A Grothendieck Universe:
Clifford Algebra Cl(16) E8 AQFT

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Abstract:

The completion of the union of all tensor products of the real Clifford algebra Cl(16)

is proposed as a Grothendieck Universe

giving a Category Theoretical description of a realistic Algebraic Quantum Field Theory

that might clarify its relationship with Path Integral Quantization

of Standard Model + Gravity Lagrangian Physics.

At present, this paper is an outline of a proposed program of research

that is not yet complete.

(References are included in the body of the paper and in linked material.)

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Daniel Murfet (Foundations for Category Theory 5 October 2006) said:
"... The most popular form of axiomatic set theory is Zermelo-Frankel (ZF) together with the Axiom of Choice (ZFC) ... this is not enough, because we need to talk about structures like the "category of all sets" which have no place in ZFC ...[more useful foundations include]
...
(a) An alternative version of set theory called NBG (due to von Neumann, Robinson, Bernays and Godel) which introduces classes to play the role of sets which are "too big" to exist in ZF
...
(b) Extend ZFC by adding a new axiom describing Grothendieck universes. Intuitively speaking, you fix a Grothendieck universe U and call elements of U sets, while calling subsets of U classes. ... This ... seems to be the only serious foundation available for modern research involving categories
...
(c) The first two options [(a) and (b)] are conversative, in that they seek to extend set theory by as little as possible to make things work. More exotically, we can introduce categories as foundational objects. This approach focuses on topoi as the fundamental logical objects (as well as the connection with the more familiar world of naive set theory). While such a foundation shows promise, it is not without its own problems ... and is probably not ready for "daily use".
...
Before we study Grothendieck universes, let us first agree on what we mean by ZFC. The first order theory ZFC has two predicate letters A,B but no function letter, or individual constants. Traditionally the variables are given by uppercase letters X1,X2, . . . (As usual, we shall use X, Y,Z to represent arbitrary variables).
We shall abbreviate A(X, Y) by X in Y and B(X, Y) by X = Y .
Intuitively e is thought of as the membership relation and the values of the variables are to be thought of as sets (in ZFC we have no concept of "class").
The proper axioms are as follows (there are an infinite number of axioms since an axiom scheme is used):

Axiom of Extensionality Two sets are the same if and only if they have the same elements ...
Axiom of Empty Set There is a set with no elements. By the previous axiom, it must be unique ...
Axiom of Pairing If x, y are sets, then there exists a set containing x, y as its only elements, which we denote {x, y}. Therefore given any set x there is
a set {x} = {x, x} containing just the set x ...
Axiom of Union For any set x, there is a set y such that the elements of y are precisely the elements of the elements of x ...
Axiom of Infinity There exists a set x such that the empty set is in x and whenever y is in x, so is y union {y} ...
Axiom of Power Set Every set has a power set. That is, for any set x there exists a set y, such that the elements of y are precisely the subsets of x ...
Axiom of Comprehension Given any set and any ... well formed formula ...
wf B(x) with x free, there is a subset of the original set containing precisely those elements x for which B(x) holds (this is an axiom schema) ... Here we make the technical assumption that the variables A,B,C do not occur in B ...
Axiom of Replacement Given any set and any mapping, formally defined as a wf B(x, y) with x, y free such that B(x, y1) and B(x, y2) implies y1 = y2, there is a set containing precisely the images of the original set's elements (this is an axiom schema) ...
Axiom of Foundation A foundation member of a set x is y in x such that
y intersect x is empty. Every nonempty set has a foundation member ...
Axiom of Choice Given any set of mutually disjoint nonempty sets, there exists at least one set that contains exactly one element in common with each of the nonempty sets.
...
Looking at the axioms, only the Axiom of Replacement can produce a set outside our universe (beginning with sets inside the universe), although one could argue that the Axiom of Infinity also "produces" the set N, which may not belong to U .
To get around the latter difficulty, we add the following axiom to ZFC ...
UA. Every set is contained in some universe ...
UA is equivalent to the existence of inaccessible cardinals,
and is therefore logically independent of ZFC
...
[This gives]... The first order theory ZFCU ...

Grothendieck Universes

Whatever foundation we use for category theory, it must somehow provide us with a notion of “big sets”. In Grothendieck’s approach, one fixes a particular set U (called the universe) and thinks of elements of U as “normal sets”, subsets of U as “classes”, and all other sets as “unimaginably massive”.
...
Definition 3. A Grothendieck universe (or just a universe) is a nonempty set U with the following properties:

U1. If x in U and y in x then y in U (that is, if x in U then x subset U).

U2. If x, y in U then {x, y} in U .

U3. If x in U , then ... power set ... P(x) in U .

U4. If I in U and {xi}_(i in I) is a family of elements of U ,
then the union over i in I of the xi belongs to U .
...

Therefore
any finite union, product and disjoint union of elements of U belongs to U .
In particular every finite subset of U belongs to U ...
by our convention U contains N, and therefore also Z,Q,R,C and
all structures built from these using the theory of sets ...”.

The Wikipedia article on Grothendieck universe said:
“... The idea of universes is due to Alexander Grothendieck,
who used them as a way of avoiding proper classes ...
There are two simple examples of Grothendieck universes:

The empty set,
and
The set of all hereditarily finite sets ... “.

Another Grothendieck universe describes a realistic E8 AQFT.
It is the completion of the union of all tensor products of Cl(16)
which I will denote as
UCl16

and which may be extraordinarily useful in using category techniques
to understand math and physics.

Since the real Clifford algebra Cl(16) = Cl(Cl(4)) = Cl(Cl(Cl(Cl(Cl(0)))))
it can be constructed by iterating the Clifford construction beginning with a point.
Since Cl(16) = Cl(8)xCl(8)
it can be used to construct, by 8-periodicity, any arbitrarily large Cl(m).

UCl16 is a hyperfinite von Neumann factor algebra, being
a real generalization of the usual complex hyperfinite II1 von Neumann factor.

Further, UCl16 inherits from its Cl(16) factors some structures that are useful
in areas including, but not limited to, physics model building
which structures can be seen in Category Theoretical terms:

- Vectors
- BiVector Lie Algebras and Lie Groups
  - Symmetric Spaces
  - Complex Domains, their Shilov boundaries, and Harmonic Analysis
- Spinors with Fermion properties
- E8 Lie Algebra
- Sl(8)xH92 Algebra (Contraction of E8)

Some other Categories useful with respect to physics model building are:

- Classical Physics Lagrangian
  - Lagrangian Spinor Fermion term
  - Lagrangian Base Manifold
  - Lagrangian MacDowell-Mansouri Gravity term
  - Lagrangian Standard Model Gauge Boson term

- Quantum Physics Hamiltonian/Heisenberg Algebra
  - Position/Momentum
  - Gravity + SM boson Creation/Annihilation
  - Fermion Creation/Annihilation

With respect to those Categories, there exist Functors

Cl(16) -> E8 -> Classical Physics Lagrangian
and
Cl(16) -> Sl(8)xH92 -> Quantum Physics Hamiltonian/Heisenberg Algebra

defined by

Cl(16) -> E8 and
E8 128 Spinors -> Lagrangian Spinor Fermion term
E8 64 Position/Momentum -> Lagrangian Base Manifold
E8 28 D4 Gravity -> Lagrangian M-M Gravity term
E8 28 D4 Standard Model -> Lagrangian SM Gauge Boson term

and

Cl(16) -> Sl(8)xH92 = Sl(8)xH(28+64) and
Sl(8) -> Position/Momentum
H28 -> Gravity + SM boson Creation/Annihilation
H64 -> Fermion Creation/Annihilation

Therefore **Path Integral quantization of Classical Physics Lagrangian** has a Category Theoretical relationship with **Quantum Physics Hamiltonian/Heisenberg Algebraic Quantum Field Theory** that may show a Categorification of Lagrangian Path Integral that is more directly related to the Standard Model + Gravity than the Chern-Simons theory whose Path Integral Quantization to a Topological Quantum Field Theory is described by Daniel Freed in Bull. AMS 46 (2009) 221-254.

Details of the physics structures mentioned above can be found in my paper Introduction to E8 Physics that is on the web at these URLs:
http://vixra.org/abs/1108.0027
http://www.valdostamuseum.org/hamsmith/E8physics2011.pdf
http://www.tony5m17h.net/E8physics2011.pdf