

A FINITE REFLECTION FORMULA FOR A POLYNOMIAL APPROXIMATION TO THE RIEMANN ZETA FUNCTION

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ABSTRACT. The Riemann zeta function can be written as the Mellin transform of the unit interval map $w(x) = \lfloor x^{-1} \rfloor (x \lfloor x^{-1} \rfloor + x - 1)$ multiplied by $s \frac{s+1}{s-1}$. A finite-sum approximation to $\zeta(s)$ denoted by $\zeta_w(N; s)$ which has real roots at $s = -1$ and $s = 0$ is examined and an associated function $\chi(N; s)$ is found which solves the reflection formula $\zeta_w(N; 1 - s) = \chi(N; s) \zeta_w(N; s)$. The integrals of $\zeta_w(N; s)$ over the interval $-1 \dots 0$ have closed-form expressions involving logarithms and exponential integrals with negative arguments. The function $\chi(N; s)$ is singular at $s = 0$ and the residue at this point changes sign from negative to positive between the values of $N = 176$ and $N = 177$. Some rather elegant graphs of $\zeta_w(N; s)$ and the reflection functions $\chi(N; s)$ are also provided. The values $\zeta_w(N; 1 - n)$ for integer values of n are found to be related to the Bernoulli numbers.

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1. THE RIEMANN ZETA FUNCTION AS THE MELLIN TRANSFORM OF A UNIT INTERVAL MAP

The Riemann zeta function can be written as the Mellin transform of the unit interval map $w(x) = [x^{-1}](x [x^{-1}] + x - 1)$ multiplied by $s \frac{s+1}{s-1}$. [2]

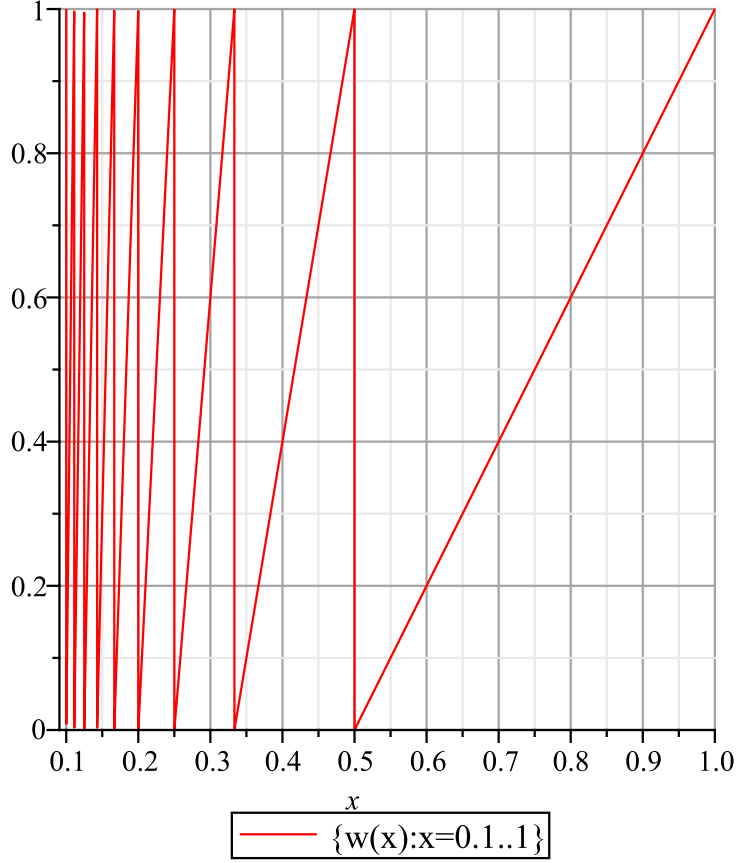


Figure 1. The Harmonic Sawtooth map

$$\begin{aligned}
 \zeta_w(s) &= \zeta(s) \forall -s \notin \mathbb{N}^* \\
 &= s \frac{s+1}{s-1} \int_0^1 [x^{-1}](x [x^{-1}] + x - 1) x^{s-1} dx \\
 &= s \frac{s+1}{s-1} \sum_{n=1}^{\infty} \int_{\frac{1}{n+1}}^{\frac{1}{n}} n(xn + x - 1) x^{s-1} dx \\
 &= \sum_{n=1}^{\infty} s \frac{s+1}{s-1} \left(-\frac{n^{1-s} - n(n+1)^{-s} - sn^{-s}}{s(s+1)} \right) \\
 &= \sum_{n=1}^{\infty} \frac{n(n+1)^{-s} - n^{1-s} + sn^{-s}}{s-1} \\
 &= \frac{1}{s-1} \sum_{n=1}^{\infty} n(n+1)^{-s} - n^{1-s} + sn^{-s}
 \end{aligned} \tag{1}$$

1.1. The Truncated Zeta Function.

The substitution $\infty \rightarrow N$ is made in the infinite sum appearing the expression for $\zeta_w(s)$ to get a finite polynomial approximation

$$\begin{aligned}\zeta_w(N; s) &= \frac{1}{s-1} \sum_{n=1}^N n(n+1)^{-s} - n^{1-s} + sn^{-s} \\ &= \frac{1}{s-1} \left(s + (N+1)^{1-s} - 1 + s \sum_{n=2}^N n^{-s} - \sum_{n=2}^{N+1} n^{-s} \right) \\ &= \frac{N}{(s-1)(N+1)^s} - \frac{\cos(\pi s) \Psi(s-1, N+1)}{\Gamma(s)} + \zeta(s) \forall s \in \mathbb{N}^*\end{aligned}\tag{2}$$

with equality in the limit except at the negative integers

$$\lim_{N \rightarrow \infty} \zeta_w(N; s) = \zeta(s) \forall -s \notin \mathbb{N}^*\tag{3}$$

The functions $\zeta_w(N; s)$ have real zeros at $s = -1$ and $s = 0$, that is

$$\lim_{s \rightarrow -1} \zeta_w(N; s) = \lim_{s \rightarrow 0} \zeta_w(N; s) = 0\tag{4}$$

One possible idea is that the functions $\zeta_w(N; s)$ can be orthonormalized over the interval $s = -1 \dots 0$ via the Gram-Schmidt process[3] and that the result might possibly shed some light on the zeroes of $\zeta(s)$. To get started in this direction, we see that the amazing Maple software package[4] is able to calculate closed-form expressions for $\int_{-1}^0 \zeta_w(N; s) ds$ utilizing the exponential integral with negative arguments

$$\text{Ei}(s) = s \left(\int_0^1 \int_0^1 e^{-sxy} dy dx \right) - \gamma - \ln(s)\tag{5}$$

The values for the first few integrals are given by

$$\begin{aligned}\int_{-1}^0 \zeta_w(1; s) ds &= 1 + \frac{\text{Ei}(-\ln(4))}{2} - \frac{\text{Ei}(\ln(2))}{2} \\ \int_{-1}^0 \zeta_w(2; s) ds &= \frac{3 + 2\text{Ei}(-\ln(9))\ln(2) - 2\text{Ei}(-\ln(3))\ln(2) + 3\ln(2)}{3\ln(2)} \\ \int_{-1}^0 \zeta_w(3; s) ds &= \frac{3\text{Ei}(-2\ln(4))\ln(2)\ln(3) + 8\ln(2) + 4\ln(2)\ln(3) + 4\ln(3) - 3\text{Ei}(-\ln(4))\ln(3)\ln(2)}{4\ln(2)\ln(3)} \\ \int_{-1}^0 \zeta_w(4; s) ds &= \frac{8\text{Ei}(-2\ln(5))\ln(2)\ln(3) + 20\ln(2) + 25\ln(3) - 8\text{Ei}(-\ln(5))\ln(2)\ln(3) + 10\ln(2)\ln(3)}{10\ln(2)\ln(3)}\end{aligned}\tag{6}$$

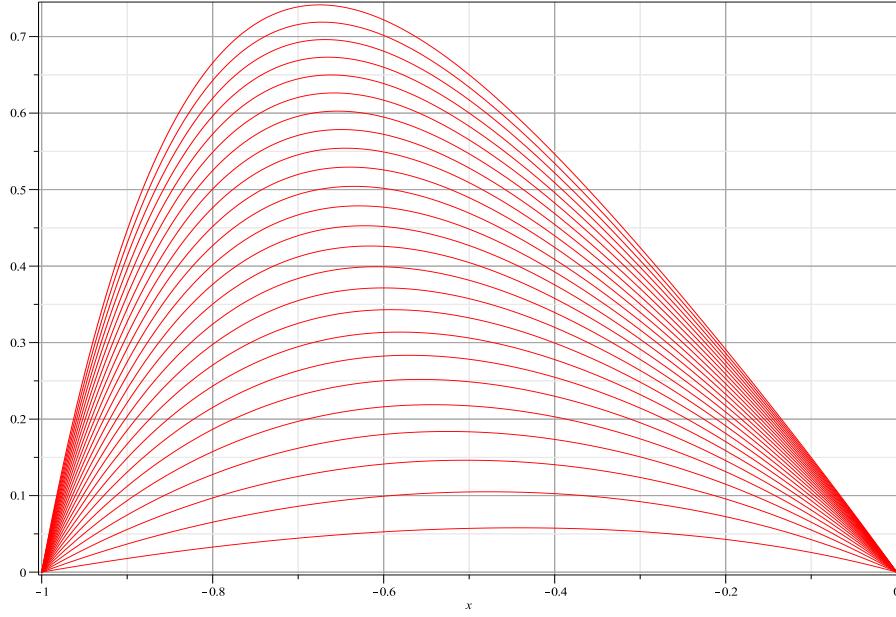


Figure 2. $\{\zeta_w(N; s): s = -1 \dots 0, N = 1 \dots 25\}$

The following table lists the values of $\zeta_w(N; 1-n)$ for $n = 2 \dots 12$.

$$\left[\begin{array}{c}
 0 \\
 -\frac{1}{6}N - \frac{1}{6}N^2 \\
 -\frac{1}{4}N - \frac{1}{2}N^2 - \frac{1}{4}N^3 \\
 -\frac{7}{30}N - \frac{4}{5}N^2 - \frac{13}{15}N^3 - \frac{3}{10}N^4 \\
 -\frac{1}{6}N - \frac{11}{12}N^2 - \frac{5}{3}N^3 - \frac{5}{4}N^4 - \frac{1}{3}N^5 \\
 -\frac{5}{42}N - \frac{6}{7}N^2 - \frac{97}{42}N^3 - \frac{20}{7}N^4 - \frac{23}{14}N^5 - \frac{5}{14}N^6 \\
 -\frac{1}{8}N - \frac{19}{24}N^2 - \frac{21}{8}N^3 - \frac{14}{3}N^4 - \frac{35}{8}N^5 - \frac{49}{24}N^6 - \frac{3}{8}N^7 \\
 -\frac{13}{90}N - \frac{8}{9}N^2 - \frac{26}{9}N^3 - \frac{56}{9}N^4 - \frac{371}{45}N^5 - \frac{56}{9}N^6 - \frac{22}{9}N^7 - \frac{7}{18}N^8 \\
 -\frac{1}{10}N - \frac{21}{20}N^2 - \frac{18}{5}N^3 - \frac{79}{10}N^4 - \frac{63}{5}N^5 - \frac{133}{10}N^6 - \frac{42}{5}N^7 - \frac{57}{20}N^8 - \frac{2}{5}N^9 \\
 -\frac{1}{66}N - \frac{10}{11}N^2 - \frac{101}{22}N^3 - \frac{120}{11}N^4 - \frac{199}{11}N^5 - \frac{252}{11}N^6 - \frac{221}{11}N^7 - \frac{120}{11}N^8 - \frac{215}{66}N^9 - \frac{9}{22}N^{10} \\
 -\frac{1}{12}N - \frac{1}{2}N^2 - \frac{55}{12}N^3 - \frac{121}{8}N^4 - \frac{55}{2}N^5 - \frac{110}{3}N^6 - \frac{77}{2}N^7 - \frac{231}{8}N^8 - \frac{55}{4}N^9 - \frac{11}{3}N^{10} - \frac{5}{12}N^{11}
 \end{array} \right] \quad (7)$$

1.1.1. The Reflection Formula.

There is a reflection equation for the finite-sum approximation $\zeta_w(N; s)$ which is similar to the well-known formula $\zeta(1-s) = \chi(s)\zeta(s)$ with $\chi(s) = 2(2\pi)^{-s} \cos(\frac{\pi s}{2})\Gamma(s)$. The solution to

$$\zeta_w(N; 1-s) = \chi(N; s)\zeta_w(N; s) \quad (8)$$

is given by the expression

$$\begin{aligned} \chi(N; s) &= \frac{\zeta_w(N; 1-s)}{\zeta_w(N; s)} \\ &= \frac{\sum_{n=1}^N \frac{-n^s + (n+1)^{s-1}n + n^{s-1} - n^{s-1}s}{s}}{\sum_{n=1}^N \frac{-n^{1-s} + (n+1)^{-s}n + n^{-s}s}{s-1}} \\ &= -\frac{(s-1)\sum_{n=1}^N \frac{-n^s + (n+1)^{s-1}n + n^{s-1} - n^{s-1}s}{s}}{s\sum_{n=1}^N \frac{-n^{1-s} + (n+1)^{-s}n + n^{-s}s}{s-1}} \end{aligned}$$

which satisfies

$$\chi(N; 1-s) = \chi(N; s)^{-1} \quad (9)$$

The functions $\chi(N; s)$, indexed by N , have singularities at $s=0$. Let

$$\begin{aligned} a(N) &= \sum_{n=1}^N n(\ln(n+1) - \ln(n)) \\ b(N) &= \sum_{n=1}^N \frac{\ln(n)n^2 - \ln(n+1)n^2 - \ln(n)}{n(n+1)} \\ c(N) &= \frac{1}{2} \sum_{n=1}^N n(\ln(n+1))^2 - \ln(n)^2 \end{aligned} \quad (10)$$

then the residue at the singular point $s=0$ is given by the expression

$$\begin{aligned} \operatorname{Res}_{s=0}(\chi(N; s)) &= -\operatorname{Res}_{s=1}(\chi(N; s)^{-1}) \\ &= \frac{1 + \gamma + \Psi(n+2) - \frac{2}{N+1} + b(N) - \frac{N(\ln(\Gamma(N+1)) - c(N))}{(N-a(N))(N+1)}}{a(N) - N} \\ &= \frac{1 + \gamma + \Psi(n+2) - \frac{2}{N+1} + \sum_{n=1}^N \frac{\ln(n)n^2 - \ln(n+1)n^2 - \ln(n)}{n(n+1)} - \frac{N(\ln(\Gamma(N+1)) - \frac{1}{2}\sum_{n=1}^N n(\ln(n+1)^2 - \ln(n)^2))}{(N - \sum_{n=1}^N n(\ln(n+1) - \ln(n)))(N+1)}}{(\sum_{n=1}^N n(\ln(n+1) - \ln(n))) - N} \end{aligned} \quad (11)$$

which has the limit

$$\lim_{N \rightarrow \infty} \operatorname{Res}_{s=0}(\chi(N; s)) = 1 \quad (12)$$

We also have the residue of the reciprocal at $s=2$

$$\operatorname{Res}_{s=2}(\chi(N; s)^{-1}) = \frac{\frac{2N}{(N+1)^2} - 2\Psi(1, N+1) + 2\zeta(2)}{\frac{(N+1)^2}{2} - \frac{N}{2} - \frac{1}{2} - \sum_{n=1}^N n(\ln(n+1) + \ln(n+1)n - \ln(n) - n\ln(n))} \quad (13)$$

which vanishes as N tends to infinity

$$\lim_{N \rightarrow \infty} \operatorname{Res}(\chi(N; s)^{-1}) = 0 \quad (14)$$

As can be seen in the figures below, the residue at $s=0$ changes sign from negative to positive between the values of $N = 176$ and $N = 177$.

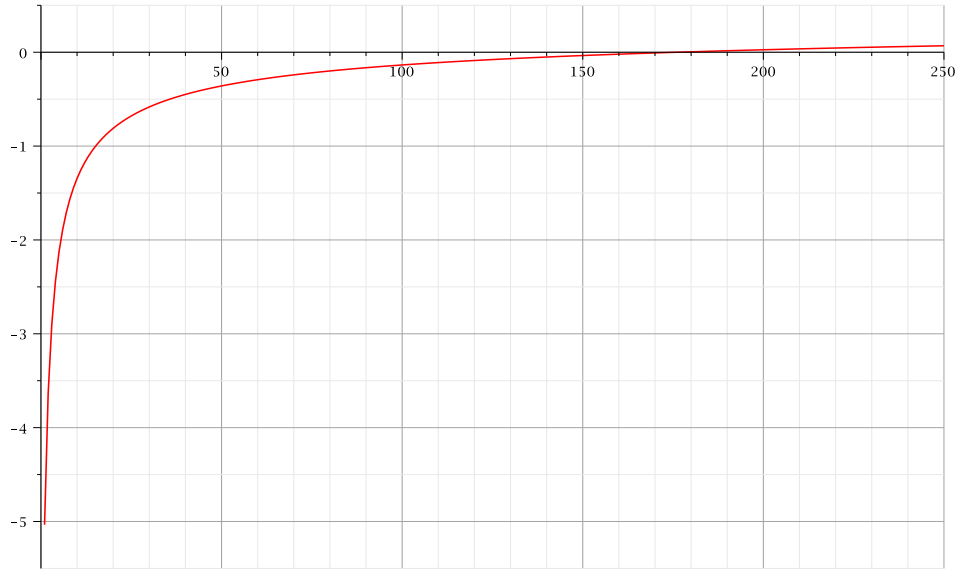


Figure 3. $\{\operatorname{Res}(\chi(N; s))_{s=0}: N = 1 \dots 250\}$

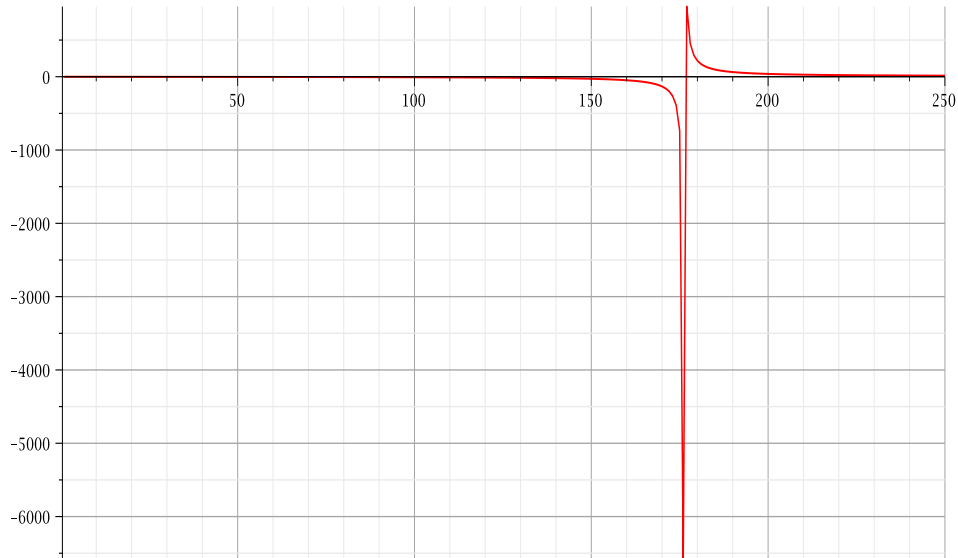


Figure 4. $\{\operatorname{Res}(\chi(N; s))^{-1}_{s=0}: N = 1 \dots 250\}$

For any positive integer N , we have the limits

$$\begin{aligned}
\lim_{s \rightarrow 0} \chi(N; s) &= \infty \\
\lim_{s \rightarrow 0} \frac{d^n}{ds^n} \chi(N; s) &= \infty \\
\lim_{s \rightarrow \frac{1}{2}} \chi(N; s) &= 1 \\
\lim_{s \rightarrow 1} \chi(N; s) &= 0 \\
\lim_{s \rightarrow 2} \chi(N; s) &= 0 \\
\lim_{s \rightarrow 1} \frac{d}{ds} \chi(N; s) &= 0
\end{aligned} \tag{15}$$

The line $\operatorname{Re}(s) = \frac{1}{2}$ has a constant modulus

$$\left| \chi\left(N; \frac{1}{2} + is\right) \right| = 1 \tag{16}$$

There is also the complex conjugate symmetry

$$\chi(N; x + iy) = \overline{\chi(N; x - iy)} \tag{17}$$

If $s = n \in \mathbb{N}^*$ is a positive integer then $\chi(N; n)$ can be written as

$$\begin{aligned}
\chi(N; n) &= \frac{\zeta_w(N; 1-n)}{\zeta_w(N; n)} \\
&= \frac{\sum_{m=1}^N -\sum_{k=1}^{n-2} \frac{m^k}{n} \binom{n-1}{k-1}}{\frac{N}{(n-1)(N+1)^n} - \frac{\cos(\pi n)\Psi(n-1, N+1)}{\Gamma(n)} + \zeta(n)} \\
&= \frac{-\sum_{m=1}^N \frac{1}{n} ((n-1)m^{n-1} + m^n - (m+1)^{n-1}m)}{\frac{N}{(n-1)(N+1)^n} - \frac{\cos(\pi n)\Psi(n-1, N+1)}{\Gamma(n)} + \zeta(n)}
\end{aligned} \tag{18}$$

where $\binom{n-1}{k-1}$ is of course a binomial. The Bernoulli numbers[1] make an appearance since

$$\chi(N; 2n)\zeta_w(N; 2n) = B_{2n}(N+1)^2 \frac{(2n+1)}{2} + \dots \tag{19}$$

The denominator of $\chi(N; n)$ has the limits

$$\begin{aligned}
\lim_{N \rightarrow \infty} \zeta_w(N; n) &= \zeta(n) \\
\lim_{n \rightarrow \infty} \zeta_w(N; n) &= 1
\end{aligned} \tag{20}$$

Another interesting formula gives the limit at $s = 1$ of the quotient of successive functions

$$\begin{aligned}
\lim_{s=1} \frac{\chi(N+1; s)}{\chi(N; s)} &= \frac{(N+2)N(N+1-a(N+1))}{(N+1)^2(N-a(N))} \\
&= \frac{(N+2)N(N+1 - \sum_{n=1}^{N+1} n(\ln(n+1) - \ln(n)))}{(N+1)^2(N - \sum_{n=1}^N n(\ln(n+1) - \ln(n)))}
\end{aligned} \tag{21}$$

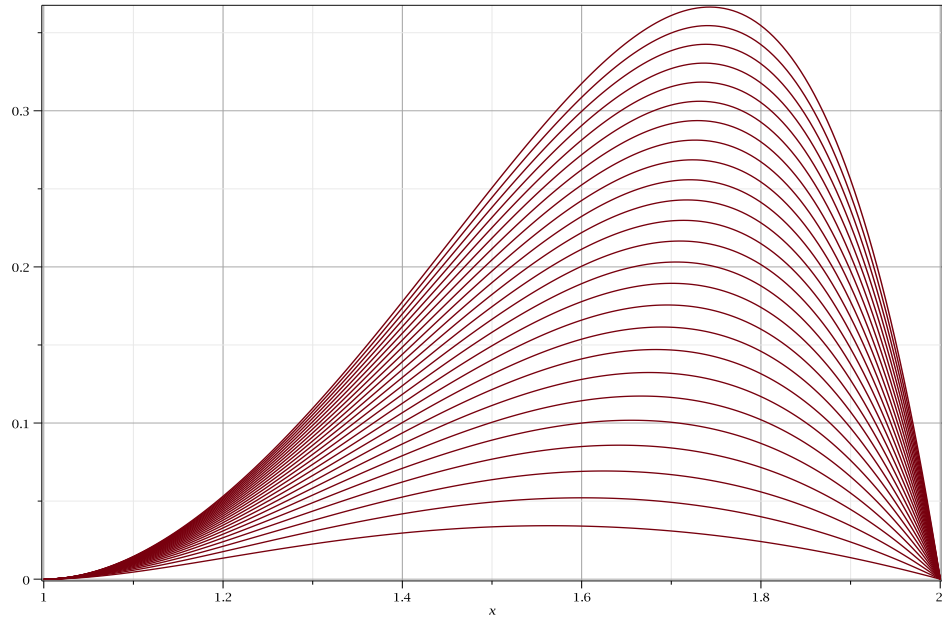


Figure 5. $\{\chi(N; s): s = 1 \dots 2, N = 1 \dots 25\}$

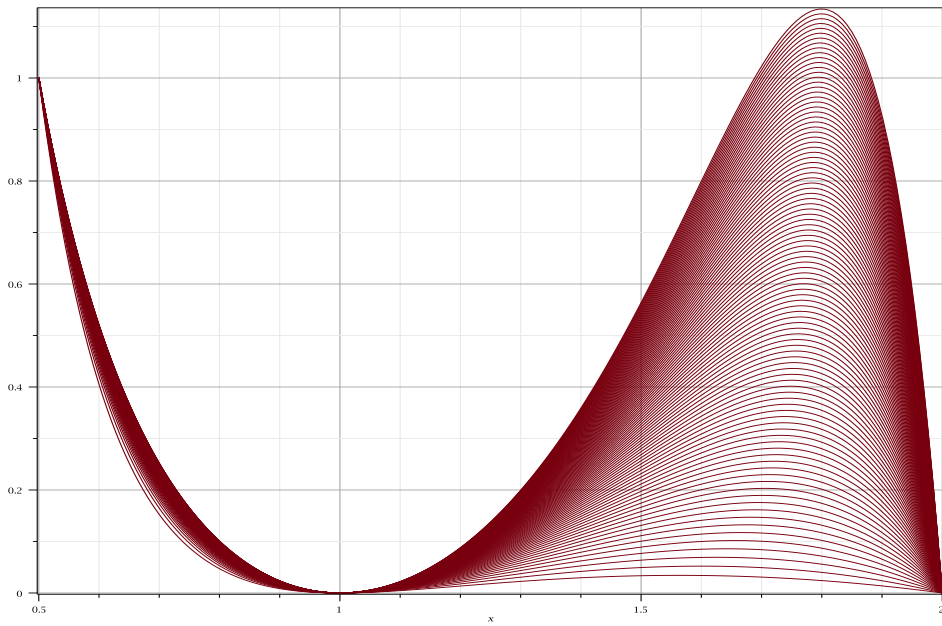


Figure 6. $\{\chi(N; s): s = \frac{1}{2} \dots 2, N = 1 \dots 100\}$

Let

$$\nu(s) = \chi(\infty; s) = \frac{\zeta(1-s)}{\zeta(s)} \quad (22)$$

Then the residue at the even negative integers is

$$\operatorname{Res}_{s=-n}(\nu(s)) = \begin{cases} \frac{\zeta(1-n)}{\frac{d}{ds}\zeta(s)|_{s=-n}} & n \text{ even} \\ 0 & n \text{ odd} \end{cases} \quad (23)$$

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