# Anisotropic to Isotropic Phase Transitions in the Early Universe

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We attempt to develop a minimal formalism to describe an anisotropic to isotropic transition in the early Universe. Assuming an underlying theory that violates Lorentz invariance, we start with a Dirac like equation which does not exhibit Lorentz invariance and write down transformations that restore it to its covariant form. It is proposed that these transformations can be visualized as waves traveling in an anisotropic media. The transformation $it/\hbar \to \beta$ is then utilized to transit to a statistical thermodynamics system and the partition function then gives a better insight into the character of this transition. The statistical system hence realized is a two level system with each state doubly degenerate. We propose that modeling the transition this way can help explain matter antimatter asymmetry of the Universe.

## I. INTRODUCTION

The idea that the Universe is homogeneous, isotropic and that space-time is Lorentz invariant are important pillars of theoretical physics. Whereas the cosmological principal assumes the Universe to be homogeneous and isotropic, Lorentz invariance is required to be a symmetry of any relativistic quantum field theory. These requirements have robust footings, but there can possibly be scenarios where these ideas are not sufficient to describe the dynamics of a system. Temperature fluctuations in the Cosmic Microwave Background (CMB) radiation indicate that the assumptions made by the cosmological principal are not perfect. There is no conclusive evidence of Lorentz violation to date but this has been a topic of considerable interest and the Standard Model Extension (SME) has been constructed which includes various terms that violate observer Lorentz invariance [1]. Limits have been placed on the coefficients of various terms in the SME as well [2]. Another important question is the matter-antimatter asymmetry of the Universe which is not completely resolved. Sakharov, in 1967 derived three conditions (baryon violation, C and CP violation and out of thermal equilibrium) for a theory to satisfy in order to explain the baryon asymmetry of the Universe.

In this paper we intend to describe the evolution of a theory that violates Lorentz invariance to a theory that preserves it. The fields that are involved in the Lorentz violating theory can be viewed in analogy with fields traveling in an anisotropic medium. When the system evolves from the anisotropic to isotropic phase the symmetry of the theory is restored and the partition function formalism can be used to better understand how this transition takes place. This formalism, we propose, can help explain the matter-antimatter asymmetry of the Universe. The paper is organized as follows: In section II and III we describe these transformations and propose a way to interpret them as plane wave transitions into anisotropic media. In section IV the partition function is used to get a better insight into how the transformations in section II occur and we conclude in section V.

## II. TRANSFORMATIONS LEADING TO COVARIANT DIRAC EQUATION

In this section we outline a set of transformations that lead to the Dirac equation for a QED (Quantum Electrodynamics) like theory with no interaction terms. We start with a Dirac-like equation which involves four fields ($\chi_a, \chi_b, \chi_c, \chi_d$). These fields can be redefined in a simple way such that the covariant form of the Dirac equation is restored along with a mass term. We assume a minimal scenario and consider just the kinetic terms for the fields in the underlying theory. If we start with the following equation ($\hbar = c = 1$):

$$i\bar{\chi_a}\gamma^0\partial_0\chi_a + i\bar{\chi_b}\gamma^1\partial_1\chi_b + i\bar{\chi_c}\gamma^2\partial_2\chi_c + i\bar{\chi_d}\gamma^3\partial_3\chi_d = 0, \tag{1}$$

and transform each of the $\chi$ fields in the following manner,

$$\begin{aligned} &\chi_a(x) \to e^{i\alpha m\gamma^0 x_0}\psi(x), \;\; \chi_b(x) \to e^{i\beta m\gamma^1 x_1}\psi(x) \\ &\chi_c(x) \to e^{i\delta m\gamma^2 x_2}\psi(x), \;\; \chi_d(x) \to e^{i\sigma m\gamma^3 x_3}\psi(x), \end{aligned} \tag{2}$$

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we get the Dirac equation in covariant form, along with a mass term (using, for e.g., $e^{i\beta m\gamma^1 x_1}\gamma_0 = \gamma_0 e^{-i\beta m\gamma^1 x_1}$),

$$\overline{\psi}(i\gamma^\mu\partial_\mu - (\alpha+\beta+\delta+\sigma)m)\psi = 0, \tag{3}$$

where $\alpha,\ \beta,\ \delta$ and $\sigma$ are real positive constants. For plane wave solution for particles, $\psi = e^{-ip.x}u(p)$, the above redefinition for the field $\chi_a$, for example, is a solution of the following equation:

$$\frac{\partial}{\partial t}\chi_a(x) = -i(E - \alpha m\gamma_0)\chi_a(x), \tag{4}$$

with similar equations for the other fields. Equation (4), is similar to equation (27) in reference [3] which is a solution of the differential equation governing linear elastic motions in an anisotropic medium (with a constant matrix, see section III of the reference). With $\alpha = 0$ the left hand side is just the Hamiltonian with the plane wave its eigenstate.

Note that the manner in which we can transform equation (1) to (3) is not unique and there are various ways to do this with different combinations of the $\chi$ fields along with the field $\psi$. A mass term $(m\overline{\chi}\chi)$ for the $\chi$ fields could have been added to equation (1), but the redefinitions (2) can be used to eliminate it. So, if we want our resulting equation to describe a massive fermion, these fields should be massless or cannot have mass term of the form $m\overline{\chi}\chi$. This argument will be further corroborated with the results we present in section IV. The transformation matrices in equation (2) are not all unitary, the matrix $e^{i\alpha m\gamma^0 x_0}$ is unitary while the rest $(e^{i\beta m\gamma^i x_i})$ are hermitian.

The fields in equation (1) can be considered as independent degrees of freedom satisfying equation (4) in an underlying theory that violates Lorentz invariance. The transformations (2) can, therefore, be seen as reducing the degrees of freedom of the theory from four to one. In such an underlying theory, various interaction terms can be written for these fields. Since we intend to obtain the free Dirac equation, we have considered only kinetic terms involving the fields $\chi$. A quadratic term involving different $\chi$ fields $(m\overline{\chi}_i\chi_j)$ can be added to equation (1) but this leads to a term that violates Lorentz invariance in the resulting Dirac equation. A quartic term $(c\overline{\chi}_i\chi_i\overline{\chi}_j\chi_j)$ is possible and would result in a dimension 6 operator for the field $\psi$ with the constant $c$ suppressed by the square of a cutoff scale. So, with the restriction that the resulting Dirac equation only contains terms that are Lorentz scalars the number of terms we can write for the $\chi$ fields can be limited. In other words we impose Lorentz symmetry in the resulting equation so that various terms vanish or have very small coefficients. Several such terms have been considered in the SME [1] and limits on their coefficients are also available [2]. Our case is different from the SME since we assume that the underlying theory explicitly violates Lorentz invariance whereas in the SME Lorentz violation occurs spontaneously, i.e., when tensors from an underlying string theory acquire vacuum expectation values (see reference [1] for details).

## III. VISUALIZING FIELD REDEFINITIONS

Space-time dependent field redefinitions in the usual Dirac Lagrangian result in violation of Lorentz invariance. For example, the field redefinition $\psi \rightarrow e^{-ia^\mu x_\mu}\chi$ leads to the Lorentz violating terms in the Lagrangian [1]. This transformation amounts to shifting the four momentum of the field. It can also be viewed in analogy with plane waves entering another medium of a different refractive index which results in a change in the wave number of the transmitted wave. Similarly, transformations (2) can be interpreted as transitions of a wave from an anisotropic to isotropic medium or vice versa as done in the Stroh's matrix formalism [3]. For plane wave solutions of $\psi$, the $\chi$ fields have propagative, exponentially decaying and increasing solutions (for example, $e^{\pm imx},\ e^{\pm mx}$). This wave behavior is similar to that in an anisotropic medium or a medium made of layers of anisotropic medium. The eigenvalues of the Dirac matrices being the wave numbers of these waves in this case. The coefficients in the exponent relates to how fast the wave oscillates, decays and/or increases exponentially. The transfer matrix in Stroh's formalism describe the properties of the material and in this case can possibly represent the properties of the anisotropic phase from which the transition to the isotropic phase occurs.

Therefore, we can visualize a global and local transformation as transitions of plane waves to different types of media. The wave function of a particle $(E > V)$ which comes across a potential barrier of a finite width and height undergoes a phase rotation $(e^{ikl}\psi)$ upon transmission. If the width of the barrier extends to infinity, the wave function can be viewed as undergoing a position dependent phase rotation $(e^{ikx}\psi)$. The transformations (2) can similarly be seen as a plane wave entering an anisotropic medium. Another phenomenon called birefringence in optics can be used to explain why these four fields map on to the same field $\psi$. Birefringence results in a plane wave splitting into two distinct waves inside a medium having different refractive indices along different directions in a crystal. These analogies can serve as crude sketches to visualize how the transformations in equation (2) can occur.

In the usual symmetry breaking mechanism a Higgs field acquires a vacuum expectation value (VEV) and the resulting mass term does not respect the symmetry of the underlying group. For example, in the Standard Model, due to its chiral nature, a Higgs field is introduced in order to manifest gauge invariance. Once the Higgs field acquires a VEV the mass term only respects the symmetry of the resulting group which is $\mathrm{U(1)_{EM}}$. In our case the mass term arises after symmetry of the Dirac equation is restored. Consider the simple case where we have one field $\chi_a$ in addition to the field $\psi$:

$$i\bar{\chi_a}\gamma^0\partial_0\chi_a + i\bar{\psi}\gamma^i\partial_i\psi = 0, \tag{5}$$

and this field transforms to the field $\psi$ as $\chi_a(x) \rightarrow e^{i\alpha m\gamma^0 x_0}\psi(x)$, leading to the Dirac equation. In order to discuss the symmetries of the above equation let's assume that the two independent degrees of freedom are described by the above equation. Equation (5) then has two independent global U(1) symmetries and the resulting equation has one. In fact, there is a list of symmetries of equation (5) not possessed by (3), for example invariance under local transformations, $\chi_a \rightarrow e^{ib^i\theta(x_i)}\chi_a'$ $(i, j = 1, 2, 3)$, where $b_i$ can be a constant vector, the matrix $\gamma_0$ or any matrix that commutes with $\gamma_0$ (e.g., $\sigma_{ij},\ \gamma_5\gamma_i$). This implies invariance under global and local SO(3) transformations (rotations of the fields $\chi_a$ but not boosts). Similarly, $\psi \rightarrow e^{iA\ \theta(t)}\psi'$ is a symmetry, where $A$ can be a constant or the matrix $i\gamma_0\gamma_5$ which commutes with the three Dirac matrices $\gamma_i$. After the transformation $\chi_a \rightarrow e^{im\gamma_0 t}\psi$ the equation is no more invariant under these symmetries and the SO(1,3) symmetry of the Dirac equation is restored along with a global U(1) symmetry.

## IV. PARTITION FUNCTION AS A TRANSFER MATRIX

In the early Universe, a transition from a Lorentz asymmetric to a symmetric phase could possibly induce transformations of the form (2). Let's again consider the simple example in equation (5). For this case the eigenvalues of the Dirac matrix $\gamma_0$ define the wave numbers of the waves traveling in the anisotropic medium. The direction of anisotropy in this case is the temporal direction, which means that the time evolution of these waves is not like usual plane waves. It is not straight forward to visualize the fields, the dynamics of whom are described by the anisotropy of space time, but we can use the partition function method to get a better insight into this. We can, by using this formalism, calculate the temperature at which the transformations in equation (2) occur.

We next perform a transition to a thermodynamics system by making the transformation $it \rightarrow \beta$, where $\beta = 1/k_B T$ [4]. The partition function is then given by the trace of the transformation matrix $e^{im\gamma_0 t}$,

$$Z = \mathrm{Tr}(e^{m\beta\gamma_0}) = 2e^{\beta m} + 2e^{-\beta m}. \tag{6}$$

This partition function is similar to that of a two-level system of spin 1/2 particles localized on a lattice and placed in a magnetic field with each state, in this case, having a degeneracy of two. The lower energy state corresponding to spin parallel to the field ($E = -m, Z_1 = e^{\beta m}$). In this case the doubly degenerate states correspond to spins up and down of the particle or anti-particle. For $N$ distinguishable particles the partition function is $Z^N$, $N$ here is the total number of particles and antiparticles of a particular species. So, we are modeling our system as being on a lattice with the spin along the field as representing a particle and spin opposite to the field representing an antiparticle.

The evolution of this system with temperature represents the time evolution of the system in equation (1). In other words the partition function describes the evolution of these waves from anisotropic to isotropic phase as the temperature decreases. For a two level system the orientation of the dipole moments becomes completely random for large enough temperatures so that there is no net magnetization. In our case we can introduce another quantity, namely a gravitational dipole, which would imply that the four states (particle/antiparticle, spin up/down) of $N$ such particles at high enough temperatures orient themselves in a way that the system is massless. This just serves as an analogy and does not mean that the masses are orientating themselves the same way as dipoles would do in space. The anisotropic character can be seen as mimicking the behavior of the field in a two level system. The population of a particular energy level is given by,

$$n_{p(\overline{p})} = \frac{Ne^{\pm\beta m}}{e^{\beta m} + e^{-\beta m}}. \tag{7}$$

Which shows that the number density of particles and antiparticles vary in a different way with respect to temperature. In the early Universe, therefore the anisotropic character of space-time seems to play an important role such that particles and anti-particles behave in different manners. As the temperature decreases the number density of the anti-particles decreases and is vanishingly small for small temperatures ($\sim e^{-2\beta m}$). When the decoupling temperature is attained there is a difference in the number density of the particles and antiparticles as described by equation (7). This leads to an excess of particles over antiparticles. The decoupling temperature of a particular species of particle with mass $m$ and which is non-relativistic is given by, $k_B T \lesssim 2m$. Below this temperature the particles annihilate to photons but the photons do not have enough energy to produce the pair. This can be used to get the ratio of antiparticles over particles (matter radiation decoupling). For $\beta m \approx 0.5$, we get,

$$\frac{n_p - n_{\overline{p}}}{n_p} \approx 0.6\ . \tag{8}$$

Which implies an excess of particles over antiparticles and thus can serve as another possible way to explain the matter anti-

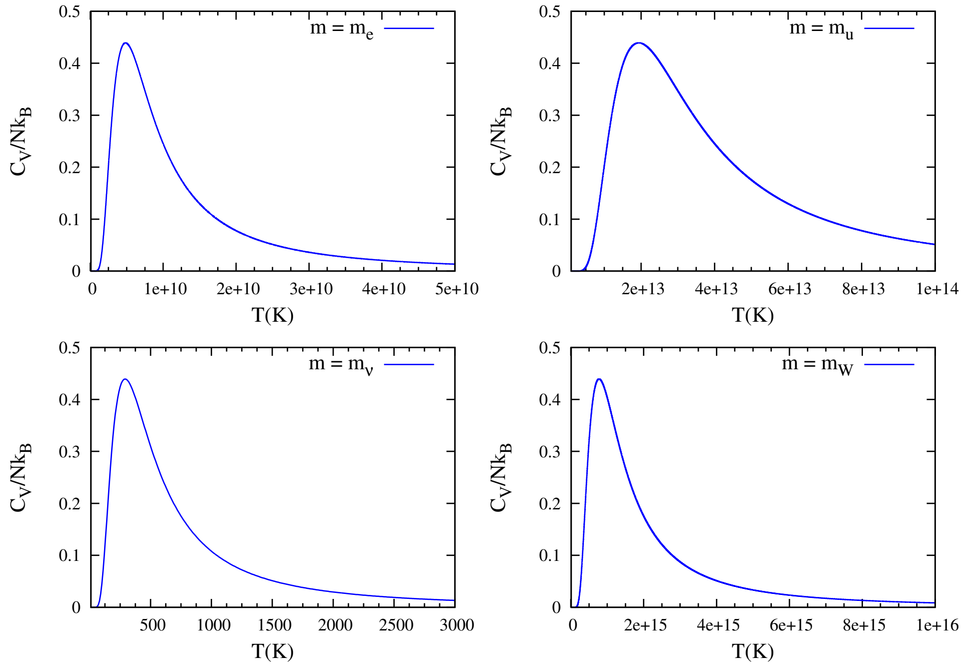


FIG. 1. Plot of heat capacity $C_V$ for the mass of electron, up quark, neutrino and W boson. The maximum of the heat capacity of the electron occurs at $4.8 \times 10^9$K, for the up quarks is $1.9 \times 10^{13}$K, for neutrinos is 291K and for the W bosons is $7.8 \times 10^{14}$K. We use $k_B = 8.6 \times 10^{-5}$ eV/K and $m_\nu = 0.03$ eV.

matter asymmetry of the Universe. This number is very large compared to the one predicted by standard cosmology ($\sim 10^{-9}$). The above expression yields this order for $\beta m \approx 10^{-9}$ which implies a large temperature. For electrons this would imply a temperature of the order $10^{18}$K which is large and the electrons are relativistic. So if we assume that the decoupling takes place at a higher temperature, the baryon asymmetry can be explained. Even without this assumption the conditions proposed by Sakharov can also enhance the number of particles over the antiparticles. Sakharov's conditions involve the interaction dynamics of the fields in the early Universe whereas in our case the statistical system serves more as a model describing the dynamics of space-time to a more ordered phase.

Statistical mechanics, therefore, enables us to visualize this transition in a rather lucid way. In a two level system the net magnetization at any given temperature is analogous to the excess of particles over antiparticles in the early Universe. The time evolution of this anisotropic to isotropic transition is modeled on the evolution of a statistical thermodynamics system with particles on a lattice placed in a magnetic field. The particles on the lattice are localized, static and have no mutual interaction. The free energy of the system is given by:

$$F = -Nk_B T \ln\{4\cosh[\mathrm{m}\beta]\} \tag{9}$$

From this we can calculate the entropy $S$, heat capacity $C_V$ and mean energy $U$ of the system:

$$\begin{aligned} S &= -\left(\frac{\partial F}{\partial T}\right)_V \\ &= Nk_B \ln\{4\cosh[m\beta]\} - mk_B\beta \tanh[m\beta] \end{aligned} \tag{10}$$

$$U = F + TS = -Nm \tanh[m\beta] \tag{11}$$

$$C_V = \left(\frac{\partial U}{\partial T}\right)_V = Nk_B m^2 \beta^2 \,\mathrm{sech}^2[m\beta] \tag{12}$$

In Fig.1, the peaks in the heat capacity represent phase transition of a particular particle species. These are second order phase transitions and the peak in the heat capacity is usually referred to as the Schottky anomaly [5]. Note that the phase transition we model our system on is a magnetic one. So, modeling the complex system in the early Universe on a lattice with spin 1/2 particles can reduce the complications of the actual system by a considerable amount.

The Schottky anomaly of such a magnetic system, therefore, represents phase transitions in the early Universe. For a particular species of particles the Schottky anomaly shows a peak around $mc^2 \approx kT$. The phase transition for the electrons occurs at the

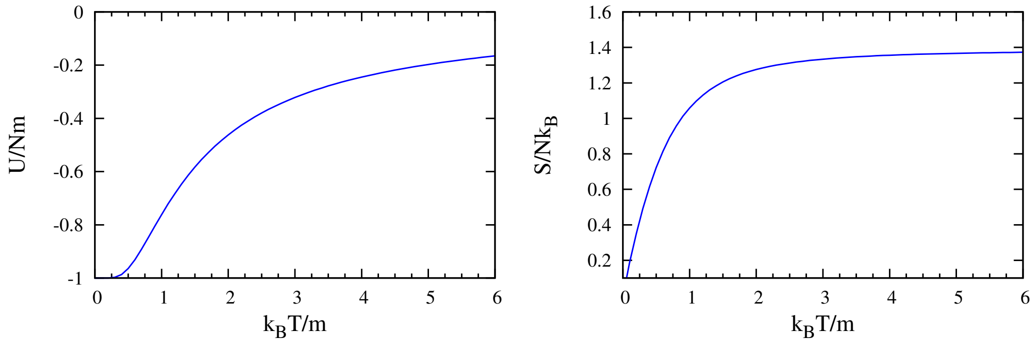


FIG. 2. Plot of entropy and energy for a particle of mass m. For large enough temperatures the energy of the system approaches zero and the entropy approaches the limiting value of $Nk_B\ln 4$.

temperature where nuclei start forming in the early Universe. For the quarks the transition temperature refers to confinement into protons and neutrons. Similarly, W boson's transition occurs at the electroweak breaking scale. The W boson, being a spin 1 particle, is not described by the Dirac equation, but the heat capacity entails this feature of showing a phase transition for the energy scale relevant to the mass of a particle.

The curve for neutrinos implies that the transition temperature for neutrinos is around 291 K, which means that the density of antineutrinos from the big bang for present neutrino background temperatures ($\sim$ 2 K) is not negligible. The ratio of antineutrinos over neutrinos for $T = 2$ K, is $n_{\overline{\nu}}/n_\nu \sim 10^{-15000}$ ($m_\nu = 2$ eV) and for an even lower neutrino mass $m_\nu = 0.1$ eV the ratio is $n_{\overline{\nu}}/n_\nu \sim 10^{-500}$, which for other more massive particles is much smaller. A cosmic neutrino and antineutrino background is one of the predictions of standard cosmology but is still unobserved. This model predicts an antineutrino background much less than the neutrino one.

In Fig.2, the plots of mean energy and entropy are shown in dimensionless units. In the massless limit for fermions the entropy attains its maximum value of $Nk_B\ln 4$. The plots show that the energy of the system approaches zero as the temperature approaches infinity. This situation is analogous to the spins being completely random at high temperatures for the two level system. The same way that the magnetic energy of the system on the lattice is zero at high temperatures, the mass of this system is zero in the very early Universe. As the temperature decreases the energy of the system attains it minimum value ($U = -Nm$) and the particles become massive at the temperature less than the value given by the peak of the heat capacity. The entropy for high temperatures asymptotically approaches its maximum value of $Nk_B\ln 4$.

According to the statistical thermodynamics model that describes this transition, as this phase transition occurs antiparticles will start changing into particles and as can be seen from the figure the system will move towards all spins aligned parallel with the "field", i.e., towards being particles. From Fig.2 we can see that the energy of the system starts attaining the minimum value as the temperature decreases where all particles are aligned with the field and are "particles". The plot of entropy vs. temperature also represents an important feature of these transformations. The entropy decreases with decreasing temperature and this represents the transition to a more ordered phase using equations (2). The plots of energy of the system U in Fig.2 show that the system will eventually settle down to the lowest energy state which in this case means that the system will have almost all particles with negligible number of antiparticles. In short, the plot of the heat capacity reflects the phase transitions, the plot of energy U represents the transition from massless to massive states and the plot of entropy represents the transition of space time to a more ordered phase.

Finally, we would like to point out that the occurrence of the Schottky anomaly has motivated the study of negative temperatures [6]. Note that the partition functions is invariant under the transformation $T \to -T$ but the equations for the free energy, entropy and energy are not. The existence of negative temperatures has been observed in experiments. Negative temperatures, for example, can be realized in a system of spins if the direction of the magnetic field is suddenly reversed for a system of spins initially aligned with the magnetic field [5]. Similarly, as described in reference [6] the allowed states of the system must have an upper limit. Whereas this is not the case for the actual particles in the early Universe, the statistical mechanics system on which it can be modeled on has this property. A negative temperature system would eventually settle down to the lower energy state ($U = Nm$) which in our case would mean that the Universe would ends up having more antiparticles than particles. This is yet another interesting insight we get by modeling the early Universe on a two state system.

## V. CONCLUSIONS

We analyzed transformations that restore the Dirac equation to its covariant form from an underlying theory that violates Lorentz invariance. These transformations, we suggest, can be interpreted as waves traveling in an anisotropic medium. The partition function formalism then, enabled us to model these transformations on the evolution of a system of spin 1/2 particles on a lattice placed in a magnetic field. Symmetry breaking in this case takes place in this lattice, the partition function of which characterizes the transition. We showed that this model can describe the anisotropic to isotropic phase transitions in the early Universe. Three important features of the early Universe are depicted in this model: (1) The heat capacity shows the occurrence of phase transitions. (2) The mean energy of the system shows how the particles become massive from being massless. (3) The plot of entropy shows that the transition to a Lorentz symmetric phase occurred from an asymmetric one. At any given temperature the net magnetization measures the excess of particles over antiparticles. We then suggest that this model can be used to explain the matter antimatter asymmetry of the Universe.

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