Linear and angular momentum spaces with Majorana matrices

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Abstract

The Majorana matrices are the Dirac Gamma matrices times the imaginary unit. They are 4x4 real matrices in the Majorana representation. The Dirac equation for the free fermion is written only with Majorana matrices.

We show that the Majorana spinor is an irreducible representation of the restricted Lorentz group. The Fourier-Majorana and Hankel-Majorana transforms are defined and related to the linear and angular momentums of free fermion fields.

Contents

1 Introduction

The Majorana matrices, $i\gamma^\mu$, are the Dirac Gamma matrices times the imaginary unit. In the Majorana bases, the Majorana matrices are 4×4 real matrices and the Majorana spinors are 4 dimensional real vectors.

The Dirac equation for the free fermion is written only with Majorana matrices: $(i\gamma^\mu\partial_\mu - m)\Psi(x) = 0$. The solution can be a Majorana spinor. Due to this fact, Ettore Majorana noted in 1937 that *"it is perfectly, and most naturally, possible to formulate a theory of elementary neutral particles which do not have negative (energy) states"* [1]. The Majorana fermions, the hypothetical particles represented by Majorana spinors, are their own anti-particles and, therefore, neutral. There are applications of Majorana fermions in neutrino physics, dark matter searches, the fractional quantum Hall effect and superconductivity [2].

The Dirac spinors are 4 dimensional complex vectors. They represent particles different from their anti-particles. In 1967, David Hestenes tried to found a geometric role for the imaginary unit in Dirac spinors: *"As the increasing theoretical importance of antiparticle conjugation tends to show, the appearance of this $(-1)^{1/2}$ is no triviality. We submit that the $(-1)^{1/2}$ in Dirac's equation can be interpreted geometrically and (...) is inseparable from spin"* [3]. In 2008 he was still working on his theory [4].

Weyl spinors are 2 dimensional complex vectors and are irreducible representations of the restricted Lorentz group. They represent massless particles in one helicity state that are different from their anti-particles. The Electroweak theory is based on Weyl spinors, used as building blocks of any kind of spinor [5]. In this construction, the imaginary unit is implicitly considered inseparable from spin, although its interpretation is not usually discussed.

The generalization of the Dirac Gamma matrices algebra to other dimensions and metrics is called Clifford algebra. In the context of Clifford Algebras, there are people working on the geometric square roots of -1 [6] and on the generalizations of the Fourier transform [7], with applications to image processing.

Our goal is to show that the kinematic properties of a free fermion can be described by the Majorana spinors verifying Dirac equation. In chapter 2 we define the Majorana Matrices and spinors. In chapter 3 we show that the Majorana spinor is an irreducible representation of the restricted Lorentz group. In 4 and 5 we define the Fourier-Majorana and Hankel-Majorana transforms of a Majorana spinor. In 6, using the solutions to the Dirac equation, we show that the Majorana transforms are related to the linear and angular momentums of a free fermion. In 7, we extend the Majorana transforms to include the energy.

2 Majorana Matrices and Spinors

The Majorana matrices, represented by the symbol $i\gamma^\mu$, $\mu = 0, 1, 2, 3$, are the Dirac Gamma matrices, γ^μ, times the imaginary unit. The notation maintains explicit the relation between the Majorana and Dirac Gamma matrices.

Definition 2.1. The Majorana matrices, $i\gamma^\mu$, are 4×4 unitary matrices with anticommutator $\{i\gamma^\mu, i\gamma^\nu\}$:

$$(i\gamma^\mu)(i\gamma^\nu) + (i\gamma^\nu)(i\gamma^\mu) = -2g^{\mu\nu},\ \ \mu,\nu = 0,1,2,3 \tag{2.1}$$

Where $g = diag(1,-1,-1,-1)$ is the Minkowski metric. The pseudo-scalar is $i\gamma^5 \equiv -\gamma^0\gamma^1\gamma^2\gamma^3$.

The product of 2 Dirac Gamma matrices is minus the product of 2 corresponding Majorana matrices: $\gamma^\mu\gamma^\nu = -i\gamma^\mu i\gamma^\nu$.

Definition 2.2. $\Gamma_A \equiv \{i\gamma^0, i\gamma^5, \gamma^0\gamma^5, i\gamma^5\gamma^0\gamma^j\}$, $\Gamma_S \equiv \{1, \gamma^0\gamma^j, i\gamma^j, \gamma^5\gamma^j\}$, $\Gamma \equiv \Gamma_S \cup \Gamma_A$, where $j = 1,2,3$.

All matrices in Γ either commute or anti-commute with each other. Since the Majorana matrices are unitary, the matrices in Γ_A are skew-hermitian while in Γ_S are hermitian.

Definition 2.3. The set of matrices that anti-commute with a matrix A is $\Omega(A) = \{B \in \Gamma : AB = -BA\}$.

Proposition 2.4. *$\Omega(A) \cap \Gamma_S$ is not empty for $A \in \Gamma \setminus \{1\}$.*

Corollary. *The matrices $A \in \Gamma \setminus \{1\}$ have null trace: $tr(A) = tr(BAB) = -tr(A)$, for $B \in \Omega(A) \cap \Gamma_S$.*

Proposition 2.5. *Γ is a basis for the space of 4×4 complex matrices.*

Proof. There are only 16 linearly independent 4×4 complex matrices.

Let $B \equiv \sum_{i=1}^{16} a_i A_i$, where a_i are coefficients and $A_i \in \Gamma$ are different elements of the set for each i. We have $tr(A_j^\dagger B) = 4a_j$, for $j = 1, ..., 16$. Then, $B = 0$ implies that all the coefficients are null and so all the elements in Γ are linearly independent. □

Proposition 2.6. *For all commuting matrices $A, B \in \Gamma \setminus \{1\}$, $AB = BA$: all matrices in $\Gamma \setminus \{1, A, B, AB\}$ anti-commute with A or B. That is, $\Omega(A) \cup \Omega(B) = \Gamma \setminus \{1, A, B, AB\}$.*

Definition 2.7. $\Gamma_2 = \{\pm 1,\ \pm i\gamma^\mu,\ \pm\gamma^0\gamma^j,\ \pm i\gamma^5\gamma^0\gamma^j,\ \pm\gamma^\mu\gamma^5,\ \pm i\gamma^5\}$, with $\mu = 0,1,2,3$ and $j = 1,2,3$, is the group of 32 Majorana matrices products.

Definition 2.8. A 4×4 unitary representation of the Majorana matrices, M, is a map from the group Γ_2 to the space of 4×4 unitary matrices, verifying:

$$\{M(\gamma^\mu), M(\gamma^\nu)\} = -2g^{\mu\nu},\ \ \mu,\nu = 0,1,2,3 \tag{2.2}$$

$$M(k_1)M(k_2) = M(k_1 k_2),\ \ k_1, k_2 \in \Gamma_2 \tag{2.3}$$

Proposition 2.9. *The 4×4 unitary representations of the Majorana matrices are related by unitary similarity transformations.*

Proof. Let A and B be 4×4 unitary representations of the Majorana matrices. Let a and b be 4 dimensional complex vectors, verifying:

$$a = \frac{1 + A(\gamma^1\gamma^0)}{2}\frac{1 + A(\gamma^2\gamma^5)}{2}a,\ \ a^\dagger a = 1 \tag{2.4}$$

$$b = \frac{1 + B(\gamma^1\gamma^0)}{2}\frac{1 + B(\gamma^2\gamma^5)}{2}b,\ \ b^\dagger b = 1 \tag{2.5}$$

We define the matrix U as:

$$U = \frac{1}{8} \sum_{g \in \Gamma_2} B(g^{-1}) b \ a^\dagger A(g) \tag{2.6}$$

For all $h \in \Gamma$, it verifies $UA(h) = B(h)U$:

$$\begin{aligned} UA(h) &= \frac{1}{8} \sum_{g \in \Gamma_2} B(g^{-1}) b \ a^\dagger A(gh) \qquad (2.7)\\ &= \frac{1}{8} \sum_{l \in \Gamma_2} B(hl^{-1}) b \ a^\dagger A(l) = B(h)U \qquad (2.8)\end{aligned}$$

Consequently, $U^\dagger U A(h) = A(h) U^\dagger U$. Since Γ is a basis, then $U^\dagger U$ must be equal to the identity matrix times a coefficient. To check what the coefficient is:

$$\begin{aligned} tr(U^\dagger U) &= \frac{1}{64} \sum_{l,g \in \Gamma_2} b^\dagger B(gl^{-1}) b \ a^\dagger A(lg^{-1}) a \qquad (2.9)\\ &= \frac{1}{2} \sum_{k \in \Gamma_2} b^\dagger B(k^{-1}) b \ a^\dagger A(k) a \qquad (2.10)\end{aligned}$$

From proposition 2.6, we have that $a^\dagger A(k) a = b^\dagger B(k^{-1}) b = 0$, for all $k \neq \pm 1, \pm\gamma^1\gamma^0, \pm\gamma^2\gamma^5, \pm i\gamma^3$. For the eight remaining k, $a^\dagger A(k) a = b^\dagger B(k^{-1}) b = \pm 1$. Then, $tr(U^\dagger U) = 4$ which implies that U is unitary. $\square$

The Majorana matrices are themselves a 4×4 unitary representation of the Majorana matrices. Therefore, choosing a 4×4 unitary representation of the Majorana matrices is the same as choosing an orthonormal basis.

In the Majorana bases, the Majorana matrices are 4×4 real orthogonal matrices. An example of the Majorana matrices in a particular Majorana basis is:

$$\begin{aligned} i\gamma^1 &= \begin{bmatrix} +1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & +1 \end{bmatrix} \quad i\gamma^2 = \begin{bmatrix} 0 & 0 & +1 & 0 \\ 0 & 0 & 0 & +1 \\ +1 & 0 & 0 & 0 \\ 0 & +1 & 0 & 0 \end{bmatrix} \quad i\gamma^3 = \begin{bmatrix} 0 & +1 & 0 & 0 \\ +1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{bmatrix} \\ i\gamma^0 &= \begin{bmatrix} 0 & 0 & +1 & 0 \\ 0 & 0 & 0 & +1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix} \quad i\gamma^5 = \begin{bmatrix} 0 & -1 & 0 & 0 \\ +1 & 0 & 0 & 0 \\ 0 & 0 & 0 & +1 \\ 0 & 0 & -1 & 0 \end{bmatrix} \quad = -\gamma^0\gamma^1\gamma^2\gamma^3 \end{aligned} \tag{2.11}$$

Definition 2.10. The Dirac spinor is a 4 dimensional complex vector.

The space of Dirac spinors is a 4 dimensional complex vector space.

Definition 2.11. Let U be an unitary matrix such that $Ui\gamma^\mu U^\dagger$ is real, for $\mu = 0, 1, 2, 3$.

The Majorana spinor u is a Dirac spinor verifying the Majorana condition:

$$U^\dagger U^* u^* = u \tag{2.12}$$

Where * denotes complex conjugation.

The space of Majorana spinors is a 4 dimensional real vector space. Note that a linear combinations of Majorana spinors with complex coefficients do not verify the Majorana condition. The Majorana spinor, in the Majorana bases, is a 4 dimensional real vector.

The linear transformations from and to Majorana spinors, are generated by the linear combinations with real coefficients of the 16 matrices in the basis Γ.

If p, q are Lorentz vectors, we define $\not{p} = \gamma^\mu p_\mu$ and $p \cdot q = p^\mu q_\nu$. Given a mass $m \geq 0$, we define:

$$\vec{p}^i = p^i, \ i = 1, 2, 3 \tag{2.13}$$
$$\vec{\not{p}} = \vec{\gamma} \cdot \vec{p} \tag{2.14}$$
$$E_p = \sqrt{\vec{p}^2 + m^2} \tag{2.15}$$
$$\not{p} = \gamma^0 p^0 - \vec{\gamma} \cdot \vec{p} \tag{2.16}$$
$$[\not{p}] = \gamma^0 E_p - \vec{\gamma} \cdot \vec{p} \tag{2.17}$$

Note that $\not{p}$ is not necessarily on-shell, while $[\not{p}]$ is on-shell, that is $([\not{p}])^2 = m^2$. Both E_p and $[\not{p}]$ do not depend on p^0.

3 Majorana representation of the Lorentz group

3.1 Lorentz group

We define some symbols for the sets we will use:

Definition 3.1. $\mathbf{M}(n, \mathbb{R})$ is the set of $n \times n$ real matrices.
$GL(n, \mathbb{R})$ is the group of $n \times n$ real invertible matrices.
$O(n)$ is the group of $n \times n$ real orthogonal matrices.
$SO(n)$ is the group of $n \times n$ real orthogonal matrices with determinant 1.
$SPD(n)$ is the set of $n \times n$ real symmetric positive definite matrices.

Definition 3.2. The Lorentz group, $O(1,3) = \{\Lambda \in \mathbf{M}(4, \mathbb{R}) : \Lambda^T g \Lambda = g\}$, is the set of matrices that leave the metric, $g = diag(1, -1, -1, -1)$, invariant. Their elements are the Lorentz matrices.

Proposition 3.3. *$O(1,3)$ is a group, with the matrix product as the group operation.*

Proof. 1) The matrix product is associative.
2) The identity matrix $1 \in O(1,3)$: $1^T g 1 = g$.
3) From the equation defining $O(1,3)$, we get that $det^2(\Lambda) = 1$ and so Λ is invertible. Multiplying the equation by $(\Lambda^{-1})^T$ on the left and Λ^{-1} on the right, we get:

$$g = (\Lambda^{-1})^T g \Lambda^{-1} \tag{3.1}$$

The inverse matrix $\Lambda^{-1} \in O(1,3)$. 4) For $\Lambda_1, \Lambda_2 \in O(1,3)$, the product $\Lambda_1 \Lambda_2 \in O(1,3)$:

$$\Lambda_2^T \Lambda_1^T g \Lambda_1 \Lambda_2 = \Lambda_2^T g \Lambda_2 = g \tag{3.2}$$

□

Note. If $\Lambda \in O(1,3)$, also the transpose matrix $\Lambda^T \in O(1,3)$. Multiplying eq. (3.1) by Λ^T on the left and Λ on the right, we get $\Lambda g \Lambda^T = g$.

Definition 3.4. The discrete Lorentz group is the subgroup $\Delta = \{1, g, -g, -1\}$, where 1 is the identity matrix and g the metric.

The parity and time-reversal transformation matrices are g and $-g$.

Definition 3.5. The restricted Lorentz group is the subset

$$SO^+(1,3) = \{\Lambda \in O(1,3) : det(\Lambda) = 1,\ \Lambda_{00} \geq 1\} \tag{3.3}$$

It is also called the special, or proper ($det(\Lambda) = 1$) orthochronous ($\Lambda_{00} \geq 1$), Lorentz group.

Proposition 3.6. *The restricted Lorentz group $SO^+(1,3)$ is a group.*

Proof. 1) $SO^+(1,3)$ is a subset of a group. It includes the identity matrix, $1 \in SO^+(1,3)$.

2) If $\Lambda \in O^(1,3)$, let $v_\Lambda = \sqrt{\sum_{i=1}^3 \Lambda_{0i}^2}$. We have:

$$(\Lambda g \Lambda^T)_{00} = \Lambda_{00}^2 - v_\Lambda^2 = 1 \implies \Lambda_{00}^2 \geq 1,\ \Lambda_{00}^2 \geq v_\Lambda^2 \tag{3.4}$$

If $\Lambda \in SO^+(1,3)$, then $\Lambda_{00} > v_\Lambda$.

Given $\Lambda, \Lambda' \in SO^+(1,3)$, the product $\Lambda\Lambda'^{-1} \in SO^+(1,3)$:

$$det(\Lambda_1\Lambda_2^{-1}) = \frac{det(\Lambda_1)}{det(\Lambda_2)} = 1 \tag{3.5}$$

$$(\Lambda\Lambda'^{-1})_{00} = (\Lambda g \Lambda'^T g)_{00} = \Lambda_{00}\Lambda'_{00} - \sum_{i=1}^{3} \Lambda_{0i}\Lambda'_{0i} > \Lambda_{00}\Lambda'_{00} - v_\Lambda v_{\Lambda'} > 0 \tag{3.6}$$

Since $(\Lambda\Lambda')_{00}^2 \geq 1$, then $(\Lambda\Lambda')_{00} > 0 \implies (\Lambda\Lambda')_{00} \geq 1$. □

We will now see how a Lorentz matrix can be factorized in matrices of the restricted and discrete Lorentz groups.

Proposition 3.7. *For all Lorentz matrices $\Lambda \in O(1,3)$, there is an unique discrete Lorentz group matrix $d \in \Delta$ and an unique restricted orthogonal Lorentz matrix $\Lambda' \in SO^+(1,3)$, such that:*

$$\Lambda = d\Lambda' \tag{3.7}$$

Proof. All $\Lambda \in O(1,3)$ verify $det(\Lambda) = \pm 1$ and $a\Lambda_{00} \geq 1$, with $a = \pm 1$. In each case there is an unique $d \in \Delta$, such that $\Lambda' = d\Lambda \in SO^+(1,3)$:

$$+a = +det(\Lambda) = 1 \implies d = +1,\ \Lambda' = +1\Lambda \tag{3.8}$$
$$+a = -det(\Lambda) = 1 \implies d = +g,\ \Lambda' = +g\Lambda \tag{3.9}$$
$$-a = +det(\Lambda) = 1 \implies d = -1,\ \Lambda' = -1\Lambda \tag{3.10}$$
$$-a = -det(\Lambda) = 1 \implies d = -g,\ \Lambda' = -g\Lambda \tag{3.11}$$

□

Remark 3.8. *Every real invertible matrix can be uniquely factored as the product of an orthogonal matrix and a symmetric positive definite matrix.*

Lemma 3.9. *For all Lorentz matrices $\Lambda \in O(1,3)$, there is an unique orthogonal matrix $\Theta \in O(4)$ and an unique symmetric positive definite matrix $\Pi \in SPD(4)$, such that $\Lambda = \Theta\Pi$. These are Lorentz matrices $\Theta, \Pi \in O(3,1)$.*

Proof. From remark 3.8, for all $\Lambda \in O(3,1)$, there are unique $\Theta \in O(4)$ and unique $\Pi \in SPD(4)$ such that $\Lambda = \Theta\Pi$. This implies:

$$\Theta\Pi g \Pi^T \Theta^T = \Theta\Pi g \Pi \Theta^T = g \tag{3.12}$$

$$\Pi^T \Theta^T g \Theta \Pi = \Pi \Theta^T g \Theta \Pi = g \tag{3.13}$$

We multiply the first equation by $\Pi\Theta^T$ on the left and by $\Theta\Pi$ on the right. Using the second equation, we get:

$$\Pi^2 g \Pi^2 = g \tag{3.14}$$

For all symmetric positive definite matrix Π, there is a unique symmetric matrix B such that $\Pi = e^B$. From the above equation:

$$e^{2B} g e^{2B} = e^{2B} e^{g2Bg} g = g \tag{3.15}$$

Since B is unique, we get $B = -gBg$, $e^B g e^B = g$ and:

$$\Theta\Pi g \Pi^T \Theta^T = \Theta g \Theta^T = g \tag{3.16}$$

So, $\Theta, \Pi \in O(3,1)$. □

Now we will study the symmetric positive definite matrix Π.

Lemma 3.10. *All symmetric positive definite Lorentz matrices $\Pi \in SPD(4) \cap O(1,3)$ are restricted Lorentz matrices $\Pi \in SO^+(1,3)$.*

Proof. $\Pi \in SPD(4)$ verifies $det(\Pi) > 0$ and $u^T \Pi u > 0$ for all non-zero real vectors u. Choosing $u_0 = 1$ and $u_i = 0,\ i = 1,2,3$, we obtain $\Pi_{00} > 0$.

$\Pi \in O(1,3)$ verifies $det^2(\Pi) = 1$ and $\Pi_{00}^2 \geq 1$.

$\Pi \in SPD(4) \cap O(1,3)$ verifies $det(\Pi) = 1$ and $\Pi_{00} \geq 1$. □

The symmetric positive definite Lorentz matrices represent the Lorentz boost transformations.

It follows the study of the orthogonal restricted Lorentz matrices.

Lemma 3.11. *The set of orthogonal restricted Lorentz matrices $O(4) \cap SO^+(1,3)$ is a group. For all $\Theta \in O(4) \cap SO^+(1,3)$, there is an unique 3×3, determinant 1, orthogonal matrix $\theta \in SO(3)$ such that:*

$$\Theta = \begin{bmatrix} 1 & 0\,0\,0 \\ 0 & \\ 0 & \theta \\ 0 & \end{bmatrix} \tag{3.17}$$

Proof. The intersection of 2 subgroups is a subgroup. $O(4)$ and $SO^+(1,3)$ are subgroups of the group of real invertible 4×4 matrices. Let the orthogonal restricted Lorentz matrix Θ be given by:

$$\Theta = \begin{bmatrix} a & v^T \\ u & \theta \end{bmatrix} \tag{3.18}$$

Where a is a real number, u, v are real 3 dimensional column vectors and ϕ is a 3×3 real matrix.

From the Lorentz group condition:

$$\Theta^T g \Theta g = 1 \implies a^2 - u^T u = 1 \tag{3.19}$$
$$\Theta g \Theta^T g = 1 \implies a^2 - v^T v = 1 \tag{3.20}$$

From the orthogonality:

$$\Theta^T \Theta = 1 \implies a^2 + u^T u = 1 \tag{3.21}$$
$$\Theta \Theta^T = 1 \implies a^2 + v^T v = 1 \tag{3.22}$$

We get $v = u = 0$ and $\theta^T\theta = 1$. From the proper and orthochronous conditions, we get $a = 1$ and $det(\theta) = 1$. □

The orthogonal restricted Lorentz matrices represent the spacial rotations.

Theorem 3.12. *All $\Lambda \in O(1,3)$ can be factored uniquely in the product of a discrete Lorentz group matrix $d \in \Delta$, a spacial rotation $\Theta' \in O(4)\cap SO^+(1,3)$ and a Lorentz boost $\Pi \in SPD(4)\cap SO^+(1,3)$.*

In the particular case of a restricted Lorentz matrix $\Lambda \in SO^+(1,3)$, $d = 1$.

Proof. From propositions 3.9 and 3.10, there are unique $\Theta \in O(4)\cap O(1,3)$ and $\Pi \in SPD(4)\cap SO^+(1,3)$, such that $\Lambda = \Theta\Pi$.

From proposition 3.7, there are unique $d \in \Delta$ and $\Theta' \in O(4)\cap SO^+(1,3)$, such that $\Theta = d\Theta'$.

Since $\Theta', \Pi \in SO^+(1,3)$, the product $\Theta'\Pi \in SO^+(1,3)$.

In the particular case $\Lambda \in SO^+(1,3)$, since the factors are unique, $d = 1$.

From proposition 3.7, there are unique $d' \in \Delta$ and $\Lambda' \in SO^+(1,3)$, such that $\Lambda = d'\Lambda'$. From the uniqueness of the factors, we have $d' = d$ and $\Lambda' = \Theta'\Pi$.

In the particular case $\Lambda \in SO^+(1,3)$, these are the unique factors $d' = 1 \in \Delta$ and $\Lambda' = \Lambda \in SO^+(1,3)$ that satisfy $\Lambda = d'\Lambda'$. □

3.2 Restricted Lorentz group

Definition 3.13. The boost generators are the matrices:

$$K_1 = \begin{bmatrix} 0&1&0&0 \\ 1&0&0&0 \\ 0&0&0&0 \\ 0&0&0&0 \end{bmatrix}, K_2 = \begin{bmatrix} 0&0&1&0 \\ 0&0&0&0 \\ 1&0&0&0 \\ 0&0&0&0 \end{bmatrix}, K_3 = \begin{bmatrix} 0&0&0&1 \\ 0&0&0&0 \\ 0&0&0&0 \\ 1&0&0&0 \end{bmatrix} \tag{3.23}$$

The rotation generators are the matrices:

$$J_1 = \begin{bmatrix} 0&0&0&0 \\ 0&0&0&0 \\ 0&0&0&-1 \\ 0&0&1&0 \end{bmatrix}, J_2 = \begin{bmatrix} 0&0&0&0 \\ 0&0&0&-1 \\ 0&0&0&0 \\ 0&1&0&0 \end{bmatrix}, J_3 = \begin{bmatrix} 0&0&0&0 \\ 0&0&-1&0 \\ 0&1&0&0 \\ 0&0&0&0 \end{bmatrix} \tag{3.24}$$

They verify the Lie algebra:

$$[J_i, J_j] = -\,\epsilon_{ijk} J_k \tag{3.25}$$
$$[J_i, K_j] = -\,\epsilon_{ijk} K_k \tag{3.26}$$
$$[K_i, K_j] = \epsilon_{ijk} J_k \tag{3.27}$$

Where $i, j, k = 1, 2, 3$ and ϵ is the Levi-Civita symbol.

Remark 3.14. *Let A and B be square matrices. The Baker-Campbell-Hausdorff formula [8], expresses the product $e^A e^B$ as an exponential of a series of nested commutators of A and B.*

Lemma 3.15. *The restricted Lorentz group is the set of exponentials of the linear combinations of generators. That is $SO^+(1,3) = E$, where:*

$$E = \{e^{\theta^i J_i + b^i K_i},\ \theta^i, b^i \in \mathbb{R},\ i = 1, 2, 3\} \tag{3.28}$$

Proof. Using proposition 3.10, any Lorentz boost $\Pi \in SPD(4) \cap SO^+(1,3)$ can be given by $\Theta = e^{b^i K_i}$, for unique $b^i \in \mathbb{R},\ i = 1, 2, 3$.

For all orthogonal matrix with determinant 1, θ, there is a skew-symmetric matrix A such that $\theta = e^A$. From proposition 3.11, any spacial rotation $\Theta \in O(4) \cap SO^+(1,3)$ can be given by $\Theta = e^{\theta^i J_i}$, for some $\theta^i \in \mathbb{R},\ i = 1, 2, 3$.

From theorem 3.12, for all $\Lambda \in SO^+(3)$, there are $\theta^i, b^i \in \mathbb{R}$ such that:

$$\Lambda = e^{\theta^i J_i} e^{b^i K_i} \tag{3.29}$$

From the Baker-Campbell-Hausdorff formula in remark 3.14 and the fact that a series of nested commutators of the generators can be expressed by a linear combination of generators, we get $e^{\theta^i J_i} e^{b^i K_i} = e^{\phi^i J_i + c^i K_i}$ for some $\phi^i, c^i \in \mathbb{R}$, and so $\Lambda \in E$.

For all $\theta^i, b^i \in \mathbb{R}$, we have:

$$\Lambda_n \equiv (e^{\frac{\theta^i J_i}{n}} e^{\frac{b^i K_i}{n}})^n \tag{3.30}$$
$$e^{\theta^i J_i + b^i K_i} = \lim_{n \to \infty} \Lambda_n \tag{3.31}$$

Λ_n is the n times product of restricted Lorentz matrices, which verifies $det(\Lambda_n) = 1$ and $(\Lambda_n)_{00} \geq 1$, even in the limit $n \to \infty$. So, $e^{\theta^i J_i + b^i K_i} \in SO^+(1,3)$. □

3.3 Majorana Spinor representation

Definition 3.16. A representation (M_G, V) of a group G is defined by:

1) the representation space V, which is a vector space;

2) the representation map $M : G \to GL(V)$ from the group elements to the automorphisms of the representation space, verifying for $\Lambda_1, \Lambda_2 \in G$:

$$M(\Lambda_1) M(\Lambda_2) = M(\Lambda_1 \Lambda_2) \tag{3.32}$$

Two examples of representations of the restricted Lorentz group are the real scalar ($M(\Lambda) = 1$ and $V = \mathbb{R}$) and the real Lorentz vector ($M(\Lambda) = \Lambda$ and as representation space the real Lorentz vectors $V = \mathbb{R}^4$) representations, where $\Lambda \in SO^+(1,3)$.

Definition 3.17. The Majorana spinor representation is defined by:
1) the representation space V is the space of Majorana spinors;
2) For $\theta^i, b^i \in \mathbb{R},\ i = 1, 2, 3$, the representation map is:

$$M(e^{\theta^i J_i + b^i K_i}) = e^{(\theta^i i\gamma^5 + b^i)\frac{1}{2}\gamma^0\gamma^i} \tag{3.33}$$

Where $i\gamma^5 = -\gamma^0\gamma^1\gamma^2\gamma^3$.

Proposition 3.18. *The Majorana spinor representation is a representation of the restricted Lorentz group.*

Proof. The commutation relations of the Lorentz generators J_i and K_i are verified by their correspondent Majorana matrices $J'_i = \frac{1}{2}i\gamma^5\gamma^0\gamma^i$ and $K'_i = \frac{1}{2}\gamma^0\gamma^i$:

$$[J'_i, J'_j] = [\frac{1}{2}i\gamma^5\gamma^0\gamma^i, \frac{1}{2}i\gamma^5\gamma^0\gamma^j] = -\epsilon_{ijk}\frac{1}{2}i\gamma^5\gamma^0\gamma^k = -\epsilon_{ijk}J'_k \tag{3.34}$$

$$[J'_i, K'_j] = [\frac{1}{2}i\gamma^5\gamma^0\gamma^i, \frac{1}{2}\gamma^0\gamma^j] = -\epsilon_{ijk}\frac{1}{2}\gamma^0\gamma^k = -\epsilon_{ijk}K'_k \tag{3.35}$$

$$[K'_i, K'_j] = [\frac{1}{2}\gamma^0\gamma^i, \frac{1}{2}\gamma^0\gamma^j] = \epsilon_{ijk}\frac{1}{2}i\gamma^5\gamma^0\gamma^k = \epsilon_{ijk}J'_k \tag{3.36}$$

From the Baker-Campbell-Hausdorff formula in remark 3.14, we get for $\Lambda_1, \Lambda_2 \in SO^+(1,3)$:

$$M(\Lambda_1)M(\Lambda_2) = M(\Lambda_1\Lambda_2) \tag{3.37}$$

□

Definition 3.19. Let W be a subspace of V. (M_G, W) is a subrepresentation of (M_G, V) if W is invariant under the group action, that is, for all $w \in W$: $(M(g)w) \in W$, for all $g \in G$.

Definition 3.20. $W^\perp$ is the orthogonal complement of the subspace W of the vector space V if:
1) all $v \in V$ can be expressed as $v = w + x$, where $w \in W$ and $x \in W^\perp$;
2) if $w \in W$ and $x \in W^\perp$, then $x^\dagger w = 0$.

Definition 3.21. The representation (M_G, V) is semi-simple if for all subrepresentation (M_G, W) of (M_G, V) , $(M_G, W^\perp)$ is also a subrepresentation of (M_G, V), where $W^\perp$ is the orthogonal complement of the subspace W.

Lemma 3.22. *Consider a representation (M_G, V) of a group G. For all $g \in G$, if there is $h \in G$ such that $M(h) = M^\dagger(g)$, then the representation (M_G, V) is semi-simple.*

Proof. Let (M_G, W) be a subrepresentation of (M_G, V). $W^\perp$ is the orthogonal complement of W.

For all $x \in W^\perp$, $w \in W$ and $g \in G$, $(M(g)x)^\dagger w = x^\dagger(M^\dagger(g)w)$.

Since W is invariant and there is $h \in G$, such that $M(h) = M^\dagger(g)$, then $w' \equiv (M^\dagger(g)w) \in W$.

Since $x \in W^\perp$ and $w' \in W$, then $x^\dagger w' = 0$.

This implies that if x is in the orthogonal complement of W ($x \in W^\perp$), also $M(g)x$ is in the orthogonal complement of W ($M(g)x \in W^\perp$), for all $g \in G$. □

Proposition 3.23. *The Majorana spinor representation of the restricted Lorentz group is semi-simple.*

Proof. In the Majorana spinor representation, for all $\theta^i, b^i \in \mathbb{R},\ i = 1, 2, 3$:

$$M^\dagger(e^{\theta^i J_i + b^i K_i}) = e^{(-\theta^i i\gamma^5 + b^i)\frac{1}{2}\gamma^0\gamma^i} = M(e^{-\theta^i J_i + b^i K_i}) \tag{3.38}$$

Since $-\theta \in \mathbb{R}$, from lemma 3.22, the Majorana spinor representation is semi-simple. $\square$

Definition 3.24. A representation (M_G, V) is irreducible if their only sub-representations are the trivial sub-representations: (M_G, V) and $(M_G, \{0\})$, where $\{0\}$ is the null space.

Lemma 3.25. *Consider a semi-simple representation (M_G, V) of a group G. If the set of hermitian automorphisms of V that square to 1 and commute with $M(g)$, for all $g \in G$, is $\{+1, -1\}$, then the representation (M_G, V) is irreducible (1 is the identity matrix).*

Proof. Let (M_G, W) and $(M_G, W^\perp)$ be sub-representations of (M_G, V), where $W^\perp$, the orthogonal complement of W.

There is an automorphism $P : V \to V$, such that, for $w, w' \in W,\ x, x' \in W^\perp$, $P(w + x) = (w - x)$. $P^2 = 1$ and P is hermitian:

$$(w' + x')^\dagger (P(w + x)) = w'^\dagger w - x'^\dagger x = (P(w' + x'))^\dagger (w + x) \tag{3.39}$$

Let $w' \equiv M(g)w \in W$ and $x' \equiv M(g)x \in W^\perp$:

$$M(\Lambda)P(w + x) = M(\Lambda)(w - x) = (w' - x') \tag{3.40}$$
$$PM(\Lambda)(w + x) = P(w' + x') = (w' - x') \tag{3.41}$$

Which implies that P commutes with $M(g)$ for all $g \in G$.

If $P = +1$, then $W = V$:

$$+(w + x) = P(w + x) = (w - x) \implies x = 0 \tag{3.42}$$

If $P = -1$, then W is the null space:

$$-(w + x) = P(w + x) = (w - x) \implies w = 0 \tag{3.43}$$

$\square$

Proposition 3.26. *The Majorana spinor representation of the restricted Lorentz group is irreducible.*

Proof. The hermitian linear transformations from and to Majorana spinors, are generated by the linear combinations with real coefficients of the 10 matrices in the basis $\Gamma_S \equiv \{1, \gamma^0\gamma^j, i\gamma^j, \gamma^5\gamma^j\}$, where $j = 1, 2, 3$.

The only matrix in Γ_S commuting with $M(\Lambda)$, for all $\Lambda \in SO^+(1, 3)$ is the identity matrix. Therefore, the set of hermitian automorphisms of the Majorana spinors that square to 1 and commute with $M(\Lambda)$, for all $\Lambda \in SO^+(1, 3)$, is $\{+1, -1\}$. Applying proposition 3.23 and lemma 3.25 the proposition is proved. $\square$

3.4 Discussion on the Dirac Spinor representation decomposition

In the Dirac spinor representation the Dirac spinors are the representation space, the representation map M is the same of the Majorana spinor representation. The matrix $\gamma^5 = -i(i\gamma^5)$ is an hermitian automorphism of the Dirac spinors that squares to 1 and commutes with $M(\Lambda)$, for all $\Lambda \in SO^+(1,3)$.

The subspaces of Dirac spinors $u_\pm$, related with Weyl spinors, verifying $u_\pm = \frac{1\pm\gamma^5}{2}u_\pm$ are invariant: $M(\Lambda)u_\pm = \frac{1\pm\gamma^5}{2}M(\Lambda)u_\pm$, for all $\Lambda \in SO^+(1,3)$. So, the Dirac spinor representation is reducible. The invariant spinors $u_\pm$ are the representation spaces of the subrepresentations.

The subsets of Dirac spinors $v_\pm$, related with Majorana spinors, verifying in the Majorana bases $v_\pm^* = \pm v_\pm$ are also linear subspaces for linear combinations with real coefficients. Since in the Majorana bases, $M(\Lambda)$ is real, then $v_\pm$ are also invariant subspaces over the real numbers. $v_\pm$ are the representation spaces of two subrepresentations of the Dirac spinor representation.

Note that for linear combinations with complex coefficients, the spinors $v_\pm$ do not form a subspace, that is, they are only subspaces over the real numbers, not over the complex numbers.

The reader should not be confused by the fact that usually in the literature, the Dirac spinor is only decomposed in Weyl spinors, with no mention to the Majorana spinors. The reason for this is that in many successful theories, from Quantum Mechanics to Quantum Field Theory, the vector spaces are over the complex numbers.

Some readers might have the respectful belief that modern Physics must be based on vector spaces over the complex numbers. Such belief, even if it is founded, should not confuse them when judging the correctness of proposition 3.26, because the fact is that the representation theory is valid for vector spaces over the real numbers, complex numbers or any other field.

The Jordan-Holder theorem implies that if a semi-simple representation of a subgroup H of a group G is decomposable in two non-trivial irreducible subrepresentations, then this decomposition is unique up to an isomorphism. But it does not imply that the isomorphism is valid when considering the larger group G. For instance, if we consider the subgroup of rotations $SO^+(1,3)\cap O(4)$, then the subspaces of Dirac spinors $w_\pm$ verifying $w_\pm = \frac{1\pm\gamma^0}{2}w_\pm$ are invariant: $M(\Lambda)w_\pm = \frac{1\pm\gamma^0}{2}M(\Lambda)w_\pm$, for all $\Lambda \in SO^+(1,3)\cap O(4)$. It can be shown that the subrepresentations where the representation spaces are $w_\pm$ or $u_\pm$, are irreducible. Although there is an isomorphism between the subrepresentations $w_\pm$ and $u_\pm$, through the correspondence between γ^0 and γ^5, valid for the subgroup of rotations, the isomorphism is no longer valid when considering the restricted Lorentz group.

The restricted Lorentz group is a subgroup of the group of symmetries of Quantum Field Theory. The Jordan-Holder theorem implies that there is an isomorphism between the Weyl and Majorana spinor representations of the restricted Lorentz group. But Weyl fermions have charge and no mass, Majorana fermions have mass and no charge, so they are not isomorphic in Quantum Field Theory.

4 Linear Momentum of Majorana spinors

Definition 4.1. $L^2(\mathbb{R}^n)$ is the Hilbert space of real functions of n real variables whose square is Lebesgue integrable in $\mathbb{R}^n$. The internal product is:

$$< f, g >\equiv \int d^n x f(x) g(x),\ \ f, g \in L^2(\mathbb{R}^n) \tag{4.1}$$

Remark 4.2. *The Fourier transform is injective in the space of complex square integrable functions.*

If $f \in L^2(\mathbb{R}^n)$, then $f_s, f_c \in L^2(\mathbb{R}^n)$:

$$f_c(p) \equiv \int d^n x\ \ \cos(p \cdot x) f(x) \tag{4.2}$$

$$f_s(p) \equiv \int d^n x\ \ \sin(p \cdot x) f(x) \tag{4.3}$$

Also, the Dirac delta δ^n is well defined:

$$\delta^n(x) \equiv \int \frac{d^n p}{(2\pi)^n} \cos(p \cdot x) \tag{4.4}$$

$$f(0) = \int d^n x\ \delta^n(x) f(x) \tag{4.5}$$

The domain of integration is $\mathbb{R}^n$.

Remark 4.3. *The derivative ∂_i, $i = 1, ..., n$, is a skew-symmetric operator of the Hilbert space $L^2(\mathbb{R}^n)$:*

$$\int d^n x (\partial_i f(x)) g(x) = -\int d^n x f(x)(\partial_i g(x)),\ \ f, g \in L^2(\mathbb{R}^n) \tag{4.6}$$

Definition 4.4. $L^2_4(\mathbb{R}^n)$ is the Hilbert space of Majorana spinors whose 4 real components in the Majorana bases are in $L^2(\mathbb{R}^n)$. The internal product is:

$$< \Psi, \Phi >\equiv \int d^n x\ \Psi^\dagger(x)\Phi(x),\ \ \Psi, \Phi \in L^2_4(\mathbb{R}^n) \tag{4.7}$$

Definition 4.5. The Fourier-Majorana Transform $\psi(\vec{p})$ of a Majorana spinor $\Psi(\vec{x}) \in L^2_4(\mathbb{R}^3)$ is the Majorana spinor:

$$\psi(\vec{p}) \equiv \int d^3\vec{x}\ O(\vec{p}, \vec{x})\Psi(\vec{x}) \tag{4.8}$$

$$O(\vec{p}, \vec{x}) \equiv e^{-i\gamma^0 \vec{p}\cdot\vec{x}} \frac{\not{p}\gamma^0 + m}{\sqrt{E_p + m}\sqrt{2E_p}} \tag{4.9}$$

Where $m \geq 0$ $p^0 = E_p = \sqrt{\vec{p}^2 + m^2}$.

Proposition 4.6. *The Fourier-Majorana Transform $\psi(\vec{p})$ of a Majorana spinor $\Psi(\vec{x}) \in L^2_4(\mathbb{R}^3)$ is also in the Hilbert space $L^2_4(\mathbb{R}^3)$.*

Proof. In the Majorana bases, $O(\vec{p},\vec{x})$ and $\Psi(\vec{x})$ are real and so is $\psi(\vec{p})$.

We have:

$$|[\frac{\not{p}\gamma^0+m}{\sqrt{E_p+m}\sqrt{2E_p}}]_{ij}|^2 \leq \frac{E_p+m}{2E_p} \leq 1,\ \ i,j=1,2,3,4 \tag{4.10}$$

$$|\psi_i(\vec{p})|^2 \leq \sum_{j=1}^{4} |\int d^3\vec{x}\ \cos(\vec{p}\cdot\vec{x})\Psi_j(\vec{x})|^2 + |\int d^3\vec{x}\ \sin(\vec{p}\cdot\vec{x})\Psi_j(\vec{x})|^2 \tag{4.11}$$

From remark 4.2, we have that both $\int d^3\vec{x}\ \cos(\vec{p}\cdot\vec{x})\Psi_j(\vec{x})$ and $\int d^3\vec{x}\ \sin(\vec{p}\cdot\vec{x})\Psi_j(\vec{x})$ are square integrable and therefore $|\psi_i(\vec{p})|^2$ is square integrable. □

Proposition 4.7. *The inverse Fourier-Majorana transform of $\psi(\vec{p})$ is:*

$$\Psi(\vec{x}) = \int \frac{d^3\vec{p}}{(2\pi)^3} O^\dagger(\vec{p},\vec{x})\psi(\vec{p}) \tag{4.12}$$

$$O^\dagger(\vec{p},\vec{x}) = \frac{\not{p}\gamma^0+m}{\sqrt{E_p+m}\sqrt{2E_p}} e^{i\gamma^0\vec{p}\cdot\vec{x}} \tag{4.13}$$

$O^\dagger$ is the hermitian conjugate of O.

Proof. The matrix $O^\dagger(\vec{p},\vec{x})$ verifies:

$$O^\dagger(\vec{p},\vec{x}) = \frac{\not{p}}{m} O^\dagger(\vec{p},\vec{x})\gamma^0 \tag{4.14}$$

$$i\gamma^0(i\vec{\not{\partial}}-m)O^\dagger(\vec{p},\vec{x}) = -\gamma^0\vec{\not{p}}O^\dagger(\vec{p},\vec{x})i\gamma^0 - \gamma^0\not{p}O^\dagger(\vec{p},\vec{x})i\gamma^0 \tag{4.15}$$

$$= -O^\dagger(\vec{p},\vec{x})i\gamma^0 E_p \tag{4.16}$$

From remark 4.3, the operator $i\gamma^0(i\vec{\not{\partial}}-m)$ is skew-hermitian, implying:

$$\int d^3\vec{x}O(\vec{q},\vec{x})O^\dagger(\vec{p},\vec{x})i\gamma^0 E_p = \int d^3\vec{x}i\gamma^0 E_q O(\vec{q},\vec{x})O^\dagger(\vec{p},\vec{x}) \tag{4.17}$$

Noting that $E_p+E_q>0$, this implies that:

$$\int d^3\vec{x}e^{-i\gamma^0\vec{q}\cdot\vec{x}} \frac{\vec{\not{q}}\gamma^0(E_p+m)+\vec{\not{p}}\gamma^0(E_q+m)}{\sqrt{E_q+m}\sqrt{2E_q}\sqrt{E_p+m}\sqrt{2E_p}} e^{i\gamma^0\vec{p}\cdot\vec{x}} = 0 \tag{4.18}$$

Therefore, we get:

$$\int d^3\vec{x}O(\vec{q},\vec{x})O^\dagger(\vec{p},\vec{x}) = \int d^3\vec{x}e^{-i\gamma^0\vec{q}\cdot\vec{x}} \frac{(E_p+m)(E_q+m)+\vec{\not{q}}\gamma^0\vec{\not{p}}\gamma^0}{\sqrt{E_q+m}\sqrt{2E_q}\sqrt{E_p+m}\sqrt{2E_p}} e^{i\gamma^0\vec{p}\cdot\vec{x}} \tag{4.19}$$

$$= \int d^3\vec{x}e^{-i\gamma^0(\vec{q}-\vec{p})\cdot\vec{x}} \frac{(E_p+m)(E_q+m)+\vec{\not{q}}\gamma^0\vec{\not{p}}\gamma^0}{\sqrt{E_q+m}\sqrt{2E_q}\sqrt{E_p+m}\sqrt{2E_p}} \tag{4.20}$$

$$= (2\pi)^3\delta^3(\vec{q}-\vec{p})\frac{(E_p+m)(E_p+m)+\vec{p}^2}{(E_p+m)2E_p} \tag{4.21}$$

$$= (2\pi)^3\delta^3(\vec{q}-\vec{p})\frac{(E_p+m)(E_p+m)+(E_p+m)(E_p-m)}{(E_p+m)2E_p} \tag{4.22}$$

$$= (2\pi)^3\delta^3(\vec{q}-\vec{p}) \tag{4.23}$$

The other way around:

$$\int \frac{d^3\vec{p}}{(2\pi)^3} O^\dagger(\vec{p},\vec{y}) O(\vec{p},\vec{x}) = \int \frac{d^3\vec{p}}{(2\pi)^3} \frac{\not{p}\gamma^0 + m}{\sqrt{E_p+m}\sqrt{2E_p}} e^{i\gamma^0 \vec{p}\cdot(\vec{y}-\vec{x})} \frac{\not{p}\gamma^0 + m}{\sqrt{E_p+m}\sqrt{2E_p}} \tag{4.24}$$

$$= \int \frac{d^3\vec{p}}{(2\pi)^3} e^{i\frac{\not{p}}{m}\vec{p}\cdot(\vec{y}-\vec{x})} \frac{\not{p}\gamma^0}{E_p} \tag{4.25}$$

$$= \int \frac{d^3\vec{p}}{(2\pi)^3} \cos(\vec{p}\cdot(\vec{y}-\vec{x})) + \tag{4.26}$$

$$+ \int \frac{d^3\vec{p}}{(2\pi)^3} (-\cos(\vec{p}\cdot(\vec{y}-\vec{x})) \frac{\vec{\not{p}}\gamma^0}{E_p} + \sin(\vec{p}\cdot(\vec{y}-\vec{x})) \frac{mi\gamma^0}{E_p} \tag{4.27}$$

$$= \delta^3(\vec{y}-\vec{x}) \tag{4.28}$$

Note that both $\cos(\vec{p}\cdot(\vec{y}-\vec{x}))\frac{\vec{\not{p}}\gamma^0}{E_p}$ and $\sin(\vec{p}\cdot(\vec{y}-\vec{x}))\frac{mi\gamma^0}{E_p}$ are odd in $\vec{p}$ and therefore do not contribute to the integral. □

5 Angular momentum of Majorana spinors

5.1 Majorana Spin

Definition 5.1. The Majorana spin operators $\frac{1}{2}\sigma^k$ are defined as:

$$\frac{1}{2}\sigma^k \equiv \frac{1}{2}\gamma^k\gamma^5, \quad k = 1,2,3 \tag{5.1}$$

They verify the angular momentum algebra:

$$[\frac{1}{2}\sigma^i, \frac{1}{2}\sigma^j] = i\gamma^0 \epsilon^{ijk} \frac{1}{2}\sigma^k \tag{5.2}$$

Where ϵ^{ijk} is the Levi-Civita symbol. Note that $i\gamma^0$ commutes with σ^k and squares to -1, so it plays the role of the imaginary unit in the angular momentum algebra.

The eigenstates of $\frac{1}{2}\sigma^3$ are the Majorana spinors ψ verifying:

$$\psi_\pm = \frac{1 \pm \sigma^3}{2} \psi_\pm \tag{5.3}$$

The eigenvalues are $\frac{1}{2}\sigma^3\psi_\pm = \pm\frac{1}{2}\psi_\pm$.

5.2 Majorana orbital angular momentum

Definition 5.2. A set S of elements of an Hilbert space H with internal product $<,>$, is an orthonormal basis if:

1) For all $a \in S$: $< a, a >= 1$;
2) (orthogonality) For all $a, b \in S$, with $a \neq b$: $< a, b >= 0$;
3) (completeness) For all $f, g \in H$, $< g, f >= \sum_{a\in S} < g, a >< a, f >$.

Definition 5.3. Let $\vec{x} \in \mathbb{R}^3$. The spherical coordinates parametrization is:

$$\vec{x} = r(\sin(\theta)\sin(\varphi)\vec{e_1} + \sin(\theta)\sin(\varphi)\vec{e_2} + \cos(\theta)\vec{e_3}) \tag{5.4}$$

where $\{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$ is an orthonormal basis of $\mathbb{R}^3$ and $r \in [0, +\infty[$ $\theta \in [0, \pi]$, $\varphi \in [-\pi, \pi]$.

Definition 5.4. $L^2(S^2)$ is the Hilbert space of real functions with domain $S^2 \equiv \{\vec{x} \in \mathbb{R}^3 : |\vec{x}| = 1\}$, whose square is Lebesgue integrable in S^2. The internal product is:

$$< f, g >\equiv \int d(\cos\theta) d\varphi f(\theta, \varphi) g(\theta, \varphi),\ \ f, g \in L^2(S^2) \tag{5.5}$$

Definition 5.5. $L^2_4(S^2)$ is the Hilbert space of Majorana spinors whose 4 real components in the Majorana bases are in $L^2(S^2)$. The internal product is:

$$< \Psi, \Phi >\equiv \int d(\cos\theta) d\varphi\ \ \Psi^\dagger(\theta, \varphi)\Phi(\theta, \varphi),\ \ \Psi, \Phi \in L^2_4(S^2) \tag{5.6}$$

Definition 5.6. The Majorana angular momentum operators $\vec{L}_k$ are:

$$\vec{L}_k \equiv \sum_{i,j=1,2,3} -i\gamma^0 \epsilon_{ijk} x^i \partial_j,\ \ k = 1, 2, 3 \tag{5.7}$$

Where ϵ_{ijk} is the Levi-Civita symbol.

The operators verify the angular momentum algebra:

$$[\vec{L}_i, \vec{L}_j] = i\gamma^0 \epsilon_{ijk} \vec{L}_k \tag{5.8}$$

In spherical coordinates:

$$i\gamma^0 \vec{L}_3 = \partial_\varphi \tag{5.9}$$

$$(\vec{L})^2 = -\sin(\theta)\partial_\theta \Big(\sin(\theta)\partial(\theta)\Big) - \frac{1}{\sin^2(\theta)}\partial^2_\varphi \tag{5.10}$$

Definition 5.7. The cosine spherical harmonics Y^c_{lm}, sine spherical harmonics Y^s_{lm} and associated Legendre functions of the first kind P_{lm} are:

$$Y^c_{lm}(\theta, \varphi) \equiv \sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}} P^m_l(\cos\theta)\cos(m\varphi) \tag{5.11}$$

$$Y^s_{lm}(\theta, \varphi) \equiv \sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}} P^m_l(\cos\theta)\sin(m\varphi) \tag{5.12}$$

$$P^m_l(\xi) \equiv \frac{(-1)^m}{2^l l!}(1-\xi^2)^{m/2}\frac{\mathrm{d}^{l+m}}{\mathrm{d}\xi^{l+m}}(\xi^2-1)^l \tag{5.13}$$

where $\theta \in [0, \pi]$, $\varphi \in [-\pi, \pi]$, $\xi \in [-1, 1]$ and l, m are integer numbers $l \geq 0$, $-l \leq m \leq l$.

The spherical harmonics verify [9]:

$$\partial_\varphi Y^c_{lm}(\theta,\varphi) = -mY^s_{lm}(\theta,\varphi) \tag{5.14}$$

$$\partial_\varphi Y^s_{lm}(\theta,\varphi) = mY^c_{lm}(\theta,\varphi) \tag{5.15}$$

$$-\Big(\sin(\theta)\partial_\theta\Big(\sin(\theta)\partial_\theta\Big)+\frac{1}{\sin^2(\theta)}\partial^2_\varphi\Big)Y^a_{lm} = l(l+1)Y^a_{lm},\ a=c,s \tag{5.16}$$

Remark 5.8. *The spherical harmonics verify $L^2(S^2)$:*

$$< Y^s_{l'm'}, Y^c_{lm} > = 0 \tag{5.17}$$

$$< Y^s_{l'm'}, Y^s_{lm} > + < Y^c_{l'm'}, Y^c_{lm} > = \delta_{l'l}\delta_{m'm} \tag{5.18}$$

For all $f,g \in L^2(S^2)$:

$$< g, f > = \sum_{a=c,s,\ l\geq 0,\ -l\leq m\leq l} < g, Y^a_{lm} >< Y^a_{lm}, f > \tag{5.19}$$

Definition 5.9. The Majorana spherical harmonics Y_{lm} are:

$$Y_{lm}(\theta,\varphi) \equiv Y^c_{lm}(\theta,\varphi) + i\gamma^0 Y^s_{lm}(\theta,\varphi) \tag{5.20}$$

$$= \sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}}P^m_l(\cos\theta)e^{i\gamma^0 m\varphi} \tag{5.21}$$

The Majorana spherical harmonics are similar to the standard Laplace spherical harmonics definition, with $i\gamma^0$ in place of i. The properties are also similar.

They verify:

$$(\vec{L}_3 - m)Y_{lm}(\vec{x}) = 0 \tag{5.22}$$

$$(\vec{L}^2 - l(l+1))Y_{lm}(\vec{x}) = 0 \tag{5.23}$$

Proposition 5.10. *The columns of the Majorana spherical harmonics matrices form an orthonormal basis of the Hilbert space $L^2_4(S^2)$.*

Proof. We apply the remark 5.8 to directly obtain:

$$\int d(\cos\theta)d\varphi Y^\dagger_{l'm'}(\theta,\varphi)Y_{lm}(\theta,\varphi) = \delta_{l'l}\delta_{m'm} \tag{5.24}$$

For all $\Phi,\Psi \in L^2_4(S^2)$:

$$< \Phi, \Psi > = \sum_{l\geq 0,\ -l\leq m\leq l} < \Phi, Y_{lm}\psi_{lm} > \tag{5.25}$$

$$\psi_{lm} \equiv \int d(\cos\theta)d\varphi Y^\dagger_{lm}(\theta,\varphi)\Psi(\theta,\varphi) \tag{5.26}$$

□

5.3 Majorana total angular momentum space

The operator $\vec{\sigma}\cdot\vec{L}$ is:

$$\vec{\sigma}\cdot\vec{L} = -i\gamma^0\epsilon_k^{ij}\sigma^k x_i\partial_j \tag{5.27}$$

$$= -\frac{[\sigma^i,\sigma^j]}{2}x_i\partial_j \tag{5.28}$$

$$= \frac{\gamma^i\gamma^j-\gamma^j\gamma^i}{2}x_i\partial_j,\ \ i,j=1,2,3 \tag{5.29}$$

In spherical coordinates:

$$i\not{\partial} = i\gamma^r(\partial_r - \frac{1}{r}\vec{\sigma}\cdot\vec{L}) \tag{5.30}$$

$$\vec{\sigma}\cdot\vec{L} = \gamma^\theta\gamma^r\partial_\theta + \gamma^\varphi\gamma^r\frac{1}{sin\theta}\partial_\varphi \tag{5.31}$$

θ and φ are the angles of $\vec{x}$ in spherical coordinates, r is the radius.

It verifies:

$$\vec{\sigma}\cdot\vec{L} = (\vec{L}+\frac{1}{2}\vec{\sigma})^2 - \vec{L}^2 - \frac{3}{4} \tag{5.32}$$

The term $\vec{L}+\frac{1}{2}\vec{\sigma}$ is the sum of two angular momentum operators of integer and one-half spin.

Remark 5.11. *Let $\vec{L}$ be an integer spin angular momentum operator, with orthonormal eigenstates $|l,m>$. Let $\frac{1}{2}\vec{\sigma}$ be a spin one-half angular momentum operator, with orthonormal eigenstates $|\frac{1}{2},s>$, where $s=\pm\frac{1}{2}$. Then, the orthonormal eigenstates of the operator $\vec{L}+\frac{1}{2}\vec{\sigma}$, are given by [9]:*

$$|j,\mu,(j+1/2)>= -\sqrt{\frac{j-\mu+1}{2j+2}}|j+1/2,\mu-1/2>|\frac{1}{2},+\frac{1}{2}> \tag{5.33}$$

$$+\sqrt{\frac{j+\mu+1}{2j+2}}|j+1/2,\mu+1/2>|\frac{1}{2},-\frac{1}{2}> \tag{5.34}$$

$$|j,\mu,(j-1/2)>= +\sqrt{\frac{j+\mu}{2j}}|j-1/2,\mu-1/2>|\frac{1}{2},+\frac{1}{2}> \tag{5.35}$$

$$+\sqrt{\frac{j-\mu-1}{2j}}|j-1/2,\mu+1/2>|\frac{1}{2},-\frac{1}{2}> \tag{5.36}$$

Where $j=\frac{1}{2},\frac{3}{2},...$ and $-j\leq\mu\leq j$. They satisfy:

$$(\vec{L}_3+\frac{\sigma^3}{2})|j,\mu,(j\pm 1/2)> = \mu|j,\mu,(j\pm 1/2)> \tag{5.37}$$

$$(\vec{L}+\frac{\vec{\sigma}}{2})^2|j,\mu,(j\pm 1/2)> = j(j+1)|j,\mu,(j\pm 1/2)> \tag{5.38}$$

$$\vec{\sigma}\cdot\vec{L}|j,\mu,(j\pm 1/2)> = -(\pm(j+1/2)+2)|j,\mu,(j\pm 1/2)> \tag{5.39}$$

$$\sigma^r\ |j,\mu,(j+1/2)> = -|j,\mu,(j-1/2)> \tag{5.40}$$

Definition 5.12. The Majorana spherical matrices are:

$$\Omega_{l\mu}(\theta,\varphi) \equiv \Big(-\sqrt{\frac{l-\mu}{2l+1}} Y_{l,\mu}(\theta,\varphi) + \sqrt{\frac{l+\mu+1}{2l+1}} Y_{l,\mu+1}(\theta,\varphi)\sigma^1 \Big) \frac{1+\sigma^3}{2} \tag{5.41}$$

$$+ \Big(\sqrt{\frac{l+\mu}{2l-1}} Y_{l-1,\mu}(\theta,\varphi)\sigma^1 + \sqrt{\frac{l-\mu-1}{2l-1}} Y_{l-1,\mu+1}(\theta,\varphi) \Big) \frac{1-\sigma^3}{2} \tag{5.42}$$

with the integers $l \geq 1$ and $-l \leq \mu \leq l$. $Y_{l\mu}$ the Majorana spherical harmonics.

Proposition 5.13. *The columns of the Majorana spherical harmonics matrices form a complete orthonormal basis of the Hilbert space $L_4^2(S^2)$.*

Proof. Using remark 5.11, after some calculations, we get:

$$\int d(\cos\theta) d\varphi \Omega^\dagger_{l'\mu'}(\theta,\varphi)\Omega_{l\mu}(\theta,\varphi) = \delta_{l'l}\delta_{\mu'\mu} \tag{5.43}$$

$$\sum_{l\geq 1,\ -l\leq\mu\leq l} \int d(\cos\theta) d\varphi\ \Phi^\dagger(\theta,\varphi)\Omega_{l\mu}(\theta,\varphi)\psi_{l\mu} = \int d(\cos\theta) d\varphi\ \Phi^\dagger(\theta,\varphi)\Psi(\theta,\varphi) \tag{5.44}$$

For all $\Phi \in L_4^2(S^2)$. □

Using remark 5.11, the Majorana spherical matrices verify:

$$(\vec{L}^3 + \frac{\sigma^3}{2})\ \Omega_{l\mu} = (\mu + \frac{1}{2})\Omega_{l\mu} \tag{5.45}$$

$$\vec{\sigma}\cdot\vec{L}\ \Omega_{l\mu} = -\Omega_{l\mu}(l\sigma^3 + 1) \tag{5.46}$$

$$\sigma^r\ \Omega_{l\mu} = -\Omega_{l\mu}\sigma^1 \tag{5.47}$$

$$i\gamma^r\Omega_{l\mu} = (-1)^\mu \Omega_{l,-\mu-1} i\gamma^5 \tag{5.48}$$

$$\vec{\sigma}\cdot\vec{L} i\gamma^r\ \Omega_{l\mu} = i\gamma^r\Omega_{l\mu}(l\sigma^3 - 1) \tag{5.49}$$

5.4 Radial Momentum Space

Remark 5.14. *The spherical Bessel functions of the first kind, $j_l : \mathbb{R}_+ \to \mathbb{R}$ with the integer $l \geq 0$, verify:*

$$(\partial_r^2 + \frac{2}{r}\partial_r - \frac{l(l+1)}{r^2}) j_l(pr) = -p^2 j_l(pr) \tag{5.50}$$

$$\int_0^{+\infty} dr\ r^2 j_l(pr) j_l(p'r) = \frac{\pi\delta(p-p')}{2p^2} \tag{5.51}$$

$$\int_0^{+\infty} \frac{dp\ 2p^2}{\pi} j_l(pr) j_l(pr') = \frac{\delta(r-r')}{r^2} \tag{5.52}$$

Where the Dirac delta δ is such that for all $f \in L^2(\mathbb{R})$:

$$f(0) = \int dx\ \delta(x) f(x) \tag{5.53}$$

Definition 5.15. The Hankel-Majorana Transform $\psi(p,l,\mu)$ of a Majorana spinor $\Psi(\vec{x}) \in L_4^2(\mathbb{R}^3)$ is the Majorana spinor:

$$\psi(p,l,\mu) \equiv \int drd(cos\theta)d\varphi r^2 \Lambda^\dagger(p,l,\mu,r,\theta,\varphi)\Psi(r,\theta,\varphi) \tag{5.54}$$

$$\Lambda(p,l,\mu,r,\theta,\varphi) \equiv \Big(pj_l(pr) + (E_p - m)j_{l-1}(pr)i\gamma^r\Big)\Omega_{l\mu}(\theta,\varphi)\frac{1+\sigma^3}{2} \tag{5.55}$$

$$+ \Big(pj_{l-1}(pr) - (E_p - m)j_l(pr)i\gamma^r\Big)\Omega_{l\mu}(\theta,\varphi)\frac{1-\sigma^3}{2} \tag{5.56}$$

Where Λ are the Hankel-Majorana matrices, $m, p \geq 0$, $E_p = \sqrt{p^2 + m^2}$ and the integers $l \geq 1, -l \leq \mu \leq l$.

Proposition 5.16. *Let $\psi(p,l,\mu)$ be the Hankel-Majorana Transform of a Majorana spinor $\Psi \in L_4^2(\mathbb{R}^3)$. The inverse Hankel-Majorana Transform of $\psi(p,l,\mu)$ is:*

$$\Psi'(r,\theta,\varphi) \equiv \sum_{l\geq 1, -l\leq\mu\leq l} \int_0^{+\infty} \frac{dp\ (E_p + m)}{E_p \pi} \Lambda(p,l,\mu,r,\theta,\varphi)\psi(p,l,\mu) \tag{5.57}$$

It verifies, for all $\Phi \in L_4^2(\mathbb{R}^3)$:

$$\int d(cos\theta)d\varphi\ dr\ r^2\ \Phi^\dagger(r,\theta,\varphi)\Psi'(r,\theta,\varphi) = \int d(cos\theta)d\varphi\ dr\ r^2 \Phi^\dagger(r,\theta,\varphi)\Psi(r,\theta,\varphi) \tag{5.58}$$

Proof. The following equation is verified:

$$i\gamma^0(i\vec{\not\partial} - m)\Lambda(p,l,\mu) = E_p\Lambda(p,l,\mu)i\gamma^0 \tag{5.59}$$

Since the operator $i\gamma^0(i\vec{\not\partial} - m)$ is skew-Hermitic the equation above implies that:

$$i\gamma^0 E_{p'} I = I i\gamma^0 E_p \tag{5.60}$$

$$I \equiv \int d(cos\theta)d\varphi\ dr\ r^2\ \Lambda^\dagger(p',l',\mu',r,\theta,\varphi)\Lambda(p,l,\mu,r,\theta,\varphi) \tag{5.61}$$

As $E_p + E_{p'} > 0$, in the integral I the terms odd in $i\gamma^r$ are null. From the orthogonality of the spherical matrices, we get that the Λ matrices are orthogonal:

$$I = \delta_{l'l}\delta_{\mu'\mu} \int d(cos\theta)d\varphi\ dr\ r^2 \tag{5.62}$$

$$\Big(p'j_l(p'r)pj_l(pr) + (E_{p'} - m)j_{l-1}(p'r)(E_p - m)j_{l-1}(pr)\frac{1+\sigma^3}{2} \tag{5.63}$$

$$+ p'j_{l-1}(p'r)pj_{l-1}(pr) + (E_{p'} - m)j_l(p'r)(E_p - m)j_l(pr)\frac{1-\sigma^3}{2}\Big) \tag{5.64}$$

$$= \delta_{l'l}\delta_{\mu'\mu}\frac{\pi\delta(p-p')}{2p^2}(E_p - m)2E_p = \delta_{l'l}\delta_{\mu'\mu}\frac{\pi E_p \delta(p-p')}{E_p + m} \tag{5.65}$$

To show completeness, using $i\gamma^r\Omega_{l\mu} = (-1)^\mu \Omega_{l,-\mu-1} i\gamma^5$, we first show that:

$$\sum_{l'\mu'} \int d(cos\theta) d\varphi\ \psi^\dagger(p,l',\mu')\Lambda^\dagger(p,l',\mu',r,\theta,\varphi)\Omega_{l\mu}(\theta,\varphi) = \tag{5.66}$$

$$= \psi^\dagger(p,l,\mu) p (j_l(pr)\frac{1+\sigma^3}{2} + j_{l-1}(pr)\frac{1-\sigma^3}{2}) \tag{5.67}$$

$$+ \psi^\dagger(p,l,-\mu-1)(-1)^\mu (E_p - m)(-j_l(pr)\frac{1-\sigma^3}{2} + j_{l-1}(pr)\frac{1+\sigma^3}{2}) i\gamma^5 \tag{5.68}$$

$$= \int d(cos\theta')d\varphi' dr'(r')^2\ \Psi^\dagger(r',\theta',\varphi')\Big(\tag{5.69}$$

$$p j_l(pr')\Big(p j_l(pr) + (E_p - m) j_{l-1}(pr) i\gamma^r\Big)\Omega_{l\mu}(\theta',\varphi')\frac{1+\sigma^3}{2} \tag{5.70}$$

$$+ p j_{l-1}(pr')\Big(p j_{l-1}(pr) - (E_p - m) j_l(pr) i\gamma^r\Big)\Omega_{l\mu}(\theta',\varphi')\frac{1-\sigma^3}{2} \tag{5.71}$$

$$(-1)^\mu (E_p - m) j_{l-1}(pr')\Big(p j_l(pr) + (E_p - m) j_{l-1}(pr) i\gamma^r\Big)\Omega_{l,-\mu-1}(\theta',\varphi')\frac{1+\sigma^3}{2} i\gamma^5 \tag{5.72}$$

$$- (-1)^\mu (E_p - m) j_l(pr')\Big(p j_{l-1}(pr) - (E_p - m) j_l(pr) i\gamma^r\Big)\Omega_{l,-\mu-1}(\theta',\varphi')\frac{1-\sigma^3}{2} i\gamma^5 \tag{5.73}$$

$$= \int d(cos\theta')d\varphi' dr'(r')^2\ \Psi^\dagger(r',\theta',\varphi')\Big(\tag{5.74}$$

$$p j_l(pr')\Big(p j_l(pr) + (E_p - m) j_{l-1}(pr) i\gamma^r\Big)\Omega_{l\mu}(\theta',\varphi')\frac{1+\sigma^3}{2} \tag{5.75}$$

$$+ p j_{l-1}(pr')\Big(p j_{l-1}(pr) - (E_p - m) j_l(pr) i\gamma^r\Big)\Omega_{l\mu}(\theta',\varphi')\frac{1-\sigma^3}{2} \tag{5.76}$$

$$(E_p - m) j_{l-1}(pr')\Big(p j_l(pr) + (E_p - m) j_{l-1}(pr) i\gamma^r\Big)\Omega_{l,\mu}(\theta',\varphi')\frac{1-\sigma^3}{2} \tag{5.77}$$

$$- (E_p - m) j_l(pr')\Big(p j_{l-1}(pr) - (E_p - m) j_l(pr) i\gamma^r\Big)\Omega_{l,\mu}(\theta',\varphi')\frac{1+\sigma^3}{2} \tag{5.78}$$

$$= \int d(cos\theta')d\varphi' dr'(r')^2\ \Psi^\dagger(r',\theta',\varphi')\Omega_{l\mu}\frac{2p^2 E_p}{E_p + m}\Big(\tag{5.79}$$

$$j_l(pr')j_l(pr)\frac{1+\sigma^3}{2} + j_{l-1}(pr')j_{l-1}(pr)\frac{1-\sigma^3}{2}\Big) \tag{5.80}$$

If we integrate on p and use the completeness of the spherical Bessel functions, we get:

$$\int d(cos\theta)d\varphi \Psi'^\dagger(r,\theta,\varphi)\Omega_{l\mu}(\theta,\varphi) = \int d(cos\theta)d\varphi \Psi^\dagger(r,\theta,\varphi)\Omega_{l\mu}(\theta,\varphi) \tag{5.81}$$

Since the columns of the spherical matrices $\Omega_{l\mu}$ are a complete basis, we have shown the completeness of the Hankel-Majorana transform:

$$\int d(cos\theta)d\varphi dr\ r^2\ \Psi'^\dagger(r,\theta,\varphi)\Phi(r,\theta,\varphi) = \int d(cos\theta)d\varphi dr\ r^2 \Psi^\dagger(r,\theta,\varphi)\Phi(r,\theta,\varphi) \tag{5.82}$$

For all $\Phi \in L_4^2(\mathbb{R}^3)$. $\square$

6 Dirac equation for the free fermion

The Dirac equation for the free fermion can be written as:

$$i\gamma^0(i\not{\partial} - m)\Psi(x) = 0 \tag{6.1}$$

Where Ψ is a spinor. Note that the equation contains only Majorana matrices. The Fourier or Hankel Transforms of the equation are:

$$(\partial_0 + i\gamma^0 E_p)\Psi(x^0, p) = 0 \tag{6.2}$$

The solutions can be written as:

$$\Psi(x) = \int \frac{d^3\vec{p}}{(2\pi)^3} \frac{\not{p}\gamma^0 + m}{\sqrt{E_p + m}\sqrt{2E_p}} e^{-i\gamma^0 p\cdot x}\psi(\vec{p}) \tag{6.3}$$

Where $p^0 = E_p$ and $\psi(\vec{p})$ is an arbitrary spinor. If $\psi(\vec{p})$ is a Majorana spinor, then the solution $\Psi(x)$ is also a Majorana spinor.

The solutions can also be written as:

$$\Psi(x^0, r, \theta, \varphi) = \sum_{l\geq 1, -l\leq\mu\leq l} \int_0^{+\infty} \frac{dp(E_p + m)}{E_p\pi} \Lambda(p, l, \mu, r, \theta, \varphi) e^{-i\gamma^0 E_p\cdot x^0}\psi(p, l, \mu) \tag{6.4}$$

Where $\psi(p, l, \mu)$ is an arbitrary spinor and Λ are the Hankel-Majorana matrices.

The set of quantum numbers $(\vec{p})$ and (p, l, μ) are related with the linear and spherical momentums of free fermions. The Majorana spin is related with the standard spin definition. For instance, to obtain the usual free electron solution, we just set $\psi_e(\vec{p}) = \frac{1+\gamma^0}{2}\psi_e(\vec{p})$ and we get:

$$\Psi_e(x) = \int \frac{d^3\vec{p}}{(2\pi)^3} \frac{\not{p} + m}{\sqrt{E_p + m}\sqrt{2E_p}} e^{-ip\cdot x} \frac{1+\gamma^0}{2}\psi_e(\vec{p}) \tag{6.5}$$

The matrix γ^0 was replaced by the identity matrix 1, due to the presence of the projector. The same thing happens with the spherical solution and with the spin.

To obtain the usual free positron solution, we just set $\psi_p(\vec{p}) = \frac{1-\gamma^0}{2}\psi_p(\vec{p})$ and the matrix γ^0 gets replaced by -1.

7 Energy-momentum space

Now we can extend our transforms to define an energy-momentum space.

Definition 7.1. Given a Majorana spinor $\Psi \in L_4^2(\mathbb{R}^4)$, the Fourier-Majorana transform in space-time is defined as:

$$\psi(p) \equiv \int d^4x O(p, x)\Psi(x) \tag{7.1}$$

Where $O(p, x)$ is:

$$O(p, x) \equiv e^{i\gamma^0 p^0 x^0} O(\vec{p}, \vec{x}) = e^{i\gamma^0 p\cdot x} \frac{[\not{p}]\gamma^0 + m}{\sqrt{E_p + m}\sqrt{2E_p}} \tag{7.2}$$

Note that E_p and $[\not{p}] = \gamma^0 E_p - \vec{\gamma}\cdot\vec{p}$ don't depend on p^0, but $p\cdot x = p^0x^0 - \vec{p}\cdot\vec{x}$ does.

Proposition 7.2. *The inverse Fourier-Majorana transform in space-time is given by:*

$$\Psi(x) = \int \frac{d^4p}{(2\pi)^4} O^\dagger(p,x)\psi(p) \tag{7.3}$$

Where $O^\dagger$ is the hermitian conjugate of O, given by:

$$O^\dagger(p,x) = O^\dagger(\vec{p},\vec{x})e^{-i\gamma^0 p^0\cdot x^0} = \frac{[\not{p}]\gamma^0 + m}{\sqrt{E_p+m}\sqrt{2E_p}}e^{-i\gamma^0 p\cdot x} \tag{7.4}$$

Proof.

$$\int \frac{d^4p}{(2\pi)^4} O^\dagger(p,y)O(p,x) = \int \frac{d^3\vec{p}}{(2\pi)^3} O^\dagger(\vec{p},\vec{y})\Big(\int \frac{dp^0}{2\pi} e^{-i\gamma^0 p^0(y^0-x^0)}\Big)O(\vec{p},\vec{x}) \tag{7.5}$$

$$= \delta(y^0 - x^0)\int \frac{d^3\vec{p}}{(2\pi)^3} O^\dagger(\vec{p},\vec{y})O(\vec{p},\vec{x}) \tag{7.6}$$

$$= \delta^4(y-x) \tag{7.7}$$

$$\int d^4x O(q,x)O^\dagger(p,x) = \int dx^0 e^{i\gamma^0 q^0 x^0}\Big(\int d^3\vec{x} O(\vec{q},\vec{x})O^\dagger(\vec{p},\vec{x})\Big)e^{-i\gamma^0 p^0 x^0} \tag{7.8}$$

$$= (2\pi)^3\delta^3(\vec{q}-\vec{p})\int dx^0 e^{i\gamma^0(q^0-p^0)x^0} \tag{7.9}$$

$$= (2\pi)^4\delta^4(q-p) \tag{7.10}$$

□

Definition 7.3. The Hankel-Majorana transform in space-time of a Majorana spinor $\Psi \in L_4^2(\mathbb{R}^4)$ is:

$$\psi'(p^0,p,l,\mu) \equiv \int dx^0 e^{i\gamma^0 p^0 x^0}\psi(x^0,p,l,\mu) \tag{7.11}$$

Where $\psi(x^0,p,l,\mu)$ is the Hankel-Majorana transform in space of Ψ.

Proposition 7.4. *Let $\psi(p^0,p,l,\mu)$ be the Hankel-Majorana Transform in space-time of a Majorana spinor $\Psi \in L_4^2(\mathbb{R}^4)$. The inverse Hankel-Majorana Transform of $\psi(p^0,p,l,\mu)$ is:*

$$\Psi'(x^0,r,\theta,\varphi) \equiv \sum_{l\geq 1, -l\leq\mu\leq l}\int_0^{+\infty}\frac{dp(E_p+m)}{E_p\pi}\int_{-\infty}^{+\infty}\frac{dp^0}{2\pi}\Lambda(p,l,\mu,r,\theta,\varphi)e^{-i\gamma^0 p^0\cdot x^0}\psi(p^0,p,l,\mu) \tag{7.12}$$

It verifies, for all $\Phi \in L_4^2(\mathbb{R}^4)$:

$$\int dx^0 d(cos\theta)d\varphi\ dr\ r^2\ \Phi^\dagger(x^0,r,\theta,\varphi)\Psi'(x^0,r,\theta,\varphi) = \tag{7.13}$$

$$= \int dx^0 d(cos\theta)d\varphi\ dr\ r^2\Phi^\dagger(x^0,r,\theta,\varphi)\Psi(x^0,r,\theta,\varphi) \tag{7.14}$$

Proof. The matrices $\Lambda(p,l,\mu,r,\theta,\varphi)e^{-i\gamma^0 p^0\cdot x^0}$ are orthogonal:

$$\int dx^0 d(cos\theta)d\varphi\ dr\ r^2\ e^{i\gamma^0 p'^0\cdot x^0}\Lambda^\dagger(p',l',\mu',r,\theta,\varphi)\Lambda(p,l,\mu,r,\theta,\varphi)e^{-i\gamma^0 p^0\cdot x^0} = \tag{7.15}$$

$$= \delta_{l'l}\delta_{\mu'\mu}\frac{\pi E_p\delta(p-p')}{E_p+m}\int dx^0 e^{i\gamma^0 p'^0\cdot x^0}e^{-i\gamma^0 p^0\cdot x^0} = \delta_{l'l}\delta_{\mu'\mu}\frac{\pi E_p\delta(p-p')}{E_p+m}2\pi\delta(p'^0-p^0) \tag{7.16}$$

To show completeness, we first show that:

$$\sum_{l'\mu'}\int_0^{+\infty}\frac{dp'\ (E'_p+m)}{E'_p\pi}\int d(cos\theta)d\varphi dr\ r^2 \tag{7.17}$$

$$\psi^\dagger(p^0,p',l',\mu')e^{i\gamma^0 p^0\cdot x^0}\Lambda^\dagger(p',l',\mu',r,\theta,\varphi)\Lambda(p,l,\mu,r,\theta,\varphi) = \tag{7.18}$$

$$= \psi^\dagger(p^0,p,l,\mu)e^{i\gamma^0 p^0\cdot x^0} \tag{7.19}$$

$$= \int dx'^0 d(cos\theta)d\varphi dr\ r^2\Psi^\dagger(x'^0,r,\theta,\varphi)\Lambda(p,l,\mu,r,\theta,\varphi)e^{-i\gamma^0 p^0 x'^0}e^{i\gamma^0 p^0\cdot x^0} \tag{7.20}$$

If we integrate on p^0, we get:

$$\int d(cos\theta)d\varphi dr\ r^2\Psi'^\dagger(x^0,r,\theta,\varphi)\Lambda(p,l,\mu,r,\theta,\varphi) = \int d(cos\theta)d\varphi dr\ r^2\Psi^\dagger(r,\theta,\varphi)\Lambda(p,l,\mu,r,\theta,\varphi) \tag{7.21}$$

Since the columns of the Hankel matrices $\Lambda(p,l,\mu,r,\theta,\varphi)$ are a complete basis, we have shown the completeness of the Hankel-Majorana transform in space-time:

$$\int dx^0 d(cos\theta)d\varphi dr\ r^2\ \Psi'^\dagger(x^0,r,\theta,\varphi)\Phi(x^0,r,\theta,\varphi) = \tag{7.22}$$

$$= \int dx^0 d(cos\theta)d\varphi dr\ r^2\Psi^\dagger(x^0,r,\theta,\varphi)\Phi(x^0,r,\theta,\varphi) \tag{7.23}$$

For all $\Phi \in L^2_4(\mathbb{R}^4)$. □

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