DISPROOF OF THE RIEMANN ZETA FUNCTION AND RIEMANN HYPOTHESIS

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Abstract

Bernhard Riemann has written down a very mysterious work "Ueber die Anzahl der Primzahlen unter einer gegebenen Grösse" since 1859. This paper of Riemann tried to show some functional equations related to prime numbers without proof. Let us investigate those functional equations together about how and where they came from. And at the same time let us find out whether or not the Riemann Zeta Function $\zeta(s) = 2^s(\pi)^{(s-1)}\sin(\pi\frac{s}{2})\Gamma(1-s)\zeta(1-s)$ really has zeroes at negative even integers $(-2, -4, -6 \ldots)$, which are called the trivial zeroes, and the nontrivial zeroes of Riemann Zeta Function which are in the critical strip $(0 < \mathfrak{R}(\mathbf{s}) < 1)$ all lie on the critical line $(\mathfrak{R}(\mathbf{s}) = \frac{1}{2})$ (or the nontrivial zeroes of Riemann Zeta Function are complex numbers of the form $(\frac{1}{2} + \propto i)$).

1.Introduction

Prime numbers are the most interesting and useful numbers. Many great mathematicians try to work with them in several ways. One of them, Bernhard Riemann, has written down a very famous work "Ueber die Anzahl der Primzahlen unter einer gegebenen Grösse" since 1859 showing a functional equation ζ(s) or Riemann Zeta Function without proof. He believed that with the assistance of his functional equation and all of the methods shown in his paper, the number of prime numbers that are

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smaller than x can be determined.

Someone believes that by using analytic continuation technique, he or she can extend a domain of a powerful analytic function, derived from two or more ordinary expressions or equations, which can help him or her reach the shore he or she tries to. One of them, Riemann, might has thought for about 150 years ago that he could extend the domain of his new analytic function, which is the composition of Riemann Zeta Function and Pi or Gamma function, to the entire complex plane by using this technique. But this technique, just like others, needs to be checked or proved for the essential conditions of the former equations and of the new functional equation itself. Until now usages of Riemann Hypothesis in mathematics and physics are still found more and more, despite the truth that it is just a "hard to solve" problem, not a proven one!

2. $\zeta(s)$, $2\sin \pi s\ \zeta(s)\prod(s-1)$, $\zeta(s) = 2^s\pi^{(s-1)}\sin\frac{\pi s}{2}\Gamma(1-s)\zeta(1-s)$ derivations, and trivial zero solution of $\zeta(s)$

2.1 Let's start from the great observation **"The Euler Product"**

$$\prod\left(1-\frac{1}{P^s}\right)^{-1} = \sum_{n=1}^{+\infty}\left(\frac{1}{n^s}\right)$$

For p = all prime numbers

n = all whole numbers = 1,2,3…,∞

Leonard Euler proved this "Euler Product Formula" in 1737.

Let us follow the proof from the series

$$\sum_{n=1}^{+\infty}\left(\frac{1}{n^s}\right) = 1+\frac{1}{2^s}+\frac{1}{3^s}+\frac{1}{4^s}+\frac{1}{5^s}+\ldots \qquad \ldots(A)$$

Multiply …(A) by $\frac{1}{2^s}$ bothsides

$$\frac{1}{2^s}\sum_{n=1}^{+\infty}\left(\frac{1}{n^s}\right) = \frac{1}{2^s}+\frac{1}{4^s}+\frac{1}{6^s}+\frac{1}{8^s}+\frac{1}{10^s}+\ldots \qquad \ldots(B)$$

Subtract… (A) by …(B) to remove all elements that have factors of 2

$$\left(1-\frac{1}{2^s}\right)\sum_{n=1}^{+\infty}\left(\frac{1}{n^s}\right) = 1+\frac{1}{3^s}+\frac{1}{5^s}+\frac{1}{7^s}+\frac{1}{9^s}+\frac{1}{11^s}+\ldots \qquad \ldots(C)$$

Multiply…(C) by $\frac{1}{3^s}$ bothsides

$$\frac{1}{3^s}(1-\frac{1}{2^s})\sum_{n=1}^{+\infty}(\frac{1}{n^s}) = \frac{1}{3^s}+\frac{1}{9^s}+\frac{1}{15^s}+\frac{1}{21^s}+\frac{1}{27^s}+\ldots \quad \ldots(D)$$

Subtract …(C) by… (D) to remove all elements that have factors of 3 or 2 or both

$$(1-\frac{1}{3^s})(1-\frac{1}{2^s})\sum_{n=1}^{+\infty}(\frac{1}{n^s}) = 1+\frac{1}{5^s}+\frac{1}{7^s}+\frac{1}{11^s}+\frac{1}{13^s}+\frac{1}{17^s}+\ldots \quad \ldots(E)$$

Repeat the process infinitely yields

$$\ldots(1-\frac{1}{11^s})(1-\frac{1}{7^s})(1-\frac{1}{5^s})(1-\frac{1}{3^s})(1-\frac{1}{2^s})\sum_{n=1}^{+\infty}(\frac{1}{n^s}) = 1$$

$$\text{Or } \sum_{n=1}^{+\infty}(\frac{1}{n^s}) = \frac{1}{\left(1-\frac{1}{2^s}\right)\left(1-\frac{1}{3^s}\right)\left(1-\frac{1}{5^s}\right)\left(1-\frac{1}{7^s}\right)\left(1-\frac{1}{11^s}\right)\ldots}$$

$$\sum_{n=1}^{+\infty}(\frac{1}{n^s}) = \prod_{p\ prime}\left[\frac{1}{(1-\frac{1}{p^s})}\right]$$

$$\sum_{n=1}^{+\infty}(\frac{1}{n^s}) = \prod_{p\ prime}(1-\frac{1}{p^s})^{-1}$$

Riemann denoted this relation $\zeta(s) = \sum_{n=1}^{+\infty}(\frac{1}{n^s}) = \prod_{p\ prime}(1-\frac{1}{p^s})^{-1}$ **would converge only when real part of s was greater than 1($\Re(s)>1$)** in his paper **“**Ueber die Anzahl der Primzahlen unter einer gegebenen Grösse**”** since 1859.

Riemann Zeta Function ζ(s) would diverge for all $s \le 1$, for example

$$\text{If } \Re(s) = 1,\ \zeta(s) = \sum_{n=1}^{+\infty}(\frac{1}{n^s}) = \frac{1}{1^s}+\frac{1}{2^s}+\frac{1}{3^s}+\frac{1}{4^s}+\ldots$$

$$\zeta(1) = \frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots$$

$$\text{By comparison test } \frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}+\frac{1}{9}+\cdots$$

$$>\frac{1}{1}+\frac{1}{2}+\frac{1}{4}+\frac{1}{4}+\frac{1}{8}+\frac{1}{8}+\frac{1}{8}+\frac{1}{8}+\frac{1}{16}+\cdots$$

$$\text{but } \frac{1}{1}+\frac{1}{2}+(\frac{1}{4}+\frac{1}{4})+(\frac{1}{8}+\frac{1}{8}+\frac{1}{8}+\frac{1}{8})+(\frac{1}{16}+\cdots)$$

$$= \frac{1}{1} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \cdots$$

$$= +\infty$$

so $\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots = +\infty$

is a harmonic and divergent series

If $\mathfrak{R}(s) = 0$, $\zeta(s) = \sum_{n=1}^{+\infty} (\frac{1}{n^s}) = \frac{1}{1^s} + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \ldots$

$$\zeta(0) = \frac{1}{1^0} + \frac{1}{2^0} + \frac{1}{3^0} + \frac{1}{4^0} + \ldots$$

$$= 1 + 1 + 1 + 1 + \ldots$$

finally diverge to ∞

If $\mathfrak{R}(s) = -1$, $\zeta(s) = \sum_{n=1}^{+\infty} (\frac{1}{n^s}) = \frac{1}{1^{-1}} + \frac{1}{2^{-1}} + \frac{1}{3^{-1}} + \frac{1}{4^{-1}} + \ldots$

$$\zeta(-1) = 1 + 2 + 3 + 4 + \ldots$$

finally diverge to ∞

2.2 Next, let us consider $\Gamma(s)$ = Gamma function

2.2.1 $\Gamma(s)$ when $s > 0$

Gamma function was first introduced by Leonhard Euler (1707-1783) in his goal to generalize the factorial to non integer values.

$$\Gamma(s) = \int_0^1 (-\log(u))^{(s-1)} du, \quad \text{for } s > 0$$

And was studied by Adrien-Marie Legendre (1752-1833)

$$\Gamma(s) = \int_0^\infty (e)^{(-u)} (u)^{(s-1)} du$$

Which would converge if real part of s was greater than 0 ($\mathfrak{R}(s) > 0$)

And can be rewritten as

$$\Gamma(s) = \frac{\Gamma(s+1)}{s}$$

or $\Gamma(s+1) = (s)\Gamma(s)$ converges for $\mathfrak{R}(s) > 0$

Let us proof using integration by parts

$$\Gamma(s+1) = \int_0^\infty (e)^{(-u)}\,(u)^{(s)}\,du \qquad \text{for } \mathfrak{R}(s) > 0$$

$$= -(u)^{(s)}(e)^{(-u)}]_0^\infty + \int_0^\infty (e)^{(-u)}\,(s)(u)^{(s-1)}\,du$$

$$= [\lim_{u\to\infty} -(u)^{(s)}(e)^{(-u)} - \lim_{u\to 0} -(u)^{(s)}(e)^{(-u)}]$$

$$+ \int_0^\infty (e)^{(-u)}\,(s)(u)^{(s-1)}\,du$$

$$= [\frac{-\infty}{\infty} + \frac{0}{1}] + s\int_0^\infty (e)^{(-u)}\,(u)^{(s-1)}\,du$$

Use L'Hospital'Rule to find $\frac{-\infty}{\infty}$ (indeterminate form)

$$\lim_{u\to\infty} \frac{-(u)^{(s)}}{(e)^{(u)}} = \lim_{u\to\infty} \frac{(-1)(s)(u)^{(s-1)}}{(e)^{(u)}}$$

Repeat differentiation until $(u)^{(s)} \to (u)^{(0)}$

Then $$\lim_{u\to\infty} \frac{-(u)^{(s)}}{(e)^{(u)}} = \lim_{u\to\infty} \frac{(-1)(s)(s-1)\ldots(u)^{(0)}}{(e)^{(u)}}$$

$$= \frac{(-1)(s)(s-1)\ldots(1)}{(\infty)}$$

$$= 0$$

Thus $$\Gamma(s+1) = 0 + s\int_0^\infty (e)^{(-u)}\,(u)^{(s-1)}\,du$$

So $$\Gamma(s+1) = s\,\Gamma(s) \qquad \mathfrak{R}(s) > 0$$

Find $\Gamma(2)$

From $$\Gamma(s+1) = s\,\Gamma(s)$$

$$\Gamma(1+1) = 1\,\Gamma(1)$$

$$\mathbf{\Gamma(2)} = 1$$

Find $\Gamma(1)$

$$\Gamma(s+1) = \int_0^\infty (e)^{(-u)}\,(u)^{(s)}\,du$$

$$\Gamma(0+1) = \int_0^\infty (e)^{(-u)}\,(u)^{(0)}\,du$$

$$= -(e)^{(-u)}]_0^\infty$$

$$= \lim_{u\to\infty} -(e)^{(-u)} - \lim_{u\to 0} -(e)^{(-u)}$$

$$= -0 + 1$$

$$\Gamma(1) = 1$$

Find $\Gamma(\frac{1}{2})$

From $\Gamma(s) = \int_0^{\infty} (e)^{(-u)}\,(u)^{(s-1)}\,du$

$$\Gamma(\tfrac{1}{2}) = \int_0^{\infty} (e)^{(-u)}\,(u)^{(\frac{1}{2}-1)}\,du$$

$$= -(u)^{(-\frac{1}{2})}(e)^{(-u)}]_0^{\infty} + \int_0^{\infty} (e)^{(-u)}\,(-\tfrac{1}{2})(u)^{(-\frac{1}{2}-1)}\,du$$

$$= [\lim_{u\to\infty} -(u)^{(-\frac{1}{2})}(e)^{(-u)} - \lim_{u\to 0} -(u)^{(-\frac{1}{2})}(e)^{(-u)}]$$

$$+ \int_0^{\infty} (e)^{(-u)}\,(-\tfrac{1}{2})(u)^{(-\frac{1}{2}-1)}\,du$$

$$= [-0 + 0] + (-\tfrac{1}{2}) \int_0^{\infty} (e)^{(-u)}.\,(u)^{(-\frac{1}{2}-1)}\,du$$

$$= (-\tfrac{1}{2})\Gamma(-\tfrac{1}{2})$$

From Euler Reflection Formula

$$\Gamma(s)\,\Gamma(1-s) = \frac{\pi}{\sin\pi s} \quad , \quad 0 < s < 1$$

$$\Gamma(\tfrac{1}{2})\,\Gamma(1-\tfrac{1}{2}) = \frac{\pi}{\sin\frac{\pi}{2}} = \pi$$

$$\Gamma(\tfrac{1}{2}) = \sqrt{\pi}$$

$$= 1.772$$

And for s = positive integers = 1, 2, 3.....

The relation between gamma function and factorial can be found from

$$\Gamma(s+1) = s\,\Gamma(s) \quad , \quad \mathbf{\Re(s) > 0}$$

$$= s(s-1)\Gamma(s-1)$$

$$= s(s-1)(s-2)\ldots(1)\,\Gamma(1)$$

$= \ s! \quad$ for s = positive integers

2.2.2 $\Gamma(s)$ when $s = 0$

Find $\Gamma(\mathbf{0})$

From Euler Reflection Formula

$$\Gamma(s)\,\Gamma(1-s) = \frac{\pi}{\sin \pi s}$$

$$\lim_{s\to 0}\Gamma(s)\,\Gamma(1-s) = \lim_{s\to 0}\frac{\pi}{\sin \pi s}$$

$$\Gamma(0)\,\Gamma(1) = \frac{\pi}{\sin 0} = \infty$$

And $\Gamma(1) = 1$

so $\Gamma(0) = \infty$

2.2.3 $\Gamma(s)$ when $s < 0$

By substitution $\Re(s) < 0$ into equation above yields Γ(s+1) which will equal to $(s)\Gamma(s)$**for every $\Re(s) <0$ (negative integers, or negative non integers).**

Let us proof by integration by parts

$$\Gamma(s) = \int_0^\infty (u)^{(s-1)}.(e)^{(-u)}\,du \quad , \quad \Re(\mathbf{s}) < \mathbf{0}, \ s = -a$$

$$\Gamma(s+1) = \int_0^\infty (u)^{(s)}.(e)^{(-u)}\,du$$

$$= [-(u)^{(-a)}(e)^{(-u)}]_0^\infty + \int_0^\infty (e)^{(-u)}\,(-a)(u)^{(-a-1)}\,du$$

$$= [\lim_{u\to\infty} -(u)^{(-a)}(e)^{(-u)} - \lim_{u\to 0} -(u)^{(-a)}(e)^{(-u)}]$$

$$+ \int_0^\infty (e)^{(-u)}\,(-a)(u)^{(-a-1)}\,du$$

$$= [-0+0] + \ s\int_0^\infty (e)^{(-u)}\,(u)^{(s-1)}\,du$$

$$= (s)\Gamma(s) \quad , \quad \Re(\mathbf{s}) < \mathbf{0}$$

Find $\Gamma(-\frac{1}{2})$

From $\Gamma(-\frac{1}{2}+1) = (-\frac{1}{2})\Gamma(-\frac{1}{2})$

$\Gamma(-\frac{1}{2}) = (-2)\,\Gamma(\frac{1}{2})$

$= (-2)\sqrt{\pi}$

$= -3.545$

Find $\Gamma(-1)$

From $\Gamma(0) = \int_0^{\infty}(\mathrm{u})^{(-1)}.(\mathrm{e})^{(-\mathrm{u})}\,\mathrm{du}$

$= -(\mathrm{u})^{(-1)}(\mathrm{e})^{(-\mathrm{u})}]_0^{\infty} + \int_0^{\infty}(\mathrm{e})^{(-\mathrm{u})}.(-1)(\mathrm{u})^{(-2)}\,\mathrm{du}$

$= [\lim_{u\to\infty} -(\mathrm{u})^{(-1)}(\mathrm{e})^{(-\mathrm{u})} - \lim_{u\to 0} -(\mathrm{u})^{(-1)}(\mathrm{e})^{(-\mathrm{u})}]$

$+ \int_0^{\infty}(\mathrm{e})^{(-\mathrm{u})}.(-1)(\mathrm{u})^{(-2)}\,\mathrm{du}$

$= [-0+0] + (-1)\int_0^{\infty}(\mathrm{e})^{(-\mathrm{u})}.(\mathrm{u})^{(-2)}\,\mathrm{du}$

$= (-1)\int_0^{\infty}(\mathrm{e})^{(-\mathrm{u})}.(\mathrm{u})^{(-2)}\,\mathrm{du}$

$\infty = (-1)\Gamma(-1)$

$\Gamma(-1) = -\infty$

Find $\Gamma(-\frac{3}{2})$

From $\Gamma(-\frac{1}{2}) = \int_0^{\infty}(\mathrm{u})^{\left(-\frac{1}{2}-1\right)}(\mathrm{e})^{(-\mathrm{u})}\,\mathrm{du}$

$= -(\mathrm{u})^{\left(-\frac{1}{2}-1\right)}(\mathrm{e})^{(-\mathrm{u})}]_0^{\infty} + \int_0^{\infty}(\mathrm{e})^{(-\mathrm{u})}\left(-\frac{1}{2}-1\right)(\mathrm{u})^{(\left(-\frac{1}{2}-1\right)-1)}\,\mathrm{du}$

$= [\lim_{u\to\infty} -(\mathrm{u})^{(-\frac{1}{2}-1)}(\mathrm{e})^{(-\mathrm{u})} - \lim_{u\to 0} -(\mathrm{u})^{(-\frac{1}{2}-1)}(\mathrm{e})^{(-\mathrm{u})}]$

$+ \int_0^{\infty}(\mathrm{e})^{(-\mathrm{u})}\left(-\frac{1}{2}-1\right)(\mathrm{u})^{(\left(-\frac{1}{2}-1\right)-1)}\,\mathrm{du}$

$= [-0+0] + \left(-\frac{3}{2}\right)\int_0^{\infty}(\mathrm{e})^{(-\mathrm{u})}(\mathrm{u})^{(\left(-\frac{3}{2}\right)-1)}\,\mathrm{du}$

$$= \left(-\frac{3}{2}\right)\Gamma\left(-\frac{3}{2}\right)$$

$$\Gamma\left(-\frac{3}{2}\right) \quad = \left(-\frac{2}{3}\right)\Gamma(-\frac{1}{2})$$

$$= \left(-\frac{2}{3}\right)(-2)\sqrt{\pi}$$

$$= \quad 2.363$$

Find $\Gamma(-\mathbf{2})$

From $\quad \Gamma(-1) = \int_0^\infty (\mathrm{u})^{(-2)}(\mathrm{e})^{(-\mathrm{u})}\,\mathrm{du}$

$$= -(\mathrm{u})^{(-2)}(\mathrm{e})^{(-\mathrm{u})}\Big]_0^\infty + \int_0^\infty (\mathrm{e})^{(-\mathrm{u})}(-2)(\mathrm{u})^{(-3)}\,\mathrm{du}$$

$$= [\lim_{u\to\infty} -(\mathrm{u})^{(-2)}(\mathrm{e})^{(-\mathrm{u})} - \lim_{u\to 0} -(\mathrm{u})^{(-2)}(\mathrm{e})^{(-\mathrm{u})}]$$

$$+ \int_0^\infty (\mathrm{e})^{(-\mathrm{u})}(-2)(\mathrm{u})^{(-3)}\,\mathrm{du}$$

$$= [-0+0] \;\; +(-2)\int_0^\infty (\mathrm{e})^{(-\mathrm{u})}(\mathrm{u})^{(-3)}\,\mathrm{du}$$

$$-\infty \quad = (-2)\Gamma(-2)$$

$$\Gamma(-2) \quad = \infty$$

Next for $\mathbf{s =}$ zero, positive, negative integers or non integers

From $\quad \Gamma(\mathrm{s}) = \int_0^\infty (\mathrm{u})^{(\mathrm{s}-1)}(\mathrm{e})^{(-\mathrm{u})}\,\mathrm{du}$

$$\Gamma(1-\mathrm{s}) = \int_0^\infty (\mathrm{u})^{((1-s)-1)}(\mathrm{e})^{(-\mathrm{u})}\,\mathrm{du}$$

$$= -(\mathrm{u})^{(-s)}(\mathrm{e})^{(-\mathrm{u})}\Big]_0^\infty + \int_0^\infty (\mathrm{e})^{(-\mathrm{u})}((1-s)-1)(\mathrm{u})^{((1-\mathrm{s})-2)}\,\mathrm{du}$$

$$= [\lim_{u\to\infty} -(\mathrm{u})^{(-s)}(\mathrm{e})^{(-\mathrm{u})} - \lim_{u\to 0} -(\mathrm{u})^{(-s)}(\mathrm{e})^{(-\mathrm{u})}]$$

$$+ \int_0^\infty (\mathrm{e})^{(-\mathrm{u})}((1-s)-1)(\mathrm{u})^{((1-\mathrm{s})-2)}\,\mathrm{du}$$

$$= [-0+0] + ((1-s)-1)\Gamma((1-s)-1)$$

$$= [(1-\mathrm{s})-1]\Gamma[(1-\mathrm{s})-1]$$

So $\Gamma(1-s) = (-s)\Gamma(-s)$

(s = zero, positive, negative integers or non integers)

And $\Gamma(1+s) = (s)\Gamma(s)$

(s = zero, positive, negative integers or non integers)

2.3 Consider $\prod(s) =$ Pi function

Pi function was denoted by Carl Friedrich Gauss since 1813

$$\prod(s) = \int_{0^+}^{+\infty} (e)^{(-u)} (u)^{(s)} du$$

The relation between Pi and Gamma functions are

$$\prod(s-1) = \int_{0}^{\infty} (e)^{(-u)} (u)^{(s-1)} du \qquad \dots (1)$$

$$= \Gamma(s)$$

Which would converge if real part of s was greater than 0, ($\Re(s) > 0$)

2.4 How to find the product of $\zeta(s)\prod(s-1)$ and corresponding value of $\Re(s)$

From equation ... (1)

$+\infty \geq u \geq 0$

Let $u = nx$

Then $+\infty \geq x \geq 0$

Multiply equation ... (1) by $\frac{1}{n^s}$ both sides

$$\left(\frac{1}{n^s}\right)\prod(s-1) = \left(\frac{1}{n^s}\right) \int_{0^+}^{+\infty} (e)^{(-u)} (u)^{(s-1)} du$$

$$= \int_{0^+}^{+\infty} (e)^{(-nx)} (nx)^{(s-1)} (n)^{(-s)} dnx$$

$$= \int_{0^+}^{+\infty} (e)^{(-nx)} (nx)^{(s-1)} (n)^{(-s)} ndx$$

$$= \int_{0^+}^{+\infty} (e)^{(-nx)} (nx)^{(s-1)} (n)^{-(s-1)} dx$$

$$= \int_{0^+}^{+\infty} (e)^{(-nx)} (x)^{(s-1)}\, dx \qquad ...(1.1)$$

To make sure that the result of the product of $(\mathbf{nx})^{(s-1)}$by $(\mathbf{n})^{-(s-1)}$ of equation ...(1.1) will exactly be $(\mathbf{x})^{(s-1)}$ without $(\mathbf{n})^{(s-1)}$left, the value of s from $(\frac{1}{n^s})$ of ζ(s) (which $1 < \Re(\mathbf{s}) \leq +\infty$) , and from $(\mathbf{u})^{(s-1)}$ of $\prod(\text{s}-1)$ (which $0 < \Re(\mathbf{s}) \leq +\infty$) must be the same. So the real part of s of the product $(\frac{1}{n^s})\prod(\text{s}-1)$ must be any numbers which are larger than 1 or ($1 < \Re(\mathbf{s}) \leq +\infty$).

Then try to make infinitely summation of $(\frac{1}{n^s})\prod(\text{s}-1)$

for $\Re(\mathbf{s}) > 1$

$$\sum_{n=1}^{+\infty} (\frac{1}{n^s})\prod(\text{s}-1) = \sum_{n=1}^{+\infty} \int_{0^+}^{+\infty} (e)^{(-nx)} (x)^{(s-1)}\, dx \qquad ... (1.2)$$

Or $$\zeta(s)\prod(\text{s}-1) = \sum_{n=1}^{+\infty} \int_{0^+}^{+\infty} (e)^{(-nx)} (x)^{(s-1)}\, dx \qquad ... (1.3)$$

And from
$$(e)^{(-nx)} = (e^{(-x)})^{(n)}$$
$$= (e^{(-x)})^{(n)} [\frac{(e)^{(-x)}}{(e)^{(-x)}}]$$
$$= (e^{(-x)})^{(n-1)}(e)^{(-x)}$$

Then
$$\zeta(s)\prod(\text{s}-1) = \sum_{n=1}^{+\infty} \int_{0^+}^{+\infty} (e^{(-x)})^{(n-1)}(e)^{(-x)} (x)^{(s-1)}dx$$
$$= \int_{0^+}^{+\infty} \sum_{n=1}^{+\infty} (e^{(-x)})^{(n-1)}(e)^{(-x)} (x)^{(s-1)}dx \qquad ... (1.4)$$

, for $\Re(\mathbf{s}) > 1$

Let $$\sum_{n=1}^{+\infty} (e^{(-x)})^{(n-1)} = \sum_{n=1}^{+\infty} ar^{(n-1)}$$

From **Geometric Series**

$$\sum_{n=1}^{+\infty} ar^{(n-1)} = \lim_{n\to\infty} Sn$$

$$= \lim_{n\to\infty} \frac{(a-ar^{n})}{(1-r)}$$

$$= \lim_{n\to\infty} [\frac{a}{(1-r)} - \frac{ar^{n}}{(1-r)}] \quad , a = 1 \quad , r = (e)^{(-x)} (<1)$$

$$, 0 \leq x \leq +\infty$$

But $\lim_{n\to\infty} \frac{ar^{n}}{(1-r)} = \lim_{n\to\infty} \frac{(e^{(-x)})^{n}}{(1-e^{(-x)})} = 0$

So $\sum_{n=1}^{+\infty} ar^{(n-1)} = \frac{a}{(1-r)}$ $\quad$ $, a = 1 \ , r = (e)^{(-x)} (<1) ,0\leq x \leq +\infty$

And then $\sum_{n=1}^{+\infty} (e^{(-x)})^{(n-1)} = \frac{1}{(1-e^{(-x)})}$

Thus $\zeta(s)\prod(s-1) = \int_{0^+}^{+\infty} \frac{(e)^{(-x)}(x)^{(s-1)}}{(1-e^{(-x)})} dx$

$$= \int_{0^+}^{+\infty} \frac{(x)^{(s-1)}}{(1-e^{(-x)})(e^{x})} dx$$

or $\zeta(s)\prod(s-1) = \int_{0^+}^{+\infty} \frac{(x)^{(s-1)}}{(e^{x}-1)} dx \ , \quad \mathfrak{R}(\mathbf{s}) > 1 \quad ... (2)$

2.5 Riemann’s attempt to extend the analytic equation $\zeta(s)\prod(s-1)$ to the negative side of real axis, formation of the equation

$$2\sin \pi s\, \zeta(s)\prod(s-1) = i \int_{+\infty}^{+\infty} \frac{(-x)^{(s-1)}}{(e^{x}-1)} dx$$

Riemann substituted $(-x)$ into $(x)^{(s-1)}$ of integral ... (2), and took consideration in positive sense around a domain $(+\infty,+\infty)$

but $\int_{+\infty}^{+\infty} \frac{(-x)^{(s-1)}}{(e^{x}-1)} dx = \int_{0^+}^{+\infty} \frac{(-x)^{(s-1)}}{(e^{x}-1)} dx + \int_{+\infty}^{0^+} \frac{(-x)^{(s-1)}}{(e^{x}-1)} dx$

$$\int_{+\infty}^{+\infty} \frac{(-x)^{(s-1)}}{(e^{x}-1)} dx = \int_{0^+}^{+\infty} \frac{(-x)^{(s-1)}}{(e^{x}-1)} dx - \int_{0^+}^{+\infty} \frac{(-x)^{(s-1)}}{(e^{x}-1)} dx$$

$$= 0$$

That means the overall value of the equation $\zeta(s)\prod(s-1) = \int_{0^+}^{+\infty} \frac{(x)^{(s-1)}}{(e^{x}-1)} dx$ after extending to $\int_{+\infty}^{+\infty} \frac{(-x)^{(s-1)}}{(e^{x}-1)} dx$ is always equal to zero independent from the values of s of $\zeta(s)$ or $\prod(s-1)$.

Now, let us go further from the equation above

$$\int_{+\infty}^{+\infty} \frac{(-x)^{(s-1)}}{(e^x-1)} dx \; = \; 0$$

$$= \int_{0^+}^{+\infty} \frac{(-1)^{(s-1)}\;(x)^{(s-1)}}{(e^x-1)} dx - \int_{0^+}^{+\infty} \frac{(-1)^{(s-1)}\,(x)^{(s-1)}}{(e^x-1)} dx$$

$$= \frac{(-1)^{(s)}}{(-1)} \int_{0^+}^{+\infty} \frac{(x)^{(s-1)}}{(e^x-1)} dx \; - \; \frac{(-1)^{(s)}}{(-1)} \int_{0^+}^{+\infty} \frac{(x)^{(s-1)}}{(e^x-1)} dx$$

From **Euler's Formula**

$$(e)^{\pm i\pi} \quad = -1$$

$$(\cos\pi \; \pm i\sin\pi) \; = \; -1$$

$$\cos\pi \; = \; -1 \;, \quad \sin\pi \; = \quad 0$$

Hence $\int_{+\infty}^{+\infty} \frac{(-x)^{(s-1)}}{(e^x-1)} dx$

$$= \frac{(-1)^{(s)}}{(-1)} \int_{0^+}^{+\infty} \frac{(x)^{(s-1)}}{(e^x-1)} dx \; - \; \frac{(-1)^{(s)}}{(-1)} \int_{0^+}^{+\infty} \frac{(x)^{(s-1)}}{(e^x-1)} dx$$

$$= \frac{(e^{i\pi})^{(s)}}{(-1)} \int_{0^+}^{+\infty} \frac{(x)^{(s-1)}}{(e^x-1)} dx \; - \; \frac{(e^{-i\pi})^{(s)}}{(-1)} \int_{0^+}^{+\infty} \frac{(x)^{(s-1)}}{(e^x-1)} dx$$

$$= [\frac{(e^{i\pi})^{(s)}}{(-1)} - \frac{(e^{-i\pi})^{(s)}}{(-1)}] \int_{0^+}^{+\infty} \frac{(x)^{(s-1)}}{(e^x-1)} dx$$

$$= [-\left(e^{i\pi}\right)^{(s)} + \left(e^{-i\pi}\right)^{(s)}] \int_{0^+}^{+\infty} \frac{(x)^{(s-1)}}{(e^x-1)} dx$$

$$= [\left(e^{-i\pi}\right)^{(s)} - \left(e^{i\pi}\right)^{(s)}] \int_{0^+}^{+\infty} \frac{(x)^{(s-1)}}{(e^x-1)} dx \qquad \dots (3)$$

$$= [(\cos\pi s - i\sin\pi s) - (\cos\pi s + i\sin\pi s)] \int_{0^+}^{+\infty} \frac{(x)^{(s-1)}}{(e^x-1)} dx$$

$$= -2i\sin\pi s \int_{0^+}^{+\infty} \frac{(x)^{(s-1)}}{(e^x-1)} dx$$

$$= -2i\sin\pi s\; \zeta(s) \prod(s-1) \qquad \dots (4)$$

Or $\int_{+\infty}^{+\infty} \frac{(-x)^{(s-1)}}{(e^x-1)} dx \; = -2i\sin\pi s\; \zeta(s) \prod(s-1)$

Multiply by i both sides

$$i\int_{+\infty}^{+\infty}\frac{(-x)^{(s-1)}}{(e^{x}-1)}dx \quad = -2(i)^{2}\sin \pi s\ \zeta(s)\prod(s-1)$$

$$= -2(-1)\sin \pi s\ \zeta(s)\prod(s-1)$$

Or $\quad 2\sin \pi s\ \zeta(s)\prod(s-1) = i\int_{+\infty}^{+\infty}\frac{(-x)^{(s-1)}}{(e^{x}-1)}dx = 0 \quad \ldots (5)$

That means the overall values of the equation $2\sin \pi s\ \zeta(s)\prod(s-1) = i\int_{+\infty}^{+\infty}\frac{(-x)^{(s-1)}}{(e^{x}-1)}dx$ is always equal to zero independent from the values of s of $\sin\pi s$, $\zeta(s)$ or $\prod(s-1)$.

Then Riemann observed the many valued function from above equation

$$(-x)^{(s-1)} = (e)^{(s-1)\mathrm{Log}(-x)}$$

and said that the logarithm of (-x) was determined to be real only $\int_{+\infty}^{+\infty}\frac{(-x)^{(s-1)}}{(e^{x}-1)}dx$ would be valuable if $x < 0$ in contrary with the domain $(+\infty, +\infty)$ of the integral. This was very confused.

Another confusion is that Riemann did not change (x) of the denominator $(e^{x}-1)$ of his equation $\int\frac{(-x)^{(s-1)}}{(e^{x}-1)}dx$ to $(-x)$ simultaneously while he changed (x) of the numerator $(x)^{(s-1)}$ to $(-x)$. Actually (x) of both denominator and numerator come from the same function $\prod(s-1)$, so they have to be changed at the same time.

I do not really know what was in his mind, but if one looks carefully at the first page of **his original paper "Ueber die Anzahl' der Primzahlen unter einer gegebenen Grösse",** you can see the traces of confusion and hesitation which caused him to change the boundary of the integral $\int\frac{(-x)^{(s-1)}}{(e^{x}-1)}dx$ from $(+\infty, +\infty)$ to $(-\infty, +\infty)$ and back to $(+\infty, +\infty)$ again.

2.5.1 Firstly, he might try to extend the functional equation ζ(s) ∏(s−1) to the negative values along the x-axis (which means that he was trying to consider the integral on the domain (−∞, +∞).

From equation ... (1.4)

$$\zeta(s)\prod(s-1) = \int_0^{+\infty} \sum_{n=1}^{+\infty} (e^{(-x)})^{(n-1)}(e)^{(-x)}\,(x)^{(s-1)}dx$$

Riemann extended it to negative values along x-axis

$$\zeta(s)\prod(s-1) = \int_0^{+\infty} \sum_{n=1}^{+\infty} (e^{(-x)})^{(n-1)}(e)^{(-x)}\,(x)^{(s-1)}dx$$

$$+\int_{-\infty}^{0} \sum_{n=1}^{+\infty} (e^{(-x)})^{(n-1)}(e)^{(-x)}\,(x)^{(s-1)}dx$$

Consider $\int_{-\infty}^{0} \sum_{n=1}^{+\infty} (e^{(-x)})^{(n-1)}(e)^{(-x)}\,(x)^{(s-1)}dx$

Let $\sum_{n=1}^{+\infty} (e^{(-x)})^{(n-1)} = \sum_{n=1}^{+\infty} ar^{(n-1)}$

From **Geometric Series**

$$\sum_{n=1}^{+\infty} ar^{(n-1)} = \lim_{n\to\infty} Sn$$

$$= \lim_{n\to\infty} \frac{(a-ar^{n})}{(1-r)}$$

$$= \lim_{n\to\infty} \left[\frac{a}{(1-r)} - \frac{ar^{n}}{(1-r)}\right] \qquad , a = 1\ , r = (e)^{(-x)}\ (>1)$$

$$, -\infty \le x \le 0$$

But $\lim_{n\to\infty} \frac{ar^{n}}{(1-r)} = \lim_{n\to\infty} \frac{(e^{(-x)})^{n}}{(1-e^{(-x)})} = \infty$

So $\sum_{n=1}^{+\infty} ar^{(n-1)} = \sum_{n=1}^{+\infty} (e^{(-x)})^{(n-1)}$

$$= -\infty \qquad , a = 1\ , r = (e)^{(-x)}\ (>1)\ , -\infty \le x \le 0$$

Thus $\int_{-\infty}^{0} \sum_{n=1}^{+\infty} (e^{(-x)})^{(n-1)}(e)^{(-x)}\,(x)^{(s-1)}dx$

$$= \int_{-\infty}^{0} (-\infty)(e)^{(-x)}\,(x)^{(s-1)}dx$$

diverge to $-\infty$ for $-\infty \leq x \leq 0$

Then $\zeta(s)\prod(s-1) = \int_0^{+\infty} \sum_{n=1}^{+\infty}(e^{(-x)})^{(n-1)}(e)^{(-x)}(x)^{(s-1)}dx$

$+\int_{-\infty}^{0} \sum_{n=1}^{+\infty}(e^{(-x)})^{(n-1)}(e)^{(-x)}(x)^{(s-1)}dx$

diverge to $-\infty$ for $-\infty \leq x \leq 0$

So extending $\zeta(s)\prod(s-1) = \int_0^{+\infty} \sum_{n=1}^{+\infty}(e^{(-x)})^{(n-1)}(e)^{(-x)}(x)^{(s-1)}dx$ to

$$\int_0^{+\infty} \sum_{n=1}^{+\infty}(e^{(-x)})^{(n-1)}(e)^{(-x)}(x)^{(s-1)}dx+\int_{-\infty}^{0} \sum_{n=1}^{+\infty}(e^{(-x)})^{(n-1)}(e)^{(-x)}(x)^{(s-1)}dx$$

is undefined (diverge to $-\infty$).

2.5.2 Secondly, he might try to take integration along a closed curve C covered the domain ($+\infty$, $+\infty$), which by famous **Cauchy's theorem** " if two different paths connect the same two points, and a function is holomorphic everywhere "in between" the two paths, then the two path integrals of the function will be the same." And briefly, "the path integral along a Jordan curve of a function, holomorphic in the interior of the curve, is zero."

$$\oint_c f(u)du = 0$$

if a and b are two points on Jordan curve c

then $\oint_c f(u)du = \int_a^b f(u)du + \int_b^a f(u)du$

$= 0$

And let us consider improper integral when $b \rightarrow +\infty$, $a = 0^+$

Then $\oint_c f(u)du = \lim_{b\rightarrow+\infty} \int_{0^+}^{b} f(u)du + \lim_{b\rightarrow+\infty} \int_{b}^{0^+} f(u)du$

Or $\oint_c f(u)du = \int_{0^+}^{\infty} f(u)du + \int_{\infty}^{0^+} f(u)du$

And for $f(u)du = \frac{(x)^{(s-1)}}{(e^x-1)}dx$

We got $\int_{+\infty}^{+\infty} \frac{(x)^{(s-1)}}{(e^x-1)}dx = \int_{0^+}^{+\infty} \frac{(x)^{(s-1)}}{(e^x-1)}dx + \int_{+\infty}^{0^+} \frac{(x)^{(s-1)}}{(e^x-1)}dx$

$$= \int_{0^+}^{+\infty} \frac{(x)^{(s-1)}}{(e^x-1)}dx - \int_{0^+}^{+\infty} \frac{(x)^{(s-1)}}{(e^x-1)}dx$$

$$= 0$$

$$= \int_{0^+}^{+\infty} \frac{(1)^{(s)}}{(1)} \frac{(x)^{(s-1)}}{(e^x-1)}dx - \int_{0^+}^{+\infty} \frac{(1)^{(s)}}{(1)} \frac{(x)^{(s-1)}}{(e^x-1)}dx$$

$$= (1)^{(s)} \int_{0^+}^{+\infty} \frac{(x)^{(s-1)}}{(e^x-1)}dx - (1)^{(s)} \int_{0^+}^{+\infty} \frac{(x)^{(s-1)}}{(e^x-1)}dx$$

From **Euler's Formula** again

$$(e)^{\pm i\pi} = -1$$

$$(-e)^{\pm i\pi} = 1$$

$$(\cos\pi \pm i\sin\pi) = -1$$

$$\cos\pi = -1 \text{ , } \sin\pi = 0$$

Hence $\int_{+\infty}^{+\infty} \frac{(x)^{(s-1)}}{(e^x-1)}dx = 0$

$$= (1)^{(s)} \int_{0^+}^{+\infty} \frac{(x)^{(s-1)}}{(e^x-1)}dx - (1)^{(s)} \int_{0^+}^{+\infty} \frac{(x)^{(s-1)}}{(e^x-1)}dx$$

$$= \left(-e^{i\pi}\right)^{(s)} \int_{0^+}^{+\infty} \frac{(x)^{(s-1)}}{(e^x-1)}dx - \left(-e^{-i\pi}\right)^{(s)} \int_{0^+}^{+\infty} \frac{(x)^{(s-1)}}{(e^x-1)}dx$$

$$= [\left(-e^{i\pi}\right)^{(s)} - \left(-e^{-i\pi}\right)^{(s)}] \int_{0^+}^{+\infty} \frac{(x)^{(s-1)}}{(e^x-1)}dx \quad ...(6)$$

$$= [(-\cos\pi s - i\sin\pi s) - (-\cos\pi s + i\sin\pi s)] \int_{0^+}^{+\infty} \frac{(x)^{(s-1)}}{(e^x-1)}dx$$

$$= -2i\sin\pi s \int_{0^+}^{+\infty} \frac{(x)^{(s-1)}}{(e^x-1)}dx \quad ... (7)$$

Or $\int_{+\infty}^{+\infty} \frac{(x)^{(s-1)}}{(e^x-1)}dx = -2i\sin\pi s \int_{0^+}^{+\infty} \frac{(x)^{(s-1)}}{(e^x-1)}dx$

$$= -2i\sin \pi s\ \zeta(s) \prod(s-1)$$

$$= 0$$

Multiply by i both sides

$$i \int_{+\infty}^{+\infty} \frac{(x)^{(s-1)}}{(e^x-1)} dx = -2(i)^2 \sin \pi s\ \zeta(s) \prod(s-1)$$

$$= 2\sin \pi s\ \zeta(s) \prod(s-1)$$

Or $2\sin \pi s\ \zeta(s) \prod(s-1) = i \int_{+\infty}^{+\infty} \frac{(x)^{(s-1)}}{(e^x-1)} dx = 0$ ... (8)

That means the overall values of the equation $2\sin \pi s\ \zeta(s)\prod(s-1) = i \int_{+\infty}^{+\infty} \frac{(x)^{(s-1)}}{(e^x-1)} dx$ is always equal to zero independent from the values of s (of $\sin\pi s$, $\zeta(s)$ or $\prod(s-1)$).

Now look at the many valued function again

$$(x)^{(s-1)} = (e)^{(s-1)\mathrm{Log}(x)}$$

The logarithm of x is determined to be real when x is positive number within the domain $(+\infty, +\infty)$.

2.6 Can we really get trivial zeroes ($-2, -4, -6, \ldots$) from Riemann Zeta Function $\zeta(s) = 2^s(\pi)^{(s-1)}\sin(\pi\frac{s}{2})\Gamma(1-s)\zeta(1-s)$?

To answer this question, we have to study two functional equations and their relationship.

1. $(\pi)^{(-\frac{s}{2})}\Gamma(\frac{s}{2})\zeta(s) = (\pi)^{-(\frac{1-s}{2})}\Gamma(\frac{1-s}{2})\zeta(1-s)$

2. $(\pi)^{-(s)}\Gamma(s)\zeta(s) = (\pi)^{-(1-s)}\Gamma(1-s)\zeta(1-s)$

Firstly, you should pay attention to interesting fact which is hidden in those equations.

2.6.1 Let us start from changing of $\prod(s-1)$ of equation ...(1) to $\prod(\frac{s}{2}-1)$.

From $\prod(s-1) = \int_{0^+}^{+\infty} (e)^{(-u)} (u)^{(s-1)}\, du$ ...(1)

Or $\Gamma(s) = \int_{0^+}^{+\infty} (e)^{(-u)} (u)^{(s-1)}\, du$

Which converges when $\Re(s) > 0$, $+\infty \geq u \geq 0$

Thus $\prod\left(\frac{s}{2}-1\right) = \int_{0^+}^{+\infty} (e)^{(-u)} (u)^{\left(\frac{s}{2}-1\right)} du$

Multiply by $\left(\frac{1}{n^s}\right)(\pi)^{\left(-\frac{s}{2}\right)}$ bothsides and let $u = nn\pi x$ (as Riemann tried to)

$$\left(\frac{1}{n^s}\right)(\pi)^{\left(-\frac{s}{2}\right)}\prod\left(\frac{s}{2}-1\right) = \int_{0^+}^{+\infty} \frac{1}{(\pi)^{\left(\frac{s}{2}\right)}}\left(\frac{1}{n^s}\right)(e)^{(-u)}\, (u)^{\left(\frac{s}{2}-1\right)} du$$

$$= \int_{0^+}^{+\infty} \frac{1}{(\pi)^{\left(\frac{s}{2}\right)}}\left(\frac{1}{(nn)^{\left(\frac{s}{2}\right)}}\right)(e)^{(-nn\pi x)}\, (nn\pi x)^{\left(\frac{s}{2}-1\right)} d(nn\pi x)$$

$$= \int_{0^+}^{+\infty} \frac{(nn\pi x)^{\left(\frac{s}{2}-1\right)}}{(nn\pi)^{\left(\frac{s}{2}\right)}}(e)^{(-nn\pi x)} nn\pi\, dx$$

$$= \int_{0^+}^{+\infty} \frac{(nn\pi x)^{\left(\frac{s}{2}-1\right)}}{(nn\pi)^{\left(\frac{s}{2}-1\right)}}(e)^{(-nn\pi x)}\, dx$$

$$= \int_{0^+}^{+\infty} (e)^{(-nn\pi x)}(x)^{\left(\frac{s}{2}-1\right)} dx$$

Take infinite summation both sides

$$\sum_{n=1}^{+\infty} \left(\frac{1}{n^s}\right)(\pi)^{\left(-\frac{s}{2}\right)}\prod\left(\frac{s}{2}-1\right) = \sum_{n=1}^{+\infty} \int_{0^+}^{+\infty} (e)^{(-nn\pi x)}(x)^{\left(\frac{s}{2}-1\right)} dx$$

$$= \int_{0^+}^{+\infty} \sum_{n=1}^{+\infty} (e)^{(-nn\pi x)}(x)^{\left(\frac{s}{2}-1\right)} dx$$

But **Riemann** denoted $\sum_{n=1}^{+\infty} (e)^{(-nn\pi x)} = \psi(x)$

so $(\pi)^{\left(-\frac{s}{2}\right)}\prod\left(\frac{s}{2}-1\right)\zeta(s) = \int_{0^+}^{+\infty} \psi(x)(x)^{\left(\frac{s}{2}-1\right)} dx$ ...(9)

Let's consider the value of $\psi(x)$

Evaluate $\psi(x)$ by Euler's Formula for $[0 \leq x \leq +\infty)$

$$\psi(x) = \sum_{n=1}^{+\infty} (e)^{(-nn\pi x)}$$

$$= \sum_{n=1}^{+\infty} [(e)^{(-nn\pi)}]^{(x)}$$

$= \sum_{n=1}^{+\infty} \ [(\mathrm{e})^{(-1)(\mathrm{nn}\pi)}]^{(x)}$

$= \sum_{n=1}^{+\infty} \ [(\mathrm{e})^{(i)^2(\pi)(\mathrm{nn})}]^{(x)}$

$= \sum_{n=1}^{+\infty} \ [(\mathrm{e})^{(i)(i\pi)(\mathrm{nn})}]^{(x)}$

$= \sum_{n=1}^{+\infty} \ [[(\mathrm{e})^{(i\pi)}]^{(\mathrm{nn})(i)}]^{(x)}$

$= \sum_{n=1}^{+\infty} \ [[(\cos\pi + \mathrm{i}\sin\pi)^{(\mathrm{nn})}]^{(i)}]^{(x)}$

$= \sum_{n=1}^{+\infty} \ [[(-1)^{(\mathrm{nn})}]^{(i)}]^{(x)}$

$= [[(-1)^{(1)(1)} + (-1)^{(2)(2)} + (-1)^{(3)(3)} + \cdots + (-1)^{(\infty)(\infty)}]^{(i)}]^{(x)}$

$= [[(-1)^{(1)} + (-1)^{(4)} + (-1)^{(9)} + \cdots + (-1)^{(\infty)(\infty)}]^{(i)}]^{(x)}$

$= [[(-1) + (1) + (-1) + \cdots + (1)]^{(i)}]^{(x)}$

$= [(0)^{(i)}]^{(x)}$

$= 0$ ...(9.1)

So $\prod\left(\frac{s}{2}-1\right)(\pi)^{\left(-\frac{s}{2}\right)}\zeta(\mathrm{s}) \ = \ \int_{0^+}^{+\infty} \psi(\mathrm{x})(x)^{\left(\frac{s}{2}-1\right)}\,\mathrm{dx}$

$= \ \int_{0^+}^{+\infty} (0)(x)^{\left(\frac{s}{2}-1\right)}\,\mathrm{dx}$

$= \ 0$ ...(9.2)

Thus $\Gamma(\frac{s}{2})(\pi)^{\left(-\frac{s}{2}\right)}\zeta(\mathrm{s}\,) \ = \ \int_{0^+}^{+\infty} \psi(\mathrm{x})(x)^{\left(\frac{s}{2}-1\right)}\,\mathrm{dx}$ is always $= 0$

2.6.2 From $\Gamma(\mathrm{s}) \ = \ \int_{0^+}^{+\infty}(\mathrm{e})^{(-\mathrm{u})}\,(\mathrm{u})^{(\mathrm{s}-1)}\,\mathrm{du}$

$$\left(\frac{1}{\mathrm{n}^{(1-s)}}\right)(\pi)^{\left(\frac{1-s}{2}\right)}\Gamma\left(\frac{1-s}{2}\right) = \int_{0^+}^{+\infty} \frac{(\mathrm{e})^{(-\mathrm{nn}\pi\mathrm{x})}}{(\mathrm{nn})^{\left(\frac{1-s}{2}\right)}(\pi)^{\left(\frac{1-s}{2}\right)}}(\mathrm{nn}\pi\mathrm{x})^{\left(\frac{1-s}{2}-1\right)}\mathrm{nn}\pi\,\mathrm{d}x$$

$$= \int_{0^+}^{+\infty}(\mathrm{e})^{(-\mathrm{nn}\pi x)}\,(x)^{\left(\frac{1-s}{2}-1\right)}\,\mathrm{d}x)$$

$$\sum_{n=1}^{+\infty}\left(\frac{1}{\mathrm{n}^{(1-s)}}\right)(\pi)^{-\left(\frac{1-s}{2}\right)}\Gamma\left(\frac{1-s}{2}\right) = \sum_{n=1}^{+\infty}\int_{0^+}^{+\infty}(\mathrm{e})^{(-\mathrm{nn}\pi x)}(x)^{\left(\frac{1-s}{2}-1\right)}\,\mathrm{d}x$$

$$= \int_{0^+}^{+\infty}\sum_{\mathrm{n}=1}^{+\infty}(\mathrm{e})^{(-\mathrm{nn}\pi x)}(x)^{\left(\frac{1-s}{2}-1\right)}\,\mathrm{d}x$$

But **Riemann** denote $\sum_{n=1}^{+\infty} (e)^{(-nn\pi x)} = \psi(x)$

And from $\psi(x) = 0$ as proof above

so $\Gamma\left(\frac{1-s}{2}\right)(\pi)^{-\left(\frac{1-s}{2}\right)}\zeta(1-s) = \int_{0^+}^{+\infty} \psi(x)(x)^{\left(\frac{1-s}{2}-1\right)}\,dx$

$= \int_{0^+}^{+\infty} (0)(x)^{\left(\frac{1-s}{2}-1\right)}\,dx$

$= 0$

Thus $\Gamma(\frac{1-s}{2})(\pi)^{-\left(\frac{1-s}{2}\right)}\zeta(1-s) = \int_{0^+}^{+\infty} \psi(x)(x)^{\left(\frac{1-s}{2}-1\right)}\,dx = 0$ (for any s)

Consider if $\psi(x) \neq 0$, then

$(\pi)^{-\left(\frac{1-s}{2}\right)}\Gamma\left(\frac{1-s}{2}\right)\zeta(1-s) \neq (\pi)^{\left(-\frac{s}{2}\right)}\Gamma\left(\frac{s}{2}\right)\zeta(s)$ except when $s = \frac{1}{2}$

But exactly $\psi(x) = 0$, so

$(\pi)^{-\left(\frac{1-s}{2}\right)}\Gamma\left(\frac{1-s}{2}\right)\zeta(1-s) = (\pi)^{\left(-\frac{s}{2}\right)}\Gamma\left(\frac{s}{2}\right)\zeta(s) = 0$ (for any s)

2.6.3 From $\Gamma(s) = \int_{0^+}^{+\infty} (e)^{(-u)}\,(u)^{(s-1)}\,du$

Let $u = n\pi x$

$\left(\frac{1}{n^{(1-s)}}\right)(\pi)^{-(1-s)}\Gamma(1-s) = \int_{0^+}^{+\infty} \frac{(e)^{(-n\pi x)}}{(n)^{(1-s)}(\pi)^{(1-s)}}(n\pi x)^{(1-s-1)} n\pi\,dx$

$= \int_{0^+}^{+\infty} (e)^{(-n\pi x)}\,(x)^{(1-s-1)}\,dx)$

$\sum_{n=1}^{+\infty}\left(\frac{1}{n^{(1-s)}}\right)(\pi)^{-(1-s)}\Gamma(1-s) = \sum_{n=1}^{+\infty}\int_{0^+}^{+\infty} (e)^{(-n\pi x)}(x)^{(1-s-1)}\,dx$

$= \int_{0^+}^{+\infty}\sum_{n=1}^{+\infty} (e)^{(-n\pi x)}(x)^{(1-s-1)}\,dx$

denote $\sum_{n=1}^{+\infty} (e)^{(-n\pi x)} = \phi(x)$

so $(\pi)^{-(1-s)}\Gamma(1-s)\zeta(1-s) = \int_{0^+}^{+\infty} \phi(x)(x)^{(1-s-1)}\,dx$

Let's consider the value of $\phi(x)$. Evaluate by Euler's Formula.

$\phi(x) = \sum_{n=1}^{+\infty} (e)^{(-n\pi x)}$

$= \sum_{n=1}^{+\infty} [(e)^{(-n\pi)}]^{(x)}$

$= \sum_{n=1}^{+\infty}\ [(\mathrm{e})^{(-1)(\mathrm{n}\pi)}]^{(x)}$

$= \sum_{n=1}^{+\infty}\ [(\mathrm{e})^{(i)^2(\pi)(\mathrm{n})}]^{(x)}$

$= \sum_{n=1}^{+\infty}\ [(\mathrm{e})^{(i)(i\pi)(\mathrm{n})}]^{(x)}$

$= \sum_{n=1}^{+\infty}\ [[(\mathrm{e})^{(i\pi)}]^{(\mathrm{n})(i)}]^{(x)}$

$= \sum_{n=1}^{+\infty}\ [[(\cos\pi + \mathrm{i}\sin\pi)^{(\mathrm{n})}]^{(i)}]^{(x)}$

$= \sum_{n=1}^{+\infty}\ [[(-1)^{(\mathrm{n})}]^{(i)}]^{(x)}$

$= [[(-1)^{(1)} + (-1)^{(2)} + (-1)^{(3)} + \cdots + (-1)^{(\infty)}]^{(i)}]^{(x)}$

$= [[(-1) + (1) + (-1) + \cdots + (1)]^{(i)}]^{(x)}$

$= [(0)^{(i)}]^{(x)}$

$= 0$

So $(\pi)^{-(1-s)}\Gamma(1-s)\zeta(1-\mathrm{s}) = \int_{0^+}^{+\infty} \phi(\mathrm{x})(x)^{(1-s-1)}\,\mathrm{dx}$

$= \int_{0^+}^{+\infty} (0)(x)^{(1-s-1)}\,\mathrm{dx}$

$= \int_{0^+}^{+\infty} (0)\,\mathrm{dx}$

$= 0$

Thus $(\pi)^{-(1-s)}\Gamma(1-s)\zeta(1-\mathrm{s}) = \int_{0^+}^{+\infty} \phi(\mathrm{x})(x)^{(1-s-1)} = 0$ (for any s)

2.6.4 From $\Gamma(\mathrm{s}) = \int_{0^+}^{+\infty} (\mathrm{e})^{(-\mathrm{u})}\,(\mathrm{u})^{(\mathrm{s}-1)}\ \mathrm{du}$

Let $\mathrm{u} = \mathrm{n}\pi x$

$\left(\frac{1}{n^{(s)}}\right)(\pi)^{-(s)}\Gamma(\mathrm{s}) = \int_{0^+}^{+\infty} \frac{(\mathrm{e})^{(-n\pi x)}}{(n)^{(s)}(\pi)^{(s)}}(n\pi x)^{(s-1)}\mathrm{n}\pi\,\mathrm{d}x$

$= \int_{0^+}^{+\infty} (\mathrm{e})^{(-\mathrm{n}\pi x)}\,(x)^{(s-1)}\,\mathrm{d}x)$

$\sum_{n=1}^{+\infty}\ \left(\frac{1}{n^{(s)}}\right)(\pi)^{-(s)}\Gamma(\mathrm{s}) = \sum_{n=1}^{+\infty}\ \int_{0^+}^{+\infty} (\mathrm{e})^{(-\mathrm{n}\pi x)}(x)^{(s-1)}\,\mathrm{d}x$

$= \int_{0^+}^{+\infty} \sum_{n=1}^{+\infty}\ (\mathrm{e})^{(-\mathrm{n}\pi x)}(x)^{(s-1)}\,\mathrm{d}x$

denote $\sum_{n=1}^{+\infty}\ (e)^{(-n\pi x)} \ = \ \phi(x) \ = \ 0$

so $(\pi)^{-(s)}\Gamma(s)\zeta(s) \ = \ \int_{0^+}^{+\infty} \phi(x)(x)^{(s-1)}\,dx$

$= \ \int_{0^+}^{+\infty}(0)(x)^{(s-1)}\,dx$

$= \ \int_{0^+}^{+\infty}(0)\,dx$

$= \ 0$

Thus $(\pi)^{-(s)}\Gamma(s)\zeta(s) = \int_{0^+}^{+\infty} \phi(x)(x)^{(s-1)}\,dx \ = 0$ (for any s)

Consider if $\phi(x) \ \neq \ 0$, then

$(\pi)^{-(1-s)}\Gamma(1-s)\zeta(1-s) \ \neq \ (\pi)^{-(s)}\Gamma(s)\zeta(s)$ except when $s = \frac{1}{2}$

But exactly $\phi(x) \ = \ 0$, so

$(\pi)^{-(1-s)}\Gamma(1-s)\zeta(1-s) \ = \ (\pi)^{-(s)}\Gamma(s)\zeta(s) \ = \ 0$ (for any s)

Next from $(\pi)^{-(1-s)}\Gamma(1-s)\zeta(1-s) \ = \ \int_{0^+}^{+\infty} \phi(x)(x)^{(1-s-1)}\,dx$.

If we try to extend $\int_{0^+}^{+\infty} \phi(x)(x)^{(1-s-1)}\,dx$ to $\int_{+\infty}^{+\infty} \phi(x)(x)^{(1-s-1)}\,dx$ by taking integration along a closed curve C covered the domain $(+\infty\ ,+\infty)$, then by famous **Cauchy's theorem** we get

$$\int_{+\infty}^{+\infty} \phi(x)(x)^{(1-s-1)}\,dx = \int_{0^+}^{0^+} \phi(x)(x)^{(1-s-1)}\,dx \ + \int_{+\infty}^{0^+} \phi(x)(x)^{(1-s-1)}\,dx$$

$$0 \quad = \int_{0^+}^{+\infty} \phi(x)(x)^{(1-s-1)}\,dx \ - \int_{0^+}^{+\infty} \phi(x)(x)^{(1-s-1)}\,dx$$

$$0 \quad = [\frac{(1)^{(1-s)}}{(1)} - \frac{(1)^{(1-s)}}{(1)}]\int_{0^+}^{+\infty} \phi(x)(x)^{(1-s-1)}\,dx$$

$$0 \quad = -2i\sin\pi(1-s)\ \pi^{-(1-s)}\Gamma(1-s)\zeta(1-s)$$

for any values of s

But $\sin\pi(1-s) \ = \ (\sin\pi\cos\pi s - \cos\pi\sin\pi s) \ = \ \sin\pi s$

So $\quad 0 \quad = -2i\sin\pi s\ \pi^{-(1-s)}\Gamma(1-s)\zeta(1-s)$

And $\sin\pi s = 2\sin\frac{\pi s}{2}\cos\frac{\pi s}{2}$

So $0 = -2i2\sin\frac{\pi s}{2}\cos\frac{\pi s}{2}\ \pi^{-(1-s)}\Gamma(1-s)\zeta(1-\mathrm{s})$ ... (9.3)

for any values of s

And from

$$(\pi)^{-\left(\frac{1-s}{2}\right)}\Gamma\left(\frac{1-s}{2}\right)\zeta(1-\mathrm{s}) = \int_{0^+}^{+\infty}\phi(\mathrm{x})(x)^{\left(\frac{1-s}{2}-1\right)}\,\mathrm{dx} = 0$$

Next extend $\int_{0^+}^{+\infty}\phi(\mathrm{x})(x)^{\left(\frac{1-s}{2}-1\right)}\,\mathrm{dx}$ to $\int_{+\infty}^{+\infty}\phi(\mathrm{x})(x)^{\left(\frac{1-s}{2}-1\right)}\,\mathrm{dx}$ by taking integration along a closed curve C covered the domain $(+\infty, +\infty)$, then by famous **Cauchy's theorem** we get

$$\int_{+\infty}^{+\infty}\phi(\mathrm{x})(x)^{\left(\frac{1-s}{2}-1\right)}\,\mathrm{dx} = \int_{0^+}^{+\infty}\phi(\mathrm{x})(x)^{\left(\frac{1-s}{2}-1\right)}\,\mathrm{dx} - \int_{0^+}^{+\infty}\phi(\mathrm{x})(x)^{\left(\frac{1-s}{2}-1\right)}\,\mathrm{dx}$$

$$0 = \left[\frac{(1)^{\left(\frac{1-s}{2}\right)}}{(1)} - \frac{(1)^{\left(\frac{1-s}{2}\right)}}{(1)}\right]\int_{0^+}^{+\infty}\phi(\mathrm{x})(x)^{\left(\frac{1-s}{2}-1\right)}\,\mathrm{dx}$$

$$0 = -2i\sin\pi\left(\frac{1-s}{2}\right)\pi^{-\left(\frac{1-s}{2}\right)}\Gamma\left(\frac{1-s}{2}\right)\zeta(1-\mathrm{s})$$

for any values of s

But $\sin\pi\left(\frac{1-s}{2}\right) = \left(\sin\frac{\pi}{2}\cos\frac{\pi s}{2} - \cos\frac{\pi}{2}\sin\frac{\pi s}{2}\right) = \cos\frac{\pi s}{2}$

So $0 = -2i\cos\frac{\pi s}{2}\pi^{-\left(\frac{1-s}{2}\right)}\Gamma\left(\frac{1-s}{2}\right)\zeta(1-\mathrm{s})$ .... (9.4)

for any values of s

Thus from ....(9.3) and ....(9.4)

$$-2i2\sin\frac{\pi s}{2}\cos\frac{\pi s}{2}\pi^{-(1-s)}\Gamma(1-s)\zeta(1-\mathrm{s})$$

$$= -2i\cos\frac{\pi s}{2}\pi^{-\left(\frac{1-s}{2}\right)}\Gamma\left(\frac{1-s}{2}\right)\zeta(1-\mathrm{s})$$

$= 0$ (for any values of s)

Then divide by $-2i\cos\frac{\pi s}{2}$ both sides yields

$$2\sin\frac{\pi s}{2}\pi^{-(1-s)}\Gamma(1-s)\zeta(1-s) = \pi^{-\left(\frac{1-s}{2}\right)}\Gamma\left(\frac{1-s}{2}\right)\zeta(1-s)$$

$$= 0 \text{ for any values of s}$$

And from $(\pi)^{-(s)}\Gamma(s)\zeta(s) = \int_{0^+}^{+\infty} \phi(x)(x)^{(s-1)}\,dx$

$$= \int_{0^+}^{+\infty}(0)(x)^{(s-1)}\,dx$$

$$= 0$$

Thus the function $(\pi)^{-(s)}\Gamma(s)\zeta(s)$ is always equal to zero for any values of s .

And hence $(\pi)^{-(s)}\Gamma(s)\zeta(s) = 2\sin\frac{\pi s}{2}\pi^{-(1-s)}\Gamma(1-s)\zeta(1-s)$ ... (9.5)

$$= 0 \text{ (for any values of s)}$$

but $\Gamma(s)$ alone $\neq 0$ and $(\pi)^{-(s)}$alone $\neq 0$

so $$\zeta(s) = \frac{0}{(\pi)^{-(s)}\Gamma(s)}$$

$$= 0 \text{ (for any values of s)}$$

Finally $\zeta(s) = 2\sin\frac{\pi s}{2}\pi^{(s-1)}\Gamma(1-s)\zeta(1-s)$ ... (9.6)

$$= 0 \text{ (for any values of s)}$$

If you need the exact equation $\zeta(s) = 2^s\sin\frac{\pi s}{2}\pi^{(s-1)}\Gamma(1-s)\zeta(1-s)$, you can get it by multiplying $(2)^{-(1-s)}$ to the L.H.S. of the equation

$$(\pi)^{-(1-s)}\Gamma(1-s)\zeta(s) = \int_{0^+}^{+\infty} \phi(x)(x)^{(1-s-1)}\,dx = 0 \text{ (for any values of s)}$$

to become $(2\pi)^{-(1-s)}\Gamma(1-s)\zeta(s) = \int_{0^+}^{+\infty} \frac{\phi(x)(x)^{(1-s-1)}}{(2)^{(1-s)}}dx = 0$ (for any values of s), and then extend R.H.S. by famous **Cauchy's theorem** to

$$\int_{+\infty}^{+\infty} \frac{\phi(x)(x)^{(1-s-1)}}{(2)^{(1-s)}}dx = \int_{0^+}^{+\infty} \frac{\phi(x)(x)^{(1-s-1)}}{(2)^{(1-s)}}dx + \int_{+\infty}^{0^+} \frac{\phi(x)(x)^{(1-s-1)}}{(2)^{(1-s)}}dx$$

$$= \int_{0^+}^{+\infty} \frac{\phi(x)(x)^{(1-s-1)}}{(2)^{(1-s)}}dx - \int_{0^+}^{+\infty} \frac{\phi(x)(x)^{(1-s-1)}}{(2)^{(1-s)}}dx$$

$$\int_{+\infty}^{+\infty} \frac{\phi(x)(x)^{(1-s-1)}}{(2)^{(1-s)}}dx = [\frac{(1)^{(1-s)}}{(1)} - \frac{(1)^{(1-s)}}{(1)}]\int_{0^+}^{+\infty} \frac{\phi(x)(x)^{(1-s-1)}}{(2)^{(1-s)}}dx$$

$$= -2i\sin\pi(1-s)(2\pi)^{-(1-s)}\Gamma(1-s)\zeta(1-\mathrm{s})$$

$$= -2i\sin\pi s(2\pi)^{-(1-s)}\Gamma(1-s)\zeta(1-\mathrm{s})$$

$$= -2i2\sin\frac{\pi s}{2}\cos\frac{\pi s}{2}(2\pi)^{-(1-s)}\Gamma(1-s)\zeta(1-\mathrm{s}) \quad ...(9.7)$$

$$= 0 \text{ (for any values of s)}$$

Then follow the previous process, and finally you will get

$$\zeta(\mathrm{s}) = 2^s\pi^{(s-1)}\sin\frac{\pi s}{2}\Gamma(1-s)\zeta(1-\mathrm{s}) \qquad ...(9.8)$$

$$= 0 \text{ (for any values of s)}$$

Because $\Gamma(1-s)$ has simple pole at s $= 1, 2, 3, \ldots$, it looks as if the Functional equation $\zeta(\mathrm{s}) = 2^s\sin\frac{\pi s}{2}\pi^{(s-1)}\Gamma(1-s)\zeta(1-\mathrm{s})$ will be equal to zero if and only if the value of s of $\sin\frac{\pi s}{2}$ is equal to $-2, -4, -6, \ldots$, which are the trivial zeroes of ζ(s) as many people think. But actually $\zeta(\mathrm{s}) = 2^s\sin\frac{\pi s}{2}\pi^{(s-1)}\Gamma(1-s)\zeta(1-\mathrm{s}) = 0$ for any values of s not only $-2, -4, -6, \ldots$

I would like to specify that the value of ζ(s) from equation $\zeta(\mathrm{s}) = \sum_{n=1}^{\infty}\left(\frac{1}{n^s}\right)$ (which is up to the value of s) is <u>not</u> the same as the value of ζ(s) from functional equation $\zeta(\mathrm{s}) = 2^s\sin\frac{\pi s}{2}\pi^{(s-1)}\Gamma(1-s)\zeta(1-\mathrm{s}) = 0$ (which is always equal to zero, independent from value of s).

3. Integral of the remaining complex quantities

Next Riemann tried to find the integral of the remaining complex quantities in negative sense around the domain. He mentioned that the integrand had discontinuities only where x was equal to the whole multiple of $\pm 2\pi i$, if the real part of s was negative (integer). And the integral was thus equal to the sum of the integrals taken in negative sense around these values. The integral around the value n2πi was = $(-2\pi ni)^{(s-1)}(-2\pi i)$, then

Riemann denoted

$$2\sin \pi s\ \zeta(s) \prod(s-1) = (2\pi)^{(s)} \sum(n)^{(s-1)} [(-i)^{(s-1)} + (i)^{(s-1)}]$$

Let us proof together,

Last time when Riemann talked about positive sense around a domain, he worked with values of x on (+∞, +∞). This time he talked about negative sense around that domain and worked with x which were imaginary numbers = ±n2πi.

From ...(5) $\quad 2\sin \pi s\ \zeta(s)\prod(s-1) = i\int_{+\infty}^{+\infty} \frac{(x)^{(s-1)}}{(e^x-1)} dx = 0$

$$0 = i\int_{0}^{+\infty} \frac{(x)^{(s-1)}}{(e^x-1)} dx + i\int_{+\infty}^{0} \frac{(x)^{(s-1)}}{(e^x-1)} dx$$

$$= i\int_{0^+}^{+\infty} \frac{(x)^{(s-1)}}{(e^x-1)} dx - i\int_{0^+}^{+\infty} \frac{(x)^{(s-1)}}{(e^x-1)} dx$$

$$= i\int_{0^+}^{+\infty} \frac{(x)^{(s-1)}}{(1-e^{(-x)})(e^x)} dx - i\int_{0^+}^{+\infty} \frac{(x)^{(s-1)}}{(1-e^{(-x)})(e^x)} dx$$

$$= i\int_{0^+}^{+\infty} \frac{(x)^{(s-1)}(e)^{(-x)}}{(1-e^{(-x)})} dx - i\int_{0^+}^{+\infty} \frac{(x)^{(s-1)}(e)^{(-x)}}{(1-e^{(-x)})} dx$$

For $x = \pm x_n = \pm n2\pi i$

$$0 = i\int_{0^+}^{+\infty} (x_n)^{(s-1)}[\frac{(e^{-(x_n)})}{(1-e^{-(x_n)})}]\, dx_n - i\int_{0^+}^{+\infty} (-x_n)^{(s-1)}[\frac{(e^{-(-x_n)})}{(1-e^{-(-x_n)})}]\, d(-x_n)$$

From **Riemann Sum**

$\int_{0}^{+\infty} f(x)\, dx = \sum_{n=1}^{+\infty} f(s_n)\, \Delta x_n$ for $x_{n+1} \geq s_n \geq x_n$

$= \sum_{n=1}^{+\infty} f(x_n)\, \Delta x_n$ if $x_n = n2\pi i$ = right-hand end point on $[(x_n) - (x_{n+1})]$ of the interval [0 , +∞).

$\int_{0}^{+\infty} f(-x)\, dx = \sum_{n=1}^{+\infty} f(-s_n)\, \Delta x_n$ for $(-x_{n-1}) \leq (-s_n) \leq (-x_n)$

$= \sum_{n=1}^{+\infty} f(-x_n)\, \Delta(-x_n)$ if $(-x_n) = (-n2\pi i) =$

right-hand end point on $[(-x_{n-1})-(-x_n)]$ of the interval $[0,+\infty)$

Thus by **Riemann Sum**

$$i\int_{0^+}^{+\infty}(x_n)^{(s-1)}[\frac{(e^{-(x_n)})}{(1-e^{-(x_n)})}]\,\mathrm{d}x_n - i\int_{0^+}^{+\infty}(-x_n)^{(s-1)}[\frac{(e^{-(-x_n)})}{(1-e^{-(-x_n)})}]\,\mathrm{d}(-x_n) = 0$$

$$0 = i\int_{0^+}^{+\infty}(x_n)^{(s-1)}\sum_{n=1}^{+\infty}\left(e^{-(x_n)}\right)^{(n-1)} e^{-(x_n)}\mathrm{d}x_n$$

$$-i\int_{0^+}^{+\infty}(-x_n)^{(s-1)}\sum_{n=1}^{+\infty}\left(e^{-(-x_n)}\right)^{(n-1)} e^{-(-x_n)}\mathrm{d}(-x_n)$$

$$= i\int_{0^+}^{+\infty}(x_n)^{(s-1)}\sum_{n=1}^{+\infty}\left(e^{-(x_n)}\right)^{(n)}\mathrm{d}x_n$$

$$-i\int_{0^+}^{+\infty}(-x_n)^{(s-1)}\sum_{n=1}^{+\infty}\left(e^{-(-x_n)}\right)^{(n)}\mathrm{d}(-x_n)$$

$$= i\int_{0^+}^{+\infty}(n2\pi i)^{(s-1)}\sum_{n=1}^{+\infty}\left(e^{-(2\pi i)}\right)^{(nn)}\mathrm{d}x_n$$

$$-i\int_{0^+}^{+\infty}(-n2\pi i)^{(s-1)}\sum_{n=1}^{+\infty}\left(e^{-(-2\pi i)}\right)^{(nn)}\mathrm{d}(-x_n)$$

$$= i\sum_{n=1}^{+\infty}(n2\pi i)^{(s-1)}\,[\cos 2\pi - i\sin 2\pi]^{(nn)}[2\pi i]$$

$$-i\sum_{n=1}^{+\infty}(-n2\pi i)^{(s-1)}\,[\cos 2\pi + i\sin 2\pi]^{(nn)}[-2\pi i]$$

$$= i\sum_{n=1}^{+\infty}(n2\pi i)^{(s-1)}\,[1]^{(nn)}[2\pi i]$$

$$-i\sum_{n=1}^{+\infty}(-n2\pi i)^{(s-1)}\,[1]^{(nn)}[-2\pi i]$$

$$= i\sum_{n=1}^{+\infty}(n2\pi i)^{(s-1)}\,[2\pi i]$$

$$-i\sum_{n=1}^{+\infty}(-n2\pi i)^{(s-1)}\,[-2\pi i]$$

$$= i\sum_{n=1}^{+\infty}(\mathrm{n}2\pi i)^{(s-1)}\,[2\pi i] - i\sum_{n=1}^{+\infty}(-\mathrm{n}2\pi i)^{(s-1)}\,[-2\pi i]$$

$$= i\sum_{n=1}^{+\infty}(i)^{(s)}(2\pi)^{(s)}\,(\mathrm{n})^{(s-1)} - i\sum_{n=1}^{+\infty}(-i)^{(s)}(2\pi)^{(s)}\,(\mathrm{n})^{(s-1)}$$

$$= i(i)^{(s)}(2\pi)^{(s)}\sum_{n=1}^{+\infty}(\mathrm{n})^{(s-1)} - i(-i)^{(s)}(2\pi)^{(s)}\sum_{n=1}^{+\infty}(\mathrm{n})^{(s-1)}$$

$$= i\frac{i}{i}(i)^{(s)}(2\pi)^{(s)}\sum_{n=1}^{+\infty}(\mathrm{n})^{(s-1)} + (-i)\frac{(-i)}{(-i)}(-i)^{(s)}(2\pi)^{(s)}\sum_{n=1}^{+\infty}(\mathrm{n})^{(s-1)}$$

$$0 = -1[\ (i)^{(s-1)}(2\pi)^{(s)} \sum_{n=1}^{+\infty}(n)^{(s-1)}$$

$$+(-i)^{(s-1)}(2\pi)^{(s)} \sum_{n=1}^{+\infty}(n)^{(s-1)}]$$

$$= (i)^{(s-1)}(2\pi)^{(s)} \sum_{n=1}^{+\infty}(n)^{(s-1)}$$

$$+(-i)^{(s-1)}(2\pi)^{(s)} \sum_{n=1}^{+\infty}(n)^{(s-1)}$$

$$= \ (2\pi)^{(s)} \sum_{n=1}^{+\infty}(n)^{(s-1)}\,[(-i)^{(s-1)} + (i)^{(s-1)}]$$

The result is exactly the same as that of Riemann

$$2\sin \pi s\ \zeta(s) \prod(s-1) \ = (2\pi)^{(s)} \sum(n)^{(s-1)}\,[(-i)^{\ (s-1)} + (i)^{\ (s-1)}]$$

$$= \ 0$$

4. Finding nontrivial zeroes on critical line ($s = \frac{1}{2} + ti$)

From ... (9.2) $\prod\left(\frac{s}{2}-1\right)(\pi)^{\left(-\frac{s}{2}\right)}\zeta(s\ ) = \int_{0^+}^{+\infty} \psi(x)(x)^{\left(\frac{s}{2}-1\right)}\,dx \ = 0,$

independent from the values of s.

Hence we can not go on anymore with this functional equation $\prod\left(\frac{s}{2}-1\right)(\pi)^{\left(-\frac{s}{2}\right)}\zeta(s) \ = \ \int_{0^+}^{+\infty} \psi(x)(x)^{\left(\frac{s}{2}-1\right)}\,dx \ = \ 0$ (multiply by zero cause the integrand to be zero, so the integral is zero). And so we have nothing to do further with the equation $\prod\left(\frac{s}{2}\right)(s-1)(\pi)^{\left(-\frac{s}{2}\right)}\zeta(s) = \xi(t)$ denoted by Riemann too. Independent from the values of s or t. But if someone tries to follow this Riemann's Hypothesis, he or she will be unavoidable facing with the mysterious and doubtful equations below

4.1. $\prod\left(\frac{s}{2}-1\right)(\pi)^{\left(-\frac{s}{2}\right)}\zeta(s) = \frac{1}{s(s-1)} + \int_{0^+}^{+\infty} \psi(x)[(x)^{\left(\frac{s}{2}-1\right)} + (x)^{-\left(\frac{1+s}{2}\right)}\ ]dx$

4.2. $\xi(t) = \prod\left(\frac{s}{2}\right)(s-1)(\pi)^{\left(-\frac{s}{2}\right)}\zeta(s)$

Let's see what's gone wrong **(only for proof of the steps of derivation of those equations by Riemann, and to do so, we have to suppose unwillingly that ψ(x) is not zero although that is not true at all).**

Proof of the falsity on 4.1 and 4.2

4.1. $\prod\left(\frac{s}{2}-1\right)(\pi)^{\left(-\frac{s}{2}\right)}\zeta(s) \;=\; \frac{1}{s(s-1)} + \int_{0^+}^{+\infty} \psi(x)\left[(x)^{\left(\frac{s}{2}-1\right)} + (x)^{-\left(\frac{1+s}{2}\right)}\right]dx$

This equation is not true. Let us prove together,

From

$$\prod\left(\tfrac{s}{2}-1\right)(\pi)^{\left(-\frac{s}{2}\right)}\zeta(s) = \int_{0^+}^{+\infty} \psi(x)(x)^{\left(\frac{s}{2}-1\right)}\,dx \qquad ...(9)$$

$$= \int_{1}^{+\infty} \psi(x)(x)^{\left(\frac{s}{2}-1\right)}\,dx + \int_{0^+}^{1} \psi(x)(x)^{\left(\frac{s}{2}-1\right)}\,dx$$

Suppose $\psi(x) \neq 0$, (not as it really be, actually $\psi(x) = 0$).

From $(2\psi(x)+1) \;=\; (x)^{\left(-\frac{1}{2}\right)}\left(2\psi\left(\frac{1}{x}\right)+1\right)$ (Jacobi, Fund. S.184)

$$\prod\left(\tfrac{s}{2}-1\right)(\pi)^{\left(-\frac{s}{2}\right)}\zeta(s) \;=\; \int_{1}^{+\infty} \psi(x)(x)^{\left(\frac{s}{2}-1\right)}\,dx + \int_{0^+}^{1} \psi\left(\frac{1}{x}\right)(x)^{\left(\frac{s-3}{2}\right)}\,dx$$

$$+\frac{1}{2}\int_{0^+}^{1}\left[(x)^{\left(\frac{s-3}{2}\right)} - (x)^{\left(\frac{s}{2}-1\right)}\right]dx$$

$$= \int_{1}^{+\infty} \psi(x)(x)^{\left(\frac{s}{2}-1\right)}\,dx + \int_{0^+}^{1} \psi\left(\frac{1}{x}\right)(x)^{\left(\frac{s-3}{2}\right)}\,dx$$

$$+\frac{1}{2}\left[\frac{(x)^{\left(\frac{s-1}{2}\right)}}{\left(\frac{s-1}{2}\right)}\right]_{0^+}^{1} - \frac{1}{2}\left[\frac{(x)^{\left(\frac{s}{2}\right)}}{\left(\frac{s}{2}\right)}\right]_{0^+}^{1}$$

$$= \int_{1}^{+\infty} \psi(x)(x)^{\left(\frac{s}{2}-1\right)}\,dx + \int_{0^+}^{1} \psi\left(\frac{1}{x}\right)(x)^{\left(\frac{s-3}{2}\right)}\,dx$$

$$+\frac{1}{2}\left[\frac{(1-0)}{\left(\frac{s-1}{2}\right)}\right] - \frac{1}{2}\left[\frac{(1-0)}{\left(\frac{s}{2}\right)}\right]$$

$$= \int_{1}^{+\infty} \psi(x)(x)^{\left(\frac{s}{2}-1\right)}\,dx + \int_{0^+}^{1} \psi\left(\frac{1}{x}\right)(x)^{\left(\frac{s-3}{2}\right)}\,dx$$

$$+\frac{1}{(s)(s-1)}$$

Finally we get

$\prod\left(\frac{s}{2}-1\right)(\pi)^{\left(-\frac{s}{2}\right)}\zeta(s)$

$$= \frac{1}{(s)(s-1)} + \int_{1}^{+\infty} \psi(x)(x)^{\left(\frac{s}{2}-1\right)}\,dx + \int_{0^+}^{1} \psi\left(\frac{1}{x}\right)(x)^{\left(\frac{s-3}{2}\right)}\,dx \qquad \ldots (10)$$

But from original Riemann's papers (1859)

$$\prod\left(\frac{s}{2}-1\right)(\pi)^{\left(-\frac{s}{2}\right)}\zeta(s)$$

$$= \int_{1}^{+\infty} \psi(x)(x)^{\left(\frac{s}{2}-1\right)}\,dx + \int_{0^+}^{1} \psi\left(\frac{1}{x}\right)(x)^{\left(\frac{s-3}{2}\right)}\,dx + \frac{1}{2}\int_{0^+}^{1}\left[(x)^{\left(\frac{s-3}{2}\right)} - (x)^{\left(\frac{s}{2}-1\right)}\right]dx$$

$$= \frac{1}{(s)(s-1)} + \int_{1}^{+\infty} \psi(x)\left[(x)^{\left(\frac{s}{2}-1\right)} + (x)^{-\left(\frac{1+s}{2}\right)}\right]\,dx \qquad \ldots (11)$$

please look more carefully at the last expressions of equations…(10) and …(11) in comparison. The integral $\int_{0^+}^{1} \psi\left(\frac{1}{x}\right)(x)^{\left(\frac{s-3}{2}\right)}\,dx$ of equations … (10) and $\int_{1}^{+\infty} \psi(x)(x)^{-\left(\frac{1+s}{2}\right)}\,dx$ of equation… (11)(of Riemann) look much different. Actually $\int_{0^+}^{1} \psi\left(\frac{1}{x}\right)(x)^{\left(\frac{s-3}{2}\right)}\,dx = \int_{0^+}^{1} \psi(x)(x)^{\left(\frac{s}{2}-1\right)}\,dx - \frac{1}{(s)(s-1)}$. It is impossible to prove that the two integrals are equal (if we suppose $\psi(x) \neq 0$, but do not worry about this anymore because multiplying the integrands of the two integrals by $\psi(x) = 0$ have already caused the two integrals to be zeroes). However, in spite of multiplication by $\psi(x) = 0$, the mathematical right one is equation …(10), or

$$\prod\left(\frac{s}{2}-1\right)(\pi)^{\left(-\frac{s}{2}\right)}\zeta(s)$$

$$= \int_{0^+}^{+\infty} \psi(x)(x)^{\left(\frac{s}{2}-1\right)}\,dx \qquad \ldots (9)$$

$$= \frac{1}{(s)(s-1)} + \int_{1}^{+\infty} \psi(x)(x)^{\left(\frac{s}{2}-1\right)}\,dx + \int_{0^+}^{1} \psi\left(\frac{1}{x}\right)(x)^{\left(\frac{s-3}{2}\right)}\,dx \qquad \ldots (10)$$

$$= 0$$

4.2. $\xi(t) = \prod\left(\frac{s}{2}\right)(s-1)(\pi)^{\left(-\frac{s}{2}\right)}\zeta(s)$

This equation is not true because there is a missing term. Let's prove.

From $\prod\left(\frac{s}{2}-1\right)(\pi)^{\left(-\frac{s}{2}\right)}\zeta(s) = \int_{0^+}^{+\infty} \psi(x)(x)^{\left(\frac{s}{2}-1\right)}\,dx \qquad \ldots (9)$

Suppose $\psi(x) \neq 0$ (not as it really be, actually $\psi(x) = 0$).

Let us follow this equation of Riemann (althought the RHS. of ...(11) are not true as proof above), what we want here is only to prove from how and from where his new functional equation was derived. If his equation came from the wrong sources (former equations) or from wrong method (derivation), then further using of it would be inappropriate.

Now, from the (wrong) equation,

$$\prod\left(\tfrac{s}{2}-1\right)(\pi)^{\left(-\frac{s}{2}\right)}\zeta(s)$$

$$= \frac{1}{(s)(s-1)} + \int_1^{+\infty} \psi(x)\left[(x)^{\left(\frac{s}{2}-1\right)} + (x)^{-\left(\frac{1+s}{2}\right)}\right] dx \qquad \text{... (11)}$$

Multiply equation ...(11) by $\left(\frac{s}{2}\right)(s-1)$ bothsides and set $s = \frac{1}{2} + it$ (as Riemann did)

$$\prod\left(\tfrac{s}{2}-1\right)\left(\frac{s}{2}\right)(s-1)(\pi)^{\left(-\frac{s}{2}\right)}\zeta(s)$$

$$= \frac{\left(\frac{s}{2}\right)(s-1)}{(s)(s-1)} + \left(\frac{s}{2}\right)(s-1)\int_1^{+\infty} \psi(x)\left[(x)^{\left(\frac{s}{2}-1\right)} + (x)^{-\left(\frac{1+s}{2}\right)}\right] dx$$

$$= \frac{1}{2} + \frac{\left(\frac{1}{2}+it\right)\left(\frac{1}{2}+it-1\right)}{2}\int_1^{+\infty} \psi(x)\left[(x)^{\left(\frac{1}{4}+\frac{it}{2}-1\right)} + (x)^{-\left(\frac{1+\frac{1}{2}+it}{2}\right)}\right] dx$$

$$= \frac{1}{2} - \frac{\left(tt+\frac{1}{4}\right)}{2}\int_1^{+\infty} \psi(x)\left[(x)^{\left(-\frac{3}{4}\right)}(x)^{\left(\frac{it}{2}\right)} + (x)^{\left(-\frac{3}{4}\right)}(x)^{\left(-\frac{it}{2}\right)}\right] dx$$

$$= \frac{1}{2} - \frac{\left(tt+\frac{1}{4}\right)}{2}\int_1^{+\infty} \psi(x)(x)^{\left(-\frac{3}{4}\right)}\left[(e)^{\left(\frac{it}{2}\log x\right)} + (e)^{\left(-\frac{it}{2}\log x\right)}\right] dx$$

$$= \frac{1}{2} - \frac{\left(tt+\frac{1}{4}\right)}{2}\int_1^{+\infty} \psi(x)(x)^{\left(-\frac{3}{4}\right)}\left[\left(\cos\left(\tfrac{1}{2}t\log x\right) + i\sin\left(\tfrac{1}{2}t\log x\right)\right)\right.$$
$$\left. + \left(\cos\left(\tfrac{1}{2}t\log x\right) - i\sin\left(\tfrac{1}{2}t\log x\right)\right)\right] dx$$

$$= \frac{1}{2} - \frac{\left(tt+\frac{1}{4}\right)}{2}\int_1^{+\infty} \psi(x)(x)^{\left(-\frac{3}{4}\right)}\left(2\cos\left(\tfrac{1}{2}t\log x\right)\right) dx$$

$= \frac{1}{2} - \left(tt + \frac{1}{4}\right) \int_{1}^{+\infty} \psi(x)(x)^{\left(-\frac{3}{4}\right)} \cos(\frac{1}{2}\text{tlogx})\, dx$

$= \xi(t)$

You can see, there are two mistakes of Riemann here.

1. The (right) equation $\prod\left(\frac{s}{2}-1\right)\left(\frac{s}{2}\right)(s-1)(\pi)^{\left(-\frac{s}{2}\right)}\zeta(s) = \xi(t)$ is different from the (wrong) equation of Riemann $\prod\left(\frac{s}{2}\right)(s-1)(\pi)^{\left(-\frac{s}{2}\right)}\zeta(s) = \xi(t)$. It is wrong to write $\prod\left(\frac{s}{2}\right)$ ... instead of $\prod\left(\frac{s}{2}-\mathbf{1}\right)\left(\frac{s}{2}\right)$ .... However both equations are useless because $\prod\left(\frac{s}{2}-1\right)\left(\frac{s}{2}\right)(s-1)(\pi)^{\left(-\frac{s}{2}\right)}\zeta(s)$

$=\left(\frac{s}{2}\right)(s-1)\int_{0^+}^{+\infty} \psi(x)(x)^{\left(\frac{s}{2}-1\right)} dx = 0$, independent from the value of s or t.

2. $\frac{1}{2} - \left(tt + \frac{1}{4}\right) \int_{1}^{+\infty} \psi(x)(x)^{\left(-\frac{3}{4}\right)} \cos(\frac{1}{2}\text{tlogx})\, dx$ is wrong and useless because it comes from $\frac{1}{(s)(s-1)} + \int_{1}^{+\infty} \psi(x)[(x)^{\left(\frac{s}{2}-1\right)} + (x)^{-\left(\frac{1+s}{2}\right)}]\, dx$ for $s = \frac{1}{2} + it$ which is wrong as proof above.

So $\quad \xi(t) = \prod\left(\frac{s}{2}-1\right)\left(\frac{s}{2}\right)(s-1)(\pi)^{\left(-\frac{s}{2}\right)}\zeta(s)$, for $s = \frac{1}{2} + it$

$= \frac{1}{2} - \left(tt + \frac{1}{4}\right) \int_{1}^{+\infty} \psi(x)(x)^{\left(-\frac{3}{4}\right)} \cos(\frac{1}{2}\text{tlogx})\, dx$

$= 0$ is wrong and useless equation.

Hence the number of roots of (wrong) equation

$$\xi(t) = \frac{1}{2} - \left(tt + \frac{1}{4}\right) \int_{1}^{+\infty} \psi(x)(x)^{\left(-\frac{3}{4}\right)} \cos(\frac{1}{2}\text{tlogx})\, dx = 0$$

is undefined (not approximately = $\left(\frac{T}{2\pi}\log\frac{T}{2\pi} - \frac{T}{2\pi}\right)$), and meaningless .

And the integral $\int d\log\xi(t) = \int d\log(0)$ is undefined (not equal to $(T\log\frac{T}{2\pi} - T)i$), and meaningless too .

It is useless to denote $\xi(\alpha) = 0$ and $\log\xi(t)$ as $\sum\log\left(1-\frac{tt}{\alpha\alpha}\right)$+$\log\xi(0)$ because $\xi(t) = \frac{1}{2} - \left(tt + \frac{1}{4}\right) \int_{1}^{+\infty} \psi(x)(x)^{\left(-\frac{3}{4}\right)} \cos(\frac{1}{2}\text{tlogx})\, dx = 0$ is wrong equation, so $\log\xi(t) = \log(0)$ is undefined, and meaningless.

5. Determination of the number of prime numbers that are smaller than x

Next, Riemann tried to determine the number of prime numbers that are smaller than x with the assistance of all the methods he had derived before.

From the **identity by Riemann**

$$\log\zeta(s) = -\sum \log(1-(p)^{-s})$$

$$= \sum p^{-s} + \tfrac{1}{2}\sum p^{-2s} + \tfrac{1}{3}\sum p^{-3s} + \cdots \qquad ...(12)$$

This is not true at all. Let's see what has gone wrong.

For p = prime numbers

n = all whole numbers = 1,2,3…,∞

Riemann denoted that

$$\zeta(s) = \prod_{p\ prime}(1-\tfrac{1}{p^s})^{-1} = \sum_{n=1}^{+\infty}(\tfrac{1}{n^s}) \quad , \quad \mathfrak{R}(\mathbf{s}) > 1$$

$$\log\zeta(s) = \log\prod_{p\ prime}(1-\tfrac{1}{p^s})^{-1}$$

$$= \log[(1-(2)^{-s})^{-1}.(1-(3)^{-s})^{-1}.(1-(5)^{-s})^{-1}....]$$

$$= \log(1-(2)^{-s})^{-1} + \log(1-(3)^{-s})^{-1} + \log(1-(5)^{-s})^{-1} + ...$$

$$= -[\log(1-(2)^{-s}) + \log(1-(3)^{-s}) + \log(1-(5)^{-s}) + \cdots]$$

$$= \sum \log\left(1-(p)^{(-s)}\right)^{(-1)}$$

But from **Geometric Series**

For a = 1 , r = $(p)^{(-s)}$

$$\sum_{n=0}^{\infty}(ar)^n = \lim_{n\to\infty} S_n$$

$$= \frac{a}{(1-r)}$$

$$= \frac{1}{(1-(p)^{(-s)})}$$

$$= \left(1-(p)^{(-s)}\right)^{(-1)}$$

So $\sum_{n=0}^{\infty} (p^{-s})^n = \left(1-(p)^{(-s)}\right)^{(-1)}$

Or $\log\left(1-(p)^{(-s)}\right)^{(-1)} = \log(\sum_{n=0}^{\infty} (p^{-s})^n$

Then $\sum \log\left(1-(p)^{(-s)}\right)^{(-1)} = \sum \log(\sum_{n=0}^{\infty} (p^{-s})^n$

And from $\log \zeta(\mathrm{s}) = \sum \log\left(1-(p)^{(-s)}\right)^{(-1)}$

Thus we get

$$\log \zeta(\mathrm{s}) = \sum \log(\sum_{n=0}^{\infty} (p^{-s})^n$$

$$= \log(\sum_{n=0}^{\infty}(2)^{(-s)n}) + \log(\sum_{n=0}^{\infty}(3)^{(-s)n}) + \log(\sum_{n=0}^{\infty}(5)^{(-s)n}) + \ldots$$

One can replace $(p^{-s})^n$ by $s\int_{p^n}^{\infty}(x)^{-(s+1)}\mathrm{dx}$.

Let's prove together

$$s\int_{p^n}^{\infty} (x)^{-(s+1)}\,\mathrm{dx} = \frac{(\mathrm{s}\,(\mathrm{x})^{(-\mathrm{s})})}{(-s)}\Big]_{p^n}^{\infty}$$

$$= -\frac{1}{(\mathrm{x})^{(\mathrm{s})}}\Big]_{p^n}^{\infty}$$

$$= -\left(\frac{1}{\infty} - \frac{1}{(p^n)^s}\right)$$

$$= (p)^{(-s)n}$$

Hence something has to be changed from that of Riemann's work especially the replacements of the following expressions.

$(2)^{(-s)n}$ **by** $s\int_{2^n}^{\infty}(x)^{-(s+1)}\mathrm{dx}$,

$(3)^{(-s)n}$ **by** $s\int_{3^n}^{\infty}(x)^{-(s+1)}\mathrm{dx}$,

…

Those should be put in stead of Riemann's replacements below (which are incorrect as proof above).

$(p)^{(-s)}$ **by** $s\int_{p}^{\infty}(x)^{-(s+1)}\mathrm{dx}$,

$(p)^{(-2s)}$ **by** $s\int_{p^2}^{\infty}(x)^{-(s+1)}\mathrm{dx}$,

…

And then

$$\log \zeta(s) = \log\left(\sum_{n=0}^{\infty}(2)^{(-s)n}\right) + \log\left(\sum_{n=0}^{\infty}(3)^{(-s)n}\right) + \cdots$$

$$= \log\left(\sum_{n=0}^{\infty} s\int_{2^n}^{\infty}(x)^{-(s+1)}\,\mathrm{dx}\right) + \log\left(\sum_{n=0}^{\infty} s\int_{3^n}^{\infty}(x)^{-(s+1)}\,\mathrm{dx}\right) + \cdots$$

So the concept of Riemann to denote $\mathrm{F(x)} + \frac{1}{2}\mathrm{F}(x^{\frac{1}{2}}) + \frac{1}{3}\mathrm{F}(x^{\frac{1}{3}}) + \ldots$ by f(x) is useless and has to be reformed to the new appropriated one.

Hope all of my papers are clear enough to point out or give proof of the original **Riemann's Hypothesis** such as

1. "All zeroes of the function ξ(t) are real". This is not true because

$$\xi(\mathrm{t}) = \prod\left(\tfrac{s}{2}-1\right)\left(\frac{s}{2}\right)(s-1)(\pi)^{\left(-\frac{s}{2}\right)}\zeta(\mathrm{s}), \text{ for s} = \frac{1}{2} + it$$

$$= \frac{1}{2} - \left(tt + \frac{1}{4}\right)\int_{1}^{+\infty}\psi(\mathrm{x})(x)^{\left(-\frac{3}{4}\right)}\cos\left(\frac{1}{2}\mathrm{tlogx}\right)\mathrm{dx}$$

is always equal to zero, independent from the values of t (or s) and derives from wrong equation as proof above (pages 32– 37). So roots (all zeroes) of

$$\xi(\mathrm{t}) = \prod\left(\tfrac{s}{2}-1\right)\left(\frac{s}{2}\right)(s-1)(\pi)^{\left(-\frac{s}{2}\right)}\zeta(\mathrm{s}), \text{ for s} = \frac{1}{2} + it$$

$$= \frac{1}{2} - \left(tt + \frac{1}{4}\right)\int_{1}^{+\infty}\psi(\mathrm{x})(x)^{\left(-\frac{3}{4}\right)}\cos\left(\frac{1}{2}\mathrm{tlogx}\right)\mathrm{dx}$$

are undefined and can not be found by this (wrong) equation.

2. "The function (functional equation) ζ(s) has zeroes at the negative even integers −2, −4, … and one refers to them as the trivial zeroes". This is not true as proof above(pages 19−29).

3. "The nontrivial zeroes of ζ(s) have real part equal to $\frac{1}{2}$ or the nontrivial zeroes are complex numbers $= \frac{1}{2} + i \propto$ where $\propto$ is a zero of ξ(t)". This is not true because $\xi(\mathrm{t}) = \prod\left(\tfrac{s}{2}-1\right)\left(\frac{s}{2}\right)(s-1)(\pi)^{\left(-\frac{s}{2}\right)}\zeta(\mathrm{s})$,

for $s = \frac{1}{2} + it$ or $\xi(t) = \frac{1}{2} - \left(tt + \frac{1}{4}\right)\int_1^{+\infty} \psi(x)(x)^{\left(-\frac{3}{4}\right)} \cos(\frac{1}{2}\text{tlogx})\, dx$

is always equal to zero, independent from the values of t (or s) and derived from wrong equation as proof above (pages $32 - 37$). So zeroes of $\xi(t) = 0$ are undefined, and can not be found by this wrong equation and the nontrivial zeroes of ζ(s) are undefined and can not be found by this way too.

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