The connection between the Riemann Hypothesis and model theory

Cristian Dumitrescu

Abstract. In this paper, I present a connection between the Riemann Hypothesis and model theory, and this connection leads to a possible proof of the Riemann Hypothesis.

Keywords. Riemann Hypothesis, Robin's reformulation, Littlewood's reformulation, Vaught's theorem, saturated models, model theory.

Mathematical Subject Classification Number: 11M99.

Introduction.

The Riemann Hypothesis has been an open problem for a long time. This is an attempt to give a proof of the hypothesis based, mainly on model theory, more precisely, a beautiful theorem of Vaught. Also essential here are known results that were first given by Littlewood and Robin.

Section 1. Preliminary facts.

The following reformulations have been known for some time and the proofs of the statements in this section can be found in many references.

Robin's reformulation of RH [6]**.** The Riemann Hypothesis is true if and only if there is an n_0 (and in fact $n_0 = 5041$) such that $\sigma(n)/n < e^{\gamma} \cdot \log(\log(n))$, for all $n > n_0$ (here $\sigma(n)$ is the sum of divisors function).

Littlewood's reformulation of RH [1]**.** The Riemann Hypothesis is equivalent to the statement that for every $\varepsilon > 0$, we have $M(x) = O(x^{(1/2 + \varepsilon)})$, when $x \to \propto$ (here $M(x)$ is the Mertens' function).

We write (R) for the statement in Robin's reformulation (Robin inequalities). We also write (L) for the statement in Littlewood's reformulation (that is $M(x) = O(x^{(1/2 + \varepsilon)})$, when $x \to \propto$).

We can conclude that the statement:

"there is an n_0 such that $\sigma(n)/n < e^{\gamma} \cdot \log(\log(n))$, for all $n > n_0$ "

is equivalent to the statement:

"for every $\varepsilon > 0$, we have $M(x) = O(x^{(1/2 + \varepsilon)})$, when $x \to \propto$".

We will write (R) $\Leftrightarrow$ (L) for this equivalence (which is a known result).

Observation 1. We also note that in Littlewood's reformulation we can take ε rational and the constant involved in $M(x) = O(x^{(1/2 + \varepsilon)})$ also rational, and find an equivalent statement. In other words, we have:

Littlewood's reformulation of RH (with ε and K rational). The Riemann Hypothesis is equivalent to the statement that for every rational $\varepsilon > 0$, we have $M(x) = O(x^{(1/2 + \varepsilon)})$, when $x \to \propto$ (here M(x) is the Mertens' function and the constant $K > 0$ involved in $M(x) = O(x^{(1/2 + \varepsilon)})$ can also be taken rational).

Lemma. Littlewood's reformulation of RH is equivalent to Littlewood's reformulation of RH with rational $K > 0$ and rational $\varepsilon > 0$.

Proof. In the following, when we refer to the constant K or K', we have in mind the constant involved in relations of the type $M(x) = O(x^{(1/2 + \varepsilon)})$.

First, we will prove that Littlewood's reformulation of RH with rational $\varepsilon > 0$ and rational $K > 0$ implies Littlewood's reformulation of RH.

We note that given an $\varepsilon > 0$, we can choose a rational ε' such that $0 < \varepsilon' < \varepsilon$, then we have: $(M(x) = O(x^{(1/2 + \varepsilon')})$ when $x \to \propto) \Rightarrow (M(x) = O(x^{(1/2 + \varepsilon)})$ when $x \to \propto)$. This means that Littlewood's reformulation with rational $\varepsilon > 0$ and rational $K > 0$ implies Littlewood's reformulation.

Next, we will prove that Littlewood's reformulation of RH implies Littlewood's reformulation of RH with rational $\varepsilon > 0$ and rational $K > 0$.

We are given a rational $\varepsilon' > 0$. We can choose a rational $K' > 0$ such that $0 < K < K'$, then we have: $(M(x) = O(x^{(1/2 + \varepsilon')})$, involving the constant $K > 0$, when $x \to \propto) \Rightarrow (M(x) = O(x^{(1/2 + \varepsilon')})$, involving the rational constant $K' > 0$, when $x \to \propto)$. Littlewood's reformulation implies Littlewood's reformulation with rational $\varepsilon > 0$ and rational $K > 0$.
QED.

Section 2. Model Theory.

The following theorems will be used in our results (for a brief introduction to model theory, see the appendix).

Definition. A model M is said to be $\aleph_0$- saturated if for every enumerable infinite set Φ

of formulas $\varphi(x)$ in the diagram language of M, if for every finite subset of formulas φ_1, φ_2, φ_3, $\dots\varphi_n \in \Phi$ the sentence $\exists x\, (\varphi_1(x) \wedge \varphi_2(x) \wedge \varphi_3(x) \wedge \dots \wedge \varphi_n(x))$ is true in M, then the infinitely long sentence $\exists x\, (\bigwedge_{\varphi\in\Phi} \varphi(x))$ is also true in M.
Essential will be the following theorem and corollary:

Theorem (Vaught) [4]. Let Γ be a theory in a countable language L. Then the following conditions are equivalent:

(i) Γ has a countable $\aleph_0$ - saturated model.
(ii) For each finite m there are only countably many m - types over Γ.

Corollary [4]. If Γ is a consistent theory in a countable language and Γ has, up to isomorphism, only countably many countable models, then Γ has a countable $\aleph_0$ - saturated model.

Observation 2. In the theory Γ of algebraically closed fields of characteristic 0, the countable models of Γ are, up to isomorphism, the algebraically closed fields $\mathbf{Q}_\alpha$, for α finite or infinite, where $\mathbf{Q}_\alpha$ has transcendence degree α over $\mathbf{Q}$. It can be proved that $\mathbf{Q}_\omega$ is $\aleph_0$ - saturated (where $\mathbf{Q}_\omega$ has infinite transcendence degree over $\mathbf{Q}$).

We assume familiarity with the notions of algebraic dependence and independence, transcendence bases and transcendence degree. We know that there is an independent set A such that $\mathbf{Q}_\omega$ is algebraic over $\mathbf{Q}(\mathbf{A})$.

We define $\mathbf{N^*} = \mathbf{N} \cup$ A. We note that $\mathbf{Q}_\omega$ is algebraic over $\mathbf{Q}(\mathbf{N^*})$. We can then construct $\mathbf{R^{**}}$ from $\mathbf{Q}_\omega$ in a similar manner in which $\mathbf{R}$ is constructed from $\mathbf{Q}$. In this nonstandard model $\mathbf{R^{**}}$ we can extend all the mathematical notions, relations, operations and functions that we work with in the standard model $\mathbf{R}$. This will be implicitly assumed in the following, for example when we talk about the sum of divisors function $\sigma(n)$ when $n \in \mathbf{N^*}$, or the Mertens function M(n) when $n \in \mathbf{N^*}$. Basically, it is important to note that this extension can be worked out.

Definitions. We write $\mathbf{N}$ for the natural numbers, $\mathbf{Q}$ for the rationals. We also write (R*) for the statement in Robin's reformulation in $\mathbf{Q}_\omega$, in other words (R*) will be the statement:

"for any $n \in \mathbf{N^*}$ *and* $n > 5041$ *the relations* $\sigma(n)/n < e^\gamma \cdot \log(\log(n))$ *are all satisfied."*

We also write (L*) for the statement in Littlewood's reformulation in $\mathbf{Q}_\omega$, in other words (L*) will be the statement:

"for every $\varepsilon > 0$, $\varepsilon \in \mathbf{Q}_\omega$ *there is a* $K > 0$, $K \in \mathbf{Q}_\omega$, *such that* $/M(x)\ / < K \cdot x^{(1/2 + \varepsilon)})$ *for every x in* $\mathbf{N^*}$*"*

We note that $\mathbf{Q}_\omega$ cannot contain only standard elements. In other words, in relation to order, $\mathbf{Q}_\omega$ contains infinite and infinitesimal elements. When K is infinite, the inequalities above are all trivially satisfied for any rational $\varepsilon \in \mathbf{Q}$. Only when ε is infinitesimal and K infinite, the inequalities above are not trivial.

Godel's Completeness Theorem [5]. If K is a set of first - order sentences over some fixed language, then K has a model if and only if K is consistent.

Section 3. The main theorems.

Theorem 1. In the countable $\aleph_0$ - saturated model $\mathbf{Q}_\omega$ (which has infinite transcendence degree over $\mathbf{Q}$), Littlewood's reformulation (L*) is a true statement.

Proof. Now we consider the statement (L*):

*"for every $\varepsilon > 0$, $\varepsilon \in \mathbf{Q}_\omega$ there is a $K > 0$, $K \in \mathbf{Q}_\omega$, such that $|M(x)| < K \cdot x^{(1/2 + \varepsilon)}$) for every x in **N***"*

We consider the formulas $\varphi_{x,\varepsilon}(K)$, and by definition the formula $\varphi_{x,\varepsilon}(K)$ will mean $(|M(x)| < K \cdot x^{(1/2 + \varepsilon)})$. We note here that K is a free variable (that is why $\varphi_{x,\varepsilon}(K)$ are formulas, not sentences). If we consider all the formulas $\varphi_{x,\varepsilon}(K)$, when x is a natural number (standard or nonstandard, in **N***) and ε is a positive (extended) rational number (standard or nonstandard, in other words, in $\mathbf{Q}_\omega$), we still have an enumerable infinite set of formulas. We consider the conjunction of all these formulas, and we write S(K), so by definition S(K) will mean:

$$(\bigwedge_{x \text{ natural and } \varepsilon \text{ positive rational}} \varphi_{x,\varepsilon}(K)). \qquad (*).$$

We notice that if the sentence $(\exists K)(\bigwedge_{x \text{ natural and } \varepsilon \text{ positive rational}} \varphi_{x,\varepsilon}(K))$ is satisfied, then (L*) is true in our model.

We notice that any finite conjunction of statements of (*), as described above, is satisfied in our model. The proof of this is based on the fact that any finite set elements from **R**** has a maximum and a minimum, and the Archimedean property (its multiplicative form still holds in the nonstandard model). Any finite subset of statements from (*), as described above, involves a finite set of natural numbers $\{x_1, x_2, x_3, \ldots..x_p\}$ and a finite set of (extended) rationals $\{\varepsilon_1, \varepsilon_2, \varepsilon_3, \ldots..\varepsilon_q\}$. Among the values $|M(x_1)|$, $|M(x_2)|$, $|M(x_3)|$, …… $|M(x_p)|$, there is an i such that $|M(x_i)|$ takes a maximum value (among the finite set of values above). Without limiting generality, we can consider $x_1 < x_2 < x_3 < \ldots..< x_p$, and also $\varepsilon_1 < \varepsilon_2 < \varepsilon_3 < \ldots..< \varepsilon_q$. Obviously, we can find a K such that $(|M(x_i)| < K \cdot x_1^{(1/2 + \varepsilon_1)})$, we just take $K > |M(x_i)|/x_1^{(1/2 + \varepsilon_1)}$. The inequality $|M(x_i)| < K \cdot x_1^{(1/2 + \varepsilon_1)}$) implies all the other inequalities involved in the finite subset of statements from (*) considered above. As a consequence, there is a K which is equal to the value K found above such that the inequality $|M(x_i)| < K \cdot x_1^{(1/2 + \varepsilon_1)}$ is satisfied, and all the other formulas from the finite subset of statements from (*) (as chosen above) are satisfied.

We proved that for any S'(K) that contains only a finite conjunction of formulas of the form $\varphi_{x,\varepsilon}$ (K), the sentences:

$\exists$K (S'(K)) is a true sentence.

From the saturation property, we can conclude that the sentence $(\exists K)(\bigwedge_{x \text{ natural and } \varepsilon \text{ positive rational}} \varphi_{x,\varepsilon}(K))$ is a true sentence. That means that in $\mathbf{Q}_\omega$, (L*) is a true statement. **QED**.

Corollary to Theorem 1. If the equivalence (R*) $\Leftrightarrow$ (L*) is true in $\mathbf{Q}_\omega$ (we can also work in **R****) then all (R), (L), and the Riemann Hypothesis are true in the standard model.

Proof. If we assume that there exists a counterexample to (R) in the standard model, then this would also be a counterexample to (R*) in $\mathbf{Q}_\omega$ (this is obvious). From theorem 1, since we assumed that (R*) $\Leftrightarrow$ (L*) , we reach a contradiction, because we know that (L*) is true in $\mathbf{Q}_\omega$, therefore the assumption that a counterexample to (R) exists is false. As a consequence, (R) , (L) and the Riemann Hypothesis are all true. **QED.**

Theorem 2. The equivalence (R*) $\Leftrightarrow$ (L*) is a true statement in $\mathbf{Q}_\omega$ (we can also work in **R****).

Proof. The basic idea is to construct (extend) all the relevant notions, relations, operations and functions in **R****, all the mathematical objects suitably extended, in order to reconstruct the known proof of the equivalence (R) $\Leftrightarrow$ (L) in the standard model **R,** and transfer it in **R****. It is essential to note that we work in exactly the same system of axioms as the system in which the known result (R) $\Leftrightarrow$ (L) was proved. We want to make sure that any theorem in the standard model (involving **N**, **Q**, and **R**) is also a theorem in R** (involving **N*,** $\mathbf{Q}_\omega$, and **R****). In $\mathbf{Q}_\omega$ and **R**** there are also other true statements related to saturation, but in principle, all the true statements in the standard system are conserved in the saturated model. In this case, (R) $\Leftrightarrow$ (L) (known result) will imply (R*) $\Leftrightarrow$ (L*). If, when applying Vaught's theorem, the theory Γ is powerful enough, such that we can prove (R) $\Leftrightarrow$ (L) in it, then these conditions are satisfied, and we can prove (R*) $\Leftrightarrow$ (L*) in $\mathbf{Q}_\omega$, or **R****. **QED.**

Observation 3. Robinson's enlargements also have a similar goal in mind, but while his construction is purely logical in nature, our construction is mostly algebraic in nature. The initial version of this article relied on elementary extensions. The problem is that, under those assumptions, nothing guarantees that **Q*** is still countable, so the proof would fail. I am grateful to Professor Feferman, Professor Haskell, and Professor Scanlon for their observations and suggestions (and the correction of several errors in the first version of the article). Any other errors still present in this article (if any) belong to the author (Cristian Dumitrescu), but the observations of the model theory experts above corrected some errors present in the first version of this article.

Conclusions. The whole construction is based on the analogy between the constructions: $\mathbf{N} \to \mathbf{Z} \to \mathbf{Q} \to \mathbf{R}$ and $\mathbf{N^*} \to \mathbf{Z^*} \to \mathbf{Q_\omega} \to \mathbf{R^{**}}$, and Vaught's theorem (when the theory Γ is powerful enough), which guarantees the existence of the countable $\aleph_0$ - saturated model $\mathbf{Q_\omega}$. The two theorems above, and the corollary imply that the Riemann Hypothesis is true.

Appendix. In this appendix, we will briefly present some facts about model theory. Model theory is a combination of universal algebra and logic. We have a set L of symbols for operations, constants and relations, called a language.

Example. L = {+, ·, 0, 1, <}. The language L can be finite or countable. A model M for the language L is an object of the form $M = < A, +_M, \cdot_M, 0_M, 1_M, <_M >$. A is a non - empty set, called the set of elements of M, and $+_M$ and $\cdot_M$,are binary operations on $A \times A$ into A, 0_M and 1_M are elements of A, and $<_M$ is a binary relation on A. Examples. The field of rationals < Q, +, ·, 0, 1, > is a model for the language {+, ·, 0, 1}. The ordered field < Q, +, ·, 0, 1, <, > is a model for the language {+, ·, 0, 1, <}.

Many facts about models can be expressed in first order logic. In addition to the operation, relation, and constant symbols of L, first order logic has an infinite list of variables, the equality symbol =, the connectives $\wedge$ (and), $\vee$ (or), $\neg$ (not), and the quantifiers $\forall$ (for all), $\exists$ (there exists). Certain finite sequences of symbols are counted as terms, formulas, sentences. Every variable or constant is a term. If t, u are terms, so are $t + u$, $t \cdot u$. If t and u are terms, then $t = u$, $t < u$, are formulas. If φ, ψ are formulas and v is a variable, then $\neg\varphi$, $\varphi \wedge \psi$, $\varphi \vee \psi$, $\forall v\, \varphi$, $\exists v\, \varphi$ are formulas. A sentence is a formula all of whose variables are bound by quantifiers. For example, the sentence $\forall x\, (x = 0 \vee \exists y\, (x \cdot y) = 1)$ states that every non - zero element has a right inverse. The central notion in model theory is that of a sentence φ being true in a model M. This relation between models and sentences is defined by induction on the subformulas of φ. For example, the sentence $\forall x\, (x = 0 \vee \exists y\, (x \cdot y) = 1)$ is true in the field of rationals, but not in the ring of integers. A set of sentences is called a theory. M is a model of a theory T, if for every sentence $\varphi \in T$ is true in M.

Examples. The theory of rings is the familiar finite list of ring axioms. The theory of real closed fields is a set of sentences, consisting of axioms for ordered fields, the axiom stating that every positive element has a square root, and for each odd n an axiom stating that every polynomial of degree n has a root. For each model M, the theory Th(M) is the set of all sentences true in M. Two classical theorems in model theory are the compactness theorem and the Lowenheim - Skolem - Tarski theorem.

The Compactness Theorem.[2][3][5] If every finite subset of a set of sentences has a model, then T has a model.

Lowenheim - Skolem - Tarski Theorem. [2] If T has at least one infinite model, then T has a model of every infinite cardinality.

Almost all the deeper results in model theory depend on the construction of a model.

The diagram of a language for M is obtained by adding to L a new constant symbol for each element of A. the elementary diagram of M, written as Diag(M), is the set of all sentences in the diagram language of M which are true in M. The difference between Th(M) and Diag(M) is that Diag(M) has new symbols for the elements of M, while Th(M) does not. There are many other concepts that are fundamental in model theory, like elementary chains, ultraproducts, saturation, but we will stop here with this brief introduction (saturation and the theorem on the existence of a saturated model is presented in section 2, and is fundamental in this work, and we wanted to take this brief introduction to the point where the reader can understand the concept of saturation).

References:

1. H. M. Edwards, *“Riemann’s Zeta Function“*, Dover Publications, Inc., 2001.

2. H. J. Keisler, *“Model Theory”*, Actes, Congres intern. math., 1970, Tome 1, p. 141-150.

3. H. R. Lewis, C. H. Papadimitriou, *“Elements of the Theory of Computation”*, Prentice Hall Software Series, 1981.

4. J. Donald Monk, *“Mathematical Logic”*, Springer - Verlag, 1976.

5. S. Naranong, *“A Model Theoretic Introduction to Nonstandard Analysis”*, 2010 http://www.math.tamu.edu/~saichu/ModelTheoreticNonstandard.pdf

6. G. Robin, *“Sur L‘Ordre Maximum de la Fonction Somme des Diviseurs”, Seminaire Delange-Pisot-Poitou, Theorie des nombres (1981 - 1982), Progress in Mathematics 38 (1983), 233 - 244.*

7. A. Robinson, *“Non-standard Analysis”*, Princeton University Press, 1996.

Cristian Dumitrescu,
119 Young St., Ap. 11,
Kitchener, Ontario N2H 4Z3,
Canada.

Email: cristiand43@gmail.com
cristiand41@hotmail.com

Tel : (519) 574-7026