The main paradox of KAM-theory, exact solution of Arnold-diffusion for restricted 3-bodies problem (a case of Archimedes spiral)

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Abstract: Here are presented a key points of criticism of KAM (Kolmogorov-Arnold-Moser) theory in the application of main results to the field of celestial mechanics, especially in the case of restricted 3-bodies problem.

The main paradox of KAM-theory is that appropriate Hamilton formalism should be valid for the KAM dynamical systems, but Hamilton formalism could not be applied for *restricted* 3-bodies problem (which is proved to have only the Jacobian-type integral of motion, but the integrals of energy, momentum are not invariants).

Finally, we obtain the Jacobian-type integral of motion in the specific case of *Arnold-diffusion* for *restricted* 3-bodies problem (orbit of 2-massive bodies is assumed to be like *Archimedes spiral*). Besides, we should especially note that there is no analogue of Jacobian-type integral of motion in the case of *photogravitational* restricted 3-bodies problem if we take into consideration even a small *Yarkovsky* effect.

Key Words: KAM (Kolmogorov-Arnold-Moser) theory, Hamilton formalism, Yarkovsky effect, photogravitational restricted three body problem, Jacobian-type integral of motion

1. Introduction.

Here are presented a key points of criticism about some *initial* assumptions in KAM-(*Kolmogorov-Arnold-Moser*)-theory [1-2] when the central KAM-theorem is known to be applied for researches of stability of Solar system in terms of *restricted* 3-bodies problem [3], especially in terms of *photogravitational* restricted 3-bodies problem [4] with additional influence of *Yarkovsky* effect of non-gravitational nature [5].

KAM is the theory of stability of dynamical systems which should solve a very specific question in regard to the stability of orbits of so-called "small bodies" in Solar system [1-2], in terms of *restricted* 3-bodies problem: indeed, dynamics of all the planets is assumed to satisfy to restrictions of *restricted* 3-bodies problem (*such as infinitesimal masses, negligible deviations of the main orbital elements, etc.*).

Nevertheless, KAM also is known to assume the appropriate Hamilton formalism in proof of the central KAM-theorem [1-2]: the dynamical system is assumed to be *Hamilton* system as well as all the mathematical operations over such a dynamical system are assumed to be associated with a proper Hamilton system.

According to the Bruns theorem [6-7], there is no other invariants except well-known 10 integrals for 3-bodies problem (*including integral of energy, momentum, etc.*), this is a classical example of Hamilton system. But in case of *restricted* 3-bodies problem, there is no other invariants except only one, Jacobian-type integral of motion [3].

Such a contradiction is the main paradox of KAM-theory: it adopts all the restrictions of *restricted* 3-bodies problem, but nevertheless it proves to use the Hamilton formalism, which assumes the conservation of all other invariants (*the integral of energy, momentum, etc.*).

2. Equations of motion.

Let us consider the system of ODE for photogravitational restricted 3-bodies problem under the influence of Yarkovsky effect, at given initial conditions [5].

We consider three bodies of masses m_1, m_2 and m such that $m_1 > m_2$ and m is an infinitesimal mass. The two primaries m_1 and m_2 are sources of radiation; q_1 and q_2 are factors characterizing the radiation effects of the two primaries respectively.

We assume that m_2 is an *oblate* spheroid. The effect of *oblateness* is denoted by the factor A_2. Let r_i ($i = 1, 2$) be the distances between the centre of mass of the bodies m_1 and m_2 and the centre of mass of body m [5]. The unit of mass is chosen so that the sum of the masses of finite bodies is equal to 1. We suppose that $m_1 = 1 - \mu$ and $m_2 = \mu$, where μ is the ratio of the mass of the smaller primary to the total mass of the primaries and $0 \le \mu \le ½$. The unit of distance is taken as the distance between the primaries. The unit of time is chosen so that the gravitational constant is equal to 1.

The three dimensional restricted 3-bodies problem (*we take also into consideration the influence of Yarkovsky effect*), with an *oblate* primary m_2 and both primaries radiating, could be presented in barycentric rotating co-ordinate system by the equations of motion below [5]:

$$\ddot{x} - 2n\dot{y} = \frac{\partial \Omega}{\partial x} + Y_x(t) ,$$

$$\ddot{y} + 2n\dot{x} = \frac{\partial \Omega}{\partial y} + Y_y(t) , \qquad (2.1)$$

$$\ddot{z} = \frac{\partial \Omega}{\partial z} + Y_z(t) ,$$

$$\Omega = \frac{n^2}{2}\left(x^2 + y^2\right) + \frac{q_1(1-\mu)}{r_1} + \frac{q_2 \mu}{r_2}\left[1 + \frac{A_2}{2r_2^2}\cdot\left(1 - \frac{3z^2}{r_2^2}\right)\right] , \qquad (2.2)$$

- where $Y_x(t)$, $Y_y(t)$, $Y_z(t)$ – are the projecting of *Yarkovsky effect* acceleration $\boldsymbol{Y}(t)$ onto the appropriate axis *Ox*, *Oy*, *Oz*,

- besides, where

$$n^2 = 1 + \frac{3}{2} A_2 ,$$

- is the angular velocity of the rotating coordinate system and A_2 - is the *oblateness* coefficient. Here

$$A_2 = \frac{AE^2 - AP^2}{5R^2} ,$$

- where *AE* is the equatorial radius, *AP* is the polar radius and *R* is the distance between primaries. Besides, we should note that

$$r_1^2 = (x+\mu)^2 + y^2 + z^2 ,$$

$$r_2^2 = (x-1+\mu)^2 + y^2 + z^2 ,$$

- are the distances of infinitesimal mass from the primaries [5].

We neglect the relativistic Poynting-Robertson effect which may be treated as a perturbation for cosmic dust (*or for small particles, less than 1 cm in diameter*), see Chernikov [8], as well as we neglect the effect of variable masses of 3-bodies [9].

The possible ways of simplifying of equations (2.1):

- if we assume effect of *oblateness* is zero, $A_2 = 0$ ($\Rightarrow n = 1$), it means m_2 is *non-oblate* spheroid (we will consider only such a case below);
- if we assume $q_1 = q_2 = 1$, it means the case of restricted 3-bodies problem.

3. Arnold-diffusion.

The equations of restricted 3-bodies problem are proved to describe the system with *non-Hamilton* formalism. The additional obvious proof could be found in the structure of system (2.1) if we attentively analyze the right part of equations (2.1):

$$\ldots \; - \; 2n\,\dot{y} \; = \; \ldots \,,$$

$$\ldots \; + \; 2n\,\dot{x} \; = \; \ldots \,,$$

- but any components of velocity must be excepted for Hamilton system in the final expressions for balance of momentum [3]. This is axiom for the Hamilton systems: the Hamilton systems are assumed to be the systems *without diffusion*.

That's why Arnold [1] was the 1-st who suggested to consider Hamilton system *with weak diffusion* in celestial mechanics: such a suggestion was very modern, original correction for KAM methodology in regard to restricted 3-bodies problem.

It means that such a dynamical systems should have a weak *Arnold-diffusion* [1]: the classical invariants of such a system don't remain the same (*the integral of energy, momentum, etc.*), but all of them are subjected to a negligible diffusion during a large time-period. Besides, for the restricted 3-bodies problem is proved to have had a new stable invariant = Jacobian-type integral of motion [3].

4. Exact solution (Arnold-diffusion).

Let us consider an example of *negligible Arnold-diffusion* [1]:

Regarding the orbit of 2-massive bodies (which are rotating around their common centre of masses on Kepler's trajectories) - let us assume such orbit to be like *Archimedes spiral* (Fig.1). Also let us assume Yarkovsky effect is zero.

It means that the unit of distance, which was taken as the distance between 2-massive bodies, is not being constant but being corrected by multiplying on the expression, which exponentially negligible depends on parameter of time.

As it was proved in [10], the function Ω in (2.1) should not be depending on time for large time-period.

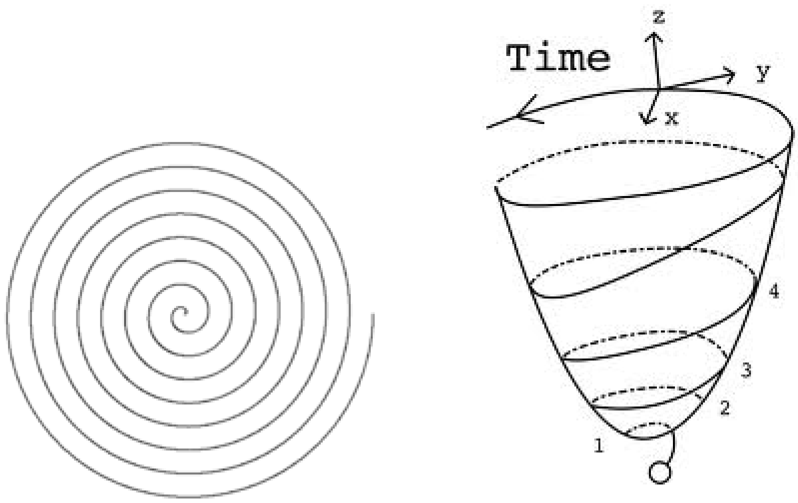

Fig.1. Archimedes spiral.

According to [3], we could obtain from the equations of system (2.1) a Jacobian-type integral of motion:

$$(\dot{x})^2 + (\dot{y})^2 + (\dot{z})^2 = 2\Omega(x, y, z) + C \tag{4.1}$$

- where *C* is so-called Jacobian constant, but Ω should not be depending on time for large time-period [10].

Thus, we obtain for the components of solution $\{x_i\} = \{x(t), y(t), z(t)\}$ after a large time-period, $x_i' = x_i \cdot \exp\{\alpha t\}$ ($i = 1, 2, 3$):

$$(\dot{x}\cdot\exp\alpha t + \alpha\exp\alpha t\cdot x(t))^2 + (\dot{y}\cdot\exp\alpha t + \alpha\exp\alpha t\cdot y(t))^2 + \\ + (\dot{z}\cdot\exp\alpha t + \alpha\exp\alpha t\cdot z(t))^2 = 2\Omega(x,y,z) + C \ , \tag{4.2}$$

- where α - is a coefficient of Arnold-diffusion, $\alpha \to 0$.

If we substract (4.1) from (4.2), we should obtain

$$\alpha\,(x(t)^2 + y(t)^2 + z(t)^2) = -\frac{d\,(x(t)^2 + y(t)^2 + z(t)^2)}{d\,t}$$

- thus

$$x(t)^2 + y(t)^2 + z(t)^2 \equiv r^2(t) = C_r\cdot\exp(-\alpha\cdot t) \tag{4.3}$$

- where C_r - is a constant which equals to the square of internal initial radius $r_o(t)$ of such an orbit.

So, we could conclude from the expression (4.3) that the distance between the body with infinitesimal mass and the centre of mass of the system should also be changing, but to the opposite direction to the changing of the distance between 2-massive bodies:

- if 2-massive bodies are to be distant from each other after large time-period *t*, the third body should move towards the centre of mass of the system;
- if 2-massive bodies are to be close to each other after a large time-period *t*, the third body should move to the opposite direction from the centre of mass of the system.

Additionally, we should especially note obvious fact: in the case of *photogravitational* restricted 3-bodies problem with Yarkovsky effect [5] there is no analogue of Jacobian-type integral for ODE system of motion (2.1).

5. Conclusion.

We discussed a key points of criticism of KAM (Kolmogorov-Arnold-Moser) theory in the application to the field of celestial mechanics, especially in the case of restricted 3-bodies problem. The main paradox of KAM-theory is that appropriate Hamilton formalism should be valid for the KAM dynamical systems, but Hamilton formalism could not be applied for *restricted* 3-bodies problem.

Nevertheless, KAM-theory tried to predict the stability for Solar system during a large time-period, despite of the fact that central KAM-theorem adopts all the restrictions of *restricted* 3-bodies problem (which was chosen as a basis for the modelling of Solar system). Such a paradox could be succesfully solved if we consider Solar system as dynamical system with Arnold diffusion.

Finally, we obtain the Jacobian-type integral of motion in the specific case of Arnold-diffusion for restricted 3-bodies problem (orbit of 2-massive bodies is assumed to be like *Archimedes spiral*).

In according to the results of Nekhoroshev [10], we obtain from the Jacobian-type integral of motion that the distance between the body with infinitesimal mass and the centre of mass of the system should also be changing, but to the opposite direction to the changing of the distance between 2-massive bodies:

- if 2-massive bodies are to be distant from each other after large time-period *t*, the third body should move towards the centre of mass of the system;

- if 2-massive bodies are to be close to each other after a large time-period *t*, the third body should move to the opposite direction from the centre of mass of the system.

Besides, we should especially note that there is no analogue of Jacobian-type integral of motion in the case of *photogravitational* restricted 3-bodies problem if we take into consideration even a small *Yarkovsky* effect [5].

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