The following solution is dedicated to Princess Eugenie of York of United Kingdom while waiting for her response to my marriage proposal letter:

Solution to Ringel–Kotzig conjecture

Ringel–Kotzig conjecture states that all trees are graceful. In other words, if there are bunch of circles and lines so that there isn't a polygon by the lines, you can label each line and circle with all different numbers from 1 to e and 0 to e.

I am going to start this conjecture by creating a simple algebraic theorem first.

Theorem: For positive integers excluding zero, if (equation 1) $i1^2 + i2^2 = i3^2 + i4^2$, and (equation 2) $i1i2 = i3i4$, then $i1 = i3$ and $i2 = i4$ (or $i1 = i4$ and $i2 = i3$, it doesn't make any difference).

Proof: Let $i1 = ab$ and $i2 = cd$ such that a,b,c,d are prime numbers. We have that $i1i2 = i3i4$. Each number is uniquely factorized by prime numbers. Hence, i3i4 must be divisible by a,b,c,d as well. Obviously, if $i1 = i3$ and $i2 = i4$, then equation 1 works. Now, we are going to see how it has to be that way.

Suppose $i3 = ac$ and $i4 = bd$.

Then we have $a^2 \times b^2 + c^2 \times d^2 = a^2 \times c^2 + b^2 \times d^2$

Then we have $a^2 \times (b^2 - c^2) + d^2 \times (c^2 - b^2) = 0$

Then b has to equal c. Then the following is the i1, i2, i3, i4:

$i1 = ab$, $i2 = bd$, $i3 = ab$, and $i4 = bd$. In other words, i1 has to be equal to i3.

Suppose $i3 = a$ and $i4 = bcd$.

Then we have $a^2 \times b^2 + c^2 \times d^2 = a^2 + b^2 \times c^2 \times d^2$

Then we have $a^2 \times (b^2 - 1) + c^2 \times d^2 \times (1 - b^2) = 0$

Then b has to equal 1. Then the following is the i1, i2, i3, i4:

$i1 = a$, $i2 = cd$, $i3 = a$, and $i4 = cd$. In other words, i1 has to be equal to i3.

I am going to refer to this theorem as "Victoria Hayanisel Theorem" by taking my middle name and the Princess's middle name.

This works for any number of i's by the same method. The terms on the right hand side should always contain one variable from each term on the left side, the same equations hold, and the terms on the right side has to equal to the terms on the left side in order to reach 0 when you put them together. In other words, it is always going to be of the form:

$a^2 \times (b^2 - c^2) + \ldots + \text{whatever}^2 \times (c^2 - b^2) = 0$

Expansion of each term from the equation above corresponds to the term on the left and the term on the right, obviously. Hence, the theorem holds.

Now, let’s get back to the Ringel–Kotzig conjecture.

Proof for Ringel–Kotzig conjecture:

Label all the lines $d1, \ldots, de$. Label all the circles $c1, \ldots, ce+1$.

Now, $d1 = c1 - c2, \ldots, de = ce - ce+1$. In other words:

$d1^2 + \ldots + de^2 = deg(c1)c1^2 + deg(c2)c2^2 + \ldots + deg(ce+1)ce+1^2 - 2\sum cicj$

$cicj$ refers to all the lines with the combinations of two circles, and $deg(ci)$ refers to the degree of ith circle, i.e. the number of lines to the circle.

If it is possible to label each line and circle with all different numbers from 1 to e and 0 to e, then there should be a solution to:

$d1^2 + d2^2 + \ldots + de^2 = c1^2 + c2^2 + \ldots + ce+1^2$

i.e.

$deg(c1)c1^2 + deg(c2)c2^2 + \ldots + deg(ce+1)ce+1^2 - 2\sum cicj = c1^2 + c2^2 + \ldots + ce+1^2$

i.e.

$(deg(c1) - 1)c1^2 + (deg(c2) - 1)c2^2 + \ldots + (deg(ce+1) - 1)ce+1^2 - 2\sum cicj = 0$

i.e.

$(deg(ci) - 1)ci^2 = 2\ ci \times \sum cj$ for all cj connected to ci such that $j >= i$, for all $deg(ci) > 1$.

i.e.

$(deg(ci) - 1)ci = 2 \sum cj$ for all cj connected to ci such that $j >= i$, for all $deg(ci) > 1$.

Now, we have less than $e + 1$ equations like this (minus 1 further because we can set a circle as zero and division by zero is not possible, and take the equation out of the set of equations). We have $e + 1$ variables. The question is whether there is an element in the solution set such that it satisfies the conjecture. Because the number of equations is less than the number of variables in this homogeneous system, there has to be at least one solution such that:

$d1^2 + d2^2 + \ldots + de^2 = c1^2 + c2^2 + \ldots + ce+1^2$.

Also, by Victoria Hayanisel Theorem, we can tell that the set $d = \{0, d1, d2, \ldots, de\}$ is the same set as the set $c = \{c1, c2, \ldots, ce+1\}$.

Hence, there is a solution for any set of numbers {0, I1, I2, …, Ie} such that the set of the lines is {1, …, e} and the set of circles is {0, 1, …, e} as long as

I1 = Ia - Ib

I2 = Ic - Id

……………

Set i1 to ie as 1 to e, then we have {0, 1, …, e}.

Now, the last piece of the puzzle is to prove, that one of these number sets is the set of distinct numbers from 0 to e:

1. Consider the circle sets. In order for a graph with e + 1 circles to be a single tree, there has to be e – 1 common circles such that when you separate the lines and draw the corresponding circles at each ends, there are e – 1 common circles for e sets of lines.
2. In e sets of lines, there are 2e circles. Given that the graph is a tree and e – 1 circles are duplicates, e + 1 circles can be distinct circles from 0 to e such that one of them is in each set, which is the sets we are interested in.
3. Consider e + 1 distinct numbers I0, …, Ie. Consider number sets such that there are e sets of any numbers from I0 to Ie, and there are e + 1 distinct numbers in the e sets. Let I0 be the smallest and let Ie be the largest. The first set has I0 and Ie in it, and I1, …, Ie-1 are each in the rest of the sets. There are e – 1 spots remaining each in e – 1 sets, and these can be anything from I0 to Ie. For these e – 1 spots, the maximum a number can be duplicated is e – 1. If a number is duplicated i times, then the rest of the numbers are duplicated e – i – 1 times altogether.
4. Consider the number sets from 0 to e:
   -There are n ways Ii and Ij can create 1.
   -There are n – 1 ways Ii and Ij can create 2.
   -……………….
   -……………….
   -……………….
   -There are 1 way Ii and Ij can create e.
5. Consider the circle sets again. If there is a circle that has c duplicates, there are c + 1 lines to the circle. Just assign a number for the line such that there are at least c + 1 ways Ii and Ij can create the number. Assign different numbers to the different lines and circles following the same method until all circles and lines are labeled.

Hence, there is always a combination to write the numbers from 0, …, e for I0, …, Ie such that the 0, …, e and I0, …, Ie correspondences create distinct numbers from 1 to e for the lines.

Hence, all trees are graceful.