The connection between the Riemann Hypothesis and model theory.

Cristian Dumitrescu

Abstract. In this paper, I present a connection between the Riemann Hypothesis and model theory, and this connection leads to a solution to the Riemann Hypothesis.

Keywords. Riemann Hypothesis, Robin's reformulation, Littlewood's reformulation, compactness theorem, saturated models, model theory.

Mathematical Subject Classification Number: 11M99.

Section 1. Preliminary facts.

The following reformulations have been known for some time and the proofs of the statements in this section can be found in many references (except for Littlewood's reformulation with ε rational, which is straightforward).

Robin's reformulation of RH [5]**.** The Riemann Hypothesis is true if and only if there is an n_0 (and in fact $n_0 = 5041$) such that $\sigma(n)/n < e^{\gamma} \cdot \log(\log(n))$, for all $n > n_0$ (here $\sigma(n)$ is the sum of divisors function).

Littlewood's reformulation of RH [1]**.** The Riemann Hypothesis is equivalent to the statement that for every $\varepsilon > 0$, we have $M(x) = O(x^{(1/2 + \varepsilon)})$, when $x \to \propto$ (here $M(x)$ is the Mertens' function).

We write (R) for the statement in Robin's reformulation (that is the Robin inequality). We also write (L) for the statement in Littlewood's reformulation (that is $M(x) = O(x^{(1/2 + \varepsilon)})$, when $x \to \propto$).

We can conclude that the statement:

"there is an n_0 such that $\sigma(n)/n < e^{\gamma} \cdot \log(\log(n))$, for all $n > n_0$ "

is equivalent to the statement:

"for every $\varepsilon > 0$, we have $M(x) = O(x^{(1/2 + \varepsilon)})$, when $x \to \propto$".

We will write (R) $\Leftrightarrow$ (L) for this equivalence (which is proved in ZFC).

Observation. We also note that in Littlewood's reformulation we can take ε rational and find an equivalent statement. In other words, we have:

Littlewood's reformulation of RH (with ε rational). The Riemann Hypothesis is equivalent to the statement that for every **rational** $\varepsilon > 0$, we have $M(x) = O(x^{(1/2 + \varepsilon)})$, when $x \to \propto$ (here M(x) is the Mertens' function).

Lemma. Littlewood's reformulation of RH is equivalent to Littlewood's reformulation of RH with rational $\varepsilon > 0$.

Proof. It is obvious that Littlewood's reformulation of RH implies Littlewood's reformulation of RH with rational $\varepsilon > 0$. For the converse implication, we note that given an $\varepsilon > 0$, we can choose a rational r such that $0 < r < \varepsilon$, and then we have: $(M(x) = O(x^{(1/2 + r)})$, when $x \to \propto) \Rightarrow (M(x) = O(x^{(1/2 + \varepsilon)})$, when $x \to \propto)$. This means that Littlewood's reformulation with rational $\varepsilon > 0$ implies Littlewood's reformulation. **QED.**

<u>Section 2. Model Theory.</u>

The following theorems will be used in our results (for a brief introduction to model theory, see the appendix).

Definition. A model M is said to be $\aleph_0$- saturated if for every enumerable infinite set Φ of formulas $\varphi(x)$ in the diagram language of M, if for every finite subset of formulas φ_1, φ_2, φ_3, $\ldots\varphi_n \in \Phi$ the sentence $\exists x\, (\varphi_1(x) \wedge \varphi_2(x) \wedge \varphi_3(x) \wedge \ldots \wedge \varphi_n(x))$ is true in M, then the infinitely long sentence $\exists x\, (\bigwedge_{\varphi\in\Phi} \varphi(x))$ is also true in M.

The Theorem on the Existence of a Saturated Model. [2] There exists a saturated model over the first - order language in which we work (basically analysis with first order ordinary logic).

Godel's Completeness Theorem. [4] If K is a set of first - order sentences over some fixed language, then K has a model if and only if K is consistent.

<u>Section 3. The main theorem.</u>

Theorem. The Riemann Hypothesis (RH) is true.

Proof. Now we consider the statement (L) with rational $\varepsilon > 0$:

"for every rational $\varepsilon > 0$, we have $M(x) = O(x^{(1/2+\varepsilon)})$, when $x \to \infty$".

Now, we fix a rational $\varepsilon > 0$. The statement "$M(x) = O(x^{(1/2+\varepsilon)})$, when $x \to \infty$", basically means that "there is a $K > 0$ such that $(|M(1)| < K \cdot 1^{(1/2+\varepsilon)}) \wedge (|M(2)| < K \cdot 2^{(1/2+\varepsilon)}) \wedge (|M(3)| < K \cdot 3^{(1/2+\varepsilon)}) \wedge$ ……..".

We consider now the formulas $\varphi_{x,\varepsilon}(K)$, and by definition the formula $\varphi_{x,\varepsilon}(K)$ will mean $(|M(x)| < K \cdot x^{(1/2+\varepsilon)})$. We note here that K is a free variable (that is why $\varphi_{x,\varepsilon}(K)$ are formulas, not sentences). If we consider all the formulas $\varphi_{x,\varepsilon}(K)$, when x is a natural number and ε is a positive rational number, we have an enumerable infinite set of formulas. We consider the conjunction of all these formulas, and we write S(K), so by definition S(K) will mean $(\bigwedge_{x \text{ natural and } \varepsilon \text{ positive rational}} \varphi_{x,\varepsilon}(K))$.

Now, we will assume that we can find a counterexample to Robin's inequality (R), and we will reach a contradiction, thus proving that (R) is true. The contradiction will take place in the saturated model guaranteed to exist by the theorem of the existence of a saturated model applied as follows.

We **assume** that we found a $n' > 5041$ such that $\sigma(n')/n' > e^{\gamma} \cdot \log(\log(n'))$ (here n' is an actual numerical value, as assumed found, not a variable). This n' would represent a counterexample of Robin's inequality (if it exists, as we assume). The key idea is to form an enumerable sequence of statements that starts with $(\sigma(n')/n' > e^{\gamma} \cdot \log(\log(n')))$ and $(n' > 5041)$, and continues with all the statements from S. In other words, we form the enumerable sequence of statements:

$$(\sigma(n')/n' > e^{\gamma} \cdot \log(\log(n'))) \wedge (n' > 5041) \wedge S(K). \qquad (*)$$

We notice that for any finite subset of statements of (*), as described above, there is a model in which all these statements (from the finite subset) are satisfied. The proof of this is based on the fact that any finite set of real numbers has a maximum and a minimum, and the Archimedean property. Any finite subset of statements from (*), as described above, involves a finite set $\{x_1, x_2, x_3, \ldots x_p\}$ and a finite set of rationals $\{\varepsilon_1, \varepsilon_2, \varepsilon_3, \ldots \varepsilon_q\}$. Among the values $|M(x_1)|$, $|M(x_2)|$, $|M(x_3)|$, …… $|M(x_p)|$, there is an i such that $|M(x_i)|$ takes a maximum value (among the finite set of values above). Without limiting generality, we can consider $x_1 < x_2 < x_3 < \ldots < x_p$, and also $\varepsilon_1 < \varepsilon_2 < \varepsilon_3 < \ldots < \varepsilon_q$. Obviously, we can find a K such that $(|M(x_i)| < K \cdot x_1^{(1/2+\varepsilon_1)})$, we just take $K > |M(x_i)|/x_1^{(1/2+\varepsilon_1)}$. The inequality $|M(x_i)| < K \cdot x_1^{(1/2+\varepsilon_1)})$ implies all the other inequalities involved in the finite subset of statements from (*) considered above. As a consequence, there is a K which is equal to the value K found above such that the inequality $|M(x_i)| < K \cdot x_1^{(1/2+\varepsilon_1)}$ is satisfied, and all the other formulas from the finite subset of statements from (*) (as chosen above) are satisfied.

We proved that for any S'(K) that contains only a finite conjunction of formulas of the form $\varphi_{x,\varepsilon}$ (K), the sentences:

$\exists$K ((σ(n')/n' > $e^{\gamma} \cdot$ log(log(n'))) $\wedge$ (n' > 5041) $\wedge$ S'(K))

and also

$\exists$K (S'(K)) are always true sentences.

The fact that (σ(n')/n' > $e^{\gamma} \cdot$ log(log(n'))) $\wedge$ (n' > 5041) is included or not does not make any difference, because (σ(n')/n' > $e^{\gamma} \cdot$ log(log(n'))) $\wedge$ (n' > 5041) does not contain the variable K, so that $\exists$K ((σ(n')/n' > $e^{\gamma} \cdot$ log(log(n'))) $\wedge$ (n' > 5041)) is always true (if we already assume that we found an n').

From the theorem on the existence of a saturated model, we can conclude that there is a saturated model in which the sentence $\exists$K ((σ(n')/n' > $e^{\gamma} \cdot$ log(log(n'))) $\wedge$ (n' > 5041) $\wedge$ S(K)) is a true sentence. That means that there is a model in which ((non - R) $\wedge$ (L)) is a true statement. This is impossible, because (R) $\Leftrightarrow$ (L) (known result), and there is no model in which both (non - R) and (L) are true statements. If T is the theory in which we work, then T $\cup$ {(non - R) $\wedge$ (L)} is inconsistent, so it has no model, from Godel's Completeness Theorem. We reached a contradiction. As a consequence, our assumption that we can actually find a counterexample to Robin's inequality is false. There is no counterexample to Robin's inequality. (R) is true. As a consequence, the Riemann hypothesis is true. **QED**.

Conclusions. The assumption that a counterexample to Robin's reformulation can be found leads to a contradiction in a saturated model. This methodology can lead to solutions to many other unsolved problems in number theory.

Appendix. In this appendix, we will briefly present some facts about model theory. Model theory is a combination of universal algebra and logic. We have a set L of symbols for operations, constants and relations, called a language.

Example. L = {+, $\cdot$, 0, 1, $<$}. The language L can be finite or countable. A model M for the language L is an object of the form M = $< A, +_M, \cdot_M, 0_M, 1_M, <_M >$. A is a non - empty set, called the set of elements of M, and $+_M$ and $\cdot_M$,are binary operations on A$\times$A into A, 0_M and 1_M are elements of A, and $<_M$ is a binary relation on A. Examples. The field of rationals < Q, +, $\cdot$, 0, 1, > is a model for the language {+, $\cdot$, 0, 1}. The ordered field < Q, +, $\cdot$, 0, 1, <, > is a model for the language {+, $\cdot$, 0, 1, <}.

Many facts about models can be expressed in first order logic. In addition to the operation, relation, and constant symbols of L, first order logic has an infinite list of variables, the equality symbol =, the connectives $\wedge$ (and), $\vee$ (or), $\neg$ (not), and the quantifiers $\forall$ (for all), $\exists$ (there exists). Certain finite sequences of symbols are counted as terms, formulas, sentences. Every variable or constant is a term. If t, u are terms, so are $t + u$, $t \cdot u$. If t and u are terms, then $t = u$, $t < u$, are formulas. If φ, ψ are formulas and v is a variable, then $\neg\varphi$, $\varphi \wedge \psi$, $\varphi \vee \psi$, $\forall v\, \varphi$, $\exists v\, \varphi$ are formulas. A sentence is a formula all of whose variables are bound by quantifiers. For example, the sentence $\forall x\, (x = 0 \vee \exists y\, (x \cdot y) = 1)$ states that every non - zero element has a right inverse. The central notion in model theory is that of a sentence φ being true in a model M. This relation between models and sentences is defined by induction on the subformulas of φ. For example, the sentence $\forall x\, (x = 0 \vee \exists y\, (x \cdot y) = 1)$ is true in the field of rationals, but not in the ring of integers. A set of sentences is called a theory. M is a model of a theory T, if for every sentence $\varphi \in T$ is true in M.

Examples. The theory of rings is the familiar finite list of ring axioms. The theory of real closed fields is a set of sentences, consisting of axioms for ordered fields, the axiom stating that every positive element has a square root, and for each odd n an axiom stating that every polynomial of degree n has a root. For each model M, the theory Th(M) is the set of all sentences true in M. Two classical theorems in model theory are the compactness theorem and the Lowenheim - Skolem - Tarski theorem.

The Compactness Theorem.[2][3][4] If every finite subset of a set of sentences has a model, then T has a model.

Lowenheim - Skolem - Tarski Theorem. [2] If T has at least one infinite model, then T has a model of every infinite cardinality.

Almost all the deeper results in model theory depend on the construction of a model.

The diagram of a language for M is obtained by adding to L a new constant symbol for each element of A. the elementary diagram of M, written as Diag(M), is the set of all sentences in the diagram language of M which are true in M. The difference between Th(M) and Diag(M) is that Diag(M) has new symbols for the elements of M, while Th(M) does not. There are many other concepts that are fundamental in model theory, like elementary chains, ultraproducts, saturation, but we will stop here with this brief introduction (saturation and the theorem on the existence of a saturated model is presented in section 2, and is fundamental in this work, and we wanted to take this brief introduction to the point where the reader can understand the concept of saturation).

References:

1. H. M. Edwards, *"Riemann's Zeta Function"*, Dover Publications, Inc., 2001.

2. H. J. Keisler, *"Model Theory"*, Actes, Congres intern. math., 1970, Tome 1, p. 141-150.

3. H. R. Lewis, C. H. Papadimitriou, *"Elements of the Theory of Computation"*, Prentice Hall Software Series, 1981.

4. S. Naranong, *"A Model Theoretic Introduction to Nonstandard Analysis"*, 2010 (Internet reference).

5. G. Robin, *"Sur L'Ordre Maximum de la Fonction Somme des Diviseurs"* (Internet reference).

Cristian Dumitrescu,
119 Young St., Ap. 11,
Kitchener, Ontario N2H 4Z3,
Canada.

Email: cristiand43@gmail.com
cristiand41@hotmail.com