A new microsimplicial homology theory

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Abstract

A homology theory based on both near-standard and non-near-standard microsimplices is constructed. Its basic properties, including Eilenberg-Steenrod axioms for homology and continuity with respect to resolutions of spaces, are proved.

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Contents

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1 Introduction

The problem of constructing good homology theories dates back to the days of Čech and Vietoris. Čech homology has a plethora of good properties such as continuity for compact spaces and shape invariance, but Čech homology fails to be exact. For compact spaces, there is a little-known remedy to problem of inexactness of Čech homology, namely the use of non-standard coefficients, or equivalently, as proved by Garavaglia [3], the use of McCord's homology [9].

McCord homology is based on hyperfinite chains of near-standard microsimplices, and as we proved in Korppi [6], it coincides with Čech homology with non-standard coefficients and compact supports for regular spaces, if the non-standard model used is rich enough.

In this paper we modify McCord's construction of a homology theory so that we base our homology theory on both near-standard and non-near-standard microsimplices. Our approach extends the good properties of McCord's homology, such as exactness, continuity and shape invariance, for non-compact spaces. Our homology theory coincides with McCord's homology for pairs of compact spaces, but it fails to have compact supports for certain non-compact spaces.

Another, more popular way to circumvent the problems of inexactness of Čech homology is to use Steenrod- style homology theories, such as strong homology, as constructed in Mardesic [7]. However, strong homology fails to be continuous, and it is only strong shape invariant.

Hence, Steenrod-style homology theories indeed have their uses, but our homology theory has good properties not shared with Steenrod-style homology theories.

We prove that our homology theory satisfies all the Eilenberg-Steenrod axioms in an aribtrary small subcategory of the category whose objects are pairs (X, A), where X is paracompact and A is closed in X, and whose morphisms are continuous. All the other axioms hold in an arbitrary small subcategory of the pairs of completely regular spaces (X, A), A closed in X, but for the excision axiom there are some complications, see Lemma 21.

We also prove the following properties of our homology theory.

- Our homology theory finds quasi-components.
- For a polyhedral pair (X, A), our homology coincides with ${}^*H$, the non-standard version of the simplicial homology functor.
- Our homology theory is continuous with respect to resolutions.
- Our homology theory satisfies the strong excision axiom for (X, A), X paracompact and A closed in X.

- Our homology theory is shape invariant.

The two restrictions are that our homology theory will be defined only for non-standard coefficients ${}^*G$, for an Abelian group G, and that one cannot define the homology simultaneously for a proper class of pairs of spaces, but must restrict oneself to an arbitrary small subcategory.

At the moment we do not have concrete applications of our homology theory, but we think that proving the existence of a homology theory with such strong properties is an interesting result in itself.

Non-standard universe

Throughout the paper, we assume that the reader is proficient with the methods of non-standard analysis as presented in Robinson [10].

For our specific approach to the non-standard universe, see Korppi [5], Section 2.

In a nutshell, our approach is as follows: We let S be an arbitrary set of mathematical objects, which contains all the objects we are interested in. We let $M = R(\kappa)$ for a big enough cardinal number κ such that M contains S and the auxiliary constructions we use in our proofs. Then we let the non-standard universe $({}^*M, {}^*\in)$ be a $|R(\kappa)|$-saturated elementary extension of $(M, \in)$.

Our proofs will work only for objects in S, but when there is no risk of confusion, we do not explicitly mention this.

For $X \in M$, we denote the non-standard counterpart of X in ${}^*M$ by ${}^*X$. When there is no risk of confusion, or more particularly, when X plays the role of an element rather than a set, we suppress the * from the notation. If X is a set, we do not distinguish ${}^*X$ and $\{x \in {}^*M \mid x \,{}^*\!\in {}^*X\}$ in the notation.

By transfer we mean the principle that if ϕ is a first-order formula in the language of set theory and $x_1, \ldots, x_n \in M$, then $M \models \phi(x_1, \ldots, x_n)$ if and only if ${}^*M \models \phi({}^*x_1, \ldots, {}^*x_n)$.

2 Normal covers

Let X a completely regular space. Following Mardesic and Segal [8], Appendix 1.3, we say that an open cover $\mathcal{U}$ of X is normal, if there is a partition of unity subordinated to it.

By a partition of unity we mean a collection $\phi_U \colon X \to I$ of continuous functions such that for all $x \in X$ we have $\sum_{U \in \mathcal{U}} \phi_U(x) = 1$ and for all $U \in \mathcal{U}$ we have $\{x \in X \mid \phi_U(x) > 0\} \subset U$. The sum $\sum$ is interpreted as the least upper bound of finite subsums.

Theorem 2 in Mardesic and Segal [8], App. 1.3 states the following:

Lemma 1 *An open cover $\mathcal{U}$ is normal if and only if there exists a map $f \colon X \to M$ into a metric space and an open covering $\mathcal{V}$ of M such that $f^{-1}\mathcal{V}$ refines $\mathcal{U}$.*

As a corollary to the lemma we have the following:

Corollary 2
1. *Each normal cover has a star-refinement that is a normal cover.*
2. *Assume that $\mathcal{U}$ is an open cover of X and $\mathcal{V}$ is a normal cover such that $\mathcal{V}$ refines $\mathcal{U}$. Then $\mathcal{U}$ is a normal cover.*

Proof: (1) Let $\mathcal{U}$ be a normal cover of X. By Lemma 1, there exists a metric space M, a map $f \colon X \to M$ and an open cover $\mathcal{W}$ of M such that $f^{-1}\mathcal{W}$ refines $\mathcal{U}$. Since M is metric, and hence paracompact, there exists $\mathcal{W}'$ that is a star-refinement of $\mathcal{W}$. Now $f^{-1}\mathcal{W}'$ is a star-refinement $f^{-1}\mathcal{W}$, which, in turn, is a refinement of $\mathcal{U}$. Hence $f^{-1}\mathcal{W}'$ is a star-refinement of $\mathcal{U}$.

(2) By Lemma 1, there exists a metric space M, a map $f \colon X \to M$ and an open cover $\mathcal{W}$ of M such that $f^{-1}\mathcal{W}$ refines $\mathcal{V}$. Now $f^{-1}\mathcal{W}$ refines $\mathcal{U}$, and the claim follows from Lemma 1. □

Theorem 4 in Mardesic and Segal [8], App. 1.3 states the following:

Lemma 3 *Any two normal coverings $\mathcal{U}$ and $\mathcal{U}'$ admit a normal covering $\mathcal{U}''$ which refines them both.*

We will be using also the following:

Lemma 4 *If X is completely regular and U is an open neighbourhood of x, then the cover $(X \setminus x, U)$ is normal.*

Proof: Since X is completely regular, there exists $f\colon X \to I$ such that $f(x) = 1$, and $f(y) = 0$ if $y \notin U$. Now the choice $f_U = f$, $f_{X \setminus x} = 1 - f$ is the required partition of unity.$\square$

Lemma 5 *Let A be a subspace of X. If $\mathcal{U}$ is a normal cover of X, then $\mathcal{U} \cap A = \{A \cap U | U \in \mathcal{U}\}$ is a normal cover of A.*

Proof: Since each metric space is paracompact, we can, by Lemma 1, replace $\mathcal{U}$ with its locally finite normal refinement. By Corollary 2 (2), this does not compromise the validity of the lemma.

Let $(\phi_{U \in \mathcal{U}})$ be a partition of unity for the normal cover $\mathcal{U}$. For each $U' \in \mathcal{U} \cap A$, let $\phi'_{U'} = \sum_{U''} \phi_{U''} | A$, where the sum is taken over $U'' \in \mathcal{U}$ such that $U' = A \cap U''$. One easily checks that $(\phi'_{U'})$ is a partition of unity for $\mathcal{U} \cap A$.$\square$

3 Infinitesimal closeness

Let X and Y be completely regular spaces. Let $x, y \in {}^*X$. We say that x is infinitesimally close to y, $x \sim y$, if for every normal cover $\mathcal{U}$ of X, there exists $U \in {}^*\mathcal{U}$ such that $x, y \in U$.

Let $x \in X$. We say that $y \in {}^*X$ is Robinson-close to x, if for all neighbourhoods U of x we have that $y \in {}^*U$.

Next we prove the basic properties of $\sim$.

Lemma 6 *We have the following:*

1. *$\sim$ is an equivalence relation in ${}^*X$.*

2. *Let $x \in X$, $y \in {}^*X$. Then x is Robinson-close to y if and only if $x \sim y$.*

3. *Let $f\colon X \to Y$. Then f is continuous if and only if for all $x \sim x' \in {}^*X$ we have that ${}^*f(x) \sim {}^*f(x')$.*

Proof: We prove (1). Reflexivity and symmetry are trivial. We prove transitivity. Let $x, y, z \in {}^*X$, $x \sim y$, $y \sim z$. Let $\mathcal{U}$ be a normal cover of X. By Corollary 2, the cover $\mathcal{U}$ has a star-refinement $\mathcal{V}$. Now there exist $V, V' \in {}^*\mathcal{V}$ such that $x, y \in V$, $y, z \in V'$. In particular $y \in V \cap V' \neq \emptyset$. Since $\mathcal{V}$ is a star-refinement of $\mathcal{U}$, we have that $V \cup V' \subset U$ for some $U \in {}^*\mathcal{U}$. But now $x, z \in U$. Hence $x \sim z$.

We prove (2). Assume $x \in X$, $y \in {}^*X$, $x \sim y$. Let U be a open neighbourhood of x. Since X is completely regular, $\mathcal{U} = (U, X \setminus x)$ is a normal cover of X, and thus x, y belong to a same element of ${}^*\mathcal{U}$. Since $x \notin {}^*(X \setminus x)$, we must have $x, y \in {}^*U$. Thus y is Robinson-close to x.

Assume that $x \in X$, $y \in {}^*X$, x is Robinson-close to y. Let $\mathcal{U}$ be a normal cover of X. Let $U \in \mathcal{U}$ such that $x \in U$. Since x is Robinson-close to y, we have $y \in {}^*U$. But now $x, y \in {}^*U \in {}^*\mathcal{U}$. Thus, $x \sim y$.

We prove (3). Let $f\colon X \to Y$ be continuous. Let $x \sim x' \in {}^*X$. Let $\mathcal{U}$ be a normal cover of Y. Now $\mathcal{U} \cap fX$ is a normal cover of fX by Lemma 5, and hence $f^{-1}\mathcal{U}$ is a normal cover of X, and thus $x, x' \in U \in {}^*(f^{-1}\mathcal{U})$. But now ${}^*f(x), {}^*f(x') \in {}^*(f)U \subset U'$ for some $U' \in {}^*\mathcal{U}$. Thus ${}^*f(x) \sim {}^*f(y)$.

Assume then that $f\colon X \to Y$ is such that $x \sim x'$ implies ${}^*f(x) \sim {}^*f(x')$ for all $x, x' \in {}^*X$. Since ${}^*f$ takes each standard point of X to a standard point of Y, we have the following: If $x \in X$, $x' \in {}^*X$, x Robinson-close to x', then ${}^*f(x)$ is Robinson-close to ${}^*f(x')$. By Robinson [10], Theorem 4.2.7, this implies that f is continuous.$\square$

We say that $B \subset {}^*X$ is small, if for each normal cover $\mathcal{U}$ of X there is $U \in {}^*\mathcal{U}$ such that $B \subset U$.

Lemma 7 *$B \subset {}^*X$ is small if and only for all $x, y \in B$ we have $x \sim y$.*

Proof: only if is trivial. We prove *if*. Assume B is such that for all $x, y \in B$ we have $x \sim y$. Let $\mathcal{U}$ be a normal cover of X, and let $\mathcal{V}$ be its star-refinement. Let $x \in B$. Now the set B is contained in the star of x in ${}^*\mathcal{V}$, and hence in an element of ${}^*\mathcal{U}$.$\square$

Remark 8 *Let X be a completely regular space, and $A \subset X$ a subspace. Let $x, y \in {}^*A$. Denote by $\sim_A$ and $\sim_X$ the infinitesimal closeness relations* *in A and X, respectively. Then it is possible that $x \sim_X y$, but not $x \sim_A y$. Consider for example the case where $X = \bar{\mathbb{N}}$, the one-point compactification of $\mathbb{N}$, and $A = \mathbb{N}$. If $x \neq y \in {}^*\mathbb{N}$, $x, y > \mathbb{N}$, then $x \sim_A y$ does not hold, but $x \sim_X y$ holds.*

Following Mardesic [7], Remark 6.39, we say that $A \subset X$ is normally embedded, if every normal cover $\mathcal{V}$ of A admits a normal cover $\mathcal{U}$ of X such that $\mathcal{U}|A$ refines $\mathcal{V}$. According to Mardesic's remark, if X is collectionwise normal, then every closed subspace A of X is normally embedded. By Seebach-Steen [12], Figure 18, each paracompact space is collectionwise normal.

However, we have the following:

Lemma 9 *Let X be a completely regular space, and $A \subset X$ a subspace. Let $x, y \in {}^*A$. Then*

1. *$x \sim_A y$ implies $x \sim_X y$*
2. *If A is normally embedded in X, then $x \sim_A y$ if and only if $x \sim_X y$.*
3. *If X is paracompact and A is closed in X, then $x \sim_A y$ if and only if $x \sim_X y$.*

Proof: (1) holds, since each normal cover of X restricts to a normal cover of A by Lemma 5.

(2) holds by the definition of normally embedded.

(3) holds by (2) and the discussion preceeding this lemma. □

Definition 10 *Let X be a completely regular space, and $\mathcal{U}$ a* *normal cover *of ${}^*X$. We say that a subset $A \subset {}^*X$ is $\mathcal{U}$-small, if there exists $U \in \mathcal{U}$ such that $A \subset U$.*

Lemma 11 *We have the following:*

1. *$A \subset {}^*X$ is small if and only if A is ${}^*\mathcal{U}$-small for all normal covers $\mathcal{U}$ of X.*
2. *If $A \subset {}^*X$ is $\mathcal{U}$-small for some* *open cover *of X refining the nonstandard version of each standard normal cover, then A is small.*

3. *If $A \subset {}^*X$ is small and internal, then there exists* *normal cover $\mathcal{U}$ *refining the non-standard version of each standard normal cover of X such that A is $\mathcal{U}$-small.*

4. *If $(A_i)_{i\in I}$ is an internal collection of small subsets of ${}^*X$, then there exists a* *normal cover $\mathcal{U}$ *refining the non-standard version of each standard normal cover of X such that each A_i is $\mathcal{U}$-small.*

Proof: (1) and (2) are immediate from the definitions. (3) is a special case of (4), so it is enough to prove (4).

For every finite collection of standard normal covers $\mathcal{U}_1, \ldots, \mathcal{U}_n$ there exists a standard normal cover $\mathcal{U}'$ refining each $\mathcal{U}_j$, and each A_i is ${}^*\mathcal{U}'$-small.

Hence, by saturation, there exists a **normal cover* $\mathcal{U}$ refining the non-standard version of every standard normal cover such that each A_i is $\mathcal{U}$-small.□

4 The construction of the homology theory

Here we construct the homology theory. Our construction is essentially the same as the construction of McCord homology[9], except that we use the concept of infinitesimally close formulated above, and we do not restrict ourselves to near-standard microsimplices.

Let G be an Abelian group.

Let X be a completely regular space. Let $O'_n X$ be the set of all ordered $n+1$-tuples of points of ${}^*X$, called n-simplices.

Let $C'_n(X; {}^*G)$ be the set of all hyperfinitely supported internal functions $c\colon O'_n X \to {}^*G$. We regard $C'_n(X; {}^*G)$ as an abelian group with pointwise addition.

Denote by $gs \in C'_n(X; {}^*G)$ the function such that $gs(s) = g$, $gs(s') = 0$, if $s \neq s'$.

Lemma 12 *Let H be an internal Abelian group. Let $(f_s\colon {}^*G \to H)$, $s \in O'_n X$, be an internal collection of homomorphisms. Then there is a unique internal homomorphism $f\colon C'_n(X; {}^*G) \to H$ such that $f(gs) = f_s(g)$ for all $s \in O'_n X$.*

Proof: We define f by $c \mapsto {}^*\sum_{s \in O'_n} f_s(c(s))$. Since c is hyperfinitely supported, f is well-defined.

Let $c, c' \in C'_n(X; {}^*G)$. Now $f(c) + f(c') = {}^*\sum_{s \in O'_n} f_s(c(s)) + f_s(c'(s)) = {}^*\sum_{s \in O'_n} f_s(c(s) + c'(s)) = {}^*\sum_{x \in O'_n} f_s((c + c')(s)) = f(c + c')$. Thus f is a homomorphism.

One readily checks that $f(gs) = f_s(g)$.

We check that f is unique, as required. Assume that f' satisfies also $f'(gs) = f_s(g)$ for all $s \in O'_n X,$, $g \in G$. Now each $c \in C'_n(X; {}^*G)$ can be represented as an internal hyperfinite sum ${}^*\sum g_i s_i$, and $f'({}^*\sum g_i s_i) = {}^*\sum f'(g_i s_i) = {}^*\sum f_{s_i}(g_i) = f({}^*\sum g_i s_i)$. Thus f is unique as required.□

We define $\Delta'_n \colon C'_n(X; {}^*G) \to C'_{n-1}(X; {}^*G)$ as follows: $gs \mapsto {}^*\sum_{i=0}^{n} (-1)^i g(\hat{s}_i)$, where $\hat{s}_i$ has been obtained from s by omitting the ith index. For for an arbitrary $c \in C_n(X; {}^*G)$, we extend Δ'_n using the previous lemma.

Lemma 13 $\Delta'_{n-1} \circ \Delta'_n = 0$.

Proof: By the uniqueness in Lemma 12, it is enough to check the property for the elements of the type gs. But for these, the proof is an elementary computation.□

Let $f \colon X \to Y$. We define $C'_n(f) \colon C'_n(X; {}^*G) \to C'_n(Y; {}^*G)$ for simplices by $C'_n(f)(gs) = g\, {}^*f(s)$, and extend to arbitrary chains by Lemma 12.

Lemma 14 *If* $f \colon X \to Y$ *and* $f' \colon Y \to Z$*, then* $C'_n(f'f; {}^*G) = C'_n(f'; {}^*G) C'_n(f; {}^*G)$.

Proof: For each simplex s of X, we have that $C'_n(f'; {}^*G) C'_n(f; {}^*G) gs = gf'fs$. Thus, by uniqueness in Lemma 12, we have $C'_n(f'; {}^*G) C'_n(f; {}^*G) = C'_n(f'f; {}^*G)$.□

Lemma 15 *If* $f \colon X \to Y$*, then* $\Delta'_n C'_n(f; {}^*G) = C'_{n-1}(f; {}^*G) \Delta'_n$.

Proof: Again, it is enough to check that the maps are the same for chains of the type gs. But the image of gs for both maps is $\sum (-1)^i ({}^*f)(g\hat{s}_i)$.□

Let $O_nX \subset O'_nX$ be the set of those simplices whose vertices form a small set.

Let $C_n(X; {}^*G) \subset C'_n(X; {}^*G)$ be defined as follows: $c \in C_n(X; {}^*G)$ if and only if $c(s) \neq 0$ implies that $s \in O_nX$.

Lemma 16 *$\Delta'_n\colon C'_n(X;G) \to C'_{n-1}(X; {}^*G)$ restricts to a map $\Delta_n\colon C_n(X;G) \to C_{n-1}(X; {}^*G)$.*

$C'_n(f)\colon C'_n(X; {}^*G) \to C'_n(Y; {}^*G)$ restricts to a map $C_n(f)\colon C_n(X; {}^*G) \to C_n(Y; {}^*G)$.

Proof: To prove the first part, we must check that if $c \in C_n(X; {}^*G)$, then the support of $\Delta'_n(c)$ contains only small simplices.

Let $c = {}^*\sum g_i s_i$. Then $\Delta'_n({}^*\sum g_i s_i) = {}^*\sum \Delta'_n(g_i s_i)$, where the support of $\Delta'_n(g_i s_i)$ contains only faces of s_i, which are small as subsets of a small s_i. Thus we have the first part.

Then the second part. Similarly, we must check that if $c \in C'_n(X; {}^*G)$, then the support of $C'_n(f)(c)$ contains only small simplices.

If $c \in C'_n(X; {}^*G)$, let $c = {}^*\sum g_i s_i$. Then $C'_n(f; {}^*G)(c) = {}^*\sum g_i f(s_i)$. By Lemmas 6 and 7, $f(s_i)$ is small for each small s_i. □

Now let (X, A) be a pair of completely regular spaces, and let $i_A\colon A \to X$ be the inclusion. We let

$$C_n(X, A; {}^*G) = \frac{C_n(X; {}^*G)}{C_n(i_A; {}^*G)C_n(A; {}^*G)}$$

.

Lemma 17 *If $f\colon (X, A) \to (Y, B)$, then $C_n(f; {}^*G)\colon C_n(X; {}^*G) \to C_n(Y; {}^*G)$ induces $C_n(f; {}^*G)\colon C_n(X, A; {}^*G) \to C_n(Y, B; {}^*G)$.*

Furthermore $\Delta_n\colon C_n(X; {}^*G) \to C_{n-1}(X; {}^*G)$ induces $\Delta_n\colon C_n(X, A; {}^*G) \to C_{n-1}(X, A; {}^*G)$.

Proof: Follows by a well-known homology theoretical argument, since C_n commutes with maps and the boundary operator.□

Definition 18 *Let $n \in \mathbb{N}$. We let $H_n(X, A; {}^*G)$ be the homology of the chain complex $(C_n(X, A; {}^*G), \Delta_n)$. Let now $f\colon (X, A) \to (Y, B)$ be a map of pairs. Since the maps $C_n(f)\colon C_n(X, A; {}^*G) \to C_n(Y, B; {}^*G)$, commute with Δ_n, they form a chain map, and induce the map $H_n(f; {}^*G)$ in homology. Furthermore, the connecting homomorphism ∂ of the short exact sequence $0 \to C(A; {}^*G) \to C(X; {}^*G) \to C(X, A; {}^*G) \to 0$ gives us the connecting homomorphism of the long exact sequence commuting with maps $H_n(f; {}^*G)$.*

Corollary 19 *Directly from the definitions, we get Eilenberg-Steenrod ([2], Section I.3) axioms 1-4.*

Proof: Since the homology is computed from chain complexes, and $(X, A) \mapsto C_n(X, A)$ is functorial, also the homology is functorial. The homology is exact, since a short exact sequence of chain complexes induces a long exact homology sequence.$\square$

5 Excision and dimension axioms

Lemma 20 *If P is a one-point space, then $H_0(P; {}^*G) = {}^*G$ and $H_n(P; {}^*G) = 0$, if $n > 0$.*

Proof: In each dimension, the $C_n(P; {}^*G) = {}^*G$, where each element is of the type gs for the only ordered n-simplex s of P. If n is odd or 0, then $\Delta_n = 0$. If $n > 0$ is even, then $\Delta_n(gs) = gs'$, where s' is the only $n-1$ dimensional simplex of P. Thus, for $n > 0$ even, Δ_n is an isomorphism. Thus, the lemma follows.$\square$

Lemma 21 *Let (X, A) be a pair of completely regular spaces such that A closed and normally embedded in X, and $U \subset A$ such that $\operatorname{cl} U \subset \operatorname{int} A$, and $A \setminus U$ is normally embedded in A, and $X \setminus U$ is normally embedded in X. Furthermore, we assume that the cover the cover $(\operatorname{int} A, X \setminus \operatorname{cl} U)$ is normal.*

Let $i\colon (X \setminus U, A \setminus U) \to (X, A)$ be the inclusion. Then $C_n(i)\colon C_n(X \setminus U, A \setminus U) \to C_n(X, A)$ is an isomorphism.

Hence, excision axiom holds for pairs (X, A), where X is paracompact and A is closed in X.

Proof: We prove that $C_n(i)$ is a monomorphism. Assume that c is a chain of $C_n(X \setminus U)$ such that $C_n(i)c = 0$ in $C_n(X, A)$. Then each simplex s in the support of c is such that all the vertices of s are in ${}^*A$, and s is small in ${}^*A$. Thus, the vertices of s are in ${}^*A \setminus {}^*U$, and they are small in ${}^*A \setminus {}^*U$, since $A \setminus U$ is normally embedded in A. Thus, c is a chain of $A \setminus U$, and thus $c = 0 \in C_x(X \setminus U, A \setminus U)$. Thus $C_n(i)$ is a monomorphism.

We prove that $C_n(i)$ is an epimorphism. Assume that $[c]$ is a chain of $C_n(X, A)$, where c is a chain of $C_n(X)$. Write $c = c' + c''$, where the support S' of c' contains only simplices with at least one vertex in ${}^*U$, and the support of c'' contains only simplices with no vertex in ${}^*U$. Since the cover $(\operatorname{int} A, X \setminus \operatorname{cl} U)$ is normal, each simplex of S' is contained in ${}^*(\operatorname{int} A)$. Hence c' is a chain of A. Thus $[c] = [c' + c''] = [c'']$.

Trivially, $c'' \in C_n(X \setminus U)$, and $C_n(i)(c'') = [c''] = [c]$. Thus, $C_n(i)$ is an epimorphism.□

6 Homotopy axiom

Definition 22 *Let X be a completely regular space, and let $\mathcal{V}$ be an open cover of X. Let T be a compact connected space. Let $\mathcal{U}$ be an open cover of $X \times T$. We say that $\mathcal{U}$ is stacked over $\mathcal{V}$, if the following hold:*

- *Each element of $\mathcal{U}$ is of the form $V \times T'$, where T' is open in T, and $V \in \mathcal{V}$.*
- *Given $V \in \mathcal{V}$ and $t \in T$ there exists an open subset $T' \subset T$ such that $t \in T'$ and $V \times T' \in \mathcal{U}$.*

For the rest of this section we assume that T is a compact and connected space, and X is a completely regular space.

Lemma 23 *Let X be a completely regular space, and $\mathcal{U}$ an normal cover of $X \times T$. Then there exists $\mathcal{V}$ and a normal open cover $\mathcal{U}'$ of $X \times I$ such that $\mathcal{U}'$ is stacked over normal $\mathcal{V}$, and $\mathcal{U}'$ refines $\mathcal{U}$.*

Proof: Mardesic [7], Lemma 6.35, and the discussion preceeding the lemma.□

Lemma 24 *Assume that x, y belong to a same quasi-component of X. Then, for every open cover $\mathcal{U}$ of X there is $n \in \mathbb{N}$ and a sequence of points $x = t_0, \ldots, t_n = y$ such that $t_i, t_{i+1} \in U$ for some $U \in \mathcal{U}$.*

Proof: Let $x, y \in X$ belong to a same quasi-component and let $\mathcal{U}$ be an open cover of X. Let $X' \subset X$ be the set of points x' for which there is $n \in \mathbb{N}$ and a sequence $x = t_0, \ldots, t_n = x'$ such that t_i, t_{i+1} belong to a same $U \in \mathcal{U}$. One easily sees that the set X' is open, and the complement of X' is open.

Since $(X', X \setminus X')$ is a partition of X with open sets, $y \in X'$. Hence the required sequence exists.□

Let T_0 and T_1 be two points in T.

Lemma 25 *Let $\mathcal{U}$ be a normal cover of $X \times T$ stacked over $\mathcal{V}$.*

Let A be another completely regular space. Let $\mathcal{U}_0$ be a normal cover of $A \times T$ stacked over $\mathcal{V}_0$.

Let $\mathcal{V}'$ be a finite subset of $\mathcal{V}$ and $\mathcal{V}'_0$ be a finite subset of $\mathcal{V}_0$. Then there exists a finite sequence of points $T_0 = t_0, \ldots, t_n = T_1$ such that each of the sets $V \times \{t_i, t_{i+1}\}$, $V \in \mathcal{V}'$, is contained in some member of $\mathcal{U}$ and each of the sets $V_0 \times \{t_i, t_{i+1}\}$, $V_0 \in \mathcal{V}'_0$, is contained in some member of $\mathcal{U}_0$.

Proof: For each $V \in \mathcal{V}'$ choose a cover $\mathcal{T}_V$ of T such that $V \times T' \in \mathcal{U}$ for all $T' \in \mathcal{T}_V$. For each $V_0 \in \mathcal{V}'_0$ choose a cover $\mathcal{T}_{V_0}$ of T such that $V \times T' \in \mathcal{U}_0$ for all $T' \in \mathcal{T}_{V_0}$

Let $\mathcal{T}$ be a common refinement of all sets $\mathcal{T}_V$, $V \in \mathcal{V}'$, and $\mathcal{T}_{V_0}$, $V_0 \in \mathcal{V}_0$. Let $T_0 = t_0, \ldots, t_n = T_1$ be the sequence of the points given by the previous lemma for the cover $\mathcal{T}$. Note that since T is connected, T_0 and T_1 both lie in the only quasi-component of X.

Now, if $V \in \mathcal{V}'$, then $V \times \{t_i, t_{i+1}\} \subset V \times T'$, where $T' \in \mathcal{T}$, and consequently $V \times \{t_i, t_{i+1}\} \subset V \times T''$ for some $T'' \in \mathcal{T}_V$. Thus $V \times \{t_i, t_{i+1}\} \subset U$ for some $U \in \mathcal{U}$.

The claim for A is proved in a similar fashion. □

Lemma 26 *Let $\mathcal{U}$ be a non-standard normal cover of ${}^*(X \times T)$ refining the non-standard version each standard normal cover, $\mathcal{U}$ stacked over $\mathcal{V}$, which* *is a non-standard normal cover of ${}^*X$ refining the non-standard version of each standard normal cover.*

Let A be another completely regular space. Let $\mathcal{U}_0$ be a non-standard normal cover of ${}^*(A \times T)$ refining the non-standard version of each standard normal cover, $\mathcal{U}_0$ stacked over $\mathcal{V}_0$, which is a non-standard normal cover of ${}^*A$ refining the non-standard version of each standard normal cover.

Let $\mathcal{V}'$ be a hyperfinite subset of $\mathcal{V}$ and let $\mathcal{V}'_0$ be a hyperfinite subset of $\mathcal{V}_0$. Then there exists a hyperfinite sequence of points of ${}^*T$, $T_0 = t_0, \ldots, t_n = T_1$ such that each of the sets $V \times \{t_i, t_{i+1}\}$, $V \in \mathcal{V}'$, is contained in some member of $\mathcal{V}$ and each of the sets $V_0 \times \{t_i, t_{i+1}\}$, $V_0 \in \mathcal{V}'_0$, is contained in some member of $\mathcal{V}_0$.

Proof: Transfer and the previous lemma.□

Lemma 27 *We have the following:*

- *If $\mathcal{U}$ is a normal cover of $X \times T$, stacked over $\mathcal{V}$, then $\mathcal{V}$ is a normal cover of X.*

- *Let $\mathcal{W}$ be a non-standard open cover of ${}^*(X \times T)$ refining the non-standard version of each standard [normal] open cover such that $\mathcal{W}$ is stacked over non-standard $\mathcal{W}'$. Then the cover $\mathcal{W}'$ refines the non-standard version of each standard [normal] open cover of X.*

Proof: (1) Lemma 5.

(2) First we prove the lemma without the normality addition.

Let $\mathcal{W}$ and $\mathcal{W}'$ be as in the statement of the lemma. Let $\mathcal{U}$ be a standard open cover of X, and let $\mathcal{V}$ be a cover of $X \times T$ defined so that the members of $\mathcal{V}$ are the sets $U \times T$, where $U \in \mathcal{U}$. Now $\mathcal{W}$ refines ${}^*\mathcal{V}$, and hence for all $W' \in \mathcal{W}'$ we have $W' \times T_0 \subset U \times T$ for some $U \in {}^*\mathcal{U}$. Hence $\mathcal{W}'$ refines ${}^*\mathcal{U}$. Since $\mathcal{U}$ was arbitrary, the first part of the lemma follows.

The rest of the lemma follows, since if $\mathcal{U}$ is normal, so is $\mathcal{V}$, since $\mathcal{V}$ is the inverse image of $\mathcal{U}$ in the projection $X \times T \to X$. □

Lemma 28 *Let J be a hyperfinite set, and $(X_j)_{j \in J}$ be an internal collection of subsets of X such that each of the sets X_j is small.*

Let A be another completely regular space, let J_0 be hyperfinite set, and let $(A_{j_0})_{j_0 \in J_0}$ be an internal collection of subsets of A such that each of the sets A_{j_0} is small.

Then there exists an internal hyperfinite sequence of points of ${}^*T$, $T_0 = t_0, \ldots, t_n = T_1$ such that each of the sets $X_j \times \{t_i, t_{i+1}\}$ is small in ${}^*(X \times T)$ and each of the sets $A_{j_0} \times \{t_i, t_{i+1}\}$ is small in ${}^*(A \times T)$.

Proof: Let $\mathcal{U}$ be a normal cover of $X \times T$ stacked over $\mathcal{V}$. Now, each of the sets X_j is contained in some $V \in {}^*\mathcal{V}$. Given a finite set of normal covers of $X \times T$, by Lemma 23, there exists a stacked normal cover refining them all. By saturation, there exists a non-standard normal cover $\mathcal{W}$ of ${}^*(X \times T)$ refining ${}^*\mathcal{W}''$ for each standard normal cover $\mathcal{W}''$ of $(X \times T)$ such that $\mathcal{W}$ is stacked over a non-standard normal cover $\mathcal{W}'$ of ${}^*X$, and such that each X_j is contained in some member of $\mathcal{W}'$.

Similarly, there exists a non-standard normal cover $\mathcal{W}_0$ of ${}^*(A \times T)$ refining ${}^*\mathcal{W}_0''$ for each standard normal cover $\mathcal{W}_0''$ of $(A \times T)$ such that $\mathcal{W}_0$ is stacked over a non-standard normal cover $\mathcal{W}_0'$ of ${}^*A$, and such that each A_{j_0} is contained in some member of $\mathcal{W}_0'$.

By transferred Axiom of Choice, let $(W_j)_{j \in J}$ be an internal collection of elements of $\mathcal{W}'$ such that each $X_j \subset W_j$. Similarly, let $(W_{0j_0})_{j_0 \in J_0}$ be an internal collection of elements of $\mathcal{W}_0'$ such that each $A_{j_0} \subset W_{j_0}$.

Now let $T_0 = t_0, \ldots, t_n = T_1$ be as in Lemma 26 for the collections $(W_j)_{j \in J}$, $(W_{0j_0})_{j \in J_0}$ and the covers $\mathcal{W}$, $\mathcal{W}_0$.

Now each $X_j \times \{t_i, t_{i+1}\}$ is contained in $W_j \times \{t_i, t_{i+1}\}$, which, in turn, is contained in some member of $\mathcal{W}$. Since $\mathcal{W}$ refines the non-standard version of each standard normal cover of $X \times T$, each member of $\mathcal{W}$ is small. Hence, each set $X_j \times \{t_i, t_{i+1}\}$ is small.

Similarly, each set $A_{j_0} \times \{t_i, t_{i+1}\}$ is small. □

Lemma 29 *Let (K, L) be a pair of simplicial complexes. Let $\{0, 1\}$ be two points not in K. Let (K', L') be a simplicial complex such that the vertices of K' are of the type $(v, 0), (v, 1)$, v is a vertex of K, (similarly for L'), and s is a simplex of K' if and only if s is contained in the set $s' \times \{0, 1\}$ for some simplex s' of K (similarly for L').*

Let c be a cycle in (K, L). If c_0 is obtained from c by replacing each vertex

v with $(v,0)$, and c_1 is obtained from c by replacing each vertex v with $(v,1)$, then c_0 and c_1 are homologuous.

Proof: We apply the theorem of acyclic carriers, see Eilenberg-Steenrod [2], Theorem VI.5.8. The carrier of a simplex s of K is $s \times \{0,1\}$, which is acyclic as a simplex.□

Let (X,A) be a pair of completely regular spaces, let $[c] \in C_n(X,A;{}^*G)$ be a cycle, and let $c' \in C_{n-1}(A;{}^*G)$ be the element $\Delta(c)$. Now the simplices of c form a hyperfinite collection $(s_j)_{j\in J}$ of small sets of X and the simplices of c' form a hyperfinite collection $(s'_j)_{j\in J_0}$ of small sets of A.

By Lemma 28, there exists a hyperfinite sequence of points $T_0 = t_0,\ldots,t_N = T_1$ of ${}^*T$ such that each of the sets $s_j \times \{t_i,t_{t+1}\}$ is small in X and each of the sets $s'_j \times \{t_i,t_{i+1}\}$ is small in A.

Denote by c_i, $[c_i] \in C_n(X\times T, A\times T;{}^*G)$, the chain that has been obtained from c by replacing each vertex v by (v,t_i).

Denote by $c'_i \in C_{n-1}(A\times T;{}^*G)$ the chain that has been obtained from c' by replacing each vertex v' by (v',t_i).

Lemma 30 *The cycles $[c_i]$ and $[c_{i+1}]$ are homologuous.*

Proof: Consider the simplicial complex K_i whose vertices are (v,t_i), (v,t_{i+1}), where v runs through the vertices of the support of c_i. s is a simplex of K_i, if s is contained in a set $s'' \times \{t_i,t_{i+1}\}$ for some simplex s'' of the support of c_i. Let L_i be the subcomplex of K_i such that the vertices of L_i are (v',t_i), (v',t_{i+1}), where v' runs through the vertices of c'_i. s' is a simplex of L_i, if s' is contained in a set $s'' \times \{t_i,t_{i+1}\}$ for some simplex s'' of the support of c'_i.

Now $[c_i]$ and $[c_{i+1}]$ are cycles of ${}^*(C_n)(K_i,L_i)$, and they are homologuous by transferred Lemma 29 in ${}^*(C)(K_i,L_i)$. They are homologuous in $C_n(X\times T, A\times T;{}^*G)$, since each simplex of K_i is small in ${}^*(X\times T)$ and each simplex of L_i is small in ${}^*(A\times T)$. Hence, if b_i is the chain killing $[c_i]-[c_{i+1}]$ in ${}^*(C)(K_i,L_i)$, then b_i is a chain of $C_{n+1}(X\times T, A\times T;{}^*G)$.□

Lemma 31 *The cycles $[c_0]$ and $[c_N]$ are homologous.*

Proof: For each i, let b_i be a chain killing $[c_{i+1}] - [c_i]$, $i = 0, \ldots, N-1$, $N \in {}^*\mathbb{N}$, given by the previous lemma.

We prove that the collection $(b_i)_{i=0,\ldots,N-1}$ can be chosen to be internal. The collection $(K_i, L_i)_{i=0,\ldots,N_1}$ from the proof of the previous lemma is internal, as well as the collections $(c_i)_{i=0,\ldots,N}$, $(c'_i)_{i=0,\ldots,N}$. Hence, the collection $(B_i)_{i=0,\ldots,N-1}$ is internal, where B_i is the set of chains of $({}^*C)_{n+1}(K_i, L_i; {}^*G)$ killing $c_{i+1} - c_i$. Now, use the transferred Axiom of Choice to choose the internal collection $(b_i)_{i=0,\ldots,N-1}$, $b_i \in B_i$.

Now $\sum_{i=0}^{N-1} b_i$ kills $[c_N] - [c_0]$.□

Theorem 32 *Let $f_0\colon (X, A) \to (X \times T, A \times T)$, $f_0(x) = (x, T_0)$, and $f_1\colon (X, A) \to (X \times T, A \times T)$, $f_1(x) = (x, T_1)$. Then f_0 and f_1 induce the same map in homology.*

Proof: Let $[c]$ be a cycle in $C_n(X, A; {}^*G)$. Choose a sequence of points $T_0 = t_0, \ldots, t_N = T_1$ as in the discussion preceeding Lemma 30. Now $[c_0] = C_n(f_0; {}^*G)[c]$, and $[c_N] = C_n(f_1; {}^*G)[c]$. By the previous lemma, $[c_0]$ and $[c_N]$ are homologous, that is $H_n(f_0; {}^*G)([c]) = H_n(f_1; {}^*G)([c])$.□

Corollary 33 *H satisfies the homotopy axiom for completely regular pairs (X, A).*

Proof: By the well-known trick, the previous theorem with $T = I$ implies the homotopy axiom.□

Remark 34 *We actually proved the homotopy axiom in a slightly stronger form: Instead of I, any compact and connected space can be used as the space along which the homotopy runs.*

Remark 35 *In McCord's paper[9], the proof of the homotopy axiom is correct but somewhat handwavish. Our construction above gives also a rigorous proof for the homotopy axiom of McCord homology, once one cheks that if the support of c consists of near-standard points, then the support of each b_i in the proof of Lemma 31 consists of near-standard points. But each point in the support of b_i is of the type (x, t), $x \in {}^*X$ near-standard, $t \in {}^*T$, and these points are near-standard by the next lemma. (Note that, since T is compact, ever point of ${}^*T$ is near-standard.)*

Lemma 36 *Assume that $x \in {}^*X$ and $y \in {}^*Y$ are near-standard. Then the point $(x, y) \in {}^*X \times {}^*Y$ is near-standard.*

Proof: Let $x' \in X$ such that $x \sim x'$, and $y' \in Y$ such that $y \sim y'$. We claim that $(x, y) \sim (x', y')$, where (x', y') is standard.

Let U be a neighbourhood of (x', y') in $X \times Y$. Let $V = U_X \times U_Y$ be a neighbourhood of (x', y') contained in U, where U_X is open in X, and U_Y is open in Y. Since $x' \sim x$, $y' \sim y$, we have that $x \in {}^*U_X$, $y \in {}^*U_Y$, and thus $(x, y) \in {}^*(U_X \times U_Y) \subset {}^*U$.□

Remark 37 *Now, we have checked that H is a homology theory for pairs (X, A) of completely regular spaces, A closed in X, except that in the excision axiom there are some normal embedding requirements.*

7 Quasi-components

Let X be a topological space. We recall that $x, y \in X$ belong to a same quasi-component, if for every partition U, V of X with open sets, $x, y \in U$ or $x, y \in V$.

Lemma 38 *Assume that x, y belong to a same quasi-component of X. Then, there exists $N \in {}^*\mathbb{N}$, and an internal hyperfinite sequence $x = t_0, \ldots, t_N = y$ of points of ${}^*X$ such that for all i, t_i, t_{i+1} are infinitesimally close.*

Proof: Let $\mathfrak{U}$ be a non-standard normal cover refining each standard normal cover. Then the claim is Lemma 24 transferred.□

Lemma 39 *Let H be a homology theory (in the sense of Eilenberg-Steenrod axioms) such that if $H(X)$ has been defined, then also $H(X')$ has been defined for all clopen subsets X' of X together with $H(i)$ for the inclusion $X' \to X$.*

Let X be a topological space, and let x, y be points of X in different quasi-components. Let P be an one-point space, and let $i_x \colon P \to X$, $i_y \colon P \to X$ be the maps mapping the point P to x and y, respectively.

Then $\operatorname{im} H_0(i_x) \cap \operatorname{im} H_0(i_y) = 0$.

Proof: Let U, V be a partition of X with clopen subsets of X such that $x \in U$, $y \in V$. From the Mayer-Vietoris sequence of the triple (X, U, V) we see that $H_0(X) = H_0(U) \oplus H_0(V)$.

Now $\operatorname{im} H_0(i_x) \subset H_0(U)$ and $\operatorname{im} H_0(i_y) \subset H_0(V)$. Hence $\operatorname{im} H_0(i_x) \cap \operatorname{im} H_0(i_y) \subset H_0(U) \cap H_0(V) = 0$.□

Lemma 40 *Assume that $x, y \in X$. Let i_x, i_y be as in the previous lemma. If x, y belong to a same quasi-component, then $H_0(i_x; {}^*G) = H_0(i_y; {}^*G)$.*

Proof: Let s be the 0-simplex in P. Consider $d = C_0(i_y; {}^*G)(gs) - C_0(i_x; {}^*G)(gs)$, and let the points $t_0, \ldots, t_N$ be as in Lemma 38. Now the chain $\sum_{i=0}^{N-1} g(t_{i+1} - t_i)$ kills d. Hence $H_0(i_x; {}^*G) = H_0(i_y; {}^*G)$.□

Theorem 41 *Our homology theory H finds quasi-components.*

Proof: Combine the Lemmas 39 and 40.□

Lemma 42 *Assume that X is connected. Then $H_0(X; {}^*G) = {}^*G$. Furthermore, if P is the one-point space, $x \in X$, and $i_x \colon P \mapsto x$, then $H_0(i; {}^*G)$ is an isomorphism.*

Proof: Let $\mathfrak{U}$ be a non-standard normal cover of X refining each standard normal cover. Let $x \in X$ be a fixed point.

Let $c = {}^*\sum g_i v_i \in C_0(X; {}^*G)$. For each v_i choose a hyperfinite sequence $v_i = t_0^i, \ldots, t_N^i = x$ such that any two consequent points belong to a same member of $\mathfrak{U}$. (Such sequence exists by transferred Lemma 24.) Let $c_i = {}^*\sum_j g_i(t_j^i, t_{j+1}^i)$. Now c_i kills $g_i x - g_i v_i$. Thus, the support of $\partial({}^*\sum_i c_i) + c$ is the point x. That is, any chain in $C_0(X; {}^*G)$ is homologuous to a chain with x as a support. Thus $H_0(X; {}^*G)$ is generated by the chains $[gx]$, $g \in {}^*G$.

To complete the proof, we must show that if $g_0 \neq g_1 \in {}^*G$, then $[g_0 x] \neq [g_1 x]$. But this is a consequence of Eilenberg-Steenrod [2], Chap 1, Theorem 7.6.

The *furthermore* part follows from the previous argument, since $H(i_x)([gP]) = [gx]$ for all $g \in {}^*G$. □

Lemma 43 *Assume that X is a Hausdorff space such that there exist $x, y \in X$ such that x, y are not contained in any compact and connected subset of X. Let P be the one-point space, and let $i_x \colon P \mapsto x$, $i_y \colon P \mapsto y$.*

Let H be a homology theory such that the following hold:

- *If X is an admissible space then so are all clopen subsets $X' \subset X$, together with the embedding $X' \hookrightarrow X$.*
- *If X is an admissible space, then so are all compact subsets $X' \subset X$ together with the embedding $X' \hookrightarrow X$.*
- *H has compact supports.*

Then $\operatorname{im} H_0(i_x) \cap \operatorname{im} H_0(i_y) = 0$.

Proof: Let X' be a compact subset of X containing x, y. Now the connected component C of X' containing x is compact and connected, and hence $y \notin C$. Since X' is compact, its connected components coincide with its quasi-components, and hence x, y lie in different quasi-components of X'. Hence, by Lemma 39, $\operatorname{im} H_0(i_x^{X'}) \cap \operatorname{im} H_0(i_y^{X'}) = 0$, where $i_x^{X'}$ is obtained from i_x by restricting its target to X' and $i_y^{X'}$ similarly from i_y.

Since H has compact supports, $H_0(X) = \operatorname{colim} H_0(X')$, where X' runs over all compact subsets of X. Thus, for $H_0(i_x) \cap H_0(i_y) \neq 0$, we should have $\operatorname{im} H_0(i_x^{X'}) \cap \operatorname{im} H_0(i_y^{X'}) \neq 0$ for some compact $X' \subset X$, but due to the first paragraph, this is impossible.□

Corollary 44 *Our homology theory H does not have compact supports, if S contains all Borel subsets of the plane.*

Proof: There exists a paracompact connected Hausdorff space X such that there are $x, y \in X$ such that x, y do not lie in any compact connected subset of X. See for example the Example 119 in Seebach-Steen [12] (Which is a Borel subset of the plane).

Thus $P \colon P \mapsto x$ induces an isomorphism in the 0th homology by Lemma 42. However, if H had compact supports, this would be impossible by Lemma 44.□

Remark 45 *We do not know whether the compactness assumption of T in Theorem 32 is essential.*

Remark 46 *For compact pairs, our homology theory is defined similarly to McCord homology. Since McCord homology has compact supports (Korppi, [6], Theorem 13), we see that our homology with compact supports coincides with McCord homology.*

Example 47 *Let* $X \subset \mathbb{R}^2$, $X = \{(0,0),(0,1)\} \cup \bigcup_{i\in\mathbb{N}} 1/i \times I$ *and* A=$\{(0,0),(0,1)\}$*. Now* $(0,0)$ *and* $(0,1)$ *belong to the same quasi-component of* X*, and hence* $i_0\colon P \mapsto (0,0)$ *and* $i_1\colon P \mapsto (0,1)$ *induce the same map in homology. Because of the exact sequence* $H_1(X,A;{}^*G) \xrightarrow{\partial} H_0(A;{}^*G) \to H_0(X;{}^*G)$*, there exists* $[z] \in H_1(X,A;{}^*G)$ *such that* $\partial([z]) = [(0,1)-(0,0)]$*.*

We remark that i_0 *and* i_1 *induce the same map also in Cech homology, but because of inexactness of Cech homology, the element* $[z]$ *does not exist in Cech homology.*

8 Non-standard homology and lim

Theorem 48 *Let* (C_n, Δ_n) *be a chain complex of inverse systems* (I, C_i, π_{ji}) *and level homomorphisms. Then* $H_n(\lim_{i\in I} {}^*C_n, \lim_{i\in I} {}^*\Delta_n)$ *and* $\lim_{i\in I} H_n({}^*C_n, {}^*\Delta_n)$ *are isomorphic. The isomorphism is given by the formula* $(z_i)_{i\in I} \mapsto ([z_i])_{i\in I}$*, where* $(z_i)_{i\in I}$ *is a cycle of* $\lim_{i\in I} {}^*C_n$*.*

Proof: Let $(Z_n)_i$ be the group of cycles in $(C_n)_i$, $(B_n)_i$ be the group of boundaries in $(C_n)_i$ and let $(H_n)_i = H_n(C_i)$. Since "being a cycle" and "being a boundary" can be expressed in the first-order logic, ${}^*(B_n)_i$ and ${}^*(Z_n)_i$ are the groups of cycles and boundaries, respectively, in $({}^*(C_n)_i, {}^*(\Delta_n)_i)$, $i \in I$.

Hence, we get a short exact sequence

$$0 \to {}^*(B_n)_i \to {}^*(Z_n)_i \to {}^*(Z_n/B_n)_i \to 0.$$

By Theorem 7 in Korppi[5], we have that $\lim^1_{i\in I} {}^*(B_n)_i = 0$, and thus we get a short exact sequence

$$0 \to \lim_{i\in I} {}^*(B_n)_i \to \lim_{i\in I} {}^*(Z_n)_i \to \lim_{i\in I} {}^*(Z_n/B_n)_i \to 0.$$

Furthermore $\lim_{i\in I} {}^*(Z_n/B_n)_i = \lim_{i\in I}({}^*H_n)_i$.

To complete the proof, we must thus show that $\lim_{i\in I}({}^*B_n)_i$ equals $\operatorname{im}\lim_{i\in I}({}^*\Delta_n)_i$ and $\lim_{i\in I} {}^*(Z_n)_i$ equals $\ker\lim_{i\in I} {}^*\Delta_n$.

Consider the exact sequence

$$0 \to ({}^*Z_{n+1})_i \to ({}^*C_{n+1})_i \overset{{}^*(\Delta_{n+1})_i}{\to} ({}^*B_n)_i \to 0.$$

Since $\lim^1_{i\in I}({}^*Z_{n+1})_i = 0$ (by Korppi [5], Theorem 7), we get an exact sequence

$$0 \to \lim_{i\in I}({}^*Z_{n+1})_i \to \lim_{i\in I}({}^*C_{n+1})_i \overset{\lim_{i\in I}\ {}^*(\Delta_{n+1})_i}{\to} \lim_{i\in I}({}^*B_n)_i \to 0.$$

Hence $\lim_{i\in I}({}^*B_n)_i = \operatorname{im}\lim_{i\in I}\ {}^*(\Delta_{n+1})_i$ and $\lim_{i\in I}\ {}^*(Z_n)_i = \ker\lim_{i\in I}\ {}^*\Delta_n$. the proof of of the fact that $\lim_{i\in I} H_n({}^*C_n, {}^*\Delta_n)$ and $H_n(\lim_{i\in I}\ {}^*C_n, \lim_{i\in I}\ {}^*\Delta_n)$ are isomorphic is complete.

.□

9 Homology of simplicial pairs

In this section, we do not assume that the simplicial complexes are either locally finite or finite-dimensional. We assume that the geometrical realization $|K|$ of K is equipped with the weak topology.

Definition 49 *Let K be a simplicial complex. We define the Vietoris complex K_V of K is so that the points of K_V are the points of $|K|$, and s is a simplex of K_V if s is contained in the open star of some vertex v of K.*

First, we observe that K and the nerve K_N of the covering $\mathfrak{U}$ of $|K|$ with open stars of vertices are naturally isomorphic, and hence their homologies coincide. Furthermore, by Dowker [1], Theorem 1a and the discussion in the beginning of Chapter 5, the homologies of K_N and K_V are isomorphic; the isomorphism is natural by Dowker [1], Lemma 4a.

Lemma 50 *Let K be a simplicial complex, and let $\mathfrak{U}$ be a normal open cover of $|K|$.*

1. *There exists a subdivision K' of K such that each geometrical simplex of K' is contained in some member of $\mathfrak{U}$.*

2. *There exists a subdivision K'' of K such that the star of each vertex is contained in some member of $\mathfrak{U}$.*

Proof: See Whitehead [13], Theorem 35 and the discussion in Spanier [11], after Theorem 3.3.14. □

Let K be a simplicial complex, and K' a subdivision of K. We identify $|K'|$ and $|K|$.

Lemma 51 *Let K be a simplicial complex, and let K' be a subdivision of K. Then $i\colon K'_V \hookrightarrow K_V$ induces an isomorphism in homology.*

Proof: S' be the set of open stars of vertices of K' and S be the set of open stars of vertices of K.

Let $(|K|, S', \in)$ and $(|K|, S, \in)$ be relations as in Dowker [1]. Let $f\colon (|K|, S', \in) \to (|K|, S, \in)$ be a map such that $f(x) = x$ for $x \in |K|$, and $s \subset f(s)$ for $s \in S'$. Then, by Dowker [1], Lemma 4a, Theorem 1 and Section 1, $g_*\omega' = \omega i_*$, where $\omega\colon H(K_V; G) \to H(K_N; G)$ is an isomorphism, $\omega'\colon H(K'_V; G) \to H(K_N; G)$ is an isomorphism and $g\colon K'_N \to K_N$ is a map that takes $s \in S'$ into gs such that $s \subset gs$.

Now K_N and K (resp. K'_N and K') are naturally isomorphic, and in the natural isomorphism, to g there corresponds a map $h\colon K' \to K$ such that for all simplices s of K and all vertices $v \in K'$ it holds that $h(v)$ is a vertex of a simplex s if $v \in |s|$. Hence h is a simplicial approximation of $\mathrm{id}\colon |K'| \to |K|$, and h induces isomorphisms in homology. Hence g induces isomorphisms in homology. Since ω and ω' are isomorphisms, also i_* is an isomorphism. □

Let $\mathcal{U}$ be an open cover of a topological space X. The vietoris complex $V_{\mathcal{U}}$ of $\mathcal{U}$ is a simplicial complex such that the vertices of $V_{\mathcal{U}}$ are the points of X, and s is a simplex if s is contained in $U \in \mathcal{U}$.

Lemma 52 $C(|K|; {}^*G) = \lim({}^*C)({}^*V_{\mathcal{U}}; {}^*G) = \bigcap({}^*C)({}^*V_{\mathcal{U}}; {}^*G)$*, where* lim *is taken over normal covers $\mathcal{U}$ of $|K|$, the relation in the inverse system is refinement.*

Proof: $C(|K|; {}^*G)$ is the subset of $C'(|K|; {}^*G)$ consisting of chains with only small simplices in support. Hence, $c \in C(|K|; {}^*G)$ implies $c \in ({}^*C)({}^*V_{\mathcal{U}}; {}^*G)$ for all normal covers $\mathcal{U}$ of $|K|$.

Assume then that $c \in \lim({}^*C)({}^*V_{\mathcal{U}})$. First, we note that if $\mathcal{V}$ refines $\mathcal{U}$, then the map $({}^*C)({}^*i; {}^*G)$, induced by the inclusion $i\colon V_{\mathcal{V}} \hookrightarrow V_{\mathcal{U}}$ is a monomorphism. To complete the proof, we must thus show that if $c \in \lim({}^*C)({}^*V_{\mathfrak{U}}; {}^*G) = \bigcap_{\mathfrak{U}}({}^*C)({}^*V_{\mathfrak{U}}; {}^*G)$, then each simplex in the support of c is small. But this holds, since each simplex s in the support of c is contained in $U \in {}^*\mathfrak{U}$ for every normal cover $\mathfrak{U}$ of $|K|$.□

Lemma 53 *There is an isomorphism* $H(|K|; {}^*G) \to \lim({}^*H)({}^*V_{\mathfrak{U}}; {}^*G)$*, where* $V_{\mathfrak{U}}$ *is the vietoris complex of the cover* $\mathfrak{U}$*, and* lim *is taken over normal covers* $\mathfrak{U}$ *of* $|K|$*. The isomorphism is given by the formula* $z \mapsto ([z])$*, where* z *is a cycle in* $C(|K|; {}^*G)$*.*

Proof: By Lemma 52 $H(|K|; {}^*G) = ({}^*H) \lim({}^*C)({}^*V_{\mathfrak{U}}; {}^*G)$, which is isomorphic to $\lim({}^*H)({}^*V_{\mathfrak{U}}; {}^*G)$ by Theorem 48. By Theorem 48, he isomorphism is given by the formula $z \mapsto ([z])$, where z is a cycle in $C(|K|; {}^*G)$. □

Lemma 54 *The inclusion* $C(|K|; {}^*G) \hookrightarrow ({}^*C)({}^*K_V; {}^*G)$ *induces isomorphisms in homology.*

Proof: By Lemma 50 (2), covers with stars of vertices in subdivisions are cofinal in the set of all normal covers. Thus, in Lemma 53 the limit can be taken over the sets K'_V, where K' is a subdivision of K.

But, if K' is a subdivision of K, then the map induced by the inclusion $({}^*H)({}^*K'_V; {}^*G) \to ({}^*H)({}^*K_V; {}^*G)$ is an isomorphism. Hence $\lim({}^*H)({}^*K'_V; {}^*G)$ and $({}^*H)({}^*K_V; {}^*G)$ are isomorphic; the isomorphism is given by the formula $([z_{K'_V}]) \mapsto [z_{K_V}]$, where $([z_{K'_V}])$ is a collection of homology classes of cycles in $\lim({}^*H)({}^*K'_V; {}^*G)$.

Hence, by Lemma 53, the $H(|K|; {}^*G)$ and $({}^*H)({}^*K_V; {}^*G)$ are isomorphic; by Lemma 53 and the above paragraph, the isomorphism is the one taking a cycle z of $H(|K|; {}^*G)$ to $[z]$ in $({}^*H)({}^*K_V; {}^*G)$. □

Theorem 55 *The inclusion* $C(|K, L|; {}^*G) \hookrightarrow ({}^*C)({}^*K_V, {}^*L_V; {}^*G)$ *induces isomorphisms in homology.*

Hence $H(|K, L|; {}^*G) = {}^*(H(K, L; G))$

Proof: Since both H on the left and ${}^*H$ on the right are exact, the first result follows from the previous lemma and the five-lemma applied to the exact homology sequences.

The last part follows, since by Dowker [1], Theorem 1a, and the discussion in the beginning of Chapter 5, $H(K_V, L_V; G) = H(K_N, L_N; G)$ (by Dowker [1], Lemma 4a, the correspondence is natural), and trivially, $H(K_N, L_N; G) = H(K, L; G)$; (one easily sees that this corresponcence is natural.) $\Box$

Example 56 *Let X be a disjoint union of $S^1_n = S^1$, $n \in \mathbb{N}$. Then $H_0(X; {}^*\mathbb{Z}) = {}^*\bigoplus_{n \in {}^*\mathbb{N}} {}^*\mathbb{Z}$, $H_1(X; {}^*\mathbb{Z}) = {}^*\bigoplus_{n \in {}^*\mathbb{N}} {}^*\mathbb{Z}$, and $H_n(X; {}^*\mathbb{Z}) = 0$ for $n \geq 2$ by the previous theorem.*

10 Non-standard characterization of resolutions

Let $(I, X_i, \pi_{ii'})$ be an inverse system of topological spaces. Let X be a topological space together with maps $\pi_i \colon X \to X_i$, $i \in I$ commuting with the maps $\pi_{ii'}$. Following Mardesic, [7], Chapter 6.2, we say that $(I, X_i, \pi_{ii'})$ is a resolution of X, if the following hold:

- (P1) For every normal cover $\mathcal{U}$ of X there exists $i \in I$ and a normal cover $\mathcal{V}$ of X_i such that $\pi_i^{-1}\mathcal{V}$ refines $\mathcal{U}$.
- (P2) For every $i \in I$ and every normal cover $\mathcal{V}$ of X_i there exists $i' \in I$ such that $\pi_{ii'} X_{i'} \subset \mathrm{St}(\pi_i(X), \mathcal{V})$.

Let $J = \{i \in {}^*I \mid i > I\}$. For $x, y \in {}^*X_j$, $j \in J$ we say that $x \sim y$ if and only if $\pi_{ij}(x) \sim \pi_{ij}(y)$ for all $i \in I$. We say that $A \subset X_j$ is small if and only if $\pi_{ij} A$ is small for all $i \in I$.

Remark 57 *Let $j > I$. Then $\sim$ in X_j will, in general, be dependent on the resolution (I, X_i, π_{ij}).*

Lemma 58 *Let $j \in J$, and $x, y \in {}^*X_j$. We have that $x \sim y$ if and only if for each $i \in I$ and each normal cover $\mathcal{U}$ of X_i there exists $U \in {}^*(\pi_{ij}^{-1}\mathcal{U})$ such that $x, y \in U$.*

Proof: Trivial from the definition.$\Box$

Lemma 59 *$\sim$ is an equivalence relation in X_j. Furthermore, a subset A in X_j is small if and only if $x \sim y$ for all $x, y \in A$.*

Proof: To prove that $\sim$ is an equvalence relation, it is enough to check transitivity. Symmetry and reflexivity are trivial. Let $i \in I$, and let $\mathcal{U}$ be a normal cover of X_i. Let $\mathcal{V}$ be a star-refinement of $\mathcal{U}$. Now, due to the previous lemma and considering the covers ${}^*(\pi_{ij}^{-1}\mathcal{U})$ and ${}^*(\pi_{ij}^{-1}\mathcal{V})$, the proofs of the claims is similar to the corresponding proofs in Lemmas 6 and 7.□

Lemma 60 *The following are equivalent:*

1. *(P1)*
2. *For all $x, y \in {}^*X$ we have $x \sim y$ if and only if ${}^*\pi_i(x) \sim {}^*\pi_i(y)$ for all $i \in I$.*
3. *For all $x, y \in {}^*X$ we have $x \sim y$ if and only if ${}^*\pi_j(x) \sim {}^*\pi_j(y)$ for all $j \in J$.*
4. *For all $x, y \in {}^*X$ we have $x \sim y$ if and only if ${}^*\pi_j(x) \sim {}^*\pi_j(y)$ for some $j \in J$.*

Proof: (2) implies (3) by the definition of $\sim$ for non-standard indices. (3) implies (4) a fortiori. (4) implies (2) by the definition of $\sim$ for non-standard indices.

We prove that (1) implies (2). Assume (P1). Let $i \in I$. Since π_i is continuous, we have that $x \sim y$ implies ${}^*\pi_i(x) \sim {}^*\pi_i(y)$.

Assume then that $x \not\sim y$. Then, by the definition of $\sim$ there exists a normal cover $\mathcal{U}$ of X such that x, y do not belong to any same member of ${}^*\mathcal{U}$. By (P1) there exists $i \in I$ and and a normal cover $\mathcal{V}$ of X_i such that $\pi_i^{-1}\mathcal{V}$ refines $\mathcal{U}$. Hence ${}^*\pi_i(x)$ and ${}^*\pi_i(y)$ do not belong to any same member of ${}^*\mathcal{V}$. Thus (1) implies (2).

We prove that (2) implies (1), that is, not (1) implies not (2). Let $\mathcal{U}$ be an open cover of X such that for all $i \in I$ and for no open cover $\mathcal{V}$ of X_i the cover $\pi_i^{-1}\mathcal{V}$ is a refinement of $\mathcal{U}$. Let $\mathcal{U}'$ be a normal cover that is a star-refinement of $\mathcal{U}$.

Let P be a finite set of pairs $(i, \mathcal{V})$ such that $i \in I$ and $\mathcal{V}$ is a normal cover of X_i. Choose $i' > i$ for all i in pairs P, a normal cover $\mathcal{V}'$ of $X_{i'}$ refining each

$\pi_{ii'}^{-1}\mathcal{V}$. Now $\pi_{i'}^{-1}\mathcal{V}'$ is not a refinement of $\mathcal{U}$. Hence, there exists $V \in \pi_{i'}^{-1}\mathcal{V}'$ such that V is not contained in any member of $\mathcal{U}$. Let $x \in V$. Now V is not contained in $\mathrm{st}(x, \mathcal{U}')$. Hence, we can choose $y \in V$, $y \notin \mathrm{st}(x, \mathcal{U}')$. Hence, x, y do not belong to a same member of $\mathcal{U}'$.

Hence by saturation, there exist $x, y \in {}^*X$ such that ${}^*\pi_i(x)$ and ${}^*\pi_i(y)$ belong to a same member of ${}^*\mathcal{V}$ for all $i \in I$ and for all normal covers $\mathcal{V}$ of X_i, but x, y do not belong to a same member of ${}^*\mathcal{U}'$. Hence $x \not\sim y$, but ${}^*\pi_i(x) \sim {}^*\pi_i(y)$ for all $i \in I$. Hence, not (1) implies not (2), and (2) implies (1).□

Lemma 61 *The following are equivalent:*

1. *(P2)*
2. *For every $i \in I$ and $j \in J$ we have the following: Every point $x \in {}^*\pi_{ij}{}^*X_j$ is infinitesimally close to some $x' \in {}^*\pi_i{}^*X$.*
3. *For every $i \in I$ and some $j \in J$ we have the following: Every point $x \in {}^*\pi_{ij}{}^*X_j$ is infinitesimally close to some $x' \in {}^*\pi_i{}^*X$.*
4. *For every $j \in J$ and every point $x \in {}^*X_j$ we have that $x \sim x'$ for some $x' \in {}^*\pi_j{}^*X$.*
5. *For some $j \in J$ and every point $x \in {}^*X_j$ we have that $x \sim x'$ for some $x' \in {}^*\pi_j{}^*X$.*

Proof: (4) implies (5) and (2) implies (3) a fortiori.

We prove that (1) implies (2). Let i in I and, j in J and let $x \in {}^*\pi_{ij}{}^*X_j$. Assume (P2). Let $\mathcal{U}$ be a normal cover of X_i. Let $i' > i$ be standard such that $\pi_{ii'}X_{i'} \subset \mathrm{St}(\pi_i(X), \mathcal{U})$. Since $\pi_{ij}(X_j) \subset \pi_{ii'}X_{i'}$, we have that x is $\mathcal{U}$-close to some $x_{\mathcal{U}} \in {}^*\pi_i{}^*X$.

Thus, given any finite family $(\mathcal{U}_k)$ of normal covers of X_i, they have a common refinement $\mathcal{U}$, and there exists $x_{\mathcal{U}} \in {}^*\pi_i{}^*X$ that is ${}^*\mathcal{U}_k$-close to x for all k. Thus, by saturation there exists a non-standard normal cover $\mathcal{W}$ of ${}^*X_i$ refining the non-standard version of each standard normal cover, and $y \in {}^*\pi_i{}^*X$ such that x is $\mathcal{W}$-close to y, and hence $x \sim y$. Thus (1) implies (2).

We prove that (3) implies (5) so that the j in both claims is the same. This will also give (2) implies (4).

Let $j \in J$, and let $x \in {}^*X_j$. Let $i \in I$, and let $\mathcal{U}$ be a normal cover of X_i. By (2), there exists $x_{i,\mathcal{U}} \in {}^*\pi_j(X)$ such that x and $x_{i,\mathcal{U}}$ are ${}^*\pi_{ij}^{-1}\mathcal{U}$-close.

Thus, given a finite set P of pairs $(i, \mathcal{U})$ such that $\mathcal{U}$ is a normal cover of X_i, there exists $i' > i$ for all i in P, and a normal cover $\mathcal{U}'$ of $X_{i'}$ such that $\mathcal{U}'$ refines each $\pi_{ii'}^{-1}\mathcal{U}$. Hence $x_{i',\mathcal{U}'}$ and x are ${}^*\pi_{ij}^{-1}\,{}^*\mathcal{U}$-close for each $(i, \mathcal{U}) \in P$. Thus, by saturation, there exists $x' \in {}^*\pi_j X$ such that x and x' are ${}^*\pi_{ij}^{-1}\,{}^*\mathcal{V}$ close for every normal cover $\mathcal{V}$ of X_i and every $i \in I$, and hence $x \sim x'$ by Lemma 57. Hence (2) implies (4) and (3) implies (5).

We prove that (5) implies (1). Let $j \in J$. Assume that (P2) does not hold, that is, there exists $i \in I$ and a normal cover $\mathcal{U}$ of X_i such that for every $i' > i$ the set $\pi_{ii'}X_{i'}$ is not contained in $\mathrm{St}(\pi_i X, \mathcal{U})$. For every $X_{i'}$, $i' > i$ choose $x_{i'} \in X_{i'}$ such that $\pi_{ii'}x_{i'} \notin \mathrm{St}(\pi_i X, \mathcal{U})$. Now, since $(x_{i'})$ is a standard sequence, it extends to all indices in J, and thus ${}^*x_j$ exists. By transferring the claim $\pi_{ii'}x_{i'} \notin \mathrm{St}(\pi_i X, \mathcal{U})$, we see that no point of ${}^*\pi_j\,{}^*X$ is ${}^*\pi_{ij}^{-1}\,{}^*\mathcal{U}$-close to ${}^*x_j$. Thus, not (1) implies not (5), and (5) implies (1).

Now we have (2) implies (3), (4) implies (5), (1) implies (2), (3) implies (5), (2) implies (4) and (5) implies (1). Hence the theorem holds. □

Lemma 62 *Given $j \in J$ there exists internal $f\colon {}^*X_j \to \pi_j\,{}^*X$ such that*

1. *$x \sim y$ if and only if $f(x) \sim f(y)$.*
2. *$x \sim f(x)$*
3. *If $x \in \pi_j\,{}^*X$, then $f(x) = x$.*

Proof: Let P be a finite set of pairs $(i, \mathcal{U})$, where $i \in I$, and $\mathcal{U}$ is a normal cover of X_i. Let I_0 be the set of i such that $(i, \mathcal{U}) \in P$ for some $\mathcal{U}$.

Let $i' > I_0$, and let $\mathcal{V}$ be a normal cover of $X_{i'}$ such that $\mathcal{V}$ refines $p_{ii'}^{-1}\mathcal{U}$ for each $\mathcal{U}$ in P. Let now $f_P\colon X_j \to \pi_j\,{}^*X$ be a retraction (no continuity assumptions) such that x and $f_P(x)$ belong to a same element of $p_{i'j}^{-1}\,{}^*\mathcal{V}$. That is, $x, f_P(x)$ belong to a same element of $p_{ij}^{-1}\,{}^*\mathcal{U}$ for all $(i, \mathcal{U}) \in P$. Such $f_P U$ exists by the transferred Axiom of Choice and the fact that for each element x of X_j there is an element y_x of $\pi_j\,{}^*X$ such that $x \sim y_x$; hence x and y_x belong to a same element of $p_{i'j}^{-1}\,{}^*\mathcal{V}$.

By saturation there exists $f\colon X_j \to \pi_j\,{}^*X$ such that f is a retraction, and for each standard i and $\mathcal{U}$ a normal cover of X_i, $f(x)$ and x are ${}^*\pi_{ij}^{-1}\,{}^*\mathcal{U}$-close. Hence $f(x) \sim x$. Now we have (2) and (3), and (1) follows by transitivity of $\sim$. $\square$

11 H is continuous with respect to resolutions for single spaces

Let (I, X_i, π_{ij}) be a resolution of X like in the previous section. Now the maps π_i induce a map $p\colon H(X;\,{}^*G) \to \lim H(X_i;\,{}^*G)$. In this section we prove that p is an isomorphism.

Our proof exploits ideas in the proof of the continuity of McCord homology for compact pairs in Garavaglia [3].

For $j \in J$, we denote by $C'_n(X_j;\,{}^*G)$ the group of hyperfinite sums of $n+1$-tuples (called simplices) of points of X_j, and by $C_n(X_j;\,{}^*G)$ the subgroup of $C'_n(X_j;\,{}^*G)$ such that all the simplices are small. We easily see that the maps π_{ij} induce maps $C_n(\pi_{ij};\,{}^*G)$.

We denote by $C_n^{\mathcal{U}}(X_i;\,{}^*G)$ the subgroup of $C'_n(X_i;\,{}^*G)$ such that all simplices in $C_n^{\mathcal{U}}(X_i;\,{}^*G)$ are $\mathcal{U}$-small. If $c, c' \in C_n^{\mathcal{U}}(X_i;\,{}^*G)$ are homologous by a boundary in $C_{n+1}^{\mathcal{U}}(X_i;\,{}^*G)$, we say that c, c' are $\mathcal{U}$-homologous.

Remark 63 *Assume that each X_i is polyhedral and $j > I$. We remark that the homology of $C(X_j;\,{}^*G)$ is in general different from the simplicial $({}^*H)(X_j;\,{}^*G)$.*

Lemma 64 *Given a collection of cycles $(c_i)_{i\in I}$, $c_i \in C_n(X_i;\,{}^*G)$ such that $C_n(\pi_{ii'})(c_{i'})$ and c_i are homologous for all $i < i' \in I$, there exists $j \in J$ and $c \in C_n(X_j;\,{}^*G)$ such that $C_n(\pi_{ij})(c)$ and c_i are homologous for all $i \in I$.*

Proof: Let P be a finite set of pairs $(i, \mathcal{U})$, where $i \in I$ and $\mathcal{U}$ is an open cover of X_i. Let I' be the set of indices i such that $(i, \mathcal{U}) \in P$ for some $\mathcal{U}$.

Then there exists $i' \in I$ such that $i' > i$ for all pairs $i \in I'$. Consider $c_{i'}$. There exists a $b_i \in C_{n+1}(X_i;\,{}^*G)$, $i \in I'$ such that $\partial b_i = \pi_{ii'} c'_i - c_i$. Thus, $b_i \in C_{n+1}^{\mathcal{U}}(X_i;\,{}^*G)$ for all $\mathcal{U}$ such that $(i, \mathcal{U}) \in P$. Furthermore, $\pi_{ii'} c_{i'} \in C_n^{\mathcal{U}}(X_i;\,{}^*G)$ for the same $\mathcal{U}$.

Thus, by saturation, there exists $j \in J$ and $c_j \in C'_n(X_j;\, {}^*G)$ such that all simplices of $\pi_{ij}c_j$ are small for all $i \in I$, and for every normal cover $\mathfrak{U}$ of X_i a $b_i^{\mathfrak{U}} \in C_{n+1}^{\mathfrak{U}}(X_i;\, {}^*G)$ such that $\partial b_i^{\mathfrak{U}} = \pi_{ij}c_j - c_i$. Fix i. By using the saturation principle for the collection $(b_i^{\mathfrak{U}})_{\mathfrak{U}}$ one finds $b_i \in C_{n+1}(X_i;\, {}^*G)$ killing $\pi_{ij}c_j - c_i$. □

Lemma 65 *Assume that $c \in C_n(X;\, {}^*G)$ such that for all $i \in I$, $\pi_i(c) = \partial b_i$ for some $b_i \in C_{n+1}(X_i;\, {}^*G)$. Then there exists $j \in J$ and $c' \in C_{n+1}(X_j;\, {}^*G)$ such that $\partial(c') = \pi_j c$.*

Proof: Let P be a finite set of pairs $(i, \mathfrak{U})$, where $\mathfrak{U}$ is a normal cover of X_i. Then there exists $i' > i$, i in P, and $b_{i'}$ such that $\partial b_{i'} = \pi_{i'}c$, and the simplices of $b_{i'}$ are ${}^*\pi_{ii'}^{-1}\, {}^*\mathfrak{U}$ small for all $(i, \mathfrak{U})$ in P.

Hence, by saturation, there exists $i_0 > I$ and b_{i_0} such that $\partial b_{i_0} = \pi_{i_0}c$ and each simplex of b_{i_0} is ${}^*\pi_{ii_0}^{-1}\, {}^*\mathfrak{U}$-small for all $(i, \mathfrak{U})$.□

Lemma 66 *Let $j \in J$.*

Given a cycle c in $C_n(X_j;\, {}^*G)$ there exists a homologous cycle in $C_n(\pi_j X;\, {}^*G)$.

Given a cycle c' in $C_n(\pi_j X;\, {}^*G)$, that bounds in $C_n(X_j;\, {}^*G)$ then it bounds in $C_n(\pi_j X;\, {}^*G)$.

Proof: Let c be as in the statement of the lemma. Let f be as in Lemma 62. Let K'' be a hyperfinite simplicial complex such that the vertices of K'' are vertices of simplices of c, and the simplices of K'' are the sets contained in the simplices of c. Let K' be the hyperfinite simplicial complex fK''. (I.e. the vertices of K' are fv, v a vertice of K'', and s is a simplex of K' iff s is contained in a set fs', s' a simplex of K''.)

Let K be a hyperfinite simplicial complex such that the set of vertices of K is $K'^{\mathrm{vert}} \cup K''^{\mathrm{vert}}$, and s is a simplex of K iff s is contained in a set $s' \cup fs'$, s' a simplex of K''. One readily checks that all the simplices of K are small.

Now $\mathrm{id}_{K''}\colon K'' \to K$ and $f|\colon K'' \to K$ are a pair of maps with an acyclic carrier (the carrier of a simplex s is $s \cup fs$). Hence, by transferred Eilenberg-Steenrod [2], Theorem VI.5.8, the maps are **chain homotopic*, and they induce the same map in homology. Hence c and fc are homologuous cycles. Hence we got the first part.

Then the last part. Let c' be as in the statement of the lemma, and let $c'' \in C_{n+1}(X_j; {}^*G)$ be such that $\partial c'' = c'$. Now fc'' is a chain of $C_{n+1}({}^*\pi_j X; {}^*G)$ such that $\partial f c'' = c'$. Hence we got the last part.□

Lemma 67 *Let K, L be hyperfinite simplicial complexes and $f\colon K \to L$ an internal simplicial map such that $f^{-1}s$ is a simplex of K and f maps $f^{-1}s$ onto s for each simplex s of L. Then f induces isomorphisms in homology.*

Proof: Let K, L be finite simplicial complexes, and $f\colon K \to L$ a simplicial such that $f^{-1}s$ is a simplex of K and f maps $f^{-1}s$ onto s for each simplex s of L. We prove that f induces isomorphisms in homology.

We prove the claim by induction w.r.t. the number n of simplices in L.

If $n = 1$, K and L are contractible, and the claim holds.

Assume then that the claim is true for $n-1$, and let L have n simplices. Let s'' be a maximal-dimensional simplex of L, let s be a simplicial complex consisting of s'' and its faces, and let s' be the simplicial complex corresponding to the simplicial boundary of s''. Let L' be the simplicial complex L minus s plus s'. Let K' be the simplicial complex K minus $f^{-1}s$ plus $f^{-1}s'$.

Now, by the inductive assumption, $f|$ induces isomorphisms in homology for $K' \to L'$, $f^{-1}s' \to s'$ and since $f^{-1}s$ and s are contractible, $f|$ induces an isomorphisms in homology for $f^{-1}s \to s$. Hence, by the exactness of Mayer-Vietoris -sequences of K', $f^{-1}s$ and L', s and the five-lemma, f induces isomorphisms in homology.

The lemma follows by transfer. □

Corollary 68 *We have the following:*

1. *Given a cycle c in $C_n(X; {}^*G)$ such that $\pi_j c$ bounds in $C_n(\pi_j X; {}^*G)$, then c bounds in $C_n(X; {}^*G)$.*

2. *Given a cycle c' in $C_n(\pi_j X; {}^*G)$ there exists a cycle c in $C_n(X; {}^*G)$ such that c' and $\pi_j c$ are homologous.*

Proof: We prove first (1). Let c be as in the statement of the corollary. Let K be a simplicial complex consisting of all the simplices and their faces that have a non-zero coefficient in c. Let b kill $\pi_j c$. Let L be the simplicial complex consisting of all simplices and their faces that have a non-zero coefficient in b. Now $\pi_j|K$ is a simplicial map $K \to L$.

Let K' be a simplicial complex defined as follows: For each vertex vertex v in $L \setminus \pi_j K$, choose an arbitrary vertex v' in X such as $\pi_j^{-1} v = v'$. By the transferred Axiom of Choice, the set of vertices v' can be chosen internal. Let the vertices of K' be the union of vertices of K and the vertices v'. The set $s \subset K'$ is a simplex of K' if it is contained in the inverse image of a simplex of L. Since K' is hyperfinite, each simplex of K' is hyperfinite-dimensional. One readily checks that K is a subcomplex of K', and by Lemma 60 (4), each simplex of K' is small in ${}^*X$. Now $\pi_j|\colon K' \to L$ is as in the statement of Lemma 67, and thus, by Lemma 67, c bounds in $({}^*C)_{n+1}(K'; {}^*G)$, and hence in $C_{n+1}(X; {}^*G)$.

Then, we prove (2). Let c' be as in the statement of the lemma. Let L be the simplicial complex consisting of all simplices and their faces that have a non-zero coefficient in c'. For each vertex v of L choose an arbitrary π_j-inverse image v', and let K be the complex whose vertices are the points v', and s is a simplex of K, if s is contained in a π_j-inverse image of a simplex of L. Similar to point (1), K can be chosen internal, each simplex in K is small and hyperfinite-dimensional. Now $\pi_j|\colon K \to L$ is as in the statement of Lemma 67, and thus, by Lemma 67, the required c exists in $({}^*C)_n(K, {}^*G)$, and hence in $C_n(X, {}^*G)$.□

Theorem 69 *H is continuous with respect to resolutions for single spaces.*

Proof: Combine the lemmas in this section: By Corollary 68, $H(\pi_j)\colon H(X, {}^*G) \to H(\pi_j X, {}^*G)$ is an isomorphism for all $j \in J$. By Lemma 66, the inclusion $k\colon {}^*\pi_j {}^*X \to X_j$ induces an isomorphism $H(k)\colon H(\pi_j X, {}^*G) \to H(X_j, {}^*G)$ for all $j \in J$. Now, by Lemmas 64 and 65, the maps π_i induce an isomorphism $H(X; {}^*G) \to \lim H(X_i; {}^*G)$. □

12 Example

Since our homology coincides with McCord homology and hence with Cech homology (with non-standard coefficients) for compact spaces, interesting examples are non-compact.

Let X' be the disjoint union of unit intervals I_j, $j \in \mathbb{N}$. Let $X = X'/\equiv$, where the zero points are glued together. Let X have otherwise the weak topology, but let all neighbourhoods of 0 contain all but finite of the 1-points of the intervals.

One easily sees that if H' is a homology theory with compact supports, then $H'_n(X) = 0$ for $n > 0$.

We let for $i \in \mathbb{N}$, $X_i = X/\equiv_i$, where $x \equiv_i 0$, if x is a 1-point of I_j, where $j > i$.

One easily sees that X_i, with the natural projections, form a resolution of X. Since each X_i is simplicial, by Theorem 55, $H(X_i; {}^*G) = {}^*(H_s(X_i; G))$, where H_s is the simplicial homology functor.

Hence $H_0(X_i; {}^*G) = {}^*G$, $H_1(X_i; {}^*G) = {}^*(\bigoplus_{i'>i} G)$ and $H_n(X_i; {}^*G) = 0$, $n > 1$.

Since $H(X; {}^*G) = \lim H(X_i; {}^*G)$ by Theorem 69, we have that $H_0(X) = {}^*G$, $H_1(X) = \{x \in {}^*(\bigoplus_{j\in\mathbb{N}} G) \mid x_j = 0 \text{ for all standard } j\}$ and $H_n(X; {}^*G) = 0$, $n > 1$.

13 Continuity for pairs

Let (X, A) be a pair of spaces, and let (X_i, A_i), $i \in I$, $(\pi_{ii'})$, $i' > i$, be an inverse system of pairs of spaces. Let $\pi_i \colon (X, A) \to (X_i, A_i)$, $i \in I$, be maps such that $\pi_i \colon X \to X_i$, $i \in I$, is a resolution of X, and $\pi_i| \colon A \to A_i$, $i \in I$, is a resolution of A.

Lemma 70 *The homology sequence,*

$$\cdots \to \lim H_{n+1}(X_i, A_i; {}^*G) \stackrel{\lim \partial}{\to} \lim H_n(A_i; {}^*G) \stackrel{\lim a_*}{\to} \lim H_n(X_i; {}^*G) \stackrel{\lim b_*}{\to}$$

$$\lim H_n(X_i, A_i; {}^*G) \to \dots$$

is exact, where a is the inclusion of A_i to X_i, and b is the projection of X_i to (X_i, A_i).

Proof: Obviously, the composition of any two maps in the sequence is zero.

We show the exactness at the place $\lim H_n(X_i, A_i; {}^*G)$. The other two places are similar.

Let $([x_i]) \in \lim H_n(X_i, A_i; {}^*G)$, $\partial([x_i]) = 0$, where each x_i is a chain in $C_n(X_i; {}^*G)$. Let P be a finite set of elements $(i, \mathcal{U}, \mathcal{V})$, where $i \in I$, $\mathcal{U}$ is a normal cover of X_i and $\mathcal{V}$ is a normal cover of A_i. Now there exists $i' > i$ for all i in P. Furthermore, there exists a cycle $x'_{i'}$ in $C(X_{i'}; {}^*G)$ such that $x'_{i'} - x_{i'} = a + \Delta(x)$ for some $a \in C(A_{i'}; {}^*G)$, $x \in C_{n+1}(X_{i'}; {}^*G)$.

Hence, the following holds: Given P, there exists $i' > i$ for all i in P and a cycle $x'_{i'} \in C'(X_{i'}; {}^*G)$ such that all simplices of $\pi_{ii'}x'_{i'}$ are $\mathcal{U}$-small for all $\mathcal{U}$ such that $(i, \mathcal{U}, \mathcal{V}) \in P$, and $x_i - \pi_{ii'}x'_{i'} = \Delta(x) + a$ for $x \in C^{\mathcal{U}}_{n+1}(X_i; {}^*G)$, $a \in C^{\mathcal{V}}(A_i; {}^*G)$.

Hence, by saturation, there exists $i_0 > I$ and a cycle $x'_{i_0} \in C'(X_{i_0}; {}^*G)$ such that all simplices of $\pi_{ii_0}x'_{i_0}$ are small for all $i \in I$, and for all $i \in I$ and for all normal covers $\mathcal{V}$ of A_i, $\mathcal{U}$ of X_i there exist $a \in C^{\mathcal{V}}(A_i; {}^*G)$, $x \in C^{\mathcal{U}}_{n+1}(X_i; {}^*G)$ such that $x_i - \pi_{ii_0}x'_{i_0} = \Delta(x) + a$.

Let $i \in I$ be fixed. Let P be a finite collection of pairs $(\mathcal{U}, \mathcal{V})$, where $\mathcal{U}$ is a normal cover of X_i and $\mathcal{V}$ is a normal cover of A_i. Let $\mathcal{U}'$ be a normal cover of X_i refining each $\mathcal{U}$ in P and $\mathcal{V}'$ be a normal cover of A_i refining each $\mathcal{V}$ in P. Now there exist $a \in C^{\mathcal{V}'}_n(A_i; {}^*G)$ and $x \in C^{\mathcal{U}'}_{n+1}(X_i; {}^*G)$ such that $\pi_{ii_0}x'_{i_0} - x_i = a + \Delta(x)$. Then $a \in C^{\mathcal{V}}_n(A_i; {}^*G)$ and $x \in C^{\mathcal{U}}_{n+1}(X_i; {}^*G)$ for all $(\mathcal{U}, \mathcal{V})$ in P.

Hence, by saturation, there exist $a' \in C_n(A_i; {}^*G)$, $x' \in C_{n+1}(X_i; {}^*G)$ such that $\pi_{ii_0}x'_{i_0} - x_i = \Delta(x') + a'$. Hence, $(\pi_{ii_0}x'_{i_0})$, $i \in I$, is the chain required to show that the homology sequence is exact at the place $\lim H_n(X_i, A_i; {}^*G)$.□

Theorem 71 $\lim H(X_i, A_i; {}^*G) = H(X, A; {}^*G)$.

Proof: By Theorem 69, $\lim H(X_i; {}^*G) = H(X; {}^*G)$ and $\lim H(A_i; {}^*G) = H(A; {}^*G)$. Now, the theorem follows by a five-lemma argument applied to exact sequences; by Lemma 70, the lim-sequence is exact. □

14 Strong Excision

Let (X, A) be a pair of spaces such that X is paracompact and A is a closed subset of X.

Let (X, A_i) $i \in I$, be an inverse system, where A_i runs through closed neighbourhoods of A, and the system projections are inclusions.

The inclusions $A \to A_i$ form a resolution of A: (P1) is satisfied, since A is normally embedded in A_i, and (P2) is satisfied, since if $\mathcal{U}$ is an open cover of A_i, $\mathrm{St}(A; \mathcal{U})$ is a neighbourhood of A, hence it contains a closed neighbourhood of A.

Similarly, $(X/A, A)$ has a resolution $(X/A, A_i/A)$, $i \in I$.

Lemma 72 *The space X/A is paracompact.*

Proof: Let $\mathcal{U}$ be an open cover of X/A. Without loss of generality we may assume that $A \subset U$ for only one $U \in \mathcal{U}$. Let $p\colon X \to X/A$ be the natural projection. Let $\mathcal{V}$ be a locally finite refinement of $p^{-1}\mathcal{U}$. Let $\mathcal{W} \subset \mathcal{V}$ be the subset consisting of those W that intersect A. Now $\mathcal{U}'' = \mathcal{V} \setminus \mathcal{W} \cup \bigcup \mathcal{W}$ is a locally finite cover of X, and $p\mathcal{U}''$ is a cover of X/A refining $\mathcal{U}$. Let U' be a neighbourhood of A such that the closure of U' is contained in $p \bigcup \mathcal{W}$. Now $\mathcal{U}'$ is formed so that $p \bigcup \mathcal{W}$ is a member of $\mathcal{U}'$, and for the sets $U'' \in p(\mathcal{V} \setminus \mathcal{W})$, the sets $U'' \setminus \mathrm{cl}\, U'$ are members of $\mathcal{U}'$. Now $\mathcal{U}'$ is a locally finite refinement of $\mathcal{U}$.□

Theorem 73 *The projection $p\colon (X, A) \to (X/A, A)$ induces isomorphisms in homology.*

Proof: By Theorem 71, it is enough to show that $\lim H(X, A_i; {}^*G) = \lim H(X/A, A_i/A; {}^*G)$ but to show that, it is enough to show that the projection $p\colon (X, A_i) \to (X/A, A_i/A)$ induces isomorphisms in homology for each i.

Let U be a neighbourhood of A such that $\mathrm{cl}\, U \subset \mathrm{int}\, A_i$. Now, by Lemma 21, $H(X, A_i; {}^*G) = H(X \setminus U, A_i \setminus U; {}^*G) = H(X/A, A_i/A; {}^*G)$.□

15 Shape invariance

For the notation and terminology used in this section, we refer to Mardesic-Segal [8].

Let S be the set of mathematical objects we are interested in, as discussed in Korppi, [5], Section 2. Let Sh$|S$ be subcategory of Sh consisting of those objects that lie in S. We may assume that Sh$|S$ is a full subcategory of Sh.

Lemma 74 *The k-th homology pro group pro-$H_k(-; {}^*G)$ is a functor $Sh|S \to$ pro-Ab for every $k \in \mathbb{N}$ and every Abelian group G. Here H is the homology theory developed in this article. (For the pro- -notation, see Mardesic-Segal [8], discussion preceeding Theorem II.3.1.)*

In particular, the pro-groups pro-$H_k(X; {}^*G)$ are shape invariants.

Proof: Mardesic-Segal [8] Theorem II.3.1, is a similar result, but for singular homology instead of our homology. Exactly the same proof (the proof is in the the discussion preceeding Mardesic-Segal [8] Theorem II.3.1) works in our case, when one replaces references to singular homology with references to the homology theory developed in this article, and one handles only HPol-expansions in S. It can be assumed that for each space in S, S contains at least one HPol-expansion. □

Hence the group $\lim$ pro-$H_k(X; {}^*G) = \lim H_k(X_i; {}^*G)$, where $(X_i), i \in I$ is a HPol-expansion of X, is shape invariant. In particular, $\lim$ pro-$H_k(X; {}^*G)$ is independent of the HPol-expansion used.

By Mardesic-Segal, Theorem I.6.2, resolutions are HTop-expansions. By Mardesic-Segal [8], I.6.7, every topological space X admits a polyhedral resolution, which is a HPol-expansion. We may assume that S contains a polyhedral resolution for each space in S.

Let $(X_i), i \in I$ be a polyhedral resolution of X in S. Then $\lim$ pro-$H_k(X; {}^*G) = \lim H_k(X_i; {}^*G) = H_k(X; {}^*G)$ (the last equality follows from Theorem 69) is shape invariant.

Thus, we have proved:

Theorem 75 *H is shape invariant.*

16 Relationship with Čech homology

Theorem 76 *Let (X, A) be a pair such that X is paracompact and A is closed in X. Then there exists a monomorphism $\check{H}(X, A; G) \to H(X, A; {}^*G)$.*

Proof: Since X is paracompact, all of its open covers are normal and A is normally embedded in X.

Let $(\mathcal{U}, \mathcal{V})$ be a pair such that $\mathcal{U}$ is an open cover of X and $\mathcal{V}$ is a subset $\mathcal{U}$ covering A. Denote by $V_{\mathcal{U}}$ the simplicial complex such that the vertices of $V_{\mathcal{U}}$ are the points of X, and s is a simplex if s is contained in $U \in \mathcal{U}$. Let $V_{\mathcal{V}}$ be the simplicial complex such that the vertices of $V_{\mathcal{V}}$ are the points of A, and s is a simplex of $V_{\mathcal{V}}$, if s is contained in $V \in \mathcal{V}$.

Let $i = {}^*()\colon H(V_{\mathcal{U}}, V_{\mathcal{V}}; G) \to {}^*(H(V_{\mathcal{U}}, V_{\mathcal{V}}; G))$ be the map $[\sum k_j s_j] \mapsto [\sum k_j s_j]$. Now i is a monomorphism, in fact, it is an elementary embedding.

Hence, i induces a map $\lim i\colon \lim H(V_{\mathcal{U}}, V_{\mathcal{V}}; G) \to \lim {}^*(H(V_{\mathcal{U}}, V_{\mathcal{V}}; G))$, where $(\mathcal{U}, \mathcal{V})$ runs through pairs such that $\mathcal{U}$ is an open cover of X, and $\mathcal{V} \subset \mathcal{U}$ covers A. The $<$-relation in the inverse systems is refinement.

One sees easily that $\lim i$ is a monomorphism.

By Dowker [1], Theorem 2a, $\check{H}(X, A; G) = \lim H(V_{\mathcal{U}}, V_{\mathcal{V}}; G)$.

By Theorem 48, $\lim {}^*(H(V_{\mathcal{U}}, V_{\mathcal{V}}; G)) = H \lim {}^*(C(V_{\mathcal{U}}, V_{\mathcal{V}}; G))$. But $\lim {}^*(C_n(V_{\mathcal{U}}, V_{\mathcal{V}}; G)) = \bigcap {}^*(C_n(V_{\mathcal{U}}, V_{\mathcal{V}}; G)) = C_n(X, A; {}^*G)$, the last equality follows, since each cover of X is normal, and A is normally embedded in X. Hence $\lim {}^*(H(V_{\mathcal{U}}, V_{\mathcal{V}}; G)) = H(X, A; {}^*G)$.

With these identifications, $\lim i\colon \check{H}(X, A; G) \to H(X, A; {}^*G)$.□

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