The Hilbert book model.

in concise format.

The HBM is a simple Higgsless model of fundamental physics that is strictly based on the axioms of traditional quantum logic. It uses a sequence of instances of an extension of a quaternionic separable Hilbert space that each represents a static status quo of the whole universe.

The Hilbert Book Model

THE HILBERT BOOK MODEL

This book is still in preparation

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ACKNOWLEDGEMENTS

The nice thing about laws of physics is that they repeat themselves. Otherwise they would not be noticed. The task of physicists is to notice the repetition.

Contents

Preface

This book presents a concise and upgraded version of another book that also treats the Hilbert Book Model, but that in addition acts as a grab-bag of other subjects that are more or less related to the Hilbert Book Model. Only the tale is kept as well. It is contained in part IV of this book.

I started the Hilbert Book Model during my studies in physics in the sixties on the Technical University of Eindhoven (TUE).

In the first two years the lectures concerned only classical physics. In the third year quantum physics was introduced. I had great difficulty in understanding why the methodology of doing physics changed drastically. So I went to the teacher, which was an old nearly retired professor and asked him:

"Why is quantum mechanics done so differently from classical mechanics?".

His answer was short. He stated":

"The reason is that quantum mechanics is based on the superposition principle".

I quickly realized that this was part of the methodology and could not be the reason of the difference in methodology. So I went back and told him my concern. He told me that he could not give me a better answer and if I wanted a useful answer I should research that myself. So, I first went to the library, but the university was quite new and its library only contained rather old second hand books, which they got from other institutions. Next I went to the city's book shops. I finally found a booklet from P. Mittelstaedt: (Philosophische Probleme der modernen Physik, BI Hochschultaschenbücher, Band 50, 1963) that contained a chapter on quantum logic. I concluded that this produced the answer that I was looking for. Small particles obey a kind of logic that differs from classical logic. As a result their dynamic behavior differs from the behavior of larger objects.

I searched further and encountered papers from Garret Birkhoff and John von Neumann that explained the correspondence between quantum logic and separable Hilbert spaces. That produced an acceptable answer to my question.

The lectures also told me that observables were related to eigenvalues of Hermitian operators. These eigenvalues are real numbers. However, it was clearly visible that nature has a 3+1D structure. So I tried to solve that discrepancy as well. After a few days of puzzling I discovered a new number system that had this 3+1D structure and I called them compound numbers. I went back to my professor and asked him why such compound numbers were not used in physics. Again he could not give a reasonable answer.

When I asked the same question to a much younger assistant professor he told me that these numbers were discovered more than a century earlier by William Rowan Hamilton when he was walking with his wife over a bridge in Dublin. He was so glad about his discovery that he carved the formula that treats the multiplication of these numbers into the sidewall of the bridge. The inscription has faded away, but it is now molded in bronze and fixed to the same wall. The numbers are known as quaternions. So, I went to the library and searched for papers on quaternions.

In those years C. Piron wrote his papers on quaternionic Hilbert spaces. That information completed my insight in this subject. I finalized my physics study with an internal paper on quaternionic Hilbert spaces.

The university was specialized in applied physics and not in theoretical physics. This did not stimulate me to proceed with the subject. Next, I went into a career in industry where I used my knowledge of physics in helping to analyze intensified imaging and in assisting with the design of night vision equipment and X-ray image intensifiers. That put me with my nose on the notion of quanta.

The image intensifiers did not show radiation. Instead they showed clouds of impinging quanta. In those times I had not much opportunity to deliberate on that fact. However, after my retirement I started to rethink the matter. That was the instant that the Hilbert Book Model continued further.

Thus, in a few words: The Hilbert Book Model tries to explain the existence of quanta. It does that by starting from traditional quantum logic.

The Hilbert Book Model is a Higgsless model of physics. It explains how elementary particles get their mass.

Before you start reading the book, as a challenge, you may try to solve the QPAD game. It is located in part V.

PART I

The fundaments

Abstract

The fundaments of quantum physics are still not well established. This book tries to find the cracks in these fundaments and suggests repair procedures. This leads to unconventional solutions and a new model of physics. The model is strictly based on the axioms of traditional quantum logic. However, in order to proceed from that point, it is necessary to extend this base, such that it also incorporates the equivalents of physical fields. This results in a model that can represent a static status quo of the whole universe.

The most revolutionary introduction is the representation of dynamics by a sequence of such static models in the form of a sequence of extended separable Hilbert spaces. Together, this embodies a repair of fundaments that does not affect the building.

The model reveals the reason of existence of fields. This starts with fields that nature uses in order to couple observations to the items that are observed. These fields are quaternionic probability amplitude distributions (QPAD's). The coupling of QPAD's is the reason of existence of particles.

The properties of the particles act as sources for another kind of fields. These are the fields that are known as physical fields. These secondary fields act in the next level of binding.

In this way the QPAD's and the elementary particles form the fundament of the whole building that we call physics. The QPAD's also form the base of the explanation of the origins of curvature and inertia.

1 FUNDAMENTS

The most basic fundaments consist of quantum logic, its lattice isomorphic companion, the separable Hilbert space and the extensions of these basic models such that they incorporate fields.

1.1 Logic model

The author has decided to base the Hilbert Book Model on a consistent set of axioms. It is often disputed whether a model of physics can be strictly based on a set of axioms. Still, what can be smarter than founding a model of physics on the axioms of classical logic?

Since in 1936 John von Neumann[1] wrote his introductory paper on quantum logic the scientific community knows that nature cheats with classical logic. Garret Birkhoff and John von Neumann showed that nature is not complying with the laws of classical logic. Instead nature uses a logic in which exactly one of the laws is weakened when it is compared to classical logic. As in all situations where rules are weakened, this leads to a kind of anarchy. In those areas where the behavior of nature differs from classical logic, its composition is a lot more complicated. That area is the site of the very small items. Actually, that area is in its principles a lot more fascinating than the cosmos. The cosmos conforms, as far as we know, nicely to classical logic. In scientific circles the weakened logic that is discussed here is named traditional quantum logic.

As a consequence the Hilbert Book Model will be strictly based on the axioms of traditional quantum logic. However, this choice immediately reveals its constraints. Traditional quantum logic is not a nice playground for the mathematics that characterizes the formulation of most physical laws. Lucky enough, von Neumann encountered the same problem and together with Garret Birkhoff[2] he detected that the set of propositions of quantum logic is lattice isomorphic with the set of closed subspaces of an infinite dimensional separable Hilbert space. Some decades later Constantin Piron[3] proved that the inner product of the Hilbert space must by defined by numbers that are taken a division ring. Suitable division rings are the real numbers, the complex numbers and the quaternions[4]. The Hilbert Book Model takes the choice with the widest possibilities. It uses quaternionic Hilbert spaces. Quaternions play a decisive role in the Hilbert Book Model. Higher dimension hyper-complex numbers may suit as eigenvalues of operators or as values of physical fields, but for the moment the HBM can do without these numbers. Instead, quaternions will be used for those purposes.

So, now we have a double model that connects logic with a flexible mathematical toolkit. But, this solution does not solve all restrictions. Neither quantum logic nor the separable Hilbert space can handle physical fields and they also cannot handle dynamics.

This is exposed by the fact that the Schrödinger picture and the Heisenberg picture are both valid views of quantum physical systems, despite the fact that these views attribute the time pa-

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[1] http://en.wikipedia.org/wiki/John_von_Neumann#Quantum_logics
[2] http://en.wikipedia.org/wiki/John_von_Neumann#Lattice_theory
[3] C. Piron 1964; _Axiomatique quantique_
[4] http://en.wikipedia.org/wiki/Quaternion

rameter to different actors. It means that the progression parameter is a characteristic of the whole representation.

An extension of the basic models helps cure the first restriction. In addition it also solves another problem. The separable Hilbert space only tolerates eigenspaces of operators that contain a countable number of eigenvalues. Thus, the Hilbert space does not know continuums. A solution can be found in the Gelfand triple[5] of the Hilbert space. This sandwich contains the Hilbert space as a member and at the same time it provides operators that have a continuum as their eigenspace.

This would introduce a new problem because the continuum fits far more eigenvalues than the eigenspace of the operator in the separable Hilbert space can offer. This problem makes think of a similar situation that occurs when a number N of linear equations must be solved, while N is larger than the number M of variables that are contained in these equations. Let the equations be available in the form:

$$\sum_{m=1}^{M} a_{mn} x_{mn} = b_n \; ; n = 1 \cdots N$$

Usually, such situations are solved by assuming that a stochastic inaccuracy exists between the values b_n in the too large result set $\{b_n\}_{n=1..N}$ and the actual results $\{c_n\}_{n=1..N}$. The actual results would offer a consistent set of equations that can be reduced to M equations. With other words these actual results make the original equations interdependent.

Translated to our problem, this solution comes down to linking the eigenvalues of operators in the separable Hilbert space to corresponding values in the continuum eigenspace of operators in the Gelfand triple by using a probability amplitude distribution as the connection between the source and the target[6]. The HBM chooses a quaternionic probability amplitude distribution (QPAD), rather than a complex probability amplitude distribution (CPAD). As will be shown, the choice for a QPAD has significant and favorable consequences.

The attachment of QPAD's extends the separable Hilbert space and connects it in a special way to its Gelfand triple. Due to the isomorphism of the lattice structures, the quantum logic is extended in a similar way. This leads to a reformulation of quantum logic propositions that makes them incorporate stochastically inaccurate observations instead of precise observations. The logic that is extended in this way will be called extended quantum logic. The separable Hilbert space that is extended in this way will be called extended separable Hilbert space.

The implementation of physical fields via the attachment of QPAD's to eigenvectors in the separable Hilbert space is a crucial departure from common physical methodology. Common quantum physics uses complex probability amplitude distributions (CPAD's), rather than QPAD's[7]. Quantum Field Theory[8], in the form of QED[9] or QCD[10], implements physical fields in a quite different manner.

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[5] http://en.wikipedia.org/wiki/Gelfand_triple

[6] The particle acts as the source of information, while the observer is the receiver.

[7] http://en.wikipedia.org/wiki/Probability_amplitude

The choice for QPAD's appears very advantageous. The real part of the QPAD can be interpreted as a "charge" density distribution. Similarly the imaginary part of the QPAD can be interpreted as a "current" density distribution. The squared modulus of the value of the QPAD can be interpreted as the probability of the presence of the carrier of the "charge". The "charge" can be any property of the carrier or it represents a collection of the properties of the carrier. In this way, when the wave function is represented by a QPAD the equation of motion becomes a continuity equation[11].

Since the QPAD's attach Hilbert eigenvectors to a continuum the carriers can be interpreted as tiny patches of that continuum. The transport of these patches can be responsible for the local compression or decompression of the continuum space. With other words, via this mechanism QPAD's may influence the local curvature.

QPAD's use a quaternion as their parameter. Usually these parameters are values of a quaternionic distribution that uses the quaternionic number space as its parameter space. In that way the quaternionic distribution can act as a curved coordinate system. The quaternionic distribution represents the parameter space of the QPAD.

In most cases where quaternionic distributions are used, the fact that quaternions possess two independent sign selections is ignored. The first sign selection, the conjugation, inverts the sign of all three imaginary base vectors. The second sign selection, the reflection, inverts the sign of a single imaginary base vector. The reflection can be taken in three independent directions. When relevant these directions are color coded. The sign selections in a quaternionic distribution are all similar. Individually, the conjugation and the reflection switch the handedness of the external vector product in the product of two quaternions that are taken from the same quaternionic distribution. The sign selection of the parameter space is usually taken as the reference for the sign selections of the quaternionic distributions. When a quaternionic distribution has the same sign selection as its parameter space has, then it will be called a base quaternionic distribution. Eight sign flavors correspond with each base quaternionic distribution. This adds a significant amount of functionality to quaternionic distributions.

For each QPAD, the mixture of conjugation and colored reflections produces eight different sign flavors. When colors are neglected, still four different flavors result. In quantum physics these sign flavors play a crucial role. In conventional physics this role is hidden in complex probability amplitude distributions (CPAD's), alpha, beta and gamma matrices and in spinors. Colors are added as an aftermath.

The extension of the primitive models cures several restrictions including the lack of support for fields, but one is left. Both the extended quantum logic and the extended separable Hilbert space can only represent a static status quo. The Hilbert space does not have an operator that de-

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[8] http://en.wikipedia.org/wiki/Quantum_field_theory

[9] http://en.wikipedia.org/wiki/Quantum_electrodynamics

[10] http://en.wikipedia.org/wiki/Quantum_chromodynamics

[11] Also called balance equation.

livers progression as an eigenvalue. Instead progression can be attached as a parameter to the whole Hilbert space including the Gelfand triple and the attached QPAD's.

Now implementing dynamics becomes a simple action. The whole Hilbert Book Model consists of an ordered sequence of sandwiches that each includes the Gelfand triple including its Hilbert space and the attached QPAD's. The progression parameter acts as page number of the book.

In the resulting Hilbert Book Model the progression is made in universe wide steps.
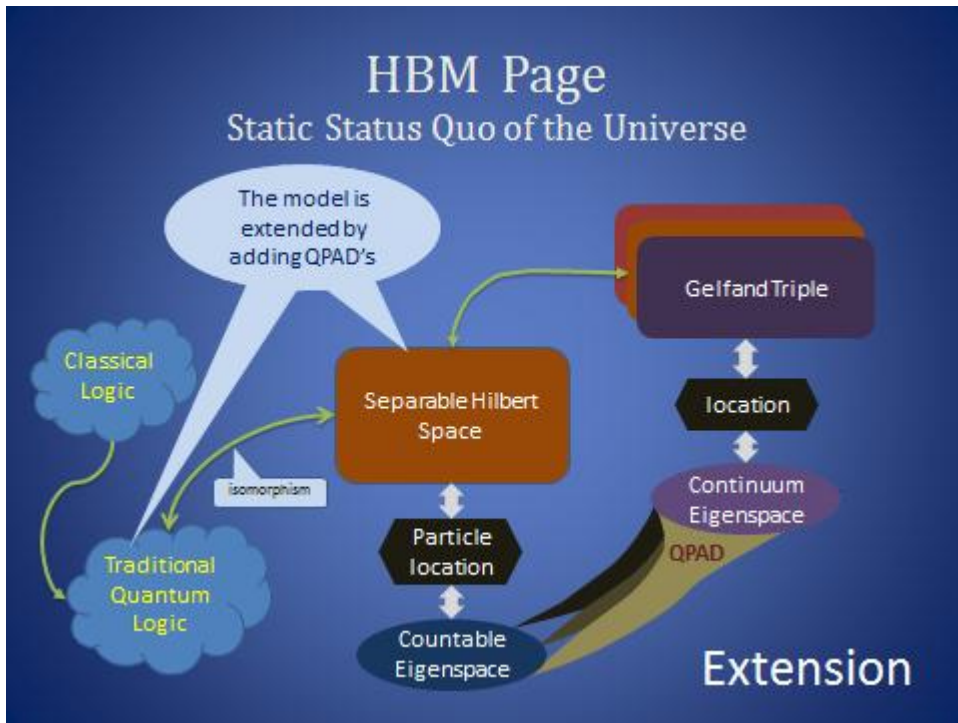
Figure 1: Structure of the Hilbert Book Model

1.2   Virtual

Primary QPAD's are quaternionic distributions of the probability of presence of *virtual* "charge" carriers. This "charge" may stand for an ensemble of properties.

The coupling of primary QPAD's results in elementary particles[12]. The properties that characterize this coupling form the sources of secondary QPAD's. Secondary QPAD's have *actual* charges as their sources and particles as their charge carriers. Secondary QPAD's concern a single property of the carrier. They are known as the physical field that relates to that property. Their presence can be observed.

The Hilbert Book Model does not use the notion of a virtual particle. Instead the role of primary QPAD's is used for this purpose.

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[12] See: Particle physics

Primary QPAD's cannot be observed directly. Their existence can only be derived from the existence of secondary QPAD's.

In the Hilbert Book Model the "implementation" of forces via the exchange of virtual particles is replaced by the mutual influencing of the corresponding QPAD's. This influence is instantiated via the fact that the concerned primary QPAD's superpose and that their currents feed/supply other QPAD's.

Interaction is implemented via QPAD's that act as a wave function. Wave function QPAD's which with respect to their parameter space exhibit an isotropic sign flavor[13], will provide the weak interaction. Other wave function QPAD's provide the strong interaction.

1.3   QPAD-sphere

QPAD's are quaternionic amplitude distributions and can be interpreted as a combination of a scalar "charge" density distribution and a vectorial "current" density distribution. The currents consist of uniformly moving charge carriers. When the wave function of a particle is represented by a primary QPAD, then this gives a special interpretation of that wave function. A very special kind of primary QPAD is a local background QPAD[14]. It represents the local superposition of the tails of the wave functions of distant elementary particles.

The QPAD's that act as wave functions may be imagined in a space that glues the eigenspace of the location operator that resides in the Gelfand triple to the Hilbert eigenvectors of the particle location operator that resides in the separable Hilbert space. This QPAD-sphere in which these QPAD's exist may be the location of the warm bath that some physicists use as the base of their theories. This QPAD-sphere contains streams of space patches that are superfluous in the eigenspace in the Gelfand triple and that fail in the corresponding eigenspace in the separable Hilbert space. Both eigenspaces are considered to be affine spaces. Coupled QPAD's act as pumps that circulate space patches in the QPAD-sphere[15].

The reason of the streaming process becomes understandable when the dimensions of the eigenspaces are reduced to one. The eigenspace that resides in the Gelfand triple is a continuum and can be represented by a **circle** whose topology is specified by real numbers. The eigenspace in the separable Hilbert space is countable and consists necessarily out of far less positions. The simplest solution is that these places are specified via rational numbers.

Now let this situation be managed by a mathemagician that must re-compute the situation at regular instances. He gets his location information from the continuum eigenspace and must find the proper rational number for each particle. He does this by allowing a stochastic inaccuracy between his real input location and his rational result location. Thus each particle possesses a normal distribution of potential locations. The mathemagician solves his problem by stealing potential

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[13] See: Quaternionic distributions

[14] See: Special QPAD's

[15] See: figure 2, the QPAD-sphere

positions from distant particles and adding them close to the center of the normal distribution of the potential positions of the local particle that he currently investigates. This process pumps potential positions around. Since every action of the mathematician is independent from previous or later actions, the process behaves as a Poisson process.
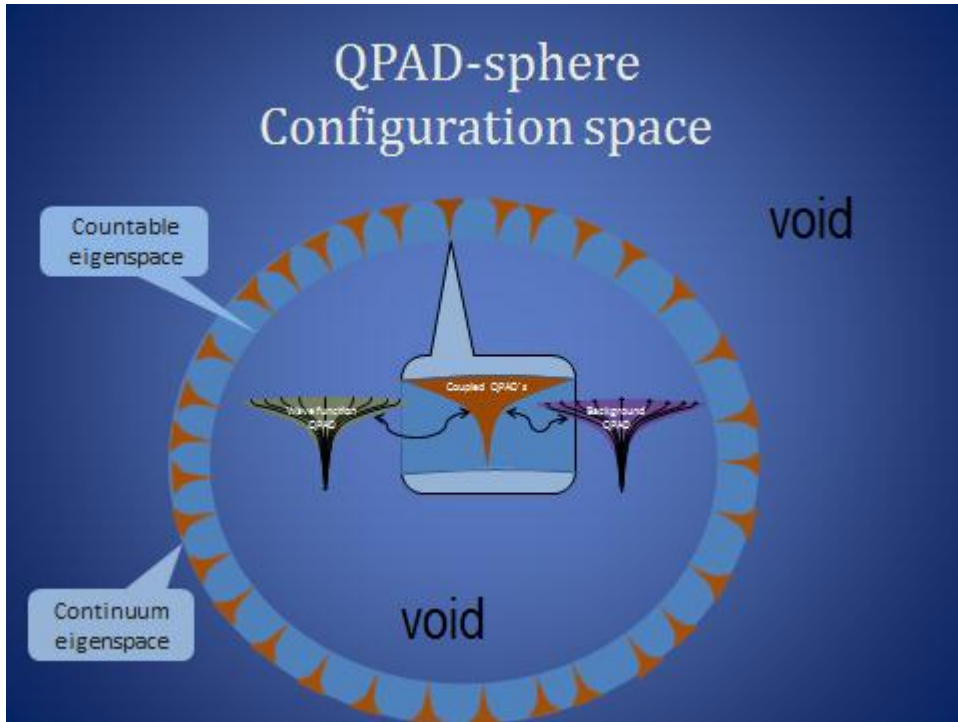
Figure 2: The QPAD-sphere

The next step is best visualized in two-dimensional affine eigenspaces. In this case results a **sphere** with a thin atmosphere in which thermal streams of eigenspace patches circulate such that at the location of particles the atmosphere is compressed. Like the air in the earth's atmosphere the eigenspace patches are circulating. Each particle has its own wave function QPAD, which is denser on its center than on its tail. The circulation takes place due to the fact that eigenspace patches are taken from the tails of the wave functions of distant particles and added to the QPAD of the local particle. The patches are retrieved from QPAD tails low in the atmosphere and they are emitted high in the atmosphere. If they spread far, then they lose temperature and thus height. In this way the local compression of this virtual atmosphere explains the local curvature of the eigenspace.

Now jump back to the full set of dimensions.

A coupled local wave function pumps space patches taken from the tails of other wave functions to the source at its center and spreads them over its surround. In this way it reclaims space patches that were supplied by distant sources and supplies them in the form of local space patches. The result of the stream of space patches is a local **space curvature**. Thus the local background

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QPAD acts as a drain where the local coupled wave function acts as a source of space patches. The process that does this can be characterized as a Poisson process.

1.4 History

In its first years, the development of quantum physics occurred violently. Little attention was paid to a solid and consistent foundation. The development could be characterized as delving in unknown grounds. Obtaining results that would support applications was preferred above a deep understanding of the fundamentals.

The first successful results were found by Schrödinger and Heisenberg. They both used a quantization procedure that converted a common classical equation of motion into a quantum mechanical equation of motion. Schrödinger used a wave function that varied as a function of its time parameter, while operators do not depend on time. Heisenberg represented the operators by matrices and made them time dependent, while their target vectors were considered to be independent of time. This led to the distinction between the Schrödinger picture and the Heisenberg picture.

Somewhat later John von Neumann and others integrated both views in one model that was based on Hilbert spaces. Von Neumann also laid the connection of the model with quantum logic. However, that connection was ignored in later developments. Due to the restrictions that are posed by separable Hilbert spaces, the development of quantum physics moved to other types of Hilbert spaces.

Due to this integration, it becomes clear that the Schrödinger picture and the Heisenberg picture represent two different views of the same situation. It appears to be unimportant were time is put as a parameter. The important thing is that the time parameter acts as a progression indicator. This observation indicates that the validity of the progression parameter covers the whole Hilbert space. With other words, the Hilbert space itself represents a static status quo.

In those days quaternions played no role. The vector spaces and functions that were used all applied complex numbers and observables were represented with self-adjoint operators. These operators are restricted to real eigenvalues.

Quaternions were discovered by the Irish mathematician Sir William Rowan Hamilton[16] in 1843. They were very popular during no more than two decades and after that they got forgotten. Only in the sixties of the twentieth century, due to the discovery of Constantin Piron that a separable Hilbert space ultimately uses quaternions for its inner product, a short upswing of quaternions occurred. But quickly thereafter they fell into oblivion again. Currently most scientists never encountered quaternions. The functionality of quaternions is taken over by complex numbers and a combination of scalars and vectors and by a combination of Clifford algebras, Grassmann algebras, Jordan algebras, alpha-, beta- and gamma-matrices and by spinors. The probability amplitude functions were taken to be complex rather than quaternionic. Except for the quaternion functionality that is hidden in the α, β, γ matrices, hardly any attention was given to the possible sign selections of quaternion imaginary base vectors and as a consequence the sign flavors of quaterni-

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[16] http://en.wikipedia.org/wiki/William_Rowan_Hamilton

onic distributions stay undetected. So, much of the typical functionality of quaternions still stays obscured.

The approach taken by quantum field theory departed significantly from the earlier generated foundation of quantum physics that relied on its isomorphism with quantum logic. Both QED and QCD put the quantum scene in non-separable Hilbert spaces. Only the wave function is seen as a (complex) probability amplitude distribution. Spinors and gamma matrices are used to simulate quaternion behavior. Physical fields are seen as something quite different from wave functions.

The influence of Lorentz transformations gives scientists the impression that space and time do not fit in a quaternion but instead in a spacetime quantity that features a Minkowski signature. Length contraction, time dilation and space curvature have made it improbable that progression would be seen as a universe wide parameter.

These developments cause a significant deviation between the approach that is taken in contemporary physics and the line according which the Hilbert Book Model is developed.

1.5 Criticism

Due to its unorthodox approach and controversial methods the Hilbert Book Model has drawn some criticism

1.5.1 Model

Question:
The separable Hilbert space has clearly some nasty restrictions. Why can quantum physics not be completely done in the realm of a rigged Hilbert space?
Answer:
In that case there is no fundamental reason for the introduction of fields in QP. It will also not be possible to base QP on traditional quantum logic (TQL), because the isomorphism that exists between TQL and separable Hilbert spaces (SHS's) does not exist between TQL and a rigged Hilbert space (RHS). See figure 1.
The quaternionic probability amplitude distributions (QPAD's) on which fields are based, link the SHS with its Gelfand triple {Φ;SHS;Φ'}, which is a RHS. However, the QPAD's are not part of the SHS and not part of the RHS. The HBM can be pictured as:

$$TQL \Leftrightarrow SHS \Rightarrow \{QPAD's\} \Rightarrow RHS \equiv \{\Phi;SHS;\Phi'\} \tag{1}$$

While the isomorphism $\Leftrightarrow$ is replaced by incongruence $\Leftarrow\neq\Rightarrow$ in

$$TQL \Leftarrow\neq\Rightarrow RHS \tag{2}$$

1.5.2 Quaternions

Remark1:
A tensor product between to quaternionic Hilbert spaces cannot be constructed. So it is better to stay with complex Hilbert spaces.
Remark2:

The notion of covariant derivative, which is an important concept on quantum field theory, offers problems with quaternionic distributions, so it is better to stay with a complex representation.

This is due to the fact that for quaternionic distributions in general:

$$\nabla(f\,g) \neq f\,\nabla\,g\,+\,(\nabla f)\,g \tag{1}$$

\Response:
The HBM proves that solutions exist that do not apply these concepts.

In fact the subject can be reversed:

If a methodology is in conflict with a quaternionic approach, then it must not be applied as a general methodology in quantum physics. Such method can only be applied in special, one dimensional cases.

1.6 Consequence

The application of the HBM requests from physicists that they give up some of the conventional methodology and learn new tricks.

PART II

The Hilbert Book Model

Abstract

The extension of the separable Hilbert space by a set of QPAD's enables the interpretation of equations of movement as continuity equations. Exploring this fact leads to a complete set of equations that describe all known elementary particles that are contained in the standard model. The equations enable the computation of the coupling factors from the configuration of the constituting fields. The properties of the elementary particles, including the coupling factors are related to the local curvature and in this way to the notions of mass and electric charge. Further the notion of a QPAD-sphere shows that QPAD play an important role in the establishment of curvature and inertia.

2   INGREDIENTS

The most intriguing ingredients of the model are quaternions, quaternionic distributions and QPAD's.

2.1   Role of the particle locator operator

The particle locator operator $\mathfrak{S}$ is one of the operators for which the eigenvectors are coupled to a background continuum that is related to the eigenspace of a corresponding position operator that resides in the Gelfand triple.

The background continuum may be curved. It means that this background continuum is a quaternionic function of the eigenvalues of that position operator. This function has the same sign flavor as its parameter space. Without curvature its parameters and the corresponding values are equal.

For each eigenvector of the particle locator operator the background continuum acts as parameter space for the QPAD that connects this eigenvector with the eigenspace of the corresponding position operator that resides in the Gelfand triple. The QPAD can be visualized as a fuzzy funnel that drops stochastically inaccurate observation values onto the particle.

This position operator has a canonical conjugate, which is the corresponding momentum operator. A connection of the particle eigenvector with the eigenspace of the momentum operator runs via a different QPAD. Without any curvature the two QPAD's would be each other's Fourier transform.

2.2   QPAD's

All elementary particles correspond to different eigenvectors of the particle locator operator $\mathfrak{S}$. Each elementary particle has its own QPAD that acts as its wave function.

If we want to categorize particles, then we must categorize their QPAD's. But, we also must take the QPAD in account to which the wave function QPAD is coupled. Some QPAD's stay uncoupled.

If a QPAD has a well-defined location in configuration space, then it does not have a well-defined location in the canonical conjugate space. So we better use both locations together.

A possible strategy is to use the superposition of the QPAD and its Fourier transform. This solution distinguishes QPAD's that, apart from a scalar, are invariant under Fourier transformation[17].

An important category of invariants is formed by QPAD's that have the shape $f(r) = C/r$, where C is a constant and r is the distance from the central location[18][19].

Further it is sensible to introduce for each category the notion of an average QPAD. Determining the average QPAD involves integration over the full parameter space.

In fact the procedure for determining the average QPAD produces a representation of the superposition of all QPAD's in the parameter space.

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[17] See Appendix; Functions invariant under Fourier transform
[18] http://en.wikipedia.org/wiki/Hankel_transform#Some_Hankel_transform_pairs
[19] Also see http://en.wikipedia.org/wiki/Bertrand's_theorem

Another procedure constructs the superposition of all tails of QPAD's at a certain location. This produces a background QPAD. For each location such a background QPAD exists, but it may differ per location. However, it can only differ very marginally for neighboring locations.

2.3 QPAD vizualization

If I was a good artist and I was asked to give an artist impression of the magic wand of a magician, then I would not paint a dull rod or a stick with a star at the end. Instead I would draw a very thin glass rod that has a sparkling fuzzy ball at its tip. A static view of that ball would be like:
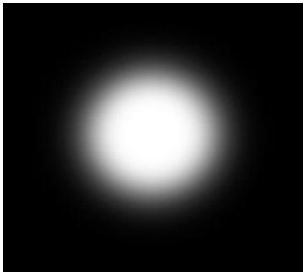
Figure 3: Typical isotropic QPAD

A dynamic view would show how the sparkles move inside out (source) or outside in (drain). This fuzzy ball is how a simple QPAD may look like when we could see it. In a static QPAD, uniformly moving virtual charge carriers replace the sparkles. The movement need neither be parallel nor spherical, but it must be uniform. The carriers may be interpreted as tiny patches that are taken from the continuum background that forms the parameter space of the QPAD.

Even when they are attenuated or spatially or temporally spread by a binomial process, Poisson processes create a result that has a Poisson distribution. The output of an efficient Poisson process has a density distribution that comes close to a Gaussian distribution. A typical example QPAD might show such a Gaussian density distribution. When actual electrical charges would be distributed this way, then this distribution creates a potential that has the shape of an error function. Already at a short distance from its center this function decreases very close to a 1/r dependence. At that distance the local potential would be the same as when a single large charge was put at the center. Instead our example has virtual charges. So it represents a single virtual charge at the center (or an ensemble of properties of the carrier of that single charge). The raised potential is also virtual. Still it describes our example QPAD.

Thus the source that creates the above described fuzzy ball may be characterized as a Poisson process.

2.4 Special QPAD's

QPAD's exist in two categories:
- A primary QPAD concerns virtual charges.
- A secondary QPAD concerns actual charges.

The actual charges concern properties that characterize the coupling of two primary QPAD's.

We will consider a special ensemble of primary QPAD's $\{\psi_i(r, q_i)\}$.

- The $\psi_i(r, 0)$ are normalized: $\int_V |\psi_i(r, 0)|^2 \, dV = 1$.
- The $\psi_i(r, 0)$ must be spherically symmetric.
- From a given minimal distance their modulus must decrease with radius r as $1/r$.

The special QPAD's are Fourier transform invariant and conform to Bertrand's theorem[20].

2.4.1   The average QPAD

The ensemble $\{\psi_i(r, 0)\}$ of the special QPAD's has an average $\Psi(r, 0)$

2.4.2   The background QPAD

The ensemble $\{\psi_i(r, q_i)\}$ is distributed randomly over the center points $\{q_i\}$ in an affine parameter space. At a given point P in this space the superposition of all $\{\psi_i(r, P)\}$ will be constructed.

This superposition will be renormalized and then indicated by $\Phi(r, P)$.

Thus,

$$\int_V |\Phi(r, P)|^2 \, dV = 1 \tag{1}$$

In this superposition the largest contribution comes from the $\psi_i(r, q_i)$ for which the q_i is farthest from P. Further the directions of the imaginary part are reversed with respect to the directions in the $\psi_i(r, q_i)$.

Especially at long distances, all differences are smoothed away via an averaging process.

The result is that:

$$\Phi(r, P) = \Psi^*(r, P) \tag{2}$$

We will interpret $\Phi(r, 0)$ as the background QPAD.

The approach taken here, shows similarity with the approach of Denis Sciama in his paper: "On the origin of inertia"[21].

Every sign flavor might have its own background QPAD.

2.4.2.1   Uniform movement

Due to its construction the location as well as the shape of the background QPAD is very stable. In the HBM the background QPAD is reconstructed at every subsequent page. As long as the whole QPAD remains static, it can be reconstructed at a displaced location. Thus, as long as this

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[20] http://en.wikipedia.org/wiki/Bertrand%27s_theorem
[21] See: http://www.adsabs.harvard.edu/abs/1953MNRAS.113...34S

movement is contained in its current density distribution, the background QPAD can move freely in a uniform way.

In principle every location has its own background QPAD. However, it makes only sense to couple the locations of particles with the local background QPAD. If this coupling is strong then it acts as a sticky resistance against acceleration of the coupled particle.

2.4.3   Isotropy

Source QPAD's are isotropic. Drain QPAD's are also isotropic and are the conjugate of a corresponding source QPAD.

Anisotropic QPAD's are hybrids. In one or two dimensions they correspond to source QPAD's. In the other dimensions they correspond to a drain QPAD.

Oscillating QPAD's oscillate between source modes and drain modes. They usually can be described by spherical or linear harmonic (quantum) oscillators. A special case of the oscillating QPAD is a plain wave.

2.5   Inertia

Inertia[22] is based on the fact that all particles in universe influence a selected local particle. Since this influence is isotropic it usually does not disturb the particle. This condition holds as long as the particle is located stationary or moves uniformly. In that case the connected QPAD is static. However, when the particle accelerates, then this goes together with the existence of an extra field that becomes part of the particle's QPAD and that counteracts the acceleration.

The background QPAD that is coupled to the wave function QPAD at the right side of the equation of motion[23] represents the influence of the universe on the local particle. It represents the superposition of all tails of the wave functions of particles that exist in universe. For that reason it is the source of inertia. This is shown in the elementary coupling equation.

$$\nabla \psi^x = m \, \psi^y \qquad\qquad (1)$$

The QPAD's ψ^x and ψ^y are sign flavors of the same base QPAD ψ, which on its turn has the same sign flavor as the coordinate system that acts as parameter space.

For elementary fermions the coupled field ψ^y equals the isotropic background field Φ.

For electrons the wave function ψ^x equals $\Psi = \Phi^*$

The formula (1) holds for all massive elementary particle types, for elementary fermions as well as for the elementary bosons that couple to other sign flavors[24] than Φ.

With other words, for every QPAD sign flavor ψ^x that acts as a wave function of an elementary particle exists a background QPAD ψ^y [25]. The corresponding particle type is fully characterized by the pair $\{\psi^x, \psi^y\}$

These facts are in detail treated in the section on elementary particles.

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[22] Appendix; The universe of items

[23] See Elementary particles

[24] See Quaternions; Sign selections

[25] See: the QPAD game

According to equation (1) the wave function QPAD is rather strongly coupled to the local background QPAD. The equation describes the situation of an independently moving particle.

In atoms the electrons oscillate around the nucleus. There these particle move more freely. They are still coupled to the centrally located background QPAD, but the coupling is rather loose. Equation (1) does not describe that situation. Instead of the Dirac equation, the Klein-Gordon equation fits more appropriately.

The background QPAD's play the role that is thought for the Higgs field.

2.6   Coupling and curvature

When a local wave function QPAD is coupled to a local background QPAD, then the background QPAD can be considered as a source that supplies the local wave function QPAD. On its turn the local wave function QPAD acts as a drain. Thus, the coupled system pumps space taken from the rest of the universe to the location of the local wave function QPAD. These flowing space patches form part of the space that the eigenspace E_G of the location operator that resides in the Gelfand triple has extra with respect to the eigenspace E_H of the particle location operator that resides in the separable Hilbert space. Otherwise said: the eigenspace E_G compresses via the wave function QPAD to the eigenspace E_H and expands back via the coupled background QPAD to eigenspace E_G. The observer that uses E_H as his observation space, experiences a local compression of his observation space at the location of the observed QPAD. This can be expressed by a local curvature of the observation space.

As extra detail can be said that the source and the drain act as Poisson processes that cause a Gaussian distribution of the patches of space that flow in/out the drain/source location.

The moving space patches may be interpreted as virtual carriers of the properties that characterize the coupling event.

The coupling properties themselves act as sources of secondary QPAD's. These are known as physical fields. In the Kerr-Newman metric equation these properties act as sources of curvature.

2.7   Hyper-complex numbers

Hyper-complex numbers form categories that are ordered with respect to their dimension. The dimension D takes the form $D = 2^n$, where n is a non-negative integer. A hyper-complex number of dimension D can be obtained from a pair of hyper-complex numbers of dimension $D - 1$ via a construction algorithm. Several construction algorithms exist. The most popular is the Cayley-Dickson construction[26]. A less known construction algorithm is the 2^n-on construction of Warren Smith[27]. This construction delivers numbers that in the higher dimensions retain better arithmetic capabilities. Up and including the octonions the two construction algorithms deliver the same numbers. The sedions differ from the 2^4-ons.

In their lower m dimensions the 2^n-ons behave similarly to the 2^m-ons.

The 2^n-ons have n independent imaginary base vectors. As a consequence the 2^n-ons feature n independent sign selections.

Both construction methods ignore these sign selections. Sign selections play a crucial role in this paper.

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[26] http://en.wikipedia.org/wiki/Cayley%E2%80%93Dickson_construction
[27] Appendix; 2^n-ons, See http://www.math.temple.edu/~wds/homepage/nce2.pdf

2.8 Quaternions

A quaternion is a 1+3 dimensional hyper-complex number. It has a one dimensional real part and a three dimensional imaginary part. As a result, it can be seen as the combination of a real scalar and a three dimensional vector.

$$q = q_0 + \boldsymbol{i}q_1 + \boldsymbol{j}q_2 + \boldsymbol{k}q_3 \tag{1}$$

The quaternions form a division ring[28]. According to the Frobenius theorem[29], the only finite-dimensional division algebras over the reals are the reals themselves, the complex numbers, and the quaternions.

The coefficients $\{q_m\}$ are real numbers. Bi-quaternions exist that have complex coefficients, but these do not form a division ring.
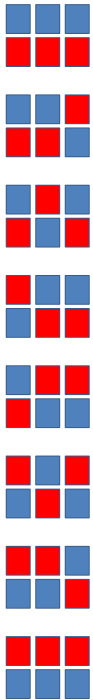
N

R

G

B

$\bar{R}$

$\bar{G}$

$\bar{B}$

W

2.8.1 Sign selections

The quaternions possess two independent sign selections. The conjugation $q \Leftrightarrow q^*$ inverts the sign of all imaginary base vectors. It acts isotropic.

$$q^* = q_0 - \boldsymbol{i}q_1 - \boldsymbol{j}q_2 - \boldsymbol{k}q_3 \tag{1}$$

The reflection $q \Leftrightarrow q^1$ inverts a single imaginary base vector and for that reason it acts anisotropic.

$$q^1 = q_0 + \boldsymbol{i}q_1 + \boldsymbol{j}q_2 - \boldsymbol{k}q_3 \tag{2}$$

Here, the base vector $\boldsymbol{k}$ is selected arbitrarily.

These two sign selections can be mixed. They generate four sign states.

When the three independent directions in which reflections can occur are also

taken into account, then eight different sign selections are possible. These sign selections are color coded as is shown in figure 4.

Individually the conjugation and the reflection both flip the handedness of the external vector product of the imaginary part when both factors use the same sign selections.

Figure 4: Sign selections.

[28] http://en.wikipedia.org/wiki/Division_ring

[29] http://en.wikipedia.org/wiki/Frobenius_theorem_(real_division_algebras)

2.8.2   Habits

The addition works as in all division rings, however the product of two quaternions does not commute.

2.8.2.1   Product rule

The product rule is best expressed in the form of real scalars and 3D vectors:

$$ab = a_0 b_0 - \langle \boldsymbol{a}, \boldsymbol{b} \rangle + a_0 \boldsymbol{b} + \boldsymbol{a} b_0 + \boldsymbol{a} \times \boldsymbol{b} \tag{1}$$

$$\langle \boldsymbol{a}, \boldsymbol{b} \rangle = a_1 b_1 + a_2 b_2 + a_3 b_3 \tag{2}$$

$$\boldsymbol{a} \times \boldsymbol{b} = \boldsymbol{i}(a_2 b_3 - a_3 b_2) + \boldsymbol{j}(a_3 b_1 - a_1 b_3) + \boldsymbol{k}(a_1 b_2 - a_2 b_1) \tag{3}$$

$$\boldsymbol{i}\,\boldsymbol{j} = \pm \boldsymbol{k} \tag{4}$$

The norm or modulus is defined by:

$$|a| = \sqrt{a_0 a_0 + \langle \boldsymbol{a}, \boldsymbol{a} \rangle} \tag{5}$$

2.9   Quaternionic distributions

Several forms of quaternionic distributions exist. Two forms are relevant for the HBM.

A curved coordinate system can be related to a flat coordinate system via a quaternionic distribution. The flat coordinate system plays the role of parameter space. On its turn the curved coordinate system can also play the role of a parameter space. It does that for quaternionic probability amplitude distributions (QPAD's).

A quaternionic probability amplitude distribution [30] is a quaternionic distribution. Its value can be split in a real part that can be interpreted as a charge density distribution and an imaginary part that can be interpreted as a current density distribution. The squared modulus of the value can be interpreted as the probability density of the presence of the carrier of the charge. The charge can be any property of the carrier or it stands for the ensemble of the properties of the carrier.

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[30] http://en.wikipedia.org/wiki/Probability_amplitude treats complex probability amplitude distributions.

2.9.1 Sign flavors

The quaternions that form the values of a quaternionic distribution must all feature the same set of sign selections. This fact attaches a sign flavor to each quaternionic distribution. Quaternionic distributions come in four sign flavors[31]: $\psi^{\circledcirc}$, $\psi^{\textcircled{1}}, \psi^{\textcircled{2}}$ and $\psi^{\textcircled{3}}$. However, $\psi^{\textcircled{1}}$ and $\psi^{\textcircled{2}}$ are direction dependent and appear in corresponding color versions. If the directions of the reflections are of interest, then eight different sign flavors exist. We indicate color by an extra index: $i = R, G, B$. See figure 4.

We will use the symbol ψ or $\psi^{\circledcirc}$ for the sign flavor of the quaternionic distribution that has the same sign flavor as its parameter space. The superscripts indicate the number of base vectors that changed sign.

We will use

$$\psi^{\textcircled{3}} = \psi^* \tag{1}$$

And with the same symbolic:

$$\psi_i^{\textcircled{1}} = \psi_i^1, \qquad i = R, G, B \tag{2}$$

$$\psi^{\circledcirc} = \psi \tag{3}$$

Often the symbols ψ and ψ^* will be used instead of the symbols $\psi^{\circledcirc}$ and $\psi^{\textcircled{3}}$.

2.9.1.1 QD multiplication

What happens when quaternions from different sign flavors will be multiplied?

1. First a reference sign flavor is selected.
2. This sign flavor is taken to be the sign flavor of the distribution that will receive the result.
3. The factors are first brought to this reference sign selection.
4. In this process nothing changes in the values of the quaternions.
5. After that the multiplication takes place.
6. The result is delivered in the reference sign flavor.

With other words the multiplication takes place with the handedness that is defined in the target distribution.

2.9.2 Differentiation and Fourier transform

A quaternionic distribution f(q) can be differentiated[32].

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[31] The notion of "sign flavor" is used because for elementary particles "flavor" already has a different meaning.

[32] For more details, see Appendix; Quaternionic distributions,

$$g(q) = \nabla_0 f_0(q) \mp \langle \boldsymbol{\nabla}, \boldsymbol{f}(q) \rangle \pm \nabla_0 \boldsymbol{f}(q) + \boldsymbol{\nabla} f_0(q) \pm (\pm \boldsymbol{\nabla} \times \boldsymbol{f}(q))$$

The colored $\mp$ and $\pm$ signs refer to the influence of conjugation of $f(q)$ on quaternionic multiplication. The $\pm$ sign refers to the influence of reflection of $f(q)$.

In this section, the parameter q is supposed to be taken from a non-curved parameter space. With that precondition, in Fourier space differentiation becomes multiplication with the canonical conjugate coordinate k and therefore the equivalent equation becomes:

$$\tilde{g}(k) = k\tilde{f}(k) = k_0 \tilde{f}_0(k) \mp \langle \mathbf{k}, \tilde{\boldsymbol{f}}(k) \rangle \pm k_0 \tilde{\boldsymbol{f}}(k) + \mathbf{k}\tilde{f}_0(k) \pm \left(\pm \mathbf{k} \times \tilde{\boldsymbol{f}}(k) \right) \tag{2}$$

For the imaginary parts holds:

$$\mathbf{g}(q) = \pm \nabla_0 \boldsymbol{f}(q) + \boldsymbol{\nabla} f_0(q) \pm \left(\pm \boldsymbol{\nabla} \times \boldsymbol{f}(q) \right) \tag{3}$$

$$\tilde{\mathbf{g}}(k) = \pm k_0 \tilde{\boldsymbol{f}}(k) + \mathbf{k}\tilde{f}_0(k) \pm \left(\pm \mathbf{k} \times \tilde{\boldsymbol{f}}(k) \right) \tag{4}$$

2.9.3 Spinors and matrices

In contemporary physics complex probability amplitude distributions (CPAD's) are used rather than QPAD's. Spinors and matrices are used to simulate QPAD behavior for CPAD's.

A spinor $[\psi]$ is a 1×4 matrix consisting of CPAD's that represent the sign flavors of a QPAD. Sometimes the spinor is represented as a 1×2 matrix.

The $\boldsymbol{\alpha}$ and β matrices influence the elements of spinor $[\psi]$.

$$\alpha_1 = \begin{bmatrix} 0 & \boldsymbol{i} \\ -\boldsymbol{i} & 0 \end{bmatrix} \tag{3}$$

$$\alpha_2 = \begin{bmatrix} 0 & \boldsymbol{j} \\ -\boldsymbol{j} & 0 \end{bmatrix} \tag{4}$$

$$\alpha_3 = \begin{bmatrix} 0 & \boldsymbol{k} \\ -\boldsymbol{k} & 0 \end{bmatrix} \tag{5}$$

$$\beta = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \tag{6}$$

$\boldsymbol{i}, \boldsymbol{j}$ and $\boldsymbol{k}$ represent imaginary base vectors of the simulated quaternion. β represents the conjugation action for the spinor.

A relation exist between $\alpha_1, \alpha_2, \alpha_3$ and the Pauli[33] matrices $\sigma_1, \sigma_2, \sigma_3$:

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[33] http://en.wikipedia.org/wiki/Pauli_matrices

$$\sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \qquad \sigma_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \qquad \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \tag{7}$$

$$1 \mapsto I, \qquad \boldsymbol{i} \mapsto \sigma_1, \qquad \boldsymbol{j} \mapsto \sigma_2, \qquad \boldsymbol{k} \mapsto \sigma_3 \tag{8}$$

This combination is usually represented in the form of gamma matrices.
In Dirac representation, the four [contravariant](#) gamma matrices are

$$\gamma^0 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}, \qquad \gamma^1 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix}, \tag{9}$$

$$\gamma^2 = \begin{bmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{bmatrix}, \qquad \gamma^3 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

It is useful to define the product of the four gamma matrices as follows:

$$\gamma^5 = i\,\gamma^0\,\gamma^1\,\gamma^2\,\gamma^3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \tag{10}$$

The gamma matrices as specified here are appropriate for acting on Dirac spinors written in the Dirac basis; in fact, the Dirac basis is defined by these matrices. In the Dirac basis[34]:

$$\gamma^0 = \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix}, \qquad \gamma^k = \begin{bmatrix} 0 & \sigma^k \\ -\sigma^k & 0 \end{bmatrix}, \qquad \gamma^5 = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} \tag{11}$$

This corresponds with $\alpha_k = \gamma^k, \beta = \gamma^5$.
Apart from the Dirac basis, a Weyl basis exists

$$\gamma^0 = \gamma^\beta = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}, \qquad \gamma^k = \begin{bmatrix} 0 & \sigma^k \\ -\sigma^k & 0 \end{bmatrix}, \qquad \gamma^5 = \begin{bmatrix} -I & 0 \\ 0 & I \end{bmatrix} \tag{12}$$

The Weyl basis has the advantage that its [chiral projections](#)[35] take a simple form:

$$\psi_L = \tfrac{1}{2}(1 - \gamma^5)[\psi] = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}[\psi] \tag{13}$$

$$\psi_R = \tfrac{1}{2}(1 + \gamma^5)[\psi] = \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix}[\psi] \tag{14}$$

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[34] http://en.wikipedia.org/wiki/Gamma_matrices#Dirac_basis
[35] http://en.wikipedia.org/wiki/Chirality_(physics)

$$[\psi^*] = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} [\psi] \tag{15}$$

2.9.4 Continuity equation

When applied to a quaternionic probability amplitude distribution (QPAD), the equation for the differentiation leads to a continuity equation.

When $\rho_0(q)$ is interpreted as a charge density distribution, then the conservation of the corresponding charge[36] is given by the continuity equation:

$$\text{Total change within } V = \text{flow into } V + \text{production inside } V \tag{1}$$

$$\frac{d}{dt} \int_V \rho_0 \, dV = \oint_S \widehat{\boldsymbol{n}} \rho_0 \frac{\boldsymbol{v}}{c} \, dS + \int_V s_0 \, dV \tag{2}$$

$$\int_V \nabla_0 \rho_0 \, dV = \int_V \langle \boldsymbol{\nabla}, \boldsymbol{\rho} \rangle \, dV + \int_V s_0 \, dV \tag{3}$$

Here $\widehat{\boldsymbol{n}}$ is the normal vector pointing outward the surrounding surface S, $\boldsymbol{v}(t, \boldsymbol{q})$ is the velocity at which the charge density $\rho_0(t, \boldsymbol{q})$ enters volume V and s_0 is the source density inside V. In the above formula $\boldsymbol{\rho}$ stands for

$$\boldsymbol{\rho} = \rho_0 \boldsymbol{v}/c \tag{4}$$

It is the flux (flow per unit area and unit time) of ρ_0 .

The combination of $\rho_0(t, \boldsymbol{q})$ and $\boldsymbol{\rho}(t, \boldsymbol{q})$ is a quaternionic skew field $\rho(t, \boldsymbol{q})$ and can be seen as a probability amplitude distribution (QPAD).

$$\rho \overset{\text{def}}{=} \rho_0 + \boldsymbol{\rho} \tag{5}$$

$\rho(t, \boldsymbol{q})\rho^*(t, \boldsymbol{q})$ can be seen as an overall probability density distribution of the presence of the carrier of the charge. $\rho_0(t, \boldsymbol{q})$ is a charge density distribution. $\boldsymbol{\rho}(t, \boldsymbol{q})$ is the current density distribution.

The conversion from formula (2) to formula (3) uses the Gauss theorem[37]. This results in the law of charge conservation:

$$s_0(t, \boldsymbol{q}) = \nabla_0 \rho_0(t, \boldsymbol{q}) \mp \langle \boldsymbol{\nabla}, \big(\rho_0(t, \boldsymbol{q})\boldsymbol{v}(t, \boldsymbol{q}) + \boldsymbol{\nabla} \times \boldsymbol{a}(t, \boldsymbol{q})\big) \rangle \tag{6}$$

$$= \nabla_0 \rho_0(t, \boldsymbol{q}) \mp \langle \boldsymbol{\nabla}, \boldsymbol{\rho}(t, \boldsymbol{q}) + \boldsymbol{A}(t, \boldsymbol{q}) \rangle$$

$$= \nabla_0 \rho_0(t, \boldsymbol{q}) \mp \langle \boldsymbol{v}(t, \boldsymbol{q}), \boldsymbol{\nabla}\rho_0(t, \boldsymbol{q}) \rangle \mp \langle \boldsymbol{\nabla}, \boldsymbol{v}(t, \boldsymbol{q}) \rangle \rho_0(t, \boldsymbol{q})$$

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[36] Also see Noether's laws: http://en.wikipedia.org/wiki/Noether%27s_theorem

[37] http://en.wikipedia.org/wiki/Divergence_theorem

$$\mp \langle \boldsymbol{\nabla}, \boldsymbol{A}(t, \boldsymbol{q}) \rangle$$

The blue colored $\pm$ indicates quaternionic sign selection through conjugation of the field $\rho(t, \boldsymbol{q})$. The field $\boldsymbol{a}(t, \boldsymbol{q})$ is an arbitrary differentiable vector function.

$$\langle \boldsymbol{\nabla}, \boldsymbol{\nabla} \times \boldsymbol{a}(t, \boldsymbol{q}) \rangle = 0 \tag{7}$$

$\boldsymbol{A}(t, \boldsymbol{q}) \stackrel{\text{def}}{=} \boldsymbol{\nabla} \times \boldsymbol{a}(t, \boldsymbol{q})$ is always divergence free. In the following we will neglect $\boldsymbol{A}(t, \boldsymbol{q})$.

Equation (6) represents a balance equation for charge density. What this charge actually is, will be left in the middle. It can be one of the properties of the carrier or it can represent the full ensemble of the properties of the carrier.

This only treats the real part of the full equation. The full equation runs:

$$s(t, \boldsymbol{q}) = \nabla \rho(t, \boldsymbol{q}) = s_0(t, \boldsymbol{q}) + \boldsymbol{s}(t, \boldsymbol{q}) \tag{8}$$

$$= \nabla_0 \rho_0(t, \boldsymbol{q}) \mp \langle \boldsymbol{\nabla}, \boldsymbol{\rho}(t, \boldsymbol{q}) \rangle \pm \nabla_0 \boldsymbol{\rho}(t, \boldsymbol{q}) + \boldsymbol{\nabla} \rho_0(t, \boldsymbol{q}) \pm \left(\pm \boldsymbol{\nabla} \times \boldsymbol{\rho}(t, \boldsymbol{q}) \right)$$

$$= \nabla_0 \rho_0(t, \boldsymbol{q}) \mp \langle \boldsymbol{v}(t, \boldsymbol{q}), \boldsymbol{\nabla} \rho_0(t, \boldsymbol{q}) \rangle \mp \langle \boldsymbol{\nabla}, \boldsymbol{v}(t, \boldsymbol{q}) \rangle \rho_0(t, \boldsymbol{q})$$

$$\pm \nabla_0 \boldsymbol{v}(t, \boldsymbol{q}) + \nabla_0 \rho_0(t, \boldsymbol{q}) + \boldsymbol{\nabla} \rho_0(t, \boldsymbol{q})$$

$$\pm \left(\pm \left(\rho_0(t, \boldsymbol{q}) \, \boldsymbol{\nabla} \times \boldsymbol{v}(t, \boldsymbol{q}) - \boldsymbol{v}(t, \boldsymbol{q}) \times \boldsymbol{\nabla} \rho_0(t, \boldsymbol{q}) \right) \right)$$

$$s_0(t, \boldsymbol{q}) = 2 \nabla_0 \rho_0(t, \boldsymbol{q}) \mp \langle \boldsymbol{v}(\boldsymbol{q}), \boldsymbol{\nabla} \rho_0(t, \boldsymbol{q}) \rangle \mp \langle \boldsymbol{\nabla}, \boldsymbol{v}(t, \boldsymbol{q}) \rangle \rho_0(t, \boldsymbol{q}) \tag{9}$$

$$\boldsymbol{s}(t, \boldsymbol{q}) = \pm \nabla_0 \boldsymbol{v}(t, \boldsymbol{q}) \pm \boldsymbol{\nabla} \rho_0(t, \boldsymbol{q}) \tag{10}$$

$$\pm \left(\pm \left(\rho_0(t, \boldsymbol{q}) \, \boldsymbol{\nabla} \times \boldsymbol{v}(t, \boldsymbol{q}) - \boldsymbol{v}(t, \boldsymbol{q}) \times \boldsymbol{\nabla} \rho_0(t, \boldsymbol{q}) \right) \right)$$

The red sign selection indicates a change of handedness by changing the sign of one of the imaginary base vectors. Conjugation also causes a switch of handedness. It changes the sign of all three imaginary base vectors.

3   PARTICLE PHYSICS

This chapter treats the **first level coupling**. The result of that coupling are first level particles. These particles are solely created out of coupled primary QPAD's and annihilate back into primary QPAD's. The zero-level of coupling stands for no coupling.

This section will not separate different particle generations.

Graphically the first level coupling is presented in more detail in the slide show.

3.1   Elementary fermions

First level particles appear to obey a special kind of continuity equation. In this continuity equation the source/drain term is represented by the coupled background QPAD.

3.1.1   Dirac equation

The best known equation of motion for elementary fermions is the Dirac equation. It is written using spinors and matrices.

The Dirac equation for a free moving electron or positron is known as:

$$\nabla_0[\psi] + \boldsymbol{\nabla}\boldsymbol{\alpha}[\psi] = m\beta[\psi] \tag{1}$$

The Dirac matrices $\boldsymbol{\alpha}$ and β give the spinor $[\psi]$ the function of a pair of QPAD's.

This spinor equation can be converted into two quaternionic equations that act on the QPAD's ψ_R and ψ_L:

$$\nabla_0\psi_R + \boldsymbol{\nabla}\psi_R = m\psi_L \tag{2}$$

$$\nabla_0\psi_L - \boldsymbol{\nabla}\psi_L = m\psi_R \tag{3}$$

In the mass term the coupling factor m couples ψ_L and ψ_R. When $m = 0$ then ψ_L and ψ_R are not coupled. Further:

$$\psi_R = \psi_L^* = \psi_0 + \boldsymbol{\psi} \tag{4}$$

In the left term of equations (2) and (3), ψ_R and ψ_L represent the wave function of the particle. In the right side appears the background field. In that sense $\{\psi_R,\psi_L\}$ and $\{\psi_L,\psi_R\}$ represent each other's antiparticle.

Reformulating these equations gives

$$\nabla\psi^{\textcircled{0}} = m\,\psi^{\textcircled{3}} \tag{5}$$

$$\nabla_0(\psi_0 + \boldsymbol{\psi}) + \boldsymbol{\nabla}(\psi_0 + \boldsymbol{\psi}) = m(\psi_0 - \boldsymbol{\psi}) \tag{6}$$

For the conjugated equation holds

$$\nabla^* \psi^{\textcircled{3}} = m \, \psi^{\textcircled{0}} \tag{7}$$

$$\nabla_0(\psi_0 - \boldsymbol{\psi}) - \boldsymbol{\nabla}(\psi_0 - \boldsymbol{\psi}) = m(\psi_0 + \boldsymbol{\psi}) \tag{8}$$

This implements the reverse flip. The corresponding particle is the antiparticle.

$$\{\psi^{\textcircled{0}}, \psi^{\textcircled{3}}\} \leftrightarrow \{\psi^{\textcircled{3}}, \psi^{\textcircled{0}}\} \tag{9}$$

Both flips switch the handedness.

Equations (5) and (7) are each other's quaternionic conjugate. However, it must be noticed that all terms are conjugated, including the nabla operator, but the parameter space stays untouched. Thus equation (7) differs from equation (5).

Summing the equations gives via

$$\boldsymbol{\nabla}\boldsymbol{\psi} = \boldsymbol{\nabla} \times \boldsymbol{\psi} - \langle \boldsymbol{\nabla}, \boldsymbol{\psi} \rangle \tag{10}$$

the result

$$\nabla_0 \psi_0 - \langle \boldsymbol{\nabla}, \boldsymbol{\psi} \rangle = m \, \psi_0 \tag{11}$$

The difference gives

$$\nabla_0 \boldsymbol{\psi} + \boldsymbol{\nabla}\psi_0 + \boldsymbol{\nabla} \times \boldsymbol{\psi} = -m \, \boldsymbol{\psi} \tag{12}$$

Just reversing the sign flavors does not work. For the same QPAD ψ, the corresponding equation will contain extra terms:

$$\nabla\psi^{\textcircled{3}} = \nabla_0(\psi_0 - \boldsymbol{\psi}) + \boldsymbol{\nabla}(\psi_0 - \boldsymbol{\psi}) = \nabla_0\psi_0 - \nabla_0\boldsymbol{\psi} + \boldsymbol{\nabla}\psi_0 - \boldsymbol{\nabla}\boldsymbol{\psi} \tag{13}$$

$$= (m \, \psi_0 + \langle \boldsymbol{\nabla}, \boldsymbol{\psi} \rangle) - (-m \, \boldsymbol{\psi} - \boldsymbol{\nabla}\psi_0) + \boldsymbol{\nabla}\psi_0 - (\boldsymbol{\nabla} \times \boldsymbol{\psi} - \langle \boldsymbol{\nabla}, \boldsymbol{\psi} \rangle)$$

$$= m \, \psi + 2\langle \boldsymbol{\nabla}, \boldsymbol{\psi} \rangle + 2\boldsymbol{\nabla}\psi_0$$

Thus if the reverse equation fits, then it will concern another QPAD configuration ψ' that will not fit the original equation.

The pair $\{\psi'^{\textcircled{3}}, \psi'^{\textcircled{0}}\}$ that fits equation:

$$\nabla\psi'^{\textcircled{3}} = m \, \psi'^{\textcircled{0}} \tag{14}$$

represents a different particle than the electron $\{\psi^{\textcircled{0}}, \psi^{\textcircled{3}}\}$, which obeys equation (5). It also differs from the positron $\{\psi^{\textcircled{3}}, \psi^{\textcircled{0}}\}$, which obeys equation (7).

Where the electron couples to the general background QPAD $\psi^{③}$, the new particle couples to the conjugate $\psi^{⓪}$ of the general background QPAD $\psi^{③}$.

3.1.1.1 The coupling factor

Multiplying both sides of the equation of motion for the electron:

$$\nabla\psi^{⓪} = m\,\psi^{③} \tag{1}$$

with $\psi^{⓪}$ and then integrate over the full parameter space gives:

$$\int_V \psi^{⓪}\nabla\psi^{⓪}\,dV = m\int_V \psi^{⓪}\psi^{③}\,dV = m\int_V \left|\psi^{⓪}\right|^2 dV = m \tag{2}$$

Thus, the coupling factor m can be computed from the QPAD ψ.

3.1.2   The Majorana equation

The Majorana equation deviates from the Dirac equation in that is applies another sign flavor of the wave function QPAD ψ. That other sign flavor is still coupled to the general background QPAD $\psi^{③}$.

$$\nabla\psi^{①} = m_n\,\psi^{③} \tag{1}$$

The conjugated equation defines the anti-particle.

$$\nabla^{*}\psi^{②} = m_n\,\psi^{⓪} \tag{2}$$

The particle is represented by the ordered pair $\{\psi^{①}, \psi^{③}\}$. The corresponding flip does not switch the handedness.

The Majorana equation is thought to hold for neutrinos, which are neutral. Equation (2) will then hold for the anti-neutrino.

The coupling coefficient m_n for the neutrino follows from:

$$\int_V \psi^{⓪}\nabla\psi^{①}\,dV = m_n \int_V \psi^{⓪}\psi^{③}\,dV = m_n \int_V \left|\psi^{⓪}\right|^2 dV = m_n \tag{3}$$

3.1.3   The next particle type

We have exploited:

$$\nabla\psi^{⓪} = m\,\psi^{③} \tag{1}$$

and

$$\nabla\psi^{①} = m_n\,\psi^{③} \tag{2}$$

The next possibility would be:

$$\nabla\psi^{②} = m_d\,\psi^{③} \tag{3}$$

The conjugated equation is:

$$\nabla^*\psi^{①} = m_d\,\psi^{⓪} \tag{4}$$

The particle is represented by the ordered pair $\{\psi^{②}, \psi^{③}\}$. The corresponding flip does switch the handedness. Again the wave function QPAD is coupled to the general background QPAD $\psi^{③}$.

Like the electron, this particle will have charge, but its charge will be three times lower, because only one instead of three imaginary base vectors cause the switch in handedness. Of course, this is an opportunistic interpretation, but it seems to fit when we assume that the particle is a down quark with charge equal to -⅓e.

The formula for the coupling factor m_d is:

$$\int_V \psi^{⓪}\nabla\psi^{②}\,dV = m_d \int_V \psi^{⓪}\psi^{③}\,dV = m_d \int_V \left|\psi^{⓪}\right|^2 dV = m_d \tag{5}$$

The coupling that constitutes the down quark is anisotropic. This fact introduces a new kind if charge, which is called color charge. The color is related to the direction of the reflection of the wave function QPAD.

In summary the down quarks have the following properties:

- Location
    - Position
    - Momentum
- Electric charge
- Spin
- Color charge

3.2 Massles bosons

This paragraph treats the zero-level of coupling.

3.2.1 No coupling

The last possible form in which the wave function couples to the background field $\psi^{③}$ is:

$$\nabla\psi^{③} = m\,\psi^{③} \tag{1}$$

The formula for the coupling factor m is:

$$\int_V \psi^{\circledcirc} \nabla \psi^{\circled3} \, dV = m \int_V \psi^{\circledcirc} \psi^{\circled3} \, dV = m \int_V \left| \psi^{\circledcirc} \right|^2 dV = m \tag{2}$$

$$\int_V \psi^{\circledcirc} \nabla \psi^{\circled3} \, dV = \tfrac{1}{2} \int_V \nabla |\psi|^2 \, dV = 0 \tag{3}$$

Presence does not leak. So,

$$m = 0. \tag{4}$$

With other words a QPAD does not couple to itself.

3.2.2   The free QPAD

When for sign flavor ψ^x the coupling factor m is zero then:

$$\nabla \psi^x = 0 \tag{1}$$

$$\nabla_0 \psi_0^x = \langle \boldsymbol{\nabla}, \boldsymbol{\psi}^x \rangle \tag{2}$$

$$\boldsymbol{\nabla} \times \boldsymbol{\psi}^x + \boldsymbol{\nabla} \psi_0^x + \nabla_0 \boldsymbol{\psi}^x = \mathbf{0} \tag{3}$$

It means that a change $\nabla_0 \boldsymbol{\psi}^x$ in the speed of the current goes together with a rotation $\boldsymbol{B}$ of the current

$$\boldsymbol{B} = \boldsymbol{\nabla} \times \boldsymbol{\psi}^x \tag{4}$$

and/or a new field $\mathfrak{E}$:

$$\mathfrak{E} = -\boldsymbol{\nabla} \psi_0^x \tag{5}$$

For comparison, in the equations of Maxwell[38] the field $\boldsymbol{E}$ is defined as:

$$\boldsymbol{E} = -\boldsymbol{\nabla} \psi_0^x - \nabla_0 \boldsymbol{\psi}^x = \mathfrak{E} - \nabla_0 \boldsymbol{\psi}^x \tag{6}$$

In those equations $\boldsymbol{E}$ is the electric field and $\boldsymbol{B}$ is the magnetic field. However here these fields have a more general meaning.

Thus equation (3) means:

$$\boldsymbol{B} = \boldsymbol{E} \tag{7}$$

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[38]http://en.wikipedia.org/wiki/Maxwell%27s_equations#Potential_formulation

More interesting is the corollary

$$\nabla_0 \boldsymbol{B} = \nabla_0 \boldsymbol{E} \tag{8}$$

$$\nabla_0 \boldsymbol{B} = \boldsymbol{\nabla} \times \nabla_0 \boldsymbol{\psi}^x = \boldsymbol{\nabla} \times \mathfrak{E} - \boldsymbol{\nabla} \times \boldsymbol{\nabla} \times \boldsymbol{\psi}^x = -\boldsymbol{\nabla} \times \boldsymbol{\nabla} \times \boldsymbol{\psi}^x \tag{9}$$

$$= \boldsymbol{\nabla}^2 \boldsymbol{\psi}^x - \boldsymbol{\nabla} \langle \boldsymbol{\nabla}, \boldsymbol{\psi}^x \rangle = \boldsymbol{\nabla}^2 \boldsymbol{\psi}^x - \boldsymbol{\nabla} \nabla_0 \psi_0^x$$

$$\nabla_0 \boldsymbol{E} = -\boldsymbol{\nabla} \nabla_0 \psi_0^x - \nabla_0^2 \boldsymbol{\psi}^x \tag{10}$$

Thus

$$\boldsymbol{\nabla}^2 \boldsymbol{\psi}^x = -\nabla_0^2 \boldsymbol{\psi}^x \tag{11}$$

Or

$$\boldsymbol{\nabla}^2 \boldsymbol{\psi}^x = \boldsymbol{0} \tag{12}$$

Further

$$\nabla_0^2 \psi_0^x = \langle \boldsymbol{\nabla}, \nabla_0 \boldsymbol{\psi}^x \rangle = -\langle \boldsymbol{\nabla}, \boldsymbol{\nabla} \times \boldsymbol{\psi}^x \rangle - \langle \boldsymbol{\nabla}, \boldsymbol{\nabla} \psi_0^x \rangle = -\boldsymbol{\nabla}^2 \psi_0^x \tag{13}$$

Thus:

$$\nabla^2 \psi^x = 0 \tag{14}$$

With other words a free (= not coupled) ψ^x is either harmonic or it is static. The static condition corresponds to the ground state.

This holds for all QPAD's.

3.3   Reflection

We have now exhausted all possibilities for coupling a QPAD sign flavor to the general background QPAD $\psi^{③}$. Above we could link the analyzed particles to electrons, neutrinos and down quarks. Their antiparticles were treated as well. The investigated particles are elementary fermions. (*We did not state why they are fermions.*)

We also analyzed the situation that a QPAD is coupled to itself. That situation leads to zero coupling factor, which means no coupling. The free QPAD's are bosons and correspond to photons or gluons. (*We did not state why these "particles" are bosons.*)

The ordered pair {ψ^x, ψ^y } represents a category of elementary particle types.
For antiparticles all participating fields and the nabla operator conjugate.

We can now try to establish the apparent rules of the game. The rules are:

- If the wave function QPAD is coupled to the general background QPAD then the particle is a fermion. Otherwise, it is a boson.

- If the coupling takes place between two QPAD sign flavors with different handedness, then the corresponding particle is charged.
- The charge depends on the number and on the direction of the base vectors that differ.
- The count for each difference is ±⅓e.

No elementary particle type exists that obeys the above rules and at the same time features electric charge ⅔e. Such a particle may exist as a composite. Thus, according to these rules the up-quarks are not elementary fermions. For that reason, they do not belong to the standard model!

The elementary particles that are not yet covered are W_+ and W_- bosons and Z bosons. We like to proceed in a similar way, but the coupling with the background QPAD is used up. Now let us try other couplings.

We already encountered one, the ordered pair of sign flavors $\{\psi^{③}, \psi^{⓪}\}$ that obeys

$$\nabla\psi^{③} = m\,\psi^{⓪} \tag{1}$$

The coupling changes the handedness, so the particle is charged. It has much in common with the positron. Still it is not the anti-particle of the electron, because its equation of motion differs. It might exist, but then it probably hides behind the positron.

3.4 Limits of the model

The scheme does not distinguish between generations of elementary particle types.

It must be stated that the reason of being a fermion as it is applied here, differs strongly from the usual fermion/boson assignment. As a consequence also the notion of spin will differ. This approach is due to the fact that the elementary particles are defined as a pair of coupled QPAD's and not as a single wave function QPAD. Only the ordered pair will define the value of the spin and the fact that the particle is a fermion.

We adopt the existing convention that fermions go together with half integer valued spin. Here it will not be explained why that relation exists. With other words, having half integer spin and being a fermion is related on the one hand to experimental results and on the other hand to the ordered pair of coupled QPAD's that represents the particle. In the HBM a particle is a fermion when its wave function is coupled to the general background QPAD. What spin actually is, is not explained in this model. We just accept the existing convention. The same holds for the electrical charges.

In short, this model does not explain why particles get their electric charge or spin. The model only explains the origin and the habits of the coupling factor and it explains how the values of the electric charge and spin relate with the ordered coupled QPAD pair that represents the particle.

Later, it will be explained how the coupling factor relates to the mass of the particle.

3.5 Anisotropic coupling fields

We have explored all particles that make use of the isotropic background QPAD $\psi^{③}$. These particles appear to be fermions. Next we like to explore particles that couple to anisotropic backgrounds. These particles will appear to be bosons. It means that they all have integer valued spin. It is not explained why these particles are bosons or have full integer spin.

3.5.1 The cross-sign flavor equations

These equations describe the situation that a flip is made from a $\psi_i^{①}$ field to a $\psi_i^{②}$ field or vice versa. The direction related index i might or might not play a role.

$$\nabla \psi_i^{②} = m_{W_+} \psi_i^{①} \tag{1}$$

The conjugated equation is:

$$\nabla^* \psi_i^{①} = m_{W_-} \psi_i^{②} \tag{2}$$

Another form is

$$\nabla \psi_i^{①} = m_{W_-} \psi_i^{②} \tag{3}$$

The conjugated equation is:

$$\nabla^* \psi_i^{②} = m_{W_+} \psi_i^{①} \tag{4}$$

The sign flavor switch affects three imaginary base vectors and flips the handedness. As a consequence the particles have a full electric charge. It concerns two particles, the W^- and the W^+ bosons. These bosons carry electrical charges.

The W_+ and W_- bosons are considered to be each other's antiparticle. It is also possible that they hide between each other's antiparticle.

$$\psi_i^{②} \nabla \psi_i^{②} = m_{W_+} \psi_i^{②} \psi_i^{①} \tag{5}$$

$$\int_V \left(\psi_i^{②} \nabla \psi_i^{②} \right) dV = m_{W_+} \int_V \left(\psi_i^{②} \psi_i^{①} \right) dV = m_{W_+} g \tag{6}$$

$$\psi_i^{①} \nabla \psi_i^{①} = m_{W_-} \psi_i^{①} \psi_i^{②} \tag{7}$$

$$\int_V \left(\psi_i^{①} \nabla \psi_i^{①} \right) dV = m_{W_-} \int_V \left(\psi_i^{②} \psi_i^{①} \right) dV = m_{W_-} g \tag{8}$$

3.5.2 The Z boson

The particle that obeys:

$$\nabla\psi^{③} = m_Z\,\psi^{①} \tag{1}$$

Is a neutral boson.

$$\int_V \left(\psi_i^{②}\,\nabla\psi^{①}\right)\,dV = m_Z \int_V \left(\psi_i^{②}\,\psi_i^{①}\right)\,dV = m_Z\,g \tag{2}$$

Another possibility is:

$$\nabla\psi^{⓪} = m_{Z?}\,\psi^{②} \tag{3}$$

Again these particles may hide between each other's anti-particle.

3.6 General form

The general form of the equation for particle $\{\psi^x, \psi^y\}$ is:

$$\nabla\psi^x = m\,\psi^y \tag{1}$$

For the antiparticle:

$$\nabla^*\psi^{x*} = m\,\psi^{y*} \tag{2}$$

For all particles hold:

$$\nabla_0\psi_0 - \langle\boldsymbol{\nabla}, \boldsymbol{\psi}^x\rangle = m\,\psi_0 \tag{3}$$

$$\boldsymbol{\nabla} \times \boldsymbol{\psi}^x + \boldsymbol{\nabla}\psi_0^x + \nabla_0\boldsymbol{\psi}^x = m\,\boldsymbol{\psi}^y \tag{4}$$

The Fourier transform equivalents are

$$P_0\tilde{\psi}_0 - \langle\boldsymbol{P}, \tilde{\boldsymbol{\psi}}^x\rangle = m\,\tilde{\psi}_0 \tag{5}$$

$$\boldsymbol{P} \times \tilde{\boldsymbol{\psi}}^x + \boldsymbol{P}\tilde{\psi}_0 + P_0\tilde{\boldsymbol{\psi}}^x = m\,\tilde{\boldsymbol{\psi}}^y \tag{6}$$

P_0 is generally known as the Hamiltonian. $\boldsymbol{P}$ is the momentum.

$$\int_V \psi\psi^{③}\,dV = 1 \tag{7}$$

$$\tag{8}$$

$$\int_V \psi^{①}\psi^{②} \, dV = g$$

The factor g is real and non-negative.

Further, the equation for coupling factor m is:

$$\int_V (\psi^{y*}\nabla\psi^{x}) \, dV \; = m \int_V (\psi^{y*}\psi^{y}) \, dV = m \int_V |\psi^{y}|^2 \, dV \tag{9}$$

An equivalent of the Lagrangian may look like

$$\mathcal{L} \overset{?}{=} \psi^{y*}\,\nabla\psi^{x} - m\,\psi^{y*}\,\psi^{y} \tag{10}$$

3.7   Shadow particles

Several particle types have properties and behavior that is similar to the properties and behavior of antiparticle types of other particle types.

For example $\{\psi^{③}, \psi^{⓪}\}$ is hidden behind the positron which is the antiparticle of the electron $\{\psi^{⓪}, \psi^{③}\}$. The W_+ boson $\{\psi_i^{②}, \psi_i^{①}\}$ hides behind the antiparticle of the W_- boson $\{\psi_i^{①}, \psi_i^{②}\}$ and vice versa.

The boson $\{\psi_i^{⓪}, \psi_i^{②}\}$ hides behind the antiparticle of $\{\psi_i^{③}, \psi_i^{①}\}$ and vice versa.

3.8   Resulting particles

Under the resulting particles fall 12 multi-color couplings. It are anisotropic QPAD's that are coupled to other anisotropic particles that only differ with respect to their color. They show great resemblance with mesons.

Further, a set of particles exist that are shadow particles of down-quarks or anti-down-quarks.

3.9   Anti-particles

Just like the universe is filled with a huge number of particles, it is also filled with a huge number of anti-particles. Otherwise the anti-particles would not sense the same kind of inertia that particles do.

This anti-world has its own kind of general background QPAD.

We christen this version the "background anti-QPAD".

It represents the local superposition of the tails of the wave functions of all anti-particles in universe.

4 Origin of curvature

The primary QPAD's cause a local pressure in the QPAD-sphere. This only occurs when the QPAD is coupled to another QPAD. Both QPAD's must be sign flavors of the same base QPAD.

On its turn the local pressure causes a local space curvature.

4.1 Physical fields

Primary QPAD's are not observable. The properties that characterize an elementary coupling can be interpreted as sources/drains of new QPAD's that we will christen secondary QPAD's.

These secondary QPAD's become noticeable as corresponding physical fields.

With other words, these physical fields use the properties of primary couplings as their sources/drains.

Wave function QPAD's and background QPAD's transport tiny patches of the parameter space. Since these patches are added to or taken from the parameter space, these patches can be interpreted as tiny virtual sources or drains. Thus, where the carriers that are transported by wave functions are tiny virtual sources/drains, the sources/drains of physical fields are actual sources/drains.

4.2 Curvature and inertia

All primary couplings affect the local curvature

Only the wave function QPAD's that couple to a background QPAD will experience inertia

This also holds for anti-particles and their background anti-QPAD

4.2.1 Inertia versus anti-particle

Besides of the fact that all particles possess corresponding anti-particles, each wave function category seems to correspond with a corresponding background QPAD, which in some cases is the conjugate of the wave function QPAD. However, fermions and their anti-particles seem to prefer isotropic background QPAD's. Fermions share the same (isotropic) background QPAD and anti-fermions do the same.

4.2.2 Inertia of W and Z Bosons

W and Z bosons use their "own" background QPAD, which is anisotropic.

The background QPAD of a W_- boson has the form of the wave function QPAD of a W_+ boson, which has the form of an anti-W_- boson.

The background QPAD of a W_+ boson has the form of the wave function QPAD of a W_- boson, which has the form of an anti-W_+ boson.

The background QPAD of a Z boson has the form of the form of the wave function QPAD of the anti-Z boson.

4.3 Effect of primary coupling

The coupling may compress or decompress the local parameter space.

Above, 56 different kinds of primary coupling are discerned.

Together with the balance equations that differ for particles and anti-particles and shadow particles this defines a large number of ways of how the local parameter space is affected.

5   Higher level coupling

The primary QPAD's cause a local pressure in the QPAD-sphere
On its turn that local pressure causes the local space curvature.

The streams of space patches that result after the primary couplings will be used in higher level interactions. It means that these resulting currents may still influence higher level coupling.

Hypothesis:
In these interactions the properties of the primary couplings are conserved

5.1   The Kerr-Newman equation

The Kerr-Newman metric equation gives a rough impression on how this works. It uses the sources/drains of physical fields and not the primary couplings. For that reason it delivers only a course approximation.

The Kerr-Newman equation describes the effects of physical fields on curvature for elementary particles as well as for black holes

5.2   Role of secondary QPAD's

The properties that characterize primary couplings act as sources/drains of secondary QPAD's

The primary couplings are responsible for affecting the local curvature, but it looks as if the secondary QPAD's have this role

This is a false impression!

5.3   Higgs

This story does not include a Higgs particle or a Higgs field. However, the Hilbert Book model uses a background field that acts as a partner in the coupling of that background field to the wave function of an elementary fermion. That background field takes the role of the Higgs field. It implements inertia. This simplified picture ignores that every coupled QPAD in some way affects curvature.

6 HADRONS

The HBM cannot discern generations of elementary particles. For that reason the elementary fermions are treated per category.

Symbol e or e_- means electron

Symbol p or e_+ means positron

Symbol n means neutrino

Symbol d means down-quark

Symbol u means up-quark. But in the HBM up-quarks are no elementary particles.

6.1   Second level coupling

This chapter treats the second level of coupling. It treats couplings between elementary particles. This coupling uses the secondary QPAD's that are generated by the properties of the first level coupling. The Hilbert Book Model delivers the reason of existence of these properties; coupling factor, electric charge, angular momentum (spin) and possibly color charge of elementary particles. In higher level couplings these properties are conserved.

The primary couplings influence the local curvature. However, the primary QPAD's are not observable. Only the secondary QPAD's become observable. Thus, it looks as if the secondary QPAD's or their sources/drains are responsible for affecting curvature. This, of course, is a false expression.

Still the currents in the coupled QPAD's that remain after coupling may still influence higher level coupling.

The curvature is also a binding ingredient for the next levels of coupling. The formulas that describe the influence of the conserved properties on the curvature inform what will happen. Currently the best available formula is the Kerr-Newman metric formula. The Kerr-Newman equation works on the base of locally existent properties (the sources or drains of the secondary QPAD's).

At a fixed instant of proper time, thus inside a single page of the Hilbert Book Model, the Kerr-Newman equation is a static equation.

If in a subsequent HBM page the controlling properties have changed, then those new properties define the new configuration.

The first level coupling that constitutes elementary fermions uses a background field that is the partner in the coupling of that background field to the wave function of the elementary fermion.

The background field represents the superposition of the tails of the wave functions of all massive particles that exist in the universe. In this way inertia gets its implementation.

Many primary couplings do not involve coupling to the general background QPAD. Thus the corresponding coupling does not so much inflict inertia. However, the QPAD's that are coupled to the wave function QPAD's may glue particles together.

Hadrons are the first of the next levels of binding products.

The Kerr-Newman equation shows an abnormality at the place where black holes get their horizon. Whether or not a hadron possesses a horizon is in this respect unimportant. The properties of the elementary particles that are bound together in order to form the hadron are sources of secondary QPAD's. The static versions of these secondary QPAD's reach beyond a possible horizon. To the outside world the superposition of these fields signal the properties of the hadron.

The primary QPAD's that constitute the elementary particles and that on their turn constitute the hadron, also reach beyond the potential horizon.

6.2   Interaction

The common way of treating interactions in quantum field theory is to apply a methodology called covariant derivation. This methodology works well in a complex representation, but fails in a quaternionic approach. With other words it works in special one-dimensional cases but it is not well suited for multidimensional cases. So, the HBM must find another approach in order to implement interactions. The HBM also does not consider the existence of virtual particles. However, this is compensated by the availability of primary QPAD's. These primary QPAD's describe the probability of presence of virtual carriers of charges, where the charges can be ensembles of properties of that carrier. Superpositions of tails of these QPAD's can construct temporary couplings. These couplings will in fact represent temporary virtual particles. This deliberation leads to the acceptance of the superposition of the tails of QPAD's as the medium that implements the feature of interaction.

6.3   Rules

The second level of coupling has its own set of rules.

- The total color of the composite must be white[39].
- The properties of the constituting particles will be conserved.
- However, mass (coupling factor) may be exchanged against field energy.
- Field energy is transported via oscillating QPAD's.
- Like the fermions, hadrons exist in generations.

The properties of the elementary particles, including color, play a significant role in the Pauli principle.
Coupling factor, electric charge, angular momentum (spin) and color charge are sources of secondary fields.
Location (position or momentum) are not sources of secondary fields.

6.4   Up-quarks

In the HBM the up-quarks cannot be constructed from a simple coupling of QPAD sign flavors. Therefore the HBM does not consider them to be elementary particles. Instead up-quarks are

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[39] In observable particles, which are particles that may be detected in measuring machines like the LHC, the color is always white.

composed of down-quarks and W+ particles or from down-quarks, positrons and neutrinos. Further, the up-quarks annihilate into these constituents.

Like down-quarks the up-quarks have color. This attribute relates to the direction of the reflection of the constituting down-quark.

Apart from up-quarks also anti-up-quarks exist.

6.5   Mesons

Mesons are composed out of quarks and anti-quarks. The following meson categories exist.

- (d$\underline{d}$+u$\underline{u}$)/√2
- (d$\underline{d}$-u$\underline{u}$)/√2
- u$\underline{d}$
- d$\underline{u}$

However, it might well be that the mixed color elementary particles are in fact mesons.

6.6   Baryons

Baryons are composed out of triples of quarks. The following baryons exist[40].

Symbol	configuration	name	Isospin I_3
Δ^{--}	ddd	delta	$-^3/_2$
Δ^-	ddu	neutron	$-^1/_2$
Δ^+	uud	proton	$^1/_2$
Δ^{++}	uuu	delta	$^3/_2$

For the anti-baryons the possibilities are:

- $\underline{ddd}$
- $\underline{ddu}$
- $\underline{uud}$
- $\underline{uuu}$

I relation to the Pauli principle the versions with multiple u or multiple d can only exist due to the fact that the constituting quarks have different spin and/or different color charge.
t

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[40] The generations are ignored!

7   COSMOLOGY

Cosmology concerns all particles with a mass above the limiting mass $\mathfrak{M}$.

7.1   Higher order couplings

It is assumed that during higher order couplings the constituting elementary particles keep their basic properties;

- coupling factor,
- electric charge
- angular momentum.

The properties that characterize the coupling event in elementary particles are sources of secondary fields. These fields are known as physical fields. For example the electric charge is a source for electromagnetic fields. The coupling factor is a source for the gravitation field. The spin also causes a field.

A secondary field is a specialized QPAD that has one of the properties of the elementary particle as its isolated source.

These secondary fields play a major role in the higher order couplings. The reason for this fact is that the sources of the secondary fields influence the curvature of the parameter space.

The composite particles can be considered to have wave functions that are formed by the superposition of the wave functions of the constituting particles. However, at least part of these particles consist of coupled pairs of QPAD's in which one is a wave function QPAD and the other is a background QPAD. With other words, a composite particle is a coupling between a superposition of a number of wave function QPAD's and a superposition of a number of background QPAD's.

It means that the wave function of the composite is a superposition of a set of QPAD's that have different sign flavors. However, the same holds for the superposition of the background QPAD's.

This fact would mean that higher order coupling is not well described by simple wave equations as those that describe elementary particles. Instead it may be better described by an equation that describes the dependence of the local curvature on the locally existing coupling properties.

An equation that does something like that is the Kerr-Newman metric equation. However, the Kerr-Newman equation produces an abnormality when the limit[41]

$$m \geq \sqrt{r_Q^2 + (S/cm)^2}$$

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[41] Appendix; Kerr-Newton limit

is passed. Above that limit the particle is encapsulated and guarded by a skin in the form of a horizon. Below that limit the particle is naked.

~~By selecting an adapted coordinate system the above geometric abnormality fades away[42].~~

7.2   Curvature

7.2.1   Hilbert Book Model ingredients

Each page of the Hilbert Book Model consists of three quite independent ingredients.

> Ingredient 1: The quantum logic, or equivalently, its lattice isomorphic companion; the set of closed subspaces of an infinite dimensional separable Hilbert space

> Ingredient 2: A background coordinate system that is taken from the continuum eigenspace of an operator that resides in the Gelfand triple of the separable Hilbert space.

> Ingredient 3: A set of QPAD's that each couple an eigenvector of a particle locator operator that resides in the Hilbert space to the background coordinate system.

Couplings between QPAD's that lead to elementary particles are characterized by three categories of properties:

- Coupling factor
- Electric charge
- Angular momentum

These properties influence the curvature that affects the third ingredient.

The way that these properties influence curvature is described by metric equations, such as the Kerr-Newman metric formula.

The three ingredients have their own properties and habits. For example the QPAD's may feature a maximum speed of information transfer, while the curvature of the background coordinate system acts instantaneously on changes of the controlling properties or takes a shortcut over flat space rather than over curved space..

7.2.2   Macroscopic black hole features

A black hole can be considered as a geometric abnormality. Since light is the carrier of it, information can pass nor leave the skin of a black hole. No distant observer can ever see that a macroscopic black hole (MBH) absorbs something. Still intelligent observers know that an observed MBH has grown to its current size. The observer derives that information from features in the surround of the MBH. However, these features must already have received enough information about the properties of the MBH. Otherwise, the intelligent observer could not have derived his knowledge from those features. Thus the features got the message about the properties of the MBH by a messenger that goes far faster than light can go.

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[42] See below: Coordinate system.

A possible explanation is the fact that the spread of the influence of the three properties; coupling factor, electric charge and angular momentum have on curvature acts instantaneously. This influence runs over the whole extent of the universe.

An alternative explanation is that the spread of curvature runs with the speed c of information transfer, but that it does not run over curved space, but instead it runs over flat space. This means that the spread still takes time, but uses a shortcut with respect to what light can achieve.

In comparison the transport of information runs via QPAD's and is limited by the maximum speed in that medium. That is the speed of light. These facts can be explained by the difference between the habits of the corresponding media.

For a part, the features that are described here for black holes also hold for other geometric abnormalities that have a much smaller scale.

7.2.3 Curvature

The coordinate system that is taken from the eigenspace of an operator that resides in the Gelfand triple is not applied directly. Instead a quaternionic distribution that uses the values of the flat coordinate system that is taken from the Gelfand triple as its parameters is used as the observed coordinate system.

Curvature can be described by the combination of a preselected coordinate system that defines location in a non-curved space and a local metric that describes the curvature in terms of that coordinate system. As is described above, the flat coordinate system is taken from the Gelfand triple.

7.2.4 Coordinate system

Several coordinate systems are possible. The most common coordinate systems for a non-curved three dimensional space are:

- Cartesian coordinates
- Spherical coordinates

Alternatives for spherical coordinates are:

- Schwarzschild coordinates[43]
- Kruskal-Szekeres coordinates[44]
- Lemaitre coordinates[45]
- Eddington–Finkelstein coordinates[46]

The advantage of the alternative coordinates is that they avoid unnecessary singularities. However, these alternatives are only relevant for situations in which the Schwarzschild radius plays a

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[43] http://en.wikipedia.org/wiki/Schwarzschild_coordinates
[44] http://en.wikipedia.org/wiki/Kruskal-Szekeres_coordinates
[45] http://en.wikipedia.org/wiki/Lemaitre_coordinates
[46] http://en.wikipedia.org/wiki/Eddington%E2%80%93Finkelstein_coordinates

significant role. This is certainly the case for black holes and their environment, but it becomes irrelevant in the realm of some elementary particles.

7.2.5   Metric

The currently best suitable local metric equation for our purposes is the Kerr-Newman metric[47]. It uses three local properties. These properties are:

- The coupling factor m
- The electric charge Q
- The angular momentum J

The angular momentum J includes the spin s.

This metric uses the sum of a category of properties that are collected within the observed sphere. However, in principle the summation produces different centers of activity for different property categories. Thus, these centers need not be at the same location. However, for large enough selected radius r and applied to black holes or single particles, these centers coincide.

The simplest interpretation of the Kerr-Newman metric can be taken on the surface of a sphere that has a selected radius r.

The formula uses three characteristic radii. The largest characteristic radius plays the most prominent role.

This fact introduces the notion of geo-cavity.

7.2.6   Scales

The charge-to-mass ratio Q/M is typically larger in smaller systems[48]. For most astrophysical systems,

$$Q/M \ll 1, \tag{1}$$

while for a Millikan oil drop,

$$Q/M \approx 10^6. \tag{2}$$

Going all the way down to elementary particles, the value for the electron is

$$Q/M \approx 10^{21}. \tag{3}$$

To achieve balance we require that Newton's gravitational force f_N has the same magnitude as Coulomb's force f_C, that is

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[47] Appendix;Metric tensor field;Local metric equation

[48] For deeper investigation, see: http://arxiv.org/abs/0802.2914

$$|f_N| = |f_C| \qquad (4)$$

To be more specific, let us assume that $Q = e$ where e is the elementary charge. We then adjust the mass M to the value for which the forces are balanced. This gives the Stoney mass[49]

$$M = m_S = \frac{e}{\sqrt{G 4\pi\epsilon_0}} = 1.85921 \times 10^{-9} \text{ kg} \approx 2 \text{ } \mu g \qquad (5)$$

It is only one order of magnitude lower than the Planck mass

$$m_P = \sqrt{\hbar c / G} = 2.17644(11) \times 10^{-8} \text{ kg} \approx 20 \text{ } \mu g \qquad (6)$$

The ratio between them is given by the square root of the fine structure constant,

$$\alpha = \frac{e^2}{c\hbar 4\pi\epsilon_0} = (137.035999679)^{-1} \qquad (7)$$

$$\frac{m_S}{m_P} = \sqrt{\alpha} \approx 0.1 \qquad (8)$$

Thus, in case of electric charges, the Coulomb forces are nearly in balance with the gravitational forces at the Planck scale. However, at subatomic scale this picture is disturbed by the spin.

For subatomic systems there is an additional phenomenon which comes into play. In fact, according to general relativity, the gravitational field tends to become dominated by the spin at distances of the order of the Compton wavelength. The relevant quantity which governs this behavior is the ratio S/M^2 where S is the (spin) angular momentum. For an electron,

$$S/M^2 \approx 10^{44}. \qquad (9)$$

As a consequence, the gravitational field becomes dominated by gravitomagnetic effects in the subatomic domain. This fact has important consequences for the electromagnetic fields of spinning charged particles.

The four known gravitational and electromagnetic multi-pole moments of the electron are:
- the mass m_e,
- the spin $S_e = \hbar/2$,
- the charge e
- the magnetic moment $\mu = \frac{e S_e}{m_e}$

The spin is a gravitomagnetic dipole moment, i.e. a gravitational analogue of the magnetic dipole moment.

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[49] http://en.wikipedia.org/wiki/Natural_units

$$\frac{S_e}{m_e} \gg e \gg m_e \tag{10}$$

The corresponding Kerr-Newman field is therefore dominated by the spin in the subatomic domain. In particular, it has no event horizon and it has no ergo-region. (The ergo-region is a region of space-time located outside the event horizon of a rotating black hole where no object even if traveling at the speed of light, can remain stationary.)

An important conclusion is that gravity tends to become spin dominated in the subatomic domain.

The Kerr-Newman metric formula indicates that small particles that are encapsulated by a horizon are restricted by the limit:

$$m \geq \sqrt{e^2 + (S_e/m)^2} \tag{11}$$

Where m is the particle mass, e is the elementary charge and S_e is the elementary spin.
This means that protons fall above that limit.

7.3   Inside black holes

Objects that fulfill the rules for the existence of a BH horizon hide their internals. Their construction is similar to that of a massive elementary particle. That means that nothing is inside that horizon than a set of coupled QPAD's. One of these is a background QPAD. The others form a superposition. This superposition is a kind of super QPAD. It let the BH act as one particle that has the properties of the combination of the gathered fields.

Some elementary particles fulfill the requirements for a black hole. The difference between an elementary particle and a non-elementary particle is that the wave function is not a pure single sign flavor QPAD. The non-elementary wave function is a superposition of QPAD's that belong to different sign flavors.

7.4   Hadrons

The Hilbert Book Model delivers the reason of existence of the properties; coupling factor, electric charge and angular momentum (spin) of elementary particles. In higher level couplings these properties are conserved. These properties influence the local curvature. The curvature is the binding ingredient for this next level. The formulas that describe the influence of the conserved properties on the curvature control what is happening. Currently the best available formula is the Kerr-Newman metric formula.
This story does not include a Higgs particle or Higgs fields. However, the Hilbert Book model uses a background field that is one partner in the coupling of that background field to the wave function of an elementary fermion. The background field represents the superposition of the tails of the wave functions of all massive particles that exist in the universe. In this way inertia gets its implementation. The coupling event is characterized by a set of properties. These are the mentioned properties of the elementary particles.

Hadrons are the first of the next levels of binding products.

The Kerr-Newman equation shows an abnormality at the place where black holes get their horizon. Whether or not a hadron possesses a horizon is in this respect unimportant. The properties of the elementary particles that are bound together are sources of secondary fields. These fields reach beyond a possible horizon. To the outside world the superposition of these fields signal the properties of the hadron.

8 THE BUILDING

The building consists of everything that can be assembled from hadrons and fields.

8.1 Natures Music

In atoms electrons oscillate around a central point that acts as the location of the particle[50]. The centrally located nucleus fulfills the rules for black holes. In molecules the same features occur in an even more complicated configuration.

The oscillations are harmonic. The most basic harmonic oscillations are, apart from a scalar factor, invariant under Fourier transformation[51]. These oscillations have modes and usually several of these modes exist in parallel. These modes can be generated and annihilated. Generation goes together with the absorption of a more elementary particle and annihilation goes together with a corresponding emission.

8.2 Hydrogen atom

In the hydrogen atom[52] one electron encircles the nucleus. The oscillation of the electron can be described as a spherical harmonic oscillation. It can have different oscillation modes. These modes are characterized by quantum numbers. Mode switching is activated by creation and annihilation operators. The speed of the electrons is high enough such that relativity effects must be considered. Also the spin of the electron plays a role and causes a magnetic effect.

8.3 Helium atom

In the helium atom[53] two electrons encircle the nucleus. In principle the electrons behave similarly as in the hydrogen atom, however due to the Pauli principle they cannot both occupy the same oscillation mode. The electrons not only interact with the nucleus, but they also interact with each other.

8.4 Modularization

Hadrons, atoms and molecules are products of a modularization process.

Modularization[54] encapsulates properties in a higher order individual and renders the resulting individual a specific behavior. Its main purpose is that the number of relations to the outside world is reduced. Usually the module can be accessed via a series of well-defined interfaces. The whole keeps the integrity of the individual intact.

Modularized systems are far simpler than their monolithic equivalents. Modularization exploits reuse. The modularization can have far reaching consequences. That is especially the case when

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[50] Appendix; Oscillations

[51] Appendix; Functions invariant under Fourier transformation.

[52] http://en.wikipedia.org/wiki/Hydrogen_atom

[53] http://en.wikipedia.org/wiki/Helium_atom

[54] http://vixra.org/abs/1101.0064

modules can be used to create a new kind of modules. In this way nature is capable of constructing very complicated systems. On earth nature achieved the stage to be able to generate intelligent species. This tendency goes straightly against the tendency that is set by the second law of thermodynamics. These laws prescribe that disorganization, randomness and chaos will increase continuously.

8.5 Black hole

8.5.1 Classical black hole

According to classical mechanics the [no-hair theorem][55] states that, once a black hole achieves a stable condition after formation, it has only three independent physical properties:

- mass,
- charge, and
- angular momentum.

The surface gravity[56] κ may be calculated directly from [Newton's Law of Gravitation][57,] which gives the formula

$$\kappa = \frac{Gm}{r^2} \tag{2}$$

where m is the mass of the object, r is its radius, and G is the [gravitational constant][58]. If we let $\rho = m/V$ denote the mean density of the object, we can also write this as

$$\kappa = \frac{4\pi}{3} G\rho r \tag{3}$$

For fixed mean density ρ, the surface gravity κ is proportional to the radius r.

[Sciama][59] relates G to the potential that is raised by the community of particles. For fixed mean density ρ this is shown by

$$\Phi = -\int_V \frac{\rho}{r} dV = -\rho \int_V \frac{dV}{r} = \rho 2\pi R^2 \tag{4}$$

$$G \approx \frac{-c^2}{\Phi} = \frac{-c^2}{\rho 2\pi R^2} \tag{5}$$

[55] http://en.wikipedia.org/wiki/No-hair_theorem
[56] http://en.wikipedia.org/wiki/Surface_gravity
[57] http://en.wikipedia.org/wiki/Newton%27s_Law_of_Gravitation
[58] http://en.wikipedia.org/wiki/Gravitational_constant
[59] Influence;Inertia

Here R is the current radius of the universe.

8.5.2 Simple black hole

The Schwarzschild radius r_S for a non-rotating spherical black hole is

$$r_S = \frac{2Gm}{c^2} \tag{1}$$

8.5.3 General black hole

More generally holds

$$dM = \frac{\kappa}{8\pi} dA + \Omega dJ + \phi dQ \tag{1}$$

where
- M is the mass/energy,
- A is the horizon area,
- Ω is the angular velocity,
- J is the angular momentum,
- ϕ is the electrostatic potential,
- κ is the surface gravity,
- Q is the electric charge.

For a stationary black hole, the horizon has constant surface gravity.
It is not possible to form a black hole with surface gravity. $\kappa = 0$.

8.5.4 Quantum black hole

When quantum mechanical effects are taken into account, one finds that black holes emit thermal radiation (Hawking radiation) at temperature

$$T_H = \frac{\kappa}{2\pi} \tag{1}$$

A quantum black hole is characterized by an entropy S and an area A.
The entropy of a black hole is given by the equation:

$$S = \frac{c^3 kA}{4\hbar G} \tag{2}$$

The Bekenstein-Hawking Entropy of three-dimensional black holes exactly saturates the bound

$$S = \frac{kA_P}{4} \tag{3}$$

where A_P is the two-dimensional area of the black hole's event horizon in units of the Planck area,

$$A_P = l_p^2 = \frac{\hbar G}{c^3}. \tag{4}$$

In the Hilbert book model this equals the number of granules that covers the horizon of the black hole.

The horizon of the black hole is an event horizon because information cannot pass this horizon. (Near the horizon the speed of light goes to zero.)

8.5.5 Holographic principle

The holographic principle[60] states that the entropy contained in a closed surface in space equals the entropy of a black hole that has absorbed everything that is contained in this surface.

In the Hilbert book model it means that if the surface is considered as a sparsely covered horizon, then that sparse horizon contains as many granules as the densely covered horizon of the corresponding black hole.

It also means that the maximum entropy that can be contained inside a surface corresponds to a dense coverage with granules of that surface.

In the Hilbert book model, any dense or sparse horizon reflects via its contained entropy the number of granules that are contained in the corresponding volume.

We might extend this picture by stating that the number of granules in a volume corresponds with the entropy in the volume. In the Hilbert book model the number of granules corresponds to the number of Hilbert vectors that are attached to a QPAD. It also corresponds to the number of anchor points of the primary physical fields.

The eigenvectors of the particle locator operator $\mathfrak{S}$ correspond to quantum logical propositions that represent the location of physical particles. These propositions have a binary yes/no value. In the extended model these propositions get extra content via the attached QPAD's.

8.5.6 Chandrasekhar limit

The Chandrasekhar limit[61] is an upper bound on the mass of a stable white dwarf star:

$$M_{limit} = \frac{\omega_3^0 \sqrt{3\pi}}{2} \left(\frac{\hbar c}{G} \right)^{3/2} \frac{1}{(\mu_e m_H)^2} \tag{1}$$

where:

- $\hbar$ is the reduced Planck constant
- c is the speed of light
- G is the gravitational constant

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[60] http://en.wikipedia.org/wiki/Holographic_principle
[61] http://en.wikipedia.org/wiki/Chandrasekhar_limit

- μ_e is the average molecular weight per electron, which depends upon the chemical composition of the star.
- m_H is the mass of the hydrogen atom.
- $\omega_3^0 \approx 2.018236$ is a constant connected with the solution to the [Lane-Emden equation](#).

Approximately:

$$M_{limit} \propto \frac{M_P^3}{m_H^2}. \tag{2}$$

Where

$$M_P = \sqrt{\frac{\hbar c}{G}} \text{ is the Planck mass} \tag{3}$$

8.6 Similarity between black hole and massive fermion

According to the no hair theorem a black hole shows only a few properties to the outside world. These properties are sources of secondary fields.

Massive fermions have a similar behavior.

Apart from the exposed features it is impossible to observe what goes on inside the black hole.

It is possible that the internals of a black hole are to a certain extent similar to the construction of a massive fermion. It would mean that the BH is based on a set of QPAD's that each are coupled to a background QPAD.

8.7 Birth of the universe

The unit sphere of the separable Hilbert space $\mathcal{H}$ is an affine space. All unit size eigenvectors end in this sphere.

The eigenvectors of the particle locator operator are exceptional. They are surrounded by a QPAD *that installs the tendency to keep these vectors together*. The parameter of these distributions is taken from a background coordinate system. This means that also the eigenvectors of the particle locator operator possess a position in this background coordinate system. The background coordinate system is formed by the eigenspace of an operator that houses in the Gelfand triple $\mathcal{H}$ of the Hilbert space $\mathcal{H}$. The coupling between the eigenvectors of the particle locator operator and the eigenspace of the operator in the rigged Hilbert space that provides the background coordinate system is not precise. It is stochastic and of the order of the Planck-length. That is why the granules have this size.

The eigenvectors of the particle locator operator all touch a granule. The relation with quantum logic means that the Hilbert vector stands for a proposition that has a yes/no value. In case of the Hilbert vectors that are attached to the granules the yes/no value represents group membership. Thus each granule represents a bit of information.

For the eigenvectors vectors of the particle locator operator a densest packaging exists. It means that in that condition the QPAD's have shrunk to their smallest possible location difference.

Assumption 1: *In that condition, due to the properties of the QPAD's, the mutual tension works asymmetrically.*

This asymmetry means that in a surface that is formed by a set of densely packed granules the tension on one side is stronger than the surface tension at the other side. As a consequence the final configuration of a densest packaging becomes an empty bubble.

In the starting condition all eigenvectors of the particle locator operator are densely packed in one assembly.

Assumption 2: *After that moment the packaging density relaxes.*

The number of granules does not change. Thus, during this spreading the total entropy does not change.

The package may fall apart in several separated subassemblies and a large series of single or more loosely packed granules. For the single and the more loosely packed granules the corresponding QPAD's fold out. The densely packed subassemblies take again a bubble shape.

This process may occur instantly or gradually, but most probably it will be done in a sequence of these two possibilities.

First occurs a sudden change of scale between the particle locator operator in the separable Hilbert space Ӈ and the GPS operator that delivers the background coordinate system and that resides in the rigged Hilbert space Ħ. It is possible that originally the bubble covered the whole of the unit sphere of the Hilbert space Ӈ, or it may just cover a finite dimensional subspace of Ӈ. This means that the bubble contains an infinite or a finite amount of granules, which suddenly get diffused in a much larger space. That space is affine like the unit sphere of the Hilbert space Ħ. The diffusion takes place at every occupied location in the background coordinate system.

This kind of universe has no spatial origin or it must be the center of the outer horizon. With the aid of the background coordinate system, it will be possible to indicate a center of that universe. Each item in this universe has its own private information horizon. This horizon is set by the reach of the light that has been travelling since the birth of the universe. As long as this light does not reach the outer horizon that sub-universe looks isotropic. A multitude of such sub-universes exist that need not overlap. However, they all look at their border at an image of part of the start horizon. Such, sub-universes obey the [cosmological principle](http://en.wikipedia.org/wiki/Cosmological_principle)[62].

In the next phase the further expansion occurs gradually. Because the QPAD's that are attached to the granules install a tendency for the granules to stay together, a different motor must be pre-

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[62] http://en.wikipedia.org/wiki/Cosmological_principle

sent behind this expansion. This motor can be found in the fact that with increasing radius the number of pulling granules grows faster than the decrease of the forces that are executed by the fields of these granules that is caused by the increasing distance. In an affine space this is always and everywhere true. This effect is also the source of inertia.

Due to local attraction, loosely packed and single granules may reassemble in bubble shaped subassemblies. These subassemblies are known as black holes. Single granules and small aggregates of granules are known as elementary particles, nuclei or atoms.

Much larger aggregates may be formed as well but these are not densely packed. Elementary particles can be categorized according to the configuration of their private fields. The private fields determine whether the particle is matter, with other words whether it has mass or not.

Inside the bubble the fact that the granule represents matter is not recognizable. It is only recognizable when the attached QPAD gets the chance to unfold. That condition is true when the granule is not part of a densely packed subassembly.

The requirements for the birth of the universe are:
1. The existence of a particle locator operator
2. The existence of QPAD's that install the tendency to keep these eigenvectors of the particle locator operator together
3. When the large numbers of eigenvectors are densely packed, then the assembly forms a bubble, because due to the properties of the QPAD's, the mutual tension works asymmetrically
4. In advance the eigenvectors of the particle locator operator are densely packed in one bubble.
5. A non-zero probability exists that the package density will be relaxed and the package falls apart. This may happen in a two stage process
   a. A sudden reduction of scale occurs
   b. Next a force that pulls the granules further away from each other exists

In the first episode of the universe the sudden scale change took place. This ripped the original bubble apart. Next a gradual further expansion took place.

The granules that move freely can at the utmost take one space step at every progression step. When the ratio of the space step and the progression step is fixed, then this determines a maximum speed of granules. A certain type of granules takes a space step at every progression step. That type transports information at the maximum possible speed.

When the path of these information transmitting particles is a straight line, then after a while, the other types of granules no longer get messages from the birth episode of the universe. But this need not be the case.

Since the messenger has a finite speed, it brings information from the past. First of all the speedy messenger and the slow addressee may have started from different locations. Further, due to curvature of space the path of the speedy messenger may take much longer than the duration of the much straighter path that the much slower addressee has taken. The information about the past that is included in the message might be close to the episode in which the granules were combined in one large bubble.

Thus despite the fact that most of the information that is generated during the birth of the universe is long gone, still some of that information may reach particles long after the instance of birth. When this information is interpreted it gives the impression of a metric expansion of the universe[63].

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[63] http://en.wikipedia.org/wiki/Metric_expansion_of_space

9   CONCLUSION

It is quite possible to build a model of physics on a solid axiomatic foundation. It has the advantage that from the beginning the model stays consistent and trustworthy. For the Hilbert Book Model this inroad has brought some rather revolutionary deviations from contemporary physics. The way that fields are treated and how dynamics is implemented differs strongly from the ordinary course of physics. Through the switch from complex Hilbert spaces to quaternionic Hilbert spaces and the attention that is given to sign flavors of quaternionic probability amplitude distributions it becomes possible to derive unique continuity equations rather than equations of motion that reveal the properties and habits of all known elementary particles.

This step only reaches to the first level of binding. The properties of the coupling that occurs inside elementary particles form the factors that influence the local curvature. The current status of the model already indicates that the next level of particle binding will use the effects of the coupling properties on the curvature of the local geometry rather than the coupling of sign flavors of quaternionic probability amplitude distributions. It means that in the higher level binding the role of the metric equation will be greater than the role of the wave equation. This step is deterministic, while the first level of coupling is afflicted with indeterminism.

PART III

Appendix

The appendix is a toolbox and a grab bag that contains everything that can be used to build or analyze the Hilbert Book Model

1 Logic

1.1 History of quantum logic

Around 1930 John von Neumann and Garrett Birkhoff were searching for an acceptable explanation of the results of experiments that showed that the execution of an observation of a very small object can completely destroy the validity of an earlier observation of another observable of that object. The Schrödinger equation that agreed with the dynamic behaviour of the particles already existed. Not much later Heisenberg's matrix formulation became popular as well. Quite soon the conclusion was made that something was fundamentally wrong with the logic behind the behaviour of small particles. These small objects show particle behaviour as well as wave behaviour and they show quantization effects. It was found that the distribution axiom of classical logic had to be changed. Soon it became apparent that the lattice structure of classical logic must be weakened from an ortho-complementary modular form to an ortho-complementary weakly modular lattice. The quantum logic was born. The next step was to find a useful mathematical presentation of this new logic. A historic review of what happened can be found in: "Quantum Theory: von Neumann" vs. Dirac; http://www.illc.uva.nl/~seop/entries/qt-nvd/. It includes extensions of the concept of Hilbert space and application of these concepts to quantum field theory. Another source is: http://www.quantonics.com/Foulis_On_Quantum_Logic.html.

1.2 Quantum logic

Elementary particles behave non-classical. They can present themselves either as a particle or as a wave. A measurement of the particle properties of the object destroys the information that was obtained from an earlier measurement of the wave properties of that object.

With elementary particles it becomes clear that that nature obeys a different logic than our old trusted classical logic. The difference resides in the modularity axiom. That axiom is weakened. The classical logic is congruent to an orthocomplemented modular lattice. The quantum logic is congruent to an orthocomplemented weakly modulare lattice. Another name for that lattice is orthomodular lattice.

1.2.1 Lattices

A subset of the axioms of the logic characterizes it as a half ordered set. A larger subset defines it as a lattice.

A lattice is a set of elements $a, b, c, ...$ that is closed for the connections $\cap$ and $\cup$. These connections obey:

- The set is partially ordered. With each pair of elements a, b belongs an element c, such that $a \subset c$ and $b \subset c$.
- The set is a $\cap$ half lattice if with each pair of elements a, b an element c exists, such that $c = a \cap b$.
- The set is a $\cup$ half lattice if with each pair of elements a, b an element c exists, such that $c = a \cup b$.
- The set is a lattice if it is both a $\cap$ half lattice and a $\cup$ half lattice.

The following relations hold in a lattice:

$$a \cap b = b \cap a \qquad (1)$$

$$(a \cap b) \cap c = a \cap (b \cap c) \qquad (2)$$

$$a \cap (a \cup b) = a \qquad (3)$$

$$a \cup b = b \cup a \qquad (4)$$

$$(a \cup b) \cup c = a \cup (b \cup c) \qquad (5)$$

$$a \cup (a \cap b) = a \qquad (6)$$

The lattice has a partial order inclusion $\subset$:

$$a \subset b \Leftrightarrow a \subset b = a \qquad (7)$$

A complementary lattice contains two elements n and e with each element a an complementary element a' such that:

$$a \cap a' = n \qquad (8)$$

$$a \cap n = n \qquad (9)$$

$$a \cap e = a \qquad (10)$$

$$a \cup a' = e \qquad (11)$$

$$a \cup e = e \qquad (12)$$

$$a \cup n = a \qquad (13)$$

An orthocomplemented lattice contains two elements n and e and with each element a an element a'' such that:

$$a \cup a'' = e \qquad (14)$$

$$a \cap a'' = n \qquad (15)$$

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$$(a")" = a \qquad (16)$$

$$a \subset b \; ⊡ \; b" \subset a" \qquad (17)$$

e is the unity element; n is the null element of the lattice

A distributive lattice supports the distributive laws:

$$a \cap (b \cup c) = (a \cap b) \cup (a \cap c) \qquad (18)$$

$$a \cup (b \cap c) = (a \cup b) \cap (a \cup c) \qquad (19)$$

A modular lattice supports:

$$(a \cap b) \cup (a \cap c) = a \cap (b \cup (a \cap c)) \qquad (20)$$

A weak modular lattice supports instead:

There exists an element d such that

$$a \subset c \Leftrightarrow (a \cup b) \cap c = a \cup (b \cap c) \cup (d \cap c) \qquad (21)$$

where d obeys:

$$(a \cup b) \cap d = d \qquad (22)$$

$$a \cap d = n \qquad (23)$$

$$b \cap d = n \qquad (24)$$

$$[(a \subset g) \, and \, (b \subset g) \Leftrightarrow d \subset g \qquad (25)$$

In an atomic lattice holds

$$\exists_{p \in L} \forall_{x \in L} \{x \subset p \Rightarrow x = n\} \qquad (26)$$

$$\forall_{a \in L} \forall_{x \in L} \{(a < x < a \cap p) \Rightarrow (x = a \, or \, x = a \cap p)\} \qquad (27)$$

p is an atom

Both the set of propositions of quantum logic and the set of subspaces of a separable Hilbert space **Н** have the structure of an orthomodular lattice. In this respect these sets are congruent.

In Hilbert space, an atom is a pure state (a ray spanned by a single vector).

Classical logic has the structure of an orthocomplemented distributive modular and atomic lattice.

Quantum logic has the structure of an orthomodular lattice. That is an orthocomplented weakly modular and atomic lattice. The set of closed subspaces of a Hilbert space also has that structure.

1.2.2   Proposition

In Aristotelian logic a proposition is a particular kind of sentence, one which affirms or denies a predicate of a subject. Propositions have binary values. They are either true or they are false.

Propositions take forms like "*This is a particle or a wave*". In quantum logic "*This is a particle.*" is not a proposition.

In mathematical logic, propositions, also called "propositional formulas" or "statement forms", are statements that do not contain quantifiers. They are composed of well-formed formulas consisting entirely of atomic formulas, the five [logical connectives](http://en.wikipedia.org/wiki/Logical_connective)[64], and symbols of grouping (parentheses etc.). Propositional logic is one of the few areas of mathematics that is totally solved, in the sense that it has been proven internally consistent, every theorem is true, and every true statement can be proved. Predicate logic is an extension of propositional logic, which adds variables and quantifiers.

In Hilbert space a vector is either inside or not inside a closed subspace. A proper quantum logical proposition is "*Vector |f> is inside state s*".

In Hilbert space, an atomic predicate corresponds with a subspace that is spanned be a single vector.

Predicates may accept attributes and quantifiers. The predicate logic is also called first order logic. A dynamic logic can handle the fact that predicates may influence each other when atomic predicates are exchanged.

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[64] http://en.wikipedia.org/wiki/Logical_connective

1.2.3   Observation

In physics, particularly in quantum physics, a system **observable** is a property of the system state that can be determined by some sequence of physical operations. This paper distinguishes between measurements and observations.

- With an observation the state is considered as a linear combination of eigenvectors of the observable. An observation returns the statistical expectation value of the eigenvalue of the observable.
- A measurement transforms the observed state to one of the eigenvectors of the observable. What happens depends on the characteristics of the measuring equipment. The measurement can be seen as a combination of a transformation and an observation.

Depending on the characteristics of the measuring equipment a measurement and a clean observation can give the same result.

With this interpretation of the concept of observation it is possible to let states observe other states. A state might do a transformation before doing an observation but in general it fails the equipment to arrange that transformation. In nature observations are far more common than measurements.

2   Numbers

2.1   Cayley-Dickson onstruction

The Cayley-Dickson construction formula enables the generation of a quaternion from two complex numbers:

$$p = a_0 + a_1\mathbf{k} + \mathbf{i}(b_0 + b_1\mathbf{k}) \tag{1}$$

$$q = c_0 + c_1\mathbf{k} + \mathbf{i}(d_0 + d_1\mathbf{k}) \tag{2}$$

$$(a, b)\,(c, d) = (ac - db^*;\ a^*d + cb) \tag{3}$$

$$r = pq \tag{4}$$

$$r_0 = a_0c_0 - a_1c_1 - b_0d_0 - b_1d_1 \tag{5}$$

$$r_{\mathbf{k}} = a_0c_1 - a_1c_0 - b_0d_1 + b_1d_0 \tag{6}$$

$$r_{\mathbf{i}} = a_0d_0 + a_1d_1 + b_0c_0 - b_1c_1 \tag{7}$$

$$r_{\mathbf{j}} = -a_1d_0 + a_0d_1 + b_0c_1 + b_1c_0 \tag{8}$$

2.2 Warren Smith's numbers

All hyper-complex numbers are based on real numbers. Two main construction formulas for hyper-complex numbers exist. The Cayley-Dickson construction is the most widely known. The Warren-Smith construction gives best algorithmic properties at higher dimensions. Until the octonions both construction formulas deliver the same results.

The quaternions are the highest dimensional hyper-complex numbers that deliver a division ring.

2.2.1 2^n-on construction

The 2^n-ons use the following doubling formula

$$(a,b)(c,d) \ = \ (a\,c - (b\,d^*)^*, (b^*c^*)^* + (b^*(a^*((b^{-1})^*d^*)^*)^*)^*) \tag{1}$$

Up until the 16-ons the formula can be simplified to

$$(a,b)(c,d) \ = \ (a\,c - b\,d^*, c\,b \ + \ (a^*\,b^{-1})\,(b\,d)) \tag{2}$$

Up to the octonions the Cayley Dickson construction delivers the same as the 2^n-on construction. From $n>3$ the 2^n-ons are 'nicer'.

2.2.1.1 2^n-ons

Table of properties of the 2^n-ons. See www.math.temple.edu/~wds/homepage/nce2.ps.

Type	name	Lose
1-ons	*Reals.*	
2-ons	*Complex numbers*	$z^* = z$ (the * denotes conjugating); the ordering properties that both $\{z > 0, -z > 0, \text{ or } z = 0\}$ and $\{w > 0, z > 0 \text{ implies } w + z > 0, wz > 0\}$.
4-ons	*Quaternions*	commutativity $ab = ba$; the algebraic closedness property that every univariate polynomial equation has a root.
8-ons	*Octonions*	associativity $ab \cdot c = a \cdot bc$.
16-ons	(*not Sedenions!*)	right-alternativity $x \cdot yy = xy \cdot y$; right-cancellation $x = xy \cdot y^{-1}$; flexibility $x \cdot yx = xy \cdot x$; left-linearity $(b + c)a = ba + ca$; anti-automorphism $ab = ba$, $(ab)^{-1} = b^{-1}\,a^{-1}$; left-linearity $(b + c)a = ba + ca$; continuity of the map $x \rightarrow xy$; Moufang and Bol identities; diassociativity
32-		generalized-smoothness of the map $x \rightarrow xy$;

Ty pe	*na me*	Retain
ons		right-division properties that xa = b has (generically) a solution x, and the uniqueness of such an x; the "fundamental theorem of algebra" that every polynomial having a unique "asymptotically dominant monomial" must have a root; Trotter's formula: $$\lim_{n\to\infty}\left[e^{x/n}e^{y/n}\right]^n = \lim_{n\to\infty}\left(1+\frac{x+y}{n}\right)^n = e^{x+y}$$

Ty pe	*na me*	Retain
2^n- ons		Unique 2-sided multiplicative & additive identity elements 1 & 0; Norm-multiplicativity $|xy|^2 = |x|^2 \cdot |y|^2$; Norm-subadditivity $|a + b| \le |a| + |b|$; 2-sided inverse $a^{-1} = a^*/|a|^2$ (a # 0); $a^{**} = a$; $(x \pm y)* = x^* \pm y^*$; $(a^{-1})^{-1} = a$; $(a^*)^{-1} = (a^{-1})^*$; $|a|^2 = |a|^2 = a^*a$; Left-alternativity $yy \cdot x = y \cdot yx$; Left-cancellation $x = y^{-1} \cdot yx$; Right-linearity $a(b + c) = ab + ac$; r^{th} power-associativity $a^n a^m = a^{n+m}$; Scaling $s \cdot ab = sa \cdot b = as \cdot b = a \cdot sb = a \cdot bs = ab \cdot s$ (s real); Power-distributivity $(ra^n + sa^m)b = ra^n b + sa^m b$ (r, s real); Vector product properties of the imaginary part: **ab** - re(**ab**) of the product for pure-imaginary 2^n-ons **a**,**b** regarded as $(2^n - 1)$-vectors; $\langle xa,b\rangle = \langle a,x^*b\rangle$, $\langle xa,xb\rangle = |x|^2 \cdot \langle a,b\rangle$ and $\langle x,y\rangle = \langle x^*,y^*\rangle$ Numerous weakened associativity, commutativity, distributivity, antiautomorphism, and Moufang and Bol properties including 9-coordinate ``niner" versions of most of those properties; contains 2^{n-1}-ons as subalgebra.

2.2.1.1.1 The most important properties of 2^n-ons

If a,b,x,y are 2^n-ons, $n \ge 0$, and s and t are scalars (i.e. all coordinates are 0 except the real coordinate) then

unit: A unique 2^n-on 1 exists, with $1 \cdot x = x \cdot 1 = x$.
zero: A unique 2^n-on 0 exists, with $0 + x = x + 0 = x$ and $0 \cdot x = x \cdot 0 = 0$.
additive properties: x+y = y+x, (x+y)+z = x+(y+z);
−x exists with $x + (−x) = x − x = 0$.
norm: $|x|^2 = xx^* = x^*x$.
norm-multiplicativity: $|x|^2 \cdot |y|^2 = |x \cdot y|^2$.
scaling: $s \cdot x \cdot y = s \cdot x \cdot y = x \cdot s \cdot y = x \cdot s \cdot y = x \cdot y \cdot s$.
weak-linearity: $(x + s) \cdot y = x \cdot y + s \cdot y$ and $x \cdot (y + s) = x \cdot y + x \cdot s$.
right-linearity: $x \cdot (y + z) = x \cdot y + x \cdot z$.

inversion: If $x \neq 0$ then a unique x^{-1} exists, obeying $x^{-1} \cdot x = x \cdot x^{-1} = 1$. It is $x^{-1} = x \cdot |x|^{-2}$.

left-alternativity: $x \cdot xy = x^2 \cdot y$.

left-cancellation: $x \cdot x^{-1} \cdot y = y$.

effect on inner products: $\langle x \cdot a, b \rangle = \langle a, x^* \cdot b \rangle$, $\langle x, y \rangle = \langle x^*, y^* \rangle$, $\langle x^* \cdot a, x^{-1} \cdot b \rangle = \langle a, b \rangle$, and $\langle x \cdot a, x \cdot b \rangle = |x|^2 \cdot \langle a, b \rangle$.

Conjugate of inverse: $(x^{-1})^* = (x^*)^{-1}$.

Near-anticommutativity of unequal basis elements: $e_k^2 = -1$ and $e_k \cdot e_l^* = -e_l \cdot e_k^*$ if $k \neq l$.

(Note: the case $k; l > 0$ shows that unequal pure-imaginary basis elements anticommute.)

Alternative basis elements: $e_k \cdot e_l \cdot e_k = e_k \cdot e_l e_k$, $e_l \cdot e_k \cdot e_k = e_l \cdot e_k e_k$, and $e_k \cdot e_k \cdot e_l = e_k \cdot e_k e_l$.
(However, when $n \geq 4$ the 2^n-ons are not flexible i.e. it is not generally true that $x \cdot y \cdot x = x \cdot y \cdot x$ if x and y are 16-ons that are not basis elements. They also are not right-alternative.)

Quadratic identity: If x is a 2^n-on (over any field F with charF $\neq 2$), then $x^2 + |x|^2 = 2 \cdot x$ re x

Squares of imaginaries: If x is a 2^n-on with re $x = 0$ ("pure imaginary") then $x^2 = -|x|^2$ is nonpositive pure-real.

Powering preserves im$x$ direction

2.2.1.1.2 Niners

Niners are 2n-ons whose coordinates with index > 8 are zero. The index starts with 0.

9-**flexibility** $xp \cdot x = x \cdot px$, $px \cdot p = p \cdot xp$.

9-**similitude unambiguity** $xp \cdot x^{-1} = x \cdot px^{-1}$, $px \cdot p^{-1} = p \cdot xp^{-1}$.

9-**right-alternativity** $xp \cdot p = x \cdot p^2$, $px \cdot x = p \cdot x^2$.

9-**right-cancellation** $xp^{-1} \cdot p = x$, $px^{-1} \cdot x = p$.

9-**effect on inner products** $\langle x, yp \rangle = \langle xp, y \rangle$, $\langle xp, yp \rangle = |p|^2 \langle x, y \rangle$.

9-**left-linearity** $(x + y)p = xp + yp$, $(p + q)x = px + qx$.

9-**Jordan-identity** $xp \cdot xx = x(p \cdot xx)$, $py \cdot pp = p(y \cdot pp)$.

9-**coordinate-distributivity** $([x + y]z)_{0;...;8} = (xz + yz)_{0;...;8}$.

9-**coordinate-Jordan-identity** $[xy \cdot xx]_{0;...;8} = [x(y \cdot xx)]_{0;...;8}$.

9-**anticommutativity for orthogonal imaginary 2^n-ons**

If $\langle p, x \rangle = $ re $p = $ re $x = 0$ then $px = -xp$.

9-**reflection** If $|a| = 1$ and the geometric reflection operator is defined below then $-(\mathrm{refl}[a](y))_{0;...;8} = (a \cdot y^* a)_{0;...;8}$, and $-\{\mathrm{refl}[a](y)\}^*_{0;...;8} = (a^* y \cdot a^*)_{0;...;8}$, and

if either a or y is a niner then $-\mathrm{refl}[a](y) = a \cdot y^* a$ and $-\mathrm{refl}[a](y) = a^* y \cdot a^*$.

$$\mathrm{refl}[\vec{x}](\vec{t}) \stackrel{\text{def}}{=} \vec{t} - \frac{2\langle \vec{x}, \vec{t} \rangle}{|\vec{x}|^2} \vec{x} \tag{1}$$

What holds for the niners, also holds for the octonions.

2.3 Quaternions

2.3.1 Sign selections

Four possibilities exist due to the sign selections of the quaternions. One sign selection is covered by the conjugation $a \rightarrow a^*$. This selection switches the sign of all three imaginary base vectors. The other is caused by switching the sign of a single binary base vector $a \rightarrow a^{\otimes}$. For this sign selection one of the three available base vectors is selected. When relevant, then these choices are indicated by colors (r, g or b). Both methods switch the handedness (chirality). When both sign

selections combine then the superscript $a \rightarrow a^{\oplus}$ is used. This combination does not switch handedness. Also this selection is colored.

It is also possible to use the extended **quaternionic conjugation concept**:

$$a^{*} = a^{③} \tag{1}$$

$$a^{\otimes} = a^{①} \tag{2}$$

$$a^{\oplus} = a^{*\otimes} = a^{②} \tag{3}$$

The encircled number stands for the number of switched base vectors. For the single sign switch $a^{①}$, three independent direction selections are possible. We indicate these choices with r, g and b.

Similarly for the double sign switch $a^{②}$, three independent direction selections are possible. We indicate these choices also with r, g and b. This direction belongs to the non-switched direction.

Without closer description the value of $a^{①②}$ is $a^{③}$. It means that the colors are suspected to be the same.

The change from a to $a^{①}$ or $a^{③}$ cause a switch of the handedness of a.

$$a^{**} = a^{③③} = \left(a^{③}\right)^{③} = a \tag{4}$$

$$a^{①①} = a^{②②} = a \tag{5}$$

$$a^{①②} = a^{②①} = a^{③} \tag{6}$$

The effects of the quaternionic conjugation are visible in the base numbers 1, $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$:

$$1^{*} = 1 \tag{7}$$

$$\mathbf{i}^{③} = -\mathbf{i}; \mathbf{j}^{③} = -\mathbf{j}; \mathbf{k}^{③} = -\mathbf{k}; \tag{8}$$

The blue colored sign selection is given by

$$\color{blue}{\mathbf{i}^{①} = +\mathbf{i}; \mathbf{j}^{①} = +\mathbf{j}; \mathbf{k}^{①} = -\mathbf{k};} \tag{9}$$

$$\color{blue}{\mathbf{i}^{②} = -\mathbf{i}; \mathbf{j}^{②} = -\mathbf{j}; \mathbf{k}^{②} = +\mathbf{k};} \tag{10}$$

In the blue colored sign selection, $\mathbf{k}$ follows the rules of complex conjugation. This renders its direction to a special direction.

The selected color direction is called the **longitudinal** direction. The the perpendicular directions are the transverse directions. Apart from that they are mutual perpendicular and perpendicular to the longitudinal direction, they have no preferred direction.

2.3.1.1 Sign selections and quaternionic distributions

Quaternionic distributions are supposed to obey a distribution wide sign selection. Thus, the distribution is characterized by one of the eight quaternionic sign flavors.

$$\psi^{\circledcirc}, \psi^{\textcircled{1}}, \psi^{\textcircled{1}}, \psi^{\textcircled{1}}, \psi^{\textcircled{2}}, \psi^{\textcircled{2}}, \psi^{\textcircled{2}}, \text{ or } \psi^{\textcircled{1}} \tag{1}$$

Many of the elementary particles are characterized by an ordered pair of two field sign flavors. These fields are coupled with a coupling strength that is typical for the particle type. These particles obey a characteristic continuity equation[65].

2.3.1.2 Product rule

We use the quaternionic product rule.

$$ab = a_0 b_0 - \langle \boldsymbol{a}, \boldsymbol{b} \rangle + a_0 \boldsymbol{b} + \boldsymbol{a} b_0 + \boldsymbol{a} \times \boldsymbol{b} \tag{1}$$

$$\langle \boldsymbol{a}, \boldsymbol{b} \rangle = a_1 b_1 + a_2 b_2 + a_3 b_3 \tag{2}$$

$$\boldsymbol{a} \times \boldsymbol{b} = \boldsymbol{i}(a_2 b_3 - a_3 b_2) + \boldsymbol{j}(a_3 b_1 - a_1 b_3) + \boldsymbol{k}(a_1 b_2 - a_2 b_1) \tag{3}$$

2.3.1.3 Operators

The sign selections of operator $\nabla = (\nabla_0, \boldsymbol{\nabla})$ depend on the sign selections of position operator Q, which determines the sign selections for its eigenvalues $q = (q_0, \boldsymbol{q})$.

Normally we consider the sign selection for operators Q and ∇ fixed to operators $Q^{\circledcirc}$ and $\nabla^{\circledcirc}$. Sometimes we chose ∇^* instead of operator ∇.

Quaternionic sign selection are directly connected with the concepts of **parity** and **spin**.

For quaternionic functions symmetry reduces the differences that are produced by conjugation and anti-symmetry stresses the differences. The same holds for operators.

2.3.1.4 Matrices

Another possibility is to present sign selections by matrices[66].

$$\sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \qquad \sigma_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \qquad \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \tag{1}$$

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[65] Hilbert field equations; Continuity equation for charges

[66] http://www.vttoth.com/qt.htm

The σ_1 matrix switches the complex fields that together form the quaternion field.

$$\begin{bmatrix} \varphi_b \\ \varphi_a \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \varphi_a \\ \varphi_b \end{bmatrix} \tag{2}$$

The σ_2 matrix switches the real parts and the imaginary parts of the complex fields that together form the quaternion field and it switches both fields.

$$i\begin{bmatrix} -\varphi_b \\ \varphi_a \end{bmatrix} = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \begin{bmatrix} \varphi_a \\ \varphi_b \end{bmatrix} \tag{3}$$

The σ_3 matrix switches the sign of the first complex field.

$$\begin{bmatrix} -\varphi_a \\ \varphi_b \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \varphi_a \\ \varphi_b \end{bmatrix} \tag{4}$$

$$\sigma_k^2 = -i\sigma_1\sigma_1\sigma_1 = I \tag{5}$$

The Pauli matrices are involutory.
The [determinants](http://en.wikipedia.org/wiki/Determinant)[67] and [traces](http://en.wikipedia.org/wiki/Trace_of_a_matrix)[68] of the Pauli matrices are:

$$\det(\sigma_k) = -1 \tag{6}$$

$$\mathrm{Tr}(\sigma_k) = 0 \tag{7}$$

$$\alpha_k = \begin{bmatrix} 0 & \sigma_k \\ -\sigma_k & 0 \end{bmatrix} \tag{8}$$

$$\alpha_1 = \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix} \tag{9}$$

$$\alpha_2 = \begin{bmatrix} 0 & j \\ -j & 0 \end{bmatrix} \tag{10}$$

$$\alpha_3 = \begin{bmatrix} 0 & k \\ -k & 0 \end{bmatrix} \tag{11}$$

$$\beta = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \tag{12}$$

The α_k matrices together select the imaginary base vectors. The β matrix exchanges the sign of all imaginary base vectors. Thus the β matrix implements the quaternionic conjugate. The conjugation also exchanges right handedness against left handedness.

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[67] http://en.wikipedia.org/wiki/Determinant
[68] http://en.wikipedia.org/wiki/Trace_of_a_matrix

Another way of exchanging right handedness against left handedness is the exchange of the sign of one of the imaginary base vectors.

$$\begin{bmatrix} \psi_L \\ \psi_R \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \psi_R \\ \psi_L \end{bmatrix} \tag{13}$$

$$\psi^* = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \psi \tag{14}$$

The gamma matrices[69] translate directly from complex fields to fully featured quaternionic fields. In this way four sign flavors of quaternionic fields are constructed from four complex fields. This is represented by four dimensional matrices and four dimensional spinors. The equivalent of the β matrix is the γ_β matrix.

$$\begin{bmatrix} \varphi_{La} \\ \varphi_{Lb} \\ \varphi_{Ra} \\ \varphi_{Rb} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \varphi_{Ra} \\ \varphi_{Rb} \\ \varphi_{La} \\ \varphi_{Lb} \end{bmatrix} \tag{15}$$

It is false to interpret the matrices as vectors. They form a shorthand for handling spinors.

The Pauli matrix σ_1 represents the sign selection $a \rightarrow a^\otimes$, while the β matrix represents the sign selection $a \rightarrow a^*$. The other Pauli matrices and the α matrices implement the resulting part of the quaternion behavior for spinors.

2.3.2   Colored signs

In the following text, the consequences for the product of the sign choices of the conjugate ③ is indicated by blue color $\pm$. The extra consequence ① for the product of the choice of the handedness of the cross product is indicated by red color $\pm$. The mixed sign selection ② acts on both sign colors.

The handedness can be switched by changing the sign of all imaginary base vectors.

$$ij = k \;\rightarrow\; (-i)(-j) = ij = -k \tag{1}$$

The sign selections split the ring of quaternions in four different realizations.

2.3.3   Waltz details

The 16-ons lose the continuity of the map $x \Rightarrow xy$. Also, in general holds $(x\,y)x \neq x\,(y\,x)$ for 16-ons. However, for all 2^n-ons the base numbers fulfill $(e_i\,e_j)\,e_i = e_i\,(e_j\,e_i)$. All 2^n-ons feature a conjugate and an inverse. The inverse only exists for non-zero numbers. The 2^n-ons support the **number waltz**

$$c = a\,b/a. \tag{1}$$

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[69] Appendix; Gamma matrices

Often the number waltz appears as a unitary number waltz

$$c = u^* b\, u \tag{2}$$

where u is a unit size number and u^* is its conjugate $u\, u^* = 1$.

In quaternion space the **quaternion waltz** $a\, b/a$ can be written as

$$a\, b\, /\, a = exp(2\,\pi\,ĩ\,\varphi)\, b\, exp(-2\,\pi\,ĩ\,\varphi) \tag{3}$$

$$= b - \boldsymbol{b}_\perp + exp(2\,\pi\,ĩ\,\varphi)\, \boldsymbol{b}_\perp\, exp(-2\,\pi\,ĩ\,\varphi)$$

$$= b - \boldsymbol{b}_\perp + exp(4\,\pi\,ĩ\,\varphi)\boldsymbol{b}_\perp$$

$$\Delta b = (exp(4\,\pi\,ĩ\,\varphi) - 1)\boldsymbol{b}_\perp \tag{4}$$

$$= (cos(4\,\pi\,\varphi) + ĩ\, sin(4\,\pi\,\varphi) - 1)\,\boldsymbol{b}_\perp$$

$$= exp(2\,\pi\,ĩ\,\varphi)\, 2\, ĩ\, sin\,(2\,\pi\,\varphi)\,\boldsymbol{b}_\perp$$

$$\|\Delta b\| = \|2\, sin(2\,\pi\,\varphi)\,\boldsymbol{b}_\perp\| \tag{5}$$

Another way of specifying the difference is:

$$\Delta b = (a\cdot b - b\cdot a)/a = 2\cdot(\boldsymbol{a}\times\boldsymbol{b})/a \tag{6}$$

$$\|\Delta b\| = 2\,\|\boldsymbol{a}\times\boldsymbol{b}\|/\,\|a\| \tag{7}$$
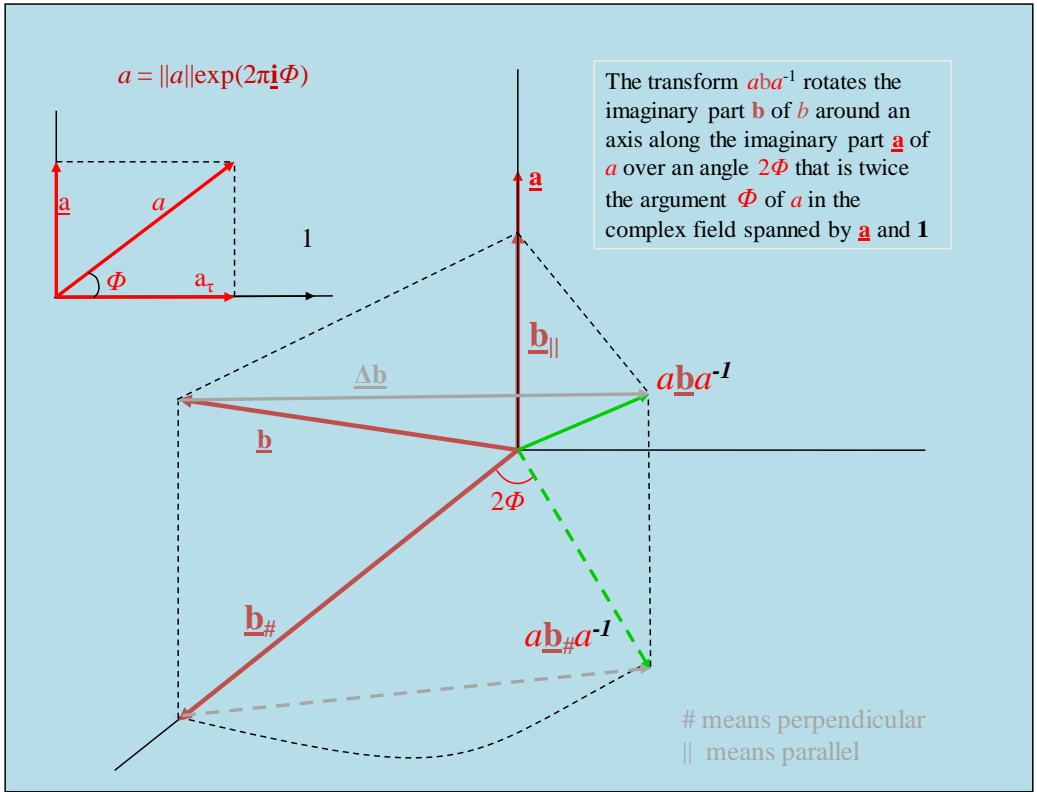
Figure 1. The rotation of a quaternion by a second quaternion.

2.3.3.1.1 Infinitesimal number transformation

The number v is close to 1. Thus $v = 1 + \Delta s$. Let us investigate the transform $c = v^* b v$.

$$c = (1 + \Delta s^*)\, b\, (1 + \Delta s) \tag{1}$$

$$= b + \Delta s^* \cdot b + b\, \Delta s + \Delta s^* b\, \Delta s$$

$$\approx b + \Delta s \ast \cdot b + b \cdot \Delta s$$

$$= b + \Delta s_0\, b + 2\, \boldsymbol{b} \times \boldsymbol{\Delta s}$$

$$\Delta b = \Delta s_0\, b + 2\, \boldsymbol{b} \times \boldsymbol{\Delta s} \tag{2}$$

This comes close to the effect of an infinitesimal number waltz, especially when $\Delta s_0 = 0$ In that case $\Delta b_0 = 0$ and $\boldsymbol{\Delta b}$ is perpendicular *to* $\boldsymbol{\Delta s}$.

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For 2^n-ons with $n > 1$, $a\,b\,/\,a$ in general does not equal b. This effect stays unnoticed when quantum mechanics sticks to a complex Hilbert space.
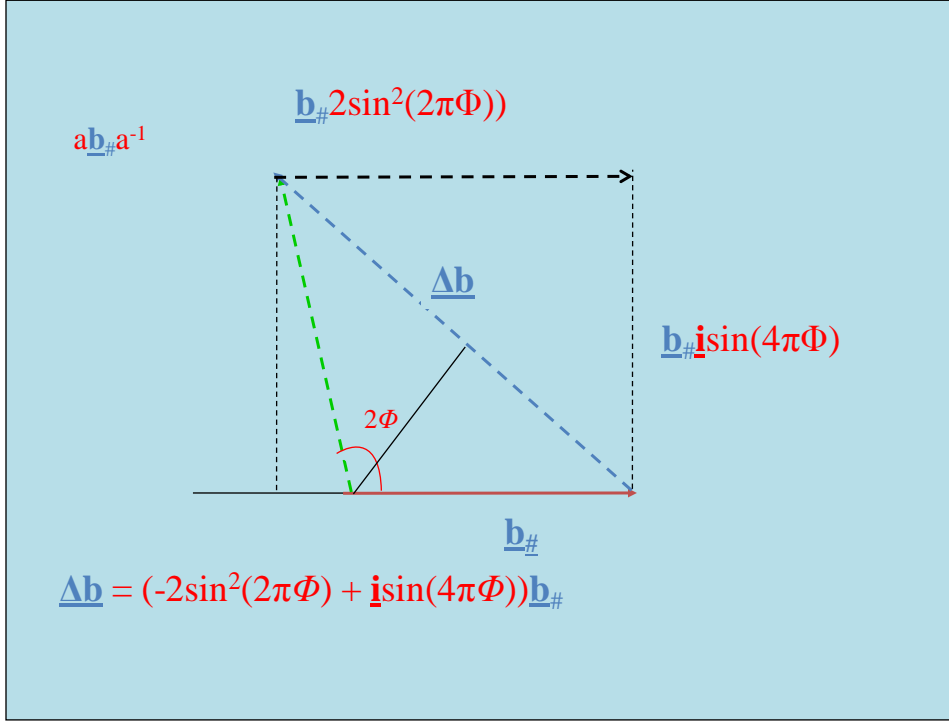

Figure 2: The difference after rotation

2.4   Quaternion coordinates

This part of the appendix describes candidates for the coordinates on the coordinate sphere.

2.4.1   Polar coordinates

The equivalent to rectangular coordinates in quaternion space is (a_τ, a_x, a_y, a_z)

$$a = a_\tau + i\,a_x + j\,a_y \pm i\,j\,a_z \tag{1}$$

The equivalent to polar coordinates in quaternion space is

$$a_\tau = \|a\|\,\cos(\psi) \tag{2}$$

$$a_x = \|a\|\,\sin(\psi)\,\sin(\theta)\,\cos(\varphi) \tag{3}$$

$$a_y = \|a\|\,\sin(\psi)\,\sin(\theta)\,\sin(\varphi) \tag{4}$$

$$a_z = \|a\| \, sin(\psi) \, cos(\theta) \tag{5}$$

$sin(\psi)$, where $\psi = (0, \pi)$, is known as the (imaginary) amplitude of the quaternion.

Angle $\theta = (0, \pi)$ is the (co-)latitude and angle $\varphi = (0, 2\pi)$ is the longitude.

For any fixed value of ψ, θ and φ parameterize a 2-sphere of radius $sin(\psi)$, except for the degenerate cases, when ψ equals 0 or π, in which case they describe a point.

This suggests the following structure of the argument **Δ**

$$a = \|a\| \, exp(ĩ \cdot \psi) \tag{6}$$

$$= \|a\| \, (cos(\psi) + ĩ \, sin(\psi)) \tag{7}$$

$$= a_\tau + \|a\| \, ĩ \, sin(\psi) = a_\tau + \boldsymbol{a} \tag{8}$$

The imaginary number ĩ may take any direction.

2.4.2   3 sphere

A 3-sphere is a compact, connected, 3-dimensional manifold without boundary. It is also simply-connected. What this means, loosely speaking, is that any loop, or circular path, on the 3-sphere can be continuously shrunk to a point without leaving the 3-sphere. The Poincaré conjecture[70] proposes that the 3-sphere is the only three dimensional manifold with these properties (up to homeomorphism)[71].

The round metric on the 3-sphere in these coordinates is given by

$$ds^2 = d\psi^2 + sin^2(\psi) \, (d\theta^2 + sin^2(\theta) d\varphi^2) \tag{1}$$

The volume form is given by

$$dV = sin^2(\psi) \, sin(\theta) \, d\psi \wedge d\theta \wedge d\varphi \tag{2}$$

The 3-dimensional volume (or **hyperarea**) of a 3-sphere of radius r is

$$2 \, \pi^2 \, r^3 \tag{3}$$

The 4-dimensional **hypervolume** (the volume of the 4-dimensional region bounded by the 3-sphere) is

$$\tfrac{1}{2} \, \pi^2 \, r^4 \tag{4}$$

The 3-sphere has constant positive sectional curvature equal to $1/r^2$.

The 3-sphere has a natural Lie group structure SU(2) given by quaternion multiplication.

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[70] http://en.wikipedia.org/wiki/Poincar%C3%A9_conjecture
[71] http://en.wikipedia.org/wiki/3-sphere

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The 3-sphere admits non-vanishing vector fields (sections of its tangent bundle). One can even find three linearly-independent and non-vanishing vector fields. These may be taken to be any left-invariant vector fields forming a basis for the Lie algebra of the 3-sphere. This implies that the 3-sphere is parallelizable. It follows that the tangent bundle of the 3-sphere is trivial.

There is an interesting action of the circle group $\mathbb{T}$ on $\mathbb{S}^3$ giving the 3-sphere the structure of a principal circle bundle known as the Hopf bundle. If one thinks of $\mathbb{S}^3$ as a subset of $\boldsymbol{C}^2$, the action is given by

$$(z_1, z_2)\, \lambda \;=\; (z_1\, \lambda, z_2\, \lambda)\; \forall_{\lambda\, \in\, \mathbb{T}}. \tag{5}$$

The orbit space of this action is homeomorphic to the two-sphere $\mathbb{S}^2$. Since $\mathbb{S}^3$ is not homeomorphic to $\mathbb{S}^2 \times \mathbb{S}^1$, the Hopf bundle is nontrivial.

2.4.3 Hopf coordinates

Another choice of hyperspherical coordinates, (η, ξ_1, ξ_2), makes use of the embedding of $\mathbb{S}^3$ in $\boldsymbol{C}^2$. In complex coordinates $(z_1, z_2) \in \boldsymbol{C}^2$ we write

$$z_1 = exp(\bar{\imath}\, \xi_1)\, sin(\eta) \tag{1}$$

$$z_2 \;=\; exp(\bar{\imath}\, \xi_2)\, cos(\eta) \tag{2}$$

Here η runs over the range 0 to $\pi/2$, and ξ_1 and ξ_2 can take any values between 0 and 2π. These coordinates are useful in the description of the 3-sphere as the Hopf bundle

$$\mathbb{S}^1 \to \mathbb{S}^3 \to \mathbb{S}^2 \tag{3}$$

For any fixed value of η between 0 and $\pi/2$, the coordinates (ξ_1, ξ_2) parameterize a 2-dimensional torus. In the degenerate cases, when η equals 0 or $\pi/2$, these coordinates describe a circle.

The round metric on the 3-sphere in these coordinates is given by

$$ds^2 = d\eta^2 + sin^2(\eta)\, (d\zeta_1^2 + cos^2(\eta)\, d\,\zeta_2^2) \tag{4}$$
and the volume form by

$$dV \;=\; sin(\eta)\, cos(\eta)\, d\eta {\wedge} d\zeta_1 {\wedge} d\zeta_2 \tag{5}$$

2.4.4 Group structure

Because the set of unit quaternions is closed under multiplication, $\mathbb{S}^3$ takes on the structure of a group. Moreover, since quaternionic multiplication is smooth, $\mathbb{S}^3$ can be regarded as a real Lie group. It is a non-abelian, compact Lie group of dimension 3. When thought of as a Lie group $\mathbb{S}^3$ is often denoted $Sp(1)$ or $U(1, \mathbb{H})$.

It turns out that the only spheres which admit a Lie group structure are $\mathbb{S}^1$, thought of as the set of unit complex numbers, and $\mathbb{S}^3$, the set of unit quaternions. One might think that $\mathbb{S}^7$, the set of unit octonions, would form a Lie group, but this fails since octonion multiplication is non-

associative. The octonionic structure does give $\mathbb{S}^7$ one important property: *[parallelizability](http://en.wikipedia.org/wiki/Parallelizability)[72]*. It turns out that the only spheres which are parallelizable are $\mathbb{S}^1$, $\mathbb{S}^3$, and $\mathbb{S}^7$.

By using a matrix representation of the quaternions, $\mathbb{H}$, one obtains a matrix representation of $\mathbb{S}^3$. One convenient choice is given by the Pauli matrices:

$$\left(a_\tau + a_x \cdot \mathbf{i} + a_y \cdot \mathbf{j} + a_z \cdot \mathbf{k}\right) = \begin{bmatrix} a_\tau + \tilde{\imath} \cdot a_x & a_y + \tilde{\imath} \cdot a_z \\ -a_y + \tilde{\imath} \cdot a_z & a_\tau - \tilde{\imath} \cdot a_x \end{bmatrix} \tag{1}$$

This map gives an injective algebra homomorphism from $\mathbb{H}$ to the set of 2×2 complex matrices. It has the property that the absolute value of a quaternion q is equal to the square root of the determinant of the matrix image of q.

The set of unit quaternions is then given by matrices of the above form with unit determinant. This matrix subgroup is precisely the special unitary group SU(2). Thus, $\mathbb{S}^3$ as a Lie group is isomorphic to SU(2).

Using our hyperspherical coordinates (η, ξ_1, ξ_2) we can then write any element of SU(2) in the form

$$\begin{bmatrix} \exp(\tilde{\imath} \cdot \xi_1) \cdot \sin(\eta) & \exp(\tilde{\imath} \cdot \xi_2) \cdot \cos(\eta) \\ -\exp(\tilde{\imath} \cdot \xi_2) \cdot \cos(\eta) & \exp(-\tilde{\imath} \cdot \xi_1) \cdot \sin(\eta) \end{bmatrix} \tag{2}$$

Another way to state this result is if we express the matrix representation of an element of SU(2) as a linear combination of the Pauli matrices. It is seen that an arbitrary element $U \in$ SU(2) can be written as

$$U = \alpha_\tau \cdot 1 + \sum_{n=x,y,z} \alpha_n \, \boldsymbol{I_n} \tag{3}$$

The condition that the determinant of U is +1 implies that the coefficients α_n are constrained to lie on a 3-sphere.

2.4.5   Versor

Any **unit quaternion** q can be written as a **versor**:

$$u = exp(\tilde{\imath} \, \psi) = cos(\psi) + \tilde{\imath} \, sin(\psi) \tag{1}$$

This is the quaternionic analogue of Euler's formula. Now the unit imaginary quaternions all lie on the unit 2-sphere in Im $\mathbb{H}$ so any such $\underline{\tilde{\imath}}$ can be written:

$$\tilde{\imath} = \boldsymbol{i} \, cos(\varphi) \, sin(\theta) + \boldsymbol{j} \, sin(\varphi) \, sin(\theta) + \boldsymbol{k} \, cos(\theta) \tag{2}$$

2.4.6   Symplectic decomposition

Quaternions can be written as the combination of two complex numbers and an imaginary number k with unit length.

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[72] http://en.wikipedia.org/wiki/Parallelizability

3 Quaternionic distributions

3.1 Sign flavors

Quaternionic distributions are quaternion valued functions of a quaternionic parameter. If not otherwise stated, the quaternionic parameter space is not curved. Quaternions feature sign selections. Inside a quaternionic distribution the quaternionic sign selections of the values are all the same. Due to the four possible sign selections of quaternions, quaternionic distributions exist in four sign flavors.

3.2 Differentiation

A quaternionic distribution f(q) can be differentiated.

$$g(q) = \nabla_0 f_0(q) \mp \langle \nabla, \boldsymbol{f}(q) \rangle \pm \nabla_0 \boldsymbol{f}(q) + \nabla f_0(q) \pm \left(\pm \nabla \times \boldsymbol{f}(q) \right) \tag{1}$$

The colored $\mp$ and $\pm$ signs refer to the influence of conjugation of $f(q)$ on quaternionic multiplication. The $\pm$ sign refers to the influence of reflection of $f(q)$.

3.3 Fourier transform

In Fourier space differentiation becomes multiplication with the canonical conjugate coordinate k and therefore the equivalent equation becomes:

$$\tilde{g}(k) = k\tilde{f}(k) = k_0 \tilde{f}_0(k) \mp \langle \mathbf{k}, \tilde{\boldsymbol{f}}(k) \rangle \pm k_0 \tilde{\boldsymbol{f}}(k) + \mathbf{k}\tilde{f}_0(k) \pm \left(\pm \mathbf{k} \times \tilde{\boldsymbol{f}}(k) \right) \tag{1}$$

For the imaginary parts holds:

$$\mathbf{g}(q) = \pm \nabla_0 \boldsymbol{f}(q) + \nabla f_0(q) \pm \left(\pm \nabla \times \boldsymbol{f}(q) \right) \tag{2}$$

$$\tilde{\mathbf{g}}(k) = \pm k_0 \tilde{\boldsymbol{f}}(k) + \mathbf{k}\tilde{f}_0(k) \pm \left(\pm \mathbf{k} \times \tilde{\boldsymbol{f}}(k) \right) \tag{3}$$

By using[73]

$$\nabla \times \nabla f_0(q) = \mathbf{0} \tag{4}$$

and

$$\langle \nabla, \nabla \times \boldsymbol{f}(q) \rangle = 0 \tag{5}$$

It can be seen that for the static part $(\nabla_0 f(q) = 0)$ holds:

$$\mathbf{g}(q) = \nabla f_0(q) \pm \left(\pm \nabla \times \boldsymbol{f}(q) \right) \tag{6}$$

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[73] Bo Thidé: http://www.plasma.uu.se/CED/Book/EMFT_Book.pdf ;Formulas:F.104, F.105

$$\tilde{\mathbf{g}}(k) = \mathbf{k}\tilde{f}_0(k) \pm \left(\pm \mathbf{k} \times \tilde{\boldsymbol{f}}(k) \right) \tag{7}$$

3.4 Helmholtz decomposition

Formula (7) of the last paragraph leads to the Helmholtz decomposition. The Helmholtz decomposition splits the **static** vector field $\boldsymbol{F}$ in a (transversal) divergence free part $\boldsymbol{F}_t$ and a (one dimensional longitudinal) rotation free part $\boldsymbol{F}_l$.

$$\boldsymbol{F} = \boldsymbol{F}_t + \boldsymbol{F}_l = \nabla \times \boldsymbol{f} - \nabla f_0 \tag{1}$$

Here f_0 is a scalar field and $\boldsymbol{f}$ is a vector field. In quaternionic terms f_0 and $\boldsymbol{f}$ are the real and the imaginary part of a quaternionic field f. $\boldsymbol{F}$ is an imaginary quaternion.[74]

The significance of the terms "longitudinal"and "transversal" can be understood by computing the local three-dimensional Fourier transform of the vector field $\boldsymbol{F}$, which we call $\widetilde{\boldsymbol{F}}$. Next decompose this field, at each point $\boldsymbol{k}$, into two components, one of which points longitudinally, i.e. parallel to $\boldsymbol{k}$, the other of which points in the transverse direction, i.e. perpendicular to $\boldsymbol{k}$.

$$\widetilde{\boldsymbol{F}}(\boldsymbol{k}) = \widetilde{\boldsymbol{F}}_l(\boldsymbol{k}) + \widetilde{\boldsymbol{F}}_t(\boldsymbol{k}) \tag{2}$$

$$\langle \boldsymbol{k}, \widetilde{\boldsymbol{F}}_t(\boldsymbol{k}) \rangle = 0 \tag{3}$$

$$\boldsymbol{k} \times \widetilde{\boldsymbol{F}}_l(\boldsymbol{k}) = \boldsymbol{0} \tag{4}$$

The Fourier transform converts gradient into multiplication and vice versa. Due to these properties the inverse Fourier transform gives:

$$\boldsymbol{F} = \boldsymbol{F}_l + \boldsymbol{F}_t \tag{5}$$

$$\langle \nabla, \boldsymbol{F}_t \rangle = 0 \tag{6}$$

$$\nabla \times \boldsymbol{F}_l = \boldsymbol{0} \tag{7}$$

So, this split indeed conforms to the Helmholtz decomposition.

This interpretation relies on idealized circumstance in which the decomposition runs along straight lines. This idealized condition is in general not provided. In normal conditions the decomposition and the interpretation via Fourier transformation only work locally and with reduced accuracy.

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[74] See next paragraph

4 Fields

4.1 The origin of physical fields.

The Hilbert Book Model is a simple Higgsless model of physics that is strictly based on traditional quantum logic and on the lattice isomorphic model; the set of subspaces of an infinite dimensional separable Hilbert space for which the inner product is specified by using quaternions[75].

This restriction results in the fact that all sets of variables are countable. At the same time most observations are taken from a continuum. As a result the set of potential observations overwhelms the set of variables[76]. The situation is comparable to the situation in which the number of equations is far larger than the number of variables that should form the result. Probably, the set of equations will appear to be inconsistent. In order to cure the situation, it is common to assume that the observations are inaccurate. The inaccuracy must be stochastic or with other words the observation result must be blurred.

Nature applies a similar solution, but instead of a simple spread function in the form of a probability density distribution, nature applies a quaternionic probability amplitude distribution (QPAD). This QPAD can be split into a real part that represents a "charge" density distribution and an imaginary part that represents a corresponding "current" density distribution. The "charge" represents the set of properties of the thing that is being observed. The parameter of the distribution represents the location at which the "charge" is observed. The squared modulus of the QPAD represents the probability density of the presence of the "charge" at the location that is specified by the parameter.

This approach transfers the dynamics of the observation into a streaming problem. The equation of motion of the "charge" becomes a continuity equation[77].

The properties of particles move according to the above principle. With each elementary particle belong one or more QPAD's that act as private fields of the particle and that determine its dynamic behavior when it moves freely. However, these fields overlap. In this way these fields and the corresponding particles interact.

A subset of the elementary particles is massless. These particles correspond to a single QPAD. That does not say that their fields cannot overlap.

All other elementary particles are identified by an ordered pair of QPAD's that are two field sign flavors of the same base field. The coordinate system, whose values are used as field parameter, has its own field sign flavor and acts as a sign flavor reference.

4.1.1   Categories of fields

Two categories of fields exist.

4.1.1.1 Primary fields

The first category consists of quaternionic probability amplitude distributions (QPAD's). The QPAD's may overlap and through this superposition they may form covering fields. The QPAD's exist in four sign flavors. The same holds for the covering fields. The QPAD's may interact.

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[75] See: http://www.crypts-of-physics.eu/HilbertBookModelEssentials.pdf

[76] A continuum has a higher cardinality than a countable set.

[77] Another name for "continuity equation" is "balance equation".

When different sign flavors interact the strength of the local interaction is characterized by a coupling factor. The members of this category will be called primary fields.

4.1.1.2 Secondary fields

The second category consists of administrator fields. These fields administer the effect of interactions on the local curvature of the positioning coordinate system. For all properties that characterize a coupling of sign flavors of primary fields an administrator field exist that registers the influence of that property during interactions on the local curvature.

One of these administrator fields is the gravitation field. It administers the influence of the strength of the coupling between sign flavors of primary fields on the local curvature.

The electromagnetic fields administer the influence of the electric charge on the local curvature.

The angular momentum including the spin also influences the local curvature. Also this effect is administered.

The members of this category will be called secondary fields or administrator fields.

All these influences can be administered by using the local metric. This generates a metric tensor field.

4.2 Example potential

The influence of local properties is represented by charges. The charge carrier may contain an assembly of charges.

Spatial [Harmonic functions](http://en.wikipedia.org/wiki/Harmonic_function)[78] are suitable charge spread functions.
For a harmonic function $f(q)$ holds:

$$\Delta f(q) = \nabla\nabla^{①}f(q) = 0 \tag{1}$$

If there is a static spherically symmetric Gaussian charge density $\rho(r)$:

$$\rho(q) = \frac{Q}{\sqrt{2\pi\sigma^2}^3}\exp(-|q|^2/(2\sigma^2)) \tag{2}$$

where Q is the total charge, then the solution $\varphi(r)$ of [Poisson's equation](http://en.wikipedia.org/wiki/Poisson%27s_equation)[79],

$$\nabla^2\varphi(q) = -\frac{\rho(q)}{\varepsilon} \tag{3}$$

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[78] http://en.wikipedia.org/wiki/Harmonic_function
[79] http://en.wikipedia.org/wiki/Poisson%27s_equation

is given by

$$\varphi(q) = \frac{Q}{4\pi\varepsilon|q|}\,\mathrm{erf}\left(\frac{|q|}{\sqrt{2}\sigma}\right) \tag{4}$$

where $\mathrm{erf}(x)$ is the error function.

In fact the quaternionic Poisson's equation represents two separate equations:

$$(\nabla_0{}^2 - \boldsymbol{\nabla}^2)\varphi_0(q) = -\frac{\rho_0(q)}{\varepsilon} \tag{5}$$

$$(\nabla_0{}^2 - \boldsymbol{\nabla}^2)\boldsymbol{\varphi}(q) = -\frac{\boldsymbol{\rho}(q)}{\varepsilon} \tag{6}$$

Note that, for $|q|$ much greater than σ, the erf function approaches unity and the potential φ (r) approaches the point charge potential $\frac{Q}{4\pi\varepsilon|q|}$, as one would expect. Furthermore the erf function approaches 1 extremely quickly as its argument increases; in practice for $|q| > 3\sigma$ the relative error is smaller than one part in a thousand[80].

The definition of the quaternionic potential $\phi(q)$ is based on the convolution of a quaternionic distribution $\rho(q)$ with the real function $\varphi(q)$ See Newton potential and Bertrand's theorem in Wikipedia. The real part $\rho_0(q)$ of the distribution $\rho(q)$ can be interpreted as a charge distribution. The imaginary part $\boldsymbol{\rho}(q)$ can be interpreted as a current distribution.
The convolution blurs the distribution such that the result becomes differentiable.

In configuration space holds:

$$\phi(q) = \rho(q) \circ \frac{1}{|q|}. \tag{7}$$

Reversely, according to Poisson's equation:

$$\rho(q) == -\Delta\phi(q) \tag{8}$$

The real part of $\phi(q)$ presents a scalar potential. The imaginary part presents a vector potential.

$$\phi(q) = \phi_0(q) + \boldsymbol{\phi}(q) \tag{9}$$

In the above section:

> The scalar potential is a blurred charge distribution.
> The vector potential is a blurred current distribution.

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[80] http://en.wikipedia.org/wiki/Poisson's_equation#Potential_of_a_Gaussian_charge_density

| Current is moving charge. |
| Mass is a form of charge. |

(The selected blurring function has striking resemblance with the ground state of the quantum harmonic oscillator[81]).

In Fourier space holds:

$$\tilde{\phi}(p) = \tilde{\rho}(p) \cdot \frac{1}{|p|} \cdot = \tilde{\phi}_0(p) + \tilde{\boldsymbol{\phi}}(p) \tag{10}$$

In Fourier space the frequency spectrum of the Hilbert distribution is multiplied with the Fourier transform of the blurring function. When this falls off when the frequencies go to infinity, then as a consequence the frequency spectrum of the potential is bounded. This is valid independent of the fact that the frequency spectrum of the Hilbert distribution is unbounded.

5 Fourier transform

5.1 Quaternionic Fourier transform split

The longitudinal Fourier transform represents only part of the full quaternionic Fourier transform. It depends on the selection of a radial line $\boldsymbol{k}(q)$ in p space that under ideal conditions runs along a straight line.

$$\mathcal{F}_{\mathbf{k}}\big(g(q)\big) = \mathcal{F}(g(q), \boldsymbol{k}(q)) \tag{1}$$

Or

$$\mathcal{F}_{\parallel}\big(g(q)\big) \overset{\text{def}}{=} \mathcal{F}\big(g_{\parallel}(q)\big) \tag{2}$$

It relates to the full quaternionic Fourier transform Ⅎ

$$\mathcal{F}\big(g(q)\big) = \tilde{g}(p) \tag{3}$$

The inverse Fourier transform runs:

$$\mathcal{F}^{-1}(\tilde{g}(p)) = g(q) \tag{4}$$

The split in longitudinal and transverse Fourier transforms corresponds to a corresponding split in the multi-dimensional Dirac delta function.

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[81] Functions and fields:Functions invariant under Fourier transformation:Ladder operator:Ground state

We consider a field $g(q)$ that equals the quaternionic differentiation of another field f with respect to a selected (ideal) coordinate system Q.

$$g(q) = \nabla_q f \tag{5}$$

We use the results of the paragraph on decomposition. We only use the static and imaginary version of field $g(q)$.

For the static imaginary part $\mathbf{g}(q)$ holds:

$$\mathbf{g}(q) = \nabla f_0(q) \pm (\pm \nabla \times \mathbf{f}(q)) = \mathbf{g}_l(q) + \mathbf{g}_t(q) \tag{6}$$

In Fourier space differentiation becomes multiplication with the canonical conjugate coordinate $\mathbf{p}$ and therefore the equivalent equation becomes:

$$\tilde{\mathbf{g}}(p) = \mathbf{p}\tilde{f}_0(p) \pm (\pm \mathbf{p} \times \tilde{\mathbf{f}}(p)) = \tilde{\mathbf{g}}_l(p) + \tilde{\mathbf{g}}_t(p) \tag{7}$$

Since

$$\nabla \times \nabla f_0(q) = \mathbf{0} \rightarrow \nabla \times \mathbf{g}_l(q) = \mathbf{0} \tag{8}$$

and

$$\langle \nabla, \nabla \times \mathbf{f}(q) \rangle = 0 \rightarrow \langle \nabla, \mathbf{g}_t(q) \rangle = 0 \tag{9}$$

Now we take

$$< q|\breve{P} f > = \hbar \cdot \nabla_q < q|f > = \hbar \cdot \nabla_q f^*(q) = \mathbf{g}(q) \tag{10}$$

$$= \int_p < q|p > \cdot < p|g >$$

The static imaginary part is

$$< q|\breve{P} f > = \hbar \cdot \nabla_q < q|f > = \hbar \cdot \nabla_q \mathbf{f}^*(q) = \mathbf{g}(q) \tag{11}$$

$$= Im\left(\int_p < q|p > \cdot < p|\mathbf{g} >\right) = \int_p Im(< q|p > \cdot < p|\mathbf{g} >)$$

$$= \int_p Im(< q|p > \cdot < p|\mathbf{g}_l >) + \int_p Im(< q|p > \cdot < p|\mathbf{g}_t >)$$

$$= \int_p Im(< q|p >\cdot \tilde{\boldsymbol{g}}_l(p)) + \int_p Im(< q|p >\cdot \tilde{\boldsymbol{g}}_t(p))$$

The left part is the longitudinal inverse Fourier transform of field $\tilde{\boldsymbol{g}}(p)$.
The right part is the transverse inverse Fourier transform of field $\tilde{\boldsymbol{g}}(p)$.
For the Fourier transform of $\boldsymbol{g}(q)$ holds the split:

$$\tilde{\boldsymbol{g}}(p) = \int_q Im(< p|q >\cdot \boldsymbol{g}_l(q)) + \int_p Im(< p|q >\cdot \boldsymbol{g}_t(q)) \qquad (12)$$

$$= \int_q Im(< p|q >\cdot \boldsymbol{g}(q))$$

The longitudinal direction is a one dimensional (radial) space. The corresponding transverse direction is tangent to a sphere in 3D. Its direction depends on the field $\boldsymbol{g}(q)$ or alternatively on the combination of field f and the selected (ideal) coordinate system $\check{Q}$.

For a weakly curved coordinate system $\check{Q}$ the formulas hold with a restricted accuracy and within a restricted region.

5.2 Alternative transverse plane

The Cayley-Dickson construction, as well as Warren Smith's construction formula shows that the transverse part can be considered as a complex number space multiplied with a fixed imaginary quaternionic base number. The selection of the imaginary base number $\boldsymbol{i}$ is arbitrary as long as it is perpendicular to $\boldsymbol{k}$. The resulting plane is spanned by axes $\boldsymbol{i}$ and $\boldsymbol{ik}$. When base number $\boldsymbol{i}$ is divided away, then a normal complex number space results.

Also here a complex Fourier transform can be defined in a way that is similar to the longitudinal Fourier transform. It must be reckoned that the sign selections for these directions differ.

5.3 Alternative approach to Fourier transform

The following draws from the work of S. Thangavelu[82].

Let us take the non-abelian group $\mathbb{H}_1$ which is $\mathbb{R} \otimes \mathbb{R} \otimes \mathbb{R}$ with the group law

$$(x, y, t)(x_0, y_0, t_0) = (x + x_0; y + y_0; t + t_0 + x y_0) \qquad (1)$$

Then it is clear that $\mathbb{H}1$ is non-abelian and the Lebesgue measure $dx\, dy\, dt$ is both left and right invariant Haar measure on $\mathbb{H}_1$. With this measure we can form the Hilbert space $L^2(\mathbb{H}_1)$. Let $\Gamma = \mathbb{Z} \otimes \mathbb{Z} \otimes \mathbb{Z}$. Then it is easy to check that Γ is a subgroup of $\mathbb{H}_1$ so that we can form the quotient M $= \Gamma/\mathbb{H}_1$ consisting of all right cosets of Γ. Functions on M are naturally identified with left Γ-invariant functions on $\mathbb{H}_1$. As the Lebesgue measure $dx\, dy\, dt$ is left Γ-invariant we can form

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[82] http://www.math.iitb.ac.in/atm/faha1/veluma.pdf

$L_2(M)$ using the Lebesgue measure restricted to M. As a set we can identify M with $[0, 1)^3$ and we just think of $L^2(M)$ as $L^2([0, 1)^3)$.

Fourier expansion in the last variable allows us to decompose $L^2(M)$ into a direct sum of orthogonal subspaces. Simply define $\mathcal{H}_k$ to be the set of all $f \in L^2(M)$ which satisfy the condition

$$f(x, y, t + s) = \exp(2\pi i k s) f(x, y, t) \tag{2}$$

Then $\mathcal{H}_k$ is orthogonal to $\mathcal{H}_j$ whenever $k \neq j$ and any $f \in L^2(M)$ has the unique expansion

$$f = \sum_{k=-\infty}^{\infty} f_k; \; f_k \in \mathcal{H}_k \tag{3}$$

In quaternionic terms, the split sees ik as imaginary quaternion $\boldsymbol{k}$ and the quaternionic Hilbert space is split in components according to the imaginary direction of $\boldsymbol{k}$, where the choice is between three mutually perpendicular directions.

For the moment, we are mainly interested in $\mathcal{H}_1$ which is a Hilbert space in its own right. It is interesting to note that functions in $\mathcal{H}_1$ are also invariant under the left action of Γ.

Our next example of a unitary operator is the following. Consider the map $J : \mathcal{H}_1 \to \mathcal{H}_1$ given by

$$J(x, y, t) = (-x, y, t - xy) \tag{4}$$

$$J^\dagger(x, y, t) = (x, -y, t - xy) \tag{5}$$

$$J^\dagger = J^{-1} \tag{6}$$

$$J^2(x, y, t) = J(-x, \cdot y; t - xy) = (-x, -y; t) \tag{7}$$

$$J^4 = I \tag{8}$$

$$J(0, 0, t) = (0, 0, t) \tag{9}$$

$$J f(x, y, t) = f(J(x, y; t)) = f(-x, \cdot y, t - xy) \tag{10}$$

5.4 Weil-Brezin transform

Next consider the Weil-Brezin transform V:

$$V f(x, y, t) = \exp(2\pi \boldsymbol{k} t) \sum_n f(x + n) \exp(2\pi \boldsymbol{k} n y) \tag{1}$$

$$\int_{y=0}^{1} |V\,f(x,y,t)|^2 dy = \int_{x=0}^{1} \sum_{n=-\infty}^{n=\infty} |f(x\,+\,n)|^2\,dx \tag{2}$$

$$\iiint_{0} |V\,f(x,y,t)|^2 dx\,dy\,dt = \int_{0}^{1} |\,f(x)|^2 dx \tag{3}$$

V is unitary.
See also <u>Zak transform</u>

5.5 Fourier transform

We define the Fourier transform $\mathcal{F}$ by:

$$\mathcal{F} = V^{\dagger} J\,V \tag{1}$$

- $\mathcal{F}^4\,f = f$; for every $f \in L^2(\mathbb{R})$
- $\mathcal{F}^2\,f(x) = f(-x)$; for almost every $x \in \mathbb{R}$
- $\|\mathcal{F}\,f\|^2 = \|f\|^2$

For $f \in L^1(\mathbb{R}) \setminus L^2(\mathbb{R})$ the Fourier transform is given by

$$\mathcal{F}f(\xi) = \int_{x \in \mathbb{R}} f(x) \cdot \exp(2\,\pi\,\boldsymbol{k}\,x) dx \tag{2}$$

If we further assume that $\mathcal{F}f \in L^1(\mathbb{R})$ then for almost every x we have

$$f(\xi) = \int_{\xi \in \mathbb{R}} \mathcal{F}f(\xi) \exp(2\,\pi\,\boldsymbol{k}\,x\,\xi) dx \tag{3}$$

5.6 Functions invariant under Fourier transform

In this section we confine to a complex part of the Hilbert space.
See <u>http://en.wikipedia.org/wiki/Hermite_polynomials</u>.
There exist two types of Hermite polynomials: $(1, 2)$

1. The probalist's Hermite polynomials:

$$H_n^{prob}(z) = (-1)^n \exp(\tfrac{1}{2}z^2)\,\frac{d^n}{dz^n}\,\exp(-\tfrac{1}{2}z^2).$$

2. The physicist's Hermite polynomials

$$H_n^{phys}(z) = (-1)^n \exp(z^2)\frac{d^n}{dx^n}\exp(-z^2) = \exp(\tfrac{1}{2}z^2)\left(z - \frac{d}{dz}\right)\exp(-\tfrac{1}{2}z^2)$$

These two definitions are *not* exactly equivalent; either is a rescaling of the other:

$$H_n^{phys}(z) = 2^{n/2}\, H_n^{prob}\left(z\sqrt{2}\right) \tag{3}$$

In the following we focus on the physicist's Hermite polynomials.

The Gaussian function φ(z) defined by

$$\varphi(x) = exp(-\pi\, z^2) \tag{4}$$

is an eigenfunction of $\mathcal{F}$. It means that its Fourier transform has the same form.

As $\mathcal{F}^4 = I$ any λ in its spectrum $\sigma\,(\mathcal{F})$ satisfies $\lambda^4 = 1$: Hence,

$$\sigma\,(\mathcal{F}) = \{1; -1; i; -i\}. \tag{5}$$

We take the Fourier transform of the expansion:

$$exp(-\tfrac{1}{2}\, z^2 + 2\, z\, c - c^2) = \sum_{n=0}^{\infty} exp(-\tfrac{1}{2}\, z^2)\, H_n(z)\, c^n/n! \tag{6}$$

First we take the Fourier transform of the left hand side:

$$\frac{1}{\sqrt{2\pi}}\int_{z=-\infty}^{\infty} exp(-\mathbf{k}\, z\, p_z)\, exp(-\tfrac{1}{2}\, z^2 + 2\, z\, c - c^2)\, dz \tag{7}$$

$$= exp(-\tfrac{1}{2}\, p_z^2 - 2\,\mathbf{k}\, p_z\, c + c^2)$$

$$= \sum_{n=0}^{\infty} exp(-\tfrac{1}{2}\, p_z^2)\, H_n(p_z)\, (-\mathbf{k}\, c)^n/n!$$

The Fourier transform of the right hand side is given by

$$\frac{1}{\sqrt{2\pi}}\sum_{n=0}^{\infty}\int_{z=-\infty}^{\infty} exp(-\mathbf{k}\, z\, p_z)\cdot exp(-\tfrac{1}{2}\, z^2)\, H_n(z)\, c^n/n!\ dz \tag{8}$$

Equating like powers of c in the transformed versions of the left- and right-hand sides gives

$$\frac{1}{\sqrt{2\pi}} \int_{z=-\infty}^{\infty} \exp(-\boldsymbol{k}\, z\, p_z) \cdot exp(-\tfrac{1}{2}\, z^2)\, H_n(z)\, c^n / n!\ dz \tag{9}$$

$$= (-\boldsymbol{k})^n \cdot \exp(-\tfrac{1}{2}\, p_z^2)\, H_n(p_z)\, \frac{c^n}{n!}$$

Let us define the Hermite functions $\psi_n(z)$

$$\psi_n(z) \overset{\text{def}}{=}\ < z|\psi_n > \,=\, c_n \exp(-\tfrac{1}{2}\, z^2)\, H_n(z) \tag{10}$$

$$|\mathcal{F}\, \psi_n > \,=\, |\psi_n > (-\boldsymbol{k})^n \tag{11}$$

with suitably chosen c_n so as to make

$$\|\psi_n\|^2 \,=\, 1 \tag{12}$$

$$c_n = \frac{1}{\sqrt{2^n n!\, \sqrt{\pi}}} \tag{13}$$

The importance of the Hermite functions lie in the following theorem.

"The Hermite functions ψ_n; $n \in \mathbb{N}$ form an orthonormal basis for $L^2(\mathbb{R})$"

Consider the operator

$$H \,=\, -\tfrac{1}{2}\frac{d^2}{dz^2} + \tfrac{1}{2}\, z^2 \tag{14}$$

Apply this to $\psi_n(z)$:

$$H \cdot \psi_n(z) \,=\, (\tfrac{1}{2} + n)\, \psi_n(z) \tag{15}$$

Thus, ψ_n is an eigenfunction of H.

Let $f \,=\, \psi_{4k+j}$ be any of the Hermite functions. Then we have

$$\sum_{n=-\infty}^{\infty} f(y + n) \cdot \exp\big(-2\,\pi\, \boldsymbol{k}\, x\, (y + n)\big) \tag{16}$$

$$= (-\boldsymbol{k})^j \sum_{n=-\infty}^{\infty} f(x + n) \, exp(2 \, \pi \, \boldsymbol{k} \, n \, y)$$

Proof: As

$$\mathcal{F} = V^\dagger J V \tag{17}$$

the equation

$$\mathcal{F}f = (-\boldsymbol{k})^j f \tag{18}$$

translates into

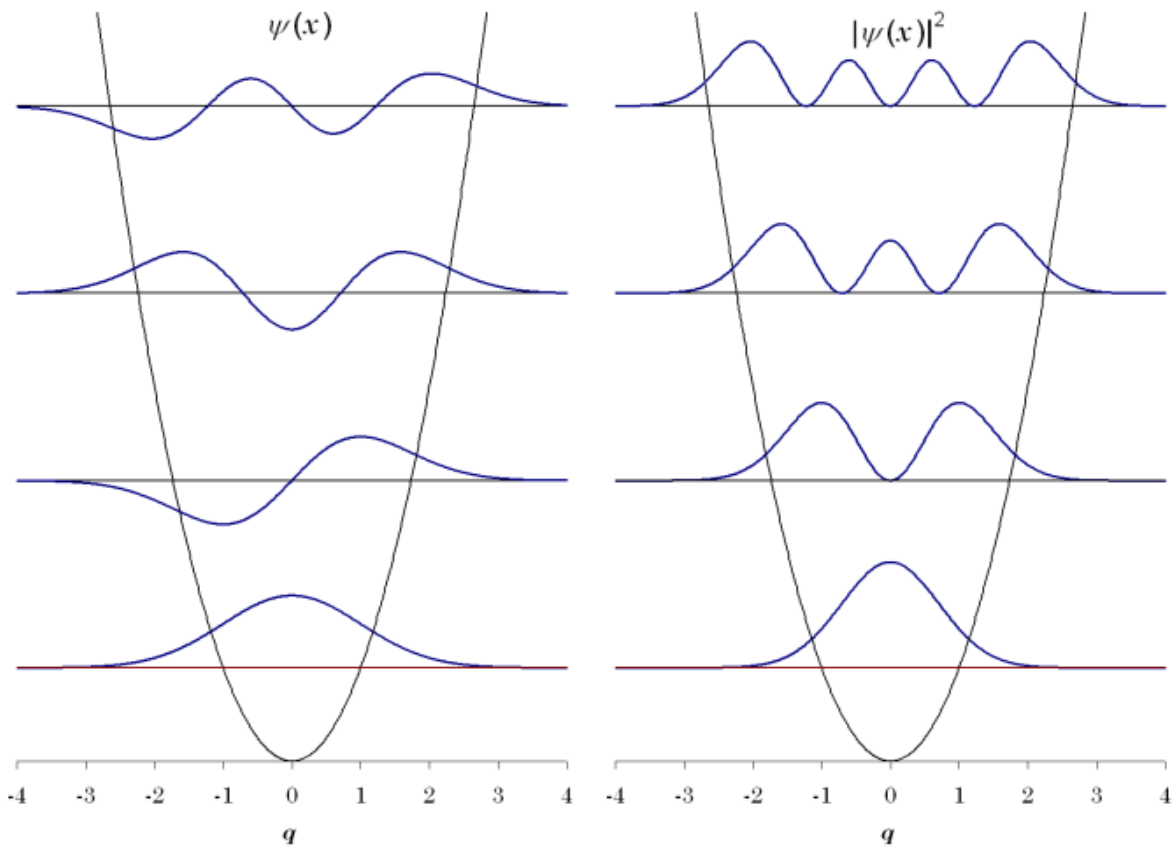$$JVf(x; y; t) = (-\boldsymbol{k})^j \, Vf(x; y; t) \tag{19}$$

With the definition of V and $t = xy$:

$$V f(x,y,t) = exp(2 \, \pi \, \boldsymbol{k} \, t) \sum_{n} f(x + n) \, exp(2 \, \pi \, \boldsymbol{k} \, n \, y) \tag{20}$$

QED.

The vectors $|\psi_n\rangle$ are eigenvectors of the Fourier transform operator with eigenvalues $(-\boldsymbol{k})^n$. The eigenfunctions $\psi_n(x)$ represent eigenvectors $|\psi_n\rangle$ that span the complex Hilbert space $\mathbf{H}_k$.
For higher n the central parts of $\psi_n(x)$ and $|\psi_n(x)|^2$ become a sinusoidal form.

$\psi(x)$ · $|\psi(x)|^2$

A **coherent state**[83] is a specific kind of state[84] of the quantum harmonic oscillator whose dynamics most closely resemble the oscillating behavior of a classical harmonic oscillator system. The ground state is a squeezed coherent state[85].

The ground state here differs from the ground state of the QPAD. That ground state equals zero in the close neighborhood of the center. The size of that neighborhood is of the order of the Planck length. Thus in this region the QPAD has the form of a stretched turban mold. It has a form similar to the second state in the picture of $|\psi(x)|^2$, thus the lowest state where $\psi(x)$ is asymmetric. Asymmetric states are better localizable than symmetric states.

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[83] http://en.wikipedia.org/wiki/Coherent_state

[84] States

[85] Canonical conjugate: Heisenberg's uncertainty

5.7   Special Fourier transform pairs

Functions that keep the same form through Fourier transformation are:

$$f(q) = \exp(-|q|^2) \tag{1}$$

$$f(q) = \frac{1}{|q|} \tag{2}$$

$$f(q) = comb(q) \tag{3}$$

The comb function consists of a set of equidistant Dirac delta functions.

Other examples of functions that are invariant under Fourier transformation are the linear and spherical harmonic oscillators and the solutions of the Laplace equation.

5.8   Complex Fourier transform invariance properties

Each even function $f(q) \iff \tilde{f}(p)$ induces a Fourier invariant:

$$h(q) = \sqrt{2\pi}\, f(q) + \tilde{f}(q). \tag{1}$$

$$\tilde{h}(q) = \sqrt{2\pi}\, h(q) \tag{2}$$

Each odd function $f(q) \iff \tilde{f}(p)$ induces a Fourier invariant:

$$h(q) = \sqrt{2\pi}\, f(q) - \tilde{f}(q). \tag{3}$$

A function $f(q)$ is invariant under Fourier transformation *if and only if* the function f satisfies the differential equation

$$\frac{\partial^2 f(q)}{\partial q^2} - t^2 f(q) = \alpha\, f(q), \text{ for some scalar } \alpha \in C. \tag{4}$$

The Fourier transform invariant functions are fixed apart from a scale factor. That scale factor can be 1, k, -1 or $-k$. k is an imaginary base number in the longitudinal direction.

Fourier-invariant functions show iso-resolution, that is, $\Delta_p = \Delta_q$ in the Heisenberg's uncertainty relation.

For proves see: http://www2.ee.ufpe.br/codec/isoresolution_vf.pdf.

5.9   Fourier transform properties

5.9.1   Parseval's theorem

Parseval's theorem runs:

$$\int f^*(q) \cdot g(q) \cdot dV_q = \int \tilde{f}^*(p) \cdot \tilde{g}(p) \cdot dV_p \tag{1}$$

This leads to

$$\int |f(q)|^2 \cdot dV_q = \int \left| \tilde{f}(p) \right|^2 \cdot dV_p \tag{2}$$

5.9.2 Convolution

Through Fourier transformation a convolution changes into a simple product and vice versa.

$$\mathcal{F}(f(q) \circ g(q)) = \tilde{f}(p) \cdot \tilde{g}(p) \tag{1}$$

5.9.3 Differentiation

Fourier transformation converts differentiation into multiplication with the canonical conjugated coordinate.

$$g(q) = \nabla f(q) \tag{1}$$

$$\tilde{g}(p) = p\tilde{f}(p) \tag{2}$$

6 Ladder operator

The Hermite functions ψ_n represent Fock states[86].

Boson ladder operators are characterized by

$$\mathcal{A}|\psi_n> = \sqrt{n}\,|\psi_{n-1}> \tag{1}$$

$$\mathcal{A}^\dagger|\psi_n> = \sqrt{n+1}\,|\psi_{n+1}> \tag{2}$$

$$\mathcal{A} = \frac{1}{\sqrt{2}}\left(c_1 \frac{d}{dq} + c_2 q\right) = -\boldsymbol{k}\,\breve{P}\sqrt{\frac{1}{2\,\hbar\,m\,\omega}} + \breve{Q}\sqrt{\frac{m\,\omega}{2\,\hbar}}$$

$$\mathcal{A}^\dagger = \frac{1}{\sqrt{2}}\left(-c_1 \frac{d}{dq} + c_2 q\right) = \boldsymbol{k}\,\breve{P}\sqrt{\frac{1}{2\,\hbar\,m\,\omega}} + \breve{Q}\sqrt{\frac{m\,\omega}{2\,\hbar}}$$

In the Heisenberg picture, the operators have the following time dependence:

$$\mathcal{A}(t) = \mathcal{A}(t_0)\exp\left(-\boldsymbol{k}\,\omega\,(t-t_0)\right) \tag{3}$$

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[86] http://en.wikipedia.org/wiki/Fock_state

$$\mathcal{A}^\dagger(t) = \mathcal{A}^\dagger(t_0) \exp(\boldsymbol{k}\,\omega\,(t - t_0)) \tag{4}$$

We can also define an enumeration operator N which has the following property:

$$N = \mathcal{A}^\dagger \mathcal{A} \tag{5}$$

$$N|\psi_n> = |\psi_n> n \tag{6}$$

In deriving the form of $\mathcal{A}^\dagger$, we have used the fact that the operators X and P_x, which represent observables, are Hermitian. These observable operators can be expressed as a linear combination of the ladder operators as

$$\check{Q}(t) = \sqrt{\frac{\hbar}{2\,m\,\omega}}\ (\mathcal{A}^\dagger(t) + \mathcal{A}(t)) \tag{7}$$

$$\check{P}(t) = \boldsymbol{k}\sqrt{\tfrac{1}{2}\hbar\,m\,\omega}\ (\mathcal{A}^\dagger(t) - \mathcal{A}(t)) \tag{8}$$

The $\check{\boldsymbol{Q}}$ and $\check{\boldsymbol{P}}$ operators obey the following identity, known as the canonical commutation relation:

$$[\check{Q}, \check{P}] = \boldsymbol{k}\,\hbar \tag{9}$$

Using the above, we can prove the identities

$$H = \hbar\,\omega\,(\mathcal{A}^\dagger \mathcal{A} + \tfrac{1}{2}) = \hbar\,\omega\,(N + \tfrac{1}{2}) \tag{10}$$

$$[\mathcal{A}^\dagger, \mathcal{A}] = 1 \tag{11}$$

Now, let $|f_E>$ denote an energy eigenstate with energy E. The inner product of any ket with itself must be non-negative, so

$$< \psi_E\,\mathcal{A}|\mathcal{A}\,\psi_E > = < \psi_E E|\,\mathcal{A}^\dagger\,\mathcal{A}\,\psi_E > \geq 0 \tag{12}$$

Expressing $\mathcal{A}^\dagger \mathcal{A}$ in terms of the Hamiltonian H:

$$< \psi_E|(H/(\hbar\,\omega) - \tfrac{1}{2})\,\psi_E > = (E/(\hbar\,\omega) - \tfrac{1}{2}) \geq 0 \tag{13}$$

so that

$$E \geq \tfrac{1}{2}\,\hbar\,\omega. \tag{14}$$

Note that when $|\mathcal{A}\,\psi_E > = |0>$ (is the zero ket i.e. a ket with length zero), the inequality is saturated, so that

$$E = \tfrac{1}{2}\hbar\,\omega \tag{15}$$

It is straightforward to check that there exists a state satisfying this condition; it is the ground state

$$|\psi_{ground} >= |\psi_{E_n} >;\ (n\ =\ 0) \tag{16}$$

Using the above identities, we can now show that the commutation relations of $\mathcal{A}$ and $\mathcal{A}^\dagger$ with H are:

$$[H,\mathcal{A}] =\ -\hbar\,\omega\,\mathcal{A} \tag{17}$$

$$[H,\mathcal{A}^\dagger] =\ \hbar\,\omega\,\mathcal{A}^\dagger \tag{18}$$

Thus, provided $|\,\mathcal{A}\ \psi_E >$ is not the zero ket,

$$|H\,\mathcal{A}\,\psi_E >=\ |[H,\mathcal{A}]\ +\ \mathcal{A}\,H\,\psi_E > \tag{19}$$

$$=\ |-\hbar\,\omega\,\mathcal{A}\ +\mathcal{A}\,E\,\psi_E >$$

$$=\ |-\hbar\,\omega\,\mathcal{A}\ +\ \mathcal{A}\,E\,\psi_E >$$

$$=\ (E\ -\ \hbar\,\omega)\,|\mathcal{A}\,\psi_E >$$

Similarly, we can show that

$$|H\,\mathcal{A}^\dagger\,\psi_E >=\ (E\ +\ \hbar\,\omega)\,|\mathcal{A}^\dagger\,\psi_E > \tag{20}$$

In other words, $\mathcal{A}$ acts on an eigenstate of energy E to produce, up to a multiplicative constant, another eigenstate of energy $E - \hbar\omega$, and $\mathcal{A}^\dagger$ acts on an eigenstate of energy E to produce an eigenstate of energy $E + \hbar\omega$. For this reason, a is called a "lowering operator", and $\mathcal{A}^\dagger\mathcal{A}$ "raising operator". The two operators together are called ladder operators. In quantum field theory, $\mathcal{A}$ and $\mathcal{A}^\dagger$ are alternatively called "annihilation" and "creation" operators because they destroy and create particles, which correspond to our quanta of energy.

Given any energy eigenstate, we can act on it with the lowering operator $\mathcal{A}$, to produce another eigenstate with $\hbar\omega$-less energy. By repeated application of the lowering operator, it seems that we can produce energy eigenstates down to $E = -\infty$. However, this would contradict our earlier requirement that $E \geq \hbar\omega/2$.

7 States

7.1 Ground state

Therefore, there must be a ground-state energy eigenstate, which we label $|f_{ground}>$, such that

$$|\mathcal{A}\,\psi_{ground} >=\ |0 >;\ \text{(zero ket)}. \tag{1}$$

In this case, subsequent applications of the lowering operator will just produce zero kets, instead of additional energy eigenstates. Furthermore, we have shown above that

$$|H \, \psi_{ground} > = \; (\tfrac{1}{2} \, \hbar \, \omega) \, |\psi_{ground} > \tag{2}$$

Finally, by acting on $|\psi_{ground} >$ with the raising operator and multiplying by suitable normalization factors, we can produce an infinite set of energy eigenstates

$$\{|\psi_{ground} >, |\psi_{E_1} >, \; |\psi_{E_2} >, \dots, |\psi_{E_n} >\}, \tag{3}$$

such that

$$|H \, \psi_{E_n} > = \; \hbar \, \omega \, (n \; + \; \tfrac{1}{2}) \, |\psi_{E_n} > \tag{4}$$

which matches the energy spectrum.

This method can also be used to quickly find the ground state wave function of the quantum harmonic oscillator.

Indeed

$$|\mathcal{A} \, \psi_{ground} > = \; |0 > \tag{5}$$

becomes

$$\psi_{ground}(x) = \psi_0(x) \; \overset{\text{def}}{=} \; <x| \, \psi_{ground} > = \; - \; \frac{\hbar}{m \, \omega} \frac{d}{dx} \, \psi_0(x) \tag{6}$$

so that

$$d \, \psi_0(x) = \psi_0(x) \; \frac{\hbar}{m \, \omega} \; x \, d \, x \; \Rightarrow \ln\big(\psi_0(x)\big) \; = \; \frac{m \, \omega}{2\hbar} \; x^2 \; + \; const \tag{7}$$

After normalization this leads to the following position space representation of the ground state wave function.

$$\psi_0(x) \; = \; \sqrt[4]{\frac{m \, \omega}{\pi\hbar}} \; e^{-\frac{m \, \omega}{2\hbar}x^2} \tag{8}$$

7.2   Coherent state

A **coherent state** is a specific kind of state[87] of the quantum harmonic oscillator[88] whose dynamics most closely resemble the oscillating behavior of a classical harmonic oscillator system.

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[87]States

[88] Functions invariant under Fourier transform

The coherent state |α> is defined to be the 'right' eigenstate of the annihilation operator $\mathcal{A}$. Formally, this reads:

$$|\mathcal{A}\,\alpha> = \alpha|\alpha> \tag{1}$$

Since $\mathcal{A}$ is not Hermitian, α is a hyper complex number that is not necessarily real, and can be represented as

$$\alpha = |\alpha|\exp(\boldsymbol{k}\,\theta) \tag{2}$$

where θ is a real number. $|\alpha|$ is the amplitude and θ is the phase of state |α>.

This formula means that a coherent state is left unchanged by the annihilation or the creation of a particle. The eigenstate of the annihilation operator has a [Poissonian](http://en.wikipedia.org/wiki/Poissonian)[89] number distribution A Poisson distribution is a necessary and sufficient condition that all annihilations are statistically independent.

The coherent state's location in the complex plane ([phase space](http://en.wikipedia.org/wiki/Phase_space)[90]) is centered at the position and momentum of a classical oscillator of the same phase θ and amplitude. As the phase increases the coherent state circles the origin and the corresponding disk neither distorts nor spreads. The disc represents Heisenberg's uncertainty. This is the most similar a quantum state can be to a single point in phase space.

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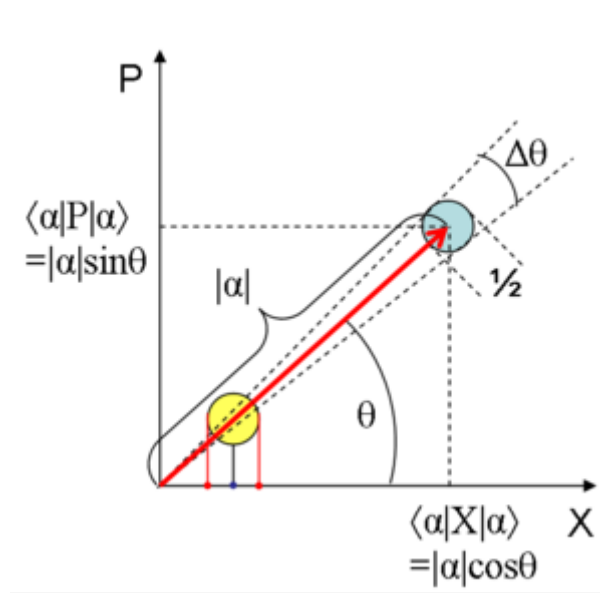
[89] http://en.wikipedia.org/wiki/Poissonian
[90] http://en.wikipedia.org/wiki/Phase_space

Phase space plot of a coherent state. This shows that the uncertainty (blur) in a coherent state is equally distributed in all directions. The horizontal and vertical axes are the X and P quadratures of the field, respectively. Oscillations that are said to be *in quadrature,* if they are separated in phase by $\pi/2$ radians. The red dots on the x-axis trace out the boundaries of the quantum noise. Further from the origin the relative contribution of the quantum noise becomes less important.

The representation of the coherent state in the basis of Fock states is:

$$|\alpha> = \exp(-\tfrac{1}{2}|\alpha|^2) \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n> = \exp(-\tfrac{1}{2}|\alpha|^2) \exp\left(\alpha \mathcal{A}^\dagger\right) |0> \tag{3}$$

where |n> are Hermite functions (eigenvectors of the Hamiltonian). This is a Poissonian distribution. The probability of detecting *n* photons is:

$$\mathcal{P}(n) = \exp(-\langle n \rangle) \frac{\langle n \rangle^n}{n!} \tag{4}$$

Similarly, the average photon number in a coherent state is

$$\langle n \rangle = \langle \mathcal{A}^\dagger \mathcal{A} \rangle = |\alpha|^2 \tag{5}$$

and the variance is

$$(\Delta n)^2 = Var\left(\mathcal{A}^\dagger \mathcal{A}\right) = |\alpha|^2 \tag{6}$$

7.3 Squeezing

The squeezing operator can squeeze a state more or less in the direction of either P or Q. The operator is defined as:

$$Sq(z) = \exp\left(\tfrac{1}{2}\left(z^* \mathcal{A} + z \mathcal{A}^\dagger\right)\right) \tag{1}$$

$$z = r \exp(\mathbf{k}\,\theta) \tag{2}$$

The ground state is a saturated squeezed coherent state where

$$\Delta p = \Delta q \text{ and } \Delta q \cdot \Delta p = \hbar/2 \tag{3}$$

8 Base transforms

Now we have discovered the following base transforms:
Position$\Longleftrightarrow$momentum:

$$< q|p > = \frac{1}{\sqrt{2\pi\hbar}} \exp(\frac{k\,q\,p}{\hbar}) \tag{1}$$

Position⟺Fock state:

$$< q|n > = \sqrt[4]{\frac{m\omega}{\pi\hbar}} \frac{1}{\sqrt{2^n\,n!}} \exp(-\frac{m\omega}{2\hbar}\,q^2)H_n(q\sqrt{\frac{m\omega}{\hbar}}) \tag{2}$$

Fock state⟺coherent state:

$$< n|z > = \frac{1}{\sqrt{n!}} z^n \exp(-\tfrac{1}{2}|z|^2) \tag{3}$$

9 Oscillations

9.1 Harmonic oscillating Hilbert field

Take the ingredients of the complex harmonic oscillator and interpret these as similar ingredients of a harmonic oscillating Hilbert field that is based on a Gaussian blur. The blur delivers the conditions of the ground state.

$$\psi_0(r) = \sqrt[4]{\frac{m\,\omega}{\pi\hbar}}\; e^{-\frac{m\,\omega}{2\hbar}r^2} \tag{1}$$

This means that the ground state corresponds with a Gaussian charge distribution. Higher states correspond to a blurred current. We indicate this current as vector potential $\boldsymbol{\varphi}$. Its time derivative $\dot{\boldsymbol{\varphi}}$ is perpendicular to $\boldsymbol{\varphi}$. The other ingredients are P, Q, $\mathcal{A}$ and $\mathcal{A}^\dagger$.

$$Q \Leftrightarrow \boldsymbol{\varphi} = \varphi_x = \sqrt{\frac{\hbar}{2m\omega}}\,(\mathcal{A}_x + \mathcal{A}_x^\dagger) \tag{2}$$

$$\mathcal{A}_x - \mathcal{A}_x^\dagger = 0 \tag{3}$$

$$P \Leftrightarrow m\dot{\boldsymbol{\varphi}} = m\dot{\varphi}_y = \sqrt{\frac{m\omega\hbar}{2}}\,(-\mathcal{A}_y + \mathcal{A}_y^\dagger) \tag{4}$$

$$\mathcal{A}_y + \mathcal{A}_y^\dagger = 0 \tag{5}$$

$$\mathcal{A} \Leftrightarrow \boldsymbol{\mathcal{A}} = i\mathcal{A}_x - ik\mathcal{A}_y = \sqrt{\frac{m\omega}{2\hbar}}\left(\boldsymbol{\varphi} + \frac{\dot{\boldsymbol{\varphi}}}{\omega}\right) = \sqrt{\frac{m\omega}{2\hbar}}\left(i\varphi_x + ik\frac{\dot{\varphi}_y}{\omega}\right) \tag{6}$$

$$\mathcal{A}^\dagger \Leftrightarrow \boldsymbol{\mathcal{A}}^\dagger = i\mathcal{A}_x^\dagger + ik\mathcal{A}_y^\dagger = \sqrt{\frac{m\omega}{2\hbar}}\left(\boldsymbol{\varphi} - \frac{\dot{\boldsymbol{\varphi}}}{\omega}\right) = \sqrt{\frac{m\omega}{2\hbar}}\left(i\varphi_x - ik\frac{\dot{\varphi}_y}{\omega}\right) \tag{7}$$

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The $\boldsymbol{\varphi}$ field and the $\dot{\boldsymbol{\varphi}}$ field are mutually perpendicular. If both fields are subjected to a synchronized quantum harmonic oscillation, then an oscillating wave results. We take the same ground state for each of the fields. These ground states correspond to a spherical symmetric Gaussian blur.

When bounds of the cavity are removed or relaxed, then the higher order modes may differ in a phase shift. The sign selections set the eigenvalues of the spin operator. The result is an elliptically polarized wave that moves in directions along $\boldsymbol{\varphi} \times \dot{\boldsymbol{\varphi}}$.

$\boldsymbol{\varphi}$ no longer stands for a single position, but instead for a Gaussian distribution of positions of virtual charges. Similarly $\dot{\boldsymbol{\varphi}}$ does not stand for a single moving particle, but for a moving Gaussian cloud of virtual charges.
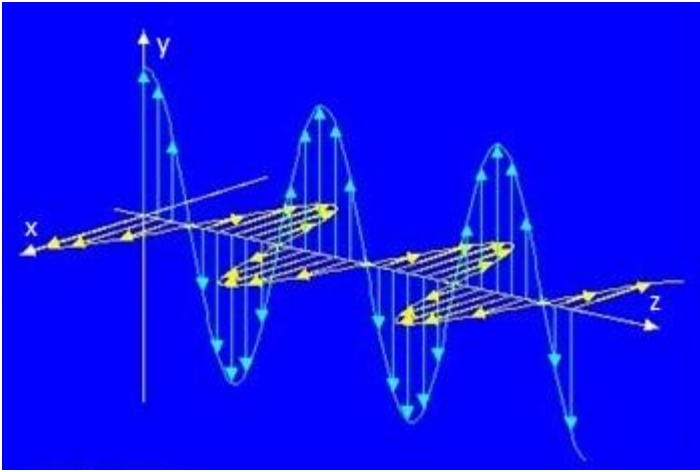
9.2   Annihilator and creator

The annihilator $\mathcal{A}$ and the creator $\mathcal{A}^{\dagger}$ are examples of boson operators. This is a consequence of their commutation relations.

$$\mathcal{A} + \mathcal{A}^{\dagger} = \alpha\,\boldsymbol{\varphi} \tag{1}$$

$$\mathcal{A} - \mathcal{A}^{\dagger} = \beta\,\dot{\boldsymbol{\varphi}} \tag{2}$$

$$\mathcal{A} = \tfrac{1}{2}\alpha\,\boldsymbol{\varphi} + \tfrac{1}{2}\beta\,\dot{\boldsymbol{\varphi}} \tag{3}$$

$$\mathcal{A}^{\dagger} = \tfrac{1}{2}\alpha\,\boldsymbol{\varphi} - \tfrac{1}{2}\beta\,\dot{\boldsymbol{\varphi}} \tag{4}$$

$$\left[\mathcal{A}(f), \mathcal{A}^{\dagger}(g)\right] = <f|g> \tag{5}$$

$$[\mathcal{A}(f), \mathcal{A}(g)] = 0 \tag{6}$$

$$\left[\mathcal{A}^\dagger(f), \mathcal{A}^\dagger(g)\right] = 0 \tag{7}$$

The corresponding fermion operators are:

$$\left\{\mathcal{B}(f), \mathcal{B}^\dagger(g)\right\} = <f|g> \tag{8}$$

$$\{\mathcal{B}(f), \mathcal{B}(g)\} = 0 \tag{9}$$

$$\left\{\mathcal{B}^\dagger(f), \mathcal{B}^\dagger(g)\right\} = 0 \tag{10}$$

The fermion operators can be represented by imaginary quaternionic base numbers:

$$\mathcal{B} + \mathcal{B}^\dagger = i \tag{11}$$

$$\mathcal{B} - \mathcal{B}^\dagger = j \tag{12}$$

$$\mathcal{B} = \tfrac{1}{2}(i + j) \tag{13}$$

$$\mathcal{B}^\dagger = \tfrac{1}{2}(i - j) \tag{14}$$

$$\left(\mathcal{B} + \mathcal{B}^\dagger\right)\left(\mathcal{B} - \mathcal{B}^\dagger\right) = \mathcal{B}\mathcal{B} - \mathcal{B}\mathcal{B}^\dagger + \mathcal{B}^\dagger\mathcal{B} - \mathcal{B}^\dagger\mathcal{B}^\dagger \tag{15}$$

$$= \mathcal{B}^\dagger\mathcal{B} - \mathcal{B}\mathcal{B}^\dagger = i\,j$$

9.3 Rotational symmetry

In case of rotational symmetry in the imaginary part of quaternion space, the exponential function must be replaced by a Bessel function. The corresponding Fourier transform then becomes a Hankel transform[91].

The spherical harmonics are eigenfunctions of the square of the orbital angular momentum operator $-i\hbar r \times \nabla$ and therefore they represent the different quantized configurations of atomic orbitals.

9.4 Spherical harmonics

The following draws from the work of S. Thangavelu[92].

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[91] http://en.wikipedia.org/wiki/Hankel_transform
[92] http://www.math.iitb.ac.in/atm/faha1/veluma.pdf

In this subsection we look for eigenfunctions of the Fourier transform which have spherical symmetry. As in the one dimensional case we consider functions of the form

$$f(x) = p(x) \exp(-\pi |x|^2) \tag{1}$$

This will be an eigenfunction of $\mathcal{F}$ if and only p satisfies

$$\int_{\mathbb{R}^n} p(x - iy) \exp(-\pi \cdot |x|^2) \, dx = \lambda \, p(y) \tag{2}$$

Here in quaternion terms x and iy represent two mutually perpendicular imaginary numbers while x and y are parallel. Thangavelu uses complex numbers. We keep as close as is possible to his text.

If (2) is true for all $y \in \mathbb{R}^n$ then we should also have

$$\int_{\mathbb{R}^n} p(x + y) \exp(-\pi \cdot |x|^2) \, dx = \lambda \, p(iy) \tag{3}$$

Integrating in polar coordinates the integral on the left is

$$\int_{r=0}^{\infty} |\mathbb{S}^{n-1}| \left(\int_{\mathbb{S}^{n-1}} p(y + r \, \omega) \, d\sigma(\omega) \right) \exp(-\pi \cdot r^2) \cdot r^{n-1} dr \tag{4}$$

where $d\sigma(\omega)$ is the normalised surface measure on the unit sphere $\mathbb{S}^{n-1}$.

If p is homogeneous of degree m then

$$p(i \, y) = i^m \, p(y) \tag{5}$$

and hence for such polynomials the equation

$$\int_{\mathbb{R}^n} p(x + y) \exp(-\pi \cdot |x|^2) \, dx = \lambda \, i^m \, p(iy) \tag{6}$$

will be satisfied for

$$\lambda = (-i)^m \tag{7}$$

if p has the mean value property

$$\int_{\mathbb{S}^{n-1}} p(y + r \, \omega) d\sigma(\omega) = p(y) \tag{8}$$

Such functions are precisely the harmonic functions satisfying

$$\Delta u = 0 \tag{9}$$

Thus we have proved:

Let

$$f(x) = p(x) \, exp(-\pi \, |x|^2) \tag{10}$$

where p is homogeneous of degree m and harmonic. Then

$$\mathcal{F} f = (-i)^m \, f \tag{11}$$

Let $\mathcal{P}^m$ stand for the finite dimensional space of homogeneous harmonic polynomials of degree m:

The above theorem says that the finite dimensional subspace of $L^2(\mathbb{R}^n)$ consisting of functions of the form

$$p(x) \exp(-\pi \, |x|^2); \; p \in \mathcal{P}^m \tag{12}$$

is invariant under the Fourier transform.
We claim that the following extension is true.
Let

$$f \in L^2(\mathbb{R}^n \tag{13}$$

be of the form

$$f(x) = p(x)g(|x|); \; p \in \mathcal{P}^m \tag{14}$$

Then

$$\mathcal{F} f(\xi) = p(\xi)g(|\xi|) \tag{15}$$

Thus the subspace of functions of the form

$$f(x) = p(x)g(|x|); \; p \in \mathcal{P}^m \tag{16}$$

is invariant under the Fourier transform.

Let

$$f \in L^2(\mathbb{R}^n) \tag{17}$$

be of the form

$$f(x) = p(x)g(|x|); \; p \in \mathcal{P}^m \tag{18}$$

Then

$$\mathcal{F}_n(f) = (-i)^m \, p \, \mathcal{F}_{n+2m} \, g \tag{19}$$

The above result is known as the Hecke-Bochner formula for the Fourier transform.

We conclude our discussion on invariant subspaces with the following result which shows that the Fourier transform of a radial function reduces to an integral transform whose kernel is a Bessel function. This relates to the Hankel transform.

Let J_α stand for the Bessel function of type $\alpha > -1$
If

$$f(x) = g(|x|) \tag{20}$$

is radial and integrable then

$$\mathcal{F}_n(f)(\xi) = c_n \int_0^\infty g(r) \cdot \frac{J_{\frac{n}{2}-1}(2\pi r |\xi|)}{(2\pi r |\xi|)^{\frac{n}{2}-1}} \cdot r^{n-1} dr \tag{21}$$

9.5   Spherical harmonic transform

Next we like to decompose 2D and 3D functions into wave-like basic patterns that have simple radial and angular structures[93]. In that case, the base functions must take the separation-of-variable form:

$$R(r)\Phi(\varphi) = \frac{1}{\sqrt{2\pi}} R(r) \exp(im\varphi) \tag{1}$$

for 2D and

$$R(r)\Theta(v)\Phi(\varphi) = R(r)\Omega(v,\varphi) \tag{2}$$

$$\Omega(v,\varphi) = Y_{lm}(v,\varphi) = \sqrt{\frac{2l+1}{4\pi} \frac{(l-m)!}{(l+m)!}} P_{lm}(v) \exp(im\varphi) \tag{3}$$

for 3D where (r,φ) and (r,v,φ) are the polar and spherical coordinates respectively. m and l are integers. $l \geq 0$ and $|m| \leq l$.

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[93]

The base functions are eigenfunctions of the Laplacian. They represent wave-like patterns. The associated angular transform is closely related to the normal Fourier transform. For polar coordinates this reduces to a simple complex 1D Fourier transform.

The radial base function is a Bessel function $J_m(kr)$ for polar coordinates and a spherical Bessel function $j_l(kr)$ for spherical coordinates. The parameter k can take either continuous or discrete values, depending on whether the region is infinite or finite. For functions defined on $(0, \infty)$, the transform with $J_m(kr)$ as integral kernel and r as weight is known as the **Hankel transform**. For functions defined on a finite interval, with zero-value boundary condition for the base functions, one gets the **Fourier-Bessel series**. For the 3D case the transform is called **Spherical Harmonic (SH) transform**.

9.6   Polar coordinates

The Laplacian in polar coordinates is:

$$\nabla^2 \psi = \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial \psi}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 \psi}{\partial \varphi^2} \tag{1}$$

The Helmholtz differential equation is

$$\nabla^2 \psi(r, \varphi) = -k^2 \psi(r, \varphi) \tag{2}$$

$$\psi(r, \varphi) = R(r)\Phi(\varphi) \tag{3}$$

$$\Phi(\varphi) = -m^2 \Phi(\varphi) \tag{4}$$

$$\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial R(r)}{\partial r} \right) = \left(\frac{m^2}{r^2} - k^2 \right) R(r) \tag{5}$$

The solution is:

$$\Phi_m(\varphi) = \exp(im\varphi) \tag{6}$$

$$R(r) = aJ_m(kr) + bY_m(kr) \tag{7}$$

J_m is the m-th order Bessel function. The Neumann function Y_m is singular at $r = 0$. Therefore $a = 1$ and $b = 0$.

In finite solutions, the boundary conditions determine what set of functions can be used as base functions. The reference in the footnote shows which choices can be relevant.

9.7   Spherical coordinates

The Laplacian in polar coordinates is:

$$\nabla^2 \psi = \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial \psi}{\partial r} \right) + \frac{1}{r^2 \sin^2 v} \frac{\partial^2 \psi}{\partial \varphi^2} + \frac{1}{r^2 \sin v} \frac{\partial}{\partial v} \left(\sin v \frac{\partial \psi}{\partial v} \right) \tag{1}$$

The Helmholtz differential equation is

$$\nabla^2 \psi(r, \varphi, v) = -k^2 \psi(r, \varphi, v) \tag{2}$$

$$\psi(r, \varphi, v) = R(r)\Omega(\varphi, v) \tag{3}$$

$$\Omega(\varphi, v) = Y_{lm}(\varphi, v) \tag{4}$$

$$Y_{lm}(v, \varphi) = \sqrt{\frac{2l+1}{4\pi} \frac{(l-m)!}{(l+m)!}} P_{lm}(v) \exp(im\varphi) \tag{5}$$

$$\frac{1}{r^2} \frac{\partial}{\partial r}\left(r^2 \frac{\partial R(r)}{\partial r}\right) = \left(\frac{l(l-1)}{r^2} - k^2\right) R(r) \tag{6}$$

A non-singular solution for $R(r)$ is:

$$R(r) = j_l(kr) \tag{7}$$

j_l is the spherical Bessel function of order l.

$$j_l(r) = \sqrt{\frac{\pi}{2r}} J_{l+\frac{1}{2}}(r) \tag{8}$$

9.8 The spherical harmonic transform

The equivalent of the Fourier transform in terms of spherical harmonics depends on the boundary conditions. For example when the analysis is done over a limited region, then the zero boundary condition will give different results than the zero derivative boundary condition[94]. An infinite range will always request a zero value of contributions when the radius goes to infinity.

$$S_{klm} = \int_{r=0}^{\infty} \int_{\varphi=0}^{2\pi} \int_{v=0}^{\pi} f(r, \varphi, v)\ \psi_{klm}^*(r, \varphi, v)\ r^2 \sin v\ dr\ d\varphi\ dv \tag{1}$$

$$f(r, \varphi, v) = \sum_{k=1}^{\infty} \sum_{l=0}^{\infty} \sum_{m=-l}^{l} S_{klm}\ \psi_{klm}(r, \varphi, v) \tag{2}$$

$$\psi_{klm}(r, \varphi, v) = j_l(kr)\ Y_{lm}(\varphi, v) \tag{3}$$

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[94] http://lmb.informatik.uni-freiburg.de/papers/download/wa_report01_08.pdf

9.9   The Fourier transform of a black hole

In its simplest form a black hole is a bubble that is covered with a blanket of ground states.

The blanket is a comb function that is convoluted with a ground state. The Fourier transform of this blanket is the product of the Fourier transform of the comb function and the Fourier transform of the ground state. Apart from a factor, the ground state is invariant under Fourier transformation. Also the comb function is invariant. Thus the Fourier transform of the blanket is a modulated comb function. The modulation does not reach far.

The most complicated component is the bubble. In its simplest form this is a pulse on the radius. If we interpret this pulse as a Dirac delta function, then the Fourier coefficients have the form:

$$\psi_{k00}(r) = j_0(kr_0) = \sqrt{\frac{\pi}{2r}} \, J_{\frac{1}{2}}(kr_0)$$

If we sum these coefficients, then we get a sampled spherical Bessel function. These spheres are blurred with the transformed blanket.

9.10   Spherical harmonics eigenvalues

See: http://en.wikipedia.org/wiki/Spherical_harmonics for more details.

Spherical harmonics are best presented in polar coordinates. There exists a corresponding polar Fourier transform. This Fourier transform also has invariant functions. Like in the rectangular case, they form the basis for spherical harmonics.

Laplace's equation in spherical coordinates is:

$$\nabla^2 f = r^{-2} \frac{\partial}{\partial r}\left(r^2 \frac{\partial f}{\partial r}\right) + \frac{1}{r^2 \sin(\theta)} \frac{\partial}{\partial \theta}\left(\sin(\theta) \frac{df}{d\theta}\right) + \frac{1}{r^2 \sin^2(\theta)} \frac{\partial^2 f}{\partial \varphi^2} = 0 \tag{1}$$

Try to find solutions in the form of the eigenfunctions of the Fourier transform.

By separation of variables, two differential equations result by imposing Laplace's equation:

$$f(r, \theta, \varphi) = R(r) \cdot Y(\theta, \varphi) \tag{2}$$

$$R^{-1} \frac{d}{dr}\left(r^2 \frac{dR}{dr}\right) = \lambda \tag{3}$$

$$\frac{1}{Y \sin(\theta)} \frac{\partial}{\partial \theta}\left(\sin(\theta) \frac{\partial Y}{\partial \theta}\right) + \frac{1}{Y \sin^2(\theta)} \frac{\partial^2 Y}{\partial \varphi^2} = -\lambda \tag{4}$$

The second equation can be simplified under the assumption that Y has the form

$$Y(\theta, \varphi) = \Theta(\theta)\Phi(\varphi) \tag{5}$$

Applying separation of variables again to the second equation gives way to the pair of differential equations

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$$\frac{1}{\Phi(\varphi)}\frac{d^2\Phi(\varphi)}{d\varphi^2} = -m^2 \tag{6}$$

$$\lambda \sin^2(\theta) + \frac{\sin(\theta)}{\Theta(\theta)}\frac{d}{d\theta}\left[\sin(\theta)\frac{d\Theta}{d\theta}\right] = m^2 \tag{7}$$

for some number m. A priori, m is a complex constant, but because Φ must be a periodic function whose period evenly divides 2π, m is necessarily an integer and Φ is a linear combination of the complex exponentials $\exp(\pm i\,m\,\theta)$. The solution function $Y(\theta,\varphi)$ is regular at the poles of the sphere, where $\theta = 0, \pi$. Imposing this regularity in the solution Θ of the second equation at the boundary points of the domain is a [Sturm–Liouville problem][95] that forces the parameter λ to be of the form $\lambda = \ell(\ell+1)$ for some non-negative integer with $\ell \geq |m|$; this is also explained below in terms of the orbital angular momentum. Furthermore, a change of variables $t = cos\theta$ transforms this equation into the Legendre equation, whose solution is a multiple of the [associated Legendre function][96]. $P_l^m(cos(\theta))$. Finally, the equation for R has solutions of the form $R(r) = A\,r^\ell + B\,r^{-\ell-1}$; requiring the solution to be regular throughout $\mathbb{R}^3$ forces $B = 0$.

Here the solution was assumed to have the special form

$$Y(\theta,\varphi) = \Theta(\theta)\,\Phi(\varphi) \tag{8}$$

For a given value of ℓ, there are $2\ell + 1$ independent solutions of this form, one for each integer m with $-\ell \leq m \leq \ell$. These angular solutions are a product of trigonometric functions, here represented as a complex exponential, and associated Legendre functions:

$$Y_l^m(\theta,\varphi) = N \exp(i\,m\,\varphi)\,P_l^m(cos(\theta)) \tag{9}$$

which fulfill

$$r^2\,\nabla^2 Y_l^m(\theta,\varphi) = -l\,(l+1)\,Y_l^m(\theta,\varphi) \tag{10}$$

Here Y_l^m is called a spherical harmonic function of degree ℓ and order m, P_l^m is an associated Legendre function, N is a normalization constant, θ represents the colatitude and φ represents the longitude. In particular, the [colatitude][97] θ, or polar angle, ranges from 0 at the North Pole to π at the South Pole, assuming the value of $\pi/2$ at the Equator, and the [longitude][98] φ, or [azimuth][99], may assume all values with $0 \leq \varphi < 2\pi$. For a fixed integer ℓ, every solution $Y(\theta,\varphi)$ of the eigenvalue problem

$$r^2\,\nabla^2 Y = -l\,(l+1)\,Y \tag{11}$$

[95] http://en.wikipedia.org/wiki/Sturm%E2%80%93Liouville_problem
[96] http://en.wikipedia.org/wiki/Associated_Legendre_function
[97] http://en.wikipedia.org/wiki/Colatitude
[98] http://en.wikipedia.org/wiki/Longitude
[99] http://en.wikipedia.org/wiki/Azimuth

is a linear combination of Y_l^m. In fact, for any such solution, $r^\ell Y(\theta, \varphi)$ is the expression in spherical coordinates of a homogeneous polynomial that is harmonic, and so counting dimensions shows that there are $2\ell + 1$ linearly independent of such polynomials.

The general solution to Laplace's equation in a ball centered at the origin is a linear combination of the spherical harmonic functions multiplied by the appropriate scale factor r^l,

$$f(r, \theta, \varphi) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} f_l^m r^l Y_l^m(\theta, \varphi) \tag{12}$$

where the f_l^m are constants and the factors $r^l Y_l^m$ are known as solid harmonics[100]. Such an expansion is valid in the ball

$$r < R = 1/\lim_{l \Rightarrow \infty} \sup |f_l^m|^{1/l} \tag{13}$$

9.11 Orbital angular momentum

In quantum mechanics, Laplace's spherical harmonics are understood in terms of the orbital angular momentum[101]

$$\boldsymbol{L} = -i \, \hbar \, \boldsymbol{x} \times \boldsymbol{\nabla} = L_x \, \boldsymbol{i} + L_y \boldsymbol{j} + L_z \, \boldsymbol{k} \tag{1}$$

The spherical harmonics are eigenfunctions of the square of the orbital angular momentum

$$\boldsymbol{L}^2 = -r^2 \, \nabla^2 + \left(r \, \frac{\partial}{\partial r} + 1 \right) r \, \frac{\partial}{\partial r} \tag{2}$$

$$= \frac{1}{\sin(\theta)} \, \frac{\partial}{\partial \theta} \, \sin(\theta) \frac{\partial}{\partial \theta} - \frac{1}{\sin^2(\theta)} \cdot \frac{\partial^2}{\partial \varphi^2}$$

Laplace's spherical harmonics are the joint eigenfunctions of the square of the orbital angular momentum and the generator of rotations about the azimuthal axis:

$$L_z = -i \cdot \hbar \cdot \left(x \frac{\partial}{\partial y} - y \, \frac{\partial}{\partial z} \right) = i \, \hbar \, \frac{\partial}{\partial \varphi} \tag{3}$$

These operators commute, and are densely defined self-adjoint operators on the Hilbert space of functions f square-integrable with respect to the normal distribution on $\mathbb{R}^3$:

$$(2\pi)^{-\frac{3}{2}} \int_{\mathbb{R}^3} |f(x)|^2 \exp(-|x|^2/2) < \infty \tag{4}$$

Furthermore, $\boldsymbol{L}^2$ is a positive operator.

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[100] http://en.wikipedia.org/wiki/Solid_harmonics
[101] http://en.wikipedia.org/wiki/Orbital_angular_momentum

If Y is a joint eigenfunction of L^2 and L_z, then by definition

$$\boldsymbol{L}^2\, Y \;=\; \lambda\, Y \tag{5}$$

$$L_z\, Y \;=\; m\, Y \tag{6}$$

for some real numbers m and λ. Here m must in fact be an integer, for Y must be periodic in the coordinate φ with period a number that evenly divides 2π. Furthermore, since

$$\boldsymbol{L}^2 \;=\; L_x^2 \;+\; L_y^2 \;+\; L_z^2 \tag{7}$$

and each of L_x, L_y, L_z are self-adjoint, it follows that $\lambda \ge m^2$.

Denote this joint eigenspace by $E_{\lambda,m}$, and define the raising and lowering operators by

$$L_+ \;=\; L_x \;+\; i\, L_y \tag{8}$$

$$L_- \;=\; L_x \;-\; i\, L_y \tag{9}$$

Then L_+ and L_- commute with $\boldsymbol{L}^2$, and the Lie algebra generated by L_+, L_-, L_z is the special linear Lie algebra, with commutation relations

$$[L_z, L_+] \;=\; L_+ \tag{10}$$

$$[L_z, L_-] \;=\; -L_- \tag{11}$$

$$[L_+, L_-] \;=\; 2\, L_z \tag{12}$$

Thus $L_+\colon E_{\lambda,m} \to E_{\lambda,m+1}$ (it is a "raising operator") and $L_-\colon E_{\lambda,m} \to E_{\lambda,m-1}$ (it is a "lowering operator"). In particular, $L_+^k\colon E_{\lambda,m} \to E_{\lambda,m+k}$ must be zero for k sufficiently large, because the inequality $\lambda \ge m^2$ must hold in each of the nontrivial joint eigenspaces. Let $Y \in E_{\lambda,m}$ be a nonzero joint eigenfunction, and let k be the least integer such that

$$L_+^k\, Y \;=\; 0 \tag{13}$$

Then, since

$$L_-\, L_+ \;=\; \boldsymbol{L}^2 \;-\; L_z^2 \;-\; L_z \tag{14}$$

it follows that

$$0 \;=\; L_-\, L_+^k\, Y \;=\; \big(\lambda - (m+k)^2 - (m+k)\big)\, Y \tag{15}$$

Thus $\lambda \;=\; \ell(\ell+1)$ for the positive integer $\ell \;=\; m+k$. \tag{16}

9.12 Spherical harmonics expansion

The Laplace spherical harmonics form a complete set of orthonormal functions and thus form an orthonormal basis of the Hilbert space of square-integrable functions. On the unit sphere, any square-integrable function can thus be expanded as a linear combination of these:

$$f(\theta, \varphi) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} f_l^m Y_l^m(\theta, \varphi) \tag{1}$$

This expansion holds in the sense of mean-square convergence — convergence in $\underline{L^2}$ of the sphere — which is to say that

$$\lim_{N \to \infty} \int_0^{2\pi} \int_0^{\pi} \left| f(\theta, \varphi) - \sum_{l=0}^{N} \sum_{m=-l}^{l} f_l^m Y_l^m(\theta, \varphi) \right|^2 \sin(\theta)\, d\theta\, d\varphi = 0 \tag{2}$$

The expansion coefficients are the analogs of Fourier coefficients, and can be obtained by multiplying the above equation by the complex conjugate of a spherical harmonic, integrating over the solid angle Ω, and utilizing the above orthogonality relationships. This is justified rigorously by basic Hilbert space theory. For the case of orthonormalized harmonics, this gives:

$$f_l^m = \int_\Omega f(\theta, \varphi)\, Y_l^{m*}(\theta, \varphi) d\Omega = \int_0^{2\pi} d\varphi \int_0^{\pi} d\theta \sin(\theta)\, f(\theta, \varphi)\, Y_l^{m*}(\theta, \varphi) \tag{3}$$

If the coefficients decay in ℓ sufficiently rapidly — for instance, exponentially — then the series also converges uniformly to f.

A real square-integrable function f can be expanded in terms of the real harmonics $Y_{\ell m}$ above as a sum

$$f(\theta, \varphi) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} f_l^m Y_l^m(\theta, \varphi) \tag{4}$$

Convergence of the series holds again in the same sense.

9.13 Spin weighted spherical harmonics

Regard the sphere $\mathbb{S}^2$ as embedded into the three-dimensional [imaginary part of the quaternionic number field]. At a point **x** on the sphere, a positively oriented orthonormal basis of tangent vectors at **x** is a pair **a**, **b** of vectors such that

$$(\boldsymbol{x}, \boldsymbol{a}) = (\boldsymbol{x}, \boldsymbol{b}) = (\boldsymbol{a}, \boldsymbol{b}) = 0 \tag{1}$$

$$(a, a) = (b, b) = 1 \tag{2}$$

$$\langle x, a \times b \rangle > 0 \tag{3}$$

where the first pair of equations states that **a** and **b** are tangent at **x**, the second pair states that **a** and **b** are unit vectors, **a** and **b** are orthogonal, and the $\{x, a, b\}$ is a right-handed basis of $\mathbb{R}^3$.
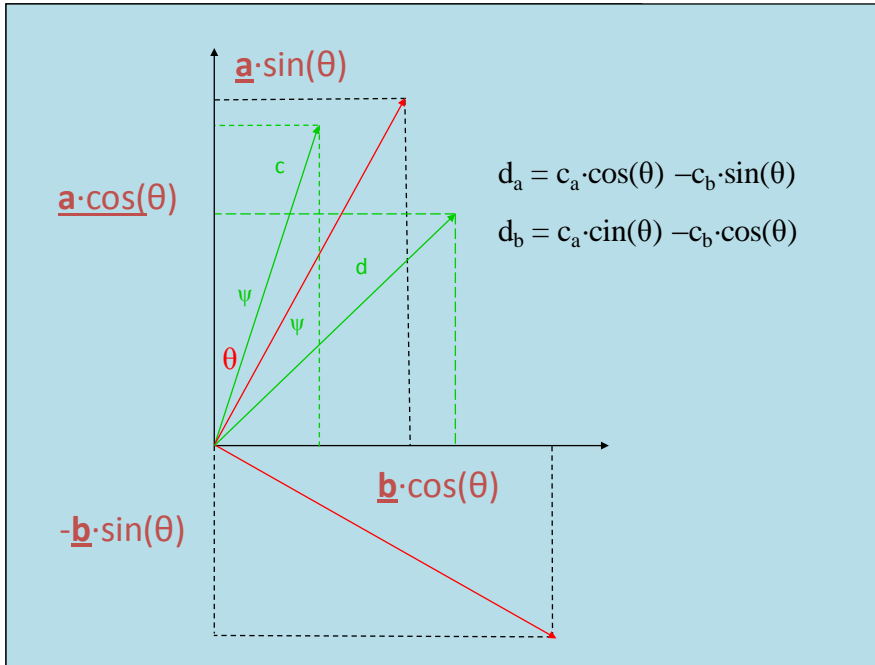

Figure 3: θ and the parameters a and b of the spin-weight function f.

A spin-weight s function f is a function accepting as input a point **x** of S^2 and a positively oriented orthonormal basis of tangent vectors at **x**, such that

$$f(x, a \cos(\theta) - b \sin(\theta), a \sin(\theta) + b \cos(\theta)) = \exp(i s \theta) f(x, a, b) \tag{4}$$

for every rotation angle θ.

Following Eastwood & Tod (1982), denote the collection of all spin-weight s functions by **B**(s). Concretely, these are understood as functions f on $C^2 \backslash \{0\}$ satisfying the following homogeneity law under complex scaling

$$\tag{5}$$

$$f(\lambda\,\mathbf{z},\bar{\lambda}\,\bar{\mathbf{z}}) = \left(\frac{\bar{\lambda}}{\lambda}\right)^{s} f(\mathbf{z},\bar{\mathbf{z}})$$

This makes sense provided s is a half-integer.

Abstractly, $\mathbf{B}(s)$ is isomorphic to the smooth vector bundle underlying the antiholomorphic vector bundle $O^{*}(2 \cdot s)$ of the Serre twist on the complex projective line $\boldsymbol{CP}^{1}$. A section of the latter bundle is a function g on $\boldsymbol{C}^{2}\backslash\{0\}$ satisfying

$$g\left(\lambda\,\mathbf{z},\bar{\lambda}\,\bar{\mathbf{z}}\right) = \left(\bar{\lambda}\right)^{2s} g(\mathbf{z},\bar{\mathbf{z}}) \tag{6}$$

Given such a g, we may produce a spin-weight s function by multiplying by a suitable power of the Hermitian form

$$P(\mathbf{z},\bar{\mathbf{z}}) = \mathbf{z}\,\bar{\mathbf{z}} \tag{7}$$

Specifically, $\mathfrak{f} = P^{-s} g$ is a spin-weight s function. The association of a spin-weighted function to an ordinary homogeneous function is an isomorphism.

9.14   Eth

The spin weight bundles $\mathbf{B}(s)$ are equipped with a differential operator ð (eth). This operator is essentially the [Dolbeault operator][102],

$$ð = \partial + \partial^{*} \tag{1}$$

Thus for $\mathfrak{f} \in \boldsymbol{B}(s)$,

$$ð\,\mathfrak{f} \equiv P^{-s+1}\,\partial(P^{s}\,\mathfrak{f}) \tag{2}$$

defines a function of spin-weight $s+1$.

9.15   Spin-weighted harmonic functions

See http://en.wikipedia.org/wiki/Spin-weighted_spherical_harmonics for more details.

Just as conventional spherical harmonics are the eigenfunctions of the Laplace-Beltrami operator on the sphere, the spin-weight s harmonics are the eigensections for the Laplace-Beltrami operator acting on the bundles $\mathcal{E}(s)$ of spin-weight s functions.

The spin-weighted harmonics can be represented as functions on a sphere once a point on the sphere has been selected to serve as the North Pole. By definition, a function η with *spin weight s* transforms under rotation about the pole via

$$\eta \;\rightarrow\; \exp(i\,s\,\psi)\,\eta \tag{1}$$

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[102] http://en.wikipedia.org/wiki/Dolbeault_operator

Working in standard spherical coordinates, we can define a particular operator ð acting on a function η as:

$$\eth\eta = -\sin^s(\theta) \left\{ \frac{\partial}{\partial\theta} + \frac{i}{\sin(\theta)} \frac{\partial}{\partial\varphi} \right\} [\sin^{-s}(\theta)\,\eta] \tag{2}$$

This gives us another function of θ and φ. [The operator ð is effectively a covariant derivative operator in the sphere.]

An important property of the new function $\eth\eta$ is that if η had spin weight s, $\eth\eta$ has spin weight $s + 1$. Thus, the operator raises the spin weight of a function by 1. Similarly, we can define an operator which will lower the spin weight of a function by 1:

$$\bar{\eth}\eta = -\sin^{-s}(\theta) \left\{ \frac{\partial}{\partial\theta} - \frac{i}{\sin(\theta)} \frac{\partial}{\partial\varphi} \right\} [(\sin^s(\theta)\,\eta] \tag{3}$$

We extend the function Y_l^m to $_sY_l^m$ according to

$$_0Y_l^m(\theta,\varphi) = Y_l^m(\theta,\varphi) \tag{4}$$

$$l = 0, 1, 2, \ldots; \quad m = -l, \ldots 0, \ldots l \tag{5}$$

The spin-weighted spherical harmonics are then defined in terms of the usual spherical harmonics as:

$$_sY_l^m = \sqrt{\frac{(l-s)!}{(l+s)!}} \, \eth^s \, Y_l^m; \, 0 \leq s \leq l \tag{6}$$

$$_sY_l^m = \sqrt{\frac{(l+s)!}{(l-s)!}} \, (-1)^s \, \bar{\eth}^s \, Y_l^m; -l \leq s \leq 0 \tag{7}$$

$$_sY_l^m = 0; \, l < |s|; \tag{8}$$

The functions $_sY_l^m$ then have the property of transforming with spin weight s.
Other important properties include the following:

$$\eth(_sY_l^m) = +\sqrt{(l-s)\,(l+s+1)} \, _{s+1}Y_l^m \tag{9}$$

$$\eth(_sY_l^m) = -\sqrt{(l+s)\,(l-s+1)} \, _{s-1}Y_l^m \tag{10}$$

10 Differentiation

A quaternionic distribution f(q) can be differentiated.

$$g(q) = \nabla_0 f_0(q) \mp \langle \nabla, \boldsymbol{f}(q) \rangle \pm \nabla_0 \boldsymbol{f}(q) + \nabla f_0(q) \pm (\pm \nabla \times \boldsymbol{f}(q)) \qquad (1)$$

The colored $\mp$ and $\pm$ signs refer to the influence of conjugation of $f(q)$ on quaternionic multiplication. The $\pm$ sign refers to the influence of reflection of $f(q)$.

10.1 Continuity equation

When applied to a quaternionic probability amplitude distribution (QPAD), the equation for the differentiation leads to a continuity equation.

When $\rho_0(q)$ is interpreted as a charge density distribution, then the conservation of the corresponding charge is given by the continuity equation:

$$\text{Total change within } V = \text{flow into } V + \text{production inside } V \qquad (1)$$

$$\frac{d}{dt} \int_V \rho_0 \, dV = \oint_S \hat{\boldsymbol{n}} \rho_0 \frac{\boldsymbol{v}}{c} \, dS + \int_V s_0 \, dV \qquad (2)$$

$$\int_V \nabla_0 \rho_0 \, dV = \int_V \langle \nabla, \boldsymbol{\rho} \rangle \, dV + \int_V s_0 \, dV \qquad (3)$$

Here $\hat{\boldsymbol{n}}$ is the normal vector pointing outward the surrounding surface S, $\boldsymbol{v}(t, \boldsymbol{q})$ is the velocity at which the charge density $\rho_0(t, \boldsymbol{q})$ enters volume V and s_0 is the source density inside V. In the above formula $\boldsymbol{\rho}$ stands for

$$\boldsymbol{\rho} = \rho_0 \boldsymbol{v}/c \qquad (4)$$

It is the flux (flow per unit area and unit time) of ρ_0 .

The combination of $\rho_0(t, \boldsymbol{q})$ and $\boldsymbol{\rho}(t, \boldsymbol{q})$ is a quaternionic skew field $\rho(t, \boldsymbol{q})$ and can be seen as a probability amplitude distribution (QPAD).

$$\rho \stackrel{\text{def}}{=} \rho_0 + \boldsymbol{\rho} \qquad (5)$$

$\rho(t, \boldsymbol{q}) \rho^*(t, \boldsymbol{q})$ can be seen as an overall probability density distribution of the presence of the carrier of the charge. $\rho_0(t, \boldsymbol{q})$ is a charge density distribution. $\boldsymbol{\rho}(t, \boldsymbol{q})$ is the current density distribution.

The conversion from formula (2) to formula (3) uses the Gauss theorem[103]. This results in the law of charge conservation:

$$s_0(t, \boldsymbol{q}) = \nabla_0 \rho_0(t, \boldsymbol{q}) \mp \langle \nabla, (\rho_0(t, \boldsymbol{q}) \boldsymbol{v}(t, \boldsymbol{q}) + \nabla \times \boldsymbol{a}(t, \boldsymbol{q})) \rangle \qquad (6)$$

$$= \nabla_0 \rho_0(t, \boldsymbol{q}) \mp \langle \nabla, \boldsymbol{\rho}(t, \boldsymbol{q}) + \boldsymbol{A}(t, \boldsymbol{q}) \rangle$$

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[103] http://en.wikipedia.org/wiki/Divergence_theorem

$$= \nabla_0 \rho_0(t, \boldsymbol{q}) \mp \langle \boldsymbol{v}(t, \boldsymbol{q}), \nabla \rho_0(t, \boldsymbol{q}) \rangle \mp \langle \nabla, \boldsymbol{v}(t, \boldsymbol{q}) \rangle \rho_0(t, \boldsymbol{q})$$

$$\mp \langle \nabla, \boldsymbol{A}(t, \boldsymbol{q}) \rangle$$

The blue colored $\pm$ indicates quaternionic sign selection through conjugation of the field $\rho(t, \boldsymbol{q})$. The field $\boldsymbol{a}(t, \boldsymbol{q})$ is an arbitrary differentiable vector function.

$$\langle \nabla, \nabla \times \boldsymbol{a}(t, \boldsymbol{q}) \rangle = 0 \tag{7}$$

$\boldsymbol{A}(t, \boldsymbol{q}) \overset{\text{def}}{=} \nabla \times \boldsymbol{a}(t, \boldsymbol{q})$ is always divergence free. In the following we will neglect $\boldsymbol{A}(t, \boldsymbol{q})$.

In Fourier space the continuity equation becomes:

$$\tilde{s}_0(t, \boldsymbol{p}) = p_0 \tilde{\rho}_0(t, \boldsymbol{p}) \mp \langle \boldsymbol{p}, \tilde{\boldsymbol{\rho}}(t, \boldsymbol{p}) \rangle \tag{8}$$

Equation (6) represents a balance equation for charge density. What this charge is will be left in the middle. It can be one of the properties of the carrier or it can represent the full ensemble of the properties of the carrier.

This only treats the real part of the full equation. The full equation runs:

$$s(t, \boldsymbol{q}) = \nabla \rho(t, \boldsymbol{q}) = s_0(t, \boldsymbol{q}) + \boldsymbol{s}(t, \boldsymbol{q}) \tag{9}$$

$$= \nabla_0 \rho_0(t, \boldsymbol{q}) \mp \langle \nabla, \boldsymbol{\rho}(t, \boldsymbol{q}) \rangle \pm \nabla_0 \boldsymbol{\rho}(t, \boldsymbol{q}) + \nabla \rho_0(t, \boldsymbol{q}) \pm \left(\pm \nabla \times \boldsymbol{\rho}(t, \boldsymbol{q}) \right)$$

$$= \nabla_0 \rho_0(t, \boldsymbol{q}) \mp \langle \boldsymbol{v}(t, \boldsymbol{q}), \nabla \rho_0(t, \boldsymbol{q}) \rangle \mp \langle \nabla, \boldsymbol{v}(t, \boldsymbol{q}) \rangle \rho_0(t, \boldsymbol{q})$$

$$\pm \nabla_0 \boldsymbol{v}(t, \boldsymbol{q}) + \nabla_0 \rho_0(t, \boldsymbol{q}) + \nabla \rho_0(t, \boldsymbol{q})$$

$$\pm \left(\pm (\rho_0(t, \boldsymbol{q}) \nabla \times \boldsymbol{v}(t, \boldsymbol{q}) - \boldsymbol{v}(t, \boldsymbol{q}) \times \nabla \rho_0(t, \boldsymbol{q}) \right)$$

$$s_0(t, \boldsymbol{q}) = 2 \nabla_0 \rho_0(t, \boldsymbol{q}) \mp \langle \boldsymbol{v}(q), \nabla \rho_0(t, \boldsymbol{q}) \rangle \mp \langle \nabla, \boldsymbol{v}(t, \boldsymbol{q}) \rangle \rho_0(t, \boldsymbol{q}) \tag{10}$$

$$\boldsymbol{s}(t, \boldsymbol{q}) = \pm \nabla_0 \boldsymbol{v}(t, \boldsymbol{q}) \pm \nabla \rho_0(t, \boldsymbol{q}) \tag{11}$$

$$\pm \left(\pm (\rho_0(t, \boldsymbol{q}) \nabla \times \boldsymbol{v}(t, \boldsymbol{q}) - \boldsymbol{v}(t, \boldsymbol{q}) \times \nabla \rho_0(t, \boldsymbol{q})) \right)$$

The red sign selection indicates a change of handedness by changing the sign of one of the imaginary base vectors. Conjugation also causes a switch of handedness. It changes the sign of all three imaginary base vectors.

10.1.1 Continuity Equations

The equation for the conservation of charge:

$$s_0(q) = \nabla_0 \rho_0(q) \mp \langle \boldsymbol{\nabla}, \boldsymbol{\rho}(q) \rangle \tag{1}$$

We can define $\mathfrak{F}(q)$:

$$\mathfrak{F}(q) \stackrel{\text{def}}{=} \nabla \boldsymbol{\rho}(q) \tag{2}$$

$$\mathfrak{F}_0(q) = \nabla_0 \boldsymbol{\rho}_0(q) \mp \langle \boldsymbol{\nabla}, \boldsymbol{\rho}(q) \rangle \tag{3}$$

$$\boldsymbol{\mathfrak{F}}(q) = \nabla \boldsymbol{\rho}_0(q) \pm \nabla_0 \boldsymbol{\rho}(q) \pm \boldsymbol{\nabla} \times \boldsymbol{\rho}(q) = \mathfrak{E}(q) + \mathfrak{B}(q) \tag{4}$$

$$\mathfrak{E}(q) = -\nabla \boldsymbol{\rho}_0(q) \mp \nabla_0 \boldsymbol{\rho}(q) \tag{5}$$

$$\mathfrak{B}(q) = \pm \boldsymbol{\nabla} \times \boldsymbol{\rho}(q) \tag{6}$$

The definition of $\mathfrak{B}(q)$ and $\mathfrak{E}(q)$ have the freedom of the [gauge transform](http://en.wikipedia.org/wiki/Gauge_fixing)[104]

$$\boldsymbol{\rho}(q) \mapsto \boldsymbol{\rho}(q) + \boldsymbol{\nabla}\boldsymbol{\phi}_0 \tag{7}$$

$$\mathfrak{E}(q) \mapsto \mathfrak{E}(q) - \boldsymbol{\nabla}\left(\nabla_0 \boldsymbol{\phi}_0(q)\right) \tag{8}$$

$$\boldsymbol{\nabla}^2 \boldsymbol{\phi}_0 = \nabla_0^2 \boldsymbol{\phi}_0 \tag{9}$$

This translates in the source free case $s_0(q) = 0$ into:

$$\nabla_0 \rho_0(q) = \pm \langle \boldsymbol{\nabla}, \boldsymbol{\rho}(q) \rangle \tag{10}$$

$$\mathfrak{F}_0(q) = \nabla_0 \boldsymbol{\rho}_0(q) \mp \langle \boldsymbol{\nabla}, \boldsymbol{\rho}(q) \rangle = 0 \tag{11}$$

In the source divergence free case $\boldsymbol{\nabla} s_0(q) = 0$ this means:

$$\nabla_0 \boldsymbol{\nabla} \boldsymbol{\rho}_0(q) = \pm \boldsymbol{\nabla} \langle \boldsymbol{\nabla}, \boldsymbol{\rho}(q) \rangle \tag{12}$$

$$\nabla_0 \boldsymbol{\nabla} \boldsymbol{\phi}_0(q) = \pm \boldsymbol{\nabla} \langle \boldsymbol{\nabla}, \boldsymbol{\phi}(q) \rangle \tag{13}$$

$$\boldsymbol{\nabla} \langle \boldsymbol{\nabla}, \boldsymbol{\phi}(q) \rangle = \boldsymbol{\nabla} \times \boldsymbol{\nabla} \times \boldsymbol{\phi}(q) + \boldsymbol{\nabla}^2 \boldsymbol{\phi}(q) \tag{14}$$

Due to the fact that there are other charges present, the divergence of the scalar potential need be in the direction of the current $\boldsymbol{\rho}(q)$, which for a spherical symmetric blur is also in the direction of the vector potential $\boldsymbol{\phi}(q)$. However, a tendency exists to minimize that difference. Thus $\nabla_0 \boldsymbol{\nabla} \boldsymbol{\phi}_0(q)$ is parallel to $\boldsymbol{\phi}(q)$. With other words:

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[104] http://en.wikipedia.org/wiki/Gauge_fixing

$$\boldsymbol{\phi}(q) \times \nabla \langle \nabla, \boldsymbol{\phi}(q) \rangle = 0 \qquad (15)$$

Reckoning the sign selections for the sign $\pm$ of the conjugation and the handedness $\pm$ of the cross product will provide four different sets of equations. This will provide four different Hilbert fields.

10.2 Discrete distribution

If $\rho(q)$ is discrete, such that

$$\rho(q) = \Sigma_i\, q_{E_i} \cdot \delta(q - q_i) \qquad (1)$$

where q_E' is a point charge at location q', then the contribution to the field $\boldsymbol{E}(q)$ that is generated by a point charge at location q_i is given by:

$$d\boldsymbol{E}(q) = q_{E_i} \cdot \frac{q_i - q}{|q_i - q|^3} = -q_{E_i} \cdot \nabla \cdot \frac{1}{|q_i - q|} \qquad (2)$$

10.3 Differential potential equations

The gradient and curl of $\phi(q)$ are related. In configuration space holds:

$$\nabla \phi(q) = \nabla_0 \phi_0(q) \mp \langle \nabla, \boldsymbol{\phi}(q) \rangle \pm \nabla_0 \boldsymbol{\phi}(q) \pm \nabla \phi_0(q) \pm \left(\pm \nabla \times \boldsymbol{\phi}(q) \right) \qquad (1)$$

$$\mathfrak{E}(q) \stackrel{\text{def}}{=} -\nabla \phi_0(q) \qquad (2)$$

$$\mathfrak{B}(q) \stackrel{\text{def}}{=} \nabla \times \phi(q) \qquad (3)$$

$$\mathfrak{F}(q) \stackrel{\text{def}}{=} \nabla \phi(q) = \mathfrak{F}_0(q) \mp \mathfrak{E}(q) \pm \mathfrak{B}(q) \pm \nabla_0 \boldsymbol{\phi}(q) \qquad (4)$$

$$\mathfrak{F}_0(q) = \nabla_0 \phi_0(q) \mp \langle \nabla, \boldsymbol{\phi}(q) \rangle \qquad (5)$$

$$\mathfrak{F}(q) = \mp \mathfrak{E}(q) \pm \mathfrak{B}(q) \pm \nabla_0 \boldsymbol{\phi}(q) \qquad (6)$$

When the field $\phi(q)$ is split into a private field $\phi_p(q)$ and a background field $\phi_b(q)$, then $\boldsymbol{\phi_p}(q)$ corresponds to the private field of the uniform moving item. When this item accelerates, then it goes together with an extra term $\nabla_0 \boldsymbol{\phi_p}(q)$. This is the reason of existence of inertia[105].

$$\langle \nabla, \mathfrak{E}(q) \rangle = -\nabla^2 \phi_0(q) = \rho_0(q) \qquad (7)$$

$$\nabla \times \mathfrak{E}(q) = 0; \text{ Rotation free field} \qquad (8)$$

$$\langle \nabla, \mathfrak{B}(q) \rangle = 0; \text{ Divergence free } B \text{ field} \qquad (9)$$

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[105] Influence; Inertia

$$\nabla \times \mathfrak{B}(q) = \nabla\langle\nabla, \boldsymbol{\phi}(q)\rangle - \nabla^2\boldsymbol{\phi}(q) = \nabla\langle\nabla, \boldsymbol{\phi}(q)\rangle + \boldsymbol{\rho}(q) + \nabla_0{}^2\boldsymbol{\phi}(q) \tag{10}$$

$$\nabla \times \mathfrak{B}(q) = \pm\nabla_0\boldsymbol{V}\phi_0(q) + \boldsymbol{\rho}(q) + \nabla_0{}^2\boldsymbol{\phi}(q) \tag{11}$$

$$= \pm\nabla_0\mathfrak{E}(q) + \boldsymbol{\rho}(q) + \nabla_0{}^2\boldsymbol{\phi}(q)$$

Since $\nabla_0\boldsymbol{\phi}(q)$is supposed to be parallel to $\nabla\phi_0(q)$, it is sensible to define $\boldsymbol{E}(q)$as the total field in longitudinal direction:

$$\boldsymbol{E}(q) = -\nabla\phi_0(q) - \nabla_0\boldsymbol{\phi}(q) = \mathfrak{E}(q) - \nabla_0\boldsymbol{\phi}(q) \tag{12}$$

And

$$\boldsymbol{B}(q) = \mathfrak{B}(q) \tag{13}$$

With this definition:

$$\nabla \times \boldsymbol{E}(q) = -\nabla_0\boldsymbol{B}(q) \tag{14}$$

$$\langle\nabla, \boldsymbol{B}(q)\rangle = 0 \tag{15}$$

$$\nabla \times \boldsymbol{B}(q) = \boldsymbol{\rho}(q) + \nabla_0\boldsymbol{E}(q) \tag{16}$$

10.3.1 In Fourier space

In Fourier space holds:

$$\widetilde{\mathfrak{F}}(p) = p_0\tilde{\phi}_0(p) - \langle\boldsymbol{p}, \tilde{\boldsymbol{\phi}}(p)\rangle \pm p_0\tilde{\boldsymbol{\phi}}(p) \pm \boldsymbol{p}\tilde{\phi}_0(p) \pm \boldsymbol{p} \times \tilde{\boldsymbol{\phi}}(p) \tag{1}$$

$$\widetilde{\mathfrak{F}}(p) = p\tilde{\phi}(p) = \widetilde{\mathfrak{F}}_0(p) \mp \widetilde{\mathfrak{E}}(p) \pm \widetilde{\mathfrak{B}}(p) \pm p_0\tilde{\boldsymbol{\phi}}(p) \tag{2}$$

$$\widetilde{\mathfrak{F}}_0(p) = p_0\tilde{\phi}_0(p) - \langle\boldsymbol{p}, \tilde{\boldsymbol{\phi}}(p)\rangle \tag{3}$$

$$\widetilde{\mathfrak{E}}(p) = -\boldsymbol{p}\tilde{\phi}_0(p) \tag{4}$$

$$\widetilde{\boldsymbol{E}}(p) = -\boldsymbol{p}\tilde{\phi}_0(p) \pm p_0\tilde{\boldsymbol{\phi}}(p)$$

$$\widetilde{\mathfrak{B}}(p) = \boldsymbol{p} \times \tilde{\boldsymbol{\phi}}(p) \tag{5}$$

$$\widetilde{\mathfrak{F}}(p) = \mp\widetilde{\mathfrak{E}}(p) \pm \widetilde{\mathfrak{B}}(p) \pm p_0\tilde{\boldsymbol{\phi}}(p) \tag{6}$$

$$\langle\boldsymbol{p}, \widetilde{\mathfrak{E}}(p)\rangle = -\boldsymbol{p}^2\tilde{\phi}_0(p) = \tilde{\rho}_0(p) \tag{7}$$

$$\boldsymbol{p} \times \widetilde{\mathfrak{E}}(p) = 0; \text{ Rotation free field} \tag{8}$$

$$\langle\boldsymbol{p}, \widetilde{\mathfrak{B}}(p)\rangle = 0; \text{ Divergence free } B \text{ field} \tag{9}$$

$$\mathbf{p} \times \mathfrak{B}(p) = \mathbf{p}\langle \mathbf{p}, \tilde{\boldsymbol{\phi}}(q)\rangle - \mathbf{p}^2\tilde{\boldsymbol{\phi}}(q) = \mathbf{p}\langle \mathbf{p}, \tilde{\boldsymbol{\phi}}(p)\rangle + \tilde{\boldsymbol{\rho}}(p) \tag{10}$$

$$\mathbf{p} \times \tilde{\mathfrak{B}}(p) = \pm p_0 \mathbf{p} \tilde{\phi}_0(p) + \tilde{\boldsymbol{\rho}}(p) = \pm p_0 \tilde{\mathfrak{E}}(p) + \tilde{\boldsymbol{\rho}}(p) \tag{11}$$

If the distribution $\rho(q)$ is differentiable, then the same equations that hold for fields $\phi(q)$ and $\tilde{\phi}(p)$ hold for the non-blurred distributions $\rho(q)$ and $\tilde{\rho}(q)$.

10.4   Maxwell equations

First it must be noted that the above derived field equations hold for general quaternionic fields.

The resemblance with physical fields holds for electromagnetic fields as well as for gravitational fields and for any fields whose blurring function approximates

$$f(q) \approx \frac{1}{|q|}.$$

In Maxwell equations, $E(\mathbf{r})$ is defined as:

$$\boldsymbol{E}(\boldsymbol{r}, t) \equiv -\nabla \phi_0(\boldsymbol{r}, t) - \frac{\partial \boldsymbol{A}(\boldsymbol{r}, t)}{\partial t} = \mathfrak{E}(\boldsymbol{r}, t) - \frac{\partial \boldsymbol{A}(\boldsymbol{r}, t)}{\partial t}$$

Further:

$$\langle \boldsymbol{\nabla}, \boldsymbol{E}(\boldsymbol{r}, t)\rangle = -\nabla^2 \phi_0(\boldsymbol{r}, t) - \frac{\partial \langle \boldsymbol{\nabla}, \boldsymbol{A}(\boldsymbol{r}, t)\rangle}{\partial t}$$

$$= \frac{\rho_0(\boldsymbol{r}, t)}{\varepsilon_0} - \frac{\partial \langle \boldsymbol{\nabla}, \boldsymbol{A}(\boldsymbol{r}, t)\rangle}{\partial t}$$

In Maxwell equations, $B(\mathbf{r})$ is defined as:

$$\boldsymbol{B}(\boldsymbol{r}, t) \equiv \boldsymbol{\nabla} \times \boldsymbol{A}(\boldsymbol{r}, t) = \mathfrak{B}(\boldsymbol{r}, t)$$

Further:

$$\boldsymbol{\nabla} \times \boldsymbol{E}(\boldsymbol{r}, t) = -\frac{\partial \boldsymbol{B}(\boldsymbol{r}, t)}{\partial t}$$

$$\langle \boldsymbol{\nabla}, \boldsymbol{B}(\boldsymbol{r}, t)\rangle = 0$$

$$\nabla \times \boldsymbol{B}(\boldsymbol{r}, t) = \mu_0 \left(\boldsymbol{j} + \varepsilon_0 \frac{\partial \boldsymbol{E}}{\partial t}\right)$$

10.4.1 Differentiable distribution

If the distribution $\rho(q)$ is differentiable, then the same equations that hold for fields $\phi(q)$ and $\tilde{\phi}(p)$ hold for the non-blurred distributions $\rho(q)$ and $\tilde{\rho}(q)$.

Using:

$$\boldsymbol{B} = \nabla \times \boldsymbol{\phi} = \boldsymbol{i}(\nabla_2 \phi_\| - \nabla_\| \phi_2) + \boldsymbol{j}(\nabla_\| \phi_1 - \nabla_1 \phi_\|) + \boldsymbol{k}(\nabla_1 \phi_2 - \nabla_2 \phi_1) \tag{1}$$

gives

$$\nabla_0 \phi_\|(q) = \mp \nabla_\| \phi_0(q) \tag{2}$$

$$\nabla_0 \phi_1(q) = \mp \left(\nabla_2 \phi_\|(q) - \nabla_\| \phi_2(q)\right) \tag{3}$$

$$\nabla_0 \phi_2(q) = \pm \left(\nabla_1 \phi_\|(q) - \nabla_\| \phi_1(q)\right) \tag{4}$$

$$\nabla_0 \phi_0(q) = \langle \nabla, \boldsymbol{\phi}(q) \rangle = \nabla_\| \phi_\|(q) + \nabla_1 \phi_1(q) + \nabla_2 \phi_2(q) \tag{5}$$

And correspondingly in Fourier space

$$p_0 \tilde{\phi}_\|(p) = \mp p_\| \tilde{\phi}_0(p) \tag{6}$$

$$p_0 \tilde{\phi}_1(p) = \pm \left(p_\| \tilde{\phi}_2(p) - p_2 \tilde{\phi}_\|(p)\right) \tag{7}$$

$$p_0 \tilde{\phi}_2(p) = \mp \left(p_\| \tilde{\phi}_1(p) - p_1 \tilde{\phi}_\|(p)\right) \tag{8}$$

$$p_0 \tilde{\phi}_0(p) = \langle \boldsymbol{p}, \tilde{\boldsymbol{\phi}}(p) \rangle = p_\| \tilde{\phi}_\|(p) + p_1 \tilde{\phi}_1(p) + p_2 \tilde{\phi}_2(p) \tag{9}$$

11 Conservation laws

11.1 Flux vector

The longitudinal direction $\boldsymbol{k}$ of $\boldsymbol{E}(q)$ and the direction $\boldsymbol{i}$ of $\boldsymbol{B}(q)$ fix two mutual perpendicular directions. This generates curiosity to the significance of the direction $\boldsymbol{k} \times \boldsymbol{i}$. With other words what happens with $\boldsymbol{E}(q) \times \boldsymbol{B}(q)$.

The **flux vector** $\mathfrak{S}(q)$ is defined as:

$$\mathfrak{S}(q) \stackrel{\text{def}}{=} \boldsymbol{E}(q) \times \boldsymbol{B}(q) \tag{1}$$

11.2 Conservation of energy

$$\langle \mathbf{\nabla}, \mathfrak{S}(q) \rangle = \langle \mathbf{B}(q), \mathbf{\nabla} \times \mathbf{E}(q) \rangle - \langle \mathbf{E}(q), \mathbf{\nabla} \times \mathbf{B}(q) \rangle \tag{1}$$

$$= -\langle \mathbf{B}(q), \nabla_0 \mathbf{B}(q) \rangle - \langle \mathbf{E}(q), \boldsymbol{\phi}(q) \rangle - \langle \mathbf{E}(q), \nabla_0 \mathbf{E}(q) \rangle$$

$$= -\tfrac{1}{2}\nabla_0 (\langle \mathbf{B}(q), \mathbf{B}(q) \rangle + \langle \mathbf{E}(q), \mathbf{E}(q) \rangle) - \langle \mathbf{E}(q), \boldsymbol{\phi}(q) \rangle$$

The **field energy density** is defined as:

$$u_{field}(q) = \tfrac{1}{2}(\langle \mathbf{B}(q), \mathbf{B}(q) \rangle + \langle \mathbf{E}(q), \mathbf{E}(q) \rangle) = u_B(q) + u_E(q) \tag{2}$$

$\mathfrak{S}(q)$ can be interpreted as the **field energy current density**.
The continuity equation for field energy density is given by:

$$\nabla_0 u_{field}(q) + \langle \mathbf{\nabla}, \mathfrak{S}(q) \rangle = -\langle \mathbf{E}(q), \boldsymbol{\phi}(q) \rangle = -\phi_0(q)\langle \mathbf{E}(q), \mathbf{v}(q) \rangle \tag{3}$$

This means that $\langle \mathbf{E}(q), \boldsymbol{\phi}(q) \rangle$ can be interpreted as a source term.
$\phi_0(q)\mathbf{E}(q)$ represents **force** per unit volume.
$\phi_0(q)\langle \mathbf{E}(q), \mathbf{v}(q) \rangle$ represents **work** per unit volume, or, in other words, the power density. It is known as the Lorentz power density and is equivalent to the time rate of change of the mechanical energy density of the charged particles that form the current $\boldsymbol{\phi}(q)$.

$$\nabla_0 u_{field}(q) + \langle \mathbf{\nabla}, \mathfrak{S}(q) \rangle = -\nabla_0 u_{mechanical}(q) \tag{4}$$

$$\nabla_0 u_{mechanical} = \langle \mathbf{E}(q), \boldsymbol{\phi}(q) \rangle = \phi_0(q)\langle \mathbf{E}(q), \mathbf{v}(q) \rangle \tag{5}$$

$$\nabla_0 \left(u_{field}(q) + u_{mechanical}(q) \right) = -\langle \mathbf{\nabla}, \mathfrak{S}(q) \rangle \tag{6}$$

$$\text{Total change within } V = \text{flow into } V + \text{production inside } V \tag{7}$$

$$u(q) = u_{field}(q) + u_{mechanical}(q) = u_B(q) + u_E(q) + u_{mechanical}(q) \tag{8}$$

$$U = U_{field} + U_{mechanical} = U_B + U_E + U_{mechanical} = \int_V u \, dV \tag{9}$$

$$\frac{d}{dt}\int_V u \, dV = \oint_S \langle \hat{\mathbf{n}}, \mathfrak{S} \rangle dS + \int_V s_0 \, dV \tag{10}$$

Here the source s_0 is zero.

How to interpret U_mechanical

$U_{mechanical}$ is the energy of the private field (wave function) of the involved particle(s).

11.4 Conservation of linear momentum

$\mathfrak{S}(q)$ can also be interpreted as the **field linear momentum density**. The time rate change of the field linear momentum density is:

$$\nabla_0 \mathfrak{S}(q) = \boldsymbol{g}_{field}(q) = \nabla_0 \, \boldsymbol{E}(q) \times \boldsymbol{B}(q) + \boldsymbol{E}(q) \times \nabla_0 \boldsymbol{B}(q) \tag{1}$$

$$= \left(\nabla \times \boldsymbol{B}(q) - \boldsymbol{\rho}(q)\right) \times \boldsymbol{B}(q) - \boldsymbol{E}(q) \times \nabla \times \boldsymbol{E}(q) \tag{2}$$

$$\boldsymbol{G}(\boldsymbol{E}) = \boldsymbol{E} \times (\nabla \times \boldsymbol{E}) = \langle \nabla \boldsymbol{E}, \boldsymbol{E} \rangle - \langle \boldsymbol{E}, \boldsymbol{E} \rangle = \tfrac{1}{2} \nabla \langle \boldsymbol{E}, \boldsymbol{E} \rangle - \langle \boldsymbol{E}, \boldsymbol{E} \rangle \tag{3}$$

$$= -\nabla(\mathbf{E}\mathbf{E}) + \tfrac{1}{2}\nabla\langle \boldsymbol{E}, \boldsymbol{E} \rangle + \langle \nabla, \boldsymbol{E} \rangle \boldsymbol{E}$$

$$= -\nabla(\mathbf{E}\mathbf{E} + \tfrac{1}{2}\mathbf{1}_3\langle \boldsymbol{E}, \boldsymbol{E} \rangle) + \langle \nabla, \boldsymbol{E} \rangle \boldsymbol{E}$$

$$\boldsymbol{G}(\boldsymbol{B}) = \boldsymbol{B} \times (\nabla \times \boldsymbol{B}) = -\nabla(\mathbf{B}\mathbf{B} + \tfrac{1}{2}\mathbf{1}_3\langle \boldsymbol{B}, \boldsymbol{B} \rangle) + \langle \nabla, \boldsymbol{B} \rangle \boldsymbol{B} \tag{4}$$

$$\boldsymbol{H}(\boldsymbol{B}) = -\nabla(\mathbf{B}\mathbf{B} + \tfrac{1}{2}\mathbf{1}_3\langle \boldsymbol{B}, \boldsymbol{B} \rangle) \tag{5}$$

$$\nabla_0 \mathfrak{S}(q) = \boldsymbol{G}(\boldsymbol{B}) + \boldsymbol{G}(\boldsymbol{E}) - \boldsymbol{\rho}(q) \times \boldsymbol{B}(q) \tag{6}$$

$$= \boldsymbol{H}(\boldsymbol{E}) + \boldsymbol{H}(\boldsymbol{B}) - \boldsymbol{\rho}(q) \times \boldsymbol{B}(q) + \langle \nabla, \boldsymbol{B} \rangle \boldsymbol{B} + \langle \nabla, \boldsymbol{E} \rangle \boldsymbol{E}$$

$$= \boldsymbol{H}(\boldsymbol{E}) + \boldsymbol{H}(\boldsymbol{B}) - \boldsymbol{\rho}(q) \times \boldsymbol{B}(q) - \rho_0(q) \, \boldsymbol{E}(q)$$

$$= \boldsymbol{H}(\boldsymbol{E}) + \boldsymbol{H}(\boldsymbol{B}) - \boldsymbol{f}(q) = \mathcal{T}(q) - \boldsymbol{f}(q)$$

$\mathcal{T}(q)$ is the linear momentum flux tensor.
The linear momentum of the field contained in volume V surrounded by surface S is:

$$\boldsymbol{P}_{field} = \int\limits_V \boldsymbol{g}_{field} \, dV = \int\limits_V \rho_0 \boldsymbol{\phi} \, dV + \int\limits_V \langle \nabla \boldsymbol{\phi}, \boldsymbol{E} \rangle \, dV + \oint\limits_S \langle \hat{\boldsymbol{n}}, \boldsymbol{E}\boldsymbol{A} \rangle dS \tag{7}$$

$$\boldsymbol{f}(q) = \boldsymbol{\rho}(q) \times \boldsymbol{B}(q) + \rho_0(q) \, \boldsymbol{E}(q) \tag{8}$$

Physically, $\boldsymbol{f}(q)$ is the Lorentz force density. It equals the time rate change of the mechanical linear momentum density $\boldsymbol{g}_{mechanical}$.

$$\boldsymbol{g}_{mechanical}(q) = \rho_{0m}(q)\boldsymbol{v}(q) \tag{9}$$

The force acted upon a single particle that is contained in a volume V is:

$$F = \int_V f \, dV = \int_V (\boldsymbol{\rho} \times \boldsymbol{B} + \rho_0 \, \boldsymbol{E}) \, dV \tag{10}$$

Brought together this gives:

$$\nabla_0 \left(\boldsymbol{g}_{field}(q) + \boldsymbol{g}_{mechanical}(q) \right) = -\langle \boldsymbol{\nabla}, \boldsymbol{\mathcal{T}}(q) \rangle \tag{11}$$

This is the continuity equation for linear momentum.

The component $\mathcal{T}_{ij}$ is the linear momentum in the i-th direction that passes a surface element in the j-th direction per unit time, per unit area.

$$\text{Total change within } V = \text{flow into } V + \text{production inside } V \tag{12}$$

$$\boldsymbol{g}(q) = \boldsymbol{g}_{field}(q) + \boldsymbol{g}_{mechanical}(q) \tag{13}$$

$$\boldsymbol{P} = \boldsymbol{P}_{field} + \boldsymbol{P}_{mechanical} = \int_V \boldsymbol{g} \, dV \tag{14}$$

$$\frac{d}{dt} \int_V \boldsymbol{g} \, dV = \oint_S \langle \hat{\boldsymbol{n}}, \boldsymbol{\mathcal{T}} \rangle dS + \int_V \boldsymbol{s}_g \, dV \tag{15}$$

Here the source $\boldsymbol{s}_g = 0$.

11.5 Conservation of angular momentum

11.5.1 *Field angular momentum*

The angular momentum relates to the linear momentum.

$$\boldsymbol{h}(\boldsymbol{q}_c) = (\boldsymbol{q} - \boldsymbol{q}_c) \times \boldsymbol{g}(q) \tag{1}$$

$$\boldsymbol{h}_{field}(\boldsymbol{q}_c) = (\boldsymbol{q} - \boldsymbol{q}_c) \times \boldsymbol{g}_{field}(q) \tag{2}$$

$$\boldsymbol{h}_{mechanical}(q) = (\boldsymbol{q} - \boldsymbol{q}_c) \times \boldsymbol{g}_{mechanical}(q) \tag{3}$$

$$\boldsymbol{\mathcal{K}}(\boldsymbol{q}_c) = (\boldsymbol{q} - \boldsymbol{q}_c) \times \boldsymbol{\mathcal{T}}(q) \tag{4}$$

This enables the balance equation for angular momentum:

$$\nabla_0 \left(\boldsymbol{h}_{field}(\boldsymbol{q}_c) + \boldsymbol{h}_{mechanical}(\boldsymbol{q}_c) \right) = -\langle \boldsymbol{\nabla}, \boldsymbol{\mathcal{K}}(\boldsymbol{q}_c) \rangle \tag{5}$$

$$\text{Total change within } V = \text{flow into } V + \text{production inside } V$$

$$\tag{6}$$

$$J = J_{field} + J_{mechanical} = \int_V h \, dV$$

$$\frac{d}{dt} \int_V h \, dV = \oint_S \langle \hat{n}, \mathcal{K} \rangle dS + \int_V s_h \, dV \qquad (7)$$

Here the source $s_h = 0$.

For a localized charge density contained within a volume V holds for the mechanical torsion:

$$\tau(q_c) = \int_V (q' - q_c) \times f(q') dV \qquad (8)$$

$$= \int_V (q' - q_c) \times \big(\rho_0(q') E(q') + j(q') \times B(q') \big) dV$$

$$= Q(q - q_c) \times (E(q) + v(q) \times B(q))$$

$$J_{field}(q_c) = J_{field}(0) + q_c \times P(q) \qquad (9)$$

Using

$$\langle \nabla a, b \rangle = n_v \frac{\partial a_\mu}{\partial q_v} b_\mu \qquad (10)$$

$$\langle b, \nabla a \rangle = n_\mu \frac{\partial a_\mu}{\partial q_v} b_\mu \qquad (11)$$

holds

$$J_{field}(0) = \int_V q' \times \mathfrak{S}(q') dV = \int_V q' \times E(q') \times \nabla \times \phi(q') \, dV \qquad (12)$$

$$= \int_V (q' \times \langle (\nabla \phi), E \rangle - \langle q' \times E, (\nabla \phi) \rangle) \, dV$$

$$= \int_V q' \times \langle (\nabla \phi), E \rangle dV$$

$$+ \int_V E \times \phi \, dV - \int_V \langle \nabla, E q' \times \phi \rangle dV + \int_V (q' \times \phi) \langle \nabla, E \rangle dV$$

11.5.2 Spin

Define the non-local spin term, *which does not depend on q'* as:

$$\boldsymbol{\Sigma}_{field} = \int_V \boldsymbol{E}(q) \times \boldsymbol{\phi}(q) dV \tag{13}$$

Notice

$$\boldsymbol{\phi}(q) \times \boldsymbol{\nabla}\phi_0(q) = \phi_0 \boldsymbol{\nabla} \times \boldsymbol{\phi}(q) + \boldsymbol{\nabla} \times \big(\phi_0(q)\boldsymbol{\phi}(q)\big)$$

And

$$\boldsymbol{L}_{field}(\boldsymbol{0}) = \int_V \boldsymbol{q}' \times \langle(\boldsymbol{\nabla}\boldsymbol{\phi}), \boldsymbol{E}\rangle dV + \int_V \boldsymbol{q}' \times \rho_0\boldsymbol{\phi}dV \tag{14}$$

Using Gauss:

$$\int_V \langle\boldsymbol{\nabla}, \boldsymbol{a}\rangle dV = \oint_S \langle\hat{\boldsymbol{n}}, \boldsymbol{a}\rangle dS \tag{15}$$

And

$$\rho_0 = \langle\boldsymbol{\nabla}, \boldsymbol{E}\rangle \tag{16}$$

Leads to:

$$\boldsymbol{J}_{field}(\boldsymbol{0}) = \boldsymbol{\Sigma}_{field} + \boldsymbol{L}_{field}(\boldsymbol{0}) + \oint_S \langle\hat{\boldsymbol{n}}, \boldsymbol{E}\boldsymbol{q}' \times \boldsymbol{\phi}\rangle dS \tag{17}$$

11.5.3 Spin discussion

The spin term is defined by:

$$\boldsymbol{\Sigma}_{field} = \int_V \boldsymbol{E}(q) \times \boldsymbol{\phi}(q) dV \tag{1}$$

In free space the charge density ρ_0 vanishes and the scalar potential ϕ_0 shows no variance. Only the vector potential $\boldsymbol{\phi}$ may vary with q_0. Thus:

$$\boldsymbol{E} = \boldsymbol{\nabla}\phi_0 - \boldsymbol{\nabla}_0\boldsymbol{\phi} \approx -\boldsymbol{\nabla}_0\boldsymbol{\phi} \tag{2}$$

$$\boldsymbol{\Sigma}_{field} \approx \int_V (\boldsymbol{\nabla}_0\boldsymbol{\phi}(q)) \times \boldsymbol{\phi}(q) dV \tag{3}$$

Depending on the selected field $\boldsymbol{\Sigma}_{field}$ has two versions that differ in their sign. These versions can be combined in a single operator:

$$\boldsymbol{\Sigma}_{field} = \begin{bmatrix} \boldsymbol{\Sigma}^+_{field} \\ \boldsymbol{\Sigma}^-_{field} \end{bmatrix} \tag{4}$$

If $\frac{\boldsymbol{\phi}(q)}{|\boldsymbol{\phi}(q)|}$ can be interpreted as tantrix (q_0)) and $\frac{\nabla_0\boldsymbol{\phi}(q)}{|\nabla_0\boldsymbol{\phi}(q)|}$ can be interpreted as the principle normal $\boldsymbol{N}(q_0)$, then $\frac{(\nabla_0\boldsymbol{\phi}(q))\times\boldsymbol{\phi}(q)}{|(\nabla_0\boldsymbol{\phi}(q))\times\boldsymbol{\phi}(q)|}$ can be interpreted as the binormal $\boldsymbol{B}(q_0)$.

From these quantities the [curvature and the torsion](#)[106] can be derived.

$$\begin{bmatrix} \dot{\boldsymbol{T}}(t) \\ \dot{\boldsymbol{N}}(t) \\ \dot{\boldsymbol{B}}(t) \end{bmatrix} = \begin{bmatrix} 0 & \kappa(t) & 0 \\ -\kappa(t) & 0 & \tau(t) \\ 0 & -\tau(t) & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{T}(t) \\ \boldsymbol{N}(t) \\ \boldsymbol{B}(t) \end{bmatrix} \tag{5}$$

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[106]Path characteristics

12 The universe of items

All particles have properties. Some of these properties expose as sources of corresponding fields. Via superposition these fields interact. Long range fields such as the gravitation field have universe wide effects.

12.1 Inertia

The influence of items in universe may decrease with distance according to some function $f(r)$ of the distance r[107]. However the number of contributing items increases with the distance. Depending on function $f(r)$ the most probable result is that the strongest influence comes from the cooperative activity of the most distant items. Due to the enormous number of items in the universe, any variation of the influences of the distant items averages away. This also holds for the density distribution of the items. So there exists a fairly uniform background influence caused by the universe of items. What will happen, can be deduced from an equivalent of [Denis Sciama's analysis](#)[108][109][110]. We will take his analysis as a guide. Sciama's analysis uses a different setting: the (observed) 3D space and coordinate time and Sciama applies Maxwell field theory. We use the coordinate space defined by an appropriate coordinate operator that resides in the Gelfand triple of the separable Hilbert space and the progression parameter t that relates to the progression step counter as our setting. A location in this coordinate space represents a location on the unit sphere of Gelfand triple. This last location is taken by the eigenvector that corresponds to the first location.

As stated before, the unit sphere of Gelfand triple is an affine space. This means that we must treat position as relative data. With other words, the eigenspace of the coordinate operator has no absolute origin. Instead of Sciama's usage of Maxwell fields we will use quaternionic field theory that is applied to quaternionic probability amplitude distributions (QPAD's).

We may specify a QPAD for usage in a continuity equation. In that case we specify in fact the combination of a charge density distribution and a current density distribution. As long as the charges and the currents stay static, the QPAD is a static object.

In the continuity equation we consider the influence of the QPAD on the whole universe. Here we consider the influence of the universe on a local charge or current. For this purpose we can use similar QPAD's and volume integrals!

At large distances, the density ρ of the contributing items can be considered to be uniformly distributed. Also any variance in strength other than the dependence on r becomes negligible because the differences between functions $\{f(r)\}$ average away. We take the average of the strength of $\{f(r)\}$ as the significant parameter. We combine it with ρ. Therefore the average of ρ can be taken out of the integral.

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[107] http://en.wikipedia.org/wiki/Bertrand's_theorem and Role of the particle locator operator

[108] http://arxiv.org/abs/physics/0609026v4.pdf

[109] http://www.adsabs.harvard.edu/abs/1953MNRAS.113...34S

[110]http://rmp.aps.org/abstract/RMP/v36/i1/p463_1

The total potential Φ at the location of the influenced subject is[111]

$$\Phi = -\int_V \frac{\rho}{r} dV = -\rho \int_V \frac{dV}{r} = 2\pi R^2 \rho \tag{1}$$

What we have here is the reverse of the definition of the potential that goes together with a charge distribution.

The integral is taken over the coordinate space volume V. Indirectly, the integral is taken over the unit sphere of the Gelfand triple. This is an affine space. The parameter r is the length of the vector from the actor to the location of the subject. The considered subject is located somewhere in the affine coordinate space. All other subjects have positions relative to that considered subject. Thus, apart from its dependence on the average value of ρ, Φ is a huge constant. Sciama relates Φ to the gravitational constant G.

$$G = {-c^2}/{\Phi}$$

As a consequence we can consider the universe as a very large rigid body. If nothing else happens then all influences compensate each other.

In contrast to Sciama, we use imaginary quaternions rather than 3D vectors. This also avoids the distracting factor i.

If the considered subject moves relative to the universe with a uniform speed $\mathbf{v}$, then a vector potential $\mathbf{A}$ is generated.

$$\mathbf{A} = -\int_V \frac{\mathbf{v}\,\rho}{c\,r} dV \tag{2}$$

Both ρ and $\mathbf{v}$ are independent of r. The product $\mathbf{v}\,\rho$ represents a current. Together with the constant c they can be taken out of the integral. Thus

$$\mathbf{A} = \Phi \cdot \mathbf{v}/c \tag{3}$$

The notions of charge and current correspond to equivalent notions in Noether's theorem[112]. Here we talk about inertia. Thus charge may symbolize mass. Or even better; it symbolizes the coupling factor that plays the role of mass.

The progression parameter t plays the role of "time". Be aware, in our setting it is the progression parameter, which is not the usual notion of time.

According to the Helmholtz theorem the field that is derived from the above potential can be split into a divergence free part and a rotation free part. The Helmholtz decomposition theorem

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[111] http://en.wikipedia.org/wiki/Newtonian_potential
[112] http://en.wikipedia.org/wiki/Noether%27s_theorem

only concerns the static versions of the derived field. It is related to the fact that the Fourier transform of a vector field can be split in a longitudinal and a transversal version. A corresponding split of the multi-dimensional Dirac delta function in a longitudinal and a transversal version exists as well.

According to Maxwell field theory as well as according to quaternionic field theory, a variation of $\mathbf{v}$ goes together with a variation of $\mathbf{A}$. On its turn this goes together with a non-zero field $\dot{\mathbf{A}}(\mathbf{r}, t)$ which is a **dynamical** part of the QPAD. Thus, with varying $\mathbf{v}$ the QPAD is no longer static.[113]

Sciama uses a Maxwell equation to explain the relation between $\partial \mathbf{v}/\partial \mathrm{t}$ and $\dot{\mathbf{A}}(\mathbf{r}, t)$. Our setting differs, but the quaternionic field theory delivers the same results.

$$E(\mathbf{r}, t) = -\boldsymbol{\nabla}\Phi(\mathbf{r}, t) - \frac{1}{c} \cdot \dot{\mathbf{A}}(\mathbf{r}, t) \tag{4}$$

$$\tilde{E}(\mathbf{k}, \omega) = -\mathbf{k} \cdot \tilde{\Phi}(\mathbf{k}, \omega) - \frac{1}{c} \cdot \omega \tilde{\mathbf{A}}(\mathbf{k}, \omega) \tag{5}$$

If we exclude the first term because it is negligible small, we get:

$$E(\mathbf{r}, t) = -\frac{\Phi}{c^2} \cdot \frac{\partial \mathbf{v}}{\partial \mathrm{t}} = G \cdot \frac{\partial \mathbf{v}}{\partial \mathrm{t}} \tag{6}$$

Remark: As soon as we turn to the dynamic version (4) an extra component $\dot{\mathbf{A}}$ of field E appears that corresponds to acceleration $\partial \mathbf{v}/\partial \mathrm{t}$.[114]

As already claimed, in our setting the component $\boldsymbol{\nabla}\Phi$ of the field E is negligible. With respect to this component the items compensate each other's influence. This means that if the influenced subject moves with uniform speed $\mathbf{v}$, then $E \approx 0$. However, a vector potential A is present due to the movement of the considered item. Like ρ and $\rho\mathbf{v}$, Φ and A together form a QPAD. Any acceleration of the considered item goes together with an extra non-zero E field. In this way the universe of items causes inertia in the form of a force that acts upon the accelerating item's charge. The item is the carrier of the charge ρ.

The situation in curved space differs. When the path of the item coincides with a **geodesic**, then it can be travelled free of extra generated fields. Thus, a uniform movement along the geodesic does not on itself generate a reaction of the universe of items. Any alteration of that uniform movement will go together with the existence of an extra field. The physical name for this reaction is **action**. It usually gets the symbol **S**.

On the other hand, as we see from inertia, any field change goes together with a corresponding acceleration. Uniform movements do cause displacement of charges. In a curved environment it

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[113] See Differentiation

[114] See http://www.plasma.uu.se/CED/Book; formula 3.25 or Appendix; Maxwell equations

changes the configuration of the QPAD. Thus, in that case, even an originally static QPAD may be affected.

We may reverse the conclusion of the analysis:

An extra field component goes together with an acceleration of the local item.

The acceleration can be seen as the consequence of a local curvature and vice versa. Thus, the extra field goes together with a local curvature.

It must be noticed that the original analysis of Sciama uses observable position space rather coordinate space and it uses a different notion of time. However, the general conclusion stays the same. Sciama's analysis is criticized because it uses infinite speed of information transfer. Since we do not work in observable position space, we do not encounter coordinate time. So for the setting of our analysis, this criticism is misplaced. Most part of the story plays in a stationary QPAD condition. As long as the movement stays uniform, the QPAD is static. Any acceleration deviates from this stationary condition. This deviation goes together with an extra field component and it goes together with a local curvature.

Coordinate time[115] relates to observations of position. It is a local player in the game, where the progression parameter is a global player.

The situation with electromagnetic fields is different, because with this field positive and negative charges compensate each other's long range influence. For that reason there exists no electromagnetic background influence. The masses of the gravitational and inertial fields only compensate each other's long range influences through geometrical circumstances. Still in combination, they create gigantic potentials.

The particles outside the information horizons also contribute to the inertia.

Thus when through uniform movement the local field configuration changes, then that change goes together with an acceleration of the local item.

12.2 Nearby items

Items that are located nearby have a different effect. In general their influence will not have its strength equal to the average strength. Further these items are not uniformly distributed. Still at macroscopic distances their influence depends on inter-distance as $f = -k/r$. As a consequence their influences form a landscape of which the effects will become sensible in the action of the fields that surround the considered item. For observers, this landscape will form a curved action space. The considered item will try to follow a geodesic through that curved space.

12.3 Rotational inertia

Besides linear inertia there exists rotational inertia. In a non-rotating universe hold near the origin $\mathbf{A} = 0$ and $\Phi = -c^2/G$. We choose units such that c=G=1. In a universe rotating slowly with angular speed ω hold

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[115] Dynamics; Relativity

$$A_x = \omega \cdot y \tag{1}$$

$$A_y = -\omega \cdot x \tag{2}$$

$$A_z = 0 \tag{3}$$

$$\Phi = -\sqrt{1 + (\omega \cdot r)^2} \tag{4}$$

A constant angular movement meets the fields that correspond to a centripetal force.

The field E has the form

$$E = \frac{\omega^2 r}{\sqrt{1 + \omega^2 r^2}} \tag{5}$$

An added uniform speed v meets the fields corresponding to a Coriolis force.

$$H = \nabla \times A = 2 \cdot \omega \tag{6}$$

$$v \times H = 2 \cdot v \times \omega \tag{7}$$

The forces are usually considered as *fictitious* but they are actually caused by inertia. Sciama treats them in section 5 of his paper. Like fields of linear inertia these rotation related fields correspond to actions of the manipulator.

12.4 Computation of the background QPAD

The same line of thinking that lead to the formula for the local potential in section 12.1-(1) can be applied to the computation of the QPAD that represents the local background field.

The ensemble $\{ \psi_i(r, q_i) \}$ is distributed randomly over the center points $\{q_i\}$ in an affine parameter space. At a given point P in this space the superposition of all $\{ \psi_i(r, P) \}$ will be constructed.
This superposition will be renormalized and then indicated by $\Phi(r, P)$.
Thus,

$$\int_V |\Phi(r, P)|^2 \, dV = 1 \tag{1}$$

In this superposition the largest contribution comes from the $\psi_i(r, q_i)$ for which the q_i is farthest from P. Further the directions of the imaginary part of $\Phi(r, P)$.are reversed with respect to the directions in the $\psi_i(r, q_i)$.
Especially at long distances, all differences are smoothed away via an averaging process.

The result is that for the average QPAD $\Psi(r, P)$:

$$\Phi(r, P) = \Psi^*(r, P) \tag{2}$$

We will interpret $\Phi(r, 0)$ as the background QPAD.

Since we are talking about quaternionic distributions it is possible that every sign flavor has its own background QPAD.

13 Path characteristics

The Frenet-Serret frame is devised for describing curved paths of particles

Let $\{\boldsymbol{\alpha}_{qt}\}_t = \boldsymbol{\alpha}(q,t)$ describe a curved path consisting of infinitesimal steps through a landscape $\{\boldsymbol{\alpha}_q\}_q = \boldsymbol{\alpha}(q)$ of imaginary quaternions $\boldsymbol{\alpha}_{qt}$, such that $||\dot{\boldsymbol{\alpha}}(q(t))|| = 1$ for all t.

The 3D Frenet-Serret frame for the above path is given by:

$$\boldsymbol{T}(q(t)) := \frac{\partial \boldsymbol{\alpha}(q(t))}{\partial t} = \boldsymbol{T}(t) = \dot{\boldsymbol{\alpha}}(t) \tag{1}$$

$$\kappa(t) := ||\dot{\boldsymbol{T}}(t)|| \tag{2}$$

$$\kappa(t) \cdot \boldsymbol{N}(t) := \dot{\boldsymbol{T}}(t) \tag{3}$$

$$\boldsymbol{B}(t) := \boldsymbol{T}(t) \times \boldsymbol{N}(t) \tag{4}$$

$$||\boldsymbol{T}(t)|| = ||\boldsymbol{N}(t)|| = ||\boldsymbol{B}(t)|| = 1 \tag{5}$$

$\boldsymbol{T}(t)$ is the **tantrix** of curve $\boldsymbol{\alpha}(q(t))$ at instance t.
$\boldsymbol{N}(t)$ is the **principal normal** of curve $\boldsymbol{\alpha}(q(t))$ at instance t. It is only defined when $\kappa(t) \neq 0$.
$\boldsymbol{B}(t)$ is the **binormal** of curve $\boldsymbol{\alpha}(q(t))$ at instance t.
$\boldsymbol{T}(t)$, $\boldsymbol{N}(t)$ and $\boldsymbol{B}(t)$ are imaginary quaternions.
$\kappa(t)$ is the curvature of curve at $\boldsymbol{\alpha}(q(t))$ at instance t.
$r(t) = 1/\kappa(t)$ is the radius of curvature at instance t.
$\tau(t)$ is the torsion of curve $\boldsymbol{\alpha}(q(t))$ at instance t.

$$\begin{bmatrix} \dot{\boldsymbol{T}}(t) \\ \dot{\boldsymbol{N}}(t) \\ \dot{\boldsymbol{B}}(t) \end{bmatrix} = \begin{bmatrix} 0 & \kappa(t) & 0 \\ -\kappa(t) & 0 & \tau(t) \\ 0 & -\tau(t) & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{T}(t) \\ \boldsymbol{N}(t) \\ \boldsymbol{B}(t) \end{bmatrix} \tag{6}$$

The Frenet-Serret curves have particular characteristics. The path may be curved and curled. The path is completely determined by its tantrix, curvature and torsion given by functions of t. Each coordinate of the quaternionic function $\boldsymbol{\alpha}(q(t))$ has its own set of characteristics. This means that for a given quaternionic function these characteristics are quaternions rather than real numbers and they are all functions of parameter t.

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13.1 Path equations

The path equations are given by

$$\dot{\boldsymbol{T}}(t) = \kappa(t) \cdot \boldsymbol{N}(t) \tag{1}$$

$$\dot{\boldsymbol{N}}(t) = -\kappa(t) \cdot \boldsymbol{T}(t) + \tau(t) \cdot \boldsymbol{B}(t) = -\kappa(t) \cdot \boldsymbol{T}(t) + \tau(t) \cdot \boldsymbol{T}(t) \times \boldsymbol{N}(t) \tag{2}$$

$$\dot{\boldsymbol{B}}(t) = -\tau(t) \cdot \boldsymbol{N}(t) = \boldsymbol{T}(t) \times \dot{\boldsymbol{N}}(t) + \dot{\boldsymbol{T}}(t) \times \boldsymbol{N}(t) \tag{3}$$

$$= \tau(t) \cdot \boldsymbol{T}(t) \times \boldsymbol{B}(t)$$

13.2 Curve length

The curve length $l(a, b)$ is defined by:

$$l(a, b) = \int_{x=a}^{x=b} |\dot{\boldsymbol{\alpha}}(q(\mathrm{x}))| \, dx \tag{1}$$

The integration over the square of the modulus delivers the **action** S of the curve.

$$S(a, b) = \int_{x=a}^{x=b} |\dot{\boldsymbol{\alpha}}(q(\mathrm{x}))|^2 \, dx \tag{2}$$

13.3 Reparameterization

The path characteristics $\kappa(t)$ and $\tau(t)$ together with the curve length and the curve action are independent of any reparameterization $s(t)$ of the progression parameter t.

A natural reparameterization is given by $s(t) = l(t_0, t)$.

This turns the curve $\boldsymbol{\alpha}(q(\mathrm{t}))$ into a **natural** curve $\boldsymbol{\gamma}(q(s))$:

$$\boldsymbol{\gamma}(q(\mathrm{s})) = \boldsymbol{\alpha}(q(\mathrm{t})) \tag{1}$$

Curves on a surface which minimize length between the endpoints are called geodesics.

The natural curve corresponds to a geodesic[116].

The consequence is that in three-dimensional space the corresponding movement obeys the geodesic equation[117]. The Lagrangian is an equivalent of this equation.

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[116] http://en.wikipedia.org/wiki/Geodesic

[117] Equations of motion; Lagrangian

14 Metric tensor field

The metric tensor is an example of a tensor field. This means that relative to a locally non-affected coordinate system[118] on the manifold, a metric tensor takes on the form of a symmetric matrix whose entries transform covariantly under changes to the coordinate system. Thus a metric tensor is a covariant symmetric tensor[119]. From the coordinate-independent point of view, a metric tensor is defined to be a non-degenerate symmetric bilinear form[120] on each tangent space that varies smoothly from point to point.

14.1 Curved path

In a [Riemannian manifold](http://en.wikipedia.org/wiki/Riemannian_manifold)[121] M with [metric tensor](http://en.wikipedia.org/wiki/Metric_tensor)[122] g, the length of a continuously differentiable curve $\gamma: [a, b] \to M$ is defined by

$$L(\gamma) = \int_a^b \sqrt{g_{\gamma(t)(\dot{\gamma}(t), \dot{\gamma}(t))}} \, dt \tag{1}$$

The distance $d(p, q)$ between two points p and q of M is defined as the [infimum](http://en.wikipedia.org/wiki/Infimum)[123] of the length taken over all continuous, piecewise continuously differentiable curves $\gamma: [a, b] \to M$ such that $\gamma(a) = p$ and $\gamma(b) = q$. With this definition of distance, geodesics in a Riemannian manifold are then the locally distance-minimizing paths, in the above sense.

The minimizing curves of L in a small enough [open set](http://en.wikipedia.org/wiki/Open_set)[124] of M can be obtained by techniques of [calculus of variations](http://en.wikipedia.org/wiki/Calculus_of_variations)[125]. Typically, one introduces the following [action](http://en.wikipedia.org/wiki/Action_(physics))[126] or [energy functional](http://en.wikipedia.org/wiki/Energy_functional)[127]

$$E(\gamma) = \tfrac{1}{2} \int_a^b g_{\gamma(t)(\dot{\gamma}(t), \dot{\gamma}(t))} \, dt \tag{2}$$

It is then enough to minimize the functional E, owing to the [Cauchy–Schwarz inequality](http://en.wikipedia.org/wiki/Cauchy%E2%80%93Schwarz_inequality)[128]

$$L(\gamma)^2 \leq 2(b - a) \, E(\gamma) \tag{3}$$

with equality if and only if $|d\gamma/dt|$ is constant.

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[118] http://en.wikipedia.org/wiki/Local_coordinate_system
[119] http://en.wikipedia.org/wiki/Symmetric_tensor
[120] http://en.wikipedia.org/wiki/Symmetric_bilinear_form
[121] http://en.wikipedia.org/wiki/Riemannian_manifold
[122] http://en.wikipedia.org/wiki/Metric_tensor
[123] http://en.wikipedia.org/wiki/Infimum
[124] http://en.wikipedia.org/wiki/Open_set
[125] http://en.wikipedia.org/wiki/Calculus_of_variations
[126] http://en.wikipedia.org/wiki/Action_(physics)
[127] http://en.wikipedia.org/wiki/Energy_functional
[128] http://en.wikipedia.org/wiki/Cauchy%E2%80%93Schwarz_inequality

The [Euler–Lagrange](#)[129] equations of motion for the functional E are then given in local coordinates by

$$\frac{d^2x^\lambda}{dt^2} + \Gamma^\lambda_{\mu\nu} \cdot \frac{dx^\mu}{dt} \cdot \frac{dx^\nu}{dt} = 0 \tag{4}$$

where $\Gamma^\lambda_{\mu\nu}$ are the [Christoffel symbols](#)[130] of the metric. This is the **geodesic equation**.

14.2 Calculus of variations

Techniques of the classical [calculus of variations](#)[131] can be applied to examine the energy functional E. The [first variation](#)[132] of energy is defined in local coordinates by

$$\delta E(\gamma)(\varphi) = \frac{\partial}{\partial t}\bigg|_{t=0} E(\gamma + t\,\varphi) \tag{1}$$

The [critical points](#)[133] of the first variation are precisely the geodesics. The second variation is defined by

$$\delta^2 E(\gamma)(\varphi,\psi) = \frac{\partial^2}{\partial t^2}\bigg|_{t=0} E(\gamma + t\,\varphi + s\psi) \tag{2}$$

In an appropriate sense, zeros of the second variation along a geodesic γ arise along [Jacobi fields](#)[134]. Jacobi fields are thus regarded as variations through geodesics.

By applying variational techniques from [classical mechanics](#)[135], one can also regard [geodesics as Hamiltonian flows](#)[136]. They are solutions of the associated [Hamilton–Jacobi equations](#)[137], with (pseudo-)Riemannian metric taken as [Hamiltonian](#)[138].

14.3 Affine geometry

A **geodesic** on a smooth manifold M with an [affine connection](#)[139] ∇ is defined as a curve $\boldsymbol{\gamma}(t)$ such that [parallel transport](#)[140] along the curve preserves the tangent vector to the curve, so

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[129] Appendix; Derivation of the one dimensional Euler Langrange equation
[130] Equations of motion; Path through field; Christoffel symbols
[131] http://en.wikipedia.org/wiki/Calculus_of_variations
[132] http://en.wikipedia.org/wiki/First_variation
[133] http://en.wikipedia.org/wiki/Critical_point_(mathematics)
[134] http://en.wikipedia.org/wiki/Jacobi_field
[135] http://en.wikipedia.org/wiki/Classical_mechanics
[136] http://en.wikipedia.org/wiki/Geodesics_as_Hamiltonian_flows
[137] http://en.wikipedia.org/wiki/Hamilton%E2%80%93Jacobi_equation
[138] http://en.wikipedia.org/wiki/Hamiltonian_mechanics
[139] http://en.wikipedia.org/wiki/Affine_connection
[140] http://en.wikipedia.org/wiki/Parallel_transport

$$\nabla_{\dot{\gamma}} \dot{\gamma}(t) = 0 \tag{1}$$

at each point along the curve, where $\dot{\gamma}$ is the derivative with respect to t. More precisely, in order to define the covariant derivative of $\dot{\gamma}$ it is necessary first to extend $\dot{\gamma}$ to a continuously differentiable imaginary Hilbert field in an open set[141]. However, the resulting value of the equation is independent of the choice of extension.

Using local coordinates[142] on M, we can write the **geodesic equation** (using the summation convention[143]) as

$$\frac{d^2 x^\lambda}{dt^2} + \Gamma^\lambda_{\mu\nu} \cdot \frac{dx^\mu}{dt} \cdot \frac{dx^\nu}{dt} = 0 \tag{2}$$

where $x^\mu(t)$ are the coordinates of the curve $\gamma(t)$ and $\Gamma^\lambda_{\mu\nu}$ are the Christoffel symbols[144] of the connection ∇. This is just an ordinary differential equation for the coordinates. It has a unique solution, given an initial position and an initial velocity.

From the point of view of classical mechanics, geodesics can be thought of as trajectories of free particles in a manifold. Indeed, the equation $\nabla_{\dot{\gamma}} \dot{\gamma}(t) = 0$ means that the acceleration of the curve has no components in the direction of the surface (and therefore it is perpendicular to the tangent plane of the surface at each point of the curve). So, the motion is completely determined by the bending of the surface. This is also the idea of the general relativity where particles move on geodesics and the bending is caused by the gravity.

14.4 Christoffel symbols

If x^i, $i = 1,2,...,n$, is a local coordinate system on a manifold M, then the tangent vectors

$$e_\mu = \frac{\partial}{\partial x_\mu}, \qquad \mu = 1, 2, \ldots, n \tag{1}$$

define a basis of the tangent space of M at each point. The Christoffel symbols $\Gamma^\lambda_{\mu\nu}$ are defined as the unique coefficients such that the equation

$$\nabla_\mu e_\nu = \Gamma^\lambda_{\mu\nu} \cdot e_\lambda \tag{2}$$

holds, where ∇_μ is the Levi-Civita connection[145] on M taken in the coordinate direction e_μ.

The Christoffel symbols can be derived from the vanishing of the covariant derivative of the metric tensor g_{ik}:

$$\tag{3}$$

[141] http://en.wikipedia.org/wiki/Open_set
[142] http://en.wikipedia.org/wiki/Local_coordinates
[143] http://en.wikipedia.org/wiki/Summation_convention
[144] http://en.wikipedia.org/wiki/Christoffel_symbol
[145] http://en.wikipedia.org/wiki/Levi-Civita_connection

$$0 \ = \ \nabla_\lambda \, g_{\mu\nu} \ = \frac{\partial g_{\mu\nu}}{\partial x_\lambda} - g_{\eta\mu} \cdot \Gamma^\eta_{\mu\lambda} - g_{\mu\eta} \cdot \Gamma^\eta_{\nu\lambda}$$

By permuting the indices, and re-summing, one can solve explicitly for the Christoffel symbols as a function of the metric tensor:

$$\Gamma^\mu_{\nu\lambda} \ = \ \tfrac{1}{2} \cdot g^{\mu\nu} \cdot (\frac{\partial g_{\eta\nu}}{\partial x^\lambda} + \frac{\partial g_{\eta\lambda}}{\partial x^\nu} - \frac{\partial g_{\nu\lambda}}{\partial x^\eta}) \tag{4}$$

where the matrix $(g^{\mu\nu})$ is an inverse of the matrix $(g_{\mu\nu})$, defined as (using the Kronecker delta, and Einstein notation for summation)

$$g^{\lambda\mu} \cdot g_{\mu\nu} = \delta^\lambda_\nu \tag{5}$$

Although the Christoffel symbols are written in the same notation as tensors with index notation, they are **not** tensors, since they do not transform like tensors under a change of coordinates.

Under a change of variable from $(x^1, \ldots, x^n)$ to $(y^1, \ldots, y^n)$, vectors transform as

$$\frac{\partial}{\partial y^i} \ = \ \frac{\partial x^k}{\partial y^i} \cdot \frac{\partial}{\partial x^k} \tag{6}$$

and so

$$\Gamma^k_{ij} \ = \ \frac{\partial x^p}{\partial y^i} \cdot \frac{\partial x^q}{\partial y^j} \cdot \Gamma^r_{pq} \cdot \frac{\partial y^k}{\partial x^r} + \frac{\partial y^k}{\partial x^m} \cdot \frac{\partial^2 x^m}{\partial y^i \partial y^j} \tag{7}$$

where the underline denotes the Christoffel symbols in the y coordinate frame. Note that the Christoffel symbol does **not** transform as a tensor, but rather as an object in the jet bundle.

At each point, there exist coordinate systems in which the Christoffel symbols vanish at the point. These are called (geodesic) normal coordinates, and are often used in Riemannian geometry.

The Christoffel symbols are most typically defined in a coordinate basis, which is the convention followed here. However, the Christoffel symbols can also be defined in an arbitrary basis of tangent vectors e_μ by

$$\nabla_{e_\mu} e_\nu \ = \ \Gamma^\lambda_{\mu\nu} \cdot e_\lambda \tag{8}$$

14.5   Local metric equation

The local metric equation relates the local value of the metric tensor field to the influence of the properties of the local particles on the local curvature.

In order to do this it requires a non-affected coordinate system and a way to qualify the influence that the local value of the particle properties have on the resulting curved coordinate system.

For example the Kerr Newman metric equation uses the per category summed property values of the local coupling factors, the electric charges of the local particles and the angular momenta of the local particles in order to relate these to the local curvature[146].

14.5.1 Kerr-Newman metric equation

The Kerr–Newman metric equation describes the geometry of spacetime in the vicinity of a rotating mass M with charge Q. The formula for this metric depends upon what coordinates or coordinate conditions are selected.

It uses three local properties. These properties are:

- The coupling factor m
- The electric charge Q
- The angular momentum J

The angular momentum J includes the spin s.

In most cases, the simplest interpretation of the Kerr-Newman metric can be taken on the surface of a sphere that has a selected radius r. This metric uses the sum of a category of properties that are collected within the observed sphere. However, the summation produces different centers of activity for different property categories. Thus, these centers need not be at the same location. However, for large enough selected radius r and applied to black holes or single particles, these centers coincide.

The formula uses three characteristic radii, whose prominence usually differs with the content of the investigated sphere.

The metric uses a non-curved coordinate system to start with. Several coordinate systems can be used. The most common coordinate systems for a non-curved three dimensional space are:

- Cartesian coordinates
- Spherical coordinates

Alternatives for spherical coordinates are:

- Schwarzschild coordinates[147]
- Kruskal-Szekeres coordinates[148]
- Lemaitre coordinates[149]
- Eddington–Finkelstein coordinates[150]

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[146] See next part.
[147] http://en.wikipedia.org/wiki/Schwarzschild_coordinates
[148] http://en.wikipedia.org/wiki/Kruskal-Szekeres_coordinates
[149] http://en.wikipedia.org/wiki/Lemaitre_coordinates
[150] http://en.wikipedia.org/wiki/Eddington%E2%80%93Finkelstein_coordinates

The advantage of the alternative coordinates is that they avoid unnecessary singularities.

14.5.1.1 Spherical coordinates

The line element $d\tau$ in spherical coordinates is given by:

$$c^2\,d\tau^2 = -\left(\frac{dr^2}{\Delta} + d\theta^2\right)\rho^2 + (c\,dt - \alpha\,sin^2(\theta)\,d\phi)^2\,\frac{\Delta}{\rho^2} \tag{1}$$

$$-\left((r^2 + \alpha^2)\,d\phi - \alpha\,c\,dt\right)^2\,\frac{sin^2(\theta)}{\rho^2}$$

where the coordinates r, θ and ϕ are the parameters of the standard spherical coordinate system. The length-scales α, ρ and Δ have been introduced for brevity.

$$\alpha = \frac{J}{M\,c} \tag{2}$$

$$\rho^2 = r^2 + \alpha^2\,cos^2(\theta) \tag{3}$$

$$\Delta = r^2 - r_s\,r + \alpha^2 + r_Q^2 \tag{4}$$

r_s is the Schwarzschild radius[151] (in meters) of the massive body, which is related to its mass M by

$$r_s = \frac{2GM}{c^2} \tag{5}$$

where G is the gravitational constant[152]. In case of a single encapsulated elementary particle, M stands for the coupling constant m.

Compare this with the Planck length, $l_{Pl} = \sqrt{\hbar G/c^3}$

The Schwarzschild radius is radius of a spherical geo-cavity with mass M. The escape speed from the surface of this geo-cavity equals the speed of light. Once a stellar remnant collapses within this radius, light cannot escape and the object is no longer visible. It is a characteristic radius associated with every quantity of mass.

r_Q is a length-scale corresponding to the electric charge Q of the mass

$$r_Q^2 = \frac{Q^2 G}{4\pi\varepsilon_0 c^4} \tag{6}$$

where $\frac{1}{4\pi\varepsilon_0}$ is Coulomb's force constant[153].

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[151] http://en.wikipedia.org/wiki/Schwarzschild_radius
[152] http://en.wikipedia.org/wiki/Gravitational_constant
[153] http://en.wikipedia.org/wiki/Coulomb%27s_law

Next for simplicity we use the dimension adapted parameter m.

The radius where the ergo region[154] of a black hole starts can be specified by:

$$r = m + \sqrt{m^2 - r_Q^2 - \alpha^2 \cos^2(\theta)} \qquad (7)$$

And the radius of the horizon by

$$r = m + \sqrt{m^2 - r_Q^2 - \alpha^2} \qquad 8)$$

where

$$\alpha = S/mc \qquad 9)$$

is the dimensionless spin parameter, q is the electric charge and m is the mass of the particle that is set by the coupling factors[155].

Between these radii lays the ergo-region. That is the place where for any item it is impossible to stand still. This is the result of a process known as frame-dragging; general relativity predicts that any rotating mass will tend to slightly "drag" along the spacetime immediately surrounding it. Any object near the rotating mass will tend to start moving in the direction of rotation.

The region where the considered item can be considered as a black hole is defined by:

$$m^2 > r_Q^2 + \alpha^2$$

Protons already fall within the region of potential black holes.

14.5.1.2   Kerr-Newman limit

The lowest mass $\mathfrak{M}$ where a horizon exists is set by

$$\mathfrak{M} \equiv m = \sqrt{r_Q^2 + (S/cm)^2}$$

Where S is the elementary spin s.

14.5.1.3   Cartesian coordinates

The Kerr–Newman metric can be expressed in "Kerr–Schild" form, using a particular set of Cartesian coordinates

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[154] http://en.wikipedia.org/wiki/Black_hole#Ergosphere

[155] .Misner, C. W., Thorne, K. S. and Wheeler, J. A., Gravitation, W. H. Freeman and Co., 1973. (Box 33.2)

$$g_{\mu\nu} = \eta_{\mu\nu} + f\, k_\mu\, k_\nu \tag{1}$$

$$f = \frac{G\, r^2}{r^4 + a^2\, z^2}\,[2\,M\,r - Q^2] \tag{2}$$

$$k_x = \frac{r\,x + a\,y}{r^2 + a^2} \tag{3}$$

$$k_y = \frac{r\,y - a\,x}{r^2 + a^2} \tag{4}$$

$$k_0 = 1 \tag{5}$$

Notice that $\boldsymbol{k}$ is a unit vector. Here M is the constant mass of the spinning object, Q is the constant charge of the spinning object, η is the Minkowski tensor, and a is a constant rotational parameter of the spinning object. It is understood that the vector $\boldsymbol{a}$ is directed along the positive z-axis. The quantity r is not the radius, but rather is implicitly defined like this:

$$1 = \frac{x^2 + y^2}{r^2 + a^2} + \frac{z^2}{r^2} \tag{6}$$

Notice that the quantity r becomes the usual radius $R = \sqrt{x^2 + y^2 + z^2}$ when the rotational parameter a approaches zero. In this form of solution, units are selected so that the speed of light is unity ($c = 1$).

In order to provide a complete solution of the Einstein–Maxwell Equations, the Kerr–Newman solution not only includes a formula for the metric tensor, but also a formula for the electromagnetic potential:

$$A_\mu = \frac{Q\, r^3}{r^4 + a^2\, z^2}\, k_\mu \tag{7}$$

At large distances from the source (R>>a), these equations reduce to the Reissner-Nordstrom metric[156] with:

$$A_\mu = \left(-\phi, A_x, A_y, A_z\right) \tag{8}$$

The static electric and magnetic fields are derived from the vector potential and the scalar potential like this:

$$\boldsymbol{E} = -\boldsymbol{\nabla}\phi \tag{9}$$

$$\boldsymbol{B} = \boldsymbol{\nabla} \times \boldsymbol{A} \tag{10}$$

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[156] http://en.wikipedia.org/wiki/Reissner%E2%80%93Nordstr%C3%B6m_metric

14.5.2 Schwarzschild metric

14.5.2.1   Schwarzschild coordinates

Specifying a metric tensor[157] is part of the definition of any Lorentzian manifold[158]. The simplest way to define this tensor is to define it in compatible local coordinate charts and verify that the same tensor is defined on the overlaps of the domains of the charts. In this article, we will only attempt to define the metric tensor in the domain of a single chart.

In a Schwarzschild chart[159] (on a static spherically symmetric spacetime), the line element ds takes the form

$$ds^2 = -\big(f(r)\big)^2 dt + \big(g(r)\big)^2 dr + r^2(d\theta^2 + \sin^2(\theta)\, d\phi^2) \qquad (1)$$

$$-\infty < t < \infty,\ r_0 < r < r_1, 0 < \theta < \pi, -\pi < \phi < \pi$$

In the Schwarzschild chart, the surfaces $t = t_0, r = r_0$ appear as round spheres (when we plot loci in polar spherical fashion), and from the form of the line element, we see that the metric restricted to any of these surfaces is

$$d\sigma = r_0^2(d\theta^2 + \sin^2(\theta)\, d\phi^2), \qquad 0 < \theta < \pi, -\pi < \phi < \pi \qquad (2)$$

That is, these nested coordinate spheres do in fact represent geometric spheres with surface area

$$A = 4\pi r_0^2 \qquad (3)$$

And Gaussian curvature
$$K = 1/r_0^2$$

That is, they are geometric round spheres. Moreover, the angular coordinates θ, ϕ are exactly the usual polar spherical angular coordinates: θ is sometimes called the colatitude and ϕ is usually called the longitude. This is essentially the defining geometric feature of the Schwarzschild chart.

With respect to the Schwarzschild chart, the Lie algebra of Killing vector fields is generated by the time-like irrotational Killing vector field ∂_t and three space-like Killing vector fields $\partial_\phi, \sin(\phi)\, \partial_\theta + \cot(\theta)\cos(\phi)\, \partial_\phi, \cos(\phi)\, \partial_\theta - \cot(\theta)\sin(\phi)\, \partial_\phi$

Here, saying that ∂_t is irrotational means that the vorticity tensor of the corresponding time-like congruence vanishes; thus, this Killing vector field is hyper-surface orthogonal. The fact that our spacetime admits an irrotational time-like Killing vector field is in fact the defining characteristic of a static spacetime. One immediate consequence is that the constant time coordinate surfaces $t = t_0$ form a family of (isometric) spatial hyper-slices. (This is not true for example in the

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[157] http://en.wikipedia.org/wiki/Metric_tensor
[158] http://en.wikipedia.org/wiki/Lorentzian_manifold
[159] http://casa.colorado.edu/~ajsh/schwp.html

Boyer-Lindquist chart for the exterior region of the Kerr vacuum, where the time-like coordinate vector is not hyper-surface orthogonal.)

It may help to add that the four Killing fields given above, considered as abstract vector fields on our Lorentzian manifold, give the truest expression of both the symmetries of a static spherically symmetric spacetime, while the particular trigonometric form which they take in our chart is the truest expression of the meaning of the term Schwarzschild chart. In particular, the three spatial Killing vector fields have exactly the same form as the three non-translational Killing vector fields in a spherically symmetric chart on E3; that is, they exhibit the notion of arbitrary Euclidean rotation about the origin or spherical symmetry.

However, note well: in general, the Schwarzschild radial coordinate does not accurately represent radial distances, i.e. distances taken along the space-like geodesic congruence which arise as the integral curves of ∂r. Rather, to find a suitable notion of 'spatial distance' between two of our nested spheres, we should integrate $g(r)dr$ along some coordinate ray from the origin:

$$\Delta\rho = \int_{r_1}^{r_2} g(r)dr \tag{4}$$

Similarly, we can regard each sphere as the locus of a spherical cloud of idealized observers, who must (in general) use rocket engines to accelerate radially outward in order to maintain their position. These are static observers, and they have world lines of form $r = r_0, \theta = \theta_0, \phi = \phi_0$, which of course have the form of vertical coordinate lines in the Schwarzschild chart.

In order to compute the proper time interval between two events on the world line of one of these observers, we must integrate $f(r)dt$ along the appropriate coordinate line:

$$\Delta\tau = \int_{t_1}^{t_2} f(r)dt \tag{5}$$

14.5.2.2 Schwarzschild metric

In Schwarzschild coordinates[160], the Schwarzschild metric has the form:

$$c^2\,d\tau^2 = \left(1 - \frac{r_s}{r}\right)c^2\,dt^2 - \left(1 - \frac{r_s}{r}\right)^{-1}dr^2 - r^2(d\theta^2 + \sin^2(\theta)\,d\phi^2) \tag{6}$$

where:
- τ is the proper time (time measured by a clock moving with the particle) in seconds,
- c is the speed of light in meters per second,
- t is the time coordinate (measured by a stationary clock at infinity) in seconds,
- r is the radial coordinate (circumference of a circle centered on the star divided by 2π) in meters,
- θ is the colatitude (angle from North) in radians,

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[160] http://en.wikipedia.org/wiki/Schwarzschild_coordinates

- φ is the longitude in radians, and
- r_s is the Schwarzschild radius (in meters) of the massive body.

14.5.2.3 Lemaître coordinates

In Schwarzschild coordinates the Schwarzschild metric has a singularity. Georges Lemaître was the first to show that this is not a real physical singularity but simply a manifestation of the fact that the static Schwarzschild coordinates cannot be realized with material bodies inside the gravitational radius[161]. Indeed inside the gravitational radius everything falls towards the center and it is impossible for a physical body to keep a constant radius.

A transformation of the Schwarzschild coordinate system from $\{t, r\}$ to the new coordinates $\{\tau, \rho\}$,

$$d\tau = dt + \frac{\sqrt{r_s/r}}{\left(1 - \frac{r_s}{r}\right)} \, dr \tag{1}$$

$$d\rho = dt + \frac{\sqrt{r/r_s}}{\left(1 - \frac{r_s}{r}\right)} \, dr \tag{2}$$

leads to the Lemaître coordinate expression of the metric,

$$ds^2 = d\tau^2 - \frac{r_s}{r} d\rho^2 - r^2(d\theta^2 + \sin^2(\theta) \, d\phi^2) \tag{3}$$

Where

$$r = r_s^{1/3} \left[\frac{3 \, (\rho - \tau)}{2}\right]^{2/3} \tag{4}$$

In Lemaître coordinates there is no singularity at the gravitational radius, which instead corresponds to the point $\frac{3 \, (\rho - \tau)}{2} = r_s$. However, there remains a genuine gravitational singularity at the centrum, where $\rho - \tau = 0$, which cannot be removed by a coordinate change.

The Lemaître coordinate system is synchronous, that is, the global time coordinate of the metric defines the proper time of co-moving observers. The radially falling bodies reach the gravitational radius and the center within finite proper time.

Along the trajectory of a radial light ray,

$$dr = \left(\pm 1 - \sqrt{r_s/r}\right) d\tau \tag{5}$$

therefore no signal can escape from inside the Schwarzschild radius, where always $dr < 0$ and the light rays emitted radially inwards and outwards both end up at the origin.

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[161] http://en.wikipedia.org/wiki/Lemaitre_coordinates

15 The action along the live path

The integrated action S_{ab} is performed over a distance along the action trail or equivalently over a period of coordination time

$$S_{ab} = -\int_a^b m \cdot c^2 \cdot ds + matter\ terms \tag{1}$$

$$= -\int_{\tau_a}^{\tau_b} m \cdot c^2 \cdot \sqrt{1 - \left(\frac{v}{c}\right)^2} \cdot d\tau + matter\ terms$$

$$= \int_{\tau_a}^{\tau_b} \mathcal{L} \cdot d\tau$$

m is the mass of the considered item.
v is the speed in Q space.
$\mathcal{L}$ is the Lagrangian.

The first line of this formula can be considered as an integral along the trail in coordinate space or equivalently over the trail in Hilbert space. The next lines concern integrals over the corresponding path in observed space combined with coordinate time. It must be noticed that these spaces have different signature.

$$\mathcal{L} = -m \cdot c^2 \cdot \frac{ds}{d\tau} + matter\ terms \tag{2}$$

In general relativity, the first term generalizes (includes) both the classical kinetic energy and interaction with the Newtonian gravitational potential. It becomes:

$$m \cdot c^2 \cdot \frac{ds}{d\tau} = -m \cdot c \cdot \sqrt{g_{\alpha\beta} \cdot \dot{q}_\alpha \cdot \dot{q}_\beta} \tag{3}$$

$g_{\alpha\beta}$ is the rank 2 symmetric metric tensor which is also the gravitational potential. Notice that a factor of c has been absorbed into the square root.

The matter terms in the Lagrangian $\mathcal{L}$ differ from those in the integrated action S_{ab}.

$$S_{ab\_matter} = -\int_a^b e \cdot A_\gamma \cdot dq^\gamma + other\ matter\ terms \tag{4}$$

The matter term in the Lagrangian due to the presence of an electromagnetic field is given by:

$$\mathcal{L} = -m \cdot c^2 \cdot \frac{ds}{d\tau} + e \cdot \dot{q}^\gamma \cdot A_\gamma + other\ matter\ terms \tag{5}$$

A_γ is the electromagnetic 4-vector potential.

15.1 Noether's theorem

When the Lagrangian does not vary with one or more of its parameters, then this corresponds with a corresponding symmetry of the system. By <u>Noether's theorem</u>[162], such symmetries of the system correspond to conservation laws[163]. In particular, the invariance of the Lagrangian with respect to time τ implies the conservation of energy.

By partial differentiation of the above Lagrangian, we find:

$$\frac{\partial \mathcal{L}_\tau(\tau, q, \dot{q})}{\partial q_i} = \frac{\partial U}{\partial q_i} = F_i \tag{5}$$

$$\frac{\partial \mathcal{L}_\tau(\tau, q, \dot{q})}{\partial \dot{q}_i} = m \cdot \dot{q}_i = p_i \tag{6}$$

where the force is $F = -\nabla U$ (the negative gradient of the potential, by definition of conservative force), and p is the momentum. By substituting these into the Euler–Lagrange equation, we obtain a system of second-order differential equations for the coordinates on the particle's trajectory,

$$F_i = \frac{d(m\dot{q}_i)}{dt} = m \cdot \ddot{q}_i = \dot{p}_i \tag{7}$$

which is Newton's second law.

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[162] http://en.wikipedia.org/wiki/Noether%27s_theorem
[163] http://en.wikipedia.org/wiki/Conservation_law

16 Quaternionic metric

For the metric holds

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = (ds\, ds) + (ds\, ds)^* \tag{1}$$

In quaternionic format this corresponds to

$$ds = q_\mu(x)dx^\mu \tag{2}$$

Where q_μ and x are quaternions. x_μ is a component of x.

$$g_{\mu\nu} = q_0^2 - \langle \boldsymbol{q}, \boldsymbol{q} \rangle \tag{3}$$

Using polar coordinates this will run:

$$x_\tau = r\, cos(\psi) \tag{4}$$

$$x_x = r\, sin(\psi)\, sin(\theta)\, cos(\varphi) \tag{5}$$

$$x_y = r\, sin(\psi)\, sin(\theta)\, sin(\varphi) \tag{6}$$

$$x_z = r\, sin(\psi)\, cos(\theta) \tag{7}$$

$$dx_\tau = dr\, cos(\psi) - r\, sin(\psi) \tag{8}$$

$$dx_z = dr\, sin(\psi)\, cos(\theta) + r\, cos(\psi)cos(\theta) - r\, sin(\psi)\, sin(\theta) \tag{9}$$

$$dx_x = dr\, sin(\psi)\, sin(\theta)\, cos(\varphi) - r\, cos(\psi)cos(\theta)\, sin(\varphi) + r\, sin(\psi)\, sin(\theta)sin(\varphi) \tag{10}$$

$$dx_y = dr\, sin(\psi)\, sin(\theta)\, sin(\varphi) + r\, cos(\psi)cos(\theta)cos(\varphi) - r\, sin(\psi)\, sin(\theta)cos(\varphi) \tag{11}$$

17 References

http://www.aflb.ensmp.fr/AFLB-283/aflb283p549.pdf

PART IV

A Tall Quantum Tale

I state you a proposition
and that proposition indicates
how the world works

1 QUANTUM TALE

1.1 Story

1.1.1 Prelude

A group of elderly Magi sit in a circle and discuss what happens around them. That is not much. The youngest of them gets bored and starts considering their discussion. The chat appears regulated, because if they start from a false proposition they will be able to draw any inference, whether true or not true, and then the conversation ends only in balderdash ad infinitum.

After some time, he has collected the rules. These rules prevent the conversations from getting out of control. He proposes these rules to his companion discussers. They are very pleased. From this moment on, every conversation runs fluently. The inventor writes his finding in a book and calls that book "Logic".

However, in their environment still little occurs that is worth a proper discussion. Since the talks no longer get out of control, most of the time passes in silence. The inventor feels bored again and therefore he tries to invent something else. He realizes that if he changes the rules in his book a little, then as a result, the discussions could be become much more interesting. He writes a new book that contains the changed rules. Next he changes the forest that exists in their neighbourhood in order to reflect the discussion rules.

After finishing this book and the forest, the situation has completely changed. Continuously, things appear in the forest around them that keep their conversations for ever alive. The writer calls the second book "Quantum Logic" and he renames his first book "Classical Logic". The toolkit that he uses to create the new structure of the forest also has a name. It is called "Mathematics".

 M   **S**

1.1.2   The encounter

An old, very experienced senior meets a young curious guy, which is full of questions about the things that he has observed during his trip through his world. The youngster asks the elder whether he can ask him a few of his most urging questions. The senior reacts positively by nicking shortly. However, already the first question of the studious guy startles him:

S: Mister, can you explain me how the world works?

The elder thinks a while very deeply and comes then with his answer:

M: That would be a hell of a job, but I can at least give it a try. Please, sit down on that stone, because this will take some time.

The lad sits down and looks expectantly to his narrator. The old man takes a breath and starts:

M: This can be done in the form of a tale. It could be done better in the form of a truck load of formulas, but I doubt that you would understand these formulas. Do you accept that I pack the story in a tale?

S: Well I like a tale much better than a truck load of formulas. I probably would not understand one of them. So please start with your tale.

The elder takes a breath and starts his tale.

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M: The world is governed by a book of laws. It must conform to these laws. There is no punishment in not following the laws, but the world cannot do anything else then operate according to the rules that are written in the book of laws.

S: Where is that book and how is it called?

M: It is in the possession of the governor of Hilbert's bush. The book's name is "The rules of quantum logic".

S: What is in that book?

M: The book contains a small set of rules that regulate what the relations are between propositions that can be made about things that live in our world.

S: What things?

M: Well, anything that has an identity and that stores the condition it is in. Let us call such a thing an item or a particle and let us use the name state for the condition it is in. Mostly the concerned things are very small. However, these things can be very large.

S: What is different with that logic? I know only one kind of logic.

M: You know the kind of logic that humans base their reasoning on. They use the rules of logic in their discussions when they start with truth and want to stay with truth. Nature uses a kind of logic that has a much richer structure. However, in that logic only one rule is different.

S: How many rules contains the book and what do these rules mean?

M: The book contains somewhat more than twenty rules and they specify the structure of the relations between the allowable propositions.

S: There are not much rules in the book! How can that book rule the world?

M: You are right about this, but these rules are very powerful.

S: Please explain that.

M: Well, the structure of the propositions is reflected in the structure of Hilbert's bush. Hilbert's bush is a huge and dense forest and is connected to our world. Via these connections Hilbert's bush controls how the world works.

S: Thus, if I visit Hilbert's bush, then I can see how the world works?

M: No, if you visit Hilbert's bush, then you can see how the world is controlled.

S: How, can I visit Hilbert's bush?

M: Well, you can join me on a virtual trip to Hilbert's bush. I will be your guide.

S: Fine. How does Hilbert's bush look?

The man describes a very strange environment. The chap follows the old man in his mind and shows astonished. However, in advance his guide warned that he would present a tale. So, he must belief what the man tells.

M: It is like a huge forest of poles. All poles have the same length and the feet of all poles are hooked at the same point in the centre of the bush. In this way the poles form an enormous sphere.

S: Where do these poles stand for?

M: The poles are the axes of a multidimensional cube that has an enormous dimension. First think of a three dimensional cube. Take a corner of it and take the three axes at that corner. You can identify the position of all points in the cube by three positions on rulers that are taken along the three axes.

Now, as in an umbrella, fold these axes together, such that they form a small bundle. Next add a large amount of axes to that bundle. Give every axis a unique label in the form of one or more numbers. Add a ruler to each of these axes. You can still define the position of each point in the multidimensional cube by stating the corresponding positions on the rulers. Next increase the number of dimensions until it reaches infinity.

The axes now form a dense ball and they all are numbered with a unique label. Finally unfold in your imagination the "umbrella" again until all axes are again perpendicular to each other. You can start counting the dimensions of the cube, but you will never finish counting.

S: Thus the poles are a plain set of axes.

M: Yes, but the space between the perpendicular axes can also be filled with poles. In this way several sets of mutually perpendicular axis poles can be found.

S: What is the function of these axis poles?

M: The axis poles have colours. Some axis poles are green poles. Together they form a base in which the position of all other poles can be expressed. Another set of axis poles are red. Also they form a base. Some of the poles are silver white. They are not necessarily axis poles. The silver white poles appear in bundles.

S: That is a strange kind of forest!

M: Indeed, but it is not the only thing that is strange about Hilbert's bush. Let me tell more about the silver white poles. The bundles of white poles represent and at the same time control the items in our world.

S: How is that arranged?

M: The items in our world are reflections of the bundles of white poles in Hilbert bush. What happens to the bundles will happen to the items.

The student tries to imagine the strange situation. Apparently two worlds exist. One in which he lives and one from where his live is controlled. He visualizes the forest in his brain.

S: What is the function of the green and red poles?

M: At their top these other poles contain a data store in the form of a label. The data stores of the green poles contain position data. They are a kind of kilometre indications that you find along our roads. Instead of a single number the stores contain all three coordinates. It works like a kind of primitive GPS system.

S: With some trouble I can understand what you paint for me.

M: The data stores of the red poles contain speed data, or better said momentum data. In this way a bundle of silver white poles can determine the current position and the momentum of the moves of its pupil in the real world.

S: Why are there two types of data poles?

M: The governor arranged it that way. In this way the bundle cannot determine both types of data at the same time. It is another detail of how the governor models our world. The stores of the poles contain the values of the properties of the type observation to which the pole belongs. Mathematicians call these values eigenvalues and the corresponding poles eigenvectors. With this trick the governor leaves us uncertain about our exact condition.

S: What are mathematicians?

M: Mathematicians are scientists that amongst other things study the mechanisms, which determine the structure and behaviour of Hilbert's forest. The creator of the forest used mathematics to give it its functionality.

S: Can white poles read data?

M: No, in fact a shepherd that takes care of the silver white bundle does that. The forest is very dense. So, the shepherd can walk on top of these poles and guard his herd of sheep. From now on, I will call the silver white poles the shepherd's sheep.

S: How does the shepherd read the data?

M: The shepherd must turn to the data pole in order to read its data. If he is close to a green pole, then he is rather far from a red pole. In fact he may be at nearly the same distance from a series of red poles. He will usually read the nearest data pole. The same holds when the shepherd looks at other colours. Thus, the governor plays a strange trick with our world.

For the insiders: This is the source for the existence of Heisenberg's uncertainty principle. It is the cause of the quantum behaviour of small particles.

S: I must say, that is a strange situation!

M: Yes, let me proceed. It will become even much stranger.

S: Please, go on.

M: The shepherd drives his sheep through Hilbert's bush. He does that guided by the smells that he receives from other silver white bundles. The smells are mixtures of perfumes that are attractive and perfumes that are repellent. The shepherd reacts on these smells.

S: What is causing these smells?

M: These smells are caused by the properties of the sheep. They hang as a blurring mist around each white pole, thus around each individual sheep. The sheep may also move inside the scope of the herd. That movement may also be caused by the influence of the emitted smells.

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S: How does the shepherd keep his sheep together?

M: Well, that happens in a particular way. The bush is so dense, that it is impossible to let the poles move. Instead at each of his steps the shepherd redefines the poles that belong to his herd. These poles turn silver white. The poles that get outside of the herd obtain their original green or red colour. The smells create a tendency to minimize action of the cheap. Further there exists another mechanism, which is called inertia.

S: What is inertia?

M: The smells invoke a sticky resistance of the system of all herds against change. Inertia represents the combined influence of all other herds. The most distant herds together form the largest part of the set of herds. So, they have the largest effect. The influence of each individual herd decreases with distance. However, the number of herds increases faster with distance. The difference between the distant herds averages away. As a consequence the distant herds form a uniform background influence.

S: What is the effect of inertia on a herd?

M: Locally the inertia produces an enormous smell pressure. A smooth uniform movement does not disturb this potential. When the herd accelerates it stirs the perfumes and in this way the inertia produces a smell that goes together with this movement.

S: I understand now how position is treated. What about time?

M: The shepherd owns a simple clock. That clock counts his steps. His steps are all the same size. When he drives his sheep around, he follows a track in Hilbert's bush. All shepherds take their steps in synchrony. In facts at each of their steps the complete forest is redefined. In this process the smells act as a guide. They store the current condition of the forest and these represent the preconditions for the new version of the forest. You can say that the smells represent potential versions of the forest. This includes potential versions of sheep. These potential sheep are virtual sheep.

S: So, compared to space, time is handled quite differently.

M: You understand it quickly and perfectly! You understand it better than the physicists of the last few centuries. Most of them were wrong with this subject. They think that time and space belong in one inseparable observable characteristic.

S: How many of these herds exist?

M: As many as there are particles in our world. So, there exist an enormous number of herds, but they are still countable. They can all be identified. All shepherds take their own track through Hilbert's bush.

S: That must make Hilbert's bush very large!

M: It is. Let me proceed. It must be obvious now that the herds influence each other's movements via their smells.

The lad reflects and pictures the forest in his mind as an enormous sphere. On top of that sphere a large number of shepherds push their own herd of silver white lights forward on curving tracks that are determined by the smells that other herds produce. At each of the shepherd's steps Hilbert's forest is reconfigured. The old man must have a strange image of the world. Nonetheless, he must have his reasons.

S: So, the shepherds play a crucial role!

M: Yes, they manipulate their own herd. However, the smells of their sheep influence for other shepherds the observation of the position and momentum of other herds.

S: How do the smells influence that observation?

M: They give the data that are transmitted in the smell an extra turn. It means that other shepherds do not get a proper impression of the position and momentum data that are sent by other herds.

S: Is there a good reason for this confusing behaviour?

M: No, there is no reason. It is just a built in habit of all sheep. On the other hand, the governor established that habit when he designed mathematics. He designed mathematics such, that Hilbert's bush and its inhabitants behave according to the rules in his book.

S: What is the consequence of this strange behaviour?

M: The consequence is that the particles in the world get the wrong impression of the position and momentum of other items. For them it appears that there exists a maximum speed. And these items think that they live in a curved space.
For the insiders: This is the source of the existence of relativity as it was discovered, but not explained by Einstein.

S: Do they think that?

M: For them, it is the truth!

S: So, I live in a curved space and for me there exists a maximum speed.

M: That is right. You properly understand how the world is controlled. As long as you do not interpret that maximum speed as the limit set by your local police officer.

S: What happens inside a herd?

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M: The sheep inside a well-shaped herd perform rhythmic movements. You could say that they are dancing. Physicists call it harmonic movements. These dances occur under the control of the shepherd. He considers them as his own possession.

S: What do you mean with a well-shaped herd?

M: A well-formed herd represents in our world a well-formed object, such as an atom.

S: Why is everything set up in such a strange way?

M: The governor of Hilbert's bush is very intelligent, but also very lazy. He does not want to create many rules, so that he does not have to write much in his law book. That is why he invented Hilbert's bush. He builds the consequences of all his rules into the structure and the dynamics of Hilbert's bush. That structure is in principle very simple. The same holds for the dynamics. In this way he does not have to take care on how the world evolves. However, this leaves an enormous freedom for what happens in the world that is controlled by Hilbert's bush. That on itself results in an enormous complexity of the world we live in. That renders the governor very, very smart and very, very lazy.

S: How did Hilbert's bush get its name?

M: Hilbert was the first human that discovered the governor's bush. So people give it his name.

S: Can everybody visit Hilbert's bush?

M: In principle yes. Everybody that possesses sufficient imagination can visit Hilbert's bush. There exist two guides. A mister Schrödinger tells the story as we did. He tells the story as if the bundle of silver white poles moves through the bush of green and red poles. The other guide, mister Heisenberg tells the story as if the bundle of white poles is stationary and the bush of green and red poles moves around. For the world it does not matter what moves. It only senses the relative motion.

S: How did intelligent creatures like us enter that world?

M: The governor installed a tendency to reduce complexity by means of modularization into his forest. When more compatible modules become available it becomes easier to construct more capable modules and more capable items from these modules. Given enough time, more and more capable items are created, which finally result in intelligent creatures. Scientists call this process evolution. It is a chaotic process, but it possesses a powerful tendency.

S: Uch. Can I tell this to my friends?

M: Yes, you can. And if you have learned to read formulas and work with them you can come back and I will tell you the same story in a cart load of formulas.

S: Thanks. I will come back when I am grown up. Can I still ask a final question?

M: You are a sauce-box, but you are smart. Go ahead.

S: What are you going to do after this?

M: I will visit a very old and very wise scientist, called Mendel. He claims that he has a cohesive explanation for all smells that shepherds react to.

S: Why is that important?

M: If his claim is right, then he has found the Holy Grail of physics.

S: Gosh!

After this the boy departs. Later he will become a good physicist.

2   Interpretation

The book of laws contains a number of axioms that define the structure of traditional quantum logic as an orthomodular lattice.

Hilbert's bush stands for an infinite dimensional separable Hilbert space that is defined over the number field of the quaternions. The set of the closed subspaces of the Hilbert space has the same lattice structure as traditional quantum logic.

The green poles represent an orthonormal base consisting of eigenvectors of the normal operator Q. This operator represents an observable quantity, which indicates the location of the item in space.

The red poles represent an orthonormal base consisting of eigenvectors of the normal operator P. This operator is the canonical conjugate of Q and represents an observable quantity, which indicates the momentum of the item.

The bundle of silver white poles and the herd of sheep represent a closed subspace of the Hilbert space that on its turn represents a particular quantum logical statement. This statement concerns a particle or a wave packet in our surroundings. Q describes the thing as a particle. P describes the thing as a wave packet.

The shepherd represents a complicated operator U_t that pushes the subspace, which is represented by his herd, around in the Hilbert space. The operator U_t may be seen as a trail of infinitesimal unitary operators. It is a function of the trail progression parameter t. The progression parameter differs from our common notion of time, which is the coordinate time.

Traditional quantum logic defines only the stationary structure of what happens in Hilbert's bush. The dynamics are introduced by the shepherds that react on the smells.

The smells correspond to physical fields. The fields transport information about the conserved quantities that characterize the movements of the item and its elements. Each type of preserved quantity has its own field type. The operators U_t react on these fields. Inertia shows how these operators reflect the actions of the fields. Any acceleration of the item goes together with a reconfiguration of the fields.

The operator U_t transforms the observation operators Q and P into respectively

$$Q_t = U_t^{-1} \cdot Q \cdot U_t$$

and

$$P_t = U_t^{-1} \cdot P \cdot U_t$$

.This distorts the correct observation and ensures that the observer experiences a speed maximum and a curved space.

The eigenvalues of Q and P and the trail progression parameter t characterize the space-time in our live space. As already indicated t is not the same as our common coordinate time.

De eigenfunctions of U_t control the (harmonic) internal movements of the particles.

The sheep represent the elements/properties of the particle.

The effect of modularization is treated in
http://www.crypts-of-physics.eu/ThereExistsATendencyInNatureToReduceComplexity.pdf.;
part four of this book

HvL

PART V

QPAD Game

1   Introduction

This game is a nice educational puzzle when you have lots of spare time. It can bring you deep insight in the standard model.

In order to know which elementary particle types exist, a small game suffices.

The game takes the presumption that all massive elementary particle types can be identified by an ordered pair $\{\psi^x, \psi^y\}$ of sign flavors of a quaternionic probability amplitude distribution (QPAD).

A quaternion offers two sign selections; a conjugation that changes the sign of three imaginary base vectors and a reflection that changes the sign of a single base vector. The sign selection stays constant throughout the whole QPAD. For a QPAD this means that four different sign flavors exist.

The coordinates that are used as parameters of the QPAD also form a quaternionic distribution (QD). For a flat coordinate system the value of the distribution equals the parameter. If the coordinate system is curved, then the values of this QPAD follow that deviation. The parameter QD is taken as a reference for comparing sign flavors. Two sign selections have an isotropic sign status. The other two sign selections are anisotropic. The conjugation and the reflection each cause a switch of the handedness of the quaternion product. The other sign selections leave the handedness untouched.

2   Instructions

2.1   Equation of motion

The ordered pair $\{\psi^x, \psi^y\}$ represents a category of elementary particle types.

Now look at the quaternionic format of the equation of free motion of elementary particles.

$$\nabla \psi^x \;=\; m\,\psi^y$$

∇ is the quaternionic nabla operator. It represents a four dimensional differentiation.

ψ^x acts as the wave function of the particle. It is a QPAD, whose sign flavor is indicated by suffix x.

m is the coupling factor. It is a real number.

ψ^y is the coupled QPAD sign flavor.

both ψ^x and ψ^y are sign flavors of the same base QPAD ψ°.

ψ° has the same sign flavor as the parameter space.

For antiparticles you must conjugate all participating fields as well as the nabla operator.

Photons and gluons have zero coupling factor.

2.2 Rules

a) When ψ^y is isotropic (zero or all three base vectors are switched), then the particle is a fermion, otherwise it is a boson.
b) If the coupling takes place between two field sign flavors with different handedness, then the corresponding particle is charged.
c) The charge depends on the number and direction of the base vectors that differ.
d) The count for each difference is $\pm\frac{1}{3}e$.
e) The procedure does not discriminate the generations

2.3 The game

The game is:

Find the particle types:

- electron,
- neutrino,
- down quark,
- W bosons,
- Z boson.

The up-quark is not in the list. According to the rules it is impossible to generate elementary particles with charge $\pm\frac{2}{3}$e. It means that up-quarks are composite particles.

2.4 Solution

The solutions is explained in part II; Elementary particles
However, you can discover it yourself.
Try it! It is fairly easy.

The most intriguing fact is that the coupling factor m can be computed from the fields $\{\psi^x, \psi^y\}$ So, no Higgs is involved there!

The properties that characterize the coupling event are sources of secondary fields. These fields are known as physical fields. In fact they are QPAD's where the property plays the role of an isolated charge.

The Philosopher Tempted.

MEDITATIONS

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The Hilbert Book Model
Author: Hans van Leunen

This book starts from the axioms of traditional quantum logic and extends this model such that it incorporates physical fields as well as dynamics.

It uses the isomorphism between the set of propositions of traditional quantum logic and the set of closed subspaces of an infinite dimensional separable Hilbert space that uses quaternions in order to specify its inner products.

The book finds solutions for the anomalies that are raised by the countability of the eigenspaces of normal quaternionic operators. It also takes the consequence of the observation that all information about nature becomes available in the form of clouds of information carrying quanta.

The book unifies all fields, such that except for the curvature field, all fields including the wave functions are considered as QPAD's. The curvature field is derived from the curvature of the superposition of all these primary fields. The curvature follows from the decomposition of this covering field in rotation free and divergence free parts.

In order to implement dynamics, the developed model applies a sequence of extended quantum logics or equivalently a sequence of extended separable Hilbert spaces. Each of the members of the sequence represents a static status quo of the universe. This leads to a new model of physics:

The Hilbert Book Model

Apart from this main subject the book contains a series of related papers.

http://vixra.org/author/Ir_J_A_J_van_Leunen
http://vixra.org/author/Ir_J_A_J_Hans_van_Leunen

http://www.crypts-of-physics.eu

The crypt is under a friend's house in Nesle, France