Hans Detlef Hüttenbach
Im Rheingarten 6
53225 Bonn
Tel.: 0228 46 87 56
E-Mail: detlef.huettenbach@computacenter.com

DISPOSING CLASSICAL FIELD THEORY

To My Little Daughter

Yes, we all shine on
Like the moon and the stars and the sun
Everyone

Instant Karma – John Lennon

SUMMARY

This article is about the concept of mass and electric charge: When the fundamental relativistic equation $E^2 = m^2c^4 + |\boldsymbol{p}|^2c^2$ is solved in the complex, this inevitably leads to an irreducible representation of the extended Lorentz group as $U(4)$ operating on the complex Clifford algebra $Cl_{\mathbb{C}}(1{,}3)$ in which mass is a complex $4x4$-spinor. Spinors are a direct consequence of taking the root of the Minkowski square distance. Doing so with the Minkowski square of differentials then gives a spinor-valued differential form. With that, classical electrodynamics is shown to be extendable into a relativistically invariant theory, in fact the simplest possible relativistically invariant one. Its symmetries reveal a unified concept of classical charge and mass. A dynamical system based on this, splits into the direct sum of a dynamical system of pure electromagnetic charges and one of purely neutral particles. In it, charged particles must be fermionic in order to conserve their net charge, and neutral non-magnetic ones are bosonic in order to be able to assign to them a positive mass. Also, it will be seen that within the Clifford algebra, the Hamiltonian of a self-interacting mechanical dynamical system of particles can be given in a closed form. I end the paper with a section on superconductivity, where it is shown that superconducting material should electromagnetically behave as opaque, dark matter.

Introduction

As Richard P. Feynman once put it, the really precious things offer astounding many views of the same topics. Time to have a fresh, unbiased look at a theory that has been overcome by one century which is to say, approximately ever since: the classical field theory.

The subject of Newtonian classical mechanics is the dynamics of a finite number of massy bodies in Euclidean space and time. The world then was thought of as consisting of individually identifiable, traceable bodies that move according to Newton's axioms and attract each other through the force of gravitation.

The great, bold step was the identification of gravitational and inert mass: they behaved the same, and therefore, they were taken as being the same.

Because of that, one could calculate the energy of the whole mechanical world (up to an inevitable constant). It all culminated into the Poisson equation:

$$(1) \quad \Delta\Phi(t,\boldsymbol{x}) = -C\varrho(t,\boldsymbol{x}), \text{ where } \Delta := \nabla\cdot\nabla := \left(\partial^2/\partial x_1^2 + \partial^2/\partial x_2^2 + \partial^2/\partial x_3^2\right),$$

Φ is the potential (energy per volume and test mass), ρ is the mass density, and C is a constant that can be set equal to 1.

This equation may be one of the most overlooked in physics: It states nothing but the duality and equivalence of field (the left hand side) and mechanical particle. Indeed, given a system of n distinct particles $q_1(t), \dots, q_n(t)$ of mass $m_1, \dots, m_n$, one can calculate $\Phi(t,x)$, and vice versa, given $\Phi(t,x)$ for a "real mechanical system" its Laplacian, $\Delta\Phi(\mathrm{t},\mathbf{x})$, in turn yields $m_1\delta(x - q_1(t)) + \cdots + m_n\delta(x - q_n(t))$, δ being the Dirac distribution, which maps continuous functions $f: \mathbb{R}^3 \to \mathbb{C}$ to

$$\delta: f \mapsto \int \delta(x) f(x) d^3x := f(0),$$

thus defining the location of a mechanical system of n particles along with their masses as a function of time. That gives velocity, momentum, force, and energy of the system as a function of time. That field equation therefore states the complete observability of any mechanical dynamical system from the outside through its gravitational potential field.

There are two further points to be made: If one ignores tracing each individual particle and studies a fluid of particles instead, then a single potential Φ or a smooth particle density $\rho(t,\boldsymbol{x})$ will no more determine the particle velocity, necessitating separate vector potentials $\boldsymbol{A}(t,\boldsymbol{x})$

and fluxes $\boldsymbol{j}(t,\boldsymbol{x}) \coloneqq \rho(t,x)\boldsymbol{v}(t,x)$ to account for that. Wouldn't that be over-determining the dynamics? Yes, it would: The introduction of flux $\boldsymbol{j}(t,\boldsymbol{x})$ e.g. leads to the possibility to have a flux $\boldsymbol{j}(t,\boldsymbol{x})$ where $\rho(t\ ,\mathbf{x})$=0. The restriction of $\boldsymbol{j}(t,\boldsymbol{x}) = 0$ for all $(t,\mathbf{x}) \in \mathbb{R}^4$ with $\rho(t,\mathbf{x})$=0, would be what in field theory is called a *gauge*.

The 2nd point is to the nature of the field Φ, which is not really a function, but rather a bilinear form $(f,\Phi) \mapsto \int f(t,\boldsymbol{x})\Delta\Phi(t,\boldsymbol{x})d^3x = -\int f(t,x)\rho(t,\boldsymbol{x})\,d^3x$

The moral to the pickiness is: The field equation in the end is proportional to the square of masses, not to the single mass itself.

Electrodynamics then evolved as seemingly distinct from mechanics. Its equations, the Maxwell equations, written a relativistically convenient form, are given by:

(2) $\Box A^\mu(t,\boldsymbol{x}) = j^\mu(t,\boldsymbol{x}),\ (0 \le \mu \le 3),$

where $\Box = (\frac{\partial^2}{\partial t^2} - \boldsymbol{\nabla}^2)$ is the so called d'Alembert operator, $\rho = j^0$ is the charge density, and $\boldsymbol{j} = (j^1, j^2, j^3) = \rho\boldsymbol{v}$ is the flux of the charge.

Because $\frac{\partial^2}{\partial t^2}$ is negatively definite, I prefer

(3) $-\Box(-A)^\mu(t,\boldsymbol{x}) = j^\mu(t,\boldsymbol{x}),\ (0 \le \mu \le 3).$

Because it is needed later, I summarize what is well-known (see the appendix for details): The kernel of the wave operator $-\Box$, i.e: the space of functions f for which $-\Box\mathrm{f} = 0$ is the space of plain waves (which are sourceless by nature), and restricting to $range(\Box)$, one can solve for $-\Box\mathrm{K} = \delta$, where $\delta\colon \mathrm{f} \mapsto \mathrm{f}(0)$ is the Dirac distribution, which gives $\mathrm{K} = \mathrm{K}_+ + \mathrm{K}_- = \mathcal{F}(\frac{1}{\mathrm{x}^\mu \mathrm{x}_\mu})\mathcal{F}^{-1}$ as Fourier transform of the function $\mathrm{x} \longmapsto \frac{1}{\mathrm{x}^\mu \mathrm{x}_\mu}$, which is a spherical wave with the origin as source propagating on the forward and the backwards light cone, hence $\mathrm{K}_+, \mathrm{K}_-$ are called forward and backward propagators, resp. . So, given a smooth function g from $range(\Box)$,, the solution of $-\Box\mathrm{f} = \mathrm{g}$ is given by $K * g \coloneqq \int K(x-y)g(y)d^4y$, * being the so called convolution.

These Maxwell equations still posed some questions: they are not Lorentz invariant, but transform as a contravariant 4-vector, where each transformation needs a special choice of gauge. Moreover, the vector field $\boldsymbol{A} \coloneqq (A^1, A^2, A^3)$ has curl, so – as shown by Poincaré – the differential 1-form $A^0dx_0 - A^1dx_1 - A^2dx_2 - A^3dx_3$ cannot be integrated into a 0-form, that is, the forces aren't conservative. Lastly, the equations can be solved by means of Fourier transformation, but that necessarily results in complex waves for A^μ, $(0 \le \mu \le 3)$. Not that complex waves are bad – quite the contrary – as they allow us to understand Huygens optics. But: if the left hand side of the equations $-\Box A^\mu(t, \boldsymbol{x})$ is complex and even phase symmetric, shouldn't the right hand side $j^\mu(t, \boldsymbol{x})$ either be complex and even phase symmetric?

At this stage Albert Einstein entered: All mechanical systems are to transform according to the laws of electromagnetism! Since Einstein, mechanics is no more ruled by the laws of gravitation, but by electromagnetism! Doesn't that mean that the nature of mass should ideally be contained within electromagnetism? As far as I know, there is no mention of Einstein having considered this. Instead, he proceeded straight with the general relativity.

Lorentz Invariant Formulation of Maxwell's Equations

Let's answer the question for the non-invariance of Maxwell's equations first:

They are not invariant due to not regarding the test charge: that also changes from one Lorentz transformation to another. If we put it back in (as done above) that gives a Hermitian or bilinear form with one 4-vector to the left, one to the right:

In order to avoid all technical complications, let's suppose $j = (j_0, \ldots, j_3)$ is smooth on $\mathbb{R}^4$, and that for each $x_0 \in \mathbb{R}^4, \mu = 0,..3$ all $j^\mu(t,\cdot)$ vanishes outside a bounded set (i.e.: is of compact support). Let $\chi: \mathbb{R}^3 \ni \boldsymbol{x} \mapsto 1[V^{-1/2}] \in \mathbb{R}$ mapping $\boldsymbol{x}$ to 1 by square root of volume, then $\Box A_\mu(t, \boldsymbol{x}) = \chi(\boldsymbol{x})\Box(A_\mu(t, \boldsymbol{x})/\chi(\boldsymbol{x}))$ and $j_\mu(t, \boldsymbol{x}) = \chi(\boldsymbol{x})\frac{j_\mu(t,\boldsymbol{x})}{\chi(\boldsymbol{x})}$. Hence, Maxwell's equations equivalently rewrite as

(4) $-\Box(-A)^\mu(t, \boldsymbol{x}) = j^\mu(t, \boldsymbol{x}),\ (0 \le \mu \le 3),$

where the j^μ are square root densities of charge flow, rather than densities, i.e.: where $\int j_\mu^2(t, \boldsymbol{x}) d^3\boldsymbol{x} = Q_\mu^2(t)$ is the square of the total charge.

This allows expressing Maxwell's equations as a Lorentz invariant bilinear form:

$$(f^\mu, j^\mu) \mapsto \int f_\mu(t, \boldsymbol{x})\, \Box A^\mu(t, \boldsymbol{x}) d^3x = \int f_\mu(t, \boldsymbol{x}) j^\mu(t, \boldsymbol{x}) d^3x,$$

So that in particular for $f^\mu = j^\mu$ with $\rho \coloneqq j^0$:

$$\int j_\mu(t, \boldsymbol{x}) \cdot \Box \mathrm{A}^\mu(t, \boldsymbol{x}) d^3x = \int \rho^2(t, y) - \boldsymbol{j}^2(t, \boldsymbol{x}) d^3x.$$

That bilinear form is Lorentz invariant, since for a Lorentz transformation Λ on the j^μ

$\int \Lambda f_\mu(t, \boldsymbol{x}) \Lambda \mathrm{j}^\mu(t, \boldsymbol{x}) d^3x = \int f_\mu(t, \boldsymbol{x}) \Lambda^t \Lambda \mathrm{j}^\mu(t, \boldsymbol{x}) d^3x = \int f_\mu(t, \boldsymbol{x}) j^\mu(t, \boldsymbol{x}) d^3x$, where $\Lambda^t = \Lambda^{-1}$ is the transpose of Λ.

Phase Invariance in Maxwell's Equations

To get at the desired phase invariance, the bilinear form $(f^\mu, j^\mu) \mapsto \int f_\mu(t,\boldsymbol{x}) j^\mu(t,\boldsymbol{x}) d^3x$ should and could be changed into the Hermititian form $(f^\mu, j^\mu) \mapsto \int \overline{f_\mu(t,\boldsymbol{x})} j^\mu(t,\boldsymbol{x}) d^3x$, where $\bar{f}$ is the complex conjugate of f. That would give:

$$\int \overline{J_\mu(t,\boldsymbol{x})} \cdot \square \mathrm{A}^\mu(t,\boldsymbol{x}) d^3x = \int |\rho|^2(t,y) - |\boldsymbol{j}|^2(t,\boldsymbol{x}) d^3x.$$

That paves the way. I would like to write A^μ as value of ρ under an operator S^2, i.e.: writing $A^\mu = S^2\mu$: $-\int \widetilde{j^\mu}^*(-\square)A^\mu = -\int \widetilde{j^\mu}^*(-\square)(S^2 j^\mu) = -\int \left((-\square)^{\frac{1}{2}} S\widetilde{j^\mu}\right)^* ((-\square)^{\frac{1}{2}} S j^\mu)$, where * denotes complex conjugation.

Dropping the integration would then give the scalar equations: $(-\square^{\frac{1}{2}}) S j^\mu = j^\mu$, $\mu = 0 \ldots, 3$. However, there are two obstacles:

The first, minor one is that the last step mandates to shovel $i\partial/\partial t$ from j^μ over to $\widetilde{j^\mu}$, while the integration is over the spatial volume, only. To solve that, we can introduce a sequence of smooth (infinitely differentiable) cut-off functions $\vartheta_n : \mathbb{R} \to [0, \infty)$ which are equal 1 on the interval $[-n, n]$ and vanish outside $[-(n+1), n+1]$. Now, $t \mapsto \int \overline{J_\mu(t,\boldsymbol{x})} \cdot \square \mathrm{A}^\mu(t,\boldsymbol{x}) d^3x$ is a distribution, and $\int \overline{\vartheta_n(t) J_\mu(t,\boldsymbol{x})} \cdot \square \mathrm{A}^\mu(t,\boldsymbol{x}) d^3x dt$ allows $i\partial/\partial t$ to be shoveled over from $\square \mathrm{A}^\mu(t,\boldsymbol{x})$ to $\vartheta_n(t) j_\mu(t,\boldsymbol{x})$ by partial integration. Then $\int \overline{\vartheta_n(t) J_\mu(t,\boldsymbol{x})} \cdot \square \mathrm{A}^\mu(t,\boldsymbol{x}) d^3x$ converges (weakly) to $\overline{J_\mu(t,\boldsymbol{x})} \cdot \square \mathrm{A}^\mu(t,\boldsymbol{x}) d^3x$ as $n \to \infty$, so that, in the distributional sense, we can indeed shovel $i\partial/\partial t$ from the $\square \mathrm{A}^\mu$ part to the left factor $\overline{J_\mu}$.

The 2nd obstacle is more involved: Whatever the root of $-\square$ is, it cannot be symmetric in the above sense, because $-\square$ is not a positive operator: Negative spectral values arise from the negative spatial derivatives, namely the Laplacian Δ. And with this, $S \neq S^*$

Instead, we have to redefine the Hermititian form: $(f^\mu, j^\mu) \mapsto \int \overline{P f_\mu(t,\boldsymbol{x})} j^\mu(t,\boldsymbol{x}) d^4x$, where P is a suitable unitary (or orthogonal) parity operator such that $-\int (P\tilde{\rho})(-\square)(S^2\rho) = -\int \left(P(-\square)^{\frac{1}{2}} S\tilde{\rho}\right)((-\square)^{\frac{1}{2}} S\rho)$ holds, in other words S and $(-\square)^{\frac{1}{2}}$ can be hoped for to be symmetric only w.r.t. this parity-twisted Hermititan form.

Now I need to solve for the root of the wave operator $-\square$, and there are – as is well-known – two different ways: By Fourier transformation, $-\square$ becomes the multiplication operator $f(y_0, \boldsymbol{y}) \mapsto (y_0^2 - |\boldsymbol{y}|^2) f(y_0, \boldsymbol{y})$, so task is to take the square root of $(y_0^2 - |\boldsymbol{y}|^2)$. The 1st,

obvious solution is $\pm\sqrt{y_0^2 - |\boldsymbol{y}|^2}$, which is the one normally taken in classical physics. But it is just a partial solution, since it still does not solve for either space and charge inversion. The other one is by the use of the Clifford algebra, as had been introduced into physics by P.A.M. Dirac (see: [1], Ch. IX): $root(y_0^2 - |\boldsymbol{y}|^2) = \gamma_0 y_0 + \gamma_1 y_1 + \gamma_2 y_2 + \gamma_3 y_3$,
where the γ_0 is a Hermititian and the y_k, $(1 \leq k \leq 3)$ anti-Hermititian 4×4-matices.

Let's derive affairs from ground up:

Definition: An *algebra A* over $\mathbb{C}$ is a complex vector space with a bilinear vector multiplication $\cdot: A \times A \to A$. The algebra *A* is said to be *associative* if the vector multiplication is associative, and it is called *unitary* if *A* contains a neutral element 1 w.r.t. the multiplication. Further, we say that two algebras *A*,*B* over $\mathbb{C}$ are isomorphic if there exists a vector space isomorphism $\iota: A \to B$ for which $x \cdot y = 0 \Leftrightarrow (\iota x) \cdot (\iota y) = 0$ holds for all $x, y \in A$. A subset $V \subset A$ is said to *generate* A if and only if a vector space basis of A can be chosen from the set $\{a_1 \cdots a_n \mid a_1, \dots, a_1 \in V,\ n \in \mathbb{N}\}$ of all products of V.
Let $Q: \mathbb{C}^4 \ni (x_0, \dots, x_3) \mapsto |x_0|^2 - (|x_1|^2 + |x_2|^2 + |x_3|^2) \in \mathbb{R}$ be the Hermititian Minkowski pseudo-scalar product which in particular is a Hermititian form.
Then the (Hermititian) *Clifford algebra* $Cl(1,3)$ is defined as unitary, associative sub-algebra of $L(X, X)$ for some vector space X for which the following conditions hold:

(1) there is a vector space embedding $\iota: \mathbb{C}^4 \to Cl(1,3)$, such that $(\iota\bar{x}) \cdot (\iota x) = Q(x)1$ for all $x \in \mathbb{C}^4$
(2) the set $\iota(\mathbb{C}^4)$ generates $Cl(1,3)$.

Now, it's not the Clifford algebra that is of my concern; my concern are the irreducible representations of that algebra in the algebra $L(X, X)$ of linear operators on some vector space X: Let me 1st define *what* that is and then explain *why*: For a complex vector space X the space $L(X, X)$ of linear operators $\phi: X \to X$ is obviously an algebra, and vice versa, since $A \ni x \mapsto y \cdot x \in A$ is a linear operator on the vector space A for each $y \in A$, every algebra A can be embedded into $L(X, X)$ for some X. Next, a projection on X is a linear mapping $\pi: X \to X$ for which $\pi^2 := \pi \circ \pi = \pi$. With this, an *irreducible* representation of $Cl(1,3)$ is an embedding $Cl(1,3) \to L(X, X)$ such that the only projections on X which commute with all elements of $Cl(1,3)$ are the identity and the zero projection.
Let me now explain *why*: given the Minkowski metrics as Hermititian quadratic form Q on $\mathbb{C}^4$ into $\mathbb{R}$, then, given that $Cl(1,3)$ is any representation into $L(X, X)$, and that the space X is

of finite dimensions (so that its Euclidian scalar product $\langle\cdot,\cdot\rangle$ and its unit vectors are well-defined), then $Q(x) = \langle\iota(x)\chi, \iota(x)\chi\rangle = \langle\phi\cdot\iota(x)\chi, \phi\cdot\iota(x)\chi\rangle$ for all unit vectors $\chi \in X$ and all unitary operators ϕ on X. That is: the unitary operators on X give a symmetry group for Q, and if that representation is even irreducible, then the dimension of X is the dimension of the internal symmetries of Q. In other words: The irreducible representation of $Cl(1,3)$ into $L(X,X)$ will disambiguate the internal symmetries of Q, and choice of a unit vector $\chi \in X$ will be exactly what physically is called a gauge (for the Lorentz invariant theory)!
Finally, let's show *that* (such a representation of $Cl(1,3)$ exists):
Let $\gamma_0, \gamma_1, \gamma_3, \gamma_3$ be the (normalized) Dirac matrices

$$\gamma^0 = 2^{-1/2}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad \gamma^1 = 2^{-1/2}\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix},$$

$$\gamma^2 = 2^{-1/2}\begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix}, \text{ and } \gamma^3 = 2^{-1/2}\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

Then these matrices define (span) a complex, unitary and associative algebra *A*, for which $\iota: \mathbb{C}^4 \ni x \mapsto x_0\gamma_0 + \cdots + x_3\gamma_3 \in A$ satisfies $(\iota\bar{x})\cdot(\iota x) = Q(x)1 \;\; \forall x \in \mathbb{C}^4$. Therefore, *A* is a representation for $Cl(1,3)$ into $L(\mathbb{C}^4, \mathbb{C}^4)$, which clearly is irreducible, and which will be denoted by $Cl(1,3)$, either. That makes $U(4)$, the unitary operators on $\mathbb{C}^4$, its symmetry group. It is as well-known as evident that $Cl(1,3)$ is of dimension $\sum_{1\le k\le 4}\binom{4}{k} = 2^4 = 16$: a vector base basis consists of factorial 1 monomials $\gamma_0, \gamma_1, \gamma_3, \gamma_3$, the unity 1 plus 4 factorial 2 duplets $\gamma_0\gamma_1, \dots \gamma_2\gamma_3$, 4 triplets, and one quadruplet $\gamma_0\cdot\gamma_1\cdot\gamma_2\cdot\gamma_3$. That makes $Cl(1,3)$ a complete, metrizable, simply connected, locally compact vector space which is as is as easy to deal with as any $\mathbb{C}^n$.
In particular, what was said about $\mathbb{C}^n$-valued functions, derivatives, tempered distributions and Fourier transforms (in the appendix) readily applies to $Cl(1,3)$-valued functions.

All that said, within $Cl(1,3)$, the root of $|E|^2 = |m|^2c^4 + |\boldsymbol{p}|^2c^2$ is given by

(5) $\quad E\gamma^0 = mc^2 - c\boldsymbol{p}\cdot\boldsymbol{\gamma} = mc^2 - c(p_1\gamma_1 + p_2\gamma_2 + p_3\gamma_3).$

Equivalently, transforming the above equation with γ^0, which is its own inverse, $E\gamma^0 = mc^2 + c\boldsymbol{p}\cdot\boldsymbol{\gamma}$ holds, which allows to rewrite it in covariant manner as: $cp_\mu\gamma^\mu = mc^2$, and at the same time proves that γ^0 must be the parity (space inversion).

(Note, that $\gamma_1, \dots, \gamma_3$ aren't Hermititian, however their products $\gamma_0\gamma_1, \dots, \gamma_0\gamma_3$ are.)

Integrability of Maxwell's Equations

Likewise, the root of $-\Box$ is $(-\Box)^{\frac{1}{2}} = i(\gamma_0\partial_0 - \gamma_1\partial_1 - \gamma_2\partial_2 - \gamma_3\partial_3) = i\gamma_\mu\partial^\mu$.
Now, what is S? Let $f = (f^0, f^1, f^2, f^3)$ be a smooth be a smooth function on space and time, $\mathbb{R}^4$ into the Clifford algebra $Cl(1,3)$. Then
(6) $\quad \omega: f \mapsto -i\gamma \cdot f dx \coloneqq -i(\gamma_0 f^0 dx^0 - \gamma_1 f^1 dx_1 - \gamma_2 f^2 dx_2 - \gamma_3 f^3 dx_3)$
is a mapping of f to the differential 1-form $\omega(f) \coloneqq -i\gamma \cdot f dx$ in the Clifford algebra $Cl(1,3)$. This is a covariant version of the Euclidian, differential 1-form which differs from it by replacing dx_μ by $\gamma_\mu dx_\mu$ and replacing the partial derivative ∂_μ by $\gamma_\mu\partial_\mu$.
By Poincaré's lemma, ω is integrable if and only if its external derivative $d\omega$ vanishes, i.e.: if and only if $d\omega = 0$. That would mean that the integral $\int f d\omega(\Gamma)$ along a continuous path Γ, connecting two points x(0) and x(1) and not touching the light cone, would depend only on the end points, $\int f d\omega(\Gamma) = S(f)\big(x(1)\big) - S(f)(x(0))$ (see [2]). The function $S(f)$ therefore is called the integral $S(f)$=∫ fdω. Taking the partial derivatives of $S(f)$ gives: $-i\gamma^\mu\partial_\mu S(f) = f$. So, the integrals $\omega(f)$ are the inverses of $-\Box^{1/2} f$. (Note that S, above, is not symmetric, but $S\gamma_0$ and $\gamma_0 S$ are.)
Now, we already have a solution for the wave operator, hence: $S(f) = i\sum\gamma^\mu\partial_\mu(K * f)$ holds (within the time like cones and outside the support of f, wich we assume to be bounded). All in all we proved two things: There is a well-defined solution for $S: f \mapsto S(f)$ in $Cl(1,3)$, and S is the integral of $\omega(f)$, which is an action integral: it is a wave representing the action of the charged particle. Plus, the path integration is covariant, depending only on the end points in every Lorentz transformed system.
Up to now, $S(f) = \int f d\omega$ is defined only on paths not touching the light cone. But it can be extended (by analytic continuation) to all points on the light cone, for which the limit exists.
Let's prove even more: The Fourier transform $\mathcal{F}(\partial^\mu f)(x) \coloneqq (2\pi)^{-2} \int e^{-ixy}(\partial^\mu f)(y) d^4y$ is given by $\mathcal{F}(\partial^\mu f)(x) = \frac{1}{-ix_\mu}(\mathcal{F}f)(x)$, and by Cauchy theorem (integrating clockwise around the pole 0): $\int \frac{e^{izs}}{z} dz = (2\pi i)^{-1} H(s)$, where H is the Heavyside function, which is defined on $\mathbb{R}$ to be ½ for s=0, 0 for s<0, and 1 for s>0. Hence, taking the Fourier inverse of $\mathcal{F}(\partial^\mu f)$ we get:

$$I^\mu := \left(\partial_\mu\right)^{-1}: f \mapsto \begin{cases} \int_{-\infty}^{x^\mu} f(x^0,..,y^\mu,..,x^3)dy^\mu \;\; for\; x^\mu < 0 \\ \left(\frac{1}{2}\right)\int_{-\infty}^{\infty} f(x^0,..,y^\mu,..,x^3)dy^\mu \;\; for\; x^\mu = 0, \\ -\int_{x^\mu}^{\infty} f(x^0,..,y^\mu,..,x^3)dy^\mu \;\; for\; x^\mu > 0 \end{cases} \quad (0 \le \mu \le 3),$$

(which in particular maps functions that vanish outside bounded sets into ones that equally vanish outside bounded sets, i.e.: the target of those functions is in the range again) see: [3, p. eq. 14.59].

Therefore, $Sj = \phi := -i\gamma^\mu I_\mu j$ if and only if $d^2\omega(\phi) = 0$. So, let's prove: $d^2\omega(\phi) = 0$:

We have:

$d^2\omega\left(-i\gamma^\mu I_\mu j\right) = \sum_{\mu<\nu} \gamma_\mu\gamma_\nu\, g_{\mu\nu}\left(\partial_\mu\partial_\nu + \partial_\nu\partial_\mu\right)\phi dx_\mu \wedge dx_\nu$, and, since j is integrable w.r.t. ω, for $\mu \neq \nu$, therefore the path integration along any small rectangle in the $\mu\nu$-plane, which is a closed curve, is zero, and with $\gamma_\mu\gamma_\nu = -\gamma_\nu\gamma_\mu$ the result follows.

Finally, we note that S is symmetric w.r.t. the Hermititian mapping

$<\cdot,\cdot>: (\phi,\psi) \mapsto \int \overline{(\gamma_0\phi)}\psi$, ie.: $<\psi, S\phi> = <S\psi, \phi>$ holds.

And with this, we can put it all together:

Multiplying Maxwell's equations $-\Box(-A)^\mu(t,\boldsymbol{x}) = j^\mu(t,\boldsymbol{x})$, $0 \le \mu \le 3$, to the left componentwise with a jet $\tilde{\jmath}^\mu$ gives:

$<\tilde{\jmath}^\mu, -\Box(-A)^\mu> = <\tilde{\jmath}^\mu, -\Box S^2 j^\mu> = <(-\Box)^{\frac{1}{2}}S\tilde{\jmath}^\mu, (-\Box)^{\frac{1}{2}}Sj^\mu>$, $0 \le \mu \le 3$, where $(-\Box)^{\frac{1}{2}} := -i\gamma_\nu\partial^\nu$, and with $A := \gamma_\mu A^\mu$, $j := \gamma_\mu j^\mu$, and $\tilde{\jmath} := j$, we can sum up, giving

$$<(-\Box)^{\frac{1}{2}}Sj, (-\Box)^{\frac{1}{2}}Sj> = <(-\Box)^{\frac{1}{2}}\tilde{\jmath}, S^2(-\Box)^{\frac{1}{2}}j>$$
$$= <(-\Box)^{\frac{1}{2}}j, (-\Box)^{\frac{1}{2}}S^2 j> = <j,j> = \bar{\jmath}_\mu j^\mu 1_4.$$

So, in particular, $-i\gamma_\nu\partial^\nu Sj = j$.

Resume

Let us resume: Starting with a jet of charged particles $j = \gamma_\mu j^\mu$ we proved that Maxwell's equations $\Box \mathrm{A}^\mu(t, \boldsymbol{x}) = j^\mu(t, \boldsymbol{x})$, $(0 \le \mu \le 3)$, are integrable w.r.t. ω, i.e.: $S(j) = -i \int \gamma_\mu j^\mu dx_\mu$ is path independent (within the time-like light cones), and we have $\mathrm{A}^\mu(t, \boldsymbol{x}) = (S^2 j(t, x))^\mu$, i.e: A^μ is the μ-th component of $S^2 j$.

This is nothing but the relativistic formulation of the action integral of a free particle system in non-relativistic classical mechanics:

In non-relativistic classical mechanics, when we have a system of n particles $q_1, \ldots, q_n$ with momenta p_k such that $\int_{t_0}^{t_1} \sum_k p_k(t) dq_k(t)$ up to an additive constant exists as a path-independent function of $t, q_1, \ldots, q_n$, then that function is the action integral of the free Lagrangian.

The difference in relativistic mechanics comes entirely from the fact that the Minkowski metrics is the (indefinite) quadratic form $ds^2 = dx^\mu dx_\mu$ (its canonical conjugated momenta have the same metrics, either). Taking its root yields the Dirac spinor terms.

So, obviously, the Maxwell equations disregard the interaction of the charged particles with themselves as an idealization, and its objective is what action that system will have on its environment.

In non-relativistic mechanics that action is by itself both instantaneous and gratis, i.e.: it does not cost the system any energy to deliver that action. However, in relativistic affairs, the action spreads at the speed of light: it is not instantly everywhere else. But does it now cost energy to propel that action? Relying on Poincaré invariance which includes invariance w.r.t. time displacement, an isolated particle at rest must still retain its energy over time. So, even in relativistic conditions the action should spread without energy cost.

Charge Inversion, Inert Mass & Neutral States

Using the Clifford algebra, we can rewrite Maxwell's equations as

(7) $i\gamma^\mu \partial^\mu S j = j,$

which can formally be rewritten into a gravitational equation just by substituting the electric charge by the (neutral) inert mass. Moreover, the $Cl(1,3)$-representation actually is a superset of what would be needed to formulate electromagnetism:

Let $\Pi_+, \Pi_-: Cl(1,3) \to L(\mathbb{C}^4, \mathbb{C}^4)$ be the projections onto the 1st two and last two rows of $Cl(1,3)$-representation with the Dirac matrices, resp., and let

$T := -\gamma_1\gamma_2\gamma_3 = i\gamma_0\gamma_5 = i\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}$, where $\gamma_5 := i\gamma_0\gamma_1\gamma_2\gamma_3$.

Since T anti-commutes with γ_0 and commutes with $\gamma_1, \gamma_2, \gamma_3$, T is the energy inversion. With this, $C := \gamma_5$ is defined as charge inversion (or equivalently the particle inversion), so that we get the fundamental relation $T = iPC.$, In particular, C is a self-adjoint conjugation, i.e. satisfies $C = C^*$ and $C \cdot C^* = 1_4$ divides $\mathbb{C}^4$ into the direct sum $\mathbb{C}^4 = C_+ \oplus C_-$ of two 2-dimensional eigenspaces of C with eigenvalue 1 and -1, respectively. Because the inverse charge of q should be $-q$, the elements of C_- must be the charges, and C_+ therefore is the space of neutral composites of charges. With π_{C_+} and π_{C_-} being the projections of $L(\mathbb{C}^4, \mathbb{C}^4)$ onto C_+ and C_- resp., that means the following for the spinor-flux $j: (t, \boldsymbol{x}) \mapsto j(t, x) \in L(\mathbb{C}^4, \mathbb{C}^4)$ splits into the sum of a (purely) charged current $\pi_{C_-}{\circ}j$ and a neutral constituent $\pi_{C_+}{\circ}j$. (A jet of charged particles will be called purely charged, if its neutral constituent is negligible.)

Starting with the basic relativistic equation $E^2 = {m_0}^2 + \boldsymbol{p}^2$ for a mechanical particle of rest mass m_0, we can solve it in $Cl(1,3)$ again, and will get a mass/energy tensor $q = m_\mu \gamma_\mu u^\mu$, where positive and negative mass can be identified with positive and negative electrical charges, if and only if charges will transform in terms of $Cl(1,3)$ equivalently to mass:

We have

$$\gamma^0 E(t,\boldsymbol{x}) := q^0(t,\boldsymbol{x}) + \gamma^1 q(t,\boldsymbol{x})u^1(t,\boldsymbol{x}) + \gamma^2 q(t,\boldsymbol{x})u^2(t,\boldsymbol{x}) + \gamma^3 q(t,\boldsymbol{x})u^3(t,\boldsymbol{x}),$$

So, reinserting c and putting $E(t,\boldsymbol{x}) = q(t,\boldsymbol{x})c^2$ we can solve for q:

(8) $q(t,\boldsymbol{x}) = q^0(t,x)(\gamma^0 - \boldsymbol{\gamma} \cdot \left(\frac{v}{c}\right))^{-1}.$

Not surprisingly, $|q| = |q^0|/(1-\left(\frac{v^2}{c^2}\right))^{1/2}$ indeed transforms as the mass under Lorentz transformation.

Moreover, $q^0 \mapsto q^0(\gamma^0 - \boldsymbol{\gamma}\cdot\left(\frac{\boldsymbol{v}}{c}\right))^{-1}$ is a transformation of trace zero: in particular that means that the net charge is conserved in a Lorentz boost: under a Lorentz boost, a particle appears to acquire positive and negative charge altogether at equal rate.

To calculate the trace, rotate the velocity onto the 1st special axis, expand

$$(\gamma^0 - \gamma^1\left(\frac{v}{c}\right))^{-1} = \gamma^0(1-\gamma^0\gamma^1\left(\frac{v}{c}\right))^{-1} = \gamma^0\left(1+\gamma^0\gamma^1\left(\frac{v}{c}\right)+\left(\gamma^0\gamma^1\left(\frac{v}{c}\right)\right)^2+\cdots\right),$$

applying $(\gamma^0\gamma^1)^2 = -1$ and $\gamma^0\gamma^0 = 1$, the terms are seen to be proportional to either γ^0 or γ^1, both of which have trace zero, so $(\gamma^0 - \gamma^1\left(\frac{\boldsymbol{v}}{c}\right))^{-1}$ has zero trace.

For a charged flux $j\colon \mathbb{R}^4 \to C_-$, we know that $-A^\mu(t,\boldsymbol{x}) = (S^2 j)^\mu(t,\boldsymbol{x})$, $(0 \le \mu \le 3)$ is the 4-vector potential, so that for a separated and real-valued charged fluxes j, j' (i.e.: j' and j are real-valued and have a disjoint spatial support with for each t), $\int j'(t,\boldsymbol{x})(S^2 j)\,(t,\boldsymbol{x})d^3x$ is the interaction energy of j and j' at time t. So, we postulate that for arbitrary, separated complex-valued fluxes j, j' their interaction energy at time t be given by: $\int \overline{j'(t,\boldsymbol{x})}(S^2 j)\,(t,\boldsymbol{x})d^3x$.

Then, up to a scaling factor κ_0, we expect the same relation for the interaction energy to hold for neutral fluxes either, and we can set $\kappa_0 \equiv 1$ (instead of $\kappa_0 \equiv -1$). This makes it possible to state the Hamiltonian for both kinds of fluxes, charged or neutral, in a closed form:

Non-Free, Interacting Electrodynamic Fields

In terms of the Clifford algebra, the Hamiltonian for a free system of (square root) mass or charge density is:

(9) $\mathcal{H}_{free} = E(t,\boldsymbol{x}) = \gamma^0 j^0(t,\boldsymbol{x}) + \gamma^0\gamma^1 j^1(t,\boldsymbol{x}) + \gamma^0\gamma^2 j^2(t,\boldsymbol{x}) + \gamma^0\gamma^3 j^3(t,\boldsymbol{x}),$

and let Sj be the action integral defined as before. Therefore, $j = (-\Box)^{1/2} Sj = i\gamma^\mu \partial_\mu Sj$ is the source of Sj, and since S propagates at the speed of light, the total action should be the integral of all actions of $j(t',\boldsymbol{x}')$ on the backward light cone $\Gamma_-(x) \coloneqq \Gamma(t,\boldsymbol{x}) \coloneqq \{(t',\boldsymbol{x}') \in \mathbb{R}^4 \mid (t-t')^2 - |\boldsymbol{x}-\boldsymbol{x}'|^2 = 0,\ t' \le t\}$.

So, the relation $\int \bar{j}\cdot(S^* Sj)\,d^3x = \int \overline{j_\mu}(-A)^\mu\,d^3x$ suggests the self-interacting Hamiltonian

(10) $\mathcal{H} = E(t, \boldsymbol{x}) = \mathcal{H}_{free} - lim_{\varepsilon \to 0} \int_{\Gamma_-(x)} \overline{Cj(y)} S^2(y, x - i\varepsilon) j(x - i\varepsilon)\, dy$

for charged and neutral particles, where it is assumed that the maximum absolute velocity is uniformly bounded $|v| < c$ and

(11) $S^2(y, x) := S^2(s, \boldsymbol{y}, t, \boldsymbol{x}) := (1/4\pi) 1/|\boldsymbol{x}(t) - \boldsymbol{y}(s)|$ with $s = t - |\boldsymbol{x} - \boldsymbol{y}|/c$

is the (forward) propagator of the spherical wave (see e.g. *[4]*, Ch. 21).
Note that the propagator is non-negative and its spinor is unity, that is, it is not spinor-dependent. Therefore, a positive mass is propagated into a positive mass.
In particular, for neutral fluxes, we get a field which in the non-relativistic limit converges to the gravitational field.

(Equation (10) could serve to extend the theory of strong interactions into classical theory: The symmetry group $SU(3) \times SU(1)$ (capturing hadrons and isospin) naturally embeds into $U(4)$, our symmetry group of Lorentz invariance.)

Let's shortly see how to get back the "ordinary" Maxwell equations from (7): Multiply from the left with a charge flow j' and integrate over space-time, bring the differential operator on $\overline{S(J')}$ over to $i\partial_\mu \gamma_\mu S(j)^\mu$ by partial integration, then drop the integral and take the limit $j' \to \sum \gamma_\mu$. That gives the Maxwell equations in terms of the Dirac matrices. Then (temporarily) drop or disregard the 2 negative energetic rows at the bottom of these matrices, which reduces the equations to ones in terms of 2x2-matrices, namely the unit matrix 1_2 and the Pauli matrices $\sigma = (\sigma_1, \sigma_2, \sigma_3)$. To finally get rid of the matrices, take the root of the square of these matrices, each, respecting the sign of the charge (and at last, transform to the density of the charge flow on either side).
In all, there are two major steps as to it: the 1st is to forget about negative energetic charges, and the 2nd is to average over oscillations (taking the square root of velocity squares), which is just what statistical mechanics is doing.
That renders classical electrodynamics as a statistical mechanics for charged particles – with a disregarded negative-energetic part. In that sense, the $Cl(1,3)$ model of electrodynamics converges to the "ordinary", spinor-free theory in the statistical limit (and the same holds for the gravitation of masses).

Spin & Superconductivity

Let’s examine parity/spin:

For a free jet of charged particles we have in terms of (square root) densities and the 4-velocity u: $\gamma_0 j_0 = \rho_0 + \gamma_1 j_1 + \gamma_2 j_2 + \gamma_3 j_3$, where the j_μ are the components of the energy – momentum (square root) density and ρ_0 is the density at rest. So, the j_μ and ρ_0 are real-valued functions (of time and space coordinates), and in addition, they all have the scalar value of electronic charge, q_e, as a common factor. But, in terms of the Clifford algebra, $Cl(1,3)$ the charge at rest is not a scalar value, but to be defined as equivalence class $\Psi_e := \left\{(q_e/\sqrt{2})\begin{pmatrix} r1_2 & 0 \\ 0 & -r1_2 \end{pmatrix} \mid r \in \mathbb{C}^2, |r| = 1\right\}$, of which $(q_e/\sqrt{2})\gamma_0$ is a representative.

So, we should multiply the above equation by γ_0, which gives in terms of the 4-vector $u = (1, \boldsymbol{u})$:

$j_0 = \gamma_0\rho_0 + \gamma_0\gamma_1\rho_0 u_1 + \gamma_0\gamma_2\rho_0 u_2 + \gamma_0\gamma_3\rho_0 u_3$.

Because ρ_0 is supposed to be real-valued, $\gamma_0\rho_0$ is a self adjoint operator, and because the $j_0 = \gamma_0 J_0 + \gamma_0\gamma_1 J_1 + \gamma_0\gamma_2 J_2 + \gamma_0\gamma_3 J_3 = \gamma_0 J_0 - \gamma_1\gamma_0 J_1 - \gamma_2\gamma_0 J_2 - \gamma_3\gamma_0 J_3$ where $J = \gamma_0 j$ as well as the $J_{\mu=}\gamma_0 j_\mu = \rho u_\mu$ are real-valued, scalar quantities, ρ_0 is the electronic charge density (at rest), and the u_μ are the components of the 4-velocity.

The three 2x2 Pauli matrices $\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\sigma_2 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}$, $\sigma_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ allow to define three Hermititian ***spin operators*** $\Xi_j := \begin{pmatrix} \sigma_j & 0_2 \\ 0_2 & \sigma_j \end{pmatrix}$, $(1 \le j \le 3)$, which all commute with the parity γ_0, and each Θ_j commutes with γ_j, but anti-commutes with the other two. For each j, γ_0 and Ξ_j have a mutual non-degenerate spectral representation, and, since the Pauli matrices all decompose into a one-dimensional eigenspaces for the eigenvalues ± 1, they uniquely define a conjugation Θ_j which commutes with γ_0 and interchanges the +1-unit-eigenvector with the negative -1-unit-eigenvector. Θ_j is called ***spin inversion***. Not only do the spin operators commute with parity γ_0, they also commute with the charge inversion γ_5, so that for each of the spin operators the pair (Ξ_j, γ_5) is a socalled “complete set of observables”. In their diagonalized spectral forms, spin inversion is the interchange of the 1st spin-up component with the 2nd spin-down component and the 3rd (spin-up) with the 4th (spin-down) component of the complex ket/bra quadruplet. The space $\mathbb{C}^4$ of d symmetric w.r.t. the ket/braquadruplets therefore decomposes into the direct sum of eigenvectors of the spin inversion with eigenvalue 1, called ***fermions***, and those of eigenvalue -1, the ***bosons***.

To see what θ_j is about, I choose x_3-axis as direction of motion. With this, the equation of motion of the free jet of particles becomes:

$$j = \gamma_0 j_0 + \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} j_3. \tag{12}$$

Under the premise that this equation is γ_5-symmetric, in order to guarantee the constancy of net charge, for a purely charged jet, it is necessary and sufficient that the gauge quadruple $(\lambda_0, \dots, \lambda_3)$ be chosen such that $\lambda_0 = \lambda_1$ and $\lambda_2 = \lambda_3$. These elements are anti-symmetric w.r.t. the spin-inversion (along the 3rd special axis). Such particles then are ***fermions***. For a neutral jet, however, $(\lambda_0, \dots, \lambda_3)$ must be chosen such that $\lambda_0 = -\lambda_1$ and $\lambda_2 = -\lambda_3$, in order that the net mass increases with velocity according to the Lorentz transformation. These tuples are symmetric w.r.t. spin inversion. Such particles are ***bosons*** (Note that due to the charge inversion symmetry of the neutral particles, one may always choose the mass to be non-negative). So, in particular, it explains why electrically charged particles are fermions, why (purely) neutral particles (with zero magnetic momentum) are bosons, why spin-up and spin-down particles come in equal ratios, and why fermions follow the exclusion principle, i.e.: avoid being in the same state.

Within the classical context above, what I termed to be spin-up and spin-down now implies that the spin-up charges should increase their charge with velocity, whereas the spin-down charges should decrease their charge. That would suggest a rough test to see whether the above notion of spin coincides with the quantum mechanical one: The two jets of silver atoms produced by the Stern-Gerlach experiment should be charged: the spin-up jet should be negative, the spin-down jet positively charged.

Let us now concentrate on purely charged jets, i.e.: jets with negligible neutral constituent: Equation (12) allows the magnetic field to be explained as the polarization of charges according to the spins:1st row up, 2nd down, 3rd up, 4th down, where the self-inductivity is proportional to the charge in motion, which is its electric “mass of inertia”. For a jet running in a ferromagnetic conductor, the ferromagnetic material itself can then be imagined as running charges internally, i.e.: the magnetic field will become the charge polarization of moving charged particles. Particularly, let j be a flux of electrons moving with constant speed u_3 along the x_3-axis. Leaving out retardation effects, its action function S is (up to an additive constant) given by

$S(t, x_1, x_2, x_3) = \left(\frac{1}{2\pi i}\right) const \cdot (t\rho_0 - \rho_0 u_3 \gamma_0 (\gamma_1 + \gamma_2)/\sqrt{x_1^2 + x_2^2}))$.

Next, superimpose that electronic current by an ionic current of speed u_3' of opposite sign along the x_3-axis, which will result in an action function which up to the velocity u_3' is the negative of S. Then, considering the mutual interaction of the (positively charged) ions with the electrons, their sum cancels, and we are left with the magnetic overall interaction (due to the velocity difference), as expected: $S(t, \boldsymbol{x}) = -const' \cdot \rho_0 u_3 \gamma_0 (\gamma_1 + \gamma_2)/\sqrt{x_1^2 + x_2^2})$). We now know two further things: Firstly, we know that this action is purely magnetic, where the magnetic field is at least roughly proportional to the j_3-term of equation (12), and secondly, we saw above the gauge quadrupel $(\lambda_0, \dots, \lambda_3)$ demands $\lambda_0 = \lambda_1$ and $\lambda_2 = \lambda_3$ which means that the spin-up and spin-down particles must come at equal rate.

At low temperature (around 4-5° K) for many ferromagnetic materials the Meißner-Ochsenfeld effect sets in, leading to a rapid decay of the internal magnetic field to zero in what thermodynamically is known to be a phase transition to a superconducting state.

What that means is that the action within the conductor will become zero. That material does therefore hide its internal electromagnetic inertia to the outside, in other words, it cannot scatter in this state with electromagnetic particles and fields. That explains why a jet of such particles does not stop at thin insulation barriers, and it explains why the resistance drops to zero. So, matter in the state of superconductivity not running a current at its surface will be electromagnetically plain dark!

Now, if we look for how the magnetic field can drop to zero in the phase transition to superconductivity, it is evident that a coupling of spin up and down charges should be made responsible. At this point we can readily dock at the BCS theory, supplying it with the necessary spin pairs, the so called Cooper pairs that this theory still appears to be missing.

In all, the existence of superconducting matter outside the electromagnetically visible edges of the galaxies, where it is relatively cold, would only show up through the light from external galaxies as an increased curvature of space time, due to the neutral, massy content of the superconductors.

APPENDIX

Solving the wave equation $\Box\boldsymbol{\phi} = \left(\frac{\partial^2}{\partial t^2} - \nabla^2\right)\boldsymbol{\phi} = \boldsymbol{\rho}$**:**

The wave equation can best be solved on a topological vector space, on which the d'Alembert operator is everywhere defined and continuous and on which the Fourier transformation is an isomorphism. That will be the space $S'(\mathbb{R}^4)$ of tempered distributions (see [5], Ch. 25, p. 264 ff.): Let $\mathbb{N}_0 = \{0,1,2\ldots\}$ be the natural numbers including 0, $\alpha = (\alpha_0, \ldots \alpha_3), \beta\epsilon\mathbb{N}_0^4$, let ∂^α denote the αth partial derivative and $x^\alpha = x_0^{\alpha_0} \cdots x_3^{\alpha_3}$, $(\alpha\epsilon\mathbb{N}_0^4)$. Then $S(\mathbb{R}^4)$ is defined as locally convex vector space of indefinitely differentiable, complex-valued functions on $\mathbb{R}^4$of rapid decrease and equipped with the seminorms:
$p_{\alpha\beta}: f \mapsto p_{\alpha\beta}(f) := \sup_{x\epsilon\mathbb{R}^4}|\partial^\alpha f(x)|\left(1 + \left|x^\beta\right|\right),\ (\alpha, \beta\epsilon\mathbb{N}_0^4)$.
This makes $S(\mathbb{R}^4)$ a locally convex space, which is even separable, hence sequentially complete, since the number of seminorms is countable (see [5], Ch. 10, Ex. IV). Armed with sequential completeness, it is straightforward to prove that $S(\mathbb{R}^4)$ is complete.
A locally convex space X is said to be separated, if for every unequal elements $x, y \in X$ there exist disjoint neighborhoods of x and y, respectively. $S(\mathbb{R}^4)$ therefore is separated as well. $S'(\mathbb{R}^4)$ then is defined as the space of continuous functionals on $S(\mathbb{R}^4)$, the dual of $S(\mathbb{R}^4)$. For $\alpha, \beta\epsilon\mathbb{N}_0^4$ let $W_{\alpha\beta}$ be the space of all $T \in S'(\mathbb{R}^4)$ such that the (semi)norm $p'_{\alpha\beta}: T \mapsto \sup_{p_{\alpha\beta}(f)\leq 1} |T(f)| < \infty$. Then $S'(\mathbb{R}^4) = \bigcup_{\alpha\beta} W_{\alpha\beta}$. The topology on $S'(\mathbb{R}^4)$ then is defined as the strongest locally convex topology such that the injections $\iota_{\alpha\beta}: W_{\alpha\beta} \to S'$) are continuous. Again, $S'(\mathbb{R}^4)$ is separable and separated, and again it is complete, since a limit of a sequence $T_k \in S'(\mathbb{R}^4)$ which is converging w.r.t. all $p'_{\alpha\beta}$ is a continuous functional on $S(\mathbb{R}^4)$.
Next, for two locally convex spaces X, Y the direct sum $X\oplus Y$ is the algebraic sum of the two spaces equipped with the finest locally convex topology for which the natural injections $i_X: X \to X\oplus Y$ and $i_Y: Y \to X\oplus Y$ are continuous.

Next, we need two lemmata:

Lemma1: Let X be a complete and separated locally convex space and $T: X \to X$ a continuous linear operator on X. Let $ker(T) := \{x \in X|Tx = 0\}$ be the kernel of T, $range(T)$ its range. Then X is isomorphic to $(X/\ker(T))\oplus ker(T)$. If moreover T is such that $T^2x = 0 \Leftrightarrow Tx = 0$, then $X = range(T)\oplus ker(T)$ holds.

Proof: Clearly, $range(T)\oplus ker(T) \subset X$, so it needs to be proved that $I: range(T)\oplus ker(T) \to X$ is onto, continuous and open. Since X is separated, its origin $\{0\}$ its closed, and so $ker(T)$ is closed as the preimage of a closed set under the continuous mapping T. ($ker(T)$ is also open in itself topological space.) Let $X/ker(T)$ be the quotient space of equivalence classes $\bar{x} = x + ker(T)$ which is given the finest locally convex topology for which the canonical projection $p: X \ni x \mapsto \bar{x} \in X/ker(T)$ is continuous. With this topology p then is also open, i.e.: p maps open sets into open ones (and therefore closed into closed ones), and since p is a (surjective) projection, $X/\ker(T)$ is open and closed. Algebraically X clearly is isomorphic to $X/ker(T)\oplus ker(T)$, and by the choice of topology on $X/ker(T)$, the isomorphism is also continuous and open. Then $\bar{T}: \bar{x} = x + \ker(T) \mapsto T(x)$ defines a surjective linear mapping onto $range(T) \subset X$, therefore $range(T)$ is closed and open (in X), so X is also topologically isomorphic to $(X/\ker(T))\oplus ker(T)$. If moreover T is such that $T^2x = 0 \Leftrightarrow Tx = 0$ holds, then $\bar{T}$ above is also injective, hence $range(T)$ isomorphic to $X/\ker(T)$, and therefore $X = range(T)\oplus ker(T)$.

Finally, let $\mathcal{F}: S(\mathbb{R}^4) \ni f \mapsto (2\pi)^{-2}\int e^{-ixy}f(x)d^4x$ be the Fourier transformation. Then $\mathcal{F}$ is an isomorphism on $S(\mathbb{R}^4)$, its transpose therefore an isomorphism on $S'(\mathbb{R}^4)$, and $\mathcal{F}^{-1}: S(\mathbb{R}^4) \ni f \mapsto (2\pi)^{-2}\int e^{ixy}f(x)d^4x \in S(\mathbb{R}^4)$ its inverse, which extends through its transpose as inverse of $\mathcal{F}$ on $S'(\mathbb{R}^4)$.
Partial integration yields: $\mathcal{F}\partial^\alpha f(y) = i^{\sum\alpha_k}y^\alpha\mathcal{F}f(y)$, therefore $\mathcal{F}(\Box \mathrm{f})(\mathrm{y}) = -(\mathrm{y}_0^2 - (\mathrm{y}_1^2 + \mathrm{y}_2^2 + \mathrm{y}_3^2))\,\mathcal{F}f(y)$, which again extends to $S'(\mathbb{R}^4)$ through transposition onto $S'(\mathbb{R}^4)$.
So, the kernel of $\Box$ in $S'(\mathbb{R}^4)$ is the Fourier inverse of the space of all $\hat{T} \in S'(\mathbb{R}^4)$ with support $supp(\hat{T}) \subset C_+ \cup C_-$ in the forward and backward light cones $C_+ \cup C_-$, and $range(\Box)$ therefore is the Fourier inverse of the closure of all $\hat{T} \in S'(\mathbb{R}^4)$ with $supp(\hat{T}) \cap C_+ \cup C_- = \{\emptyset\}$. Also, clearly $\Box \mathrm{f} \neq 0 \Rightarrow \Box^2\mathrm{f} \neq 0$ holds. Now, because of the Lemma1 above, $range(\Box)$ is closed in $S'(\mathbb{R}^4)$ and $\Box$ an isomorphism onto $range(\Box)$, that is: $\Box^{-1}$ is a continuous operator on $range(\Box)$.

Lemma2: $S(\mathbb{R}^4) \cap range(\Box)$ is dense in $range(\Box)$.

Proof: For $k \in \mathbb{N}$ let $B(k) := \{x \in \mathbb{R}^4 \mid |x| \leq k\}$, $\mathbb{I}_{B(k)}$ the characteristic function of $B(k)$, which is 1 on $B(k)$ and 0 outside, let
$\psi(x) := (\int \exp(\frac{-1}{1-|y|^2})\,d^4y)^{-1}\exp(\frac{-1}{1-|x|^2})$, $(x \in \mathbb{R}^4, |x| < 1)$, and $\psi(x) = 0$ for $|x| \geq 1$.
Then $\psi \in S(\mathbb{R}^4)$.
For $f \in S(\mathbb{R}^4)$: $\mathbb{R}^4 \ni y \mapsto f_{(y)} := f(\cdot - y) \in S(\mathbb{R}^4)$, so for $T \in S'(\mathbb{R}^4)$: $T * f: \mathbb{R}^4 \ni y \mapsto Tf_{(y)} \in \mathbb{C}$ a well-defined infinitely differentiable function of polynomial growth.
Now, for $k \in \mathbb{N}$ let $\psi_k := (2k)^4\psi(\,\cdot\,/2k)$. Then $(\mathbb{I}_{B(k)} * \psi_k)(T * \psi_k) \to_{k\to\infty} T$ for every $T \in S'(\mathbb{R}^4)$. On the same line, $T \in S'(\mathbb{R}^4)$ has support in $C_+ \cup C_-$ if and only if , given $C(k) := \{x \in \mathbb{R}^4 \mid \inf_{y \in C_+\cup C_-}|x - y| \leq k^{-1}\}$, $(k \in \mathbb{N})$: $(\mathbb{I}_{C(k)} * \psi_k)(T * \psi_k) \to_{k\to\infty} T$. So, for $T \in \mathcal{F}(range(\Box))$ and $\Omega(k) := (\mathbb{R}^4\backslash C(k)) \cap B(k)$: $(\mathbb{I}_{\Omega(k)} * \psi_k)(T * \psi_k) \to_{k\to\infty} T$.
Therefore $\mathcal{F}(range(\Box)) \cap S(\mathbb{R}^4)$ is dense in $\mathcal{F}(range(\Box))$, which proves Lemma2.

Now, for $f, g \in S(\mathbb{R}^4)$ the convolution $f * g$ is well-defined, and $\mathcal{F}(f * g) = (\mathcal{F}f)(\mathcal{F}g)$ holds. Then, by extension, $\mathcal{F}(T * f) = (\mathcal{F}T)(\mathcal{F}f)$ for all $f \in S(\mathbb{R}^4)$ and $T \in S'(\mathbb{R}^4)$.
Therefore,

$$\Box^{-1}f = \lim_{k\to\infty}\mathcal{F}^{-1}\left(\mathbb{I}_{\Omega(k)}\frac{-1}{|x_0^2| - (|x_1^2| + |x_2^2| + |x_3^2|)}\mathcal{F}f(x)\right) =$$
$$\lim_{k\to\infty}(\mathcal{F}^{-1}\mathbb{I}_{\Omega(k)}(\frac{-1}{|x_0^2| - (|x_1^2| + |x_2^2| + |x_3^2|)})) * f = (\mathcal{F}^{-1}\frac{-1}{|x_0^2| - (|x_1^2| + |x_2^2| + |x_3^2|)}) * f$$

for all $f \in S(\mathbb{R}^4) \cap range(\Box)$, and, since $\Box^{-1}$ is continuous on $range(\Box)$, that relation extends onto $range(\Box)$.

Extension: The spaces $S(\mathbb{R}^4)$ and its dual $S'(\mathbb{R}^4)$ can be generalized to the duality of $S(\mathbb{R}^4, \mathbb{C}^n)$ and $S'(\mathbb{R}^4, \mathbb{C}^n)$, where $S(\mathbb{R}^4, \mathbb{C}^n)$ is the space of smooth and rapidly decreasing functions $f: \mathbb{R}^4 \to \mathbb{C}^n$ and $n \in \mathbb{N}$: Both spaces are n-tuples of $S(\mathbb{R}^4)$ and $S'(\mathbb{R}^4)$, resp., and we can make use of the scalar product $\cdot : \mathbb{C}^n \times \mathbb{C}^n \ni (x, y) \mapsto \sum x_k y_k \in \mathbb{C}$ to write the bilinear form on $S'(\mathbb{R}^4, \mathbb{C}^n) \times S(\mathbb{R}^4, \mathbb{C}^n)$ as: $T(f) = T \cdot f$. With this, all of the above also holds for $\mathbb{C}^n$-valued functions and distributions.

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