Momentum and angular momentum spaces with Majorana matrices

Leonardo Pedro
Centro de Fisica Teorica de Particulas, Portugal
leonardo@cftp.ist.utl.pt

September 3, 2012

Abstract

The Dirac matrices (γ^μ) in Majorana representation are purely imaginary. The Majorana matrices ($i\gamma^\mu$) are 4x4 real matrices in Majorana representation. Several complete sets of Majorana matrices, solutions of equations related with the Dirac equation, are shown to exist. They define momentum, orbital angular momentum, total angular momentum and radial spaces. They are applied in the solution of the Dirac equations for the free fermion and the Hydrogen atom.

Contents

1 Introduction

In reference [1] written in 1937, Ettore Majorana, states that "it is perfectly, and most naturally, possible to formulate a theory of elementary neutral particles which do not have negative (energy) states.". In his theory, Majorana uses a basis in which the Dirac Gamma matrices have only imaginary entries.

In reference [2] written in 1967, the Oersted Medal's winner David Hestenes proposed an alternative to the Dirac equation for the charged fermions, where the imaginary unit is replaced by matrix multiplications on the right. In reference [3] written in 2008, Hestenes proposes an improved version of his original theory where Gravity and Electroweak interactions may be accounted.

The Dirac matrices, γ^μ, in Majorana representation are purely imaginary. That means that the Majorana matrices, $i\gamma^\mu$, are 4x4 real matrices.

An example of such matrices in a particular basis is:

$$\begin{aligned}
i\gamma^1 &= \begin{bmatrix} +1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & +1 \end{bmatrix} \quad i\gamma^2 = \begin{bmatrix} 0 & 0 & +1 & 0 \\ 0 & 0 & 0 & +1 \\ +1 & 0 & 0 & 0 \\ 0 & +1 & 0 & 0 \end{bmatrix} \quad i\gamma^3 = \begin{bmatrix} 0 & +1 & 0 & 0 \\ +1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{bmatrix} \\
i\gamma^0 &= \begin{bmatrix} 0 & 0 & +1 & 0 \\ 0 & 0 & 0 & +1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix} \quad i\gamma^5 = \begin{bmatrix} 0 & -1 & 0 & 0 \\ +1 & 0 & 0 & 0 \\ 0 & 0 & 0 & +1 \\ 0 & 0 & -1 & 0 \end{bmatrix} \quad = -\gamma^0\gamma^1\gamma^2\gamma^3
\end{aligned} \tag{1.1}$$

The metric, given by the anti-commutator of the matrices, is the Minkowski spacetime metric:

$$g^{\mu\nu} = -\{i\gamma^\mu, i\gamma^\nu\} = \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = diag(1, -1, -1, -1), \; \mu, \nu = 0, 1, 2, 3 \tag{1.2}$$

In fact, when working with 4x4 real matrices, we can only find a set of 5 anti-commuting matrices. This means that with 4x4 real matrices we can describe the Minkowski spacetime, but we can not describe, for instance, a 4D euclidean space.

We define $\not{p} = \gamma^\mu p_\mu$. The Dirac equation for the free fermion can be written only with real matrices:

$$i\gamma^0(i\gamma^\mu\partial_\mu - m)\Psi(x) = i\gamma^0(i\not{\partial} - m)\Psi(x) = 0 \tag{1.3}$$

And we can express Lorentz transforms only with real matrices.

The spin operators are defined as:

$$\sigma^k = \gamma^k\gamma^5 \quad k = 1, 2, 3 \tag{1.4}$$

They verify:

$$[\sigma^i, \sigma^j] = i\gamma^0\epsilon^{ij}_k\sigma^k \tag{1.5}$$

Where ϵ^{ij}_k is the Levi-Civita symbol. Note that $i\gamma^0$ commutes with σ^k and squares to -1, so it can be thought of as the imaginary unit in the spin algebra.

We will use the following conventions:

If p is a Lorentz vector:

$$(\gamma^\mu p_\mu)(\gamma^\mu p_\mu) = (\not{p})(\not{p}) = p^\mu p_\mu = p \cdot p = p^2 = (p^0)^2 - (p^1)^2 - (p^2)^2 - (p^3)^2 \tag{1.6}$$

Given a 3-vector $\vec{p}$ and a real number $m > 0$, we define:

$$\vec{p}^i = p^i, \; i = 1, 2, 3 \tag{1.7}$$
$$\vec{\not{p}} = \vec{\gamma} \cdot \vec{p} \tag{1.8}$$
$$E_p = \sqrt{\vec{p}^2 + m^2} \tag{1.9}$$
$$\not{p} = \gamma^0 E_p - \vec{\gamma} \cdot \vec{p} \tag{1.10}$$

Note that $(\not{p})^2 = m^2$. A Majorana spinor is a real 4D vector on which the Dirac matrices act. A Dirac spinor is a complex 4D vector, on which the Dirac matrices act.

The references I most used were [4] and [5].

2 Lorentz transformations

A Lorentz transformation can be represented by a tensor ${a_\rho}^\mu$ which leaves the metric invariant:

$$g^{\rho\sigma} {a_\rho}^\mu {a_\sigma}^\nu = g^{\mu\nu} \tag{2.1}$$

Let S be a Majorana matrix that verifies:

$$\gamma^\rho {a_\rho}^\mu = S^{-1} \gamma^\mu S \tag{2.2}$$

Then it verifies $\gamma^0 S^{-1} = S^\dagger \gamma^0$ and $\gamma^5 S = S \gamma^5$. In the particular case of a Lorentz boost, the S matrix is given by:

$$S_L = \frac{\not{p}\gamma^0 + m}{\sqrt{E_p + m}\sqrt{2m}} \tag{2.3}$$
$$S_L^{-1} = -\alpha^0 S_L^\dagger \alpha^0 = \frac{\gamma^0 \not{p} + m}{\sqrt{E_p + m}\sqrt{2m}} \tag{2.4}$$

where $\frac{\vec{p}}{m} = \frac{\vec{v}}{\sqrt{1-\vec{v}^2}}$,$\vec{v}$ is the boost velocity. In the particular case of a rotation, the S matrix is given by:

$$S_R = exp(i\gamma^5 \gamma^0 \gamma^i \varphi_i), \; i = 1, 2, 3 \tag{2.5}$$
$$S_R^{-1} = S_R^\dagger = -\gamma^0 S_R^\dagger \gamma^0 \tag{2.6}$$

In general, the S matrix is the product of a Lorentz boost and a rotation.

3 Momentum space

We assume that we are in a space of functions where the Fourier transform is well defined, that is:

$$\int dx cos(px) = 2\pi\delta(p) \tag{3.1}$$

$$\int \frac{dp}{2\pi} cos(px) = \delta(x) \tag{3.2}$$

Such that:

$$\int dp\ \delta(p-q)f(p) = f(q) \tag{3.3}$$

$$\int dx\ \delta(x-y)f(x) = f(y) \tag{3.4}$$

The equation which defines the momentum space is:

$$i\gamma^0(i\vec{\not\partial} - m)M(\vec{x}) = M(\vec{x})i\gamma^0 E_p \tag{3.5}$$

One solution is $M(\vec{x}) = O^\dagger(\vec{p},\vec{x})$:

$$O^\dagger(\vec{p},\vec{x}) = \frac{\not p\gamma^0 + m}{\sqrt{E_p+m}\sqrt{2E_p}} e^{i\gamma^0\vec{p}\cdot\vec{x}} \tag{3.6}$$

O is the hermitic conjugate of $O^\dagger$, given by:

$$O(\vec{p},\vec{x}) = e^{-i\gamma^0\vec{p}\cdot\vec{x}} \frac{\not p\gamma^0 + m}{\sqrt{E_p+m}\sqrt{2E_p}} \tag{3.7}$$

Where $p^0 = E_p$. Since the operator $i\gamma^0(i\vec{\not\partial} - m)$ is anti-hermitic, we have:

$$\int d^3\vec{x} O(\vec{q},\vec{x})O^\dagger(\vec{p},\vec{x}) i\gamma^0 E_p = \int d^3\vec{x} i\gamma^0 E_q O(\vec{q},\vec{x})O^\dagger(\vec{p},\vec{x}) \tag{3.8}$$

Noting that $E_p + E_q > 0$, this implies that:

$$\int d^3\vec{x} e^{-i\gamma^0\vec{q}\cdot\vec{x}} \frac{\vec{\not q}\gamma^0(E_p+m) + \vec{\not p}\gamma^0(E_q+m)}{\sqrt{E_q+m}\sqrt{2E_q}\sqrt{E_p+m}\sqrt{2E_p}} e^{i\gamma^0\vec{p}\cdot\vec{x}} = 0 \tag{3.9}$$

Therefore, we get:

$$\int d^3\vec{x} O(\vec{q},\vec{x})O^\dagger(\vec{p},\vec{x}) = \int d^3\vec{x} e^{-i\gamma^0\vec{q}\cdot\vec{x}} \frac{(E_p+m)(E_q+m) + \vec{\not q}\gamma^0\vec{\not p}\gamma^0}{\sqrt{E_q+m}\sqrt{2E_q}\sqrt{E_p+m}\sqrt{2E_p}} e^{i\gamma^0\vec{p}\cdot\vec{x}} \tag{3.10}$$

$$= \int d^3\vec{x} e^{-i\gamma^0(\vec{q}-\vec{p})\cdot\vec{x}} \frac{(E_p+m)(E_q+m) + \vec{\not q}\gamma^0\vec{\not p}\gamma^0}{\sqrt{E_q+m}\sqrt{2E_q}\sqrt{E_p+m}\sqrt{2E_p}} \tag{3.11}$$

$$= (2\pi)^3\delta^3(\vec{q}-\vec{p}) \frac{(E_p+m)(E_p+m) + \vec{p}^2}{(E_p+m)2E_p} \tag{3.12}$$

$$= (2\pi)^3\delta^3(\vec{q}-\vec{p}) \frac{(E_p+m)(E_p+m) + (E_p+m)(E_p-m)}{(E_p+m)2E_p} \tag{3.13}$$

$$= (2\pi)^3\delta^3(\vec{q}-\vec{p}) \tag{3.14}$$

To check that the momentum space is complete, we do:

$$\int \frac{d^3\vec{p}}{(2\pi)^3} O^\dagger(\vec{p},\vec{y}) O(\vec{p},\vec{x}) = \int \frac{d^3\vec{p}}{(2\pi)^3} \frac{\not{p}\gamma^0 + m}{\sqrt{E_p+m}\sqrt{2E_p}} e^{i\gamma^0 \vec{p}\cdot(\vec{y}-\vec{x})} \frac{\not{p}\gamma^0 + m}{\sqrt{E_p+m}\sqrt{2E_p}} \tag{3.15}$$

$$= \int \frac{d^3\vec{p}}{(2\pi)^3} e^{i\frac{\not{p}}{m}\vec{p}\cdot(\vec{y}-\vec{x})} \frac{\not{p}\gamma^0}{E_p} \tag{3.16}$$

$$= \int \frac{d^3\vec{p}}{(2\pi)^3} cos(\vec{p}\cdot(\vec{y}-\vec{x})) + \tag{3.17}$$

$$+ \int \frac{d^3\vec{p}}{(2\pi)^3} (-cos(\vec{p}\cdot(\vec{y}-\vec{x})) \frac{\vec{\not{p}}\gamma^0}{E_p} + sin(\vec{p}\cdot(\vec{y}-\vec{x})) \frac{mi\gamma^0}{E_p} \tag{3.18}$$

$$= \delta^3(\vec{y}-\vec{x}) \tag{3.19}$$

Note that both $cos(\vec{p}\cdot(\vec{y}-\vec{x}))\frac{\vec{\not{p}}\alpha^0}{E_p}$ and $sin(\vec{p}\cdot(\vec{y}-\vec{x}))\frac{mi\gamma^0}{E_p}$ are odd in $\vec{p}$ and therefore do not contribute to the integral.

This means that given an arbitrary matrix in coordinate space, $M(\vec{x})$, it can be given in terms of other matrix in momentum space $M(\vec{p})$:

$$M(\vec{x}) = \int \frac{d^3\vec{p}}{(2\pi)^3} O^\dagger(\vec{p},\vec{x}) M(\vec{p}) \tag{3.20}$$

Where $M(\vec{p})$ is given by:

$$M(\vec{p}) = \int d^3\vec{x} O(\vec{p},\vec{x}) M(\vec{x}) \tag{3.21}$$

This defines a transform, similar to the Fourier transform, which we will call Fourier-Majorana transform.

4 Energy-momentum space

Now we can extend our transform to define an energy-momentum space. Given a 4x4 matrix $M(x)$, the Fourier-Majorana transform (in space-time) is defined as:

$$M(p) = \int d^4x O(p,x) M(x) \tag{4.1}$$

Where O is the 4x4 matrix given by:

$$O(p,x) = e^{i\gamma^0 p^0 x^0} O(\vec{p},\vec{x}) = e^{i\gamma^0 p\cdot x} \frac{\not{p}\gamma^0 + m}{\sqrt{E_p+m}\sqrt{2E_p}} \tag{4.2}$$

Note that E_p and $\not{p}\gamma^0$ don't depend on p^0, but $p\cdot x$ does. The inverse Fourier-Majorana transform is given by:

$$M(x) = \int \frac{d^4p}{(2\pi)^4} O^\dagger(p,x) M(p) \tag{4.3}$$

Where $O^\dagger$ is the hermitic conjugate of O,given by:

$$O^\dagger(p,x) = O^\dagger(\vec{p},\vec{x})e^{-i\gamma^0 p^0\cdot x^0} = \frac{\not{p}\gamma^0 + m}{\sqrt{E_p+m}\sqrt{2E_p}}e^{-i\gamma^0 p\cdot x} \tag{4.4}$$

To prove it:

$$\int \frac{d^4p}{(2\pi)^4}O^\dagger(p,y)O(p,x) = \int \frac{d^3\vec{p}}{(2\pi)^3}O^\dagger(\vec{p},\vec{y})(\int \frac{dp^0}{2\pi}e^{-i\gamma^0 p^0(y^0-x^0)})O(\vec{p},\vec{x}) \tag{4.5}$$

$$= \delta(y^0-x^0)\int \frac{d^3\vec{p}}{(2\pi)^3}O^\dagger(\vec{p},\vec{y})O(\vec{p},\vec{x}) \tag{4.6}$$

$$= \delta^4(y-x) \tag{4.7}$$

$$\int d^4x O(q,x)O^\dagger(p,x) = \int dx^0 e^{i\gamma^0 q^0 x^0}(\int d^3\vec{x}O(\vec{q},\vec{x})O^\dagger(\vec{p},\vec{x}))e^{-i\gamma^0 p^0 x^0} \tag{4.8}$$

$$= (2\pi)^3\delta^3(\vec{q}-\vec{p})\int dx^0 e^{i\gamma^0(q^0-p^0)x^0} \tag{4.9}$$

$$= (2\pi)^4\delta^4(q-p) \tag{4.10}$$

In what follows we will call just Fourier-Majorana transform to both Fourier-Majorana transforms in space and space-time. It will be clear from the context to which we are referring to.

5 Dirac equation for the free fermion

The Dirac equation for the free fermion is:

$$i\gamma^0(i\not{\partial} - m)\Psi(x) = 0 \tag{5.1}$$

Where Ψ is a spinor, a vector of the 4D space, on which the Dirac matrices act. Note that the equation contains only real matrices.

We can make a Fourier-Majorana transform and go to momentum space:

$$i\gamma^0(i\not{\partial} - m)\Psi(x) = (-\partial_0 + i\gamma^0 i\vec{\not{\partial}} - i\gamma^0 m)\int \frac{d^3\vec{p}}{(2\pi)^3}O^\dagger(\vec{p},\vec{x})\Psi(\vec{p},x^0) \tag{5.2}$$

$$= \int \frac{d^3\vec{p}}{(2\pi)^3}(-\partial_0 - i\gamma^0\vec{\not{p}}\frac{\not{p}}{m} - i\gamma^0 m)O^\dagger(\vec{p},\vec{x})\Psi(\vec{p},x^0) \tag{5.3}$$

$$= \int \frac{d^3\vec{p}}{(2\pi)^3}(-\partial_0 + i\vec{\not{p}}\frac{E_p}{m} - i\gamma^0\frac{E_p^2}{m})O^\dagger(\vec{p},\vec{x})\Psi(\vec{p},x^0) \tag{5.4}$$

$$= \int \frac{d^3\vec{p}}{(2\pi)^3}(-\partial_0 - i\frac{\not{n}_p}{m}E_p)O^\dagger(\vec{p},\vec{x})\Psi(\vec{p},x^0) \tag{5.5}$$

$$= \int \frac{d^3\vec{p}}{(2\pi)^3}O^\dagger(\vec{p},\vec{x})(-\partial_0 - i\gamma^0 E_p)\Psi(\vec{p},x^0) \tag{5.6}$$

The Dirac equation in momentum space is then:

$$(-\partial_0 - i\gamma^0 E_p)\Psi(\vec{p}, x^0) = 0 \tag{5.7}$$

The solution is:

$$\Psi(\vec{p}, x^0) = e^{-i\gamma^0 E_p x^0}\psi(\vec{p}) \tag{5.8}$$

Making an inverse Fourier-Majorana transform we get:

$$\Psi(x) = \int \frac{d^3\vec{p}}{(2\pi)^3} \frac{\not{p}\gamma^0 + m}{\sqrt{E_p + m}\sqrt{2E_p}} e^{-i\gamma^0 p\cdot x}\psi(\vec{p}) \tag{5.9}$$

Where $p^0 = E_p$ and $\psi(\vec{p})$ is an arbitrary spinor. In Majorana representation, if $\psi(\vec{p})$ is a real spinor, then the solution $\Psi(x)$ is real.

6 Spin

A spinor verifying the Majorana condition has 4 degrees of freedom. There are 2 degrees of freedom that are consumed by the phase of the oscillation of the wave. The 2 degrees of freedom that left correspond to the spin up/down property of the spinor.

The spin vector s verifies $s^\mu s_\mu = -1$ and $s^0 = 0$. The spin operator $\not{s}\gamma^5$ commutes with $i\gamma^0$ and squares to 1. Therefore, it has eigenvalues 1 (up) and -1 (down). The eigen-vectors of $\not{s}\gamma^5$, in momentum space, can be defined as:

$$\psi(\vec{p}, s) = \frac{1 + \not{s}\gamma^5}{2}\psi(\vec{p}, s) \tag{6.1}$$

$$\psi(\vec{p}, -s) = \frac{1 - \not{s}\gamma^5}{2}\psi(\vec{p}, -s) \tag{6.2}$$

And the Majorana spinor in momentum space with a defined spin and that satisfies the Dirac equation is:

$$\Psi(x^0, \vec{p}, s) = e^{-i\gamma^0 E_p x^0}\psi(\vec{p}, s) \tag{6.3}$$

The spin operators are defined as:

$$\sigma^k = \gamma^k\gamma^5 \quad k = 1, 2, 3 \tag{6.4}$$

They verify:

$$[\sigma^i, \sigma^j] = i\gamma^0\epsilon^{ij}_k\sigma^k \tag{6.5}$$

Where ϵ^{ij}_k is the Levi-Civita symbol. Note that $i\gamma^0$ commutes with σ^k and squares to -1, so it can be thought of as the imaginary unit in the spin algebra.

7 Orbital angular momentum space

A good reference for this part is [6]. We define the angular momentum operator $\vec{L}$ as:

$$\vec{L}_k = -i\gamma^0 \epsilon_k^{ij} x_i \partial_j \tag{7.1}$$

Where ϵ_k^{ij} is the Levi-Civita symbol. Note that the usual definition for the angular momentum has i instead of $i\gamma^0$.

One solution to the equations (m stands for the angular momentum, not for the mass):

$$(L_3 - m)Y(\vec{x}) = 0 \tag{7.2}$$

$$(\vec{L}^2 - l(l+1))Y(\vec{x}) = 0 \tag{7.3}$$

Is the Majorana matrix:

$$Y_{lm}(\theta, \varphi) = \sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}} P_l^{(m)}(\cos\theta) \mathrm{e}^{i\gamma^0 m\varphi} \tag{7.4}$$

$$P_l^{(m)}(\xi) = \frac{(-1)^m}{2^l l!}(1-\xi^2)^{m/2} \frac{\mathrm{d}^{l+m}}{\mathrm{d}\xi^{l+m}}(\xi^2 - 1)^l \tag{7.5}$$

Y_{lm} are the spherical harmonics and P_l^m are the associated Legendre functions of the first kind. θ and φ are the angles of $\vec{x}$ in spherical coordinates, r is the radius. Note that the usual definition for the spherical harmonics has the i instead of $i\gamma^0$.

The orbital momentum space defined by the orthonormal states (l, m) is complete, that is:

$$\int d(cos\theta) d\varphi Y_{l'm'}^\dagger(\theta, \varphi) Y(\theta, \varphi)_{lm} = \delta_{l'l}\delta_{m'm} \tag{7.6}$$

$$\sum_{lm} Y_{lm}^\dagger(\theta', \varphi) Y(\theta', \varphi)_{lm} = \delta(cos\theta' - cos\theta)\delta(\varphi' - \varphi) \tag{7.7}$$

We don't need to show this because the definition of the spherical harmonics is similar to the standard definition, with $i\gamma^0$ in place of i.

8 Total angular momentum space

The operator $\vec{\sigma} \cdot \vec{L}$ is:

$$\vec{\sigma} \cdot \vec{L} = -i\gamma^0 \epsilon_k^{ij} \sigma^k x_i \partial_j \tag{8.1}$$

$$= -[\sigma^i, \sigma^j] x_i \partial_j \tag{8.2}$$

$$= \frac{\gamma^i\gamma^j - \gamma^j\gamma^i}{2} x_i \partial_j \tag{8.3}$$

$$i, j \in 1, 2, 3 \tag{8.4}$$

It verifies:

$$i\partial\!\!\!/ = i\gamma^r(\partial_r - \frac{1}{r}\vec{\sigma} \cdot \vec{L}) \tag{8.5}$$

$$\vec{\sigma} \cdot \vec{L} = \gamma^\theta\gamma^r\partial_\theta + \gamma^\varphi\gamma^r\frac{1}{sin\theta}\partial_\varphi \tag{8.6}$$

θ and φ are the angles of $\vec{x}$ in spherical coordinates, r is the radius.

We define the spherical matrices, in a basis independent form, as:

$$\Omega_{lm}(\theta,\varphi) = \Big(-\sqrt{\frac{l-m}{2l+1}} Y_{l,m}(\theta,\varphi) + \sqrt{\frac{l+m+1}{2l+1}} Y_{l,m+1}(\theta,\varphi)\sigma^1 \Big) \frac{1+\sigma^3}{2} \tag{8.7}$$

$$+ \Big(\sqrt{\frac{l+m}{2l-1}} Y_{l-1,m}(\theta,\varphi)\sigma^1 + \sqrt{\frac{l-m-1}{2l-1}} Y_{l-1,m+1}(\theta,\varphi) \Big) \frac{1-\sigma^3}{2} \tag{8.8}$$

with

$$l \in \{1, 2, \ldots\} \tag{8.9}$$

$$m \in \{-l, -l+1, \ldots, l-1\} \tag{8.10}$$

$$Y_{lm}(\theta,\varphi) = \sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}} P_l^{(m)}(\cos\theta) \mathrm{e}^{\alpha^3 m\varphi} \tag{8.11}$$

$$P_l^{(m)}(\xi) = \frac{(-1)^m}{2^l l!}(1-\xi^2)^{m/2}\frac{\mathrm{d}^{l+m}}{\mathrm{d}\xi^{l+m}}(\xi^2-1)^l \tag{8.12}$$

Where Y_{lm} are called the spherical harmonics and P_l^m are the associated Legendre functions of the first kind.

The spherical matrices verify:

$$(\vec{L}^3 + \frac{\sigma^3}{2})\ \Omega_{lm} = (m+\frac{1}{2})\Omega_{lm} \tag{8.13}$$

$$\vec{\sigma}\cdot\vec{L}\ \Omega_{lm} = -\Omega_{lm}(l\sigma^3+1) \tag{8.14}$$

$$\sigma^r\ \Omega_{lm} = -\Omega_{lm}\sigma^1 \tag{8.15}$$

$$i\gamma^r\Omega_{lm} = (-1)^m\Omega_{l,-m-1}(\theta,\varphi)i\gamma^5 \tag{8.16}$$

$$\vec{\sigma}\cdot\vec{L}i\gamma^r\ \Omega_{lm} = i\gamma^r\Omega_{lm}(l\sigma^3-1) \tag{8.17}$$

Now, we note that:

$$\int d(cos\theta)d\varphi\ \frac{1+\epsilon'\sigma^3}{2}\Omega^\dagger(\theta,\varphi)_{l'm'}\Omega(\theta,\varphi)_{lm}\frac{1+\epsilon\sigma^3}{2} = 0 \tag{8.18}$$

For $l', m', \epsilon' \neq l, m, \epsilon$ and $\epsilon', \epsilon = \pm 1$, because the matrices correspond to different eigenvalues of hermitic operators.

Therefore, we have orthogonality:

$$\int d(cos\theta)d\varphi\ \Omega^\dagger(\theta,\varphi)_{l'm'}\Omega(\theta,\varphi)_{lm} = \tag{8.19}$$

$$\delta_{l'l}\delta_{m'm}\sum_{\epsilon=\pm1}\int d(cos\theta)d\varphi\frac{1+\epsilon\sigma^3}{2}\Omega^\dagger(\theta,\varphi)_{lm}\Omega(\theta,\varphi)_{lm}\frac{1+\epsilon\sigma^3}{2} \tag{8.20}$$

$$= \delta_{l'l}\delta_{m'm} \tag{8.21}$$

To show the completeness, we use the fact that we know that the spherical harmonics form a complete basis.

Given a matrix $M(\theta,\varphi)$, we define it's expansion as spherical harmonics M':

$$M'(\theta,\varphi) = \sum_{l,m} Y_{lm}(\theta,\varphi)M_{lm} \tag{8.22}$$

$$M_{lm} = \int d(cos\theta)d\varphi Y^{\dagger}_{lm}(\theta,\varphi)M(\theta,\varphi) \tag{8.23}$$

The completeness of the spherical harmonics means that given any matrix $N(\theta,\varphi)$, we can replace M by M' in the convolution:

$$\int d(cos\theta)d\varphi N^{\dagger}(\theta,\varphi)M(\theta,\varphi) = \int d(cos\theta)d\varphi N^{\dagger}(\theta,\varphi)M'(\theta,\varphi) \tag{8.24}$$

This is the same as saying that M' converges weakly to M.

To show that the space of the spherical spinors Ω is complete, we need to show that N' converges weakly to N, where N' is given by:

$$N'(\theta,\varphi) = \sum_{l,m} \Omega_{lm}(\theta,\varphi)N_{lm} \tag{8.25}$$

$$N_{lm} = \int d(cos\theta)d\varphi \Omega^{\dagger}_{lm}(\theta,\varphi)N(\theta,\varphi) \tag{8.26}$$

Given an arbitrary matrix $M(\theta,\varphi)$, we have:

$$\int d(cos\theta)d\varphi {N'}^{\dagger}(\theta,\varphi)M(\theta,\varphi) = \int d(cos\theta)d\varphi {N'}^{\dagger}(\theta,\varphi)M'(\theta,\varphi) \tag{8.27}$$

$$= \sum_{l,m} \int d(cos\theta)d\varphi {N'}^{\dagger}(\theta,\varphi)Y_{lm}(\theta,\varphi)M_{lm} \tag{8.28}$$

Using the orthogonality of the spherical harmonics, we get:

$$\int d(cos\theta)d\varphi {N'}^{\dagger}(\theta,\varphi)Y_{lm}(\theta,\varphi) = \tag{8.29}$$

$$= -\sqrt{\frac{l-m}{2l+1}}N^{\dagger}_{l,m}\frac{1+\sigma^3}{2} + \sqrt{\frac{l+m}{2l+1}}N^{\dagger}_{l,m-1}\frac{1+\sigma^3}{2}\sigma^1 \tag{8.30}$$

$$+ \sqrt{\frac{l+1+m}{2l+1}}N^{\dagger}_{l+1,m}\frac{1-\sigma^3}{2}\sigma^1 + \sqrt{\frac{l-m+1}{2l+1}}N^{\dagger}_{l+1,m-1}\frac{1-\sigma^3}{2} \tag{8.31}$$

$$= \int d(cos\theta)d\varphi N^{\dagger}(\theta,\varphi) \tag{8.32}$$

$$\Big(-\sqrt{\frac{l-m}{2l+1}}\Omega_{l,m}\frac{1+\sigma^3}{2} + \sqrt{\frac{l+m}{2l+1}}\Omega_{l,m-1}\frac{1+\sigma^3}{2}\sigma^1 \tag{8.33}$$

$$+ \sqrt{\frac{l+1+m}{2l+1}}\Omega_{l+1,m}\frac{1-\sigma^3}{2}\sigma^1 + \sqrt{\frac{l-m+1}{2l+1}}\Omega_{l+1,m-1}\frac{1-\sigma^3}{2}\Big) \tag{8.34}$$

$$= \int d(cos\theta)d\varphi N^\dagger(\theta,\varphi) \tag{8.35}$$

$$\Big(- \sqrt{\frac{l-m}{2l+1}}\Omega_{l,m}\frac{1+\sigma^3}{2} + \sqrt{\frac{l+m}{2l+1}}\Omega_{l,m-1}\frac{1+\sigma^3}{2}\sigma^1 \tag{8.36}$$

$$+ \sqrt{\frac{l+1+m}{2l+1}}\Omega_{l+1,m}\frac{1-\sigma^3}{2}\sigma^1 + \sqrt{\frac{l-m+1}{2l+1}}\Omega_{l+1,m-1}\frac{1-\sigma^3}{2}\Big) \tag{8.37}$$

$$= \int d(cos\theta)d\varphi N^\dagger(\theta,\varphi)\Big(\tag{8.38}$$

$$+ \Big(\frac{l-m}{2l+1}Y_{l,m}(\theta,\varphi) - \frac{\sqrt{l-m}\sqrt{l+m+1}}{2l+1}Y_{l,m+1}(\theta,\varphi)\sigma^1\Big)\frac{1+\sigma^3}{2} \tag{8.39}$$

$$+ \Big(- \frac{\sqrt{l+m}\sqrt{l-m+1}}{2l+1}Y_{l,m-1}(\theta,\varphi)\sigma^1 + \frac{l+m}{2l+1}Y_{l,m}(\theta,\varphi)\Big)\frac{1-\sigma^3}{2} \tag{8.40}$$

$$+ \Big(\frac{l+1+m}{2l+1}Y_{l,m}(\theta,\varphi) + \frac{\sqrt{l+1+m}\sqrt{l-m}}{2l+1}Y_{l,m+1}(\theta,\varphi)\sigma^1\Big)\frac{1+\sigma^3}{2} \tag{8.41}$$

$$+ \Big(\frac{\sqrt{l-m+1}\sqrt{l+m}}{2l+1}Y_{l,m-1}(\theta,\varphi)\sigma^1 + \frac{l-m+1}{2l+1}Y_{l,m}(\theta,\varphi)\Big)\frac{1-\sigma^3}{2}\Big) \tag{8.42}$$

$$= \int d(cos\theta)d\varphi N^\dagger(\theta,\varphi)Y_{l,m} \tag{8.43}$$

Therefore, we have:

$$\int d(cos\theta)d\varphi N^\dagger(\theta,\varphi)M(\theta,\varphi) = \int d(cos\theta)d\varphi N^\dagger(\theta,\varphi)M'(\theta,\varphi) \tag{8.44}$$

$$= \int d(cos\theta)d\varphi N'^\dagger(\theta,\varphi)M'(\theta,\varphi) = \int d(cos\theta)d\varphi N'^\dagger(\theta,\varphi)M(\theta,\varphi) \tag{8.45}$$

We have proved that the spherical matrices form a complete basis:

$$\sum_{lm}\Omega(\theta',\varphi')_{lm}\Omega^\dagger(\theta,\varphi)_{lm} = \sum_{lm}\Omega(\theta',\varphi')_{lm}\Omega^\dagger(\theta,\varphi)_{lm} \tag{8.46}$$

$$= \delta(cos\theta' - cos\theta)\delta(\varphi'-\varphi) \tag{8.47}$$

9 Radial Space

We define the matrix:

$$\Lambda(p)_{lm} = \Big(\Big(pj_l(pr) + (E_p - m)j_{l-1}(pr)i\gamma^r\Big)\Omega_{lm}\frac{1+\sigma^3}{2} \tag{9.1}$$

$$+ \Big(pj_{l-1}(pr) - (E_p - m)j_l(pr)i\gamma^r\Big)\Omega_{lm}\frac{1-\sigma^3}{2}\Big) \tag{9.2}$$

Where $E_p = \sqrt{p^2+m^2}$ and j_l is the non-divergent spherical Bessel function, verifying:

$$(\partial_r^2 + \frac{2}{r}\partial_r - \frac{l(l+1)}{r^2})j_l(pr) = -p^2 j_l(pr) \tag{9.3}$$

We can check that the equation holds:

$$i\gamma^0(i\vec{\not\partial}-m)\Lambda(p)_{lm}=E_p\Lambda(p)_{lm}i\gamma^0 \tag{9.4}$$

There is the spherical Hankel transform [7]:

$$\int_0^{+\infty} dr\ r^2 j_l(pr)j_l(p'r)=\frac{\pi\delta(p-p')}{2p^2} \tag{9.5}$$

$$\int_0^{+\infty} \frac{dp\ 2p^2}{\pi}j_l(pr)j_l(r')=\frac{\delta(r-r')}{r^2} \tag{9.6}$$

We define the Hankel-Majorana transform as:

$$N'(\vec{x})=\sum_{lm}\int \frac{dp\ 2(E_p+m)}{E_p\pi}\Lambda_{lm}(p)N_{lm}(p) \tag{9.7}$$

$$N_{lm}(p)=\int d^3\vec{x}\ \Lambda^\dagger_{lm}(p)N(\vec{x}) \tag{9.8}$$

With N' converging weakly to N.

The Λ matrix can be written as:

$$\Lambda(p)_{lm}=\Big(pj_l(pr)\Omega_{lm}\frac{1+\sigma^3}{2}+(-1)^m(E_p-m)j_{l-1}(pr)\Omega_{l-m-1}\frac{1-\sigma^3}{2}i\gamma^5 \tag{9.9}$$

$$+pj_{l-1}(pr)\Omega_{lm}\frac{1-\sigma^3}{2}-(-1)^m(E_p-m)j_l(pr)\Omega_{l-m-1}\frac{1+\sigma^3}{2}i\gamma^5\Big) \tag{9.10}$$

Now we can check that:

$$\int d^3\vec{x}\Lambda^\dagger(p')_{l'm'}\Lambda(p)_{lm}=\frac{\pi E_p\delta(p'-p)\delta_{l'l}\delta_{m'm}}{2(E_p+m)} \tag{9.11}$$

That is, the Λ matrices are orthogonal.

To show completeness, we first show that:

$$\sum_{l'm'}\int d(cos\theta)d\varphi N^\dagger(p)_{l'm'}\Lambda^\dagger(p,r,\theta,\varphi)_{l'm'}\Omega_{lm}(\theta,\varphi)= \tag{9.12}$$

$$=N^\dagger_{lm}p(j_l(pr)\frac{1+\sigma^3}{2}+j_{l-1}(pr)\frac{1-\sigma^3}{2}) \tag{9.13}$$

$$+N^\dagger_{l-m-1}(-1)^m(E_p-m)(-j_l(pr)\frac{1-\sigma^3}{2}+j_{l-1}(pr)\frac{1+\sigma^3}{2})i\gamma^5 \tag{9.14}$$

$$=\int d^3\vec{x}'N^\dagger(\vec{x}')\Big(\tag{9.15}$$

$$pj_l(pr')\Big(pj_l(pr)+(E_p-m)j_{l-1}(pr)i\gamma^r\Big)\Omega_{lm}\frac{1+\sigma^3}{2} \tag{9.16}$$

$$+pj_{l-1}(pr')\Big(pj_{l-1}(pr)-(E_p-m)j_l(pr)i\gamma^r\Big)\Omega_{lm}\frac{1-\sigma^3}{2} \tag{9.17}$$

$$(-1)^m(E_p-m)j_{l-1}(pr')\Big(pj_l(pr)+(E_p-m)j_{l-1}(pr)i\gamma^r\Big)\Omega_{l-m-1}\frac{1+\sigma^3}{2}i\gamma^5 \tag{9.18}$$

$$-(-1)^m(E_p-m)j_l(pr')\Big(pj_{l-1}(pr)-(E_p-m)j_l(pr)i\gamma^r\Big)\Omega_{l-m-1}\frac{1-\sigma^3}{2}i\gamma^5 \tag{9.19}$$

$$= \int d^3\vec{x}' N^\dagger(\vec{x}')\Big(\tag{9.20}$$

$$+ pj_l(pr')\Big(pj_l(pr) + (E_p - m)j_{l-1}(pr)i\gamma^r\Big)\Omega_{lm}\frac{1+\sigma^3}{2} \tag{9.21}$$

$$+ pj_{l-1}(pr')\Big(pj_{l-1}(pr) - (E_p - m)j_l(pr)i\gamma^r\Big)\Omega_{lm}\frac{1-\sigma^3}{2} \tag{9.22}$$

$$+ (E_p - m)j_{l-1}(pr')\Big(pj_l(pr)i\gamma^r + (E_p - m)j_{l-1}(pr)\Big)\Omega_{lm}\frac{1-\sigma^3}{2} \tag{9.23}$$

$$- (E_p - m)j_l(pr')\Big(pj_{l-1}(pr)i\gamma^r - (E_p - m)j_l(pr)\Big)\Omega_{lm}\frac{1+\sigma^3}{2} \tag{9.24}$$

$$= \int d^3\vec{x}' N^\dagger(\vec{x}')\frac{p^2 E_p}{E_p + m}\Big(\tag{9.25}$$

$$j_l(pr')j_l(pr)\Omega_{lm}\frac{1+\sigma^3}{2} + j_{l-1}(pr')j_{l-1}(pr)\Omega_{lm}\frac{1-\sigma^3}{2} \tag{9.26}$$

If we integrate on p and use the completeness of the spherical Hankel transform, we get:

$$\int d(cos\theta)d\varphi N'^\dagger(r,\theta,\varphi)\Omega_{lm}(\theta,\varphi) = \int d(cos\theta)d\varphi N^\dagger(r,\theta,\varphi)\Omega_{lm}(\theta,\varphi) \tag{9.27}$$

Since the matrices Ω are complete in θ, φ, we have shown the completeness of the Hankel-Majorana transform:

$$\sum_{lm}\int \frac{dp\ 2(E_p + m)}{E_p\pi}\Lambda(p,r',\theta',\varphi')_{lm}\Lambda^\dagger(p,r,\theta,\varphi)_{lm} = \frac{\delta(r-r')}{r^2}\delta(cos\theta' - cos\theta)\delta(\varphi' - \varphi) \tag{9.28}$$

10 Hydrogen Atom

The Dirac equation for the Hydrogen atom is:

$$i\gamma^0(i\partial\!\!\!/ - eA\!\!\!/ - m)\Psi = 0 \tag{10.1}$$

With $A_i = 0$, $A_0 = -\frac{e}{r}$. The term with the potential is imaginary, therefore, the equation is complex.

We define the matrix:

$$\Lambda_{nlm\epsilon} = \Big(\frac{f_{nl\epsilon}(r)}{r} + \frac{g_{nl\epsilon}(r)}{r}i\gamma^r\Big)\Omega_{lm}\frac{1+\epsilon\sigma^3}{2} \tag{10.2}$$

If f and g are such that the following equations hold:

$$(E_{nl} + \frac{e^2}{r} - m)\frac{f_{nl\epsilon}(r)}{r} + (\partial_r + \frac{1-\epsilon l}{r})\frac{g_{nl\epsilon}(r)}{r} = 0 \tag{10.3}$$

$$(-E_{nl\epsilon} - \frac{e^2}{r} - m)\frac{g_{nl\epsilon}(r)}{r} + (\partial_r + \frac{1+\epsilon l}{r})\frac{f_{nl\epsilon}(r)}{r} = 0 \tag{10.4}$$

The last equations are the same equations as for the Dirac fermion in the hydrogen atom. That is, one can obtain the same solution for the Hydrogen atom, absent from the imaginary unit, using Majorana fermions. We will not solve these equations here, the solution can be seen in [8].

Then Λ verifies:

$$i\gamma^0(i\vec{\partial\!\!\!/} - m)\Lambda_{nlm\epsilon}\frac{1+\gamma^0}{2} = i(E_{nl} + \frac{e^2}{r})\Lambda_{nlm\epsilon}\frac{1+\gamma^0}{2} \tag{10.5}$$

The solution to Dirac equation is:

$$\Psi = \Lambda_{nlm\epsilon}e^{-i\gamma^0 E_{nl}x^0}\frac{1+\gamma^0}{2}\psi \tag{10.6}$$

Where ψ is a fixed spinor.

References

[1] E. Majorana, “Theory of the Symmetry of Electrons and Positrons,” *Nuovo Cim.* **14** (1937) 171–184. `http://www2.phys.canterbury.ac.nz/editorial/Majorana1937-Maiani2.pdf`.

[2] D. Hestenes, “Real Spinor Fields,” *J. Math. Phys.* **8** no. 4, (1967) 798808. `http://www.intalek.com/Index/Project/Research/RealSpinorFields.pdf`.

[3] D. Hestenes, “Gauge Gravity and Electroweak Theory,” `arXiv:0807.0060 [gr-qc]`.

[4] M. Srednicki, “Quantum field theory. Part 2. Spin one half,” `arXiv:hep-th/0409036 [hep-th]`. This is a draft version of Part 2 of a three-part textbook on quantum field theory.

[5] R. Casalbuoni, “Advanced Quantum Field Theory,” 2004. `http://theory.fi.infn.it/casalbuoni/lezioni99.pdf`.

[6] R. Szmytkowski, “Recurrence and differential relations for spherical spinors,” *ArXiv e-prints* (Nov., 2010) , `arXiv:1011.3433 [math-ph]`.

[7] N. Baddour, “Operational and convolution properties of three-dimensional fourier transforms in spherical polar coordinates,” *J. Opt. Soc. Am. A* **27** no. 10, (Oct, 2010) 2144–2155. `http://uottawa.academia.edu/NatalieBaddour/Papers/280366/Operational_and_Convolution_Properties_of_Three_Dimensional_Fourier_Transforms_in_Spherical_Polar_Coordinates`.

[8] P. S. Riseborough, “Advanced Quantum Mechanics,” May, 2011. `http://www.math.temple.edu/~prisebor/Advanced.pdf`.