On the $\mathbf{W}^{\pm}$ and $\mathbf{Z}^0$ Masses

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Abstract

Scalar and vector fields are coupled in a gauge invariant manner, such as to form massive vector fields. The nonlinear equations of motion admit transverse and longitudinal solutions for the $W^{\pm}$ and Z^0 bosons. Total energy and momentum are calculated, and this determines the mass ratio m_W/m_Z. There is no vacuum expectation value in this work.
PACS: 11.15.-q 12.60.Cn

1. Introduction

In the standard electroweak model, a uniform "condensate" is imposed upon the coupled scalar-vector system. This gives rise to the linear wave equation. Here, no such condensate is invoked. Instead, the coupled system yields the non-linear cubic wave equation. This equation admits exact solutions which have well-defined mass.

The U(1) model of scalar electrodynamics has been used to create massive photons.[1] However, it was only with the advent of electroweak theory that the weak constants, g and g', were thought to play a similar role. In this paper, the weak coupling between scalar and vector fields is shown to generate mass. The coupling terms occur in the standard electroweak Lagrangian. Transverse and longitudinal solutions are found for the $W^{\pm}$ and Z^0 bosons. Energy and momentum are shown to be conserved, and the total energy and momentum are calculated. The mass ratio m_W/m_Z emerges, during the course of this calculation.

The paper begins by revisiting (and revising) the U(1) model. Most of the analysis carries over to the electroweak theory.

2. $U(1)$: Equations of Motion

The U(1) Lagrangian is [2]

$$L = L(A) + L(\Phi) + L(k) \tag{1}$$

where

$$\begin{aligned} L(A) &= -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} && (2) \\ L(\Phi) &= g^{\mu\nu} (D_\mu \Phi)^* (D_\nu \Phi) && (3) \\ L(k) &= -\frac{k^2}{6g^2}\, g^{\mu\nu} k_\mu k_\nu && (4) \end{aligned}$$

The constant term $L(k)$ does not enter the field equations, but it will contribute to the energy tensor. The U(1) covariant derivatives are

$$\begin{aligned} F_{\mu\nu} &= \partial_\mu A_\nu - \partial_\nu A_\mu && (5) \\ D_\mu \Phi &= \partial_\mu \Phi + igA_\mu \Phi && (6) \end{aligned}$$

The functional derivatives

$$\begin{aligned}
\frac{\partial L}{\partial(\partial_\mu A_\nu)} &= -F^{\mu\nu} && (7)\\
\frac{\partial L}{\partial A_\nu} &= 2g^2 A^\nu \Phi^* \Phi + ig\{(\partial^\nu \Phi^*)\Phi - \Phi^* \partial^\nu \Phi\} && (8)\\
\frac{\partial L}{\partial(\partial_\mu \Phi^*)} &= \partial^\mu \Phi + igA^\mu \Phi && (9)\\
\frac{\partial L}{\partial \Phi^*} &= g^2 A_\mu A^\mu \Phi - igA^\mu \partial_\mu \Phi && (10)
\end{aligned}$$

yield coupled equations of motion

$$-\partial_\mu F^{\mu\nu} = 2g^2 A^\nu \Phi^* \Phi + ig\{(\partial^\nu \Phi^*)\Phi - \Phi^* \partial^\nu \Phi\} \tag{11}$$

and

$$\partial_\mu \partial^\mu \Phi = g^2 A_\mu A^\mu \Phi - ig\{(\partial_\mu A^\mu)\Phi + 2A^\mu \partial_\mu \Phi\} \tag{12}$$

The U(1) gauge invariance allows the transformation

$$\Phi = \frac{1}{\sqrt{2}}(\phi + i\psi) \longrightarrow \frac{1}{\sqrt{2}}\phi \tag{13}$$

where $\phi(x)$ is a real function.[3] In this unitary gauge, the equations of motion become

$$\begin{aligned}
\partial_\mu F^{\mu\nu} + g^2 A^\nu \phi^2 = 0 && (14)\\
\partial_\mu \partial^\mu \phi - g^2 A_\mu A^\mu \phi = 0 && (15)\\
(\partial_\mu A^\mu)\phi + 2A^\mu \partial_\mu \phi = 0 && (16)
\end{aligned}$$

Since $\partial_\mu \partial_\nu F^{\mu\nu} \equiv 0$, it follows from (14) that

$$\partial_\mu (A^\mu \phi^2) = 0 \tag{17}$$

which agrees with (16). These equations admit traveling solutions, if A^μ is a polarized vector. In this case,

$$A^\mu(u) = \epsilon^\mu f(u) \tag{18}$$

where the argument $u = -k_\mu x^\mu$, and ϵ^μ is a real polarization vector.[1] Such vectors satisfy

$$\partial_\mu A^\mu(u) = -k_\mu \epsilon^\mu \frac{df(u)}{du} = 0 \tag{19}$$

since $k_\mu \epsilon^\mu = 0$. From (16), it must also be true that

$$A^\mu \partial_\mu \phi = 0 \tag{20}$$

This condition is satisfied, if $\phi = \phi(u)$

$$A^\mu \partial_\mu \phi = -k_\mu \epsilon^\mu f(u) \frac{d\phi}{du} = 0 \tag{21}$$

Polarization vectors are space-like, $\epsilon_\mu \epsilon^\mu = -1$, so that

$$A_\mu A^\mu = -f^2(u) \tag{22}$$

Therefore, both equations (14) and (15) will be satisfied, if $f(u) = \phi(u)$ and

$$\partial_\mu \partial^\mu \phi(u) + g^2 \phi^3(u) = 0 \tag{23}$$

This equation, known as the cubic wave equation, is solved by the elliptic function $\mathrm{cn}(u, \frac{1}{2})$ (appendix A)

$$\phi(u) = \frac{k}{g}\,\mathrm{cn}(-k_\mu x^\mu) \tag{24}$$

The amplitude of this traveling wave is not arbitrary, but is fixed by the value of k/g. It will be shown in the following section that k is directly proportional to the mass.

3. $U(1)$: Energy and Momentum

The energy tensor for the vector field is

$$T_{\mu\nu}(A) = F_{\mu\eta} F^{\eta}{}_{\nu} + \frac{1}{4} g_{\mu\nu} F_{\eta\rho} F^{\eta\rho} \tag{25}$$

with energy and momentum densities

[1] Throughout this paper, $u = -k_\mu x^\mu = (\mathbf{k} \cdot \mathbf{x} - k^0 x^0)$ and $k^2 = k_\mu k^\mu = (k^0)^2 - \mathbf{k}^2$.

$$
\begin{aligned}
T_{00}(A) &= \frac{1}{2}\Big\{F_{01}^2 + F_{02}^2 + F_{03}^2 + F_{23}^2 + F_{31}^2 + F_{12}^2\Big\} && (26)\\
T_{0i}(A) &= F_{01}F_{i1} + F_{02}F_{i2} + F_{03}F_{i3} && (27)
\end{aligned}
$$

For the real scalar field

$$
T_{\mu\nu}(\phi) = \partial_\mu\phi\,\partial_\nu\phi + g^2 A_\mu A_\nu \phi^2 - g_{\mu\nu} L(\phi) \tag{28}
$$

with energy and momentum densities

$$
\begin{aligned}
T_{00}(\phi) &= \frac{1}{2}\Big\{(\partial_0\phi)^2 + (\nabla\phi)^2 + g^2[(A_0)^2 + \mathbf{A}^2]\phi^2\Big\} && (29)\\
T_{0i}(\phi) &= \partial_0\phi\,\partial_i\phi + g^2 A_0 A_i\,\phi^2 && (30)
\end{aligned}
$$

The constant term $L(k)$ contributes

$$
T_{\mu\nu}(k) = -\frac{k^2}{3g^2}(k_\mu k_\nu - \frac{1}{2}g_{\mu\nu}k^2) \tag{31}
$$

so that

$$
\begin{aligned}
T_{00}(k) &= -\frac{k^2}{6g^2}(k_0^2 + \mathbf{k}^2) && (32)\\
T_{0i}(k) &= -\frac{k^2}{3g^2}\,k_0 k_i && (33)
\end{aligned}
$$

The following formulas occur repeatedly in the calculations and are placed here for reference (app. A):

$$
\begin{aligned}
\phi^2 &= \frac{k^2}{g^2}\,\mathrm{cn}^2(u) && (34)\\
\Big(\frac{d\phi}{du}\Big)^2 &= \frac{k^2}{2g^2}\Big\{1 - \mathrm{cn}^4(u)\Big\} && (35)
\end{aligned}
$$

(a) Transverse polarization

For plane waves moving along the x^3-axis, $k^\mu = (k^0, k^3)$, and the linear polarization vectors are

$$\epsilon_1^\mu = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \qquad \epsilon_2^\mu = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \tag{36}$$

If the polarization is along x^1, then $A^\mu = \epsilon_1^\mu \, \phi(u)$ has the single component $A^1 = \phi(u)$. In this case, the energy contributions are

$$\begin{aligned} T_{00}(A) &= \frac{1}{2}(\partial_0 A_1)^2 + \frac{1}{2}(\partial_3 A_1)^2 = \frac{k_0^2 + k_3^2}{2}\Big(\frac{d\phi}{du}\Big)^2 \\ &= \frac{k^2}{2g^2}\frac{k_0^2 + k_3^2}{2}\Big\{1 - \mathrm{cn}^4(u)\Big\} \end{aligned} \tag{37}$$

$$\begin{aligned} T_{00}(\phi) &= \frac{1}{2}\Big\{(\partial_0\phi)^2 + (\partial_3\phi)^2 + g^2 A_1^2 \phi^2\Big\} \\ &= \frac{k^2}{2g^2}\Big\{\frac{k_0^2 + k_3^2}{2}\{1 - \mathrm{cn}^4(u)\} + (k_0^2 - k_3^2)\mathrm{cn}^4(u)\Big\} \end{aligned} \tag{38}$$

$$T_{00}(k) = -\frac{k^2}{2g^2}\frac{k_0^2 + k_3^2}{3} \tag{39}$$

Similarly, the momenta are

$$T_{03}(A) = \partial_0 A_1 \, \partial_3 A_1 = \frac{k^2}{2g^2} k_0 k_3 \{1 - \mathrm{cn}^4(u)\} \tag{40}$$

$$T_{03}(\phi) = \partial_0 \phi \, \partial_3 \phi = \frac{k^2}{2g^2} k_0 k_3 \{1 - \mathrm{cn}^4(u)\} \tag{41}$$

$$T_{03}(k) = -\frac{k^2}{2g^2}\frac{2}{3}\, k_0 k_3 \tag{42}$$

Sum terms, in order to obtain

$$T_{00} = \frac{k^2}{2g^2}\Big\{\frac{2}{3}(k_0^2 + k_3^2) - 2k_3^2 \, \mathrm{cn}^4(u)\Big\} \tag{43}$$

$$T_{03} = \frac{k^2}{2g^2}\, k_0 k_3 \Big\{\frac{4}{3} - 2\, \mathrm{cn}^4(u)\Big\} \tag{44}$$

These expressions obey the energy conservation law, $\partial_\nu T^{0\nu} = 0$. It can be shown that the momentum is conserved as well, $\partial_\nu T^{3\nu} = 0$.

The calculation of total energy and momentum follows the introduction of a volume element $l^3\, dV/V$. In order to arrive at the correct quantum expressions, the integrals must be independent of the ratio k^2/g^2. This factor is eliminated by setting

$$\frac{l^3}{V}\, dV = \frac{3\pi}{2K}\frac{g^2}{k^2k^0V}\, dV \tag{45}$$

The integrals are

$$\begin{aligned} E &= \frac{3\pi}{2K}\frac{g^2}{k^2k^0V}\int T_0{}^0\, dV \\ &= \frac{3\pi}{2K}\frac{\hbar c}{2k^0V}\int\left\{\frac{2}{3}(k_0^2+k_3^2) - 2k_3^2\,\mathrm{cn}^4(u)\right\} dV \end{aligned} \tag{46}$$

$$\begin{aligned} cp^3 &= \frac{3\pi}{2K}\frac{g^2}{k^2k^0V}\int T_0{}^3\, dV \\ &= \frac{3\pi}{2K}\frac{\hbar c}{2k^0V}\int k_0k^3\left\{\frac{4}{3} - 2\,\mathrm{cn}^4(u)\right\} dV \end{aligned} \tag{47}$$

The elliptic function $\mathrm{cn}(u, \frac{1}{2})$ admits the integral (app. A)

$$\int \mathrm{cn}^4(u)\, du = \frac{u}{3} + \frac{2}{3}\,\mathrm{sn}(u)\mathrm{cn}(u)\mathrm{dn}(u) + \text{constant} \tag{48}$$

Over one period,

$$\frac{1}{4K}\int_0^{4K} \mathrm{cn}^4(u)\, du = \frac{1}{3} \tag{49}$$

$\mathrm{sn}(0) = \mathrm{sn}(4K) = 0$. Therefore, the energy and momentum are given by

$$E = \frac{\pi}{2K}\hbar c k^0 = \hbar\omega \tag{50}$$

$$cp^3 = \frac{\pi}{2K}\hbar c k^3 = \hbar c\,\frac{2\pi}{\lambda} \tag{51}$$

(b) Longitudinal polarization

In this case, $A^\mu(u) = \epsilon^\mu\, \phi(u)$ is expressed in terms of the zero-helicity vector

$$\epsilon^{\mu}(0) = \frac{1}{k}\begin{pmatrix} k^3 \\ 0 \\ 0 \\ k^0 \end{pmatrix} \tag{52}$$

A^{μ} has two components $(A^0, A^3) = (k^3, k^0)k^{-1}\phi(u)$. The contributions to the energy density are

$$\begin{aligned}
T_{00}(A) &= \frac{1}{2}(\partial_0 A_3 - \partial_3 A_0)^2 = \frac{k^2}{2g^2}\frac{k_0^2 - k_3^2}{2}\Big\{1 - \mathrm{cn}^4(u)\Big\} && (53)\\
T_{00}(\phi) &= \frac{1}{2}\Big\{(\partial_0\phi)^2 + (\partial_3\phi)^2 + g^2(A_0^2 + A_3^2)\phi^2\Big\} \\
&= \frac{k^2}{2g^2}\frac{k_0^2 + k_3^2}{2}\Big\{1 + \mathrm{cn}^4(u)\Big\} && (54)\\
T_{00}(k) &= -\frac{k^2}{2g^2}\frac{k_0^2 + k_3^2}{3} && (55)
\end{aligned}$$

while those for the momentum are

$$\begin{aligned}
T_{03}(A) &= 0 && (56)\\
T_{03}(\phi) &= \partial_0\phi\,\partial_3\phi + g^2 A_0 A_3\,\phi^2 = \frac{k^2}{2g^2}k_0 k_3\Big\{1 + \mathrm{cn}^4(u)\Big\} && (57)\\
T_{03}(k) &= -\frac{k^2}{2g^2}\frac{2}{3}\,k_0 k_3 && (58)
\end{aligned}$$

Sum terms to find

$$\begin{aligned}
T_{00} &= \frac{k^2}{2g^2}\Big\{\frac{2}{3}k_0^2 - \frac{1}{3}k_3^2 + k_3^2\,\mathrm{cn}^4(u)\Big\} && (59)\\
T_{03} &= \frac{k^2}{2g^2}k_0 k_3\Big\{\frac{1}{3} + \mathrm{cn}^4(u)\Big\} && (60)
\end{aligned}$$

Here, too, the energy and momentum are conserved, $\partial_\nu T^{0\nu} = \partial_\nu T^{3\nu} = 0$. The integrals are

$$
\begin{aligned}
E &= \frac{3\pi}{2K}\frac{g^2}{k^2k^0V}\int T_0{}^0\,dV \\
&= \frac{3\pi}{2K}\frac{\hbar c}{2k^0V}\int \Big\{\frac{2}{3}k_0^2-\frac{1}{3}k_3^2+k_3^2\mathrm{cn}^4(u)\Big\}\,dV \\
&= \frac{\pi}{2K}\hbar c k^0 = \hbar\omega \qquad (61)\\
cp^3 &= \frac{3\pi}{2K}\frac{g^2}{k^2k^0V}\int T_0{}^3\,dV \\
&= \frac{3\pi}{2K}\frac{\hbar c}{2k^0V}\int k_0k^3\Big\{\frac{1}{3}+\mathrm{cn}^4(u)\Big\}\,dV \\
&= \frac{\pi}{2K}\hbar c k^3 = \hbar c\,\frac{2\pi}{\lambda} \qquad (62)
\end{aligned}
$$

These calculations establish the equality

$$E^2-\mathbf{p}^2=\Big(\frac{\pi}{2K}\Big)^2k^2 \qquad (63)$$

showing that k is directly proportional to the mass

$$m=\frac{\pi}{2K}\,k \qquad (64)$$

4. $U(1)\otimes SU(2)_L$: Field Equations; Energy Tensors

The Lagrangian is given by

$$L=L(W)+L(B)+L(\Phi)+L(k)+L(\text{leptons}) \qquad (65)$$

The lepton term contains the electroweak interaction and will be set aside. The focus is upon the coupling between vector and scalar fields. The covariant derivatives are [4]

$$
\begin{aligned}
G_{\mu\nu}(W) &= \partial_\mu(W^i_\nu T^i)-\partial_\nu(W^i_\mu T^i)+g\,\Sigma_{ijk}\epsilon_{ijk}W^i_\mu W^j_\nu T^k \qquad (66)\\
F_{\mu\nu}(B) &= \partial_\mu B_\nu-\partial_\nu B_\mu \qquad (67)\\
D_\mu\Phi &= \Big\{\partial_\mu+ig\,T^iW^i_\mu+ig'\frac{Y}{2}B_\mu\Big\}\Phi \\
&= \Big\{\partial_\mu+ig\,\frac{\tau^i}{2}W^i_\mu+ig'\frac{1}{2}B_\mu\Big\}\begin{pmatrix}\phi^+\\ \phi^0\end{pmatrix} \qquad (68)
\end{aligned}
$$

The final term in $G_{\mu\nu}(W)$ yields third- and fourth-order vector boson interactions and will also be ignored.[5] This leaves

$$\begin{aligned} L(W) + L(B) &= -\frac{1}{2}\mathrm{Tr}\, G_{\mu\nu}(W)G^{\mu\nu}(W) - \frac{1}{4}F_{\mu\nu}(B)F^{\mu\nu}(B) \\ &= -\frac{1}{4}\Big\{F_{\mu\nu}(A)F^{\mu\nu}(A) + F_{\mu\nu}(Z)F^{\mu\nu}(Z) \\ &\quad +F_{\mu\nu}(W^1)F^{\mu\nu}(W^1) + F_{\mu\nu}(W^2)F^{\mu\nu}(W^2)\Big\} \end{aligned} \tag{69}$$

where

$$W^3_\mu = \frac{g'}{\sqrt{g^2+g'^2}}A_\mu + \frac{g}{\sqrt{g^2+g'^2}}Z_\mu = \sin\theta A_\mu + \cos\theta Z_\mu \tag{70}$$

$$B_\mu = \frac{g}{\sqrt{g^2+g'^2}}A_\mu - \frac{g'}{\sqrt{g^2+g'^2}}Z_\mu = \cos\theta A_\mu - \sin\theta Z_\mu \tag{71}$$

The charged bosons $W^\pm_\mu = (W^1_\mu \mp iW^2_\mu)/\sqrt{2}$ are expressed in terms of the real fields W^1_μ and W^2_μ.

The expansion of

$$L(\Phi) = g^{\mu\nu}(D_\mu\Phi)^\dagger D_\nu\Phi) \tag{72}$$

is carried out in appendix B, where the functional derivatives are also found. Before writing the equations of motion, the scalar field is transformed to unitary gauge

$$\Phi = \begin{pmatrix}\phi^+\\ \phi^0\end{pmatrix} \longrightarrow \begin{pmatrix}0\\ \phi/\sqrt{2}\end{pmatrix} \tag{73}$$

where $\phi(x)$ is real. This greatly simplifies the functional derivatives

$$\frac{\partial L}{\partial(\partial_\mu A_\nu)} = -F^{\mu\nu}(A) \qquad \frac{\partial L}{\partial A_\nu} = 0 \tag{74}$$

$$\frac{\partial L}{\partial(\partial_\mu Z_\nu)} = -F^{\mu\nu}(Z) \qquad \frac{\partial L}{\partial Z_\nu} = \frac{1}{4}(g^2+g'^2)Z^\nu\phi^2 \tag{75}$$

$$\frac{\partial L}{\partial(\partial_\mu W_\nu)} = -F^{\mu\nu}(W) \qquad \frac{\partial L}{\partial W_\nu} = \frac{1}{4}g^2W^\nu\phi^2 \tag{76}$$

where W_ν is either W^1_ν or W^2_ν. The equations of motion for the vector fields are

$$\begin{aligned}
-\partial_\mu F^{\mu\nu}(A) &= 0 && (77)\\
-\partial_\mu F^{\mu\nu}(Z) &= \frac{1}{4}(g^2+g'^2)Z^\nu\phi^2 && (78)\\
-\partial_\mu F^{\mu\nu}(W) &= \frac{1}{4}g^2\,W^\nu\phi^2 && (79)
\end{aligned}$$

Since $\partial_\mu\partial_\nu F^{\mu\nu}\equiv 0$, these equations yield

$$\partial_\nu(Z^\nu\phi^2)=\partial_\nu(W^\nu\phi^2)=0 \tag{80}$$

For the scalar field,

$$\frac{\partial L}{\partial(\partial_\mu\phi^{0*})} = \partial^\mu\phi^0-\frac{i}{2}\sqrt{g^2+g'^2}\,Z^\mu\phi^0 \tag{81}$$

$$\frac{\partial L}{\partial\phi^{0*}} = \frac{i}{2}\sqrt{g^2+g'^2}\,Z^\mu\partial_\mu\phi^0+\frac{g^2}{2}\,W^{-\mu}W^+_\mu\phi^0+\frac{1}{4}(g^2+g'^2)Z^\mu Z_\mu\phi^0 \tag{82}$$

$$\frac{\partial L}{\partial(\partial_\mu\phi^{+*})} = \frac{i}{\sqrt{2}}\,gW^{+\mu}\phi^0 \tag{83}$$

$$\frac{\partial L}{\partial\phi^{+*}} = -\frac{i}{\sqrt{2}}\,gW^{+\mu}\partial_\mu\phi^0+(eA^\mu-g'\sin\theta Z^\mu)\frac{1}{\sqrt{2}}gW^+_\mu\phi^0 \tag{84}$$

The equations of motion are ($\phi^0=\phi/\sqrt{2}$)

$$\partial_\mu\partial^\mu\phi-\frac{1}{4}g^2\,W^{1\mu}W^1_\mu\,\phi-\frac{1}{4}g^2\,W^{2\mu}W^2_\mu\,\phi-\frac{1}{4}(g^2+g'^2)Z^\mu Z_\mu\,\phi=0 \tag{85}$$

and

$$(eA^\mu-g'\sin\theta Z^\mu)\,g\,W^+_\mu\phi=0 \tag{86}$$

where (80) has been used. The vector fields couple individually with ϕ, so that (86) will be satisfied.

The energy tensors are much the same as for U(1). Each vector field A_μ, W^1_μ, W^2_μ, Z_μ contributes

$$T_{\mu\nu} = F_{\mu\eta}F^{\eta}{}_{\nu} + \frac{1}{4}g_{\mu\nu}F_{\eta\rho}F^{\eta\rho} \tag{87}$$

while the scalar Lagrangian

$$L(\phi) = \frac{1}{2}g^{\mu\nu}\Big\{\partial_\mu\phi\,\partial_\nu\phi + \frac{1}{4}g^2(W^1_\mu W^1_\nu + W^2_\mu W^2_\nu)\phi^2 + \frac{1}{4}(g^2+g'^2)Z_\mu Z_\nu\phi^2\Big\} \tag{88}$$

gives

$$T_{\mu\nu}(\phi) = \partial_\mu\phi\,\partial_\nu\phi + \frac{1}{4}g^2(W^1_\mu W^1_\nu + W^2_\mu W^2_\nu)\phi^2 + \frac{1}{4}(g^2+g'^2)Z_\mu Z_\nu\phi^2 - g_{\mu\nu}L(\phi) \tag{89}$$

The constant term $T_{\mu\nu}(k)$ is treated in the following section.

5. The W$^{\pm}$ and Z^0 Bosons

The coupling between $W^{\pm}$ and ϕ is described by (79, 85)

$$\partial_\mu F^{\mu\nu}(W) + \frac{1}{4}\,g^2\,W^\nu\phi^2 = 0 \tag{90}$$

$$\partial_\mu\partial^\mu\phi - \frac{1}{4}\,g^2\,W_\mu W^\mu\phi = 0 \tag{91}$$

where $\partial_\mu W^\mu = W^\mu\partial_\mu\phi = 0$. These equations are identical to those of U(1), (14) and (15), with the replacement $g \longrightarrow g/2$. The solutions are polarized, $W^\mu(u) = \epsilon^\mu\,\phi(u)$, where

$$\phi(u) = \frac{2k_W}{g}\,\mathrm{cn}(-k_\mu x^\mu) \tag{92}$$

Similarly, the coupling between Z^0 and ϕ is given by

$$\partial_\mu F^{\mu\nu}(Z) + \frac{1}{4}(g^2+g'^2)Z^\nu\phi^2 = 0 \tag{93}$$

$$\partial_\mu\partial^\mu\phi - \frac{1}{4}(g^2+g'^2)Z_\mu Z^\mu\phi = 0 \tag{94}$$

The solutions are $Z^\mu(u) = \epsilon^\mu\,\phi(u)$, where

$$\phi(u) = \frac{2k_Z}{\sqrt{g^2 + g'^2}} \,\mathrm{cn}(-k_\mu x^\mu) \tag{95}$$

The expressions for T_{00} and T_{03} are exactly the same as those in U(1), apart from the new coupling constants. They give rise to factors k_W^2/g^2 and $k_Z^2/(g^2 + g'^2)$. Before integration can occur, the volume element must eliminate these two factors. This is possible, if they are equal

$$\frac{k_W^2}{g^2} = \frac{k_Z^2}{g^2 + g'^2} \tag{96}$$

Only then will the volume element be uniquely defined (compare (45))

$$\frac{l^3}{V}\, dV = \frac{3\pi}{8K}\frac{g^2}{k_W^2 k^0 V}\, dV = \frac{3\pi}{8K}\frac{(g^2 + g'^2)}{k_Z^2 k^0 V}\, dV \tag{97}$$

Integration may now go forward as in U(1).[2] The integrals for $W^\pm$ and Z^0, including transverse and longitudinal fields, all assume the form

$$E = \frac{l^3}{V}\int {T_0}^0\, dV = \hbar\omega \tag{98}$$

$$cp^3 = \frac{l^3}{V}\int {T_0}^3\, dV = \hbar c\,\frac{2\pi}{\lambda} \tag{99}$$

6. Concluding Remarks

The coupling theory is made tractable, with the choice of unitary gauge. It eliminates the scalar currents, leaving direct nonlinear coupling between the real scalar and vector fields. The equations of motion yield both transverse and longitudinal solutions, indicating the presence of mass.

[2]Moreover, the constant terms are identical

$$\begin{aligned} T_{\mu\nu}(k) &= -\frac{4}{3}\frac{k_W^2}{g^2}(k_\mu k_\nu - \frac{1}{2}g_{\mu\nu}k^2) \\ &= -\frac{4}{3}\frac{k_Z^2}{g^2 + g'^2}(k_\mu k_\nu - \frac{1}{2}g_{\mu\nu}k^2) \end{aligned}$$

In the U(1) model, scalar-vector coupling generates a single massive vector field, while in the electroweak theory there are three, the $W^{\pm}$ and Z^0. They satisfy relation (96), yielding the mass ratio

$$\frac{m_W}{m_Z} = \frac{k_W}{k_Z} = \frac{g}{\sqrt{g^2 + g'^2}} \tag{100}$$

In linear field theory, it is customary to include the integration volume V as part of the plane wave expansion. For example, the expression $(\hbar c/2k^0 V)^{1/2}$ appears in each boson term. This is possible because the energy tensor $T_{\mu\nu}$ is second-order in the field. However, the non-linear tensor (89) contains both second- and fourth-order terms, making such a procedure untenable. Instead, the volume is introduced as a factor, l^3/V, during the integration of $T_{\mu\nu}$.

Finally, the constant term in the energy tensor $T_{\mu\nu}(k)$ has been introduced out of necessity. Without it, the energy and momentum would not satisfy the relation $c^2\mathbf{p} = E\,\mathbf{v}$, as they must; the work would simply be incomplete.

Appendix A. Elliptic Functions [6, 7]

The elliptic functions $\mathrm{sn}(u,m)$, $\mathrm{cn}(u,m)$, and $\mathrm{dn}(u,m)$ satisfy

$$\mathrm{sn}^2(u,m) + \mathrm{cn}^2(u,m) = 1 \tag{101}$$

$$\mathrm{dn}(u,m) = \sqrt{1 - m\,\mathrm{sn}^2(u,m)} \tag{102}$$

Their derivatives are

$$\frac{d\,\mathrm{sn}(u,m)}{du} = \mathrm{cn}(u,m)\mathrm{dn}(u,m) \tag{103}$$

$$\frac{d\,\mathrm{cn}(u,m)}{du} = -\mathrm{sn}(u,m)\mathrm{dn}(u,m) \tag{104}$$

$$\frac{d\,\mathrm{dn}(u,m)}{du} = -m\,\mathrm{sn}(u,m)\mathrm{cn}(u,m) \tag{105}$$

In particular,

$$\begin{aligned}\frac{d^2\mathrm{cn}(u,m)}{du^2} &= -\mathrm{cn}(u)\mathrm{dn}^2(u) + m\,\mathrm{sn}^2(u)\mathrm{cn}(u) \\ &= (2m-1)\,\mathrm{cn}(u) - 2m\,\mathrm{cn}^3(u) \end{aligned} \tag{106}$$

If the parameter $m = \frac{1}{2}$, then

$$\frac{d^2 \mathrm{cn}(u, \frac{1}{2})}{du^2} = -\mathrm{cn}^3(u, \frac{1}{2}) \tag{107}$$

The cubic wave equation (23) yields the ordinary differential equation

$$k_\mu k^\mu \, \frac{d^2\phi(u)}{du^2} + g^2\phi^3(u) = 0 \tag{108}$$

Substitute $\phi(u) = a\,\mathrm{cn}(u, \frac{1}{2})$ to find

$$-k^2 a\,\mathrm{cn}^3(u, \frac{1}{2}) + g^2 a^3 \mathrm{cn}^3(u, \frac{1}{2}) = 0 \tag{109}$$

This equation is satisfied, if $a^2 = k^2/g^2$. The solution of (108) is

$$\phi(u) = \frac{k}{g}\,\mathrm{cn}(u, \frac{1}{2}) = \frac{k}{g}\,\mathrm{cn}(-k_\mu x^\mu) \tag{110}$$

The elliptic functions, with $m = \frac{1}{2}$, satisfy two useful identities. The first is

$$\Big(\frac{d\,\mathrm{cn}(u)}{du}\Big)^2 = \mathrm{sn}^2(u)\mathrm{dn}^2(u) = \frac{1}{2}\{1 - \mathrm{cn}^4(u)\} \tag{111}$$

The second identity is

$$\frac{d}{du}\{\mathrm{sn}(u)\mathrm{cn}(u)\mathrm{dn}(u)\} = \frac{1}{2}\{3\,\mathrm{cn}^4(u) - 1\} \tag{112}$$

and it follows that

$$\int \mathrm{cn}^4(u)\,du = \frac{u}{3} + \frac{2}{3}\,\mathrm{sn}(u)\mathrm{cn}(u)\mathrm{dn}(u) + \text{constant} \tag{113}$$

The period of an elliptic function is $4K$. (If $m = \frac{1}{2}$, then $K \doteq 1.85$.) For motion along the x^3-axis,

$$\mathrm{cn}(-k_\mu x^\mu) = \mathrm{cn}(k^3x^3 - k^0x^0) = \mathrm{cn}\,4K\Big(\frac{x^3}{\lambda} - \frac{t}{T}\Big) \tag{114}$$

It follows that

$$
\begin{aligned}
ck^0 &= \frac{4K}{T} = \frac{2K}{\pi}\, 2\pi f = \frac{2K}{\pi}\,\omega && (115)\\
k^3 &= \frac{4K}{\lambda} = \frac{2K}{\pi}\frac{2\pi}{\lambda} && (116)
\end{aligned}
$$

and

$$
\begin{aligned}
E &= \hbar\omega = \frac{\pi}{2K}\hbar c k^0 && (117)\\
cp^3 &= \hbar c\frac{2\pi}{\lambda} = \frac{\pi}{2K}\hbar c k^3 && (118)
\end{aligned}
$$

Appendix B. Scalar Lagrangian

(a) Lagrangian

$$
\begin{aligned}
L(\Phi) &= g^{\mu\nu}(D_\mu\Phi)^\dagger(D_\nu\Phi)\\
&= \partial^\mu\phi^{+*}\partial_\mu\phi^+ + \partial^\mu\phi^{0*}\partial_\mu\phi^0\\
&+\Big\{[eA^\mu + \frac{1}{2}(g\cos\theta - g'\sin\theta)Z^\mu]^2 + \frac{1}{2}g^2W^{+\mu}W^-_\mu\Big\}\phi^{+*}\phi^+\\
&+[eA^\mu - g'\sin\theta\, Z^\mu]\Big\{\frac{g}{\sqrt{2}}W^+_\mu\phi^{+*}\phi^0 + \frac{g}{\sqrt{2}}W^-_\mu\phi^{0*}\phi^+\Big\}\\
&+\Big\{\frac{1}{2}g^2W^{-\mu}W^+_\mu + \frac{1}{4}(g^2+g'^2)Z^\mu Z_\mu\Big\}\phi^{0*}\phi^0\\
&-i[eA^\mu + \frac{1}{2}(g\cos\theta - g'\sin\theta)Z^\mu](\phi^{+*}\partial_\mu\phi^+ - \partial_\mu\phi^{+*}\phi^+)\\
&-\frac{i}{\sqrt{2}}gW^{-\mu}(\phi^{0*}\partial_\mu\phi^+ - \partial_\mu\phi^{0*}\phi^+) - \frac{i}{\sqrt{2}}gW^{+\mu}(\phi^{+*}\partial_\mu\phi^0 - \partial_\mu\phi^{+*}\phi^0)\\
&+\frac{i}{2}\sqrt{g^2+g'^2}\, Z^\mu(\phi^{0*}\partial_\mu\phi^0 - \partial_\mu\phi^{0*}\phi^0) && (119)
\end{aligned}
$$

(b) Functional derivatives

$$
\begin{aligned}
&\frac{\partial L}{\partial(\partial_\mu\phi^{0*})} = \partial^\mu\phi^0 + \frac{i}{\sqrt{2}}gW^{-\mu}\phi^+ - \frac{i}{2}\sqrt{g^2+g'^2}Z^\mu\phi^0 && (120)\\
&\frac{\partial L}{\partial\phi^{0*}} = -\frac{i}{\sqrt{2}}gW^{-\mu}\partial_\mu\phi^+ + \frac{i}{2}\sqrt{g^2+g'^2}Z^\mu\partial_\mu\phi^0
\end{aligned}
$$

$$+[eA^\mu - g' \sin\theta\, Z^\mu]\frac{1}{\sqrt{2}} g W_\mu^- \phi^+$$

$$+\frac{1}{2} g^2 W^{-\mu} W_\mu^+ \phi^0 + \frac{1}{4}(g^2 + g'^2) Z^\mu Z_\mu \phi^0 \tag{121}$$

and

$$\frac{\partial L}{\partial(\partial_\mu \phi^{+*})} = \partial^\mu \phi^+ + i[eA^\mu + \frac{1}{2}(g\cos\theta - g'\sin\theta)Z^\mu]\phi^+ + \frac{i}{\sqrt{2}} g W^{+\mu}\phi^0 \tag{122}$$

$$\frac{\partial L}{\partial \phi^{+*}} = -i[eA^\mu + \frac{1}{2}(g\cos\theta - g'\sin\theta)Z^\mu]\partial_\mu \phi^+ + \frac{1}{2} g^2 W^{+\mu} W_\mu^- \phi^+$$

$$+[eA^\mu + \frac{1}{2}(g\cos\theta - g'\sin\theta)Z^\mu]^2 \phi^+$$

$$+[eA^\mu - g'\sin\theta\, Z^\mu]\frac{1}{\sqrt{2}} g W_\mu^+ \phi^0 - \frac{i}{\sqrt{2}} g W^{+\mu}\partial_\mu \phi^0 \tag{123}$$

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