THE NAVIER-STOKES EQUATIONS AND TURBULENCE

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Abstract

The Navier-Stokes differential equations describe the motion of fluids which are incompressible. The three-dimensional Navier-Stokes equations misbehave very badly although they are relatively simple-looking. The solutions could wind up being extremely unstable even with nice, smooth, reasonably harmless initial conditions. A mathematical understanding of the outrageous behaviour of these equations would dramatically alter the field of fluid mechanics. This paper describes why the three-dimensional Navier-Stokes equations are not solvable, i.e., the equations cannot be used to model turbulence, which is a three-dimensional phenomenon.

The general equations of motion for a viscous fluid were obtained by Sir George Stokes in 1845. The following is the fundamental equation (in vectorial form) governing the flow of a viscous fluid:-

$$\frac{\partial v}{\partial t} + (v.\nabla)v = -\frac{1}{p}\nabla Pe - \nabla\varphi + \frac{\eta}{p}\nabla 2v\ ,$$

where v is the velocity of the fluid (as a function of position), Pe the pressure, φ the gravitational potential, p the density and η the viscosity.

The scientist normally makes a forecast of the outcome of a flow and uses the Navier-Stokes equations to model this forecast. However, in the instance of turbulence, making this forecast will be fraught with difficulty, if it can be carried out at all. Putting it another way, if turbulence could be forecasted, predicted and described by the Navier-Stokes equations it could not be turbulence, for turbulence implies puzzlement, lack of order or pattern and lack of predictability.

The Navier-Stokes equations are nonlinear due to the acceleration terms such as $u\partial u/\partial x$. As a result, the solution to these equations may not be unique. For instance, the flow between two rotating cylinders can be solved using the Navier-Stokes equations to treat a relatively simple flow with circular streamlines; it can also be a flow with streamlines which are like a spring wound around the cylinders as a torus; there are also more complex flows which are solutions to the Navier-Stokes equations, all satisfying the identical boundary conditions.

For simple geometries, the Navier-Stokes equations can be solved with relative ease. However, the equations cannot be solved for a turbulent flow even for the simplest of examples. A turbulent flow is highly unsteady, nonlinear and three-dimensional and therefore requires that the three velocity components be specified at all points in a region of interest at some initial time, say $t = 0$. But, even for the simplest geometry, such information will be almost impossible to obtain. Therefore, the solutions for turbulent flows have to be left to the experimentalist and are not attempted by solving the Navier-Stokes equations.

References

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