A Kaluza-Cartan Theory

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Abstract

Kaluza's 1921 theory of gravity and electromagnetism using a fifth wrapped-up spatial dimension is at the root of many modern attempts to develop new physical theories. Lacking important electromagnetic fields however, and having other problems, the theory is incomplete and generally considered untenable. An alternative is presented that includes torsion. Coulomb's law in the form of the Lorentz force law is investigated starting with a non-Maxwellian definition of charge, this is shown to be related to Maxwellian charge. It is concluded that Kaluza's 5D space and torsion should go together in what is here called a Kaluza-Cartan theory in order to form a unified theory of gravity and electromagnetism. A new cylinder condition is proposed that takes torsion into account and is fully covariant. Divergence from existing theories suggests experimental tests may be possible. Two connections are used, one with, and one without torsion. The theory explores the concept of general covariance with respect to global properties that can be modelled via a non-maximal atlas.

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1 Conventions

The following conventions are adopted unless otherwise specified:

Five dimensional metrics, tensors and pseudo-tensors are given the hat symbol. Five dimensional indices, subscripts and superscripts are given capital Roman letters. So for example the five dimensional Ricci flat 5-dimensional superspace-time of Kaluza theory is given as: $\hat{g}_{AB}$, all other tensors and indices are assumed to be 4 dimensional, if a general non-specified dimensional case is not being considered, for which either convention can be used. Index raising is referred to a metric $\hat{g}_{AB}$ if 5-dimensional, and to g_{ab} if 4-dimensional. The domain of partial derivatives carries to the end of a term without need for brackets, so for example we have $\partial_a g_{db} A_c + g_{db} g_{ac} = (\partial_a (g_{db} A_c)) + (g_{db} g_{ac})$. Terms that might repeat dummy variables or are otherwise in need of clarification use additional brackets. Square brackets can be used to make dummy variables local in scope.

Space-time is given signature $(-, +, +, +)$, Kaluza space $(-, +, +, +, +)$ in keeping with [6], except where stated and an alternative from [1] is referred to. Under the Wheeler et al [6] nomenclature, the sign conventions used here as a default are $[+, +, +]$. The first dimension (index 0) is always time and the 5^{th} dimension (index 4) is always the topologically closed Kaluza dimension. Universal constants defining physical units: c = 1, and $\mathbf{G}$ as a constant. The scalar field component is labelled ϕ^2 (in keeping with the literature) only as a reminder that it is associated with a spatial dimension, and to be taken as positive. The matrix of g_{cd} can be written as $|g_{cd}|$ when considered in a particular coordinate system to emphasize a component view. The Einstein summation convention may be used without special mention.

Some familiar defining equations consistent with [1] (using Roman lower-case for the general case only for ease of reference):

$$R_{ab} = \partial_c \Gamma^c_{\ ba} - \partial_b \Gamma^c_{\ ca} + \Gamma^c_{\ ba}\Gamma^d_{\ dc} - \Gamma^c_{\ da}\Gamma^d_{\ bc} \tag{1.0.1}$$

$$G_{ab} = R_{ab} - \frac{1}{2}Rg_{ab} = 8\pi\mathbf{G}T_{ab} \tag{1.0.2}$$

For convenience we will define $\alpha = \frac{1}{8\pi\mathbf{G}}$.

$$F_{ab} = \nabla_a A_b - \nabla_b A_a = \partial_a A_b - \partial_b A_a \text{ equally } F = dA \tag{1.0.3}$$

Any 5D exterior derivatives and differential forms could also be given a hat, thus: $\hat{d}\hat{B}$. However, the primary interest here will be 4D forms. $\Box$ represents the 4D D'Alembertian.

Torsion introduces non-obvious conventions in otherwise established definitions. The order of the indices in the Christoffel symbols comes to matter, and this includes in the Ricci tensor definition and the definition of the Christoffel symbols themselves:

$$\nabla_a w_b = \partial_a w_b - \Gamma^c_{\ ab} w_c \tag{1.0.4}$$

Christoffel symbols in general will take the usual form: $\Gamma^c_{\ ab}$ or Γ^{abc} and so on. However, often we will need to distinguish a with-torsion Christoffel symbol from a without-torsion Christoffel symbol in some way. In the completely antisymmetric case, when the torsion tensor is completely anti-symmetric, we can write the without-torsion connection coefficients as: $\Gamma^c_{\ (ab)}$. But greater generality to specify the Levi-Civita connection at all times may call for the more explicit: $\{^{\,c}_{ab}\}$. More conveniently we may also refer to the Levi-Civita connection coefficients using: $\digamma^c_{\ ab}$, and a covariant derivative operator: $\triangle_a$. In order to distinguish general G_{ab} and R_{ab} etc. from the case where the torsion has been explicitly excluded from the definition we use cursive: $\mathcal{G}_{ab}$ and $\mathcal{R}_{ab}$.

2 Introduction

Kaluza's 1921 theory of gravity and electromagnetism [2][3][4] using a fifth wrapped-up spatial dimension is at the heart of many modern attempts to develop new physical theories [1][5]. From supersymmetry to string theories topologically closed small extra dimensions are used to characterize the various forces of nature. It is therefore at the root of many modern attempts and developments in theoretical physics. However it has a number of foundational problems and is often considered untenable in itself. This paper looks at these problems from a purely classical perspective, without involving quantum theory. This perhaps runs against the grain for modern physics due to the great success of quantum mechanics, but is nevertheless worth doing as an independent enterprise.

The theory assumes a (1,4)-Lorentzian Ricci flat manifold to be the underlying metric, split (analogously to the later ADM formalism) as follows:

$$\hat{g}_{AB} = \begin{bmatrix} g_{ab} + \phi^2 A_a A_b & \phi^2 A_a \\ \phi^2 A_b & \phi^2 \end{bmatrix} \tag{2.0.1}$$

Note that a common scaling factor has been set to $k = 1$ and so is not present, this will be reintroduced. By inverting this metric as a matrix (readily checked by multiplication) we get:

$$\hat{g}^{AB} = |\hat{g}_{AB}|^{-1} = \begin{bmatrix} g^{ab} & -A^a \\ -A^b & \frac{1}{\phi^2} + A_i A^i \end{bmatrix} \tag{2.0.2}$$

Maxwell's law are automatically satisfied: dF=0 follows from dd = 0. d*F= 4π*J can be set by construction. d*J=0, conservation of charge, follows also by dd=0 on most parts of the manifold.

However, in order to write the metric in this form there is a subtle assumption, that g_{ab}, which will be interpreted as the usual four dimensional space-time metric, is itself non-singular. However, this will always be the case for moderate or small values of A_x which will here be identified with the electromagnetic 4-vector potential. The raising and lowering of this 4-vector are defined in the obvious way in terms of g_{ab}. The 5D metric can be represented at every point on the Kaluza manifold in terms of this 4D metric g_{ab} (when it is non-singular), the vector potential A_x, and the scalar field ϕ^2. We have also assumed that topology is such as to allow the Hodge star to be defined. This means that near a point charge source the argument that leads to charge conservation potentially breaks down as the potential may cease to be well-defined. Whereas the Toth charge that will be defined in the sequel does not have this problem. So two different definitions of charge are given: the Maxwellian, and the Toth charge.

With values of ϕ^2 around 1 and relatively low 5-dimensional metric curvatures, we need not concern ourselves with this assumption beyond stating it on the basis that physically these parameters encompass tested theory. Given this proviso A_x is a vector and ϕ^2 is a scalar - with respect to the tensor system defined on any 4-dimensional submanifold that can take the induced metric g.

Kaluza's cylinder condition (KCC, or original KCC) is that all partial derivatives in the 5th dimension i.e. ∂_4 and $\partial_4\partial_4$ etc... of all metric components and of all tensors and their derivatives are zero. A perfect 'cylinder'. Here we extend it to torsion terms, and indeed all tensors and pseudo-tensors. This leads to constraints on g_{ab} given in [1] by three equations, the field equations of Kaluza theory, where the Einstein-Maxwell stress-energy tensor can be recognised embedded in the first equation:

$$G_{ab} = \frac{k^2\phi^2}{2}\left\{\frac{1}{4}g_{ab}F_{cd}F^{cd} - F^c_a F_{bc}\right\} - \frac{1}{\phi}\{\nabla_a(\partial_b\phi) - g_{ab}\Box\phi\} \tag{2.0.3}$$

$$\nabla^a F_{ab} = -3\frac{\partial^a\phi}{\phi}F_{ab} \tag{2.0.4}$$

$$\Box\phi = \frac{k^2\phi^3}{4}F_{ab}F^{ab} \tag{2.0.5}$$

Note that there is both a sign difference and a possible factor difference with respect to Wald [7] and Wheeler [6]. The sign difference appears to be due to the mixed use of metric sign conventions in [1] and can be ignored. A k factor is present and scaling will be investigated. These will be referred to as the first, second and third torsionless field equations, or original field equations, respectively. They are valid only in Kaluza vacuum, that is, outside of matter-charge models, and when there is no torsion. This require Kaluza's original cylinder condition.

Definition 2.0.6: Perpendicular electromagnetic solutions.

Field for which the following equation hold will be called perpendicular electromagnetic fields, and likewise those that do not satisfy this: non-perpendicular. Null solutions are perpendicular solutions with a further constraint. But perpendicular solutions will be of most interest here.

$$F_{ab}F^{ab} = 0 \tag{2.0.6}$$

By looking at field equation 3 it can be seen that if the scalar field does not vary then only a limited range of solutions result, that have perpendicular electric and magnetic fields. Eg null solutions. The second field equation then also imposes no charge sources. Here the scalar term could be allowed to vary in order to allow for non-zero $F_{ab}F^{ab}$. This falls within Kaluza's original theory. This potentially allows for more electromagnetic solutions, but there are problems to overcome: the field equations cease being necessarily electrovacuum. One inadequate and arbitrary fix is to set the scalar field term large, as is sometimes done to ensure that field equation 2 is identically zero despite scalar fluctuations. This approach will not be taken here. The stress-energy tensor under scalar field fluctuations is different from the Einstein-Maxwell tensor [6][7] and the accepted derivation of the Lorentz force law (for electrovacuums [6]) can not be assumed. A variable scalar field also implies non-conservation of Maxwell charge via field equation 3. Attempts to loosen constraints such as the KCC have also not been successful so far.

Another foundational issue of Kaluza theory is that even with a scalar field it does not have convincing sources of mass or charge built in. Field equation 2 has charge sources, but it's unlikely that realistic sources are represented by this equation. Matter (and charge) models in this work will be assumed to be regions of the Kaluza space that are *not* Ricci flat in the otherwise Ricci flat Kaluza space (leaving open for the moment which Ricci tensor exactly is to be used), just as matter/energy is analogously assumed to be in general relativity. That is, where the 5D Einstein tensor of the Kaluza space itself is non-zero.

As mentioned charge will be given a possible alternative definition, Toth charge: as 5-dimensional momentum, following a known line of reasoning [8] within Kaluza theory. This will enable a derivation of Coulomb's law, via the Lorentz force law - leading to a genuine mathematical unification of electromagnetism and gravity. As momentum, the Toth charge is of necessity locally conserved, provided there are no irregularities in the topology of the Kaluza 5th dimension. Similarly the conservation of Maxwellian charge is normally guaranteed by the existence of the potential, except that this may not be valid in extreme curvatures where the values here associated with the 4-potential may cease to be a vector.

We will also assume global hyperbolicity in the sense of the existence of a Cauchy surface as is often done in general relativity to ensure 4 dimensional causality. Though this will not necessarily guarantee 5D causality in the event that either KCC is weakened or the theory presented here breaks down due to singularities. It also requires the usual energy conditions on the resulting 4D space-time and fields, or a 5D equivalent generalization.

One leading issue is that Kaluza theory offers limited electromagnetic solutions. Non-perpendicular electrovacuums more generally are not so easily supported as changes in the scalar field may force divergence of the field equations from those of the electrovacuum (see field equation 1). This will lead to the potential failure of the Lorentz Force Law, in effect Coulomb's law. The Lorentz force law/Coulomb's law is to be derived from this theory independently of the electrovacuum solutions of general relativity. Note that in addition the derivation of the Lorentz force law within general relativity (from an assumed Einstein-Maxwell stress-energy tensor) is not without problems of principle [6]. But more importantly the stress-energy tensor that defines the electrovacuum geometry has to be assumed in classical electrodynamics within general relativity, whereas in this Kaluza-Cartan theory it is not, the Lorentz force law follows from different considerations. Thus, the program is to seek a link between the Toth charge and Maxwellian charge, so that they might be sufficiently interchangeable - as this can then be shown to lead to an alternative explanation for the Lorentz force law. To obtain the sought for range of electromagnetic solutions a particular constraint will be weakened: that the torsion tensor is necessarily vanishing. Torsion will be allowed to vary allowing greater scope for solutions. The Lorentz force law, rather than electrovacuum solutions, will be sought for Toth charges.

Some useful Christoffel symbol terms are placed in the Appendix: equations (9.1.1) onwards. And a number of other detailed workings are placed in the Appendix too. A considerable preamble is required and presented in the next two sections: Preliminary Notes, and The Geometry of Torsion.

3 Preliminary Notes

3.1 Kaluza Cylinder Condition And Charge

Throughout this work the limit of a new Kaluza Cylinder Condition (new KCC, or new cylinder condition) will be taken to be that the covariant derivative of all tensors in the Kaluza direction are to be zero, and that it depends on torsion. The 5D metric generally decomposes into 4D metric, vector potential and scalar field, at least when the embedded 4D metric is non-singular.

The new KCC by construction must be true for a subatlas covering the 5D Lorentzian manifold. But charts may exist in the maximal atlas for which these constraints are not possible. The atlas that is compliant is restricted. This means that the new KCC can be represented by a subatlas of the maximal atlas for the manifold. The set of local coordinate transformations that are compliant with this atlas (call it the Kaluza atlas) is non-maximal by design.

A further reduction in the atlas is also implied by setting c=1, and constant **G**. The Kaluza dimension as spatial. Rescaling a spatial axis would require also rescaling a time axis in order not to change c, and hence the other spatial axes would need rescaling. The existence of a single unit for space and time can be assumed, and this must be scaled in unison for all dimensions. Consistently with cgs units we can choose either centimetres or seconds. This would leave velocities (and other geometrically unitless quantities) unchanged in absolute magnitude. This doesn't prevent reflection of an axis however, and indeed reflection of the Kaluza dimension will be equivalent to a charge inversion. However, given an orientation we can also remove this.

Space-time can not be an arbitrary 4D Lorentz submanifold as it must be one that is normal to a Kaluza axis and that satisfies certain constraints. This will generally have the interpretation best visualized as a cylinder with a longitudinal space-time and a perpendicular Kaluza dimension.

We can further reduce the Kaluza atlas by removing boosts in the Kaluza dimension. Why? This requires the new Kaluza cylinder condition, postulate K7 later, which significantly reduces the possible geometries. Space-time will be taken to be a subframe within a 5D frame within the Kaluza subatlas wherein uncharged matter can be given a rest frame via a 4D Lorentz transformation. Boosting uncharged matter along the Kaluza axis will give it kinetic Toth charge.

The Kaluza atlas represents the 4D view that charge is 4D covariant. Here we require that the Toth charge coincides with Maxwellian charge in some sense. The justification for this assertion will be clarified later. Rotations into the Kaluza axis can likewise be omitted. These result in additional constraints on the Christoffel symbols associated with charts of this subatlas, and enable certain geometrical objects to be more easily interpreted in space-time. The use of this subatlas does not prevent the theory being generally covariant, but simplifies the way in which we look at the Kaluza-space through a 4D physical bias.

Preliminary Description 3.1.1: The Kaluza-Cartan space-time.

In the most general case a Kaluza-Cartan space-time will be a 5D Lorentz manifold with metric and metric torsion connection. A new Kaluza cylinder condition will be added (see postulate K7 later) and the topology of the Kaluza dimension will be closed and geometrically small. A global Kaluza direction will be defined as normal (relative to the new cylinder condition) to a 4D Lorentz submanifold. That submanifold, and all parallel submanifolds as a set, will constitute space-time. Charged particles will be those that are not restricted in movement to within space-time. The new cylinder condition will ensure that all the parallel space-times are equivalent.

A complete definition is given later in postulates K1-K7.

Definition 3.1.1: The Kaluza atlas.

(i) The Kaluza atlas is a subatlas of the maximal atlas of Kaluza-Cartan space-time, given geometrized units.

(ii) Boosts and rotations into the Kaluza dimension (as defined by the new cylinder condition) are explicitly omitted.

(iii) This represents the physical interpretation of charge as a covariant property of space-time even if it is not a covariant property of the 5D Kaluza-Cartan space.

(iv) Mathematically this is also an atlas of charts for which the partial derivatives of tensors and pseudo-tensors in the Kaluza dimension vanish.

(3.1.1)

Kinetic Toth charge is defined as the 5D momentum component in terms of the 5D Kaluza rest mass of a hypothesised particle: ie (i) its rest mass in the 5D Lorentz manifold (m_{k0}) and (ii) its proper Kaluza velocity ($dx_4/d\tau^*$) with respect to a frame in the maximal atlas that follows the particle. And equally it can be defined in terms of (i) the relativistic rest mass (m_0), relative to a projected frame where the particle is stationary in space-time, but where non-charged particles are stationary in the Kaluza dimension, and in terms of (ii) coordinate Kaluza velocity (dx_4/dt_0):

Definition 3.1.2: Toth charge (scalar).

$$Q^* = m_{k0}dx_4/d\tau^* = m_0 dx_4/dt_0 \tag{3.1.2}$$

This makes sense because mass can be written in fundamental units (i.e. in distance and time). And the velocities in question defined relative to particular frames. It is not a generally covariant definition but it is nevertheless mathematically meaningful. In the appendix (9.4.1) it is shown that this kinetic Toth charge can be treated in 4D Space-Time, and the Kaluza atlas, as a scalar: the first equation above is covariant with respect to the Kaluza atlas. It can be generalized to a 4-vector as follows, and it is also conserved:

In general relativity at the special relativistic Minkowski limit the conservation of momentum-energy/stress-energy can be given in terms of the stress-energy tensor as follows [9]:

$$\frac{\partial \hat{T}^{00}}{\partial t} + \frac{\partial \hat{T}^{i0}}{\partial x_i} = 0 \tag{3.1.3}$$

Momentum in the j direction:

$$\frac{\partial \hat{T}^{0j}}{\partial t} + \frac{\partial \hat{T}^{ij}}{\partial x_i} = 0 \tag{3.1.4}$$

This is approximately true at a weak field limit and can be applied equally to Kaluza theory, in the absence of torsion. We have a description of conservation of momentum in the 5th dimension as follows:

$$\frac{\partial \hat{T}^{04}}{\partial t} + \frac{\partial \hat{T}^{i4}}{\partial x_i} = 0 \tag{3.1.5}$$

We also have i=4 vanishing by KCC. Thus the conservation of Toth charge becomes (when generalized to different space-time frames) the property of a 4-vector current, which we know to be conserved:

$$(\hat{T}^{04}, \hat{T}^{14}, \hat{T}^{24}, \hat{T}^{34}) \tag{3.1.6}$$

$$\partial_0 \hat{T}^{04} + \partial_1 \hat{T}^{14} + \partial_2 \hat{T}^{24} + \partial_3 \hat{T}^{34} = 0 \tag{3.1.7}$$

As in relativity this can be extended to a definition that is valid even when there is curvature. Nevertheless the original Toth charge definition (3.1.2) has meaning in all Kaluza atlas frames as a scalar.

Kinetic Toth charge current is the 4-vector, induced from 5D Kaluza-Cartan space as follows (using the Kaluza atlas to ensure it is well-defined as a 4-vector):

$$J^{*a} = -\alpha \hat{G}^{a4} \tag{3.1.8}$$

Noting that,

$$\hat{\nabla}_A \hat{G}^{AB} = \hat{\nabla}_a \hat{G}^{a4} = 0 \tag{3.1.9}$$

Using Wheeler et al [6] p.131, and selecting the correct space-time (or Kaluza atlas) frame, we have:

$$Q^* = J^*_a (1, 0, 0, 0)^a \tag{3.1.10}$$

So we have a scalar, then a vector representation of relativisitic invariant charge current, and finally a 2-tensor unification with mass-energy.

In the sequel torsion will be introduced. But this will not prevent the Levi-Civita connection having meaning and application on the same manifold contemporaneously. Kinetic Toth charge is defined in the same way even when torsion is present - via the Einstein tensor without torsion, and applying conservation of mass-energy relative to the torsionless connection. The new cylinder condition is defined, on the other hand, using the *torsion* connection. Distinction is therefore necessary between connections.

The definition of kinetic charge and the conservation law of mass-energy-charge need to be written in the appropriate form when torsion is used on the same manifold:

Definition 3.1.11: Toth charge current.

Toth charge current is defined to be the 4-vector $J^{*a} = -\alpha \hat{\mathcal{G}}^{a4}$, with respect to the Kaluza atlas, and noting:

$$\hat{\triangle}_a \hat{\mathcal{G}}^{aB} = 0 \tag{3.1.11}$$

3.2 Geometrized Units Of Mass

The metric components used in [1] as the 5D Kaluza metric, defined in terms of the original KCC follow. It will be equally used here in its new context, where the geometry of this space will depend on the new KCC defined in terms of a metric torsion tensor. It is called here the Kaluza-Cartan metric to remind us of this context.

Definition 3.2.1: The 5D Kaluza-Cartan metric.

$$\hat{g}_{AB} = \begin{bmatrix} g_{ab} + k^2\phi^2 A_a A_b & k\phi^2 A_a \\ k\phi^2 A_b & \phi^2 \end{bmatrix} \tag{3.2.1}$$

This gives inverse as follows:

$$\hat{g}^{AB} = |\hat{g}_{AB}|^{-1} = \begin{bmatrix} g^{ab} & -kA^a \\ -kA^b & \frac{1}{\phi^2} + k^2 A_i A^i \end{bmatrix} \tag{3.2.2}$$

This gives (with respect to space-time) perpendicular solutions [1] under the original KCC, such that $G_{ab} = -\frac{k^2}{2}\ F_{ac}F^c_b$. Compare this with [7] where we have $G_{ab} = 2F_{ac}F^c_b$ in geometrized units we would need to have $k = 2$ or $k = -2$ for compatibility of results and formulas. Noting the sign change introduced by [1] - where it appears that the Einstein tensor was defined relative to (+, −, −, −), despite the 5D metric tensor being given in a form that can only be (−, +, +, +, +), which is confusing. Essentially the same result, but with consistent sign conventions, is achieved here in (6.2.4).

The geometrized units, Wald [7], give a relation for mass in terms of fundamental units as follows:

$$\mathbf{G} = 1 = 6.674 \times 10^{-8} cm^3 g^{-1} s^{-2} = 6.674 \times 10^{-8} cm^3 g^{-1} \times (3 \times 10^{10} cm)^{-2}$$
$$= 6.674 \times 10^{-8} cm g^{-1} \times (3 \times 10^{10})^{-2}$$

$$1g \approx 7.42 \times 10^{-29} cm \text{ for c=1, } \mathbf{G}\text{=1} \tag{3.2.3}$$

$$1g \approx G/c^2 cm \text{ for c=1, } \mathbf{G}\text{=1} \tag{3.2.4}$$

$\mathbf{G}$ and k will not be independent however. If we fix one the other is fixed too, as a consequence of requiring the Lorentz force law written in familiar form. The relation between $\mathbf{G}$ and k is given later in equation (7.4.5). Simple compatibility with Wald [7], where $k = 2$ and $\mathbf{G} = 1$, results. The sign of k is also fixed by (6.6.5).

For $k = 2$, c=1, $\mathbf{G}$=1 we have:

$$1statC = 1cm^{3/2} s^{-1} g^{1/2} = cm^{1/2} \times (7.42 \times 10^{-29} cm)^{1/2} / (3.00 \times 10^{10})$$

$$= 8.61 \times 10^{-15} cm/(3 \times 10^{10}) \approx 2.87 \times 10^{-25} cm \approx 3.87 \times 10^3 g \tag{3.2.5}$$

Using cgs (Gaussian) units and the cgs versions of $\mathbf{G}$ and c, ie $\mathbf{G} = 6.67 \times 10^{-7} cm^3 g^{-1} s^{-1}$ and $c = 3 \times 10^{10} cms^{-1}$, the charge can be written in terms of 5D proper momentum P_4 as follows:

$$1statC = 1cm^{3/2} s^{-1} g^{1/2} = 1(cm/s) cm^{1/2} g^{1/2} = \frac{c}{\sqrt{\mathbf{G}}} g.cm/s$$

$$Q^* = \frac{c}{\sqrt{\mathbf{G}}} P_4 \tag{3.2.6}$$

Generally speaking the approach here will be to do the calculations using $k = 1$ and then add in the general k term later, as and when needed, simply to ease calculation.

3.3 Consistency With Special Relativity

Toth charge is identified with 5D momentum in a space-time rest frame. This is already known in the original Kaluza theory to obey a Lorentz-like force law, but will be extended here in scope.

That this is consistent with special relativity can be investigated. What this consistency means is that the relativistic mass created by momentum in the 5th dimension is kinematically identical to the relativistic rest mass.

The additions of velocities in special relativity is not obvious. Assume a flat 5D Kaluza space (i.e without geometric curvature or torsion, thus analogously to special relativity at a flat space-time limit, a 5D Minkowski limit). Space-time can be viewed as a 4D slice (or series of parallel slices) perpendicular to the 5th Kaluza dimension that minimizes the length of any loops that are perpendicular to it. Taking a particle and an inertial frame, the relativistic rest frame where the particle is stationary with respect to space-time but moving with velocity u in the 5th dimension, and a second frame where the charge is now moving in space-time at velocity v, but still with velocity u in the 5th dimension, then the total speed squared of the particle in the second frame is according to relativistic addition of orthogonal velocities:

$$s^2 = u^2 + v^2 - u^2v^2 \tag{3.3.1}$$

The particle moving in the Kaluza dimension with velocity u, but stationary with respect to 4D space-time, will have a special relativistic 4D rest mass (m_0) normally greater than its 5D Kaluza rest mass (m_{k0}). We can see that the Kaluza rest mass definition (m_{k0}) is consistent with the orthogonal addition of velocities as follows:

$$m_0 = \frac{m_{k0}}{\sqrt{(1-u^2)}} \text{ where } u = Tanh[Sinh^{-1}(Q^*/(m_{k0}))] \tag{3.3.2}$$

$$m_{rel} = \frac{m_0}{\sqrt{(1-v^2)}} = \frac{m_{k0}}{\sqrt{(1-u^2)}} \times \frac{1}{\sqrt{(1-v^2)}} = \frac{m_{k0}}{\sqrt{(1-u^2-v^2+u^2v^2)}} \tag{3.3.3}$$

By putting $u = Tanh[Sinh^{-1}(Q^*/(m_{k0}))]$ (the conversion between unidirectional proper and coordinate velocities) into the definition of relativistic rest mass in terms of Kaluza rest mass and solving, we get that charge, whether positive or negative, is related to the relativistic rest mass according to the following formula:

$$Cosh[Sinh^{-1}(Q^*/(m_{k0}))] = m_0/m_{k0} = \frac{dt_0}{d\tau^*} \tag{3.3.4}$$

Using $k = 2$ we also have, for a typical unit charge:

$$m_e = 9.1094 \times 10^{-28} g \tag{3.3.5}$$

$$Q^* = 4.8032 \times 10^{-10} statC = 4.8032 \times 3.87 \times 10^{-10+3} g = 1.859 \times 10^{-6} g \tag{3.3.6}$$

If we take these figures and equate $m_e = m_0$ then we end up with imaginary m_{k0} and imaginary proper Kaluza velocity. Obviously to detail this the Kaluza-Cartan space-time would have to be adapted further in some way. But on the other hand it causes no causality problems provided the net result is compliant with any energy conditions being applied. And what is important in this respect is that the figures we know to be physical in 4D remain so.

Further issues are pertinent.

Observed electrons have static charge, angular momentum, a magnetic moment, and a flavor. The only thing distinguishing the electron from the muon is the flavor. The mass difference between the muon and the electron is about 105 MeV, perhaps solely due to this difference in flavor. The issue of modeling particles within a classical theory is, not surprisingly, a difficult one! Thus at this stage the idealized hypothetical charges used here, and real particles, can only be tentatively correlated.

It is possible to proceed without concern for the foundational issues of such charge models or attempting to interpret this quandary, instead simply developing the mathematics as is and seeing where it leads without judging it a priori.

3.4 Matter And Charge Models, A Disclaimer

This theory assumes some sort of particle model of matter and charge is possible, that it can be added to the original theory without significantly changing the ambient space-time solution and thus its own path, which is as approximate as in general relativity except as a limit. Secondly we might imagine that what has been described is a particle whizzing around the fifth dimension like a roller coaster on its spiralled tracks. The cylinder conditions can in fact be maintained if, instead of a 5D particle, the matter-charge source were rather a 'solid' ring, locked into place around the 5th dimension, rotating at some predetermined proper Kaluza velocity (albeit imaginary). An exact solution could even involve changes in the size of the 5th dimension. None of that is investigated here, the aim was originally just to see whether non-perpendicular solutions can be found in a variant Kaluza theory, and what constraints are needed.

It is a proviso that a physically realistic matter-charge model has not been detailed, nor formally identified with a real charge source such as an electron. The assumption then that such a hypothetical model would necessarily follow (albeit approximately) some predetermined path such as geodesics is therefore an assumption to be made about such matter-charge models - though not without analogs in other, experimentally valid classical theories. However, with the addition of torsion this becomes very much an assumption. Geodesics, or extremals, being followed by spinless particles in 4D Einstein-Cartan theory [11]. Other particles following different paths when interaction with torsion is present.

An exact differential geometrical model of such a matter-charge source is presumed too difficult to produce here, even if possible, especially given the previous discussion about imaginary masses and velocities. In addition, the fact that real charge sources are quantum mechanical may also be discouraging, though a classical limit interpretation should be possible regardless. The philosophy here has not been to provide a Lagrangian for a hypothesised charge model either, but instead to simply delimit what might constitute such models, and to weaken such constraints as much as possible.

This work assumes a limited concept of charge models and attempts to investigate whether non-perpendicular electromagnetic fields are possible in conjunction with a Lorentz force law. The theory is an attempt to replicate all the important features of classical physics, without predicting or imposing a particular model of charge as the correct one. To do this however some assumptions must in any case be made about the path of any hypothesized particle.

Geodesic Assumption: That any particle-like matter-charge approximately follow auto-parallels. (Auto-parallels being one of two analogs of geodesics used when torsion is present, but neither of which in the most general case determine the paths followed by particles).

Note that spinless particles according to [11] will actually follow the other geodesic analog: extremals. Extremals coincide with auto-parallels when torsion is completely antisymmetric. Particles with spin may interact in other ways. So when torsion is antisymmetric the assumption is that torsion-spin coupling does not significantly effect the path of the particle at least to some approximation.

In this work the original field equations of Kaluza cease to apply when such matter-charge models become part of the solution, and the new cylinder condition is applied with reference to the torsion connection. Therefore the original Kaluza field equations are only used primarily in discussion to give an introduction and motivation to the new Kaluza-Cartan theory. It is to be noted that in the following the Ricci flat condition of the original Kaluza theory's Kaluza space (ie the Ricci flat 5-space from which 4D space-time is extracted) will not be generally valid due to the presence of matter models and torsion, and the new Kaluza cylinder condition. It is also necessary to be careful as there will be two connections, and thus two separate possible definitions of vacuum. A 5-vacuum for the Levi-Civita connection is not the 5-vacuum for the torsion connection, and is not again, as with Kaluza's original theory, the 4-vacuum of space-time.

4 The Geometry Of Torsion

4.1 Introducing Torsion

5D Cartan torsion will be admitted. This will provide extra and required degrees of freedom. The new cylinder condition (see postulate K7 later) would otherwise be too restrictive.

For both 5D and 4D manifolds (i.e. dropping the hats and indices notation for a moment), torsion will be introduced into the Christoffel symbols as follows, using the notation of Hehl [11]. Metricity of the torsion tensor will be assumed [19], the reasonableness of which (in the context of general relativity with torsion) is argued for in [20] and [21]:

$$\frac{1}{2}(\Gamma^k_{ij} - \Gamma^k_{ji}) = S_{ij}{}^k \tag{4.1.1}$$

This relates to the notation of Kobayashi and Nomizu [12] and Wald [7] as follows:

$$T^i{}_{jk} = 2S_{jk}{}^i \equiv \Gamma^i_{jk} - \Gamma^i_{kj} \tag{4.1.2}$$

We have the contorsion tensor $K_{ij}{}^k$ [11] as follows, and a number of relations [11]:

$$\Gamma^k_{ij} = \frac{1}{2}g^{kd}(\partial_i g_{dj} + \partial_j g_{di} - \partial_d g_{ij}) - K_{ij}{}^k = \{^k_{ij}\} - K_{ij}{}^k \tag{4.1.3}$$

$$K_{ij}{}^k = -S_{ij}{}^k + S_j{}^k{}_i - S^k{}_{ij} = -K_i{}^k{}_j \tag{4.1.4}$$

With torsion included, the auto-parallel equation becomes [11]:

$$\frac{d^2x^k}{ds^2} + \Gamma^k_{(ij)}\frac{dx^i}{ds}\frac{dx^j}{ds} = 0 \tag{4.1.5}$$

$$\Gamma^k_{(ij)} = \{^k_{ij}\} + S^k{}_{(ij)} - S_{(j}{}^k{}_{i)} = \{^k_{ij}\} + 2S^k{}_{(ij)} \tag{4.1.6}$$

Only when torsion is completely antisymmetric is this the same as the extremals [11] which give the path of spinless particles and photons in Einstein-Cartan theory: extremals are none other than geodesics with respect to the Levi-Civita connection.

$$\frac{d^2x^k}{ds^2} + \{^k_{ij}\}\frac{dx^i}{ds}\frac{dx^j}{ds} = 0 \tag{4.1.7}$$

It is noted that Einstein-Cartan theory, that adds torsion to the dynamics of relativity theory is most probably a minimal ω-consistent extension of general relativity [13][14] and therefore the use of torsion is not only natural, but arguably a necessity on philosophical and physical grounds. That argument can also be applied here. What we have defined by this addition can be called

Kaluza-Cartan theory as it takes Kaluza's theory and adds torsion. We assume that the torsion connection is metric.

The torsionless field equations of Kaluza theory can not be used with torsion. Kaluza-Cartan theory requires adjustments for torsion.

When completely antisymmetric we have the following simplification:

$$K_{ij}{}^{k} = -S_{ij}{}^{k} \tag{4.1.8}$$

4.2 The Stress-Energy Tensor

The stress-energy tensor for a torsion bearing non-symmetric connection in Einstein-Cartan theory is usually labelled $\kappa \hat{P}_{AB}$, it need not be symmetric. In the literature the constant κ is included analogously to the 8π in general relativity. Here we will use the Einstein tensor $\hat{G}_{AB}$, taking a purely geometrical view.

The Belinfante-Rosenfeld [15] stress-energy tensor $\hat{B}$ is a symmetric adjustment of $\kappa\hat{P}$ that adjusts for spin currents as sources for Riemann-Cartan spaces. It can be defined equally for the 5D case. It is divergence free. It is the torsion equivalent according to Belinfante and Rosenfeld of the original Einstein tensor $\hat{G}$ [12] in some sense: the Einstein-Hilbert action. However the usual definition in terms of stress-energies and Noether currents, rather than the Einstein tensor, is not appropriate here. In effect repeating the Belinfante-Rosenfeld procedure, by defining the torsionless Einstein tensor in terms of torsion bearing components, yields what can be interpreted as extra spin-spin coupling term $\hat{X}_{AB}$:

$$\hat{\mathcal{G}}_{AB} = \hat{G}_{AB} + \hat{V}_{AB} + \hat{X}_{AB} \tag{4.2.1}$$

$$\hat{V}_{AB} = \frac{1}{2}\hat{\nabla}^{C}(\hat{\sigma}_{ABC} + \hat{\sigma}_{BAC} + \hat{\sigma}_{CBA}) \tag{4.2.2}$$

Where σ is defined as the spin tensor in Einstein-Cartan theory. However, here we do not start with spin (and some particle Lagrangians), but with the torsion tensor. So instead the spin tensor is defined in terms of the torsion tensor using the Einstein-Cartan equations. Here spin is explicitly defined in terms of torsion:

$$\hat{\sigma}_{ABC} = 2\hat{S}_{ABC} + 2\hat{g}_{AC}\hat{S}_{BD}{}^{D} - 2\hat{g}_{BC}\hat{S}_{AD}{}^{D} \tag{4.2.3}$$

When the torsion is completely antisymmetric we have:

$$\hat{V}_{AB} = \hat{\nabla}^{C}\hat{S}_{ABC} \tag{4.2.4}$$

We have therefore, in terms of $\hat{V}$, a conservation law pertaining to spin by definition:

$$\hat{\nabla}_A \hat{V}^{AB} = 0 \tag{4.2.5}$$

Note that the mass-energy-charge conservation law for the torsionless Einstein tensor is currently in terms of the torsionless connection, but the spin source conservation law here is in terms of the torsion-bearing connection. However, for completely antisymmetric torsion we have:

$$\hat{\nabla}_C \hat{\mathcal{G}}_{AB} = \hat{\triangle}_C \hat{\mathcal{G}}_{AB} + \hat{K}_{CA}{}^{D} \hat{\mathcal{G}}_{DB} + \hat{K}_{CB}{}^{D} \hat{\mathcal{G}}_{AD}$$

So,

$$\hat{\nabla}^A \hat{\mathcal{G}}_{AB} = 0 + 0 + \hat{K}^{A}{}_{B}{}^{D} \hat{\mathcal{G}}_{AD} = -\hat{K}_{B}{}^{AD} \hat{\mathcal{G}}_{AD}$$
$$= -\hat{K}_{B}{}^{AD} \hat{\mathcal{G}}_{DA} = +\hat{K}_{B}{}^{DA} \hat{\mathcal{G}}_{DA} = +\hat{K}_{B}{}^{AD} \hat{\mathcal{G}}_{AD} = 0 \tag{4.2.6}$$

$$\hat{\nabla}^A (\hat{G}_{AB} + \hat{X}_{AB}) = 0 \tag{4.2.7}$$

And so there is a stress-energy conservation law with respect to the torsion connection also.

Further, still assuming complete antisymmetry of torsion, by definition of the Ricci tensor:

$$\hat{R}_{AB} = \hat{\mathcal{R}}_{AB} + \hat{K}_{DA}{}^{C} \hat{K}_{BC}{}^{D} - \partial_C \hat{K}_{BA}{}^{C} - \hat{K}_{BA}{}^{C} \hat{\Gamma}^{D}{}_{DC} + \hat{K}_{DA}{}^{C} \hat{\Gamma}^{D}{}_{DC} - \hat{K}_{DB}{}^{C} \hat{\Gamma}^{D}{}_{AC}$$
$$= \hat{\mathcal{R}}_{AB} - \hat{K}_{AD}{}^{C} \hat{K}_{BC}{}^{D} - \hat{\nabla}^C \hat{S}_{ABC} \tag{4.2.8}$$

$$\hat{G}_{[AB]} = \hat{R}_{[AB]} = -\hat{\nabla}^C \hat{S}_{ABC} \tag{4.2.9}$$

$\hat{V}_{AB}$ is the antisymmetric part of $-\hat{G}_{AB}$. And $\hat{X}_{AB}$ is symmetric.

5 Kaluza-Cartan Space-Time

In this section a full outline of the theory in terms of postulates, assumptions and possible additions is presented. K1-K7 constitute the theory proper.

5.1 Kaluza-Cartan Space-Time Definition

A definition of Kaluza-Cartan Space-Time, or Kaluza-Cartan Space, follows. K7 is the new KCC:

Core Definitions and Postulates:

(K1) A Kaluza-Cartan manifold is a 5D smooth Lorentzian manifold.

(K2) One spatial dimension is topologically closed and small, the Kaluza dimension. There is a global unit vector that defines this direction and they form closed non-intersecting loops.

(K3) The other spatial dimensions and time dimension are large.

(K4) There is a connection that is a metric torsion connection with respect to the geometry, this is the torsion connection.

(K5) Kaluza-Cartan Space is assumed globally hyperbolic in the sense that there exists a 3D spatial cauchy surface plus time, extended in the obvious way via the new cylinder condition into 5D.

(K6) Kaluza-Cartan Space is oriented.

(K7) The covariant derivative (with respect to the torsion tensor) of all tensors and pseudotensors in the Kaluza direction are zero. This is the covariant derivative with lower index: $\hat{\nabla}_4$.

The new KCC given here determines that local charts are possible with vanishing partial derivatives for all tensors. That is, partial derivatives with lower index: ∂_4 are all zero. What we can do now is take a loop by following the Kaluza direction and forming a 1-dimensional submanifold for every point of space-time. By inspecting the bundles inherited by each loop submanifold we can observe that at every point they are necessarily static. As a result ∂^4 and $\hat{\nabla}^4$ must be vanishing for all tensors.

5.2 The Scalar Field And Torsion Versus Observation

In the sequel a justification (somewhat empirical - equation 7.3.1) is given for restricting the scalar field and accordingly the Kaluza atlas as follows:

(B1) $\phi^2 = 1$

The text provides the case for a constraint on the torsion tensor, that it is completely antisymmetric, this then also ensures total antisymmetry of the contorsion tensor via (4.1.8). The case is made implicitly by the fact that it is frequently required as a simplification step.

(B2) The torsion tensor is completely antisymmetric.

B2 will be shown to imply a fixed gauge on the vector potential components given by (6.1.9). That is: $A^d F_{cd} = 0$.

5.3 Secondary Definitions And Points

Some useful observations and definitions:

(S1) The Kaluza-Cartan vacuum is a Ricci flat region of a Kaluza-Cartan manifold with respect to the torsion connection definition of the Ricci tensor.

Similarly the Kaluza vacuum is a Ricci flat region with respect to the Levi-Civita connection. They are different. Here they are both defined in terms of the geometry implied by the new KCC. There is also the Kaluza space (and hence Kaluza vacuum) of the original Kaluza theory. So the definition of the vacuum, whichever may be being referenced, depends on the cylinder condition being used.

(S2) K7 can be used to decompose the entire 5D geometry into a 4D metric, a vector potential and a scalar field when curvatures are not so extreme as to lead to a singularity in the 4D metric. It defines how space-time is a parallel set of submanifolds.

(S3) Singularities resulting from an S2 break-down may indicate regions where the theory as presented breaks down.

(S4) The Kaluza vacuum (under the new KCC) typically contains fields when the 4D metric is inspected, and will most often not be Ricci flat in 4D space-time. Likewise for the Kaluza vacuum under the original KCC, and for the Kaluza-Cartan vacuum under the new KCC.

(S5) What is allowed as a physical solution needs to be delimited in some way, at the very least to avoid acausality as in classical physics. This can be done by using energy conditions as in relativity for the resultant 4D space-time, at least as long as the decomposition doesn't break-down. The cylinder condition can then extrapolate that to the whole of the Kaluza-Cartan manifold.

(S6) Matter-charge models must also be consistent with K7. A realistic elementary charge model would have imaginary Kaluza rest mass and Kaluza velocity, but this does not prevent it satisfying the cylinder condition.

(S7) K7 may well be a limit rather than a fundamental postulate, that is not dealt with further. Similiarly the lack of a scalar field.

(S8) A number of possible constraints can be experimented with and applied to matter models, the aim here is to keep the options as broad as possible. Assumptions pertaining to matter models are therefore defined separately below.

(S9) The new KCC is defined in terms of covariant derivatives instead of the usual frame dependent partial derivatives, and then a restricted Kaluza subatlas is used in which the partial derivatives are also zero. In particular it takes account of torsion.

(S10) The antisymmetric part of the torsion-bearing Einstein tensor is associated with spin sources. There is a conservation law automatic from the definition. The torsionless Einstein tensor is associated with matter and charge, and a similar conservation law follows.

5.4 Matter And Charge Models

Some notes on matter andcharge models:

(M1) Matter-charge models must be consistent with the preceding.

(M2) Simple examples could include black-hole singularities.

(M3) Net positivity of mass and energy may be imposed by the usual energy conditions or similar extensions in 5D.

(M4) No comment is passed on the fact that the proportions of mass and charge of a realistic elementary charge model yield imaginary Kaluza rest mass and Kaluza proper velocity - this paper proceeds without judging this peculiar result.

(M5) Matter-Charge models can be defined as any part of the Kaluza-Cartan manifold that are not Kaluza vacuums (according to definition S1).

(M6) A further definition of Matter-Charge-Spin models is required. This will be defined differently: when, according to definition S1, the Kaluza-Cartan manifold is not a Kaluza-Cartan vacuum.

(M7) There is a need to make an assumption: the Geodesic Assumption, that light particle-like solutions are possible and follow 5D auto-parallels. I.e. geodesics when torsion is not present. This should in any case be the case for spinless particles, as seen in 4D Einstein-Cartan theory. As applied to particles with spin we are making an assumption about spin-torsion coupling that may well be incorrect.

(M8) Quantization of charge is not dealt with - no less than quantization of energy or momentum. The proper place for this is a quantum theory, or a theory that encompasses the quantum.

5.5 An Apology

Admittedly (M4) through (M8) show that this theory is in an early state of development. Many theories pass through such a state, and complete realization from the outset can not be expected. An objective in the foregoing has been to present the postulates that are needed for the theory to be well-defined even if incomplete. Ultimately empirical observation determines validity.

6 The Field Equations

6.1 The Cylinder Condition And Scalar Field

Here we look at how K7 affects the Christoffel symbols of any coordinate system within the Kaluza atlas. Given that the Kaluza atlas is reduced in the following way: all partial derivatives of tensors in the Kaluza direction are set to zero (using $k = 1$). The Appendix (9.1.1 onwards) contains a reference for Christoffel symbol working both with and without the torsion component.

The following follow from the selection of coordinates that set the partial derivatives in the Kaluza dimension to zero; from K7, the new covariant version of the Kaluza cylinder condition used in this paper; and, from the relationship between these two and the Christoffel symbols given in Wald [7] p33 eqn (3.1.14) as applied to a number of test vectors. Note that there is no hint of symmetry of the (with torsion) Christoffel symbols suggested here. That is, these terms are forced zero by the fact that both the partial derivatives and the covariant derivatives in the Kaluza direction are zero. Cf equation (1.0.4) where the consequences of setting both the partial derivatives and the covariant derivative to zero can be seen on the Christoffel symbols. The new cylinder condition is quite strong and without torsion would lead immediately to insufficiently diverse geometry given the other postulates here.

$$0 = 2\hat{\Gamma}^A_{4c} = \sum_d \hat{g}^{Ad}(\partial_4 g_{cd} + \partial_4\phi^2 A_c A_d + \partial_c\phi^2 A_d - \partial_d\phi^2 A_c) + \hat{g}^{A4}\partial_c\phi^2 - 2\hat{K}_{4c}{}^A \tag{6.1.1}$$

$$0 = 2\hat{\Gamma}^A_{44} = 2\sum_d \hat{g}^{Ad}\partial_4\phi^2 A_d - \sum_d \hat{g}^{Ad}\partial_d\phi^2 + \hat{g}^{A4}\partial_4\phi^2 - 2\hat{K}_{44}{}^A \tag{6.1.2}$$

We have:

$$2\hat{K}_{4c}{}^A = \hat{g}^{Ad}(\partial_c\phi^2 A_d - \partial_d\phi^2 A_c) + \hat{g}^{A4}\partial_c\phi^2 \tag{6.1.3}$$

$$2\hat{K}_{44}{}^A = -\hat{g}^{Ad}\partial_d\phi^2 \tag{6.1.4}$$

Inspecting the first of these equations (6.1.1), and given that $K_{A(BC)} = 0$ (4.1.4), and further applying A=c without summing, we have a constraint on the scalar field in terms of the vector potential that further motivates postulate B1. Here, however, we make a priori use of postulate B1.

The immediate result of this is as follows (using $k = 1$):

$$2\hat{K}_{4c}{}^A = \hat{g}^{Ad}(\partial_c A_d - \partial_d A_c) \tag{6.1.5}$$

$$2\hat{K}_{44}{}^A = 0 \tag{6.1.6}$$

Giving the contorsion a very clear interpretation here in terms of the electromagnetic field.

$$\hat{K}_{4c}{}^a = \frac{1}{2}F^a_c \tag{6.1.7}$$

$$\hat{K}_{4c}{}^4 = -\frac{1}{2}A^d F_{cd} \tag{6.1.8}$$

Complete antisymmetry of contorsion imposes, by (6.1.1):

$$A^d F_{cd} = 0 \tag{6.1.9}$$

This is nothing other than a vector potential gauge for completely antisymmetric torsion as there are four unknowns in the offset for each vector potential component and four unknowns in the above equation, one for each c. This gauge constraint is noted in postulate B2.

This and similar considerations lead to a number of simplifications by looking at the details of the Christoffel symbols:

$$\hat{K}_{4a}{}^{4} = \hat{K}_{a4}{}^{4} = \hat{\Gamma}^{4}_{a4} = \hat{\Gamma}^{4}_{4a} = 0 \tag{6.1.10}$$

6.2 The First Field Equation With Torsion, $k = 1$

Looking at the Ricci tensor, but here with torsion (using equations 9.1.7 repeatedly, and the new KCC as required):

$$\begin{aligned}
\hat{R}_{ab} &= \partial_C \hat{\Gamma}^{C}_{ba} - \partial_b \hat{\Gamma}^{C}_{Ca} + \hat{\Gamma}^{C}_{ba}\hat{\Gamma}^{D}_{DC} - \hat{\Gamma}^{C}_{Da}\hat{\Gamma}^{D}_{bC} \\
\hat{R}_{ab} &= \partial_c \hat{\Gamma}^{c}_{ba} - \partial_b \hat{\Gamma}^{c}_{ca} + \hat{\Gamma}^{C}_{ba}\hat{\Gamma}^{D}_{DC} - \hat{\Gamma}^{C}_{Da}\hat{\Gamma}^{D}_{bC} \\
\hat{R}_{ab} &= \partial_c \hat{\Gamma}^{c}_{ba} - \partial_b \hat{\Gamma}^{c}_{ca} + \hat{\Gamma}^{C}_{ba}\hat{\Gamma}^{d}_{dC} - \hat{\Gamma}^{C}_{da}\hat{\Gamma}^{d}_{bC} \\
\hat{R}_{ab} &= \partial_c \hat{\Gamma}^{c}_{ba} - \partial_b \hat{\Gamma}^{c}_{ca} + \hat{\Gamma}^{c}_{ba}\hat{\Gamma}^{d}_{dc} - \hat{\Gamma}^{C}_{da}\hat{\Gamma}^{d}_{bC}
\end{aligned} \tag{6.2.1}$$

Doing the same for the without torsion definitions (using equations 9.1.6 repeatedly, and the new KCC as required). The extra term with respect to the above coming from the fact that (6.1.10) can not be applied to the torsionless Christoffel symbols, which in turn comes from the new cylinder condition:

$$\begin{aligned}
\hat{\mathcal{R}}_{ab} &= \partial_C \hat{\varGamma}^{C}_{ba} - \partial_b \hat{\varGamma}^{C}_{Ca} + \hat{\varGamma}^{C}_{ba}\hat{\varGamma}^{D}_{DC} - \hat{\varGamma}^{C}_{Da}\hat{\varGamma}^{D}_{bC} \\
\hat{\mathcal{R}}_{ab} &= \partial_c \hat{\varGamma}^{c}_{ba} - \partial_b \hat{\varGamma}^{c}_{ca} + \hat{\varGamma}^{C}_{ba}\hat{\varGamma}^{d}_{dC} - \hat{\varGamma}^{C}_{Da}\hat{\varGamma}^{D}_{bC} \\
\hat{\mathcal{R}}_{ab} &= \partial_c \hat{\varGamma}^{c}_{ba} - \partial_b \hat{\varGamma}^{c}_{ca} + \hat{\varGamma}^{c}_{ba}\hat{\varGamma}^{d}_{dc} - \hat{\varGamma}^{C}_{da}\hat{\varGamma}^{d}_{bC} - \hat{\varGamma}^{c}_{4a}\hat{\varGamma}^{4}_{bc}
\end{aligned} \tag{6.2.2}$$

In the original Kaluza theory the Ricci curvature of the 5D space is set to 0. The first field equation comes from looking at the Ricci curvature of the space-time that results. Here there is a choice: whether to base the vacuum on a Kaluza vacuum or a Kaluza-Cartan vacuum. To be consistent with the definition of matter-charge models, ie M5 and definition (3.1.11), the Kaluza vacuum is looked at first as the more obvious choice. Setting $\hat{\mathcal{R}}_{ab} = 0$,

$$\begin{aligned}
\mathcal{R}_{ab} = \mathcal{R}_{ab} - \hat{\mathcal{R}}_{ab} &= \partial_c \varGamma^{c}_{ba} - \partial_b \varGamma^{c}_{ca} + \varGamma^{c}_{ba}\varGamma^{d}_{dc} - \varGamma^{c}_{da}\varGamma^{d}_{bc} \\
&\quad -\partial_c \hat{\varGamma}^{c}_{ba} + \partial_b \hat{\varGamma}^{c}_{ca} - \hat{\varGamma}^{c}_{ba}\hat{\varGamma}^{d}_{dc} + \hat{\varGamma}^{C}_{da}\hat{\varGamma}^{d}_{bC} - \hat{\varGamma}^{c}_{4a}\hat{\varGamma}^{4}_{bc} \\
&= \partial_c \varGamma^{c}_{ba} - \partial_c \hat{\varGamma}^{c}_{ba} \quad - \partial_b \varGamma^{c}_{ca} + \partial_b \hat{\varGamma}^{c}_{ca} \\
&\quad + \varGamma^{c}_{ba}\varGamma^{d}_{dc} - \hat{\varGamma}^{c}_{ba}\hat{\varGamma}^{d}_{dc} \quad - \varGamma^{c}_{da}\varGamma^{d}_{bc} + \hat{\varGamma}^{C}_{da}\hat{\varGamma}^{d}_{bC} \quad + \hat{\varGamma}^{c}_{4a}\hat{\varGamma}^{4}_{bc} \\
&= -\frac{1}{2}\partial_c (A_b F^c_a + A_a F^c_b) + 0 - \frac{1}{2}(A_b F^c_a + A_a F^c_b)\varGamma^{d}_{dc}
\end{aligned}$$

$$+\frac{1}{2}(A_d F^c_a + A_a F^c_d)\varGamma^{d}_{bc} + \frac{1}{2}(A_b F^d_c + A_c F^d_b)\varGamma^{c}_{da} + \frac{1}{4}(A_d F^c_a + A_a F^c_d)(A_b F^d_c + A_c F^d_b)$$

$$+\frac{1}{2}F^d_b(-A_c \varGamma^{c}_{da} + \frac{1}{2}\partial_d A_a + \frac{1}{2}\partial_a A_d)$$

$$
\begin{aligned}
&+\frac{1}{2}F^c_a(-A_d\varGamma^{\ d}_{bc}+\frac{1}{2}\partial_b A_c+\frac{1}{2}\partial_c A_b)\\
&=-\frac{1}{2}\partial_c(A_bF^c_a+A_aF^c_b)-\frac{1}{2}(A_bF^c_a+A_aF^c_b)\varGamma^{\ d}_{dc}\\
&+\frac{1}{2}A_aF^c_d\varGamma^{\ d}_{bc}+\frac{1}{2}A_bF^d_c\varGamma^{\ c}_{da}+\frac{1}{4}(A_dF^c_a+A_aF^c_d)(A_bF^d_c+A_cF^d_b)\\
&+\frac{1}{4}F^d_b(\partial_d A_a+\partial_a A_d)+\frac{1}{4}F^c_a(\partial_b A_c+\partial_c A_b)
\end{aligned}
\tag{6.2.3}
$$

Taking a limit without Maxwell charge sources (see 6.3.1), and ignoring terms with more than two components at a weak field limit we can simplify this equation for a particular limit defined by, and restricted to, that simplification:

$$
\begin{aligned}
&=-\frac{1}{2}(\partial_c A_b)F^c_a-\frac{1}{2}(\partial_c A_a)F^c_b+\frac{1}{4}F^d_b(\partial_d A_a+\partial_a A_d)+\frac{1}{4}F^c_a(\partial_b A_c+\partial_c A_b)\\
&=\frac{1}{4}F^d_b(-\partial_d A_a+\partial_a A_d)+\frac{1}{4}F^c_a(\partial_b A_c-\partial_c A_b)\\
&=\frac{1}{2}F^c_aF_{bc}
\end{aligned}
\tag{6.2.4}
$$

When compared with the second and third field equations below, (6.3.1) and (6.4.1), the similarity with the original Kaluza field equations and the perpendicular solutions of the Einstein-Maxwell equations is unmistakable.

Now, doing the same with respect to M6 and (4.2.1),(4.2.5), so that we are defining vacuum as outside of matter-charge-spin models, that is, a Kaluza-Cartan vacuum according to S1 ($\hat{R}_{AB}=0$) - and using (6.2.1):

$$
\begin{aligned}
\mathcal{R}_{ab}=\mathcal{R}_{ab}-\hat{R}_{ab}&=\partial_c\varGamma^{\ c}_{ba}-\partial_b\varGamma^{\ c}_{ca}+\varGamma^{\ c}_{ba}\varGamma^{\ d}_{dc}-\varGamma^{\ c}_{da}\varGamma^{\ d}_{bc}\\
&-\partial_c\hat{\Gamma}^c_{ba}+\partial_b\hat{\Gamma}^c_{ca}-\hat{\Gamma}^c_{ba}\hat{\Gamma}^d_{dc}+\hat{\Gamma}^C_{da}\hat{\Gamma}^d_{bC}\\
&=\mathcal{R}_{ab}-\hat{\mathcal{R}}_{ab}+\partial_c\hat{K}_{ba}{}^c-\partial_b\hat{K}_{ca}{}^c+\hat{K}_{ba}{}^c\hat{\Gamma}^d_{dc}+\hat{\Gamma}^c_{ba}\hat{K}_{dc}{}^d-\hat{K}_{ba}{}^c\hat{K}_{dc}{}^d\\
&-\hat{K}_{da}{}^C\hat{\Gamma}^d_{bC}-\hat{\Gamma}^C_{da}\hat{K}_{bC}{}^d+\hat{K}_{da}{}^C\hat{K}_{bC}{}^d
\end{aligned}
$$

Simplifying for complete antisymmetry of torsion and contorsion:

$$
=\mathcal{R}_{ab}-\hat{\mathcal{R}}_{ab}+\partial_c\hat{K}_{ba}{}^c+\hat{K}_{ba}{}^c\hat{\Gamma}^d_{dc}-\hat{K}_{da}{}^C\hat{\Gamma}^d_{bC}-\hat{\Gamma}^C_{da}\hat{K}_{bC}{}^d+\hat{K}_{da}{}^C\hat{K}_{bC}{}^d
$$

We could further substitute in equation (6.2.3), but the result is perhaps not immediately informative, except to note that the limit leading previously to perpendicular electrovacuums is now more complex and involves torsion terms.

6.3 The Second Field Equation With Torsion

Rederivation of the second field equation under the new KCC, and $\hat{\mathcal{R}}_{ab} = 0$ in mind:

$$\begin{aligned}\hat{\mathcal{R}}_{a4} &= \partial_C \hat{\Gamma}^{C}_{\ 4a} - \partial_4 \hat{\Gamma}^{C}_{\ Ca} + \hat{\Gamma}^{C}_{\ 4a}\hat{\Gamma}^{D}_{\ DC} - \hat{\Gamma}^{C}_{\ Da}\hat{\Gamma}^{D}_{\ 4C} \\ &= \partial_c \hat{\Gamma}^{c}_{\ 4a} + \hat{\Gamma}^{c}_{\ 4a}\hat{\Gamma}^{d}_{\ dc} - \hat{\Gamma}^{c}_{\ da}\hat{\Gamma}^{d}_{\ 4c} \\ &= \frac{1}{2}\partial_c F^c_a + \frac{1}{2}F^c_a\Gamma^{d}_{\ dc} - (\Gamma^{c}_{\ da} + \frac{1}{2}(A_d F^c_a + A_a F^c_d))F^d_c \\ &= \frac{1}{2}\partial_c F^c_a + \frac{1}{2}F^c_a\Gamma^{d}_{\ dc} - (\Gamma^{c}_{\ da} + \frac{1}{2}A_a F^c_d)F^d_c\end{aligned}$$

Looking at this locally so that 1st derivatives are vanishing, but second derivatives remain (re-inserting general k):

$$\hat{\mathcal{R}}_{a4} = \frac{k}{2}\partial_c F^c_a \tag{6.3.1}$$

This couldn't be a clearer conception of Maxwell charge. Setting this to 0 in Kaluza vaccum assures us that Maxwell charge is locally restricted to matter-charge models, but does not do the same for matter-charge-spin models.

By definition (and the new KCC, and 9.1.7), we immediately get:

$$\hat{R}_{a4} = 0 \tag{6.3.2}$$

Whereas $\hat{R}_{4b}$ simplifies to:

$$\hat{R}_{4b} = \partial_c F^c_b + F^c_b\hat{\Gamma}^d_{dc} - F^c_d\hat{\Gamma}^d_{bc} \tag{6.3.3}$$

Which is not the clean local definition of Maxwell charge presented in (6.3.1). Although of course at a limit where first derivatives vanish it does become a charge source.

6.4 The Third Field Equation With Torsion, $k = 1$

This section shows how torsion releases the constraint of the third torsionless field equation, thus allowing non-perpendicular solutions. The constraint that the Ricci tensor be zero leads to no non-perpendicular solutions in the original Kaluza theory via the third field equation. This is caused by setting $\hat{R}_{44} = 0$ in that theory and observing the terms. The result is that (when the scalar field is constant) $0 = F_{cd}F^{cd}$ in the original Kaluza theory. The same issue arises here:

We have:

$$\begin{aligned}\hat{\mathcal{R}}_{44} &= \partial_C \hat{\Gamma}^{C}_{\ 44} - \partial_4 \hat{\Gamma}^{C}_{\ C4} + \hat{\Gamma}^{C}_{\ 44}\hat{\Gamma}^{D}_{\ DC} - \hat{\Gamma}^{C}_{\ D4}\hat{\Gamma}^{D}_{\ 4C} \\ &= 0 - 0 + 0 - \hat{\Gamma}^{C}_{\ D4}\hat{\Gamma}^{D}_{\ 4C} \\ &= -\frac{1}{4}F^c_d F^d_c \end{aligned} \tag{6.4.1}$$

The result is that whilst we can have non-perpendicular solutions, we can only have them outside of a strict Kaluza vaccum, contradicting the initial approach here based on M5. The idea that arises is that the perpendicular solutions are those which correlate with a Kaluza vacuum, whilst the non-perpendicular solutions arise in Kaluza-Cartan vacuum.

By definition (and the new KCC, and 9.1.7), we immediately get:

$$\hat{R}_{44} = 0 \tag{6.4.2}$$

6.5 Matter-Charge-Spin

As a result of the preceding an investigation into basing matter models on M6 is required. By making the electrodynamical vacuum a region of the Kaluza-Cartan manifold without matter-charge-spin, that is, a Kaluza-Cartan vacuum, certain complications and opportunities arise. The main opportunity is that we can resolve the problem of perpendicular electromagnetic solutions. Secondly it is more logical in that both the cylinder condition and the vacuum are defined in terms of the torsion connection. The main complication is that it is not at all clear that Maxwell charge is bound to matter-charge-spin models. It appears that Maxwell charge could leak into the ambient via non-perpendicular fields since in general $\hat{\mathcal{R}}_{AB} \neq 0$. Cf (6.3.1) and the presence of Maxwell charge terms present on inspection in (6.2.3). At first glance (6.3.3) seems to offer the possibility of charge sources outside matter-charge-spin models, even if within matter-charge models. As such matter-charge-spin models would appear to have no intergity unlike matter-charge models - it is important that matter, charge and spin that are sourced in a matter-charge-spin model stay there in some definable sense. For matter-energy this is governed by (4.2.7). For spin this is governed by (4.2.5), (4.2.7) and (4.2.9), in that the conserved spin component can not pass into a symmetric part of the Ricci tensor, hence certainly not a region where it is vanishing. The same must surprisingly also follow for charge by (6.1.7), as a corollary.

So in fact the problem presumed for $\hat{R}_{AB}$ is not a problem under completely antisymmetric torsion. In this way matter-charge-spin models and Kaluza-Cartan vacuum can be assigned the fundamental role initially expected for matter-charge.

Further however this is also consistent with the fact that spin is empirically important in nature.

6.6 Toth Charge

Now to investigate the relationship between Toth charge and Maxwell charge. For this we need a particular limit.

Definition 6.6.1: The Maxwell limit.

(i) The limit shall be a local limit where second derivatives of metric components are significant and first derivatives discarded.

(ii) The electromagnetic potential vector and space-time curvatures shall be weak, such that terms of the following form will be discarded despite the second derivatives:

$$A_v \partial_w \partial_x g_{yz} \to 0 \qquad A_v \partial_w \partial_x g^{yz} \to 0 \tag{6.6.1}$$

Using Appendix (9.3.3), and (i) and (ii) above, at the defined limit:

$$\hat{\mathcal{G}}^{a4} = \hat{\mathcal{R}}^{a4} - \frac{1}{2}\hat{g}^{a4}\hat{\mathcal{R}} = \hat{\mathcal{R}}^{a4} - \frac{1}{2}(-A^a)\hat{\mathcal{R}} \to \hat{\mathcal{R}}^{a4}$$

$$\hat{\mathcal{R}}^{a4} = \partial_C \hat{\Gamma}^{C4a} - \partial^4 \hat{\Gamma}^{C\ a}_{\ C} + \hat{\Gamma}^{Cba}\hat{\Gamma}^{D}_{\ DC} - \hat{\Gamma}^{C\ a}_{\ D}\hat{\Gamma}^{Db}_{\ \ C}$$

$$\hat{\mathcal{G}}^{a4} \to \hat{\mathcal{R}}^{a4} = \partial_c \hat{\Gamma}^{c4a} \tag{6.6.2}$$

Applying the local Maxwell flat space-time limit, and putting k back in, and by using Appendix equation (9.2.1) for the Christoffel symbol, we get:

$$\hat{\mathcal{R}}^{a4} \to \frac{1}{2}\partial_c k F^{ac} \tag{6.6.3}$$

And so by (3.1.11),

$$J^*_a \to -\frac{\alpha k}{2}\partial_c F_a^{\ c} \tag{6.6.4}$$

So Toth and Maxwell charges are related by a simple formula. The right hand side being Maxwell's charge current (see p.81 of [6]), and has the correct sign to identify a positive Toth charge Q^* with a positive Maxwell charge source $4\pi Q_M$, whenever $\alpha k > 0$. In the appropriate space-time frame, and Kaluza atlas frame, and at the appropriate limit, using (3.1.10):

$$4\pi Q_M = +\frac{2}{\alpha k}Q^* \tag{6.6.5}$$

7 The Lorentz Force Law

Toth [8] derives a Lorentz-like force law where there is a static scalar field and Kaluza's cylinder condition applies in the original Kaluza theory. The resulting 'charge' is the momentum term in the fifth dimension and it is not apparent how this relates to the Maxwell current, except as Toth states via 'formal equivalence'. While this result is not new, Toth's calculation is extended here for the new theory. here we make use of the Geodesic Assumption.

7.1 A Lorentz-Like Force Law

The Christoffel symbols and the geodesic equation are the symmetric ones defined in the presence of totally antisymmetric torsion. We will here initially use $k = 1$, a general k can be added in later.

$$\hat{\Gamma}^c_{(4b)} = \tfrac{1}{2}g^{cd}(\delta_4\hat{g}_{bd} + \delta_b\hat{g}_{4d} - \delta_d\hat{g}_{4b}) + \tfrac{1}{2}\hat{g}^{c4}(\delta_4\hat{g}_{b4} + \delta_b\hat{g}_{44} - \delta_4\hat{g}_{4b}) =$$
$$\tfrac{1}{2}g^{cd}[\delta_b(\phi^2 A_d) - \delta_d(\phi^2 A_b)] + \tfrac{1}{2}g^{cd}\delta_4\hat{g}_{bd} + \tfrac{1}{2}\hat{g}^{c4}\delta_b\hat{g}_{44} =$$
$$\tfrac{1}{2}\phi^2 g^{cd}[\delta_b A_d - \delta_d A_b] + \tfrac{1}{2}g^{cd}A_d\delta_b\phi^2 - \tfrac{1}{2}g^{cd}A_b\delta_d\phi^2 + \tfrac{1}{2}g^{cd}\delta_4\hat{g}_{bd} + \tfrac{1}{2}\hat{g}^{c4}\delta_b\phi^2 =$$
$$\tfrac{1}{2}\phi^2 F^c_b + \tfrac{1}{2}g^{cd}A_d\delta_b\phi^2 - \tfrac{1}{2}g^{cd}A_b\delta_d\phi^2 + \tfrac{1}{2}g^{cd}\delta_4\hat{g}_{bd} + \tfrac{1}{2}\hat{g}^{c4}\delta_b\phi^2 =$$
$$\tfrac{1}{2}\phi^2 F^c_b - \tfrac{1}{2}g^{cd}A_b\delta_d\phi^2 + \tfrac{1}{2}g^{cd}\delta_4\hat{g}_{bd} = \tfrac{1}{2}\phi^2 F^c_b \quad -\tfrac{1}{2}g^{cd}A_b\delta_d\phi^2$$

(7.1.1)

$$\hat{\Gamma}^c_{44} = \tfrac{1}{2}\hat{g}^{\mathrm{cD}}(\delta_4\hat{g}_{4D} + \delta_4\hat{g}_{4D} - \delta_D\hat{g}_{44}) = \text{-}\ \tfrac{1}{2}g^{\mathrm{cd}}\delta_d\phi^2$$

(7.1.2)

We have:

$$\hat{\Gamma}^c_{(ab)} = \tfrac{1}{2}g^{cd}(\delta_a g_{db} + \delta_b g_{da} - \delta_d g_{ab})$$
$$+\tfrac{1}{2}g^{cd}(\delta_a(\phi^2 A_d A_b) + \delta_b(\phi^2 A_a A_d) - \delta_d(\phi^2 A_a A_b)) + \tfrac{1}{2}\hat{g}^{c4}(\delta_a\hat{g}_{4b} + \delta_b\hat{g}_{4a} - \delta_4\hat{g}_{ab})$$
$$= \Gamma^c_{(ab)} \quad +\tfrac{1}{2}g^{cd}(\delta_a(\phi^2 A_d A_b) + \delta_b(\phi^2 A_a A_d) - \delta_d(\phi^2 A_a A_b))$$
$$-A^c(\delta_a\phi^2 A_b + \delta_b\phi^2 A_a)$$

(7.1.3)

So, for any coordinate system within the maximal atlas:

$$0 = \tfrac{d^2x^a}{d\tau^2} + \hat{\Gamma}^a_{(BC)}\tfrac{dx^B}{d\tau}\tfrac{dx^C}{d\tau}$$
$$= \tfrac{d^2x^a}{d\tau^2} + \hat{\Gamma}^a_{(bc)}\tfrac{dx^b}{d\tau}\tfrac{dx^c}{d\tau} + \hat{\Gamma}^a_{(4c)}\tfrac{dx^4}{d\tau}\tfrac{dx^c}{d\tau} + \hat{\Gamma}^a_{(b4)}\tfrac{dx^b}{d\tau}\tfrac{dx^4}{d\tau} + \hat{\Gamma}^a_{44}\tfrac{dx^4}{d\tau}\tfrac{dx^4}{d\tau}$$
$$= \tfrac{d^2x^a}{d\tau^2} + \hat{\Gamma}^a_{(bc)}\tfrac{dx^b}{d\tau}\tfrac{dx^c}{d\tau} + (\phi^2 F^a_b - g^{ad}A_b\delta_d\phi^2)\tfrac{dx^b}{d\tau}\tfrac{dx^4}{d\tau} - \tfrac{1}{2}g^{\mathrm{ad}}\delta_d\phi^2\tfrac{dx^4}{d\tau}\tfrac{dx^4}{d\tau}$$

(7.1.4)

Taking $\phi^2 = 1$ and the charge-to-mass ratio to be:

$$Q'/m_{k0} = \frac{dx^4}{d\tau} \tag{7.1.5}$$

We derive a Lorentz-like force law:

$$\frac{d^2x^a}{d\tau^2} + \hat{\Gamma}^a_{(bc)}\frac{dx^b}{d\tau}\frac{dx^c}{d\tau} = -(Q'/m_{k0})F^a_b\frac{dx^b}{d\tau} \tag{7.1.6}$$

Putting arbitrary k and variable ϕ back in we have:

$$\frac{d^2x^a}{d\tau^2} + \hat{\Gamma}^a_{(bc)}\frac{dx^b}{d\tau}\frac{dx^c}{d\tau} = -k(Q'/m_{k0})(\phi^2 F^a_b - g^{ad}A_b\delta_d\phi^2)\frac{dx^b}{d\tau} - \frac{1}{2}g^{\mathrm{ad}}\delta_d\phi^2\frac{dx^4}{d\tau}\frac{dx^4}{d\tau} \tag{7.1.7}$$

7.2 Constant Toth Charge

Having derived a Lorentz-like force law we look also at the fact that the momentum of the charge in the Kaluza dimension can not be changed by the action of an electromagnetic field alone. We look at this acceleration as with the Lorentz force law. We have, with torsion (and $k = 1$):

$$0 = \frac{d^2x^4}{d\tau^2} + \hat{\Gamma}^4_{(BC)}\frac{dx^B}{d\tau}\frac{dx^C}{d\tau}$$

$$= \frac{d^2x^4}{d\tau^2} + \hat{\Gamma}^4_{(bc)}\frac{dx^b}{d\tau}\frac{dx^c}{d\tau} + \hat{\Gamma}^4_{(4c)}\frac{dx^4}{d\tau}\frac{dx^c}{d\tau} + \hat{\Gamma}^4_{(b4)}\frac{dx^b}{d\tau}\frac{dx^4}{d\tau} + \hat{\Gamma}^4_{44}\frac{dx^4}{d\tau}\frac{dx^4}{d\tau}$$

$$= \frac{d^2x^4}{d\tau^2} + \hat{\Gamma}^4_{(bc)}\frac{dx^b}{d\tau}\frac{dx^c}{d\tau} + 2\hat{\Gamma}^4_{(4c)}\frac{dx^4}{d\tau}\frac{dx^c}{d\tau} + \frac{1}{2}A^d\delta_d\phi^2\frac{dx^4}{d\tau}\frac{dx^4}{d\tau} \tag{7.2.1}$$

7.3 Unitary Scalar Field And Torsion

Both equations above (7.1.7) and (7.2.1) have a term that wrecks havoc to any similarity with the Lorentz force law proper, the terms at the end. Both terms can however be eliminated by setting the scalar field to 1. This is postulate B1. This leads to torsion as the only way available to us to allow for non-perpendicular electromagnetic solutions. All the previous workings apply with torsion admitted, except the standard field equations of Kaluza theory, here called the torsionless field equations.

The two equations under B1 become (for all k):

$$\frac{d^2x^a}{d\tau^2} + \hat{\Gamma}^a_{(bc)}\frac{dx^b}{d\tau}\frac{dx^c}{d\tau} = -k(Q'/m_{k0})F^a_b\frac{dx^b}{d\tau} \tag{7.3.1}$$

$$\frac{d^2x^4}{d\tau^2} + \hat{\Gamma}^4_{(bc)}\frac{dx^b}{d\tau}\frac{dx^c}{d\tau} = -k^2(Q'/m_{k0})A_cF^c_b\frac{dx^b}{d\tau} = 0 \tag{7.3.2}$$

This certainly looks more hopeful. The more extreme terms have disappeared, the general appearance is similar to the Lorentz force law proper. The right hand side of (7.3.2) vanishes by B2 and the potenial gauge constraint. This and equation (7.2.1) equate to no change in momentum in the Kaluza direction, and small changes to any kinetic Toth charge in a flat space approximation.

7.4 The Lorentz Force Law

It is necessary to confirm that equation (7.3.1) not only looks like the Lorentz force law formally, but is indeed the Lorentz force law. Multiplying both sides of (7.3.1) by $\frac{d\tau}{d\tau'}\frac{d\tau}{d\tau'}$, where τ' is an alternative affine coordinate frame, gives:

$$\frac{d^2x^a}{d\tau'^2} + \hat{\Gamma}^a_{(bc)}\frac{dx^b}{d\tau'}\frac{dx^c}{d\tau'} = -k\frac{d\tau}{d\tau'}(Q'/m_{k0})F^a_b\frac{dx^b}{d\tau'} \tag{7.4.1}$$

Given $Q^* = Q'\frac{d\tau}{d\tau^*}$ and therefore $\frac{m_{k0}}{m_0}Q^* = Q'\frac{d\tau}{dt_0}$ by definition, we can set the frame such that $\tau' = t_0$ via the projected 4D space-time frame of the charge. And the Lorentz force is derived:

$$\frac{d^2x^a}{d\tau'^2} + \hat{\Gamma}^a_{(bc)}\frac{dx^b}{d\tau'}\frac{dx^c}{d\tau'} = -k(Q^*/m_0)F^a_b\frac{dx^b}{d\tau'} \tag{7.4.2}$$

In order to ensure the correct Lorentz force law using the conventions of Wald [7] p69, this can be rewritten as follows, using the antisymmetry of $F^a_b = -F^a{}_b$:

$$= k(Q^*/m_0)F^a{}_b\frac{dx^b}{d\tau'} \tag{7.4.3}$$

Using (6.6.5) this can be rewritten again in terms of the Maxwell charge:

$$= k(\frac{\alpha k}{2}(4\pi Q_M)/m_0)F^a{}_b\frac{dx^b}{d\tau'} \tag{7.4.4}$$

The result is that we must relate $\mathbf{G}$ and k to obtain the Lorentz force law in acceptable terms:

$$\frac{d^2x^a}{d\tau'^2} + \hat{\Gamma}^a_{(bc)}\frac{dx^b}{d\tau'}\frac{dx^c}{d\tau'} = (Q_M/m_0)F^a_b\frac{dx^b}{d\tau'}$$

$$k = 2\sqrt{\mathbf{G}} \tag{7.4.5}$$

8 Conclusion

Kaluza's 1921 theory of gravity and electromagnetism using a fifth wrapped-up spatial dimension is at the root of many modern attempts to develop new physical theories. However for a number of reasons it is sometimes considered untenable.

A new cylinder condition was imposed as with Kaluza's original theory, but one based on the covariant derivative and associated with a metric torsion connection. A generally covariant definition. A number of other constraints and definitions were provided. The result was the appearance of the missing non-perpendicular electromagnetic fields and a new definition of charge in terms of the 5D momentum. The new definition of kinetic charge and the Maxwellian charge coincide. In order to obtain non-perpendicular electromagnetic fields it was necessary to generalize matter-charge to matter-charge-spin.

Restrictions to the geometry and certain symmetries were handled by reducing the maximal atlas to a reduced Kaluza atlas that automatically handled the restrictions and symmetries without further deferment to general covariance. Physically this represents the idea that in 4D charge is a generally covariant scalar, whereas in 5D charge is entirely dependent on the frame. That this is meaningful stems from the global property of a small wrapped-up fifth spatial dimension with new cylinder condition. Mathematically the Kaluza atlas is a choice of subatlas for which the partial derivatives in the Kaluza direction are vanishing. This led to useful constraints on the Christoffel symbols for all coordinate systems in the Kaluza atlas.

Decomposition of the 5D metric into a 4D metric and a vector and scalar part was also possible.

One oustanding issue is that realistic charge models are not possible without involving imaginary numbers. However, that problem does not extend to 4D space-time and all the imaginary numbers actually disappear as soon as they are interpreted with respect to space-time. Thus no actual contradiction with experiment need arise. The 4D construction can be investigated independently of the 5D model. Barring realistic charge models which pose peculiar challenges, the postulates actually required are straight forward. It is in this sense a simple theory. In effect all we have is a 5D manifold with a covariant cylinder condition on one spatial dimension defined with respect to a completely antisymmetric metric torsion tensor.

Given certain assumptions about matter-charge-spin models, the entirety of classical electrodynamics is rederived. Gravity and electromagnetism are unified in a way not fully achieved by general relativity, Einstein-Cartan theory or Kaluza's original 5D theory.

The collocation of torsion with electromagnetism is different from other Einstein-Cartan theories where the torsion is limited to within matter models. Here torsion is an essential counterpart of electromagnetism. This promises experimental differences in the field equations. Similarly experimental differences may creep in due to high curvatures.

Due to the lack of realistic charge models and certain other considerations, this theory remains at an early stage of development, though the essential ingredients are present. Other issues that remain outstanding include the exact role of spin in the dynamics of particles, and a fuller exploration of the energy conditions and causality. Much work from Einstein-Cartan theory could probably be carried over without loss.

Why go to all this effort to unify electromagnetism and gravitation and to make electromagnetism fully geometric? Because experimental differences due to torsion in electromagnetic fields should be detectable given sufficient technology. In this respect this theory differs from both general relativity and Einstein-Cartan theory. The expected ω-consistency of Einstein-Cartan theory together with the derivation of a Lorentz force law via the Kaluza part of the theory gives a unique theoretical motivation, as does the fact that the other approaches beyond general relativity have not fulfilled their promise completely. Further, attempting to extend and unify classical theory prior to a full unification with quantum mechanics may even be a necessary step in a future unification.

9 Appendix

9.1 The Christoffel Symbols

Here we assume only the definitions of the Christoffel symbols and the new KCC. (Without torsion terms shown, k set to 1)

$$2\hat{\Gamma}^{A}_{\ BC} = \sum_d \hat{g}^{AD}(\partial_B \hat{g}_{CD} + \partial_C \hat{g}_{DB} - \partial_D \hat{g}_{BC})$$

$$= \sum_d \hat{g}^{Ad}(\partial_B \hat{g}_{Cd} + \partial_C \hat{g}_{dB} - \partial_d \hat{g}_{BC}) + \hat{g}^{A4}(\partial_B \hat{g}_{C4} + \partial_C \hat{g}_{4B} - \partial_4 \hat{g}_{BC})$$

$$2\hat{\Gamma}^{A}_{\ bc} = \sum_d \hat{g}^{Ad}(\partial_b g_{cd} + \partial_c g_{db} - \partial_d g_{bc}) + \sum_d \hat{g}^{Ad}(\partial_b \phi^2 A_c A_d + \partial_c \phi^2 A_d A_b - \partial_d \phi^2 A_b A_c) + \hat{g}^{A4}(\partial_b \phi^2 A_c + \partial_c \phi^2 A_b - \partial_4 g_{bc} - \partial_4 \phi^2 A_b A_c)$$

$$2\hat{\Gamma}^{A}_{\ 4c} = \sum_d \hat{g}^{Ad}(\partial_4 g_{cd} + \partial_4 \phi^2 A_c A_d + \partial_c \phi^2 A_d - \partial_d \phi^2 A_c) + \hat{g}^{A4}\partial_c \phi^2$$

$$2\hat{\Gamma}^{A}_{\ 44} = 2\sum_d \hat{g}^{Ad}\partial_4 \phi^2 A_d - \sum_d \hat{g}^{Ad}\partial_d \phi^2 + \hat{g}^{A4}\partial_4 \phi^2$$

(9.1.1)

The Electromagnetic Limit $\boldsymbol{\phi^2 = 1}$

$$2\hat{\Gamma}^{A}_{\ bc} = \sum_d \hat{g}^{Ad}(\partial_b g_{cd} + \partial_c g_{db} - \partial_d g_{bc}) + \sum_d \hat{g}^{Ad}(\partial_b A_c A_d + \partial_c A_d A_b - \partial_d A_b A_c) + \hat{g}^{A4}(\partial_b A_c + \partial_c A_b - \partial_4 g_{bc} - \partial_4 A_b A_c)$$

$$2\hat{\Gamma}^{A}_{\ 4c} = \sum_d \hat{g}^{Ad}(\partial_4 g_{cd} + \partial_4 A_c A_d + \partial_c A_d - \partial_d A_c)$$

$$\hat{\Gamma}^{A}_{\ 44} = \sum_d \hat{g}^{Ad}\partial_4 A_d$$

(9.1.2)

Simplifying...

$$2\hat{\Gamma}^{a}_{\ bc} = 2\Gamma^{a}_{\ bc} + \sum_d g^{ad}(A_b F_{cd} + A_c F_{bd}) + A^a \partial_4 g_{bc} + A^a \partial_4 A_b A_c$$

$$2\hat{\Gamma}^{4}_{\ bc} = -\sum_d A^d(\partial_b g_{cd} + \partial_c g_{db} - \partial_d g_{bc}) - \sum_d A^d(A_b F_{cd} + A_c F_{bd}) - (1 + \sum_i A_i A^i)(\partial_4 g_{bc} + \partial_4 A_b A_c) + (\partial_b A_c + \partial_c A_b)$$

$$2\hat{\Gamma}^{a}_{\ 4c} = \sum_d g^{ad}(\partial_4 g_{cd} + \partial_4 A_c A_d) + \sum_d g^{ad} F_{cd}$$

$$2\hat{\Gamma}^{4}_{\ 4c} = -\sum_d A^d(\partial_4 g_{cd} + \partial_4 A_c A_d) - \sum_d A^d F_{cd}$$

$$\hat{\Gamma}^{a}_{\ 44} = \sum_d g^{ad}\partial_4 A_d$$

$$\hat{\Gamma}^{4}_{\ 44} = -\sum_d A^d \partial_4 A_d$$

(9.1.3)

The Scalar Limit $A_i = 0$

$$
\begin{aligned}
2\hat{\Gamma}^{A}_{bc} &= \textstyle\sum_d \hat{g}^{Ad}(\partial_b g_{cd} + \partial_c g_{ab} - \partial_d g_{bc}) - \hat{g}^{A4}\partial_4 g_{bc} \\
2\hat{\Gamma}^{A}_{4c} &= \textstyle\sum_d \hat{g}^{Ad}\partial_4 g_{cd} + \hat{g}^{A4}\partial_c \phi^2 \\
2\hat{\Gamma}^{A}_{44} &= -\textstyle\sum_d \hat{g}^{Ad}\partial_d \phi^2 + \hat{g}^{A4}\partial_4 \phi^2
\end{aligned}
\tag{9.1.4}
$$

Simplifying...

$$
\begin{aligned}
\hat{\Gamma}^{a}_{bc} &= \Gamma^{a}_{bc} \\
2\hat{\Gamma}^{4}_{bc} &= -\tfrac{1}{\phi^2}\partial_4 g_{bc} \\
2\hat{\Gamma}^{a}_{4c} &= \textstyle\sum_d g^{ad}\partial_4 g_{cd} \\
2\hat{\Gamma}^{4}_{4c} &= \tfrac{1}{\phi^2}\partial_c \phi^2 \\
2\hat{\Gamma}^{a}_{44} &= -\textstyle\sum_d g^{ad}\partial_d \phi^2 \\
2\hat{\Gamma}^{4}_{44} &= \tfrac{1}{\phi^2}\partial_4 \phi^2
\end{aligned}
\tag{9.1.5}
$$

The Electromagnetic Limit And Cylinder Condition

By applying equations (6.1.10) and the new KCC in order to simplify terms of the electromagnetic limit with $k = 1$, again omitting the torsion contribution (noting that these Christoffel symbols are symmetric in the lower indices):

$$
\begin{aligned}
2\hat{\Gamma}^{a}_{bc} &= 2\Gamma^{a}_{bc} + (A_b F_c{}^{a} + A_c F_b{}^{a}) \\
2\hat{\Gamma}^{4}_{bc} &= -\textstyle\sum_d A^d(\partial_b g_{cd} + \partial_c g_{db} - \partial_d g_{bc}) + (\partial_b A_c + \partial_c A_b) \\
&= -2A_d \Gamma^{d}_{bc} + (\partial_b A_c + \partial_c A_b) \\
2\hat{\Gamma}^{a}_{4c} &= F_c{}^{a} \\
\hat{\Gamma}^{4}_{4c} &= \hat{\Gamma}^{a}_{44} = \hat{\Gamma}^{4}_{44} = 0
\end{aligned}
\tag{9.1.6}
$$

Now with torsion built into the Christoffel symbols, using equations (6.1.1) -(6.1.10) (noting that these Christoffel symbols are not necessarily symmetric in the lower indices):

$$
\begin{aligned}
2\hat{\Gamma}^{a}_{bc} &= g^{ad}(\partial_b g_{cd} + \partial_c g_{db} - \partial_d g_{bc}) + (A_b F_c{}^{a} + A_c F_b{}^{a}) - 2\hat{K}_{bc}{}^{a} \\
&= 2\Gamma^{a}_{bc} + (A_b F_c{}^{a} + A_c F_b{}^{a}) - 2\hat{K}_{bc}{}^{a} \\
2\hat{\Gamma}^{4}_{bc} &= -\textstyle\sum_d A^d(\partial_b g_{cd} + \partial_c g_{db} - \partial_d g_{bc}) + (\partial_b A_c + \partial_c A_b) - 2\hat{K}_{bc}{}^{4} \\
&= -2A_d \Gamma^{d}_{bc} + (\partial_b A_c + \partial_c A_b) - 2\hat{K}_{bc}{}^{4} \\
\hat{\Gamma}^{a}_{4c} &= 0 \\
\hat{\Gamma}^{4}_{4c} &= \hat{\Gamma}^{4}_{c4} = \hat{\Gamma}^{a}_{44} = \hat{\Gamma}^{4}_{44} = 0 \\
\hat{\Gamma}^{a}_{c4} &= F_c{}^{a}
\end{aligned}
\tag{9.1.7}
$$

9.2 Raised Levi-Civita Christoffel Symbols

Following the same procedure as with Christoffel symbols of the first and second kind a raised version of the Christoffel symbols can be derived, a third kind. It takes the value that would be guessed at by inspecting the elements of the Christoffel symbol (within partial derivatives and therefore not allowed as such) of the second kind and raising each element individually.

Starting from the covariant derivative of the raised metric tensor being zero:

$$\nabla^i g^{jk} = 0 = \partial^i g^{jk} + \Gamma^{jik} + \Gamma^{kij}$$

Cycling indices we have:

$$\begin{aligned} 0 &= \partial^k g^{ij} + \Gamma^{ikj} + \Gamma^{jki} \\ 0 &= \partial^j g^{ki} + \Gamma^{kji} + \Gamma^{ijk} \end{aligned}$$

Adding the first two and subtracting the third:

$$\partial^i g^{jk} + \partial^k g^{ij} - \partial^j g^{ki} = 2\Gamma^{jik}$$

So, exactly as would be guessed:

$$\Gamma^{ijk} = \frac{1}{2}(\partial^j g^{ik} + \partial^k g^{ji} - \partial^i g^{jk}) \tag{9.2.1}$$

9.3 Local Ricci Scalar Curvature, $k = 1$

Here we calculate the Ricci scalar curvature for $k = 1$ at the local limit defined by vanishing first derivatives of metric components, with Ricci curvature defined relative to the torsionless connection. Using (6.2.2),(6.3.1) and (6.4.1):

$$|\hat{\mathcal{R}}_{AB}| \to \begin{bmatrix} \partial_c \hat{\Gamma}^c_{ba} - \partial_b \hat{\Gamma}^c_{ca} & \frac{1}{2}\partial_c F_a{}^c \\ \frac{1}{2}\partial_c F_a{}^c & 0 \end{bmatrix} \tag{9.3.1}$$

The Ricci scalar can now be calculated as follows, the apparent asymmetry being caused by the limit terms:

$$|\hat{g}^{DA}||\hat{\mathcal{R}}_{AB}| = |\hat{\mathcal{R}}^D_B|$$

$$\to \begin{bmatrix} \partial_c \hat{\Gamma}^{c\,d}_{\ b} - \partial_b \hat{\Gamma}^{c\,d}_{\ c} - A^d \frac{1}{2}\partial_c F_b{}^c & \frac{1}{2}\partial_c F^{dc} \\ -A^a(\partial_c \hat{\Gamma}^c{}_{ba} - \partial_b \hat{\Gamma}^c{}_{ca}) + \frac{1}{2}\partial_c F_b{}^c(\frac{1}{\phi^2} + A_i A^i) & -A^a \frac{1}{2}\partial_c F_a{}^c \end{bmatrix} \tag{9.3.2}$$

$$\hat{\mathcal{R}} \to \partial_c \hat{\Gamma}^{c\,d}_{\ d} - \partial_d \hat{\Gamma}^{c\,d}_{\ c} - A^d \frac{1}{2}\partial_c F_d{}^c - A^a \frac{1}{2}\partial_c F_a{}^c = \partial_c \hat{\Gamma}^{c\,d}_{\ d} - \partial_d \hat{\Gamma}^{c\,d}_{\ c} - A^d \partial_c F_d{}^c$$

The last term can be discarded by using the chain rule, the local limit and the gauge equation (6.1.9):

$$\hat{\mathcal{R}} \to \partial_c \hat{\Gamma}^{c\,d}_{\ d} - \partial_d \hat{\Gamma}^{c\,d}_{\ c} \tag{9.3.3}$$

9.4 Proper Kaluza Velocity As Space-Time Scalar

This section shows that the proper velocity **W** (written as a vector) with only one component in the Kaluza dimension is invariant under 4D space-time boosts orthogonal to it.

$W_4 = dx_4/d\tau$ proper velocity in a stationary space-time frame, but following the particle

$U_4 = \frac{W_4}{\sqrt{1+{W_4}^2}}$ coordinate velocity using proper velocity formula

Using orthogonal addition of coordinate velocities formula to boost space-time frame by orthogonal coordinate velocity **V**:

$$\mathbf{V} = (V, 0, 0, 0)$$
$$\mathbf{U} = (0, 0, 0, U_4)$$

Coordinate velocity vector in new frame, using the orthogonal velocity addition formula:

$$\overline{\mathbf{U}} = \mathbf{V} + \sqrt{1-V^2}\ \mathbf{U}$$

So,

$$\overline{U}_4 = \sqrt{1-V^2}\frac{W_4}{\sqrt{1+W_4^2}}$$

Define proper velocity in new frame: $\overline{\mathbf{W}}$, using proper velocity definition:

$$\begin{aligned}\overline{W}_4 &= \frac{\overline{U}_4}{\sqrt{1-V^2-{\overline{U}_4}^2}} \\ &= \frac{\sqrt{1-V^2}\frac{W_4}{\sqrt{1+W_4^2}}}{\sqrt{1-V^2-(\sqrt{1-V^2}\frac{W_4}{\sqrt{1+W_4^2}})^2}} \\ &= \frac{W_4}{\sqrt{1+W_4^2}\sqrt{1-(\frac{W_4}{\sqrt{1+W_4^2}})^2}} \\ &= \frac{W_4}{\sqrt{1+W_4^2-W_4^2}} = W_4\end{aligned} \tag{9.4.1}$$

$\overline{W_4} = W_4$ being the result required

The proper Kaluza velocity therefore being a scalar with respect to the Kaluza atlas.

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11 References

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