Beltrami-Trkalian Vector Fields in Electrodynamics: Hidden Riches for Revealing New Physics and for Questioning the Structural Foundations of Classical Field Physics

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Introduction

When we come to examine the annals of classical hydrodynamics and electrodynamics, we find that the foundations of vector field theory have provide some key field structures whose role has repeatedly been acknowledged as instrumental in not only underpinning the structural edifice of classical continuum field physics, but in accounting for its empirical exhibits as well.

However, there is one equally important vector field configuration which, despite its ubiquitous exhibit throughout a panoply of applications in classical field physics, has strangely enough remained less understood, under-appreciated and currently consigned to linger in relative obscurity, known only to specialists in certain fields.

The following exposition seeks to remedy this situation by calling attention to the extensive but little recognized applications of this field structure in various areas of hydrodynamics and electrodynamics. Also, many of these empirical exhibits, particularly in classical electrodynamics, have continued to be accompanied by recorded energy anomalies or other related phenomena currently little understood by accepted scientific paradigms. Is one indication which strongly implies a disturbing lack of completeness in the foundation of electromagnetic theory.

The aim of this paper is to show that these deficiencies could possibly be better understood once it is acknowledged that certain exhibits of classical EM in a macroscopic context – much like those associated with the Aharonov-Bohm effect for instance, which has been originally quantum-mechanically canonized, may in fact be a function of classical topological field symmetries higher than those of the standard U(1) variety. Consequently, our inability to quantify some of the EM phenomena associated with the field structure in question, could be due to a current lack of understanding of its possible significance as a fundamental topological field archetype, universal throughout physics in general. Accordingly, in support of this contention, a smaller focus of this paper explore the speculation that our failures up to present in constructing a viable deterministic model or theory of turbulence in hydrodynamics, might also rest on the heretofore unsuspected role this field structure might play at a more archetypal level of nature.

Properties of Beltrami Flow Fields in Hydrodynamics

This vector field condition is sometimes referred to as Beltrami fluid flow, and was previously treated in a similar exposition by the author in 1995[1].

This also describes a Magnus force-free flow which is expressed by the relationship:

$$\mathbf{v} \times (\text{curl } \mathbf{v}) = 0 \tag{1}$$

As pointed out by Bjorgum and Godal [3], this relationship represents one of the eight types of possible vector fields. These are further derived from three basic field types:

a. Solenoidal vector fields for which

$$\text{div } \mathbf{v} = 0, \tag{2}$$

b. Complex-lamellar vector fields for which

$$\mathbf{v} \cdot (\text{curl } \mathbf{v}) = 0, \tag{3}$$

c. Beltrami vector field condition in (1) above.

An alternative formulation of the Beltrami condition is given as follows. For any vector field there exists the identity:

$$(\mathbf{v} \cdot \text{grad})\mathbf{v} = \text{grad}\left(\frac{v^2}{2}\right) - \mathbf{v} \times (\text{curl } \mathbf{v})$$

where ν denotes the magnitude of $\mathbf{v}$. Now, the Beltrami condition is satisfied if:

$$(\mathbf{v} \cdot \text{grad})\mathbf{v} = \text{grad}\left(\frac{\nu^2}{2}\right). \tag{4}$$

Consequently, (4) represents a necessary and sufficient Beltrami condition. Since the Beltrami flow (1) describes parallel or anti-parallel vorticity and velocity vectors, another useful formulation of the Beltrami condition is represented by the relation:

$$\text{curl } \mathbf{v} = \text{c } \mathbf{v}, \tag{5}$$

where c denotes a scalar point function of position. This factor c assumes a certain degree of importance in association with Beltrami fields. Certain formulae can be derived from (5), First, $/\mathbf{c}/ = w/\nu$, where w denotes the magnitude of the vorticity $\mathbf{w}$. By taking the scalar product of $\mathbf{v}$ with (5), we obtain:

$$\text{c} = \mathbf{v} \cdot (\text{curl } \mathbf{v})/\nu^2 \tag{6}$$

Beltrami derived a formula similar to (6) in which c is expressed by the vorticity $\mathbf{w}$ only. By introducing $\mathbf{v} = \mathbf{w}/\text{c}$ we obtain:

$$\text{c} = (\mathbf{w}/\text{c})\cdot(\text{curl } \mathbf{w}/\text{c})/(w/\text{c})^2 = \mathbf{w}\cdot(\text{curl } \mathbf{w})/w^2 . \tag{7}$$

Specialized Beltrami Fields

Now, if a Beltrami field is simultaneously complex lamellar,(1) combined with (3), then curl $\mathbf{v}$ is both parallel and perpendicular to $\mathbf{v}$. this can only happen if curl $\mathbf{v}$ is zero (that is, the field $\mathbf{v}$ is irrotational or lamellar). Hence a vector field which is simultaneously a complex-lamellar and a Beltrami field is necessarily lamellar.

If the divergence of (5) is taken we obtain:

$$\text{c}(\text{div } \mathbf{v}) + \mathbf{v} \cdot (\text{grad c}) = 0. \tag{8}$$

If a Beltrami field (1) is simultaneously solenoidal (2), then (8) reduces to:

$$\mathbf{v}\cdot(\text{grad c}) = 0. \tag{9}$$

In other words, in a solenoidal Beltrami field the vector field lines are situated in the surfaces c = const. This theorem was originally derived by Ballabh [4] for a Beltrami flow proper of an incompressible medium. For the sake of completeness, we mention that the combination of (1),(2) and (3) only leads to a Laplacian field, which is better defined by a vector field which is both solenoidal (divergenceless) and lamellar (irrotational).

Now, we consider a Beltrami field the curl of which is also a Beltrami field. For this case, in addition to (1), the field must satisfy the condition:

$$(\text{curl } \mathbf{v}) \times (\text{curl curl } \mathbf{v}) = 0, \quad \text{or} \quad \text{using (5)}, \ \mathbf{v} \times (\text{curl curl } \mathbf{v}) = 0 . \tag{10}$$

By taking the curl of (5), we obtain:

$$\text{curl curl } \mathbf{v} = (\text{grad c}) \times \mathbf{v} + \text{c}(\text{curl } \mathbf{v}). \tag{11}$$

Now, if c is uniform(i.e.,c(x,y,z) = k (a const.)), then (11) becomes:

$$\text{curl curl } \mathbf{v} = \text{c}(\text{curl } \mathbf{v}). \tag{12}$$

In the case of uniform c (grad c = 0), curl $\mathbf{v} = \mathbf{w}$ (vorticity) is also a Beltrami field, possessing the same coefficient c. This type of vector field is called a Trkalian field, after Trkal, a Russian researcher who studied Beltrami flows proper where c = const. As pointed out by Nemenyi and Prim [5], all successive curls of Trkalian fields will also represent Trkalian fields, with the same coefficient c as the original field. Also, under these conditions, taking the divergence of (5), we get:

div(curl **v**) = 0 = div(c**v**) = (grad c)·**v** + c(div **v**), or 0 = c(div **v**), showing div **v** = 0 (**v** is solenoidal). Thus, we conclude that the Trkalian field is also a special solenoidal Beltrami field.

Properties of the Coefficient c

Due to (5), the equation for c (6) becomes:

$$c = \frac{\mathbf{v}\cdot\text{curl}\,(\text{curl}\,\mathbf{v}/c)}{v^2} = \frac{c\,\mathbf{v}\cdot\text{curl}(\text{curl}\,\mathbf{v}/c)}{c\,v^2} = \frac{c\,\mathbf{v}\cdot(\text{curl}(\text{curl}\,\mathbf{v}))}{(c\,\mathbf{v})^2} \qquad (13)$$

Finally, (13) becomes:

$$1 = \frac{\mathbf{v}\cdot\text{curl}(\text{curl}\,\mathbf{v})}{(\text{curl}\,\mathbf{v})^2} \quad \text{or} \quad (\text{curl}\,\mathbf{v})^2 = \mathbf{v}\cdot\text{curl}(\text{curl}\,\mathbf{v})) \qquad (14)$$

Since the left side of (14) is necessarily positive, we conclude that a Beltrami field **v** and the curl of its curl (right side of (14)) always meet at an acute angle. If the angle is zero, then curl **v** is also a Beltrami field. In this last case, due to considerations above, **v** is a Trkalian field.

For any Beltrami field, c can also be given a geometrical interpretation. Calling **t** the unit vector along **v**, we apply Stokes theorem to a curve (*d***s**) determined by an orthogonal cross-section (*da*) of an infinitesimal vector tube. If r denotes the radius, we find (see Fig. 1):

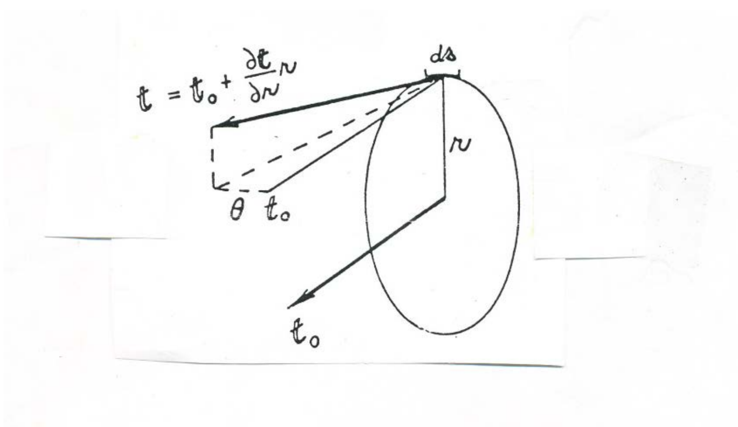


Figure 1. Diagram showing that $\theta = \frac{\partial t}{\partial r} r.\ \frac{ds}{ds}$

$$\int_{(c)} \mathbf{t}\cdot d\mathbf{s} = \int_{(s)} \text{curl}\,\mathbf{t}\cdot da = \pi\, r^2\, \mathbf{t}\cdot\text{curl}\,\mathbf{t}. \qquad (15)$$

Since **v** = ν **t**, we have for (6):

$$\frac{\mathbf{v}\cdot\text{curl}\,\mathbf{v}}{\nu^2} = \mathbf{t}\cdot\,\text{curl}\,\mathbf{t} = c \qquad (16)$$

Applying (16) to (15), we obtain:

$$c = \frac{1}{\pi r^2}\int \mathbf{t}\cdot d\mathbf{s} = \frac{1}{\pi r^2}\int(\mathbf{t}_0 + \frac{\delta\mathbf{t}}{\delta r})\cdot d\mathbf{s}$$

$$\frac{1}{\pi r^2}\int \frac{\delta\mathbf{t}}{\delta r} r\cdot\frac{d\mathbf{s}}{ds}\,ds = \frac{1}{\pi r^2}\int \theta\, d\mathbf{s} = \frac{2\theta}{r}\ . \qquad (16a)$$

Here Θ denotes the average angle θ between $\mathbf{t}_o$ and the projection of $\mathbf{t}_o + \delta\mathbf{t}/\delta r$ on the plane determined by $d\mathbf{s}$ and $\mathbf{t}$ (see Fig. 1); $\theta = \delta\mathbf{t}/\delta r\ r \cdot d\mathbf{s}/ds$. From this analysis, we see that the factor c can be considered geometrically as the torsion of neighboring vector lines **v** in any Beltrami field. Truesdell [16] calls c the "abnormality" of the vector line system of a Beltrami field.

Field Morphology in a Beltrami Configuration

In a general hydrodynamic system, the vorticity **w** is perpendicular to the velocity field **v**, creating a so-called Magnus pressure force. This force is directed along the axis of a right-hand screw as it would advance if the velocity vector rotated around the axis towards the vorticity vector. The conditions surrounding a wing which produce aerodynamic lift describe this effect precisely (see Fig. 2). However, in a Beltrami field, the vorticity and the velocity vectors are parallel or anti-parallel, resulting in a zero Magnus force. The Beltrami condition (1) is therefore an equivalent way of characterizing a force-free flow situation, and vice versa.

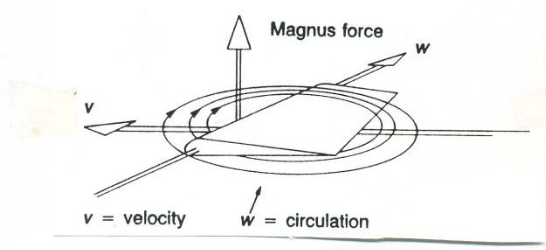


Figure 2. Indicating the Magnus force and vortices surrounding a wing.

In order to model this type of flow field geometrically, Beltrami found that it was necessary to consider a three-dimensional circular axisymmetric flow in which the velocity and vorticity field lines described a helical pattern. This helicoidal flow field was unique in that the pitch of the circular helices decreased as the radius from the central axis increased. This produces a specialized shear effect between the field lines of successively larger cylindrical tubes comprising the respective helices. In the limit of such a field, the central axis of the flow also serves as a field line (see Fig. 3).

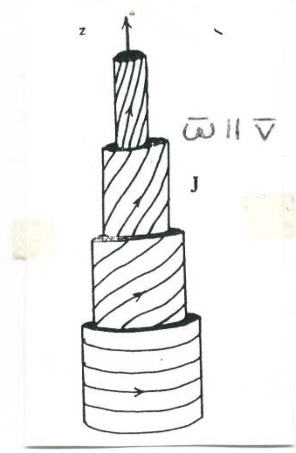


Figure 3. Classical sheared vortex configuration in a Beltrami field (axisymmetric mode)

A New Look at Vector Field Theory

Although Beltrami fields have featured prominently in hydrodynamics for over a century, only until relatively recently have they received much attention in experimental and theoretical classical electrodynamics. The reason for the omission of this link in the standard development of electric/magnetic field theory can possibly be traced to a key deficiency in the structure of vector field theory – which in turn constitutes the very foundation upon which all edifices of classical continuum field physics are constructed. Here we refer to the self-imposed limitations of standard Gibbsian vector analysis for modeling the evolution of vector fields with higher topological structural features. This deficiency can partially be attributed to the decision by the architects of vector analysis at the turn of the century (notably Gibbs and Heaviside) to remove its *quaternion-based* foundation for the purpose of modeling vector fields in simply-connected 3-dimensional Euclidean space. However, with vector fields limited to simple topologies, key topological features such as *helicity*, which feature prominently in the dynamics of the Beltrami flow (in both turbulence and plasma dynamics contexts), remain untreated by these methods, and are better understood topologically through treatment by Cartan’s calculus of differential forms. Specifically, the inherent spiral geometry exhibited by Beltrami flow cannot be clearly ascertained from use of the standard traditional methods for the construction of the general vector field.

There are two known standard methods for decomposition of any smooth (differentiable) vector field. One is that attributed to Helmholtz, which splits any vector field into a lamellar (curl-free) component, and a solenoidal (divergenceless) component. The second, which divides a general vector field into lamellar and complex-lamellar parts, is that popularized by Monge. However, the relatively recent discovery by Moses [7], shows that any smooth vector field – general or with constraints to be determined – may also be separable into *circularly-polarized* vectors. Furthermore, this third method simplifies the otherwise difficult analysis of three-dimensional classical flow fields. The Beltrami flow field, which has a natural chiral structure, is particularly amenable to this type analysis.

Beltrami flow is a particular case whose field generalization is popularly referred to as *eigenfunctions of the curl operator*. In his investigations using curl-eigenfunctions, Moses shows that the expansion of vector fields in terms of these operators, leads to a decomposition into *three* modes, as opposed to the customary two. One is the lamellar field (implying the existence of a scalar potential) with eigenvalue zero. However, the solenoidal vector field divides into two chiral circularly-polarized vector potential fields of opposite signs of polarization and eignevalues +k and –k, respectively. Not only does this new decomposition considerably simplify problems dealing with vector fields which are defined over the whole coordinate space, but Moses describes how this decomposition is rotationally invariant. That is, under a rotation of the coordinates, the vector modes which are introduced in this manner, remain individually invariant under this transformation. This allows for substantial improvement upon the traditional Helmholtz or Monge decompositions, In several instructive examples, Moses [7] displays the versatile utility of this method. For instance, in the arena of incompressible/incompressible fluid dynamics, the curl-eigenfunction method shows how fluid motion with vorticity can be described as a superposition of circularly-polarized modes only. A consequence of this approach is that the normally non-linear convection term in the Navier-Stokes equation drops out, allowing for exact solutions of this relation. These are impossible under the conditions of turbulence, which allows for only approximate solutions to be derived.

Fluid Turbulence and Beltrami Field Modes

In the specialized area of turbulent fluid-flow research, it is therefore not surprising that the Beltrami vortex flow itself, may eventually come to play a significant role. Accordingly, on the forefront of frontiers leading to a heretofore elusive deterministic theory of turbulence, it has recently been suggested [8], as a result of empirical evidence, that in regions of space, turbulent flows spontaneously organize into a coherent hierarchy of weakly interacting superimposed Beltrami flows. Detailed numerical experiments for channel flows and decaying homogeneous turbulence using spectral methods, have provided evidence for such behavior [9]. The implications for this for fluid dynamics is that every incompressible fluid flow (divergenceless velocity field), as well as every solution to the Navier-Stokes equation, is a superposition of interacting Beltrami flows. It has been further postulated that full 3-dimensional turbulent flows may exhibit regimes of both weak local interaction and strong local interaction (due for instance to vortex stretching) between their Beltrami components [8]. These facts are suggestive of a non-linear *uncertainty-principle* for macroscopic turbulent fluid flow which mediates between regions of effective description of the flow field in physical (coordinate) space and that in wave-number (momentum) space [8]. The discovery and certification of such an uncertainty principle in a macroscopic context, analogous to that canonized in quantum physics, could have astounding cross-disciplinary implications. Not only might it provide for the long sought-for key to establishing a deterministic model for turbulent fluid-flow, but might in directly provide a major breakthrough towards a deeper understanding of both the classical and quantum descriptions of nature. At any rate, the novel *circularly-polarized* decomposition of vector fields in this context might possibly pave the way to this goal.

Complex-Helical Wave Decomposition and Vortex Structures

In this regard, Melander and Hussain [10] have made significant advances in casting fresh light on the perplexing problems of vortex core dynamics and the coupling between large scales and fine scales in the vicinity of a coherent structure. Essentially, they postulated that the dynamics of coherent structures is better understood within the framework of vortex/vorticity dynamics rather than in terms of primitive variables such as pressure and velocities. On this basis this detailed analysis of helical fluid structures was accomplished by decomposing the flow field into *complex curl-eigenfunction* modes, originally inaugurated by Lesieur [11] and called *complex-helical wave decomposition.* Lesieur initially utilized this method for easier access to a spectral Fourier) analysis of turbulent flow modes. The following brief interlude on the particulars of the helical wave decomposition for description of aspects of fluid turbulence will be time well spent as this model will resurface later when considering novel electrodynamic applications of the Beltrami vector field.

We start by constructing an orthonormal basis **a,b,c** (with **c** a unit vector along the wave vector **k**, with **a** and **b** in the two-dimensional vector space orthogonal to **k**). the significance of this ansatz is that any vector function **F**(x,y,z) is divergence-free if and only if its Fourier coefficients **F**(**k**) are orthogonal to **k** (i.e., if $\mathbf{k}\cdot\mathbf{F}(\mathbf{k}) = 0$). That is, **F**(**k**) is a linear combination of **a**(**k**) and **b**(**k**). Lesieur defines the complex-helical waves as:

$$\mathbf{V}^{+}(\mathbf{k,x}) = [\mathbf{b}(\mathbf{k}) - i\mathbf{a}(\mathbf{k})]\exp(i\mathbf{k,x}) \tag{17}$$

$$\mathbf{V}^{-}(\mathbf{k,x}) = [\mathbf{b}(\mathbf{k}) + i\mathbf{a}(\mathbf{k})]\exp(i\mathbf{k,x}), \tag{18}$$

which are eigenfunctions of the curl operator to the curl operator corresponding to the eigenvalues $|\mathbf{k}| \neq \mathbf{0}$, i.e., curl $\mathbf{V}^{+} = |\mathbf{k}|\, \mathbf{V}^{+}$, and curl $\mathbf{V}^{-} = -|\mathbf{k}|\, \mathbf{V}^{-}$. These eigenfunctions are orthogonal with respect to the inner product:

$$< \mathbf{f,g} > \; = \int \mathbf{f}\cdot\mathbf{g}^{*}\, d\mathbf{x}, \tag{19}$$

Where * denotes complex conjugation and the integration extends over all space (or a periodic box). In fact, all eigenfunctions of the curl operator, including those corresponding to the eigenvalue zero, are orthogonal with respect to the inner product (19). Moreover, the set of all linearly-independent eigenfunctions fo the curl operator form a complete set of vector functions in R3 and all eigenvalues are real. The complex helical waves span the subspace of solenoidal vector functions, for the only vector field that cannot be expanded in terms of complex-helical waves is the gradient of a potential. Thus, the divergence-free velocity field $\mathbf{u}(\mathbf{x},t)$ can be expressed in terms of complex-helical waves and the gradient of a harmonic potential:

$$\begin{aligned} \mathbf{u}(\mathbf{x},t) &= \int_{k\neq 0} u^{+}(\mathbf{k},t)\cdot\mathbf{V}^{+}(\mathbf{k,x})\, d\mathbf{k} + \int_{k\neq 0} u^{-}(\mathbf{k},t)\cdot\mathbf{V}^{-}(\mathbf{k,x})\, d\mathbf{k} + \text{grad}\,\Phi \\ &= \mathbf{u_R} + \mathbf{u_L} + \text{grad}\,\Phi. \end{aligned} \tag{20}$$

Here, the first integral ($\mathbf{u}_R$) is the projection of **u** onto the vector space spanned by all eigenfunctions corresponding to positive eigenvalues (+|**k**| **)** of the curl operator (right-handed component of **u**), Likewise, the second integral (**$\mathbf{u}_L$) is a** linear combination of eigenfunctions corresponding negative eigenvalues (-|**k**|**)** of the curl operator (*left-handed component of* **u**). Also, grad Φ is the projection of **u** onto the null space of the curl operator (zero eigenvalue of the curl operator). Since **u** is solenoidal we have div(grad Φ) = 0. Assuming that the potential part of the flow is constant at infinity, we have that grad Φ is a constant vector field and can be removed by choosing an appropriate inertial frame. For the vorticity field **w**(**x,***t*) = curl **u(x,***t*), we have a similar decomposition.

In their milestone work, Melander and Hussain found that the method of complex-helical wave decomposition was instrumental in modeling both laminar as well as turbulent shear flows associated with coherent vortical structures, and revealed much new important information about this phenomenon than had been known before through standard statistical procedures. In particular, this approach plays a crucial role in the description of the resultant intermittent fine-scale structures which accompany the core vortex. Specifically, the large-scale coherent central structure is responsible for organizing nearby fine-scale turbulence into a family of highly polarized vortex threads spun azimuthally around the coherent structure.

In addition, this method shows that the polarization alternates between adjacent threads along the column. It was found that for a localized central axisymmetric vortex, the polarized structure constituting the central column are also localized and deform slowly (compared to unpolarized structures) and behave almost like solitary waves when isolated. This is because the non-linearity in the Navier-Stokes equation, much like the findings of the Moses' research, is largely suppressed between eigenmodes of the same polarity. Thus, it is found that the rapid changes in the total vorticity field result from the superposition of two slowly deforming wave-trains moving in opposite directions with different propagation velocities. The ability of the helical wave decomposition to extract the wave packets propagating in opposite directions on a vortex allows us to view the vortex evolution in terms of the motion of the polarized wave-packets and their non-linear interaction.

Beltrami Flow as Archetypal Field Structure – A Schauberger/Beltrami Connection?

One of the underlying themes of this exposition is the suggestion that the Beltrami flow field could play an important but yet dimly suspected archetypal role in organizing matter and energy at a deeper level of nature. One indication of this might be the possible non-linear uncertainty principle cited earlier, which may be a feature exhibited by Beltrami topology in turbulent fluid media. Another could be the ability of a central fluid vortex core to organize (polarize) surrounding fine-scale threadlike turbulence, as revealed by the use of curl-eigenfunction methods. However, as we will shortly see, this possible unexplored fundamental aspect of Beltrami flow geometry turns out to be indicated in other field contexts in which it manifests, such as electrodynamics. In particular, in many empirical exhibits where Beltrami flow plays a role it will be found that phenomena have been recorded that are unexplainable by orthodox field methods. Just such a result comes from an unexpected source - V. Schauberger's work which uncovered the energetic properties of water flow. In the early part of the 20th century, Schauberger demonstrated that particular vertical patterns formed by the streamlines of natural water flow, resulted in a water quality that was pure and health promoting [12]. In water pipes which were constructed with a vertical cross-section, not only was excess energy delivered to the system (process of "implosion"[12]), but the water flowed in the pipe with negative resistance [12]. Figs. 4,5 depict one embodiment of these spiral configurations – Schauberger's so-called *longitudinal vortex*. A key observation here is that this streamline geometry is virtually identical to the sheared-helical cylindrical vector field flow pattern exhibited by the Beltrami vortex. In both of these figures, the water is colder and more dense the closer the spiral approaches the central axis. This fact cannot be accounted for by standard classical fluid dynamics, which postulates that fluid temperature is independent of the density of the medium. Certainly, most laws of mechanics prohibit the occurrence of negative resistance anyway. It becomes apparent that Schauberger's work shows possible support for considering the sheared-helical (Beltrami) vortex morphology in an archetypal context.

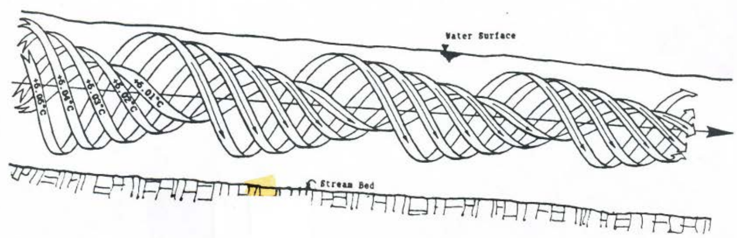


Fig. 4 **The Longitudinal Vortex**

A longitudinal vortex showing laminar flow about the central axis. The coldest water-filaments are always closest to the central axis of flow. Thermal stratification occurs even with minimal differences in water temperature. The central core water is subjected to the least turbulence and acclerates ahead, drawing the rest of the water-body in its wake.

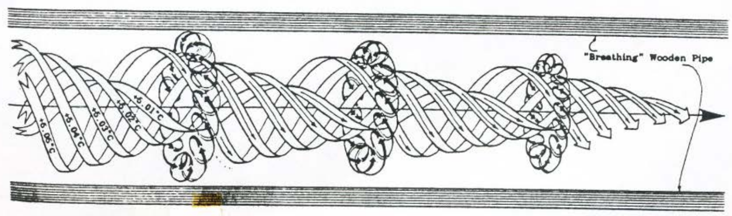


Fig. 5 **The double-spiral longitudinal vortex**

A longitudinal vortex showing the development of toroidal counter-vortices. These occur due to the interaction with the pipe-walls and have an effect similar to ball-bearings, enhancing the forward movement. Their interior rotation follows the direction of rotation and forward motion of the central vortex, whereas the direction of their exterior rotation and translatory motion are reversed. These toroidal vortices act to transfer oxygen, bacteria and other impurities to the periphery of the pipe, where, due to the accumulation of excessive oxygen, the inferior, pathogenic bacteria are destroyed and the water rendered bacteria-free.

Hypothesis of Force-Free Magnetic Fields

Up until the 20th century, helicoidal fields of the Beltrami variety had only been recognized to exist in hydrodynamic phenomena. For instance, wing-tip vortices as well as the flow engendered by atmospheric vortices such as in tornadoes, are two examples in which fluid flow approximates a force-free configuration [13]. Then, in 1901, the Norweigian physicist Birkeland observed filaments in the Earth's aurora and modeled them with his terella experiments. He postulated that the filaments were formed by the helicoidal flow of electrons around bundles of the Earth's magnetic field lines in a force-free arrangement.

Early interest in force-free magnetic fields also arose in the study of astronomy. Stars which exhibited large magnetic fields have been known to exist. In the gaseous envelopes surrounding these magnetic stars, strong magnetic fields and currents are known to exist simultaneously. Early investigations indicated that pressure gradients or gravitational fields appear to be of insufficient magnitude to counteract the reaction forces between the magnetic fields and the currents present. In 1954 Lust and Schutler [14] introduced force-free magnetic fields **(FFMF)** into a theoretical model for stellar media in order to allow intense magnetic fields to coexist with large currents in stellar matter with vanishing Lorentz force. Notice should be taken that the Lorentz force is the electrodynamic analogue of the magnus force alluded to above (see Fig. 6 and compare with Fig. 2). Taking this model for astrophysics further by postulating a Beltrami field morphology to electromagnetics, Chandrasekhar and Woltjer [15] posited that similar Lorentz force-free fields might exist which are quantified by the relation:

$$\text{curl}\,\mathbf{B} = k\,\mathbf{B}\,, \tag{21}$$

(with k a constant or function of position), in a magnetostatic field where the current density $\mathbf{j} = \text{curl}\,\mathbf{B}$, and the magnetic field induction vector **B** are everywhere parallel to each other. Similar to the hydrodynamic case,(5), currents do not have to do any work against a force-free magnetic field. For these fields to occur and to continue to exist in the envelopes of magnetic stars, it is also necessary that **FFMF** solutions be stable. Chandrasekhar and Woltjer also showed that for a given mean-square current density, exclusive of surface currents, the maximum magnetic energy per unit volume can exist in a stationary state only if the magnetic field is force free with a constant value of k. Secondly, it was also shown that for a given amount of magnetic energy, the minimum dissipation occurs for force-free fields with k

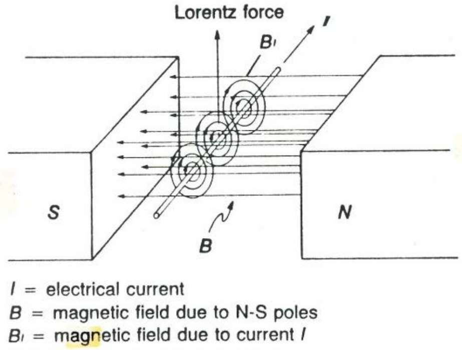
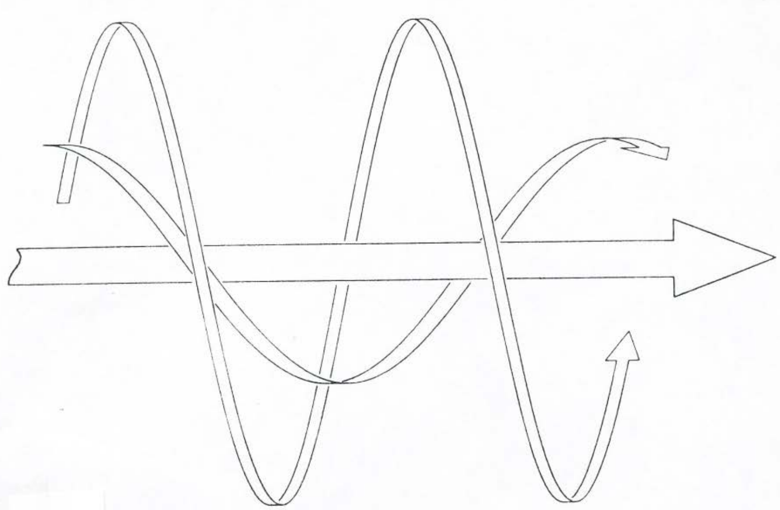


Figure 6. Illustrating electromagnetic action of the Lorentz force in a current carrying conductor

constant. Woltjer [16] further showed that for infinite conductivity, the force-free fields with constant k represent the lowest state of magnetic energy that a closed system may attain. Woltjer also demonstrated that for spherically symmetric perturbations, which are zero on the boundary, the axisymmetric fields with constant k are stable.

Additional models of **FFMF** for interstellar physics also postulated that the spiral arms of galaxies, as well as solar flares and prominences, could be constructed of such force-free fields [17]. Similar Beltrami field structures have been formulated and developed as models for many other astrophysical phenomena in recent years. For instance, the magnetic clouds ejected from the Sun which have produced major perturbations in the Earth's radiation belts during the satellite era seem to possess **FFMFs** which have budded from the solar magnetic field [18,19]. The topology of filaments and chromospheric fibrils near sunspots have also been interpreted in terms of configuration lines of force of an axisymmetric chromospheric force-free field [20]. Magnetic flux ropes in the ionosphere of Venus also appear to possess force-free topology [21]. These theoretical findings have been corroborated by satellite data on magnetic field strength in various parts of these magnetic field structures. Fig. 7 depicts the topology postulated for the **FFMF** believed common to magnetic clouds, chromospheric fibrils, and Venusian magnetic flux ropes. Immediately apparent is the signature sheared-helical Beltrami morphology for the lines of force. Notice that the magnetic field strength (thickness of field lines) increases as the central axis is approached while the outer, more nearly azimuthal field is weaker. Compare this with the Schauberger water flow geometry which is quite similar. Again, here is possibly another hint relating to an archetypal role for Beltrami morphology in nature.

Elphic and Russell: Flux Rope Observations and Models



Flux Rope Magnetic Structure

Fig. 7. Schematic representation of flux rope magnetic structure. The breadth of each arrow denotes the field strength; the central axial field is strong, while the outer, more nearly azimuthal field is weaker.

Empirical Confirmation of Beltrami FFMF in Plasma Research

Prior to 1966, the properties of **FFMFs** had only been theoretically predicted [22]. However, in the late 60's, the morphology of electromagnetic Beltrami vortices was observed during the plasma focus experiments originally designed to investigate the possibilities for plasma confinement in fusion technology. A plasma focus group, headed by W. Bostick demonstrated that the current sheath of the plasma focus is carried by contra-rotating pairs of vortex filaments which exhibit force-free morphology [23]. Under the intense discharge created in the plasma focus, it was found that these non-linear field structures cause interesting energy anomalies to occur, unaccountable by standard theory. For instance, due to the helical path followed by the current around the cylinder axis, as opposed to a linear path, these plasma vortices routinely permit the standard Alfven current limit to be surpassed, as well as transform a significant fraction of the input electrical energy into intense magnetic fields [24]. In her thought-provoking book, White [25] discusses these plasma filaments as examples which appear to violate the Second Law of Thermodynamics by increasing the energy gradient within the field, while orthodox continuum classical physics predicts the entropic dissipation of energy. Moreover, these structures violate Heaviside-Lorentz relationships by creating ion currents which capture and concentrate their own and surrounding magnetic fields. While the electron and ion currents are in the process of forming vertical filaments, a nested series of spiral paths is set up such that the generated magnetic fields cancel out in terms of in terms of their effect on the motion of the current flow. In fact, as shown in Fig. 8, all the force fields of the plasma – the local magnetic field, the current density, the fluid flow velocity, and the fluid vorticity, - are locally collinear in each filament.

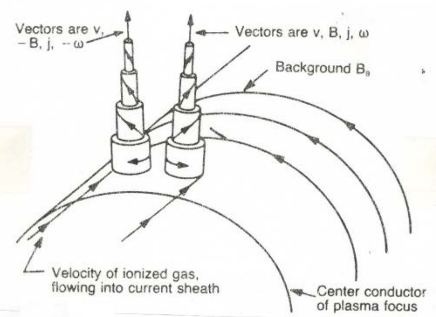


Figure 8. Dissected diagram of the vector configuration of a pair of Beltrami vortex filaments formed in the current sheath of the plasma focus. **v** is the flow velocity, **B** is the local magneitc field, **j** is the current density, ω is the vorticity, B_O is background magnetic field

In Fig. 9 it should be noted that the Poynting vector (~ **E** x **B**), which is taken to represent the power flow in an electromagnetic field, is directed parallel to the filament axis in a single vortex. Now, since this figure represents one-half of such a vortex-pair, the total Poynting vector sums to zero due to the anti-parallel magnetic fields existing in the complete structure.

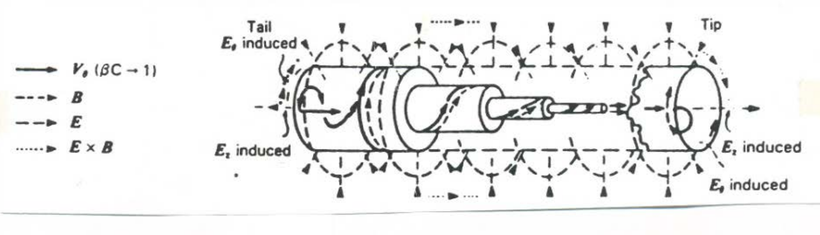


Figure 9. Morphologies of **V**, **B**, **E**, and **E** x **B** as hypothesized by Bostick for a propagating vortical-shaped electron vortex structure.

Beltrami FFMF Topology as a Model for Empirical EM Phenomena

Superconductivity is one of the best-known empirically quantified macroscopic electromagnetic phenomena whose basis is currently recognized to be quantum-mechanical. The behavior of the electric and magnetic fields under superconductivity is governed by the London equations. However, in recent times, there have been a series of papers questioning whether the originally phenomenologically theorized but now quantum-mechanically canonized London equations can be given a purely classical derivation [26]. Bostick [27,28], for instance, has claimed to show that the London equations do indeed have a classical origin that applies to superconductors and to some collisionless plasmas as well. In particular, it has been asserted that the Beltrami vortices in the plasma focus display the same paired flux-tube morphology as Type II superconductors [29,30]. Others have also pointed to this little explored connection. Frohlich [31] has shown that the hydrodynamic equations of compressible fluids, together with the London equations, lead to the macroscopic Ginzburg-Landau equation, and in the presence of many fluxoids (quantum units of flux), all relevant equations can be expressed with the aid of the velocity potential (**v**), and the macroscopic parameter (μ = electric charge/mass density), without involving either quantum phase factors or Planck's constant.

In essence, it has been asserted that Beltrami plasma vortex filaments are able to at least simulate the morphology of Type I and Type II superconductors. This occurs because the organized energy of the vortex configurations comprising the ions and electrons far exceeds the disorganized or thermal energy, and that the transition from disorganized turbulence to organized helical structures is a phase transition involving condensation without the rise of temperature. As we have pointed out, due to the feature of zero Lorentz force, the Beltrami vortex structures have less resistance than any other morphology. For ideal conditions in superconductivity, the elimination or reduction of the Lorentz force without reducing the field or the current is definitely desirable. These facts would tend to lend credence to the hypothesis of the concept that superconductivity in macroscopic classical physics can essentially exist, even though quantum effects have not been invoked to canonize the process.

Helicity of Beltrami Topology in FFMF

In all applications of the Beltrami vector field, whether in fluid or electromagnetic contexts, it is found that the existence of *helicity* is the common key. In fluid dynamics, helicity is present due to the non-zero value of the "abnormality", c. In a general context, helicity measures the topological knotted-ness of a given vector field: vorticity (**w**) in fluid dynamics, magnetic field induction (**B)** in plasma dynamics, and has been shown to be related to the mathematical Hopf invariant [32,33]. The significance of topological parameters such as helicity in engendering a key role for **FFMF** in nature. comes to the fore when considering an ideal plasma possessing small viscosity and infinite conductivity on the boundary. Here, the minimization of magnetic energy subject to the constraint of magnetic helicity conservation (invariance in the linkage of field lines), produces, through plasma self-organization a magnetic relaxation of the system into an equilibrium satisfying a Beltrami equation [34]. The following derivation showing this relationship, bears further witness to an archetypal role for the Beltrami field condition. Indeed, given the opportunity in certain environments, nature seems to have the pronounced inclination to organize itself according to force-free least-action systems.

The helicity, defined by the following relation, is an invariant for every infinitesimal flux tube surrounding a closed line of force:

$$\mathrm{H} = \int \mathbf{A}\cdot\mathbf{B}\, dv, \tag{22}$$

where **A** represents the vector potential and **B =** curl **A**., and *dv* is an infinitesimal volume element. If we minimize the magnetic energy:

$$\mathrm{W} = \frac{1}{2\mu_0}\int \mathrm{B}^2\, dv \tag{23}$$

Subject to the constraint of magnetic helicity conservation described above, then for a plasma confined by a perfectly conducting toroidal shell the equilibrium satisfies curl **B** = k **B.** From the first variation of (W –kH/2):

$$\delta\left(\mathrm{W} - \frac{\mathrm{kH}}{2}\right) = \int \frac{\mathbf{B}\cdot\delta\mathbf{B}}{\mu_0}\cdot\, dv - \frac{\mathrm{k}}{2}\int (\boldsymbol{\delta}\mathbf{A}\cdot\mathbf{B} - \mathbf{A}\cdot\delta\mathbf{B})\, dv, \tag{24}$$

Where k is a Lagrange multiplier. The first term is written as:

$$\frac{1}{\mu_0}\int \mathbf{B}\cdot\delta\mathbf{B}\,dv \;=\; \frac{1}{\mu_0}\int \mathbf{B}\cdot \mathrm{curl}\,\delta\mathbf{A}\,dv \;=\; \frac{-1}{\mu_0}\left\{\int \mathbf{B}\times\delta\mathbf{A}\cdot\mathbf{n}\,dS \;-\; \int \delta\mathbf{A}\cdot\mathrm{curl}\,\mathbf{B}\,dv\right\} \qquad (25)$$

and the second term is:

$$\int (\delta\mathbf{A}\cdot\mathbf{B} + \mathbf{A}\cdot\delta\mathbf{B})\,dv \;=\; \int(\delta\mathbf{A}\cdot\mathbf{B} + \mathbf{A}\cdot\mathrm{curl}\,\delta\mathbf{A})\,dv \;=\; 2\int(\delta\mathbf{A}\cdot\mathbf{B})\,dv \;-\; \int(\mathbf{A}\times\delta\mathbf{A})\cdot\mathbf{n}\,\boldsymbol{d}\mathrm{S}. \qquad (26)$$

Here we assume the wall is perfectly conducting. Then $\mathbf{n}\times\delta\mathbf{A} = 0$ is imposed on the wall, which corresponds to $\mathbf{n}\times\mathbf{E} = 0$, and the surface integrals of (25) and (26) vanish. Therefore (24) becomes:

$$\delta\left(\mathrm{W} - \frac{\mathrm{kH}}{2}\right) = \int \delta\mathbf{A}\cdot[\mathrm{curl}\,\mathbf{B} - \mathrm{k}\mathbf{B}]\,dv. \qquad (27)$$

For an arbitrary choice of $\delta\mathbf{A}$, (27) becomes zero when (21) is satisfied. When (27) is zero, the magnetic energy is minimized with the helicity being held constant. Thus, the state of minimum magnetic energy is a force-free equilibrium.

However, the above discussion cannot be the appropriate description of the quiescent state. In order to determine k, the invariant H for each closed field line would have to be calculated and related to its initial state. Hence, far from being universal and independent of initial conditions, the state defined by (22) depends on every detail of the initial state. To resolve this difficulty, it must be recognized that real turbulent plasmas are never perfectly conducting as in the ideal example above, but possess a certain amount of resistivity. Consequently, all topological invariants H cease to be relevant. Nevertheless, as long as the resistivity is small, the sum of all the invariants, that is, the integral of $\mathbf{A}\cdot\mathbf{B}$ over the total plasma volume is independent of any topological considerations and the need to identify particular field lines. The resulting configuration of a slightly resistive plasma after minimizing the energy subject to the constraint that total magnetic helicity be invariant is also the force-free state (21).

Transformation Properties of FFMF

As mentioned previously, misplaced focus throughout the years on the limitations inherent in the Gibbs-Heaviside vector analysis for modeling the real world, has in many ways done a disservice to formulating a proper understanding of classical vector field theory, especially when vector fields are to be associated with non-trivial topologies. In particular, many false conclusions have been repeatedly drawn, pertaining to the Beltrami **FFMF** condition by employing standard vector analysis. Such incorrect findings have in turn been partially instrumental in helping to promulgate confusion and in prejudicing physicists from considering the Beltrami vector field in any but the most superficial field contexts. One prime example clearly illustrating the confusion that has ensued, is the matter of the transformation properties of the **FFMF**. We shall consider specifically gauge invariance, parity, Lorentz-invariance and time reversal.

Originally, it was erroneously concluded that the **FFMF** condition was not invariant under parity [35]. This mistake was due to a misinterpretation caused by the ambiguity in standard vector analysis which does not clearly distinguish between a polar vector and a so-called pseudo-(or axial) vector, the latter of which is obtained as the vector product of two polar vectors (or curl of a polar vector). Later, correction of this error demonstrated the parity invariance of of the **FFMF** equation through the use of the calculus of differential forms [36]. This formalism is free from the inconsistencies noted above and is also inherently topological, independent of coordinate system or metric. In particular, it was shown [37,38] that the force-free condition (curl $\mathbf{B}$ = k $\mathbf{B}$), is parity-invariant, where k must be a pseudoscalar since it represents the ratio between a polar vector (curl $\mathbf{B}$) and an axial vector (**B**). The force-free relation is also gauge-invariant [39], and invariant under time reversal, where k is a proper pseudoscalar [40]. However, it is not necessarily Lorentz-invariant, as was shown in [41]. A summary of all correct invariance properties of the **FFMF** relation is given in [41,42].

Magnetic Field Solutions to the Force-Free Magnetic Field Equation

A divergenceless magnetic field (div $\mathbf{B}$ =0) is assumed in most classical electromagnetic field applications. Thus, this constraint is usually applied in the Beltrami **FFMF** condition (21) when determining solutions to this equation. From previous considerations, the solenoidal **FFMF** describes a Trkalian field with k=constant. Taking the curl of (21) under the constraint of solenoidal magnetic field, results in the Helmholtz equation:

$$\nabla^2\mathbf{B} \quad + \quad k^2\mathbf{B} \;=\; 0. \tag{28}$$

To solve this we consider a scalar function and its related Helmholtz equation:

$$\nabla^2\psi \quad + \quad k^2\psi \;=\; 0. \tag{29}$$

Now, from any function which satisfies (29), can be formed three independent vectors which satisfy the vector wave equation (28) [42]. They are traditionally signified by **L =** grad (ψ), **P** = curl (ψ**a**) and **T** = curl curl (ψ**a**), where **a** is an arbitrary constant vector. Thus, to find solutions to (22) we express the vector **B** as a linear superposition of the three vectors:

$$\mathbf{B} = x\,\mathrm{grad}(\psi) \quad + \quad y\,\mathrm{curl}(\psi\mathbf{a}) \quad + \quad z\,\mathrm{curl\,curl}\,(\psi\mathbf{a}), \tag{30}$$

Where x,y and z are some constants to be determined. To find the relationship between the constants, we first apply div **B** =0 to (30). This implies x = 0. Then applying curl **B** = k**B** to (30) gives:

$$\begin{aligned}\mathrm{curl}\,\mathbf{B} \;&=\; y\,\mathrm{curl\,curl}\,(\psi\mathbf{a}) \quad + \quad z\,\mathrm{curl\,curl\,curl}\,(\psi\mathbf{a})\\ &=\; y\,\mathrm{curl\,curl}\,(\psi\mathbf{a}) \quad - \; z\,\nabla^2\,\mathrm{curl}(\psi\mathbf{a}), \qquad (\text{since, curl(grad } \psi) = 0)\\ &=\; y\,\mathrm{curl\,curl}\,(\psi\mathbf{a}) \quad + \quad zk^2\,\mathrm{curl}(\psi\mathbf{a}), \qquad (\text{since } \nabla^2 = -k^2, \text{ from (29)}). \end{aligned} \tag{31}$$

Now, applying (30) again,

$$k\mathbf{B} \;=\; k\,y\,\mathrm{curl}(\psi\mathbf{a}) \quad + \quad k\,z\,\mathrm{curl\,curl}(\psi\mathbf{a}) \tag{32}$$

Equating coefficients in (31) and (32) gives y = k z. Consequently, we seek solutions of the form:

$$\mathbf{B} \;=\; k\,z\,\mathrm{curl}(\psi\mathbf{a}) \quad + \quad z\,\mathrm{curl\,curl}(\psi\mathbf{a}), \text{ or}$$

$$\mathbf{B} \;=\; B\,[k\,\mathrm{curl}\,(\psi\mathbf{a}) \quad + \quad \mathrm{curl\,curl}(\psi\mathbf{a})]. \qquad (\text{letting B = z}) \tag{33}$$

With this expression for **B**, we find the field lines for the solution assume a key geometrical relationship in addition to the previously considered axisymmetric helicoidal solutions exemplified in the vortex filaments of the plasma focus. Some researchers [43,44] have termed **P** the poloidal component and **T** the toroidal component. Accordingly,, if the equation for **B** is expressed in terms of cylindrical polar coordinates, we find that the **FFMF** solutions to (21) with k constant, are spiral field lines which lie on axisymmetrical torus surfaces, with the pitch of the spirals changing gradually from totally poloidal (surface C) to completely toroidal (surface A). Consult Fig. 10 for the associated depiction of this field morphology.

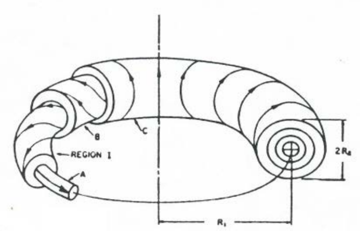


Figure 10. Illustrating the toroidal mode solution to soleniodal FFMF equation

The toroidal solution to the Beltrami equation is more than a theoretical possibility. Indeed, in addition to the contra-rotating Beltrami vortex filaments discovered by Bostick, plasma vortex-rings called *plasmoids* were also recognized experimentally. Wells [45,46] in particular, studied the plasma topology generated by a conical theta-pinch device and verified that the field geometry of these plasma vortex-rings approximated the quasi-force-free topology described above and illustrated in Fig. 10. Moreover, Chandrasekhar [10] showed that within a spherical boundary there is equipartition of the energy between poloidal and toroidal components of any force-free magnetic field.

The general solution of equation (33) in cylindrical coordinates can be written as a series of modes of the form [47]:

$$\mathbf{B} = \sum_{mn} B_{mn}\, \mathbf{b}_{mn}(r,\theta,z), \tag{34}$$

where n is a non-negative integer, and where individual modes $\mathbf{b}_{mn}$ depend upon θ and z through the phase function, $\Phi = (m\theta + nz)$. The explicit expressions for the modes generally involve a linear combination of the Bessel functions J and Neumann functions N. However, when the domain of the solution involves the axis r=0, and we restrict ourselves to an axisymmetric wave equation:

$$\frac{d}{dr}\left(r\frac{d\psi}{dr}\right) + k^2\psi = 0\,, \tag{35}$$

The solution to the scalar function is $\psi = C\, J_o(kr)$, where C is any constant.

Using this and substituting in (34) with mode m=0,n=0, and $\mathbf{a}$ =(0,0,1), we get for the field components of the magnetic induction:

$$\mathbf{B} = B_o(0,\, J_1(kr),\, J_o(kr)\,). \tag{36}$$

This is the solution originally demonstrated by Chandrasekhar and Kendall [48]. A pictorial representation of this axisymmetric **FFMF** is shown in Fig. 10.

The solution to (34) has also been found in rectangular and spherical coordinates, but due to these complicated expressions, will not be further considered here. For details, readers are referred to the excellent review work by Zaghoul and Barajas [49], as well as the comprehensive book by Marsh [50]. These references also include information about **FFMFs** when k is not constant, but a function of position or time.

Nevertheless, the solution to the **FFMF** equation in cylindrical and coordinates with certain constraints on the wave equation, most clearly illustrates the geometrical relation between the field lines. Also, the solution to the **FFMF** equation given in Beltrami’s 1889 paper and in Bjorgum and Godal, which has been experimentally verified by Wells [51],Taylor [52] and others, illustrates that this is an eigenvalue equation. It possesses a whole spectrum of solutions corresponding to various energy states with correspondingly different values of k. However, the geometry of the lowest state is that of the sheared helical structure across the cross-section of the plasma column. This is illustrated in Fig. 3 for an axisymmetric field in cylindrical coordinates, or as in Fig. 10, for the general solution in toroidal coordinates or cylindrical polar coordinates. There are two separate field configurations in the next highest energy state. At the center of the cross-section is a modified form of the first eigenvalue field, whereas the second is a helical field of opposite handedness, that is, a “reversed” field.

Regardless, in the case of the solenoidal **FFMF** with k constant (Trkalian field), the lowest state of magnetic energy which a closed system may attain always assumes the topological feature of the sheared vertical structure, either in cylindrical or vortex-ring geometry. This fact implies the stability of the Trkalian field flow. Moreover, for a plasma system in which magnetic forces are dominant, and in which there is some mechanism to dissipate the fluid motions, the **FFMF** with Trkalian flow is the only form of stable magnetic field which can decay without giving rise to material fluid motions. The solenoidal **FFMF** solution with k constant appears to be the natural end configuration.

Trkalian Field Solutions to Maxwell’s Equations

A complete set of standard time-harmonic solutions to Maxwells equations usually involve the plane wave decomposition fo the field into transverse electric (TE) and transverse magnetic (TM) parts. However, Rumsey [53] detailed a secondary method of solving the same equations which effected a decomposition of the field into left-handed And right-handed *circularly polarized* parts. For such unique field solutions to the time-harmonic Maxwell equations (ε = electric permittivity, μ = magnetic permeability):

$$\text{curl}\,\mathbf{H} = iw\varepsilon\,\mathbf{E} \tag{37a}$$

$$\text{curl}\,\mathbf{E} = iw\mu\mathbf{H}, \tag{37b}$$

instead of expressing the electric and magnetic field intensities (**E** and **H**) of equations (37ab) customarily as a superposition of plane waves, Rumsey showed how the field solutions could be expressed as circularly-polarized waves in terms of only one scalar potential function U where for instance:

$$\mathbf{E} \;=\; \mathrm{curl}\,\mathrm{curl}\,(\mathbf{z}\mathrm{U}) \;\pm\; \mathrm{k}\,\mathrm{curl}\,(\mathbf{z}\mathrm{U}), \tag{38}$$

+ or – indicating chiral circularly-polarized vectors. This is highly similar to the previously given solution to the **FFMF** equation (33), and at the same time also reminiscent of the chiral complex vector solutions found by Moses [7]. Since the wave vector is along the z-axis (38) reduces to:

$$\mathbf{E} \;=\; \mathrm{grad}\left(\frac{\mathrm{dU}}{\mathrm{dz}}\right) \pm \mathrm{k}\,\mathrm{curl}\,(\mathbf{z}\mathrm{U}) \;+\; \mathrm{k}^2(\mathbf{z}\mathrm{U}). \tag{39}$$

In the quite different physical context of fields in ideal fluid media (incompressible, inviscid, homogeneous, with external force (k) conservative) Bjorgum and Godal [2] derived a solution for the vector velocity field (**v**), for a general Trkalian flow, which bears a striking resemblance to (39):

$$\mathbf{v} \;=\; \mathrm{grad}\left(\frac{\mathrm{dH}}{\mathrm{dl}}\right) \;\pm \boldsymbol{\beta}\,\mathrm{curl}\,(\mathbf{I}\mathrm{H}) \;\;\partial + \;\; \beta(\mathbf{I}\mathrm{H}), \tag{40}$$

where **I** is the vector of propagation , and H is a scalar potential. We see that (39) and (40) are structurally equivalent if we set **I = z**, H = U, and k = β. In addition to the conclusions drawn by Rumsey in regard to the circularly-polarized solutions to (37ab), we observe that these solutions also possess the sheared vortex field topology we have alluded to in connection with any general Trkalian field solution. Whether Rumsey was aware of this connection between fluid and electrodynamics is unknown.

These facts clearly demonstrate a possible significant, albeit little-explored correlation between foundational classical electrodynamics and fluid dynamics via the general Trkalian sheared vortex topology. At any rate, a potential fruitful avenue has been opened for future exploration of this link.

Beltrami Field Relations from Time-Harmonic EM in Chiral Media

When we consider time-harmonic electrodynamics in more general media (chiral-biisotropic), A. Lakhtakia also underscored the importance of the Beltrami field condition. [54]. In particular, he found that time-harmonic EM fields in homogeneous reciprocal biisotropic media are circularly-polarized, and must be described by Beltrami vector fields.

The motivation for this work is the potential use of chiral (and maybe biisotropic) cylinders as rod antennas and scatterers. Accordingly, Lakhtakia investigated the boundary value problem relevant to the scattering of an incident oblique plane EM wave by an infinitely long homogeneous biisotropic cylinder. This medium is described by the so-called Federov representation through the monochromatic frequency-domain constitutive relations:

$$\mathbf{D} \;=\; \varepsilon\,[\mathbf{E} + \alpha\,\mathrm{curl}\,\mathbf{E}], \qquad \mathbf{B} \;=\; \mu[\mathbf{H} + \beta\,\mathrm{curl}\,\mathbf{H}], \tag{41}$$

where μ is the magnetic permeability scalar, ε is the electric permittivity scalar, and α and β are the biisotropy pseudoscalars. By applying the specialized Bohren diagonalization transformation [55] to this synthetic field composition, when **E** and **H** are substituted into the time-harmonic source-free Maxwell equations, with a time dependence assumed to be exp(*iwt):*

$$\mathrm{curl}\,\mathbf{H} \;=\; \frac{\partial \mathbf{D}}{\partial t} \quad , \qquad \mathrm{curl}\,\mathbf{E} \;=\; -\frac{\partial \mathbf{B}}{\partial \mathrm{t}} \;\; , \tag{42}$$

we get:

$$\mathbf{E} \;=\; (\mathbf{Q_1} \;+\; \mathbf{Q_2}), \qquad \mathbf{H} \;=\; i(\mathbf{Q_1}/\eta_1 \;+\; \mathbf{Q_2}/\eta_2) \tag{43}$$

where η_1 and η_2 are impedances, and $\mathbf{Q_1}$ and $\mathbf{Q_2}$ are Beltrami vector fields satisfying the relations:

$$\mathrm{curl}\,\mathbf{Q_1} \;=\; \boldsymbol{\gamma_1}\,\mathbf{Q_1}, \qquad \mathrm{curl}\,\mathbf{Q_2} \;=\; \gamma_2\,\mathbf{Q_2}, \tag{44}$$

where γ_1 and γ_2 are wave numbers which are functions of the frequency and constitutive scalar/pseudoscalar parameters.

It was found that in order to derive an accurate description of both incident and scattered EM radiation in a chiral medium, the use of Beltrami field relations is essential. Further details may be found in [56-58].

Beltrami Vector Potential Associated with TEM Standing Waves with E//B

Besides its appearance in the **FFMF** equation in plasma physics, as well as associated with time-harmonic fields in chiral media, the chiral Beltrami vector field reveals itself in recent theoretical models for classical transverse electromagnetic waves (TEM). Specifically, the existence of a general class of TEM waves has been advanced in which the electric and magnetic field vectors are parallel [59]. Interestingly, it was found that for one representation of this wave type, the magnetic vector potential (**A**) satisfies a Beltrami equation:

$$\text{curl}\,\mathbf{A} \;=\; k\mathbf{A}. \tag{45}$$

A solution of (45) is:

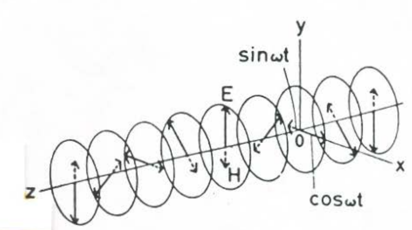
$$\mathbf{A} \;=\; a[\;\mathbf{i}\sin(kz) \;+\; \mathbf{j}\cos(kz)\;]\cos(wt), \tag{46}$$

where a is a constant. The associated electric and magnetic fields are:

$$\mathbf{E} \;=\; -\frac{1}{c}\frac{\partial\mathbf{A}}{\partial t} \;=\; \frac{wa}{c}\,[\;\mathbf{i}\sin(kz) \;+\; \mathbf{j}\cos(kz)]\sin(wt) \tag{47}$$

$$\mathbf{B} \;=\; \text{curl}\,\mathbf{A} \;=\; ka\,[\;\mathbf{i}\sin(kz) \;+\; \mathbf{j}\cos(kz)\;]\cos(wt) \tag{48}$$

One immediately sees that **E** and **B** (and also **A**) are everywhere parallel, and all are perpendicular to the propagation vector (k**z**). Consequently, every plane wave solution to (45) corresponds to two circularly polarized waves propagating oppositely to each other and combining to form a standing wave. This standing wave does not possess the standard power flow feature of linearly- or circularly-polarized waves with **E** $\perp$ **B**, since the combined Poynting vectors of the circularly-polarized waves cancel each other similar to the situation we met earlier in connection with Beltrami plasma vortex filaments. Essentially, the combination of these two waves produces a standing wave propagating non-zero magnetic helicity. In the book by Marsh [50] the relationship is shown between the helicity and energy densities for this wave as well, as the very interesting fact that any magnetostatic solution to the **FFMF** equations can be used to construct a solution to Maxwell's equations with **E**//**B**. The current author also shows the geometric relationship of these waves with respect to space and time [60]. It is noted from this analysis that, in contrast to standard linearly-polarized waves, these unique standing waves have no nodes, constant amplitude, and describe a surface od minimum area called the helicoids. See Fig.11. Moreover, such waves with these counter-intuitive properties have not only been theoretically predicted but have been experimentally realized in the so-called "twisted-mode" technique for obtaining uniform energy density in a laser cavity [61,62]. Experimental protocols to produce **E**//**B** waves are also illustrated in the paper [60].



11a. Helical distribution of linearly oscillating parallel electric and magnetic fields shown by solid and broken arrows, respectively.

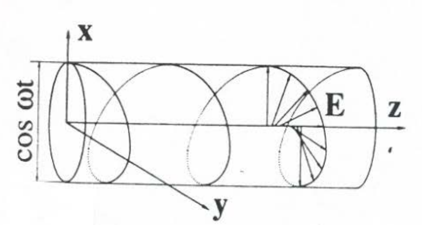


FIG 11b. Electric field distribution of a helicoidally polarized standing wave. The magnetic field **B** (not shown) is parallel to **E** and oscillates in time in quadrature with **E**.

Connections between Spinors, Hertz Potential and Beltrami Fields

a. Model of Hillion/Quinnez

In utilizing a complex three-vector (self-dual tensor) rather than a real symmetric tensor to describe the electromagnetic field, Hillion and Quinnez discussed the equivalence between the 2-spinor field and the complex electromagnetic field [63]. Using a Hertz potential [64] instead of the standard four-vector potential in this model, they derived an energy-momentum tensor out of which Beltrami-type field relations emerged. The development proceeded from the Maxwell equations in free homogeneous isotropic space:

$$\text{curl}\,\mathbf{E} = -\frac{\mu}{c}\,\partial_t\,\mathbf{H}, \qquad \text{curl}\,\mathbf{H} = \frac{\varepsilon}{c}\,\partial_t\,\mathbf{E}, \tag{49ab}$$

where $\mathbf{E}(\underline{x},t)$ and $\mathbf{H}(\underline{x},t)$ are the components of the electric and magnetic fields, ε and μ are the permittivity and permeability, respectively, c is velocity of light, ∂_t and ∂_j are the derivatives with respect to time and space and $\underline{x}$ is an arbitrary point in R3. If we introduce the complex vector:

$$\mathbf{\Lambda} = -(\sqrt{\varepsilon}\,\mathbf{B} - i\sqrt{\mu}\,\mathbf{H}), \tag{50}$$

Then the above equations (49ab) become:

$$i\,\text{curl}\,\mathbf{\Lambda} = \frac{n}{c}\,\partial_t\mathbf{\Lambda} \qquad \text{n} = (\varepsilon\mu)^{1/2}\text{, the refractive index).} \tag{51}$$

From taking the x-component of (49b) and adding *i* times the y-component of (49b), we get:

$$i\,\frac{\varepsilon}{c}\,\partial_t\,(E_x \pm i\,E_y) + \partial_z\,(H_x \pm i\,H_y) = (\partial_x \pm i\,\partial_y)\,H_z \tag{52a}$$

Doing the same with (49a) we get:

$$\partial_z\,(E_x \pm i\,E_y) - i\,\frac{\mu}{c}\,\partial_t\,(H_x \pm i\,H_y) = (\partial_x + i\,\partial_y)\,E_z \tag{52b}$$

Which becomes in terms of Λ :

$$(\partial_x + i\,\partial_y)\,\Lambda_z = (\partial_z + \frac{n}{c}\,\partial_t)(\Lambda_x - i\,\Lambda_y),$$

(53a)

$$(\partial_x - i\,\partial_y)\,\Lambda_z = (\partial_z - \frac{n}{c}\,\partial_t)(\Lambda_x - i\,\Lambda_y). \tag{53b}$$

If we now consider a set of 2-spinors $\psi^a(\underline{x}, t)$, a=1,2, with complex components $\psi^a_\alpha(\underline{x},t)$, $\alpha = 1,2$, satisfying the Pauli equation:

$$(\sigma^j\,\partial_j - \frac{n}{c}\,\partial_t)\psi^a(\underline{x},t) = 0, \qquad a=1,2, \tag{54}$$

Where σ_j are the Pauli matrices and we use the summation convention, $\sigma^j\sigma_j = \sigma_1\partial_x + \sigma_2\partial_y + \sigma_3\partial z$.
Equation (54) takes the form:

$$(\partial_x + i\,\partial_y)\psi^a{}_1 - (\partial_z + \frac{n}{c}\,\partial_t)\,\psi^a{}_2 = 0 \tag{55a}$$

$$(\partial_x - i\,\partial_y)\psi^a{}_2 + (\partial_z - \tfrac{n}{c}\,\partial_t)\psi^a{}_1 = 0. \quad (55b)$$

Now, by comparing (53ab) with (55ab) we get the following identifications:

$\Psi^1{}_1 = \Lambda_z$; $\psi^1{}_2 = \Lambda_x + i\,\Lambda_y$; $\psi^2{}_2 = -\Lambda_z$; $\psi^2{}_1 = \Lambda_x - i\,\Lambda_y$, or

$$\Lambda_x = ½(\psi^1{}_2 + \psi^2{}_1)\ ;\quad \Lambda_y = ½(\psi^1{}_2 - \psi^2{}_1)\ ;\quad \Lambda_z = ½(\psi^1{}_1 - \psi^2{}_2), \quad (56)$$

with the constraint on the spinor field:

$$\psi^1{}_1 + \psi^2{}_2 = 0. \quad (57)$$

We now introduce a complex scalar φ such that:

$$\Lambda_z = \psi^1{}_1 = (\partial^2{}_z - \left(\tfrac{n}{c}\right)^2 \partial^2{}_t)\varphi \quad (58)$$

Then using (55) and (56) we get:

$$\Lambda_x = ½[(\partial_x + i\partial_y)(\partial_z - \tfrac{n}{c}\,\partial_t)\varphi + (\partial_x - i\partial_y)(\partial_z + \tfrac{n}{c}\,\partial_t)\varphi], \quad (59a)$$

$$\Lambda_y = ½[(\partial_x + i\partial_y)(\partial_z - \tfrac{n}{c}\,\partial_t)\varphi - (\partial_x - i\partial_y)(\partial_z + \tfrac{n}{c}\,\partial_t)\varphi]. \quad (59b)$$

Epanding these we get:

$$\Lambda_x = (\partial_x\partial_z - i\tfrac{n}{c}\,\partial_y\partial_t)\varphi, \quad (60a)$$

$$\Lambda_y = (i\partial_y\partial_z - \tfrac{n}{c}\,\partial_x\partial_t)\varphi, \quad (60b)$$

Which are recognized as the x and y components of a Hertz potential $\mathbf{\Pi}$ (complex 3-vector) with the following relationship to the complex field vector:

$$\mathbf{\Lambda} = \text{curl curl}\,\mathbf{\Pi} + i\tfrac{n}{c}\partial_t\,\text{curl}\,\mathbf{\Pi}. \quad (61)$$

Thus, defining $\mathbf{\Pi}$ by $\varphi\mathbf{k}$, where k is a unit vector along the z-axis, from (58) and (59) we get:

$$\mathbf{\Pi} = \mathbf{M} + i\,\mathbf{N}, \quad (62)$$

Where **M** and **N** are, respectively the electric and magnetic Hertz vectors [65]. Thus, it follows that any electromagnetic field in a homogeneous isotropic medium in free space, away from charges and currents, can be expressed either in terms of the spinors ψ^a, or in terms of the complex Hertz vector $\mathbf{\Pi}$.

Now, in order to derive a scalar Lagrangian density, we introduce the matrix Ω and the spinors ϕ_o and $\phi^\wedge{}_o$:

$$\Omega = \begin{pmatrix} \psi 11 & \psi 21 \\ \psi 12 & \psi 22 \end{pmatrix},\qquad \phi_o = \begin{pmatrix} 1 \\ 0 \end{pmatrix},\qquad \phi^\wedge{}_o = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad (63)$$

And we define the Proca-Pauli field $\psi^a = 1,2$ by the relations $\psi^1 = \Omega\phi_o$ and $\psi^2 = \Omega\phi^\wedge{}_0$, we get the Proca-Pauli equation $\sigma^u\sigma_u\,\psi^a = 0$, a=1,2; u=0-3, which is equation (54) written in a manifestly covariant form.

Then the Lagrangian density:

$$\mathcal{L} = \tfrac{ic}{2}\textstyle\sum_{a=1}^{4}(\psi^{a+}\sigma^u\partial_u\psi^a - \partial_u\psi^{a+}\sigma^u\psi^a), \quad (64)$$

where ψ^{a+} represents the Hermitian conjugate of ψ^a, is a real scalar invariant under the proper othochronous Lorentz group $L^\uparrow{}_+$. Let j_u be the energy flow vector. Then, from (64) we get:

$$j_u = \textstyle\sum_{a=1}^{2}\psi^{a+}\sigma_u\psi^a\,,\qquad u=0\text{-}3. \quad (65)$$

the energy-momentum tensor derived from (65) is:

$$T_{uv} = \frac{ic}{2} \sum_{a=1}^{2}(\psi^{a+}\sigma_u\partial_v\psi^a - \partial_v\psi^{a+}\sigma_u\psi^a). \qquad (66)$$

In particular,:

$$T_{ov} = \frac{ic}{2} \sum_{a=1}^{2}[\, \psi^{a+}\partial_v\psi^a - (\partial_v\psi^{a+})\psi^a],$$

$$= -2nc\,(H_k\,\partial_v E^k - E_k\partial_v H^k). \qquad (67)$$

Now, the energy density T_{oo} is:

$$T_{oo} = -2nc\,(H_k\partial_o E^k - E_k\partial_o H^k). \qquad (68)$$

The surprising connection with the Beltrami relation emerges when we use (49ab) to transform (68) to:

$$T_{oo} = -c\,(\mu\, \mathbf{H}\cdot \text{curl}\, \mathbf{H} + \varepsilon\, \mathbf{E}\cdot \text{curl}\, \mathbf{E}). \qquad (69)$$

Examining this equation and returning to previous considerations, both of these terms have the form for the "abnormality" of a twice differentiable vector field. See equation (6). There we saw that if the abnormality factor of a specific vector field is non-zero, it represents a departure of that vector field from the property of having a normal congruence of surfaces. So, according to (69), the energy-momentum tensor has to do with the *vorticity* or a type of *helicity* displayed by the electromagnetic field itself. Assuming a constant value for the abnormality (k), then both fields **E** and **H** must then conform to the Beltrami equations:

$$\mathbf{H} \times \text{curl}\, \mathbf{H} = 0, \qquad \mathbf{E} \times \text{curl}\, \mathbf{E} = 0. \qquad (70)$$

This then implies:

$$\text{curl}\, \mathbf{H} = k\,\mathbf{H}, \qquad \text{curl}\, \mathbf{E} = k\mathbf{E}. \qquad (71)$$

In this application we consider EM fields in free space, consequently both **E** and **H** are solenoidal and satisfy Trkalian field relations (71). Taking the curl of (71), both vector fields satisfy Helmholtz vector wave equations:

$$\nabla^2\mathbf{H} + k^2\mathbf{H} = 0, \qquad \nabla^2\mathbf{E} + k^2\mathbf{E} = 0. \qquad (72)$$

Now, from (49ab) we get:

$$\mathbf{H} \times \frac{d\mathbf{E}}{dt} = 0 = \mathbf{E} \times \frac{d\mathbf{H}}{dt} \qquad (73a)$$

$$k\mathbf{E} = -\frac{\mu}{c}\frac{d\mathbf{H}}{dt} \qquad k\mathbf{H} = \frac{\varepsilon}{c}\frac{d\mathbf{E}}{dt}, \qquad (73b)$$

and these relations imply:

$$\frac{d^2\mathbf{H}}{dt^2} + \frac{k^2c^2}{n^2}\mathbf{H} = 0, \quad \frac{d^2\mathbf{E}}{dt^2} + \frac{k^2c^2}{n^2}\mathbf{E} = 0. \qquad (74)$$

Now, a solution of (73) and (74) takes the form:

$$\mathbf{E} = \frac{\mathbf{A}_1}{\sqrt{\varepsilon}}\cos\left(\frac{kn}{c}t\right) + \frac{\mathbf{A}_2}{\sqrt{\varepsilon}}\sin\left(\frac{kn}{c}\,t\right) \qquad (75a)$$

$$\mathbf{H} = \frac{\mathbf{A}_1}{\sqrt{\mu}}\cos\left(\frac{kn}{c}\,t\right) + \frac{\mathbf{A}_2}{\sqrt{\mu}}\sin\left(\frac{kn}{c}\,t\right), \qquad (75b)$$

As is well known, this transformation has the form of a duality transformation [66-68], where, in the above special case, the vectors $\mathbf{A_1}$ and $\mathbf{A_2}$, similar to **E** and **H**, are compelled to satisfy Trkalian field relations:

$$\text{curl}\,\mathbf{A}_1 = k\,\mathbf{A}_1, \quad \text{curl}\,\mathbf{A}_2 = k\,\mathbf{A}_2, \qquad \text{div}\,\mathbf{A}_1 = 0 = \text{div}\,\mathbf{A}_2. \tag{76}$$

Let us carefully note that the emergence of Beltrami field relations from the Hillion-Quinnez model, is not due to an ad hoc rendering of the EM field relations, but is a logically consistent result that follows from recognizing that any electromagnetic field in a homogeneous isotropic medium in free space, away from charges and currents, can be expressed either in terms of the spinors ψ^a, or in terms of the complex Hertz vector **Π.**

The possible ultimate significance these facts might have for reshaping the edifice of classical electrodynamics through the compatible incorporation of non-Abelian SU(2) symmetries such as those represented by the above spinor-Hertz potential rendering of free-space EM, can at this point only be speculated upon. Moreover, even more amazing as we shall see next, the seemingly surprising connection revealed between Beltrami vector fields and SU(2) transformation groups is not an isolated phenomenon, relevant to this one special model developed by Hillion and Quinnez. In fact, there appears to be a recurring theme of Beltrami (specifically Trkalian) field relations that emerge when EM fields are derived from a Hertz potential in the context of the general Clifford algebra formalism, which subsumes the complex unimodular group transformations that spinors encompass. Also, like many of the previous EM models we have considered that are associated with the Beltrami relation, Clifford algebra rendering of electromagnetic, such as is next examined in the Rodrigues/Vaz model, can produce fields possessing many counter-intuitive properties, similar to those of the **E** // **B** wave, and others which even appear to violate established physical laws.

b. Model of Rodrigues/Vaz

Up to now, we have examined how the Beltrami vector field relation surfaces in many electromagnetic contexts, featuring predominantly plane-wave solutions (PWS) to the free-space Maxwell equations: in conjunction with biisotropic media (Lakhtakia/Bohren), in homogeneous isotropic vacua (Hillion/Quinnez), or in the magnetostatic context exemplified by **FFMFs** associated with plasmas (Bostick, etc).

A feature common to all of the above PWS to the Maxwell equations is that they all share the attribute of exhibiting *null-field* behavior [69]. That is, for this type of field, **E** and **H** vectors are orthogonal and transverse to the propagation vector at all times and locations as well as proportional in magnitude. The factor of proportionality c, the speed of light, which is also the group velocity of the wave, is the rate at which energy is propagated through space. These relations are described by the Lorentz-Poincare field invariant scalars. Using the Hillion-Quinnez model, the two scalars are defined in the following manner, in terms of the field vectors **E** and **H**:

$$I_1 = \varepsilon\,|E|^2 - \mu\,|H|^2\,; \qquad I_2 = \mathbf{E}\cdot\mathbf{H}, \tag{77}$$

Where each of these values is invariant under any proper(orthochronous) Lorentz transformation [70]. Now, the product of the complex field vector (50) with itself is a combination of I_1 and I_2:

$$(\sqrt{\varepsilon}\,\mathbf{E} - i\sqrt{\mu}\,\mathbf{H})^2 = I_1 - i\,(\varepsilon\mu)^{1/2}\,I_2 \tag{78}$$

As we can see, it is the constraint on the spinor field, in this case (57) that leads to a zero value of I_1 and I_2 if it is also postulated that $\psi^1{}_1 = f_1g_1$, $\psi^2{}_1 = f_1g_2$, $\psi^1{}_2 = f_2g_1$, $\psi^2{}_2 = f_2g_2$, where (f_1,f_2) and (g_1,g_2) are the components of the spinors satisfying the Pauli equation. Then (57) becomes:

$$f_1g_1 + f_2g_2 = 0. \tag{79}$$

By substituting the previous definition for these spinors into (56) and using (79) it follows that Λ_j is a null vector (e.g., $\Lambda_j\Lambda^j = 0$), which, according to (78), implies $I_1 = I_2 = 0$.

Similar to the Hillion/Quinnez model, Rodrigues and Vaz defined an EM field that is a function of a specific Hertz potential:

$$\Pi = \emptyset(\mathbf{x})\,\exp(\gamma_5\Omega t)\gamma_1\gamma_2, \tag{80}$$

Where γ_1 and γ_2 are two of the four basis vectors $\gamma_0,\gamma_1,\gamma_2,\gamma_3$, in the so-called spacetime algebra $Cl(1,3)$, which satisfy the commutation relations $\gamma_u\gamma_v + \gamma_v\gamma_u = 2n_{uv}$, and $n_{uv} = \text{diag}(1,-1,-1,-1)$, u,v =0-3 and $\gamma_5 = \gamma_0\gamma_1\gamma_2\gamma_3$. In this model, the electromagnetic field tensor F^{uv} is represented by a two-form *F*, where:

$$F = ½F^{uv}\gamma_u\gamma_v. \tag{81}$$

In this model c, the velocity of light is equal to 1, and the invariants of the EM field are obtained from:

$$F^2 = F\cdot F + F\wedge F \quad \text{where,}$$

$$I_1 = F\cdot F = ½F^{uv}, \qquad I_2 = F\wedge F = -\gamma_5 F^{uv}F_{\alpha\beta}\, e_{uv\alpha\beta} \tag{82}$$

Where $e_{uv\alpha\beta}$ is the anisymmetric Levi-Civita symbol. These assume the form of the familiar invariant expressions once we recognize the even sub-algebra of the Clifford algebra $Cl(1,3) = Cl(3,0)$, the Pauli algebra, in which the pseudoscalar $\mathbf{i} = \sigma_1\sigma_2\sigma_3 = \gamma_0\gamma_1\gamma_2\gamma_3$. In the Pauli algebra we have $F = \mathbf{E} + \mathbf{i}\,\mathbf{B}$, so that:

$$F^2 = (|\mathbf{E}|^2 - |\mathbf{B}|^2) + 2\mathbf{i}\,\mathbf{E}\cdot\mathbf{B} = F\cdot F + F\wedge F \tag{83}$$

For their Hertz potential, Rodrigues and Vaz chose the factor $(t, \mathbf{x}) = \emptyset(\mathbf{x})\exp(\gamma_5\Omega t)$. Now, since $\mathbf{\Pi}$ satisfies the wave equation, we conclude that the factor $\boldsymbol{\phi}(\mathbf{x})$ in turn satisfies the Helmholtz equation:

$$\nabla^2\boldsymbol{\phi}(\mathbf{x}) + \Omega^2\boldsymbol{\phi}(\mathrm{x}) = 0. \tag{84}$$

In this case we consider the simplest solutions of (84) in spherical coordinates:

$$\emptyset(\mathbf{x}) = C\frac{\sin(\Omega r)}{r}, \quad r^2 = x^2 + y^2 + z^2. \tag{85}$$

Once again, using the Pauli algebra we express the Hertz potential as a sum of its electric and magnetic parts:

$$\mathbf{\Pi} = \mathbf{M} + \mathbf{i}\,\mathbf{N}. \tag{86}$$

In terms of these vectors the electric and magnetic field vectors are expressed as:

$$\mathbf{E} = -\partial_0(\text{curl}\,\mathbf{N}) + \text{curl curl}\,\mathbf{M}; \quad \mathbf{B} = -\partial_0(\text{curl}\,\mathbf{M}) - \text{curl curl}\,\mathbf{N}) \tag{87}$$

Using (85),(80)(86) and substituting into (87), we get for the resulting two-form F_0:

$$\begin{aligned} F_0 = \frac{C}{r^3}[& \sin(\Omega t)(\alpha\Omega r \sin\theta\sin\varphi - \beta\sin\theta\cos\theta\cos\varphi)\gamma_0\gamma_1 \\ & - \sin(\Omega t)(\alpha\Omega r\sin\theta\cos\varphi + \beta\sin\theta\cos\theta\sin\varphi)\gamma_0\gamma_2 \\ & + \sin(\Omega t)(\beta\sin^2\theta - 2\alpha)\gamma^0\gamma^3 + \cos(\Omega t)(\beta\sin^2\theta - 2\alpha)\gamma_1\gamma_2 \\ & + \cos(\Omega t)(\beta\sin\theta\cos\theta\sin\varphi + \alpha\Omega r\sin\theta\cos\varphi)\gamma_1\gamma_3 \\ & + \cos(\Omega t)(-\beta\sin\theta\cos\theta\cos\varphi + \alpha\Omega r\sin\theta\sin\varphi)\gamma_2\gamma_3], \end{aligned} \tag{88}$$

Where $\alpha = \Omega r\cos(\Omega r) - \sin(\Omega r)$ and $\beta = 3\alpha + \Omega^2 r^2\sin(\Omega r)$. Observe that F_0 is regular at the origin and vanishes at infinity. Rewriting the solution using the Pauli algebra:

$$F_0 = \mathbf{E}_0 + \mathbf{i}\,\mathbf{B}_0, \tag{89}$$

we get the result:

$$\mathbf{E}_0 = \mathbf{W}\mathbf{sin}(\Omega t), \qquad \mathbf{B}_0 = \mathbf{W}\,\mathbf{cos}(\Omega t), \tag{90}$$

with:

$$\mathbf{W} = -C\left[\frac{\alpha\Omega y}{r^3} - \frac{\beta xz}{r^5}, \; \frac{-\alpha\Omega x}{r^3} - \frac{\beta yz}{r^5}, \; \frac{\beta(x^2+y^2)}{r^5} - \frac{2\alpha}{r^3}\right].$$

We verify that div $\mathbf{W} = 0$, div $\mathbf{E}_0 = 0$, div $\mathbf{B}_0 = 0$, curl $\mathbf{E}_0 + \partial_t\mathbf{B}_0 = 0$, curl $\mathbf{B}_0 - \partial_t\mathbf{E}_0 = 0$, and the key relation:

$$\text{curl}\,\mathbf{W} = \Omega\mathbf{W}. \tag{91}$$

This is clearly a Beltrami equation, but what is more amazing is that the field result (88) describes a solution to the free-space Maxwell equations which, in contrast to standard PWS, the electric ($\mathbf{E}_0$) and magnetic ($\mathbf{B}_0$) vectors are parallel [e.g., $\mathbf{E}_0 \times \mathbf{B}_0 = 0$, where $\mathbf{E}_0 \times \mathbf{B}_0 = -\mathbf{i}(\mathbf{E}_0 \wedge \mathbf{B}_0)$], the signal (group) velocity of the wave is *subluminal* (v<c), the field invariants are *non-null*, and as (91) clearly shows, this wave is not transverse but possesses *longitudinal* components. Moreover, Rodrigues and Vaz found similar solutions to the free-space equations which describe a *superluminal* (v>c) situation [71].

Consequently, much like the Beltrami vortex filaments discussed earlier in conjunction with the magnetostatic **FFMF**, the Beltrami vector relations associated with non-luminal solutions to the free-space Maxwell equations, are directly related to physical classical field phenomena currently unexplainable by accepted paradigms. For instance, such non PWS of the free-space Maxwell equations are direct violations of the sacrosanct principle of special relativity [72], as well as exhibit other counter-intuitive properties. Yet, even more extraordinary, these non PWS are not only theoretical possibilities, but have been demonstrated to exist empirically in the form of the so-called *evanescent-mode* propagation of electromagnetic energy [72-76].

Although SU(2) groups associated with spinors are not directly incorporated into the Rodrigues/Vaz model to describe the EM field, they surprisingly turn out to be implied as an integral part of this edifice, in order to describe super- and subluminal waves with non-null behavior. In this regard, the EM field two-form is shown to be a function of the so-called Dirac-Hestenes spinors [77] through the relation:

$$F = \Psi\gamma_1\gamma_2\Psi^{\sim}, \tag{92}$$

where $\Psi^{\sim}$ represents the spinor obtained by reversion of the original spinor Ψ [78]. To see how (92) comes about, we need to understand the meaning of an *extremal* EM field. The latter is field for which the electric[magnetic] field vanishes and the magnetic[electric] field is parallel to a given spatial direction. Now, there is a well-known proven theorem discovered by Rainich [79] and reconsidered by Misner and Wheeler [80,81], that at any point of spacetime, any non-null ($F^2 \neq 0$) electromagnetic field can be transformed into an extremal field by performing a Lorentz transformation combined with a so-called *duality* transformation. A duality-transformed electromagnetic field can be most simply described as the product of the original field and the *quasi-scalar* factor (sum of scalar plus pseudoscalar): $F' = \exp(\gamma_5\lambda)F$, where λ is the "angle of rotation" in "duality space". It is known that the two Lorentz-Poincare field invariants I_1 and I_2 do not remain individually constant under a duality-rotation [82], although the combination $[(I_1)^2 + (I_2)^2]$ will stay fixed.

The relative significance of the role played by the Beltrami relation (91) in formulating this SU(2) structure of *non-luminal* electrodynamics is at present uncertain. Nevertheless, there is enough evidence to suggest that the Beltrami (specifically Trkalian) vector fields might possibly be intimately associated with higher symmetry field physics, and related to multiply-connected topologies due to their intrinsic non-zero helicity. On efinal additional study, in which longitudinal components also play a key role, indicates that this might be the case is illustrated below.

Evans/Vigier Longitudinal B(3) Field and Trkalian Vector Fields

Developed over the past decade, concurrently with both Hillion/Quinnez and Rodrigues/Vaz SU(2) EM field models, but based upon a different non-Abelian gauge group, is the so-called Evans/Vigier longitudinal B(3) field representation [89-93]. In this model, a Yang-Mills gauge field theory [94] with an internal O(3) gauge field symmetry [95] is invoked to account for various magneto-optical effects which are claimed to be a function of a third magnetic field vector component that has been termed $\mathbf{B}(3)$. O of the central theorems of O(3) electrodynamics is the $\mathbf{B}$-cyclic theorem:

$$\mathbf{B}_{(1)} \times \mathbf{B}_{(2)} = i\mathbf{B}_{(0)}\mathbf{B}_{(3)}^{*}, \tag{93}$$

A conjugate product which relates three basic magnetic field components in vacuo defined as:

$$\mathbf{B}_{(1)} = \frac{\mathbf{B}^{(0)}}{\sqrt{2}}(i\,\mathbf{i} + \mathbf{j})\exp(i\,\emptyset) \tag{93a}$$

$$\mathbf{B}_{(2)} = \frac{\mathrm{B}^{(0)}}{\sqrt{2}}(-i\,\mathbf{i} + \mathbf{j})\exp(-i\,\emptyset) \tag{93b}$$

$$\mathbf{B}_{(3)} = \mathbf{B}_{(0)}\mathbf{k}, \tag{93c}$$

where $\emptyset = \omega t - kz$, a phase factor, and **i, j, k** are the three unit vectors in the direction of the axes x, y, and z, respectively. Although the existence of the $\mathbf{B}_{(3)}$ field has been a subject of controversy both pro and con over recent years, Evans recently claimed [96] that these magnetic field components encompassed by the relations (93a-c), along with the electric field components as well as the components of the magnetic vector potential ($\mathbf{A}$), are themselves components of a Beltrami-Trkalian vector field relations (assuming the Coulomb gauge div $\mathbf{A} = 0$). This is readily verified in the case of (93ab), since they present the form of the circularly-polarized solution to the Moses eigenfunctions of the curl operator we have discussed formerly in connection with turbulence in fluid dynamics.

Associated with the above developments, is the increasing importance given to hypercomplex formalisms for modeling the symmetries in elementary particle physics and quantum vacuum morphology. As discussed in former papers [97,98], the author believe that the most appropriate algebra for describing a hypothesized vortical structure for quantum-level singularities, as well as their macroscopic counterparts (Beltrami-type fields), is the *biquaternion* algebra (hypercomplex numbers of order 8) – the Clifford algebra of order 3, represented by the Pauli algebra *Cl*(3,0) such as previously examined in the Rodrigues/Vaz model. For instance, it is known that in a macroscopic Euclidean context, biquaternions are required to describe the kinematics/dynamics for the most general twisting (screw) movement of a rigid body in space [99,100]. It is therefore suggested that the most suitable formalism for *screw-type* EM fields of the Beltrami variety should transcend a traditional vectorial treatment, encompassing a *para-vectorial* hypercomplex formalism akin to the Clifford (Dirac) algebra used effectively to describe the electron spin in a relativistic context [101].

It is a conclusion in this regard that the founders of vector field analysis were remiss in failing to take into consideration account of the significance of the Beltrami field topology in addition to the traditional solenoidal, lamellar and complex-lamellar fields. An inclusion of the thorough examination of the Beltrami condition in the development of the vector calculus, would possibly have brought attention to the important intimate association of this field configuration with non-Abelian mathematical structure. If the history of vector analysis had taken this path, it is possible that the architects of vector field theory and classical electrodynamics, would not have been so quick to indiscriminately sever its connection from the natural quaternion-based foundation. Perhaps the recent work by Hillion/Quinnez, Rodrigues/Vaz, Evans, etc. [102,103] showing the necessity of considering non-Abelian models in electromagnetism, will be instrumental in helping to set the future of classical EM theory and vector field theory in general, on a firmer foundation.

Recent Research: Implementing Beltrami EM Fields for Possible Engineering Protocols

Pursuant to much of the above speculations on connection with Beltrami field morphology to higher-symmetry electrodynamics, Greek physicist T. Raptis has done yeoman's work into the new millennium, over the last half-decade, in various papers, reports, etc, in bringing to light countless inspired possible unplumbed applications of the Beltrami field condition in various practical engineering protocols. With the rare combined cutting-edge creative sensibilities and rigor of a frontier theoretical physicist and practicing seasoned electrical engineer, he has virtually singlehandedly advanced Beltrami vector field research in many significant ways. First, his unprecedented demonstration that special eigenfunctions of the curl operator can be constructed from *any* general vector function [104, Part I] is, in this researcher's opinion a tour-de-force that could have vast ramifications not only in further sharpening the foundations of classical vector field theory, but due to its intimate connection with higher projective Moebius transformations, can possibly ultimately demonstrate the general force-free vectorial condition as essential in explicating energetic electromagnetic fields at the vacuum level of nature. Indeed, associated with this development is the long-standing recognition of the invariance of the Maxwell equations under the full 15-parameter conformal group, which of course subsumes the standard Lorentz transformations used to relativistically describe the velocity of light as independent of its source. Raptis, in his investigations, now has opened up the possibility to consider standard EM radiation as a 'deviation' from a perfect projectivity of empty space. This implies a complementary covariance principle interpreted as to suggest that every point of our "real" space can be conceived as a special projection from a much larger, possibly higher-dimensional universe. Indeed, using this general factorized dyadic continuous frame transformation, Raptis claims that understanding of both Maxwell and Yang-Mills equations could be reduced to an appropriate generalization of the Navier-Stokes equations, describing a self-quantizing relativistic superfluid structure of a primordial non-linear vacuum. Moreover, this specialized dyadic transformation unfolding this new class of Beltrami fields, is an approach which converts every vector field to a "gauge" field in a suitably "shaped" space. This, in turn, suggests the real, physical existence of the magnetic vector potential, without appeal to standard quantum codification of the field as is the case with certain phenomena like the Aharonov-Bohm effect.

Guided by this purely classical argument of vector field analysis as applied to electrodynamics, in [104, Part II], Raptis deals with more complex constructions based upon special types of magnetization. Considering the case of both and temporal modulation of helicity content of EM field solutions in Part I which may vary locally. Accordingly, he proposes a methodology and thorough field configuration analysis for a feasibility study for the purpose of developing a certain category of apparatus based on a particular formation of magnetic boundary conditions to shape the magnetic field in the form of a Beltrami field which can then be excited by an additional time-dependent excitation. Featured prominently in this embodiment is a special arrangement of magnetization current known as a *Halbach Array*, a class of magnetic cylinders particularly designed for use in high energy physics. This ambitious engineering protocol, which Raptis acknowledges has never been attempted, alluding to some thorny engineering problems which must be addressed, seeks to modulate the flux lines in each section of the cylinder into a helical shape characterizing Beltrami flows. A helicity modulation device for self-dual EM waves is also described in [105], which, through a Beltrami EM field analysis using a Hertz potential ansatz, a configuration comprised of an electric dipole/current loop apparatus is enumerated, which it is claimed might afford a novel fieldless EM communication device utilizing only the electromagnetic potentials.

In [105], he further investigates the theoretical properties of EM fields with $\mathbf{E}//\mathbf{B}$ in a hypothetical anisotropic medium with varying susceptibility. He considers a spherical region where a varying charge density followed by the associated polarization current exists and the general form of the solutions of Maxwell's equations are analyzed under the conditions that the electric field is a Beltrami field. Finally, simplified multipole solutions for such fields are presented, from which is derived the generalized susceptibility and permeability of the enclosing space. In a completely original treatment which invokes the nearly forgotten Ballabh dual curl-eigenfunction ansatz, in association with a vector-spherical harmonic analysis of the solutions, it is shown that the effective permittivity and the associated refractive index vary with the inverse radius and frequency. This implies that the equivalent velocity of light inside such a medium would analogously slow down as the center of the spherical region is approached. Surprisingly, this leads to an analogy with gravitational interactions; general relativistic arguments then become relevant and the amount of electromagnetic energy stored in the medium may become self-gravitating.

Raptis follows a similar development in [106], where, through a vector spherical harmonic treatment, demonstrates a surprising previously unnoticed possibly significant connection between Beltrami fields and the EM transmission line equations. In [104 Part III], his power-point presentation [107] and in his most recent offering [108], Raptis takes on the relatively little-explored subject of the possibility of electrovacuum fields in principle compatible with Maxwellian field theory, employing curl eigenfunctions with <u>non-constant</u> eigenvalues and non-Hertzian modes of linear wave equations. His conclusion is that such an extended class is associable with a set of underlying scalar potentials that might support localized solutions as interference patterns that could have a connection with some old and recent electromagnetic mass theories. He also cites the recent related development of Meta-fluid hydrodynamics, which, although in its infancy, could be the framework for incorporating such developments if they can be supported by experimentation on the creation of similar field structures in plasma or elsewhere.

Conclusions and Prospects

Both theoretical and empirical investigations are necessary in any study of the Beltrami field relation, since at present much more is unknown than is known about the phenomenon of Beltrami electromagnetic fields. Consequently, it is hoped that this present exploration will spur on specialists in the technical arena to focus their attention on the mysteries surrounding **FFMFs** and vortices in general – especially the properties attributed to plasma vortex filaments which apparently demonstrate a violation of the various established laws of thermodynamics. To this end, we have reviewed much past literature on the **FFMF**, and Beltrami flow fields in general, underscoring key established characteristics of these special vector fields. Beltrami field anomalies associated with the plasma configurations are in this author's opinion, not to be viewed as merely a fluke or trivial footnote in the annals of empirical science, but represent a profound enigma whose resolution may be intimately related to the very structural foundations of classical electrodynamics itself**.** Our second focus therefore, has been of a more conjectural nature, examining Beltrami vortex fields and their specialized Trkalian modes, as a possible foundation for a more expansive view of electromagnetism which incorporates a higher-symmetry non-Abelian structural edifice.

In accordance with the above notions, the current paper has focused on a visual approach with a primarily pictorial representation of Beltrami fields and their relatives. It is hoped that this emphasis on intuitive pictures will perhaps inspire researchers of fundamental field physics to consider the benefits of a return to more concrete theoretical models in the future. Certainly, in a macroscopic context, the anomalies found associated with classical vortex study applied to empirical exhibits in plasma focus entities, tornadoes and astrophysical phenomena, need reexamination.

The limited scope and space allotted to this current study, precludes a detailed investigation into all aspects of the Beltrami vector field and its applications in science. Much more needs to be done and we have barely scratched the surface of this subject. For readers who would like to further skirmish on the frontiers searching for the possibly unplumbed secrets underlying Beltrami vector fields and their value to science, the following pertinent extra reference studies are recommended.

Z. Yoshida has been instrumental in further uncovering the mathematical properties of the curl-eigenfunctions, the solution to the force-free equations, and the importance of topology in dealing with vector fields possessing non-zero helicity [109-111]. M. MacCleod used Moses curl-eigenfuncions to describe **FFMFs**, showing such fields can be defined entirely by the value of their curl transform on the unit hemisphere in transform space [112,113]. This change of viewpoint suggests an orderly approach to the classification of the properties of such fields. P. Baldwin has done some significant work exploring the properties of the solution to the Trkalian field equation for complex-dual vector fields, in terms of Monge-Clebsch potentials [114,115]. R. Kiehn has underscored the importance of topology in modeling the evolution of vector one-form potentials of rank ≥ 3. Using Cartan's calculus of differential forms and the fact that vector fields of positive helicity (such as Beltrami fields) correspond to one-form potentials of rank at least three (Pfaff dimension of 3 or 4), Kiehn has explored both turbulence in fluid dynamics and thermodynamic irreversibility for one-form potential fields of rank 4 [116], as well as helicity-torsion properties of unique EM waves with non-zero value for Lorentz-Poincare field invariants [117]. D.R. Wells [118] and T. Waite have used the Beltrami-Trkalian field relations to describe extended elementary particles [Waite] and astrophysical plasmas [Wells], whose quantum dynamics are a function of continuous vector fields. Waite, in particular [119], and in association with Barut and Zeni [120], has shown how solitonic behavior of pure electromagnetic particles (PEP), such as electrons, can arise from the assumption of a continuous fluid-like primordial aether whose dynamics conform to a Beltrami-Trkalian field relation. The toroidal topology depicted in Fig. 10 describes this aether-like fluid precisely. Considering the possible further connection of EM Hertz-Debye potentials to Beltrami fields, Benn and Kress [121] have applied the generalized Hertz-Debye potential scheme to the force-free problem, by showing that a non-constant eigenvalue of the curl operator produces an equivalent curved-space source-free Maxwell equation set. Martinez [122] used differential forms to show a previously unnoticed important connection with force-free fields and minimal surfaces. Kaiser [123] has demonstrated a Beltrami field relation in his development of scalar pulsed-beam EM wavelets associated with twisting null-congruences. The wavelets are necessarily a complex function of position, which figures significantly in their topology/geometry. It is found, for instance that the curl-eigenfield eigenvalues are also a complex function of position. Considering the possible higher algebraic symmetries associated with Beltrami fields, A. Shaarawi [124] has produced a 'charge-current' wave theory using Clifford algebra ansatz, which incorporates a Maxwell-like field equation that remarkably involves a curl-eigenfunction equation as a solution for the particle aspect of the field. Beltrami field fluid dynamics research into turbulence has been underscored in the so-called chaotic ABC flows [125,126], and Viktor Trkal's original seminal paper on incompressible fluid dynamics and curl-eignefunction with constant 'abnormality', has been translated and reprinted in [127]. Additional references concerning research into **FFMF** are those by Freire [128] and Vainshtein [129]. Key properties of Beltrami fields in hydrodynamics and magneto-hydrodynamics are indicated by Dritschel [130], McLaughlin & Pironneau [131], Montgomery et al. [132], and Marris [133-135]. Finally, Bjorgum has written a seminal paper on 3-dimensional Trkalian flows as solutions to the non-linear hydrodynamical equations [136].

With all these exhibits of the Beltrami vector field in nature previously expounded upon, some of which appear to emerge as a total surprise out of disparate empirical applications, we now return to examine in greater depth, the original thesis of this paper. This is the suggestion that the Beltrami-Trkalian vector field might possibly be an archetypal structure of field morphology which is universal and therefore ubiquitous throughout nature. One feature which would tend to support this tenet is the fact that this field structure, in whatever topological context, exemplifies the characteristics of a perfect trade-off in regards to associated aspects of field morphology – not only between kinematical parameters exhibited through the connection of geometrical relationships, but also in regards to the dynamic relationship between the two modes (poloidal) and (toroidal) in the toroidal representation (Fig. 10). Kinematically, there is recognized a 'symbiotic' push-pull feedback relationship that is characteristic of no other vector field, between the distance from the axis of the velocity vector tangent to the helical field lines in the sheared-helical configuration (Fig. 3), and the inclination angle of this vector's normal plane from the vortex axis. Referring to Fig. 12 for pictorial clarification, we observe that as the radius from the central axis increases from a to b, the corresponding angle of the normal plane with respect to the vertical axis decreases from α to β. Thus, with respect to the equally-pitched helices, $p = a \tan \alpha = b \tan \beta$, given the pitch of the helices as p, and inclination angle λ, we have $p = r \tan \lambda$. Thus, every axisymmetric (Fig. 3) [or topologically equivalent toroidal (Fig. 10)] Beltrami vector field system, determines a helicoidal velocity field describing motion on an infinite set of coaxial helices, with the limiting motions of translation [toroidal axial rotation] where $r = 0$, $p = 0$ along, and pure rotation [poloidal motion] where $r = \infty, p = \infty$, around the central [toroidal] axis. In fact, as the author has shown in [1], that the helical axisymmetric solution attributed to the lowest energy mode of the Beltrami **FFMF**, is similar to the helicoidal velocity/force-field associated with the structure known from 19th century antiquarian geometrical mechanics as the *screw-field,* generated from projective line geometry – also known as the *linear line-complex*. In the book previously noted [25], White also noted this possible correspondence. Note should also be taken contrasting this observation with the similar conjectures of Raptis in [104 Part I] stipulating that mechanics of the non-linear primordial vacuum might also be due to a higher-dimensional projective edifice. Here, Raptis' universal "projective point" (whose coordinates are formed from the exterior product between respective pairs of four homogeneous coordinates), would have all 6 Plucker coordinates zero.

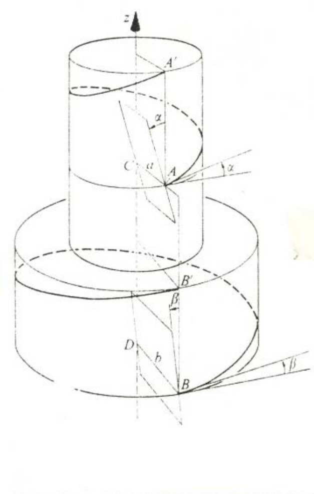


Figure 12. Two equal -pitched helices (radii a and b) of the same coaxial system. The planes at A and B are normal to them.

Also, in regards to the unique property of Trkalian fields in which all successive curls of such a field are also Trkalian fields, suggests a unique permanence or resiliency of field structure associated with no other vector field, making it rich with promise for application to fundamental vacuum EM structures. The related property in Trkalian field morphology, that both poloidal and toroidal modes *feedback* on each other in a push-pull dynamics with equipartition of mode-energy [23] in an infinite loop through the curl operation, also suggests this possibility. The Trakalian feature of chiral modes of inherent non-zero helicity, night have a direct application to explicating the associated invariances already established in fundamental field and particle physics, in which helicity/spin play a predominant role. However, such an investigation, as well as further insight into possible connection between Beltrami fields and multiply-connected topologies other than toroidal, as well as non-Abelian field symmetries, is beyond the scope of the present paper and await further research.

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