sum representation.
3. $r_3(n)$ is proportional to the so called class number function $h(-n)$ telling how many non-equivalent decompositions algebraic integers have in the quadratic algebraic extension generated by $\sqrt{-n}$.

The findings about quantum $SO(3)$ suggest a possible explanation for p-adic length scale hypothesis and preferred p-adic primes.

1. The basic idea is that the quantum matrix group which is discrete is very large for preferred p-adic primes. If cognitive representations correspond to the representations of quantum matrix group, the representational capacity of cognitive representations is high and this kind of primes are survivors in the algebraic evolution leading to algebraic extensions with increasing dimension.
2. The preferred primes correspond to a large value of $r_3(n)$. It is enough that some of their multiples do so (the 2^{2r} multiples of these do so automatically). Indeed, for Mersenne primes and integers one has $r_3(n) = 0$, which was in conflict with the original expectations. For integers $n = 2M_m$ however $r_3(n)$ is a local maximum at least for the small integers studied numerically.

3. The requirement that the notion of quantum integer applies also to algebraic integers in quadratic extensions of rationals requires that the preferred primes (p-adic primes) satisfy $p = 8k + 7$. Quite generally, for the integers $n = 2^{2r}(8k + 7)$ not representable as sum of three integers the decomposition of ordinary integers to algebraic primes in the quadratic extensions defined by $\sqrt{-n}$ is unique. Therefore also the corresponding quantum algebraic integers are unique for preferred ordinary prime if it is prime also in the algebraic extension. If this were not the case two different decompositions of one and same integer would be mapped to different quantum integers. Therefore the generalization of quantum arithmetics defined by any preferred ordinary prime, which does not split to a product of algebraic primes, is well-defined for $p = 2^{2r}(8k + 7)$.
4. This argument was for quadratic extensions but also more complex extensions defined by higher polynomials exist. The allowed extensions should allow unique decomposition of integers to algebraic primes. The prime defining the quantum arithmetics should not decompose to algebraic primes. If the algebraic evolution leadis to algebraic extensions of increasing dimension it gradually selects preferred primes as survivors.

1 Introduction

The construction of quantum counterparts for various mathematical structures of theoretical physics have been a fashion for decades. Quantum counterparts for groups, Lie algebras, coset spaces, etc... have been proposed often on purely formal grounds. In TGD framework quantum group like structures emerges via the hyper-finite factors of type II_1 (HFFs) about which WCW spinors represent a canonical example [11]. The inclusions of HFFs provide a very attractive manner to realize mathematically the notion of finite measurement resolution.

In the following a proposal for what might be called quantum integers and quantum matrix groups is discussed. Quantum integers n_q differ from their standard variants in that the map $n \rightarrow n_q$ respects prime decomposition so that one obtains quantum number theory. Also quantum rationals belonging to algebraic extension of rationals can be defined as well as their algebraic extensions. Quantum arithmetics differs from the usual one in that quantum sum is defined in such a manner that the map $n \rightarrow n_q$ commutes also with sum besides the product: $m_q +_q n_q = (m + n)_q$. Quantum matrix groups differ from their standard counterparts in that the matrix elements are not non-commutative. The matrix multiplication involving summation over products is however replaced with quantum summation.

The proposal is that these new mathematical structures allow a more understanding of the relationship between real and p-adic physics for various values of p-adic prime p, to be called l in the sequel because of its preferred physical nature resembling that of l-adic prime in l-adic cohomology. The correspondence with the ordinary quantum groups [15] is also considered and suggested to correspond to a discretization following as a correlate of finite measurement resolution.

1.1 What could be the deeper mathematics behind dualities?

Dualities certainly represent one of the great ideas of theoretical physics of the last century. The mother of all dualities might be electric-magnetic duality due to Montonen and Olive [2]. Later a proliferation, one might say even inflation, of dualities has taken place. AdS/CFT correspondence [3] is one example relating to each other perturbative QFT working in short scales and string theory working in long scales.

Also in TGD framework several dualities suggests itself. All of them seem to relate to dictotomies such as weak–strong, perturbative–non-perturbative, point like particle–string. Also number theory seems to be involved in an essential manner.

1. If $M^8 - -M^4 \times CP_2$ duality is true it is possible to regard space-times as surfaces in either M^8 or $M^4 \times CP_2$ [10]. One manner to interpret the duality would as the analog of q-p duality in wave mechanics. Surfaces in M^8 would be analogous to momentum space representation of the physical stats: space-time surfaces in M^8 would represent in some sense the points for the tangent space of the "world of classical worlds" (WCW) just like tangent for a curve gives the first approximation for the curve near a given point.

The argument supporting $M^8 - -M^4 \times CP_2$ duality involves the basic facts about classical number fields - in particular octonions and their complexification - and one can understand $M^4 \times CP_2$ in terms of number theory. The analog of the color group in M^8 picture would be the isometry group $SO(4)$ of E^4 which happens to be the symmetry group of the old fashioned hadron physics. Does this mean that $M^4 \times CP_2$ corresponds to short length scales and perturbative QCD whereas M^8 would correspond to long length scales and non-perturbative approach?

2. Second duality would relate partonic 2-surfaces and string world sheets playing a key role in the recent view about preferred extremals of Kähler action [3]. Partonic 2-surfaces are magnetic monopoles and TGD counterparts of elementary particles, which in QFT approach are regarded as point like objects. The description in terms of partonic 2-surfaces forgetting that they are parts of bigger magnetically neutral structures would correspond to perturbative QFT. The description in terms of string like objects with vanishing magnetic charge is needed in longer length scales. Electroweak symmetry breaking and color confinement would be the natural applications. The essential point is that stringy description corresponds to long length scales (strong coupling) and partonic description to short length scales (weak coupling).

   Number theory seems to be involved also now: string world sheets could be seen as hyper-complex 2-surfaces of space-time surface with hyper-quaternionic tangent space structure and partonic 2-surfaces as co-hyper complex 2-surfaces (normal space would be hyper-complex).

3. Space-time surface itself would decompose to hyper-quaternionic and co-hyperquaternionic regions and a duality also at this level is suggestive [1], [2]. The most natural candidates for dual space-time regions are regions with Minkowskian and Euclidian signatures of the induced metric with latter representing the generalized Feynman graphs. Minkowskian regions would correspond to non-pertubative long length scale description and Euclidian regions to perturbative short length scale description. This duality should relate closely to quantum measurement theory and realize the assumption that the outcomes of quantum measurements are always macroscopic long length scale effects. Again number theory is in a key role.

Real and p-adic physics and their unification to a coherent whole represent the basic pieces of physics as generalized number theory program.

1. p-Adic physics can mean two different things. p-Adic physics could mean a discretization of real physics relying on effective p-adic topology. p-Adic physics could also mean genuine p-adic physics at p-adic space-time sheets. Real continuity and smoothness is an enormous constraint on short distance physics. p-Adic continuity and smoothness pose similar constraints in short scales an therefore on real physics in long length scales if one accepts that real and space-time surfaces (partonic 2-surfaces for minimal option) intersect along rational points and possible common algebraics in preferred coordinates. p-Adic fractality implying short range chaos and long range correlations is the outcome. Therefore p-adic physics could allow to avoid the landscape problem of M-theory due to the fact that the IR limit is unpredictable although UV behavior is highly unique.

2. The recent argument [3] suggesting that the areas for partonic 2-surfaces and string world sheets could characterize Kähler action leads to the proposal that the large N_c expansion [1] in terms of the number of colors defining non-perturbative stringy approach to strong coupling phase of gauge theories could have interpretation in terms of the expansion in powers of $1/\sqrt{p}$, p the p-adic prime. This expansion would converge extremely rapidly since N_c would be of the order of the ratio of the secondary and primary p-adic length scales and therefore of the order of $\sqrt{p}$: for electron one has $p = M_{127} = 2^{127} - 1$.

3. Could there exist a duality between genuinely p-adic physics and real physics? Could the mathematics used in p-adic mass calculations- in particular canonical identification $\sum_n x_n p^n \rightarrow \sum x_n p^{-n}$ - be extended to apply to quantum TGD itself and allow to understand the non-perturbative long length scale effects in terms of short distance physics dictated by continuity and smoothness but in different number field? Could a proper generalization of the canonical identification map allow to realize concretely the real–p-adic duality?

A generalization of the canonical identification [7] and its variants is certainly needed in order to solve the problems caused by the fact that it does not respect symmetries. That the generalization might exist was suggested already by the model for Shnoll effect [1], which led to a proposal that this effect can be understand in terms of a deformation of probability distribution $f(n)$ (n non-negative integer) for random fluctuations. The deformation would replace the rational parameters characterizing the distribution with new ones obtained by mapping the parameters to new ones by using the analog of canonical identification respecting symmetries. This deformation would involve two parameters: quantum phase $q = exp(i2\pi/m)$ and preferred prime l, which need not be independent however: $m = l$, is a highly suggestive restriction.

The idea of the model of Shnoll effect was to modify the map $n \rightarrow n_q$ in such a manner that it is consistent with the prime decomposition of ordinary integers. One could even consider the notion of quantum arithmetics requiring that the map commutes with sum. This in turn suggest the generalization of the matrix groups to what might be called quantum matrix groups. The matrix elements would not be however non-commutative but obey quantum arithmetics. These quantum groups w ould be labelled by prime l and the original form of the canonical identification $l \rightarrow 1/l$ defines a group homomorphism. This form of canonical identification respecting symmetries could be applied to the linear representations of these groups. This map would be both continuous and respect symmetries.

1.2 Correspondence along common rationals and canonical identification: two manners to relate real and p-adic physics

The relationship between real and p-adic physics deserves a separate discussion.

1. The first correspondence between reals and p-adics is based on the idea that rationals are common to all number fields implying that rational points are common to both real and p-adic worlds. This requires preferred coordinates. It also leads to a fusion of different number fields along rationals and common algebraics to a larger structure having a book like structure [9, 7].

   (a) Quite generally, preferred space-time coordinates would correspond to a subset of preferred imbedding space coordinates, and the isometries of the imbedding space give rise to this kind of coordinates which are however not completely unique. This would give rise to a moduli space corresponding to different symmetry related coordinates interpreted in terms of different choices of causal diamonds (CDs).

   (b) Cognitive representation in the rational (partly algebraic) intersection of real and p-adic worlds would necessarily select certain preferred coordinates and this would affects the physics in a delicate manner. The selection of quantization axis would be basic example of this symmetry breaking. Finite measurement resolution would in turn reduce continuous symmetries to discrete ones.

   (c) Typically real and p-adic variants of given partonic 2-surface would have discrete and possibly finite set of rational points plus possible common algebraic points. The intersection of real and p-adic worlds would consist of discrete points. At more abstract level rational functions with rational coefficients used to define partonic 2-surfaces would correspond to common 2-surfaces in the intersection of real and p-adic WCW:s. As a matter fact, the quantum arithmetics would make most points algebraic numbers.

   (d) The correspondence along common rationals respects symmetries but not continuity: the graph for the p-adic norm of rational point is totally discontinuous. Most non-algebraic reals and p-adics do not correspond to each other. In particular, transcendental at both sides belong to different worlds with some exceptions like e^p which exists p-adically.

2. There is however a totally different view about real–p-adic correspondence. The predictions of p-adic mass calculations are mapped to real numbers via the canonical identification applied to the p-adic value of mass squared [7, 6]. One can imagine several forms of canonical identification but this affects very little the predictions since the convergence in powers of p for the mass squared thermal expectation is extremely fast.

3. The two views are consistent if appropriately generalized canonical identification is interpreted as a concrete duality mapping short length scale physics and long length scale physics to each other. As a matter fact, I proposed for more that 15 years ago that canonical identification could be essential element of cognition mapping external world to p-adic cognitive representations realized in short length scales and vice versa. If so, then real–p-adic duality would be a cornerstone of cognition [8]. Common rational points would relate to the intentionality which is second aspect of the p-adic real corresponence: the transformation of real to p-adic surfaces in quantum jump would be the correlate for the transformation of intention to action. The realization of intention would correspond to the correspondence along rationals and common algebraics (the more common points real and p-adic surface have, the more faithful the realization of intentional action) and the generation of cognitive representations to the canonical identification.

There are however hard technical problems involved. Maybe canonical identification should be realized at the level of imbedding space at least - or even at space-time level. Canonical identification would be locally continuous in both directions. Note that for the points with finite pinary expansion (ordinary integers) the map is two-valued. Note also that rationals can be expanded in infinite powers series with respect to p and one can ask whether one should do this or map $q = m/n$ to $I(m)/I(n)$ (the representation of rational is unique if m and n have no common factors).

The basic problem is that canonical identification in its basic form does not respect symmetries: the action of the p-adic symmetry followed by a canonical identification to reals is not equal to the canonical identification map followed by the real symmetry.

1. One can imagine modifications of the canonical identification in attempts to solve this problem. One can map rationals by $m/n \rightarrow I(m)/I(n)$. One can also express m and n as power series of p^k as $x = \sum x_n p^{nk}$ and perform the map as $x \rightarrow \sum x_n p^{-nk}$. This allows to preserve symmetries in arbitrary good measurement resolution characterizing by the power p^{-k} on real side.

2. Could one circumvent this difficulty without approximations? This kind of approach should work at least when finite measurement resolution is used meaning the replacement of the space-time surface with a set of discrete points. Could the already mentioned quantum integers provide a generalization of the notion of symmetry itself in order to circumvent ugly constructions?

1.3 Brief summary of the general vision

The basic questions of the p-adicization program are following.

1. Is there a duality between real and p-adic physics? What is its precice mathematic formulation? In particular, what is the concrete map p-adic physics in long scales (in real sense) to real physics in short scales? Can one find a rigorous mathematical formulation of the canonical identification induced by the map $p \rightarrow 1/p$ in pinary expansion of p-adic number such that it is both continuous and respects symmetries.

2. What is the origin of the p-adic length scale hypothesis suggesting that primes near power of two are physically preferred? Why Mersenne primes are especially important?

The answer to these questions proposed in this chapter relies on the following ideas inspired by the model of Shnoll effect [1]. The first piece of the puzzle is the notion of quantum arithmetics formulated in non-rigorous manner already in the model of Shnoll effect.

1. Quantum arithmetics is induced by the map of primes to quantum primes by the standard formula. Quantum integer is obtained by mapping the primes in the prime decomposition of integer to quantum primes. Quantum sum is induced by the ordinary sum by requiring that also sum commutes with the quantization.

2. The construction is especially interesting if the integer defining the quantum phase q is prime. One can introduce the notion of quantum rational defined as series in powers of the preferred prime p defining quantum phase. The coefficients of the series are quantum rationals for which neither numerator and denominator is divisible by the preferred prime.

3. p-Adic- real duality can be identified as the analog of canonical identification induced by the map $p \to 1/p$ in the pinary expansion of quantum rational. This maps maps p-adic and real physics to each other and real long distances to short ones and vice versa.

Quantum arithmetics inspires the notion of quantum matrix group as counterpart of quantum group for which matrix elements are non-commuting numbers. Now they would be ordinary numbers. Quantum classical correspondence and the notion of finite measurement resolution realized at classical level in terms of discretization suggest that these two views about quantum groups are closely related. The preferred prime p defining the quantum matrix group is identified as p-adic prime and canonical identification $p \to 1/p$ is group homomorphism so that symmetries are respected.

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2. For the quantum counterparts of $SO(3)$ ($SU(2)$/ $SU(3)$) the orthogonality conditions state that at least some multiples of the prime characterizing quantum arithmetics is sum of three (four/six) squares. For $SO(3)$ this condition is strongest and satisfied for all integers, which are not of form $n = 2^{2r}(8k + 7)$). The number $r_3(n)$ of representations as sum of squares is known and $r_3(n)$ is invariant under the scalings $n \to 2^{2r}n$. This means scaling by 2 for the integers appearing in the square sum representation.

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The findings about quantum $SO(3)$ suggest a possible explanation for p-adic length scale hypothesis and preferred p-adic primes.

1. The basic idea is that the quantum matrix group which is discrete isin some sense very large for preferred p-adic primes. If cognitive representations correspond to the representations of quantum matrix group, the representational capacity of cognitive representations is high and this kind of primes are survivors in the algebraic evolution leading to algebraic extensions with increasing dimension.

2. There is no need that the preferred primes correspond to larger value of $r_3(n)$. It is enough that some of their multiples do so. Indeed, for Mersenne primes and also integers one has $r_3(n) = 0$, which is in conflict with the original naive expectations. For integers $n = 2M_m$ however $r_3(n)$ is a local maximum at least for the small integers studied numerically.

3. The requirement that the notion of quantum integer applies also to algebraic integers in quadratic extensions of rationals requires that the preferred primes (p-adic primes) satisfy $p = 8k + 7$. Quite generally, for the integers $n = 2^{2r}(8k + 7)$ not representable as sum of three integers the decomposition of ordinary integers to algebraic primes in the quadratic extensions defined by $\sqrt{-n}$ is unique. Therefore also the corresponding quantum algebraic integers are unique for preferred ordinary prime if it is prime also in the algebraic extension. If this were not the case two different decompositions of one and same integer would be mapped to different quantum integers. Therefore the generalization of quantum arithmetics defined by any preferred ordinary prime, which does not split to a product of algebraic primes, is well-defined for $p = 2^{2r}(8k + 7)$ when quadratic extensions are considerd. This select Mersenne primes as preferred ones.

4. This argument was for quadratic extensions but also more complex extensions defined by higher polynomials exist. For these higher dimensional algebraic extensions the number of ordinary primes allowing no decomposition to ordinary primes and implying unique decomposition in possibly existing algebraic extension defined by the prime gets smaller. Hence algebraic evolution leading to algebraic extensions of increasing dimension would gradually select preferred primes and integers.

2 Quantum arithmetics and the notion of commutative quantum group

In this section the notion of quantum arithmetics as a generalization of ordinary arithmetics preserving its structure but mapping preferred integer- most naturally prime- to zero is discussed. Also the notion of quantum matrix group differening from ordinary quantum groups in that matrix elements are commuting numbers is discussed. This group forms a discrete counterpart of ordinary quantum group and its existence suggested by quantum classical correspondence.

2.1 Quantum arithmetics

The basic idea is that quantum arithmetics is isomorphic to the ordinary arithmetics of integers.

1. The multiplicative structure of ordinary integers is respected in the map taking ordinary integers to quantum integers:

$$n = kl \rightarrow n_q = k_q l_q \ . \tag{2.1}$$

   This is guaranteed if the map is induced by the map of ordinary primes to quantum primes.

2. Also the sum of quantum integers is well-defined and induces sum of the quantum rationals. Therefore the sum $+_q$ of quantum integers should reflect the summation of ordinary integers:

$$n = k+l \rightarrow n_q = k_q +_q l_q \ . \tag{2.2}$$

The basic formula for quantum integers in the case of quantum groups is

$$n_q = \frac{q^n - q^{-n}}{q - q^{-1}} \ . \tag{2.3}$$

Here q is any complex number. The generalization respective the notion of primeness is obtained by mapping only the primes p to their quantum counterparts and defining quantum integers as products of the quantum primes involved in their prime factorization.

$$\begin{aligned} p_q &= \frac{q^p - q^{-p}}{q - q^{-1}} \\ n_q &= \prod_p p_q^{n_p} \ \text{for} \ \ n = \prod_p p^{n_p} \ . \end{aligned} \tag{2.4}$$

2.1.1 Quantum counterparts of real integers

The propoed definition is just the first guess. Let us consider now some aspects of this definition to see whether it must be modified somehow.

1. The $n = 0, 1, -1$ are fixed points of $n \rightarrow n_q$ so that one can say that all these numbers are common to quantum integers for all values of q.

2. An important special case corresponds to the roots of unity: $q = e^{i2\pi/m}$. In this case primes p_1, p_2 satisfying $p_1 - p_2 \ mod \ n = 0$ are mapped to same quantum integers. If one has

$$q = exp(\frac{\eta}{m}) exp(i2\pi/m) \tag{2.5}$$

   the map is 1-1 for a non-vanishing value of η and the limit $m \rightarrow \infty$ gives ordinary integers. It seems that one must include the factor making the modulus of q different from unity if one wants 1-1 correspondence between ordinary and quantum integers guaranteing a unique definition of quantum sum.

3. Second potential problem is that p_q is negative for $n/2 \leq p \mod n \leq n$. This would mean that quantum integers can be negative. In p-adic contex this is not a problem. In real context this could be a problem if one maps a probability distribution $f(n)$ to its quantum counterpart by $n \rightarrow n_q$ unless one makes special assumption about the distribution. If this is a real problem, one can try to avoid it in a straightforward manner by including a compensating sign factor which is -1 for $n/2 \leq p \mod n \leq n$ and +1 otherwise.

   The sign factor seems to be consistent with the preservation of product structure and there seems to be no obvious reason why this definition could not be consistent with the proposed definition of quantum sum since it is just the image of the ordinary sum if m is not prime. For $\eta \neq 0$ one could say that the quantum integers define a different coordinates for integer points of the real line as algebraic numbers in the algebraic extension defined by the quantum phase.

4. If m is prime: $m = l$ (the notation is inspired by l-adicity), $l_q = 0$ holds true and all integers divisible by l are mapped to zero. If one restricts the quantum integers to the ones corresponding to $0 > n < l$, one obtains the q-analog of finite field $G(l,1)$ by defining the sum in such a manner that it is respects the sum for finite field $G(l,1)$. In this case l is mapped to zero in perfect analogy with mod l arithmetics. One can however allow arbirtary quantum integers: not however that those divisible by l_q vanish.

5. One can also consider powers $m = l^k$ of prime. Does one obtain the analog of finite field $G(p,k)$ by defining the sum so that it respects the sum of ordinary integers modulo l^k? This need not be the case since finite fields correspond to algebraic extensions rather than integers modulo l^k. Note that for $k > 1$ one does not encounter the problem with the vanishing of l_q.

2.1.2 The quantum counterparts of p-adic integers

One an also ask what might be the best manner to define the quantum counterparts of p-adic integers. Also now one needs a quantum phase. Its existence as a p-adic number poses strong constraints.

1. The root of unity must now correspond to an element of algebraic extension. Here Fermat's theorem $a^{p-1} \mod p = 1$ poses constraints since $p-1$:th root of unity exists as ordinary p-adic number. Hence $m = p-1$:th root of unity is excluded. Also the modulus of q must exist either as a p-adic number or a number in the extension of p-adic numbers. The generalization of the expression of q in the real context to p-adic context reads as

$$q = exp(mr)exp(i2\pi/m) \ , \tag{2.6}$$

   where the phase factors in the algebraic extension of p-adic integers and r is integer. If m is divisible by p the exponent exists p-adically without an extension of p-adics.

2. If m is prime: $m = l$, one obtains

$$q = exp(ml)exp(\frac{i2\pi}{l}) \ . \tag{2.7}$$

   Here the condition $0 < m < l$ is natural.

2.1.3 Quantum counterpart of pinary expansion?

Is $l_q = 0$ for $q = exp(i2\pi/l)$ a curse or blessing? The generalization of the notion of quantum integer to a power series in l turns $l_q = 0$ to a blessing as later considerations demonstrate.

1. The idea is simple: consider power series

$$x = \sum x_n l^n \tag{2.8}$$

of l with coefficients x_n which are arbitrary quantum rationals $r_q = m_q/n_q$ rather than only integers in the range $(0, l-1)$ as for ordinary pinary expansion. If m_q is divisible by l_q, one has $r_q = 0$. If $n_q = 0$, r_q is infinite so that also this option must be excluded. Somewhat loosely one can say that quantum rationals correspond to rationals not divisible by l.

2. One can define quantum arithmetics for these powers series by regarding l as a formal variable. If quantum sum is proportional to l_q it vanishes. It will be found that this could provide a very elegant manner to realize p-adic length scale cutoff without breaking of symmetries if one works in quantum rational discretization. The map $l \to 1/l$ mapping UV and IR to each other would serve as a symmetry of the theory and could relate real and p-adic physics to each other in continuous and symmetry respecting manner in the quantum intersection of real and p-adic worlds.

An attractive definition for the quantum counterparts of p-adic integers is based on the expansion in powers of l since its coefficients are not divisible by l.

1. The prime l in the expansion $\sum x_n l^n$ is interpreted as a symbolic coordinate variable and the product of two quantum integers is analogous to the product of polynomials reducing to a convolution of the coefficient using quantum sum. The coefficient of a given power of l in the product would be just the convolution of the coefficients for factors using quantum sum. In the sum coefficients would be just the quantum sums of coefficients of summands.

2. The coefficient x_n can be larger than l as ordinary integers. In the product of ordinary p-adic integers the convolution for given power of l can lead to overflow and this leads to the emergence of modulo arithmetics. As a consequence, the canonical identification $\sum x_n l^n \to \sum x_n l^{-n}$ does not respect product and sum in general. Canonical identification does not respect symmetries although it is continuous. The overflow does not happen for quantum integers. For quantum integers the image under canonical identification induced by $l \to 1/l$ respects the product and sum structures.

3. The expansion in powers of l could also have as coefficients quantum rationals for which both numerator and denominator are indivisible by l. The quantum sum however vanishes when it is proportional l_q. This might be quite essential for the definition of quantum counterparts of the matrix groups.

4. It can happen that quantum sum resulting in the product or sum of quantum integers is proportional to l_q and vanishes. This is not a catastrophe and turns out to be crucial in the definition of quantum counterparts of matrix groups with commuting elements.

Note that these numbers are algebraic numbers so that quantum integers are algebraic numbers with prime l remaining ordinary integer. Canonical identification could give rise to a correspondence between real physics and p-adic physics respecting both continuity and symmetries and mapping long real length scales to short p-adic scales and vice versa. This kind of map would allow to relate real and p-adic variants of symmetries.

This notion of quantum integer is more general than that proposed in the model of Shnoll effect [1] but gives identical predictions when the parameters characterizing the probability distribution $f(n)$ correspond contain only single term in the p-adic power expansion. The mysterious dependence of nuclear decay rates on physics of solar system in the time scale of years reduces to similar dependence for the parameters characterizing $f(n)$. Could this dependence relate directly to the fact that canonical identification maps long length scale physics to short length scales physics. Could even microscopic systems such as atomic nuclei give rise to what might be called "cognitive representations" about the physics in astrophysical length scales?

2.2 Do commutative quantum counterparts of Lie groups exist?

The proposed definition of quantum rationals involves exceptional prime l expected to define what might be called p-adic prime. In p-adic mass calculations canonical identification is based on the map $p \to 1/p$ and has several variants but quite generally these variants fail to respect symmetries.

Canonical identification for space-time coordinates fails also to be general coordinate invariant unless one has preferred coordinates.

The natural question is whether the proposed definition of quantum integers as series of powers of p-adic prime l with coefficients which are arbitrary quantum rationals not divisible by l with product defined in terms of convolution for the coefficients of the series in powers of l using quantum sum for the summands in the convolution could save the situation.

To see whether this is the case on must find whether the quantum analogues of classical matrix groups exist. To avoid confusion it should be emphasized that these quantum counterparts are distinct from the usual quantum groups having non-commutative matrix elements. Later a possible connection between these notions is discussed. In the recent case matrix elements commute but sum is replaced with quantum sum and the matrix element is interpreted as a powers series or polynomial in symbolic variable $x = l$ or $x = 1/l$, l prime such that coefficients are rationals not divisible by l.

The crucial points are the following ones.

1. All classical groups [3] are subgroups of the special linear groups [16] $SL_n(F)$, $F = R, C$, consisting of matrices with unit determinant. These groups are obtained by posing additional conditions such as the orthonormality of the rows with respect to real, complex or quaternionic inner product. Determinant defines a homomorphism mapping the product of matrices to the product of determinants in the field F.

   Could one generalize rational special linear group and its algebraic extensions by replacing the group elements by polynomials of a formal variable x, which has as its value the preferred prime l such that the coefficients of the polynomial are rational numbers not divisible by l?

   Could one perform this generalization in such a manner that the canonical identification $p \to 1/p$ maps this group to an isomorphic group?

2. The identity $det(AB) = det(A)det(B)$ and the fact that the condition $det(A) = 1$ involves at the right hand side only the unit element common to all quantum integers suggests that this generalization could exist. If one has found a set of elements satisfying the condition $det_q(A) = 1$ all quantum products satisfy the same condition and subgroup of rational special linear group is generated.

2.2.1 Quantum counterparts of special linear groups

Special linear groups [16] defined by matrices with determinant equal to 1 contain classical groups as subgroups and the conditions for their quantum counterparts are therefore the weakest possible.

1. To see that the generalization exists in the case of special linear groups one just just writes the matrix elements a_{ij} in series in powers of l

$$a_{ij} = \sum_n a_{ij}(n) l^n \quad . \tag{2.9}$$

   This expansion is very much analogous to that for the Kac-Moody algebra element and also the product and sum obey similar algebraic structgure. l is treated as a symbolic variable in the conditions stating $det_q(A) = 1$. It is essential that $det_q(A) = 1$ holds true when l is treated as a formal symbol so that each power of l gives rise to separate conditions.

2. For SL_n the definition of determinant involves sum over products of n elements. Quantum sums of these elements are in question. The question whether the quantum sum can correspond to a quantum integer which is divisible by l_q and therefore vanishes. For $q = 1$ the question is whether the sum for products of rationals, which do not have p as a factor can have p as a factor. Quite generally the situation reduces to this if ordinary sum induces quantum sum. It seems that this can be the case and the question is whether one can just assume that these terms vanish without ending up with some internal inconsistency.

3. Consider now the number of conditions involved. The number of matrix elements is in real case $N^2(k+1)$, where k is the highest power of l involved. $det(A) = 1$ condition involves powers of l

up to l^{Nk} and the total number of conditions is $kN+1$- one for each power. For higher powers of l the conditions state the vanishing of the coefficients of l^m. This is achieved elegantly in the sense of modulo arithmetics if the quantum sum involved is proportional to l_q.

The number of free parameters is

$$\# \quad = (k+1)N^2 - kN - 1 = kN(N-1) + N^2 - 1 \ . \tag{2.10}$$

For $N=2, k=0$ one obtains $\#=3$ as expected for SL(2,R). For $N=2, k=1$ one obtains $\#=5$. This can be verified by a direct calculation. Writing $a_{ij} = b_{ij} + c_{ij}p$ one obtains three conditions

$$det_q(A) = 1 \ , \ Tr_q(AB) = 0 \ , \ det_q(B) = 0 \ . \tag{2.11}$$

for the 8 parameters leaving six parameters which of course are rational numbers whose numerator and denominator are not divisible by l.

4. Complex case can be treated in similar manner. In this case the number of three parameters is $2(k+1)N^2$, the number of conditions is $2(kN+1)$ and the number of parameters is

$$\# = 2(k+1)N^2 - 2(kN+1) \ . \tag{2.12}$$

5. Since the conditions hold separately for each power of l, the formulate $det_q(AB) = det_q(A)det_q(B)$ implies that the matrices satisfying the conditions generate a subgroup of SL_n.

The result means that rational subgroups of special linear groups $SL_n(R)$ and $SL(n,C)$ quantum matrix groups characterized by prime l exist in both real and p-adic context and can be related by the map $l \rightarrow 1/l$ mapping short and length scales to each other.

It is remarkable that only the Lorentz groups $SO(2,1)$ and $SO(3,1)$ have covering groups are isomorphic to $SL(2,R)$ and $SL(2,C)$ allow these subgroups. All classical Lie groups involve additional conditions besides the condition that the determinant of the matrix equals to one and all these groups except symplectic groups fail to allow the generalization of this kind for arbitrary values of k. Therefore four-dimensional Minkowski space is in completely exceptional position.

2.2.2 Do classical Lie groups allow quantum counterparts?

In the case of classical groups one has additional conditions stating orthonormality of the rows of the matrix in real, complex, or quaternionic number field. It is quite possible that the conditions might not be satisfied always and it turns out that for G_2 and probably also for other exceptional groups this is the case.

1. Non-exceptional classical groups

It is easy to see that all non-exceptional classical groups quantum counterparts in the proposed sense for sufficiently small values of k and in the case of symplectic groups quite generally.

1. Consider first orthogonal groups $SO(N)$.

   (a) For $q=1$ there are N^2 parameters. There are N conditions stating that the rows are unit vectors and $N(N-1)/2$ conditions stating that they are orthogonal. The total number of free parameters is $\# = N(N-1)/2$.

   (b) If the highest power of l is k there are $(k+1)N^2$ parameters and $(2k+1)[N+N(N-1)/2] = (2k+1)(N+1)/2$ conditions. The number of parameters is

$$\# = N^2(k+1) - \frac{N(N+1)(2k+1)}{2} = \frac{N(N-2k+1)}{2} \ . \tag{2.13}$$

This is negative for $k > (N+1)/2$. It is quite not clear how to interpret this result. Does it mean that when one forms products of group elements satisfying the conditions the powers higher than $k_{max} = [(N+1)/2]$ vanish by quantum modulo arithmetics. Or do the conditions separate to separate conditions for factors in AB: this indeed occurs in the unitarity conditions as is easy to verify. For $SO(3)$ and $SO(2,1)$ this would give $k_{max} = 2$. For $SO(3,1)$ one would have $k_{max} = 2$ too. Note that for the covering groups $SL(2,R)$ and $SL(2,C)$ there is no restrictions of this kind.

(c) The normalization conditions for the coefficients of the highest power of a given row imply that the vector in question has vanishing length squared in quantum inner product. For $q = 1$ this implies that the coefficients vanish. The repeated application of this condition one would obtain that $k = 0$ is the only possible solution. For $q \neq 1$ the conditions can be satisfied if the quantum length squared is proportional to $l_q = 0$. It seems that this condition is absolutely essential and serves as a refined manner to realize p-adic cutoff and quantum group structure and p-adicity are extremely closely related to each other. This conclusion applies also in the case of unitary groups and symplectic groups.

(d) Complex forms of rotation groups can be treated similarly. Both the number of parameters and the number of conditions is doubled so that one obtqins $\# = N^2(k+1) - N(N+1)(2k+1) = N(N-2k+1)$ which is negative for $k > (N+1)/2$.

2. Consider next the unitary groups $U(N)$. Similar argument leads to the expression

$$\# = 2N^2(k+1) - (2k+1)N^2 = N^2 \tag{2.14}$$

so that the number of three parameters would be N^2- same as for $U(N)$. The determinant has modulus one and the additional conditions requires that this phase is trivial. This is expected to give $k+1$ conditions since the fixed phase has l-adic expansion with $k+1$ powers. Hence the number of parameters for $SU(N)$ is

$$\# = N^2 - k + 1 \tag{2.15}$$

giving the condition $k_{max} < N^2 - 1$ which is the dimension of $SU(N)$.

3. Symplectic group can be regarded as a quaternionic unitary group. The number of parameters is $4N^2(k+1)$ and the number of conditions is $(2k+1)(N+2N(N-1)) = N(2N-1)(2k+1)$ so that the number of three parameters is $\# = 4N^2(k+1)-(2k+1)N(N-1) = (2k+3)N^2+N(2k+1)$. Fixing single quaternionic phase gives 3(k+1) conditions so that the number of parameters reduces to

$$\# = (2k+3)N^2 + (2k+1)N - 3(k+1) = (k+1)(2N^2 + 2N - 3) + N(N-1) \ , \tag{2.16}$$

which is positive for all values of N and k so that also symplectic groups are in preferred position. This is rather interesting, since the infinite-dimensional variant of symplectic group associated with the $\delta M^4 \times CP_2$ is in the key role in quantum TGD and one expects that in finite measurement resolution its finite-dimensional counterparts should appear naturally.

2. Exceptional groups are exceptional

Also exceptional groups [7] [7] related closely to octonions allow an analogous treatment once the nature of the conditions on matrix elements is known explicitly. The number of conditions can be deduced from the dimension of the ordinary variant of exceptional group in the defining matrix representation to deduce the number of conditions. The following argument allows to expect that exceptional groups are indeed exceptional in the sense that they do not allow non-trivial quantum counterparts.

The general reason for this is that exceptional groups are very low dimensional subgroups of matrix groups so that for the quantum counterparts of these groups the number N_{cond} of group conditions is too large since the number of parameters is $(k+1)N^2$ in the defining matrix representation (if such exists) and the number of conditions is at least $(2k+1)N_{class}$, where N_{class} is the number of condition for the classical counterpart of the exceptional group. Note that r-linear conditions the number of conditions is proportional to $rk+1$.

One can study the automorphism group G_2 [8] of octonions as an example to demonstrate that the truth of the conjecture is plausible.

1. G_2 is a subgroup of $SO(7)$. One can consider 7-D real spinor representation so that a representation consists of real 7×7matrices so that one has $7^2=49$ parameters. One has $N(N+1)/2$ orthonormality conditions giving for $N=7$ orthonormality conditions 28 conditions. This leaves 21 parameters. Besides this one has conditions stating that the 7-dimensional analogs of the 3-dimensional scalar-3-products $A\cdot(B\times C)$ for the rows are equal 1, -1, or 0. The number of these conditions is $N(N-1)(N-2)/3!$. For $N=7$ this gives 35 conditions meaning that these conditions cannot be independent of orthonormalization conditions The number of parameters is $\#=49-35=14$ - the dimension of G_2 - so that these conditions must imply orthonormality conditions.

2. Consider now the quantum counterpart of G_2. There are $(k+1)N^2=49(k+1)$ parameters altogether. The number of cross product conditions is $(3k+1)\times 35$ since the highest power of l in the scalar-3-product is l^{3k}. This would give

$$\# = -56k+14 \ . \tag{2.17}$$

   This number is negative for $k>0$. Hence G_2 would not allow quantum variant. Could this be interpreted by saying that the breaking of G_2 to $SU(3)$ must take place and indeed occurs in quantum TGD as a consequence of associativity conditions for space-time surfaces.

3. The conjecture is that the situation is same for all exceptional groups.

The general results suggest that both the covering group of the Lorenz group of 4-D Minkowski space and the hierarchy symplectic groups have very special mathematical role and that the notions of finite measurement resolution and p-adic physics have tight connections to classical number fields, in particular to the non-associativity of octonions.

2.3 Questions

In the following some questions are introduced and discussed.

2.3.1 How to realize p-adic-real duality at the space-time level?

The concrete realization of p-adic–real duality would require a map from p-adic realm to real realm and vice-versa induced by the map $p\to 1/p$ leading from p-adic number field to real number field or vice versa.

If possible, the realization of p-adic real duality at the space-time level should not pose additional conditions on the preferred extremals themselves. Together with effective 2-dimensionality this suggests that the map from p-adic realm to real realm maps partonic 2-surfaces to partonic 2-surfaces defining at least partially the boundary data for holography.

The situation might not be so simple as this.

1. One must however also consider the possibility that its is 3-D space-like surfaces at the ends of CDs which are mapped by the duality from p-adic realm to real realm or vice versa. A possible reason is that this kind of surfaces can be easily defined as intersections $F_i(z,r\xi^2,\xi^2)=0$, $i=1,2$ of two complex valued functions F_i of compex coordinate z and radial light-like coordinate for $\delta M^4_\pm = S^2\times T_+$ and two complex coordinates ξ^i, $i=1,2$ of CP_2: the number of conditions is 4 and this gives D= 7-4=3-dimensional space-like surface as a solution. These surfaces - that is functions F_i cannot be completely free but solutions of field equations in the direction of radial coordinate, and this might pose a difficulty.

2. It is also possible that some local 4-D tangent space data at partonic 2-surfaces are needed to characterize the space-time surface. An alternative possibility is that the failure of standard form of determinism for Kähler action forces to introduce partonic 2-surfaces in various scales and the breaking of strict 2-dimensionality does not occur locally. This option would correspond at quantum level radiative corrections in shorter scales down to CP_2 scale and might be seen as aesthetically more attractive option.

3. The realization of p-adic real duality by applying the proposed form of canonical identification to quantum rational points requires preferred coordinates. For the minimum option defined by the map of partonic 2-surfaces (no 4-D tangent space data) this would mean that one must have preferred coordinates for partonic 2-surfaces. It is easy to imagine how to identify this kind of preferred complex coordinate. The complex coordinate could correspond to a preferred complex coordinate for $S^2 \subset \delta M^4_\pm$ or for a homologically non-trivial geodesic sphere of CP_2. The complex coordinates would transform linearly under the maximal compact subgroup of SO(3) *resp.* SU(3).

2.3.2 How commutative quantum groups could relate to the ordinary quantum groups?

The interesting question is whether and how the commutative quantum groups relate to ordinary quantum groups.

This kind of question is also encountered when considers what finite measurement resolution means for second quantized induced spinor fields [4]. Finite measurement resolution implies a cutoff on the number of the modes of the induced spinor fields on partonic 2-surfaces. As a consequence, the induced spinor fields at different points cannot ant-commute anymore. One can however require anti-commutativity at a discrete set of points with the number of points "more or less equal" to the number of modes. Discretization would follow naturally from finite measurement resolution in its quantum formulation.

The same line of thinking might apply to to quantum groups. The matrix elements of quantum group might be seen as quantum fields in the field of real or complex numbers or possibly p-adic number field or of its extension. Finite measurement resolution means a cutoff in the number of modes and commutativity of the matrix elements in a discrete set of points of the number field rather than for all points. Finite measurement resolution would apply already at the level of symmetry groups themselves. The condition that the commutative set of points defines a group would lead to the notion of commutative quantum group and imply p-adicity as an additional and completely universal outcome and select quantum phases $exp(i2\pi/p)$ in a preferred position. Also the generalization of canonical identification so central for quantum TGD would emerge naturally.

One must of course remember that the above considerations probably generalize so that one should not take the details of the discussion too seriously.

2.3.3 How to define quantum counterparts of coset spaces?

The notion of commutative quantum group implies also a generalization of the notion of coset space G/H of two groups G and $H \subset G$. This allows to define the quantum counterparts of the proper time constant hyperboloid and $CP_2 = SU(3)/U(2)$ as discrete spaces consisting of quantum points identifiable as representatives of cosets of the coset space of discrete quantum groups. This approach is very similar but more precise than the earlier approach in which the points in discretization had angle coordinates corresponding to roots of unity and radial coordinates with discretization defined by p-adic prime.

The infinite-dimensional "world of classical worlds" (WCW) can be seen as a union of infinite-dimensional symmetric spaces (coset spaces) [3] and the definition as a quantum coset group could make sense also now in finite measurement resolution. This kind of approach has been already suggested and might be made rigorous by constructing quantum counterparts for the coset spaces associated with the infinite-dimensional symplectic group associated with the boundary of causal diamond. The problem is that matrix group is not in question. There are however good hopes that the symplectic group could reduces to a finite-dimensional matrix group in finite measurement resolution. Maybe it is enough to achieve this reduction for matrix representations of the symplectic group.

3 Could one understand p-adic length scale hypothesis number theoretically?

p-Adic length scale hypothesis states that primes near powers of two are physically interesting. In particular, both real and Gaussian Mersenne primes seem to be fundamental and can be tentatively assigned to charged leptons and living matter in the length scales between cell membrane thickness and size of the cell nucleus. They can be also assigned to various scaled up variants of hadron physics and with leptohadron physics suggested by TGD.

How could one understand p-adic length scale hypothesis? One explanation would be in terms of evolution by quantum jumps selecting the primes that are the fittest. This would mean also selection of preferred scales for CDs, instead of integer multiples of CP_2 scale only prime multiples or possibly prime power multiples would be favored and primes near powers of two were especially fit. A possible "biological" explanation is that for the preferred primes the number of quantum states is especially large making possible to build complex sensory and cogniive representations about external world.

The proposed vision about commutative quantum groups suggests a number theoretic explanation for the p-adic length scale hypothesis consistent with the evolutionary explanation is that the quantum counterpart of symmetry groups are especially large for preferred primes. Large symmetries indeed imply large numbers of states related by symmetry transformations and high representational capacity provided by the p-adic–real duality. It is easy to make a rough test of the proposal.

1. For $SL(2,C)$ - the covering group of Lorentz group- one obtains no constraints and all quantum phases $exp(i2\pi/n)$ are allowed: this would mean that all CDs are in the same position. One must however notice that $l_q = 0$ allows additional solutions to the conditions since the determinant highest power of l need only be proportional to l_q rather than vanish. The rational $SL(2,C)$ matrices whose determinant is zero modulo l form a group and and it might be that for some values of l this group is exceptionally large.$SL(2,C)$ defines also the covering group of conformal symmetries of sphere.

2. For orthogonal, unitary, and symplectic groups only $n = l$, l prime allows $k > 0$ and genuine p-adicity. Since $SO(3,1)$, $SO(3)$, $SU(2)$ and $SU(3)$ should alow p-adicization this selects CDs with size scale characterized by prime l.

3. For orthogonal, unitary, and symplectic groups one obtains non-trivial solutions to the unitarity conditions only if the highest power of l corresponds quantum image of a vector with zero norm modulo l as follows from the basic properties of quantum arithmetics.

   (a) In the case of $SO(3)$ one has the condition

   $$\sum_{i=1}^{3} x_i^2 = k \times l \tag{3.1}$$

   Note that this condition can degenerate to a condition stating that a sum of two squares is multiple of prime.

   (b) For the covering group $SU(2)$ of $SO(3)$ one has the condition

   $$\sum_{i=1}^{4} x_i^2 = k \times l = k \times l \tag{3.2}$$

   since two complex numbers for the row of SU(2) matrix correspond to four real numbers

   (c) For $SU(3)$ one has the condition

   $$\sum_{i=1}^{6} x_i^2 = k \times l = k \times l \tag{3.3}$$

   corresponding to 3 complex numbers defining the row of SU(3) matrix.

What can one say about these conditions? The first thing to look is whether the conditions can be satisfied at all. Second thing to look is the number of solutions to the conditions.

3.1 Orthogonality conditions for $SO(3)$

The conditions for $SO(3)$ are certainly the strongest ones so that it is reasonable to study this case first.

1. One must remember that there are also integers -in particular primes- allowing representation as a sum of two squares. For instance, Fermat primes whose number is very small, allow representation $F_n = 2^+1$. More generally, Fermat's theorem on sums of two squares states that and odd prime is expressible as sum of two squares only if it satisfies $p \ mod \ 4 = 1$. The second possibility is $p \ mod \ 4 = 3$ so that roughly one half of primes satisfy the $p \ mod \ 4 = 1$ condition: Mersenne primes do not satisfy it.

   The more general condition giving sum proportional to prime is satisfied for all $n = k^2 l$, $k = 1, 2, ...$

2. For the sums of three non-vanishing squares one can use the well-known classical theorem stating that if integers n can be represented as a sum of three non-vanishing squares only if it is *not* of the form [11]

$$n = 2^{2r}(8k+7) \tag{3.4}$$

   For instance, squares of odd integers multiplied by any power of two satisfy this condition. If n satisfies (does not satisfy) this condition then nm^2 satisfies this condition for any m so that one can say that square free odd integers for which the condition $n \neq 7$ (mod 8) generate this set of integers.

   In the recent case these integers must be also divisible by prime l. Note that the integers representable as sums of three non-vanishing squares do not allow a representation using two squares. The product of odd primes $p_1 = 8m_1 + k_1$ and $p_1 = 8m_2 + k_2$ fails to satisfy the condition only if one has $k_1 = 3$ and $k_2 = 5$. The product of n primes $p_i = 8m_i + k_i$ must satisfy the condition $\prod k_i \neq 7$ (mod 8) in order to serve as a generating square free prime.

   The cold -or at least cool- shower is that Mersenne primes $M_n > 3$ do *not* satisfy the condition guaranteining representability as a sum of three squares as one sees from $2^n - 1 = (2^(n-3) - 1)8 + 7$. The integers $2^{2k+1}M_n$ satisfy the condition. One can of course ask whether Mersenne primes might be special just because they representation requires four integers so that they would correspond to the covering $SU(2)$ of $SO(3)$ instead of $SO(3)$: could this mean that Mersenne primes -and more generally primes $p = km + 7$ - must correspond to fermions?

   One must also remember that all that is needed is that sufficiently small multiples of Mersenne primes correspond to large value of $r_3(n)$.

3. If one has $\sum n_i^2 = l$ requiring

$$l = 8k+7 \tag{3.5}$$

   then the scaling $n_i \rightarrow kn_i$ gives a solution to the condition $\sum n_i^2 = k^2 l$.

4. The condition $l = 8k + 7$ is true for all Mersenne primes $M_n = 2^n - 1$, $n > 2$, since $2^n - 1 = 8 \times (2^{n-3} - 1) + 7$ in this case. Hence this condition indeed selects Mersenne primes plus some other primes as sepcial but not necessarily preferred ones for $l \ mod \ 4 = 3$ case. The list of allowed primes begins with $7, 23, 31, 47, 71, 79, 103, 127, ...$: $7, 31$, and 127 are Mersenne primes.

5. If prime near power of 2 but smaller than it is to satisfy this condition $l = 8k+7$, one must have

$$l = 2^n - 1 - 8m - 1 \ , \ n > 2 \ . \tag{3.6}$$

so that special -one might hope preferred -p-adic length scales could somehow correspond to Mersenne integers (to be distinguished from primes) from which a suitable multiple of 8 is subtracted.

3.2 Number theoretic functions $r_k(n)$ for $k = 2, 4, 6$

The number theoretical functions $r_k(n)$ telling the number of vectors with length squred equal to a given integer n are well-known for $k = 2, 3, 4, 6$ and can be used to gain information about the constraints posed by the existence of quantum groups $SO(2)$, $SO(3)$, $SU(2)$ and $SU(3)$. In the following the easy cases corresponding to $k = 2, 4, 6$ are treated first and after than the more difficult case $k = 3$ is discussed. For the auxiliary function the reader can consult to the Appendix.

3.2.1 The behavior of $r_2(n)$

$r_2(n)$ gives information not only about quantum $SO(2)$ but also about $SO(3)$ since 2-D vectors define 3-D vectors in an obvious manner. The expression for $r_2(n)$ is given by

$$r_2(n) = \sum_{d|n} \chi(d) \ , \ \chi(d) = \left(\frac{-4}{d}\right) \ . \tag{3.7}$$

For primes this gives

$$r_2(p) = \begin{cases} 2 & \text{if } p = 1 \ (\text{mod } 4) \ , \\ 0 & \text{if } p = 3 \ (\text{mod } 4) \ . \end{cases} \tag{3.8}$$

The result is expected and the two solutions for $p = 1$ (mod 4) are obtained by permuting the components of the 2-vector. In 3-D case 2-D solutions gives rise to 12 solutions as is easy to see.

3.2.2 The behavior of $r_4(n)$

The expression for $r_4(n)$ reads as

$$r_4(n) = \begin{cases} 8\sigma(n) & \text{if } n \text{ is odd} \ , \\ 24\sigma(m) & \text{if } n = 2^\nu m, \ m \text{ odd} \ . \end{cases} \tag{3.9}$$

For $n = p$ one has $\sigma(p) = p + 1$ giving

$$r_4(p) = 8(p+1) \ . \tag{3.10}$$

The behavior as a function of p is smooth and does not distinguish between different primes. Since σ is mutiplicative function it is easy to calculate the values of $r_4(n)$ if n is a small multiple of prime since one has

$$\begin{aligned} r_4((2m+1)l) &= r(l)\sigma(2m+1) \ , \\ r_4(2^s l) &= 24 r_4(l) \ . \end{aligned} \tag{3.11}$$

One has a periodicity in powers of 2 so that large values of r_4 appear at octaves of l. From the point of view of p-adic length scale hypothesis this is an encouraging sign but is not enough to distinguish preferred primes.

The asymptotic behavior of σ function is known so that it is relatively easy to estimate the behavior of $r_4(n)$. The behavior involves random looking local fluctuation which can be understood as reflective the multiplicative character implying correlation between the values associated with multiples of a given prime.

3.2.3 The behavior of $r_6(n)$

The analytic expression for $r_6(n)$ is given by

$$\begin{aligned} r_6(n) &= \sigma_{d|n}\left[16\chi(\frac{n}{d})-4\chi(d)\right]d^2 \ , \\ \chi(n) &= \left(\frac{-4}{n}\right)=\begin{cases} 0 & \text{if } n \text{ is even} \\ 1 & \text{if } n=1 \ (\text{mod } 4) \\ -1 & \text{if } n=3 \ (\text{mod } 4) \end{cases} \end{aligned} \tag{3.12}$$

For primes this gives

$$r_6(p) = \begin{cases} 12(p^2+1) & \text{for } p=1 \ (\text{mod p}) \\ 12+20p^2 & \text{for } p=3 \ (\text{mod p}) \end{cases} \tag{3.13}$$

The behavior is smooth and for primes $p = 3$ (mod 4) the parabolic growth is faster. $r_6(p)$ does not seem to distinguish between different primes.

3.3 What can one say about the behavior of $r_3(n)$?

The proportionality of $r_3(D)$ to the order of $h(-D)$ [1] of the ideal class group [10] [10] for quadratic extensions of rationals [1] inspires some conjectures.

1. The conjecture that preferred primes l correspond to large commutative quantum groups translates to a conjecture that the order of ideal class group is large for the algebraic extension generated by $\sqrt{-l}$ or more generally $\sqrt{-kl}$- at least for some values of k such as $k = 2^r$. Could suitable integer multiples primes near power of 2- in particular Mersenne primes - be such primes? Note that only integer multiple is required by the basic argument.
2. Also some kind of approximate fractal behavior $r_k(sl) \simeq r_k(l)f_k(s)$ for some values of s analogous to that encountered for $r_4(D)$ for all values of s might hold true since $k = 3$ is a critical transition dimension between $k = 2$ and $k = 3$. In particular, an approximate periodicity in octaves of primes might hold true: $r_k(2^sl) \simeq r_k(l)$: this would support p-adic length scale hypothesis and make the comutative quantum group large.

3.3.1 Expression of $r_3(p)$ in terms of class number function

To proceed one must have an explicit expression for the class number function $h(D)$ and the expression of r_3 in terms of $h(D)$.

1. For $D = -p$ defining the complex extension the general expression for $h(D)$ discussed in the Appendix gives

$$h(-p) = -\frac{1}{p}\sum_1^p r\times\left(\frac{-p}{r}\right) \ . \tag{3.14}$$

The general expression is obtained by replacing p with D. The symbols($\left(\frac{-p}{r}\right)$ are Dirchlet and Kronecerk symbols defined in the Appendix.

2. One can express $r_3(|D|)$ in terms of $h(D)$ as

$$r_3(|D|) = 12(1-(\frac{D}{2}))h(D) \ . \tag{3.15}$$

For $D = -p$ the relationship between $r_3(|D|)$ and $h(D)$ gives

$$r_3(p) \quad = \quad 12(1-(\frac{p}{2}))h(-p) \ . \tag{3.16}$$

Note that $(\frac{p}{2})$ refers to Kronecker symbol.

3. From Wolfram one finds the following expressions of $r_3(n)$ for square free integers

$$\begin{array}{ll} r_3(n) = 24h(-n) & n = 3 \ (\mathrm{mod}\ 8) \ , \\ r_3(n) = 12h(-4n) & n = 1,2,5,6 \ (\mathrm{mod}\ 8) \ , \\ r_3(n) = 0 & n = 7 \ (\mathrm{mod}\ 8) \ . \end{array} \tag{3.17}$$

4. The generating function for r_3 [17] is third power of θ function θ_3.

$$\sum_{n\geq 0} r_3(n)x^n \quad = \quad \theta_3^3(n) = 1+6x+12x^2+8x^3+6x^4+24x^5+24x^6+12x^8+30x^9+... \tag{3.18}$$

This representation follows trivially from the definition of θ function as sum $\sum_{n=-\infty}^{\infty} x^{n^2}$.

The behavior of $h(-p)$ for large primes is not easy to deduce without numerical calculations which probably get too heavy for primes of order M_{127}. The definition involves sum of p terms labeled by $r = 1,...,p$, and each term is a product is product of terms expressible as a product over the prime factors of of r with over all term being a sign factor. "Interference " effects between terms of different sign are obviously possible in this kind of situation and one might hope that for large primes these effects imply wild fluctuations of $r_3(p)$.

3.3.2 Simplified formula for $r_3(D)$

Recall that the proportionality of $r_3(|D|)$ to the ideal class number $h(D)$ is for $D < -4$ given by

$$r_3(|D|) \quad = \quad 12[1-\left(\frac{D}{2}\right)]h(D) \ . \tag{3.19}$$

The expression for the Kronecker symbol appears in the formula as well as formulas to be discussed below and reads as

$$\left(\frac{D}{2}\right) \quad = \quad \begin{cases} 0 & \text{if } D \text{ is even} \ , \\ 1 & \text{if } D = -1 \ (\mathrm{mod}\ 8) \ , \\ -1 & \text{if } D = \pm 3 \ (\mathrm{mod}\ 8) \ . \end{cases} \tag{3.20}$$

The proportionality factor vanishes for $D = 2^{2r}(8m+7)$ and equals to 12 for even values of D and to 24 for $D = \pm 3$ (mod 8).

To get more detailed information about r_3 one can begin from class number formula [2] for $D < -4$ reading as

$$h(D) \quad = \quad \frac{1}{|D|}\sum_{r=1}^{|D|} r\left(\frac{D}{r}\right) \ . \tag{3.21}$$

Each Jacobi symbol $\left(\frac{D}{r}\right)$ decomposes to a product of Legendre and Kronecker symbols $\left(\frac{D}{p_i}\right)$ in the decomposition of odd integer r to a product of primes p_i.

For $\left(\frac{D}{p_i}\right) = 1$ p_i splits into a product of primes in quadratic extension generated by $\sqrt{D}$. If it vanishes p_i is square of prime in the quadratic extension. In the recent case neither of these options are possible for the primes involved as is easy to see by using the definition of algebraic integers. Hence one has $\left(\frac{D}{p_i}\right) = -1$ for all odd primes to transform the formula for $D < -4$ to the form

$$\begin{aligned} h(D) &= \frac{1}{|D|}\sum_{r=1}^{|D|} r[\left(\frac{D}{2}\right)]^{\nu_2(r)}(-1)^{\Omega(r)-\nu_2(r)} \\ &= \frac{1}{|D|}\sum_{r=1}^{|D|} r[-\left(\frac{D}{2}\right)]^{\nu_2(r)}(-1)^{\Omega(r))} \ . \end{aligned} \tag{3.22}$$

Here $\nu_2(r)$ characterizes the power of 2 appearing in r and $\Omega(r)$ is the number of prime divisors of r with same divisor counted so many times as it appears. Hence the sign factor is same for all integers r which are obtained from the same square free integer by multiplying it by a product of even powers of primes.

Consider next various special cases.

1. For even values $D < -4$ (say $D = -2M_n$) only odd integers r contribute to the sum since the Kronecker symbols vanish for even values of r.

$$h(D = 2d) = \frac{1}{|D|}\sum_{1\leq r<|D| \quad odd} r(-1)^{\Omega(r)} \ . \tag{3.23}$$

2. For $D = \pm 1 \pmod 8$, the factors $\left(\frac{D}{2}\right) = -1$ implies that one can forget the factors of 2 altogether in this case (note that for $D = -1 \pmod 8$ $r_3(|D|)$ vanishes unlike $h(D)$).

$$h(D = \pm 1(mod\ \ 8)) = \frac{1}{|D|}\sum_{r=1}^{|D|} r(-1)^{\Omega(r))} \ . \tag{3.24}$$

3. For $D = \pm 3 \pmod 8$, the factors $\left(\frac{D}{2}\right) = 1$ implies that one has

$$h(D = \pm 3(mod\ \ 8)) = \frac{1}{|D|}\sum_{r=1}^{|D|} r(-1)^{\Omega(r)-\nu_2(r)} \ . \tag{3.25}$$

The magnitudes of the terms in the sum increase linearly but the sign factor fluctuates wildly so that the value of $h(-p)$ varies chaotically but must be divisible by p and negative since $r_3(p)$ must be a positive integer. Even in this form the calculation of $r_3(p)$ requires summation over p terms so that for M_{127} the number of terms is still huge.

3.3.3 Could thermodynamical analogy help?

For $D < -4$ $h(D)$ is expressible in terms of sign factors determined by the number of prime factors or odd prime factors modulo two for integers or odd integers $r < D$. This raises hopes that $h(D)$ could be calculated for even large values of D.

1. Consider first the case $D = \pm 1$ (mod 8)). The function $\lambda(r) = (-1)^{\Omega(r)}$ is known as Liouville function [12]. From the product expansion of zeta function in terms of "prime factors" it is easy to see that the generating function for $\lambda(r)$

$$
\begin{aligned}
\sum_n \lambda(n) n^{-s} &= \frac{\zeta(2s)}{\zeta(s)} = \frac{1}{\zeta_F(s)} \ , \\
\zeta_(s) &= \prod_p (1-p^{-s})^{-1} \ , \ \zeta_F(s) = \prod_p (1+p^{-s}) \ .
\end{aligned} \tag{3.26}
$$

Recall that $\zeta(s)$ *resp.* $\zeta_F(s)$ has a formal interpretation as partition functions for the thermodynamics of bosonic *resp.* fermionic system. This representation applies to $h(D = \pm 1(mod 8))$.

2. For $D = 2d$ the representation is obtained just by dropping away the contribution of all even integers from Liouville function and this means division of $(1+2^{-s})$ from the fermionic partition function $\zeta_F(s)$. The generating function is therefore

$$
\sum_{n \ odd} \lambda(n) n^{-s} = \prod_{p \ odd} (1+p^{-s})^{-1} = (1+2^{-s}) \frac{1}{\zeta_F(s)} \ . \tag{3.27}
$$

3. For $h(D = \pm 3(mod 8))$. One most modify the Liouville function by replacing $\Omega(r)$ by the number of odd prime factors but allow also even integers r. The generating function is now

$$
\sum_n \lambda(n) (-1)^{\nu_2(n)} n^{-s} = \frac{1}{1-2^{-s}} \prod_{p \ odd} (1+p^{-s})^{-1} = \frac{1}{1-2^{-s}} \frac{1}{\zeta_F(s)} \ . \tag{3.28}
$$

The generating functions raise the hope that it might be possible to estimate the values of the $h(D)$ numerically for large values of D using a thermodynamical analogy.

1. $h(D)$ is obtained as a kind of thermodynamical average $\langle r(-1)^{\Omega(r)} \rangle$ for particle number r weighted by a sign factor telling the number of divisors interpreted as particle number. s plays the role of the inverse of the temperature and infinite temperature limit $s = 0$ is considered. One can also interpret this number as difference of average particle number for states restricted to contain even *resp.* odd particle number identified as the number of prime divisors with 2 and even particle numbers possibly excluded.

2. The average is obtained at temperature corresponding to $s = 0$ so that $n^{-s} = 1$ holds true identically. The upper bound $r < D$ means cutoff in the partition sum and has interpretation as an upper bound on the energy $log(r)$ of many particle states defined by the prime decomposition. This means that one must replace Riemann zeta and its analogs with their cutoffs with $n \leq |D|$. Physically this is natural.

3. One must consider bosonic system all the cases considered. To get the required sign factor one must associated to the bosonic partition functions assigned with individual primes in $\zeta(s)$ the analog of chemical potential term $exp(-\mu/T)$ as the sign factor $exp(i\pi) = -1$ transforming ζ to $1/\zeta_F$ in the simplest case.

One might hope that one could calculate the partition function without explicitly constructing all the needed prime factorizations since only the number of prime factors modulo two is needed for $r \leq |D|$.

3.3.4 Expression of $r_3(p)$ in terms of Dirichlet L-function

It is known [13] that the function $r_3(D)$ is proportional to Dirichlet L-function $L(1,\chi(D))$ [5]:

$$\begin{aligned} r_3(|D|) &= \frac{12\sqrt{D}}{\pi}L(1,\chi(D))) \ , \\ L(s,\chi) &= \sum_{n>0}\frac{\chi(n,D)}{n^s} \ , \end{aligned} \tag{3.29}$$

$\chi(n,D)$ is Dirichlet character [4] which is periodic and multiplicative function - essentially a phase factor- satisfying the conditions

$$\begin{array}{ll} \chi(n,D)\neq 0 & \text{if n and D have no common divisors} > 1 \ , \\ \chi(n,D)=0 & \text{if n and D have a common divisor} > 1 \ , \\ \chi(mn,D)=\chi(m,D)\chi(n,D) \ , & \chi(m+D,D)=\chi(m,D) \ , \\ \chi(1,D)=1 \ . & \end{array} \tag{3.30}$$

1. $L(1,\chi(D))$ varies in average sense slowly but fluctuates wildly between certain bounds. One can say that there is local chaos.

   The following estimates for the bounds are given in [13]:

   $$c_1(D) \equiv k_1 log(log(D) < L_1(1,\chi(D)) < c_2(D) \equiv k_2 log(log(D)) \ . \tag{3.31}$$

   Also other bounds are represented in the article.

3.3.5 Could preferred integers correspond to the maxima of Dirichlet L-function?

The maxima of Dirichlet L-function are excellent candidates for the local maxima of $r_3(D)$ since $\sqrt{D}$ is slowly varying function.

1. As already found, Mersenne primes and integers cannot represent pronounced maxima of $r_3(n)$ since there are no representation as a sum of three squares and the proportionality constant vanishes. In this special case it does not matter whether L-function has a maximum or not.

   (a) Could just the fact that the representation in terms of three primes is not possible, select Mersenne primes $M_n > 3$ as preferred ones? For $SU(2)$, which is covering group of $SO(3)$ the representation as a sum of four squares is possible. Could it be that the spin 1/2 character of the fermionic building blocks of elementary particles means that a representation as sum of four squares is what matters. But why the non-existence of representation as a sum of three squares might make Mersenne primes so special?

   (b) Mersenne prime multiplied by odd power of two satisfies the condition and some of these square free integers might correspond to pronounced maxima.

2. Could also primes near power of 2 define maxima? Unfortunately, the calculations of [13] involve averaging, minimum, and maximum over 10^6 integers in the ranges $n\times 10^6 < D < (n+1)\times 10^6$, so that they give very slowly varying maximum and minimum.

3. Could Dirichlet function have some kind of fractal structure such that for any prime one would have approximate factorization? The naivest guesses would be $L(1,\chi_{kl}) \simeq f_1(k)L(1,\chi_l)$ with $k=2^s$. This would mean that the primes for which $D(1,\chi_p)$ is maximum would be of special importance.

4. p-Adic fractality and effective p-adic topology inspire the question whether L-function is p-adic fractal in the regions above certain primes defining effective p-adic topology $D(1,\chi_{p^k}) \simeq f_1(k)DK(1,\chi_p)$ for preferred primes.

3.3.6 Interference as a helpful physical analogy?

Could one use physical analog such as interference for the terms of varying sign appearing in L-function to gain some intuition about the situation?

1. One could interpret L-function as a number theoretic Fourier transform with D interpreted as a wave vector and one has an interference of infinite number of terms in position space whose points are labelled by positive integers defining a half -lattice with unit lattice length. The magnitude of n:th summand $1/n$ and its phase is periodic with period $D = kp$. The value of the Fourier component is finite except for $D = 0$ which corresponds to Riemann Zeta at $s = 1$. Could this means that the Fourier component behaves roughly like $1/D$ apart from an oscillating multiplicative factor.

2. The number theoretic counterparts of plane waves are special in that besides D-periodicity they are multiplicative making thema lso analogs of logarithmic waves. For ordinary Fourier components one additivity in the sense that $\Psi(k_1 + k_2) = \Psi(k_1)\Psi(k_2)$. Now one has $\Psi(k_1k_2) = \Psi(k_1)\Psi(k_2)$ so that $log(D)$ corresponds to ordinary wave vector. p-Adic fractality is an analog for periodicity in the sense of logarithmic waves so that powers rather than integer multiples of the basic scale define periodicity. Could the multiplicative nature of Dirichlet characters imply p-adic - or at least 2-adic - fractality, which also means logarithmic periodicity?

3. Could one say that for these special primes a constructive interference takes place in the sum defining the L-function. Certainly each prime represents the analog of fundamental wavelength whose multiples characterize the summands. In frequency space this would mean fundamental frequency and its sub-harmonics.

3.3.7 Period doubling as physical analogy?

1. For $k = 4$ all scales are present because of the multiplicative nature of σ function. Now only the Dirichlet characters are multiplicative which suggests that only few integers define preferred scales? Prime power multiples of the basic scale are certainly good candidates for preferred scales but amongst them must be some very special prime powers. $p = 2$ is the only even prime so that it is the first guess.

2. Could the system be chaotic or nearly chaotic in the sense of period doubling so that octaves of preferred primes interfere constructively? Why constructively? Could complete chaos -interpreted as randomness- correspond to a destructive interference and minimum of the L-function?

3. What about scalings by squares of a given prime? It seems that these scalings cannot be excluded by any simple argument. The point is that $r_3(n)$ contains also the factor $\sqrt{n}$ which must transform by integer in the scaling $n \to kn$. Therefore k must be power of square.

This leaves two extreme options. Both options are certainly testable by simple numerical calculations for small primes. For instance one can use generating function $\theta_3^3(x) = \sum r_3(n)x^n$ to kill the conjectures.

1. The first option corresponds to scalings by all integers that are squares. This option is also consistent with the condition $n \neq 2^k(8m + 7)$ since both the scaling by a square of odd prime and by a square of 2 preserve this condition since one has $n^2 = 1 \pmod 8$ for odd integers. This is also consistent with the finding that $r_3(n) = 1$ holds true only for a finite number of integers. A simple numerical calculation for the sums of 3 squares of 16 first integers demonstrates that the conjecture is wrong.

2. The second option corresponds only to the scaling by even powers of two and is clearly the minimal option. This period quadruping for n corresponds to period doubling for the components of 3-vector. A calculation of the sums of squares of the 16 first integers demonstrates that for $n = 3, 6, 9, 11, ..$ the conjecture the value of $r_3(n)$ is same so that the conjecture might hold true! If it holds true then Dirichlet L-function should suffer scaling by 2^{-r} in the scaling $n \to 2^{2r}n$. The integer solutions for n scaled by 2^r are certainly solutions for $2^{2r}n$. Quite generally, one

has $r_3(m^2n) \geq r_3(n)$ for any integer m. The non-trivial question is whether some new solutions are possible when the scaling is by 2^{2r}.

A simple argument demonstrates that there cannot be any other solutions to $\sum_{n_i=1}^{3} m_i^2 = 2^{2r}n$ than the the scaled up solutions $m_i = 2n_i$ obtained from $\sum_{n_i=1}^{3} n_i^2 = n$. This is seen by noticing that non-scaled up solutions must contain $1, 2, or 3$ integers m_i, which are odd. For this kind of integers one has $m^2 = 1 \pmod 4$ so that the sum $(\sum_i m_i^2)$= 1,2, or 3 (mod 4) whereas the the right hand side vanishes mod 4.

3. If D is interpreted as wave vector, period quadrupling could be interpreted as a presence of logarithmic wave in wave-vector space with period $2log(2)$.

3.3.8 Which preferred primes could winners in the number theoretic evolution?

Since the invariance under scalings by even powers of two holds true in strong sense, it is enough to find which square free integers satisfying the basic condition correspond to the maxima of Dirichlet function.

1. Mersenne primes (same applies to Mersenne numbers) certainly do not satisfy the condition but their odd power multiples do. The study of the situation for the smallest Mersenne primes indeed shows that for $n = 2M_k$ for $M_k = 3, 7, 31, 127$ $r_3(n)$ has a local maximum. For Mersenne integers $m = 2M_n$ with $n = 3, 5, 6, 7, 9, 12$ the ratio $r_3(n)/\sqrt(n)$ proportional to Dirichlet L-function is larger than 1.5 in the range $k \in [1, 40000]$. The maximum occurs for $n = 12$ and is equal to 2.25. $n = 3, 5, 7$ correspond to Mersenne primes and $n = 6, 9, 12$ to Mersenne integers divisible by the Mersenne primes associated with the factors of n, in particular all are divisible by $M_3 = 7$ so that $M_3 = 7$ sees to be a lucky number. For $n = 4, 8, 10, 11, 13$ the values are $(1.10, 1.06, 1.06.1.33)$. In this case M_n is divisible by $M_2 = 3$ but not divisible by higher Mersenne primes. $n = 13$ corresponds to Mersenne prime so that $n = 13$ is indeed unlucky number. One could ask whether this tendency is true also for n, when n is Mersenne integer. Checking this should be quite easy. If so then divisors of Mersenne integers would be special.

2. What matters is the existence of a large number of integer component vectors with length squared proportional to the preferred prime. The implication would be that for large values of D integers near powers of two would correspond to several closely located maxima of $h(D)$ assignable to different powers of 2.

3. The following argument favors primes of form $p = 2^{2r}(8k + 7)$ and therefore Mersenne primes.

   (a) One could generalize the quantum arithmetics in such a manner that the primes associated with algebraic integers are mapped to corresponding quantum primes. If the preferred ordinary prime does not decompose to generalized primes in the extension, there are no problems: this prime would still mapped to zero but in general new quantum primes would be transcendental numbers.

   (b) If the decomposition to primes is not unique for a general ordinary prime $(h(-p) > 1)$, problems are encountered since the quantum decompositions corresponding to two compositions to more general primes need not be identical. The manner to solve this problem would be simple in the case of quadratic extensions (but not generally): allow only the primes $p = 2^{2r}(8k + 7)$ as preferred primes mapped to zero. In a given algebraic extension only those ordinary primes which do not split to produces of new primes could define quantum extensions.

   (c) The higher the algebraic dimension of the extension of rationals, the smaller the number of preferred ordinary primes able to define the quantum arithmetics. Could this mechanism gradually select preferred primes in the number theoretical evolution by quantum jumps leading to increasingly larger algebraic extensions of rationals?

4. Note that the scaling invariance under powers of 4 does not correspond to 2-adic fractality (or equivalently continuity). 2-Adic fractality of r_3 would state that $r_3(n)$ and $r_3(n + 2^r)$ do not differ much for large enough r so that there is continuity in 2-adic topology: here $r_3(n)$ could be as real or 2-adic integer. 2-adic fractality could explain why primes near prime powers of two

since the addition of a large power 2^s to the integer kp having representation $kp = 2^r(8l + m)$ leaves this representation invariant. If $r_3(n)$ behaves as 2-adic number then for large values of 2^s the addition could give $r_3(n+m2^s) = r_3(n)+n_1 2^{s_1}$, $s_1 >> 1$ so that large primes near power of two would have large alue of r_3 which is in 2-adic sense is strongly correlated with the value of r_3 for rather small integers n. The smoothed out behavior $r_3 \propto \sqrt{n}$ as real valued function poses constraints on possible 2-adic fractality. The study of r_3 for $n = 3 + 2^r$ does not however support 2-adic fractality for smaller values of r ($r < 9$): about larger values one cannot say anything without heavy numerical calculations.

4 How quantum arithmetics affects basic TGD and TGD inspired view about life and consciousness?

The vision about real and p-adic physics as completions of rational physics or physics associated with extensions of rational numbers is central element of number theoretical universality. The physics in the extensions of rationals are assigned with the interaction of real and p-adic worlds.

1. At the level of the world of classical worlds (WCW) the points in the intersection of real and p-adic worlds are 2-surfaces defined by equations making sense both in real and p-adic sense. Rational functions with polynomials having rational (or algebraic coefficients in some extension of rationals) would define the partonic 2-surface. One can of course consider more stringent formulations obtained by replacing 2-surface with certain 3-surfaces or even by 4-surfaces.

2. At the space-time level the intersection of real and p-adic worlds corresponds to rational points common to real partonic 2-surface obeying same equations (the simplest assumption). This conforms with the vision that finite measurement resolution implies discretization at the level of partonic 2-surfaces and replaces light-like 3-surfaces and space-like 3-surfaces at the ends of causal diamonds with braids so that almost topological QFT is the outcome.

How does the replacement of rationals with quantum rationals modify quantum TGD and the TGD inspired vision about quantum biology and consciousness?

4.1 What happens to p-adic mass calculations and quantum TGD?

The basic assumption behind the p-adic mass calculations and all applications is that one can assign to a given partonic 2-surface (or even light-like 3-surface) a preferred p-adic prime (or possibly several primes).

The replacement of rationals with quantum rationals in p-adic mass calculations implies effects, which are extremely small since the difference between rationals and quantum rationals is extremely small due to the fact that the primes assignable to elementary particles are so large ($M_{127} = 2^{127} - 1$ for electron). The predictions of p-adic mass calculations remains almost as such in excellent accuracy. The bonus is the uniqueness of the canonical identification making the theory unique.

The problem of the original p-adic mass calculations is that the number of common rationals (plus possible algebraics in some extension of rationals) is same for all primes p. What is the additional criterion selecting the preferred prime assigned to the elementary particle?

Could the preferred prime correspond to the maximization of number theoretic negentropy for a quantum state involved and therefore for the partonic 2-surface by quantum classical correspondence? The solution ansatz for the modified Dirac equation indeed allows this assignment [4]: could this provide the first principle selecting the preferred p-adic prime? Here the replacement of rationals with quantum rationals improves the situation dramatically.

1. Quantum rationals are characterized by a quantum phase $q = exp(i2\pi/p)$ and thus by prime p (in the most general but not so plausible case by an integer n). The set of points shared by real and p-adic partonic 2-surfaces would be discrete also now but consist of points in the algebraic extension defined by the quantum phase $q = exp(i2\pi/p)$.

2. What is of crucial importance is that the number of common quantum rational points of partonic 2-surface and its p-adic counterpart would depend on the p-adic prime p. For some primes p

would be large and in accordance with the original intuition this suggests that the interaction between p-adic and real partonic 2-surface is stronger. This kind of prime is the natural candidate for the p-adic prime defining effective p-adic topology assignable to the partonic 2-surface and elementary particle. Quantum rationals would thus bring in the preferred prime and perhaps at the deepest possible level that one can imagine.

4.2 What happens to TGD inspired theory of consciousness and quantum biology?

The vision about rationals as common to reals and p-adics is central for TGD inspired theory of consciousness and the applications of TGD in biology.

1. One can say that life resides in the intersection of real and p-adic worlds. The basic motivation comes from the observation that number theoretical entanglement entropy can have negative values and has minimum for a unique prime [5]. Negative entanglement entropy has a natural interpretation as a genuine information and this leads to a modification of Negentropy Maximization Principle (NMP) allowing quantum jumps generating negentropic entanglement. This tendency is something completely new: NMP for ordinary entanglement entropy would force always a state function reduction leading to unentangled states and the increase of ensemble entropy.

   What happens at the level of ensemble in TGD Universe is an interesting question. The pessimistic view [5], [2] is that the generation of negentropic entanglement is accompanied by entropic entanglement somewhere else guaranteeing that second law still holds true. Living matter would be bound to pollute its environment if the pessimistic view is correct. I cannot decide whether this is so: this seems like deciding whether Riemann hypothesis is true or not or perhaps unprovable.

2. Replacing rationals with quantum rationals however modifies somewhat the overall vision about what life is. It would be quantum rationals which would be common to real and p-adic variants of the partonic 2-surface. Also now an algebraic extension of rationals would be in question so that the proposal would be only more specific. The notion of number theoretic entropy still makes sense so that the basic vision about quantum biology survives the modification.

3. The large number of common points for some prime would mean that the quantum jump transforming p-adic partonic 2-surface to its real counterpart would take place with a large probability. Using the language of TGD inspired theory of consciousness one would say that the intentional powers are strong for the conscious entity involved. This applies also to the reverse transition generating a cognitive representation if p-adic-real duality induced by the canonical identification is true. This conclusion seems to apply even in the case of elementary particles. Could even elementary particles cognize and intend in some primitive sense? Intriguingly, the secondary p-adic time scale associated with electron defining the size of corresponding CD is .1 seconds defining the fundamental 10 Hz bio-rhythm. Just an accident or something very deep: a direct connection between elementary particle level and biology perhaps?

5 Appendix: Some number theoretical functions

Explicit formulas for the number $r_k(n)$ of the solutions to the conditions $\sum_1^k x_k^2 = n$ are known and define standard number theoretical functions closely related to the quadratic algebraic extensions of rationals. The formulas for $r_k(n)$ require some knowledge about the basic number theoretical functions to be discussed first. Wikipedia contains a good overall summary about basic arithmetic functions [1] including the most important multiplicative and additive arithmetic functions.

Included are character functions which are periodic and multiplicative: examples are symbols (m/n) assigned with the names of Legendre, Jacobi, and Kronecker as well as Dirichlet character.

5.1 Characters and symbols

5.1.1 Principal character

Principal character [1] $\chi(n)$ distinguishes between three situations: n is even, $n = 1 \pmod 4$, and $n = 3 \pmod 4$ and is defined as

$$\chi(n) \quad = \quad \left(\frac{-4}{n}\right) = \begin{cases} 0 & \text{if n=0 (mod 2)} \\ +1 & \text{if } n = 1 \pmod 4 \\ -1 & \text{if } n = 3 \pmod 4 \end{cases} \tag{5.1}$$

Principal character is multiplicative and periodic with period $k = 4$.

5.1.2 Legendre and Kronecker symbols

Legendre symbol $\left(\frac{n}{p}\right)$ characterizes what happens to ordinary primes in the quadratic extensions of rationals. Legendre symbol is defined for odd integers n and odd primes p as

$$\left(\frac{n}{p}\right) \quad = \quad \begin{cases} 0 & \text{if } n = 0 \text{ (mod p)} \ , \\ +1 & \text{if } n \neq 0 \quad \text{(mod p) and } n = x^2 \text{ (mod p)} \ , \\ -1 & \text{if there is no such } x \ . \end{cases} \tag{5.2}$$

When D is so called fundamental discriminant- that is discriminant $D = b^2 - 4c$ for the equation $x^2 - bx + c = 0$ with integer coefficients b, c, Legendre symbols tells what happens to ordinary primes in the extension:

1. $\left(\frac{D}{p}\right) = 0$ tells that the prime in question divides D and that p is expressible as a square in the quadratic extension of rationals defined by $\sqrt{D}$.
2. $\left(\frac{D}{p}\right) = 1$ tells that p splits into a product of two different primes in the quadratic extension.
3. For $\left(\frac{D}{p}\right) = -1$ the splitting of p does not occur.

This explains why Legendre symbols appear in the ideal class number $h(D)$ characterizing the number of different splittings of primes in quadratic extension.

Legendre symbol can be generalized to Kronecker symbol well-defined for also for even integers D. The multiplicative nature requires only the definition of $\left(\frac{n}{2}\right)$ for arbitrary n:

$$\left(\frac{n}{2}\right) \quad = \quad \begin{cases} 0 & \text{if } n \text{ is even} \ , \\ (-1)^{\frac{n^2-1}{8}} & \text{if n is odd} \ . \end{cases} \tag{5.3}$$

Kronecker symbol for $p = 2$ tells whether the integer is even, and if odd whether $n = \pm 1 \pmod 8$ or $a = \pm 3 \pmod 8$ holds true. Note that principal character $\chi(n)$ can be regarded as Dirichlet character $\left(\frac{-4}{n}\right)$.

For $D = p$ quadratic resiprocity [14] allows to transform the formula

$$\chi_p(n) \quad = \quad (-1)^{(p-1)/2}(-1)^{(n-1)/2}\left(\frac{p}{n}\right) = (-1)^{(p-1)/2}(-1)^{(n-1)/2}\prod_{p_i|n}\left(\frac{p}{p_i}\right) \quad . \tag{5.4}$$

5.1.3 Dirichlet character

Dirichlet character [4] $\left(\frac{a}{n}\right)$ is also a multiplicative function. Dirichlet character is defined for all values of a and odd values of n and is fixed completely by the conditions

$$\chi_D(k) = \chi_D(k+D) \ , \qquad \chi_D(kl) = \chi_D(k)\chi_D(l) \ ,$$
$$\text{If } D|n \text{ then } \chi_D(n) = 0 \ , \quad \text{otherwise } \chi_D(n) \neq 0 \ . \tag{5.5}$$

Dirichlet character associated with quadratic residues is real and can be expressed as

$$\chi_D(n) \ = \ \left(\frac{n}{D}\right) = \prod_{p_i|D} \left(\frac{n}{p_i}\right) \ . \tag{5.6}$$

Here $\left(\frac{n}{p_i}\right)$ is Legendre symbol described above. Note that the primes p_i are odd. $\left(\frac{n}{1}\right) = 1$ holds true by definition.

For prime values of D Dirichet character reduces to Legendre symbol. For odd integers Dirichlet character reduces to Jacobi symbol defined as a product of the Legendre symbols associated with the prime factors. For $n = p^k$ Dirichlet character reduces to $((\frac{p}{n}))^k$ and is non-vanishing only for odd integers not divisible by p and containing only odd prime factors larger than p besides power of 2 factor.

5.2 Divisor functions

Divisor functions [6] $\sigma_k(n)$ are defined in terms of the divisors d of integer n with $d = 1$ and $d = n$ included and are also multiplicative functions. $\sigma_k(n)$ is defined as

$$\sigma_k(n) \ = \ \sum_{d|n} d^k \ , \tag{5.7}$$

and can be expressed in terms of prime factors of n as

$$\sigma_k(n) \ = \ \sum_i (p_i^k + p_i^{2k} + ... + p_i^{a_i k}) \ . \tag{5.8}$$

$\sigma_1 \equiv \sigma$ appears in the formula for $r_4(n)$.

The figures in Wikipedia [9] give an idea about the locally chaotic behavior of the sigma function.

5.3 Class number function and Dirichlet L-function

In the most interesting $k = 3$ case the situation is more complicated and more refined number theoretic notions are needed. The function $r_3(D)$ is expressible in terms of so called class number function $h(n)$ characterizing the order of the ideal class group for a quadratic extension of rationals associated with D, which can be negative. In the recent case $D = -p$ is of special interest as also $D = -kp$, especially so for $k = 2^r$. $h(n)$ in turn is expressible in terms of Dirichlet L-function so that both functions are needed.

1. Dirichlet L-function [5] can be regarded as a generalization of Riemann zeta and is also conjectured to satisfy Riemann hypothesis. Dirichlet L-function can be assigned to any Dirichlet character χ_D appearing in it as a function valued parameter and is defined as

$$L(s, \chi_D) \ = \ \sum_n \frac{\chi_D(n)}{n^s} \ . \tag{5.9}$$

For $\chi_1 = 1$ one obtains Riemann Zeta. Also L-function has expression as product of terms associated with primes converging for $Re(s) > 1$, and must be analytically continued to get an analytic function in the entire complex plane. The value of L-function at $s = 1$ is needed and for Riemann zeta this corresponds to pole. For Dirichlet zeta the value is finite and $L(1, \chi_{-n})$ indeed appears in the formula for $r_3(n)$.

2. Consider next what class number function h means.

   (a) Class number function [2] characterizes quadratic extensions defined by $\sqrt{D}$ for both positive and negative values of D. For these algebraic extensions the prime factorization in the ring of algebraic integers need not be unique. Algebraic integers are complex algebraic numbers which are not solutions of a polynomial with coefficients in Z and with leading term with unit coefficient. What is important is that they are closed under addition and multiplication. One can also defined algebraic primes. For instance, for the quadratic extension generated by $\sqrt{\pm 5}$ algebraic integers are of form $m+n\sqrt{\pm 5}$ since $\sqrt{\pm 5}$ satisfies the polynomial equation $x^2 = \pm 5$.

   Given algebraic integer n can have several prime decompositions: $n = p_1p_2 = p_3p_4$, where p_i algebraic primes. In a more advance treatment primes correspond to ideals of the algebra involved: obviously algebra of algebraic integers multiplied by a prime is closed with respect to multiplication with any algebraic integer.

   A good example about non-unique prime decomposition is $6 = 2\times 3 = (1+\sqrt{-5})(\sqrt{1-\sqrt{-5}}$ in the quadratic extension generated by $\sqrt{-5}$.

   (b) Non-uniqueness means that one has what might be called fractional ideals: two ideals I and J are equivalent if one can write $(a)J = (b)I$ where (n) is the integer ideal consisting of algebraic integers divisible by algebraic integer n. This is the counterpart for the non-uniquencess of prime decomposition. These ideals form an Abelian group known as ideal class group [10]. For algebraic fields the ideal class group is always finite.

   (c) The order of elements of the ideal class group for the quadratic extension determined by integer D can be written as

$$h(D) = \frac{1}{D}\sum_1^{|D|} r \times \left(\frac{D}{r}\right) \quad , \quad D < -4 \ . \tag{5.10}$$

   Here $\left(\frac{D}{r}\right)$ denotes the value of Dirichlet character. In the recent case D is negative.

3. It is perhaps not completely surprising that one can express $r_3(|D|)$ characterizing quadratic form in terms of $h(D)$ charactering quadratic algebraic extensions as

$$r_3(|D|) = 12(1-\left(\frac{D}{2}\right))h(D) \quad , \quad D < -4 \ . \tag{5.11}$$

   Here $\left(\frac{D}{2}\right)$ denotes Kronecker symbol.

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