On a Theoretical Proof of the Weak Equivalence Principle from within the Confines of Newtonian Gravitation

G. G. Nyambuya[†]

[†]*National University of Science & Technology, Faculty of Applied Sciences,*
School of Applied Physics, P. O. Box 939, Ascot, Bulawayo,
Republic of Zimbabwe.
Email: *physicist.ggn@gmail.com*

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Abstract

The great Italian scientist and philosopher, Galileo Galilee, is reported to have stood at the learning tower of Pisa in Italy and famously dropped objects of different masses (and compositions), thereby demonstrating that the motion of matter in a gravitational field is independent of the body's composition since these objects, despite their different masses (and compositions), their free-fall time was practically equal. By so doing, Galileo demolished a more than one and a half millennium of dogma that had been set forth by the Greek philosopher, Aristotle, namely that, in a gravitational field, heavier objects fall faster than lighter ones. Galileo's conclusion is a *posteriori* justified current *scientific dogma*, we all accept this as a durable fact of experience. Gently and modestly, this reading appears to furnish this hypothesis. We unambiguously demonstrate beyond any shadow of doubt that Newtonian gravitation implies that gravitational and inertial mass are equal.

1 Introduction

Popular physics legend told to aspirant and freshman physicists from one generation to the other holds that the great Italian scientist and philosopher, Galileo Galilee (1564 − 1642), famously stood at the learning tower of Pisa in Italy and dropped bodies of different masses (and compositions), thereby demonstrating that the motion of matter in a gravitational field is independent of the body's composition since these bodies, despite their different masses (and compositions), their free-fall time was practically equal. By so doing, the great Galileo demolished a more than one and half millennium dogma that had been set forth by the great Greek philosopher and polymath, Aristotle (384 BC − 322 BC), namely that, in a gravitational field, heavier objects fall faster than lighter ones. Galileo's conclusion is a *current scientific dogma*, we all accept this as a durable fact of experience existing beyond the point of debate or disagreement. No theoretical proof has ever been advanced to furnish this hypothesis. From within the domains and confines Newtonian gravitation, this reading modestly provides a theoretical proof to that end. Gently, we unambiguously demonstrate beyond any shadow or shred of doubt that, Newtonian gravitation directly and naturally implies that gravitational and inertial mass are equal. Thus, this equality seizes to be just an experimental fact, it becomes a theoretical fact as-well.

Providing a theoretical proof of the weak equivalence principle has far reaching implications on the very foundations of physics as one of the twin pillars of modern physics (General

Relativity) rests all if not the bulk of its weight on this principle. It would mean, the weak equivalence principle seizes to be a theory, but becomes a theorem within the structures of Newtonian gravitational physics. The weak equivalence principle is the statement that bodies of different mass and composition will – as first demonstrated by Galileo Galilee; fall at the same rate in a gravitational field. Current measurements on the equality of gravitational and inertial mass indicate that this equality holds on a level of one part to 10^{13} (see *e.g.* Will 2009; William *at al.* 2004). This has been taken as the clearest indication yet, that gravitational and inertial mass are equal if not identical in nature.

Given the implications on the foundations of physics of the ideas propagated herein, at this very point of the reading, allow us to clearly state that we do not want to be judge of our own work. As we submit this work to the judgement of a high scientific aeropag, we will do very little if anything in this reading to defend what we present, the facts must defend themselves within the consistency and internal structure of their own logic and coherence. We simple want to present facts as they stand and let our dear reader(s) be their own judge(s). This is especially important if one dares to delve onto matters that have the strong implications on the very foundations of physics. If the work is worthwhile – obviously – it will be clear at the outset to the esoteric and erudite.

Before delivering a class on the equality of inertial and gravitational mass, the intent of this class being to explain to the students the deeper and far reaching implications of this rather strange setup in *Nature*, we decided to go through the equations of Newtonian gravitation *albeit* with the important difference that we chose not to cancel out the ratio $\gamma = m_g/m_i$, we expected that somewhere somehow in the labyrinth of the calculation, this ratio will mysteriously vanish from the equations without revealing its true nature, to our delight, we ended up with equations that implied that this ratio must be unity! Invariably, this means that from a theoretical standpoint, inertia and gravitational mass are equal. We repeated this calculation over and over until we where convinced beyond the shadow and shred of doubt the of truth of this calculation. It is then that we strongly felt this result must be reported and communicated so that others may make their own analysis and independent judgement. We should say, to the best of our knowledge (or ignorance), no theoretical proof of this exists.

Judging from the tone of the reading so far, it is clear that this reading has been prepared in a manner such that it appeals at the advanced popular level in a mood that is easy to follow without any need for extra material *i.e.*, it is accessible to anyone with a basic understanding of freshman physics. This has been done so that one can easily *accept* or *refute* the conclusions drawn thereof. We accept the conclusion made herein and all we can do is hope that our readers will find it equally easy to accept this rather interesting conclusion that puts more concrete on the foundations of physics.

The synopsis of this reading is as follows. In the subsequent section, we go through an exposition were we define inertial and gravitational mass. In the section following this, we give another exposition, this time of Einstein's equivalence principle. These expositions are meant to put into perspective the idea that we seek to champion, mainly that inertial and gravitational mass are not equal, these differ from one body to the other. In section (4), we lay down the main idea of this reading. Before this is done, we first give an exposition of Newtonian gravitational albeit with the important difference that inertial and gravitational mass maintain their identity. It is seen in this section that, once one maintain's the identity of inertial and gravitational mass, the resulting equations invariably lead us to our sort for conclusion about the weak equivalence principle. In section (5), we give a general discussion and the conclusions drawn thereof.

2 Gravitational and Inertia Mass

As is well known, there is at least two distinct and important kinds of mass that enter Newtonian mechanics. The first is the *inertial mass* (m_i) which enters in Newton's second law of motion. As it was first stated by the great Sir Isaac Newton, this law states that the resultant of all the forces ($\boldsymbol{F}_{res}$) acting on a body is equal to the rate of change of motion of that body, *i.e.*:

$$\boldsymbol{F}_{res} = \frac{d\boldsymbol{p}}{dt} \quad \text{where,} \quad \boldsymbol{p} = m_i \boldsymbol{v}. \tag{1}$$

By motion, Newton meant the momentum $\boldsymbol{p}$ of the body in question. Momentum ($\boldsymbol{p}$) is the product of inertial mass (m_i) and the velocity ($\boldsymbol{v}$) of the body in question. In most cases considered in natural systems, the inertial mass of the object is a constant of motion, so this law is often stated as:

$$\boldsymbol{F}_{res} = m_i \boldsymbol{a} \quad \text{where,} \quad \boldsymbol{a} = \frac{d\boldsymbol{v}}{dt}. \tag{2}$$

The vector quantity $\boldsymbol{a}$ is the acceleration of the body in question.

The second kind of mass enters Newtonian mechanics in Newton's law of universal gravitation which states that the gravitational force drawing together two objects of gravitational mass $\mathcal{M}_g$ and m_g that are separated by a distance r is:

$$\boldsymbol{F}_g = -\frac{G\mathcal{M}_g m_g}{r^2}\hat{\boldsymbol{r}}, \tag{3}$$

where $G > 0$, is Newton's constant of universal gravitation and $\hat{\boldsymbol{r}}$ is the unit vector along the line joining the centers of mass of these objects and the negative sign is there to denote the fact that the gravitational force is a force of attraction.

As afore-stated, the weak equivalence principle due to Galileo states that test bodies fall with the same acceleration independent of their internal structure or composition: in other words, the gravitational mass appearing in (3) and inertial mass appearing (2) are the same *i.e.* $m_i \equiv m_g$. Throughout this reading, in order to distinguish between gravitational and inertial mass, we shall use the subscripts "i" and "g" respectively *i.e.* m_i and m_g. If by any chance the hypothesis $m_i \equiv m_g$ is true, then, this fact must and will come out clean from the resultant equations.

Formally, let us denote the ratio between the gravitational (m_g) and inertial mass (m_i) by:

$$\gamma = \frac{m_g}{m_i}. \tag{4}$$

From this definition, it follows that the acceleration of an object in a gravitational field which is obtained by setting $\boldsymbol{F}_{res} = \boldsymbol{F}_g$ and then dividing the resultant equation throughout by m_i, is given by:

$$\boldsymbol{a} = -\frac{\gamma G\mathcal{M}_g}{r^2}\hat{\boldsymbol{r}}. \tag{5}$$

As already stated, under the Newtonian scheme $\gamma \equiv 1$. This is a fundamental assumption of Newtonian mechanics. Although he made the first attempts to verify this from experience (see *e.g.* Capri 2005), Newton never bothered to explain this important and crucial assumption. Judging from his writings as recorded in his great master piece *"Philosophiae Naturalis Principia Mathematica"*, Newton (1726) was a very careful man who believed in facts derived from experience. His famous statement *"Hypotheses non fingo."*[1] captures very well Newton as a man that strongly believed in facts of experience. Perhaps, because this fact was a fact of experience, he saw no need to explain it, but to simple take it as experience dictates. It was Einstein (1907) that made the first attempt to explain this. He [Einstein] used this as a starting point to seek his GTR. Here in, we shall drop this assumption *i.e.* $\gamma \equiv 1$ and shortly, we shall show that – from the maths emerging from Newtonian gravitation; this hypothesis that $\gamma \equiv 1$, does hold true, it can be derived from within the confines and domains of Newtonian gravitation.

To arrive at the conclusion of this reading, we are going to deal with two important quantities, the kinetic $K(r)$ and gravitational potential energy $U(r)$. These two quantities have associated with them some mass. The kinetic energy is associated with inertial mass. To see this, we simple visit the definition of kinetic energy, it is:

$$K(r) = \int_r^\infty \boldsymbol{F}_{res} \cdot d\boldsymbol{r} = \frac{1}{2} m_i v^2, \qquad (6)$$

hence the kinetic energy is associate with inertial mass. At infinity, the kinetic energy may be zero or finite, in the above, we have assumed $K(\infty) = 0$. The gravitational potential energy is simple the gravitational potential multiplied by the gravitational charge which is the gravitational mass, *i.e.*:

$$U(r) = \int_r^\infty \boldsymbol{F}_g \cdot d\boldsymbol{r} = -\frac{G\mathcal{M}_g m_g}{r}. \qquad (7)$$

By definition $U(\infty) = 0$.

To test the weak equivalence principle, one meticulously compares the accelerations of two bodies with different compositions in an external gravitational field. Such experiments are often called Eötvös experiments after Baron Ronald *von* Eötvös ($1813 - 1871$), the Hungarian physicist whose pioneering experiments with torsion balances provided a foundation for making these tests Eötvös (1890). These tests are assumed to vindicate Einstein's ideas on general relativity. The best test of the weak equivalence principle to date has been performed by Eric Adelberger and the Eöt-Wash collaboration at the University *of* Washington in Seattle, who have used an advanced torsion balance to compare the accelerations of various pairs of materials toward the Earth (see *e.g.* Will 2009). As afore-stated in the introductory section, their accuracy is one part to about 10^{13}. This accuracy is taken as the clearest indication yet, that γ should be unity, or equal for all material bodies in the Universe. In this case, the equivalence principle is not just a Principle *of Nature*, but a true Law *of Nature*.

3 Einstein's Principle of Equivalence

At the very heart and nimbus of Einstein's GTR is the *Einstein's Equivalence Principle* (EEP); this is an idea that came to Einstein in a 1907 epiphany while writing an invited

[1] This is a Latin statement which translates to "*I feign no hypotheses*", or "*I contrive no hypotheses*"

review article on his STR. This was only two years after he had developed the STR (Einstein 1907). In an article on the fundamental ideas of relativity theory, written in 1920, Einstein recalled his epiphany in greater detail:

> *"When I was busy (in* 1907*) writing a summary of my work on the theory of special relativity, I also had to try to modify the Newtonian theory of gravitation such as to fit its laws into the theory ... At that moment I got the happiest thought of my life in the following form: The gravitational field has a relative existence only in a manner similar to the electric field generated by magneto-electric induction. Because for an observer in free-fall from the roof of a house there is during the fall – at least in his immediate vicinity – no gravitational field. This is to say, if the observer lets go of any bodies, they remain, relative to him, in a state of rest or uniform motion ..."*

This idea led him to the melodramatic conclusion that mass and gravity are intimately linked; it would take Einstein another 10 years of one the greatest intellectual struggles of 20^{th} century physics by a single mind before he finally arrived at the correct link between gravitation and the geometry of spacetime. Off cause, Newton was the first to show a link between gravitation and mass, but Einstein went further, he reasoned that gravitation was not only linked to mass, but to the geometry of spacetime, in his conclusions – ultimately, mass curves spacetime and a curved spacetime tells us of the amount and distribution of mass causing this curvature. To arrive at this conclusion that gravitation is part and parcel of the fabric of spacetime, in his own words as written in the same 1920 article mentioned above, this is how Einstein reasoned:

> *"One can also start with a space that has no gravitational field. A material point in this space, when sufficiently distant from other masses, behaves free of acceleration relative to an inertial system K. However, if one introduces a uniformly accelerated coordinate system K' relative to K (uniformly accelerated parallel translation), we can also view K' as an admissible system (at rest) and attribute the acceleration of masses relative to K' to a static gravitational field that fills the entire space that is under consideration."*

In the 1907 review article, this is what Einstein had to say:

> *"We consider two systems S_1 and S_2 ... Let S_1 be accelerated in the direction of its $x-axis$, and let g be the (temporally constant) magnitude of that acceleration. S_2 shall be at rest, but it shall be located in a homogeneous gravitational field that imparts to all objects an acceleration $-g$ in the direction of the $x-axis$. As far as we know, the physical laws with respect to S_1 do not differ from those with respect to S_2; this is based on the fact that all bodies are equally accelerated in a gravitational field. At our present state of experience we have thus no reason to assume that the systems S_1 and S_2 differ from each other in any respect, and in the discussion that follows we shall therefore assume the complete physical equivalence of a gravitational field and a corresponding acceleration of the reference system."*

Put in simple terms, the equivalence principle states that gravity and acceleration are equivalent, they are different manifestations of the same phenomena. In an environment free from gravitation, one can generate a gravitation field using inertial acceleration, and in retrospection, the effects of a gravitational acceleration is the same as that of an inertial acceleration. This idea, embellished over the years (since its conception in 1907), it has come to be known as Einstein's Equivalence Principle. Further, this idea encompasses three separate principles (see *e.g.* Will 2009): the weak equivalence principle, and the principles of Local Lorentz Invariance and Local Position Invariance, that is:

1. Test bodies fall with the same acceleration independently of their internal structure or composition. This is the Weak Equivalence Principle (WEP) first set into motion by Galileo's famous experiment at the learning tower of Pisa and the full significance of this insofar as the Fundamental Laws *of Nature* are concerned was realised by Einstein.

2. The outcome of any local non-gravitational experiment is independent of the velocity and acceleration of the freely-falling reference system in which it is performed. This is Local Lorentz Invariance (LLI).

3. The outcome of any local non-gravitational experiment is independent of where and when in the Universe it is performed. This is Local Position Invariance (LPI).

The equivalence principle naturally casts the gravitational field into a metric theory (see *e.g.* Will 2009), thus justifying Einstein's GTR.

4 Newtonian Gravitation Revisited

We shall derive here the equations of motion that emerge from Newton's gravitational theory. What makes this exercise important is that the gravitational and inertial mass keep their identity. We know that in polar coordinates $(\hat{\boldsymbol{r}}, \hat{\boldsymbol{\theta}})$, the force acting on an object:

$$\boldsymbol{F}_{res} = m_i \boldsymbol{a} = m_i(\ddot{r} - r\dot{\varphi}^2)\hat{\boldsymbol{r}} + m_i(r\ddot{\varphi} + 2\dot{r}\dot{\varphi})\hat{\boldsymbol{\theta}}, \tag{8}$$

(see any good textbook on Classical Mechanics) where a single dot represents the time derivative d/dt and likewise a double dot presents the second time derivative d^2/dt^2. The gravitational force:

$$\boldsymbol{F}_g = m_g \boldsymbol{g} = -m_g \boldsymbol{\nabla}\Phi(r), \tag{9}$$

where $\Phi(r) = -G\mathcal{M}_g/r$ is the Newtonian gravitational potential. According to Newton's second law $m_i\boldsymbol{a} = -m_g\boldsymbol{\nabla}\Phi(r)$, this implies that, $\boldsymbol{a} = \gamma\boldsymbol{g}$. Comparison of the components of $\boldsymbol{a} = (\ddot{r} - r\dot{\varphi}^2)\hat{\boldsymbol{r}} + (r\ddot{\varphi} + 2\dot{r}\dot{\varphi})\hat{\boldsymbol{\theta}}$ with $\gamma\boldsymbol{g}$, leads us to the equations:

$$\frac{d^2r}{dt^2} - r\left(\frac{d\varphi}{dt}\right)^2 = -\gamma\frac{d\Phi}{dr} = -\frac{\gamma G\mathcal{M}}{r^2}, \tag{10}$$

for the $\hat{\boldsymbol{r}}$-component and for the $\hat{\boldsymbol{\theta}}$-component we will have:

$$r\frac{d^2\varphi}{dt^2} + 2\frac{dr}{dt}\frac{d\varphi}{dt} = -\frac{\gamma}{r}\frac{d\Phi}{d\theta} = 0. \tag{11}$$

Now, taking equation (11) and dividing throughout by $r\dot{\varphi}$ one we will be led to $dJ/dt = 0$, where $J = r^2\dot{\varphi}$ is the specific angular momentum *i.e.*, the angular momentum per unit mass. This equation tells us immediately that the specific orbital angular momentum is a conserved quantity. This is a central result of Newtonian gravitation.

Now, if we make the transformation $u = 1/r$, then for $\dot{r}$ and $\ddot{r}$ we will have:

$$\frac{dr}{dt} = -J\frac{du}{d\varphi} \text{ and } \frac{d^2r}{dt^2} = -\frac{dJ}{dt}\frac{du}{d\varphi} - J^2u^2\frac{d^2u}{d\varphi^2}, \tag{12}$$

respectively. Inserting these into (10) and then dividing the resultant equation by "$-u^2 J$" and remembering (15) and also that $dr = -du/u^2$ and $r\dot{\varphi}^2 = u^3 J^2$, one is led to the result:

$$\frac{d^2u}{d\varphi^2} + u = \frac{\gamma G \mathcal{M}_g}{J^2}. \tag{13}$$

A solution to this equation gives the Newtonian equation for orbits, this solution is:

$$\frac{l}{r} = 1 + \epsilon \cos\varphi, \tag{14}$$

where l is the *semi-latus rectum* of the orbit (see *e.g.* Kibble 2009), it is a constant for the orbit and is the distance between the parent object and the test body when $\varphi = \frac{1}{2}\pi$ and ϵ is the eccentricity of the orbit. The orbit and its geometrical characteristics are shown in the schematic diagram presented in figure (1).

Now, if one were to substitute (14) into (13), they would obtain the fundamental result:

$$J^2 = \gamma G \mathcal{M}_g l. \tag{15}$$

This equation is very important in arriving at the result that we seek – this, the reader must keep in mind. What makes this equation subtly different is the appearance of the γ-factor, in Newtonian gravitation, this does not appear at all since $\gamma = 1$.

Now, pertaining the eccentricity ϵ, from the fore-knowledge of the maximum and minimum distance of a planet from its parent gravitating body, $\mathcal{R}_{max}$ and $\mathcal{R}_{min}$ respectively, or from the major and minor axis of the orbit, $\mathcal{R}_{mj}$ and $\mathcal{R}_{mn}$ respectively, one can compute the eccentricity of the orbit from the formulae:

$$\epsilon = \frac{\mathcal{R}_{max} - \mathcal{R}_{min}}{\mathcal{R}_{max} + \mathcal{R}_{min}}, \quad \text{or} \quad \epsilon = 1 - \frac{\mathcal{R}_{mn}^2}{\mathcal{R}_{mj}^2}. \tag{16}$$

At this point, it would seem as though once Newton's inverse square law had arrived at the (13), Newton's theory had explained the existence of Kepler's elliptical orbits. We shall show that Newton's theory under the assumption $\gamma \equiv 1$ can not account for planetary orbits. To explain elliptical, parabolic and hyperbolic orbits, this requires that $\gamma \neq 1$; that is, the weak equivalence principle must and can not hold.

Now, we must find a scheme from within Newtonian gravitation, a scheme that predicts the eccentricity of the orbit from orbital parameters such as the mass and the total energy of the orbiting gravitating body. Equation (16) is not a fore-hand prediction as this would require one to know forehand the values of $\mathcal{R}_{max}$, $\mathcal{R}_{min}$, $\mathcal{R}_{mj}$ and $\mathcal{R}_{mn}$.

To find a predictive equation, we have to turn our eyes to the energy equation, namely $K(r) + U(r) = E$, where $K(r) = \frac{1}{2} m_i v^2$ is the kinetic energy of the particle, and $U(r) = -G\mathcal{M}_g m_g / r$ is the gravitational potential energy of the particle inside the gravitational field and E is the total energy of the particle. The square of the speed is given by $v^2 = \dot{r}^2 + r^2\dot{\varphi}^2 = \dot{r}^2 + J^2/r^2$, so that written in terms of the transformation $u = 1/r$, we have:

$$K(r) = \frac{m_i J^2}{2}\left[\left(\frac{du}{d\varphi}\right)^2 + u^2\right] \text{ and } U(r) = -G\mathcal{M}_g m_g u. \tag{17}$$

Inserting these into the equation $K(r) + U(r) = E$, and then dividing the result equation by $m_i J^2$, one is led to:

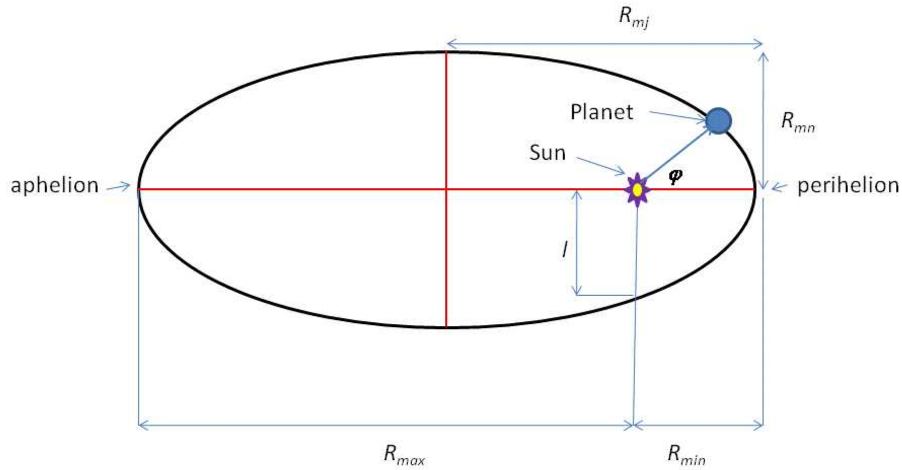

Figure (1): This diagram gives a *birds-eye-view* of the planetary orbit. The orbit is an ellipse with the Sun at one of the foci. The minor and major axis are represented by $\mathcal{R}_{mn}$ and $\mathcal{R}_{mj}$ respectively. The minimum and maximum distance of the planet from the Sun are $\mathcal{R}_{min}$ and $\mathcal{R}_{max}$ respectively. The distance $l = (1+\epsilon)\mathcal{R}_{min}$ is the distance of the planet away from the Sun when $\varphi = \frac{1}{2}\pi$: ϵ is the eccentricity of the orbit.

$$l^2\left(\frac{du}{d\varphi}\right)^2 + l^2u^2 - 2lu = \frac{2El^2}{m_iJ^2}. \tag{18}$$

Now, applying equation (15) to the right hand side of the above equation *i.e.* $l = J^2/\gamma G\mathcal{M}_g$, one is led to:

$$l^2\left(\frac{du}{d\varphi}\right)^2 + l^2u^2 - 2lu = \frac{2EJ^2}{\gamma^2G^2\mathcal{M}_g^2m_i}, \tag{19}$$

and now adding "1" on both sides so that we can complete the square on the left hand side, *i.e.* $z^2 = l^2u^2 - 2lu + 1$, we will have:

$$l^2\left(\frac{du}{d\varphi}\right)^2 + l^2u^2 - 2lu + 1 = 1 + \frac{2EJ^2}{\gamma^2G^2\mathcal{M}_g^2m_i} = \epsilon^2. \tag{20}$$

The right hand side of this equation is a constant, and is the eccentricity of the orbit. This is what we wanted to achieve to derive, that is, the eccentricity in terms of E, J and other physical parameters. Now, setting $z = lu - 1$, this implies $ldu/d\varphi = dz/d\varphi$, so that the above reduces to:

$$\left(\frac{dz}{d\varphi}\right)^2 + z^2 = 1 + \frac{2EJ^2}{\gamma^2G^2\mathcal{M}_g^2m_i} = \epsilon^2. \tag{21}$$

In this way we have derived the formula of the eccentricity in terms of forehand knowledge of the orbit, that is:

$$\epsilon^2 = 1 + \frac{2EJ^2}{\gamma^2 G^2 \mathcal{M}_g^2 m_i}. \quad (22)$$

The solution to (21) is $z = \epsilon \cos\varphi$ and this leads to the same solution as (14).

Now, the solution (22) with $\gamma \equiv 1$ is quoted in all textbooks of physics that deal with Newtonian gravitation as being the equation from which the eccentricity can be known from forehand knowledge of the properties of the orbiting bodies. It is from this equation that we shall *derive our very important result*!

From this solution *i.e.* (22) and the equation of the orbit (14), one can show that if $(\epsilon > 0)$, the resultant orbit is hyperbolic, this corresponds to $(E > 0)$; if $(\epsilon = 0)$, the resultant orbit is circular, this corresponds to $(E = 0)$; if $(\epsilon = 1)$, the resultant orbit is parabolic, this corresponds to $(E > 0)$; and if $(0 < \epsilon < 1)$, the resultant orbit is elliptical, this corresponds to $(E < 0)$. So, the total energy of the test body determines the nature of the orbit. The new result to be derived in a short-while is that, the ratio γ, also determines the nature of the orbit.

Now, what we shall do is to evaluate the total energy of the test body $E = K(r) + U(r)$. First things first, the energy is considered a constant of motion, *i.e.*, it is the same at all points of the orbit. We know that: $E(r) = \frac{1}{2}m_i v^2 - G\mathcal{M}_g m_g/r$. We also know that $v^2 = \dot{r}^2 + J^2/r^2$, inserting this into this equation, we are invariably led to:

$$E(r) = \frac{1}{2}m_i \left(\dot{r}^2 + \frac{J^2}{r^2} - \frac{2\gamma G\mathcal{M}_g r}{r^2}\right). \quad (23)$$

Now, using the fact that the total energy is a constant of motion, we can evaluate the energy at any given point of the orbit and it will have that same value everywhere. We shall choose the point $r = \mathcal{R}_{min}$ because at this point $\dot{r} = 0$. The fact that at $r = \mathcal{R}_{min}$ we must have $\dot{r} = 0$, this should not be difficult to see since $\dot{r} \propto \sin\varphi$ and given that at $r = \mathcal{R}_{min}$, we have $\varphi = 0$, hence result. We also know from (15) that $J^2 = \gamma G\mathcal{M}l$ everywhere along the orbit. Plugging these facts into the above equation, one is led to:

$$E(\mathcal{R}_{min}) = -\frac{(2\gamma - 1 - \epsilon)\gamma G\mathcal{M}_g m_i}{2\mathcal{R}_{min}}. \quad (24)$$

In (24), notice the stubborn insistence and resistance of the γ-factor, it won't just go. In the derivation of (24), one aught to remember that $l/\mathcal{R}_{min} = 1 + \epsilon$. Now, plugging this, *i.e.* $E(\mathcal{R}_{min})$, into (22), one is led to:

$$\epsilon^2 = 1 + (1 + \epsilon - 2\gamma)(1 + \epsilon). \quad (25)$$

Removing the brackets in the right handside, and simplifying, one is led to *our very important, awaited and sought for result*, namely that:

$$(\gamma - 1)(\epsilon + 1) = 0 \implies (\gamma - 1) = 0 \text{ and or } (\epsilon + 1) = 0. \quad (26)$$

The solutions to this equation are $(\gamma = 1)$ (and) or $(\epsilon = -1)$. The latter solution $(\epsilon = -1)$ is not physical whereas the former $(\gamma = 1)$ is. This means that the only solution that could make sense is $(\gamma = 1)$. That said, what makes this calculation important is the fact that $(\gamma = 1)$ is, in both Newtonian and Einsteinian gravitation, an experimental fact and not a fact derived from the theory itself. Here, we have derived this result from the soils of theoretical physics, thus giving impetus and much weight to the experiments that have been carried out to try and verify this hypothesis.

5 Discussion and Conclusion

In our modest and humble of opinion, we have shown that, from its own internal logic and consistency, Newton's theory implies that the WEP holds exactly. This puts flesh to the experiments that have been conducted to furnish this hypothesis. Einstein never proved this, but merely used this fact from the standpoint of experience to build his GTR. Newton conducted inconclusive experiments on the matter but generally believed in the validity of the equality of gravitational and inertia mass. Here, we have shown from the very soils of theoretical physics that this experimental fact is also a theoretical fact. As far as we are concerned, there exists no theoretical prove to this effect.

At this point as we close this reading, allow us to say that, we are currently working on what appears to be a new gravitational paradigm emerging from the higher symmetry solutions of he Poission-Laplace equation, namely $\boldsymbol{\nabla}^2\Phi = 4\pi G\rho$ (see Nyambuya 2010). If our current judgement were to prove in the long-run to be correct, this equality of gravitational and inertia mass appears not to hold, thus, this proof, may very much be restricted to Newtonian gravitation which is nothing but a spherical symmetric solution of the Poission-Laplace theory *i.e.* $\boldsymbol{\nabla}^2\Phi(r) = 4\pi G\rho(r)$. We do not want to say much, lest our final findings turn out to have a result contrary our current incomplete position.

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