A theorem producing the fine structure constant inverse and the quark and lepton mixing angles

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(Dated: March 30, 2012)

The value 137.036, a close approximation of the fine structure constant inverse, is shown to occur naturally in connection with a theorem employing a pair of related functions. It is also shown that the formula producing this approximation contains terms expressible using the sines squared of the experimental quark and lepton mixing angles, implying an underlying relationship between these constants. This formula places the imprecisely measured neutrino mixing angle θ_{13} at close to 8.09°, so that $\sin^2 2\theta_{13} \approx 0.0777$.

PACS numbers: 06.20.Jr

The value 137.036, a close approximation of the fine structure constant (FSC) inverse $1/\alpha$ [1], is shown to occur naturally in connection with a theorem employing a pair of related functions [2]. It is also shown that the formula producing this FSC approximation contains terms expressible using the sines squared of the experimental quark and lepton mixing angles, implying an underlying relationship between these constants.

I. TWO FUNCTION DEFINITIONS

We begin by defining the pair of related functions that the theorem will exploit. Let M and N be positive integer constants, so that

$$h(u) = \frac{M^3 - u^3}{N^3} + M^2 - u^3$$
$$j(u) = \frac{(M-u)^3}{N^3} + (M-u)^2 \quad ,$$

where u is a variable such that

$$0 < u \leq 0.1 \tag{1.1}$$

and

$$M \geq 10 \quad . \tag{1.2}$$

II. THE FSC THEOREM

We then specify and prove the theorem making use of these functions.

Theorem 1. (*The FSC Theorem.*) *Let*

$$j(y) = h(x) \tag{2.1}$$

satisfying

$$M = \frac{N^3}{3} + 1 \quad . \tag{2.2}$$

Then at

$$x = \frac{1}{M} \tag{2.3}$$

we get

$$\frac{dy}{dx} \approx \frac{1}{M^3} \quad . \tag{2.4}$$

Proof. Equation (2.1) gives

$$\frac{(M-y)^3}{N^3} + (M-y)^2 = \frac{M^3 - x^3}{N^3} + M^2 - x^3 \quad ,$$

which expands and simplifies to

$$-\frac{3M^2 y}{N^3} + \frac{3My^2}{N^3} - \frac{y^3}{N^3} - 2My + y^2 = -\frac{x^3}{N^3} - x^3 \quad ,$$

or

$$3M^2 y - 3My^2 + y^3 + 2MN^3 y - N^3 y^2 = (N^3+1)x^3 \quad .$$

It follows that

$$(3M^2 - 6My + 3y^2 + 2MN^3 - 2N^3 y)dy = 3(N^3+1)x^2 dx \quad ,$$

so that

$$\frac{dy}{dx} = \frac{3(N^3+1)x^2}{3M^2 - 6My + 3y^2 + 2MN^3 - 2N^3 y} \quad .$$

We want to remove all terms from the denominator that are small relative to M. As Eq. (2.2) requires that

$$N^3 = 3M - 3 \quad ,$$

substituting for N^3 in the denominator gives

$$\begin{aligned} \frac{dy}{dx} &= \frac{3(N^3+1)x^2}{3M^2 - 6My + 3y^2 + 2M(3M-3) - 2y(3M-3)} \\ &= \frac{3(N^3+1)x^2}{3M^2 - 6My + 3y^2 + 6M^2 - 6M - 6My + 6y} \\ &= \frac{3(N^3+1)x^2}{9M^2 - 12My + 3y^2 - 6M + 6y} \\ &= \frac{(N^3+1)x^2}{3M^2 - 4My + y^2 - 2M + 2y} \quad . \end{aligned} \tag{2.5}$$

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But Eqs. (1.1) and (1.2) require that $y \leq M/100$, so that $4My$, y^2, and $2y$ in the denominator are necessarily small compared to M. It follows that the approximation

$$\frac{dy}{dx} \approx \frac{(N^3+1)x^2}{(3M-2)M}$$

holds. But Eq. (2.2) also provides that

$$M = N^3/3 + 1 \quad ,$$

so that substituting for the first M in the denominator gives

$$\begin{aligned}\frac{dy}{dx} &\approx \frac{N^3+1}{3(N^3/3+1)-2} \times \frac{x^2}{M} \\ &\approx \frac{N^3+1}{N^3+3-2} \times \frac{x^2}{M} \\ &\approx \frac{N^3+1}{N^3+1} \times \frac{x^2}{M} \quad .\end{aligned}$$

Accordingly,

$$\frac{dy}{dx} \approx \frac{x^2}{M} \quad ,$$

so that at

$$x = \frac{1}{M}$$

we get

$$\frac{dy}{dx} \approx \frac{1}{M^3} \quad . \qquad \square$$

III. THE FINE STRUCTURE CONSTANT INVERSE

Now inspection reveals that $M = 10$ and $N = 3$ are the smallest positive integers fulfilling Eq. (2.2). For this solution Eq. (2.1) gives

$$\frac{(10-y)^3}{3^3} + (10-y)^2 = \frac{10^3 - x^3}{3^3} + 10^2 - x^3 \quad .$$

Then, by Theorem 1, at

$$x = \frac{1}{M} = 0.1$$

we get

$$\frac{dy}{dx} \approx \frac{1}{M^3} = 0.001 \quad ,$$

where Eq. (2.1) gives $y \approx 0.00003333340873$.

As M and N are minimal this M, N, and x will be termed the *minimal solution* to Eq. (2.1). By substituting $M = 10$ and $N = 3$ into Eq. (2.5), we get

$$\begin{aligned}\frac{dy}{dx} &= \frac{28x^2}{(28-y)(10-y)} \\ &\approx 0.001000004524 \quad ,\end{aligned}$$

which shows the accuracy of the above approximation.

The key point, however, is that the minimal solution to Eq. (2.1) *simultaneously* produces

$$\begin{aligned}h(x) &= \frac{M^3 - x^3}{N^3} + M^2 - x^3 \\ &= \frac{10^3 - 0.1^3}{3^3} + 10^2 - 0.1^3 \\ &= 137.036 \quad ,\end{aligned} \tag{3.1}$$

the fine structure constant inverse approximation promised at the outset, where the experimental FSC inverse is fit within seven parts per billion [1]. Hence, Theorem 1 will be termed *The FSC Theorem*, and Eq. (2.1) *The FSC Equation.* In this way this close FSC inverse approximation occurs as the natural and unique result of the analysis of the above pair of related functions, showing that 137.036 is relevant to pure mathematics independent of its role as a constant famous to physicists.

IV. THE QUARK AND LEPTON MIXING ANGLES

But it is also noteworthy that the above minimal solution is expressible using the sines squared of the experimental quark and lepton mixing angles [3][4]. Recall that for the minimal solution

$$\begin{aligned}j(y) &= \left(\frac{M-y}{3}\right)^3 + (M-y)^2 \\ &\approx \left(\frac{10}{3} - \frac{0.0000333334}{3}\right)^3 + (10 - 0.0000333334)^2 \\ &= 137.036 \quad .\end{aligned} \tag{4.1}$$

But within the limits of experimental error the four terms used above can be replicated by the sines squared of the six quark and lepton mixing angles

$$M/3 = 1/\sin^2 L12 \tag{4.2a}$$
$$y/3 = \sin^2 Q13 \tag{4.2b}$$
$$M = \sin^2 L23 \times 1/\sin^2 Q12 \tag{4.2c}$$
$$y = \sin^2 Q23 \times \sin^2 L13 \quad , \tag{4.2d}$$

so that

$$10/3 = 1/\sin^2 L12 \tag{4.3a}$$
$$0.0000333334/3 \approx \sin^2 Q13 \tag{4.3b}$$
$$10 = \sin^2 L23 \times 1/\sin^2 Q12 \tag{4.3c}$$
$$0.0000333334 \approx \sin^2 Q23 \times \sin^2 L13 \quad , \tag{4.3d}$$

and

$$\begin{aligned}j(y) &= \left(\frac{1}{\sin^2 L12} - \sin^2 Q13\right)^3 \\ &+ \left(\frac{1}{\sin^2 Q12} \times \sin^2 L23 - \sin^2 L13 \times \sin^2 Q23\right)^2 \\ &= 137.036 \quad .\end{aligned} \tag{4.4}$$

TABLE I: Quark mixing data compared against predictions.

	$\|V_{us}\|$	$\|V_{ub}\|$
Prediction	0.2236	0.003333
2010 [a]	$0.2253^{+0.0007}_{-0.0007}$	$0.00347^{+0.00016}_{-0.00012}$
Error in SD	2.4	1.1

[a] Ref. [3]. Particle Data Group 1σ global fit.

TABLE II: Lepton mixing data compared against predictions.

	$\sin^2 L12$	$\sin^2 L13$	$\sin^2 L23$
Prediction	0.3	0.0198 [a]	0.5
2011 [b]	$0.312^{+0.017}_{-0.015}$	$0.013^{+0.007}_{-0.005}$	$0.52^{+0.06}_{-0.07}$
Error in SD	0.8	1.0	0.3

[a] This prediction rests on the assumption that $Q23 = 2.35°$, which derives from [3]. See Eq. (4.5).
[b] Ref. [4]. A 1σ global fit that assumes the normal hierarchy.

Now, if we assume that

$$\sin^2 L23 = 0.5$$

then

$$\sin^2 Q12 = 0.05 \quad .$$

Moreover, given that Eq. (4.3d) implies that

$$\sin^2 L13 \approx 0.0000333334/\sin^2 Q23 \quad , \tag{4.5}$$

and we know that $Q23$ measures roughly 2.35 degrees [3], then

$$\sin^2 L13 \approx 0.0198 \tag{4.6a}$$
$$L13 \approx 8.09° \tag{4.6b}$$
$$\sin^2 2L13 \approx 0.0777 \quad . \tag{4.6c}$$

The fit of the above predictions in the quark sector can be seen in Table I, where $Q12$ and $Q13$ are primarily responsible for $|V_{us}|$ and $|V_{ub}|$, respectively. The fit of the above predictions in the leptonic sector can be seen in Table II. The largest of all these errors is 2.4 SD, for $|V_{us}|$. The fit to the recent electron-antineutrino disappearance data from Daya Bay [5] is also good, as currently it provides that

$$\sin^2 2L13 \approx 0.092 \pm 0.016(\text{stat}) \pm 0.005(\text{syst}) \quad ,$$

which is not far from the value supplied by Eq. (4.6c).

And, finally, here are all of the predicted mixing angles expressed in degrees:

$$\begin{aligned} L23 &= 45° \\ L13 &= \ \ 8.09° \\ L12 &= 33.210911° \\ Q13 &= \ \ 0.190987° \\ Q12 &= 12.920966° \quad . \end{aligned}$$

But notice that all of the above angles *except* $L13$ were predicted in 2007 by an alternate method that required that $L13$ measure about $1/73^{\text{rd}}$ of a degree. These widely differing values for $L13$ depend on whether $\sin^2 L13$ is set equal to 0.0000333334 *divided* by $\sin^2 Q23$, as in Eq. (4.5), or 0.0000333334 *multiplied* by $\sin^2 Q23$, as in [6].

V. CONCLUSION

Consider that Theorem 1 is purely mathematical: it possesses neither variables nor constants chosen for physical reasons, which means that whatever physical ordering it achieves has not been superimposed by a succession of expedient choices. And yet, the minimal solution to Eq. (2.1) produces values for the fine structure constant and four of the quark and lepton mixing angles that fit experiment closely, where collectively these are known with extraordinary precision. For the minimal solution to reproduce closely either the fine structure constant or the above four angles would be in itself powerful evidence that it has physical significance, but it closely reproduces both.

[1] P. J. Mohr, B. N. Taylor, and D.B. Newell (2011), "The 2010 CODATA Recommended Values of the Fundamental Physical Constants" (Web Version 6.0). This database was developed by J. Baker, M. Douma, and S. Kotochigova. Available: `http://physics.nist.gov/constants` [Friday, 22-Jul-2011 10:04:27 EDT]. National Institute of Standards and Technology, Gaithersburg, MD 20899.

[2] J. S. Markovitch, "The fine structure constant derived from the broken symmetry of two simple algebraic identities," (2012) `http://vixra.org/abs/1102.0012`.

[3] A. Ceccucci, Z. Ligeti, and Y. Sakai, "The CKM Quark-Mixing Matrix" in K. Nakamura *et al.* (Particle Data Group), J. Phys. **G37**:075021 (2010).

[4] T. Schwetz, M. Tortola and J. W. F. Valle, "Where we are on θ_{13}: addendum to 'Global neutrino data and recent reactor fluxes: status of three-flavour oscillation parameters'," New J. Phys. **13**, 109401 (2011) [arXiv:1108.1376 [hep-ph]].

[5] F. P. An *et al.* [DAYA-BAY Collaboration], "Observation of electron-antineutrino disappearance at Daya Bay," arXiv:1203.1669 [hep-ex].

[6] J. S. Markovitch, "A mathematical model of the quark and lepton mixing angles (2011 update)," (2011) `http://www.vixra.org/abs/1102.0021`.