The connection between the Riemann Hypothesis and model theory

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Abstract. In this paper, I present a connection between the Riemann Hypothesis and model theory, and this connection leads to a possible proof of the Riemann Hypothesis.

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Introduction.

The Riemann Hypothesis has been an open problem for a long time. This is an attempt to give a proof of the hypothesis based, mainly on model theory, more precisely, results related to countable recursively saturated models relative to a language L. Also essential here are known results that were first given by Littlewood and Robin. The proof of the Riemann Hypothesis is based on the existence of a countable recursively saturated model relative to a language L, of the Peano system of axioms of arithmetic.

Section 1. Preliminary facts.

The following reformulations have been known for some time and the proofs of the statements in this section can be found in many references.

Robin's reformulation of RH [7]**.** The Riemann Hypothesis is true if and only if there is an n_0 (and in fact $n_0 = 5041$) such that $\sigma(n)/n < e^{\gamma} \cdot \log(\log(n))$, for all $n > n_0$ (here $\sigma(n)$ is the sum of divisors function).

Littlewood's reformulation of RH [1]**.** The Riemann Hypothesis is equivalent to the statement that for every $\varepsilon > 0$, we have $M(x) = O(x^{(1/2 + \varepsilon)})$, when $x \to \propto$ (here M(x) is the Mertens' function).

We write (R) for the statement in Robin's reformulation (Robin inequalities). We also write (L) for the statement in Littlewood's reformulation (that is $M(x) = O(x^{(1/2 + \varepsilon)})$, when $x \to \propto$).

We can conclude that the statement:

"there is an n_0 such that $\sigma(n)/n < e^{\gamma} \cdot \log(\log(n))$, for all $n > n_0$ "

is equivalent to the statement:

"for every $\varepsilon > 0$, we have $M(x) = O(x^{(1/2 + \varepsilon)})$, when $x \to \propto$".

We will write (R) $\Leftrightarrow$ (L) for this equivalence (which is a known result).

Section 2. Model Theory.

The following theorems will be used in our results (for a brief introduction to model theory, see the appendix).

Definition. Let $\aleph_\alpha$ be an uncountable cardinal. A model M is said to be $\aleph_\alpha$ - saturated if for every set Φ (of fewer than $\aleph_\alpha$ formulas) of formulas $\varphi(x)$ in the diagram language of M, if for every finite subset of formulas $\varphi_1, \varphi_2, \varphi_3, \ldots\varphi_n \in \Phi$ the sentence $\exists x\ (\varphi_1(x) \wedge \varphi_2(x) \wedge \varphi_3(x) \wedge \ldots \wedge \varphi_n(x))$ is true in M, then the infinitely long sentence $\exists x\ (\bigwedge_{\varphi\in\Phi} \varphi(x))$ is also true in M.

We also note the following theorem on the existence of saturated models, due to Keisler.

Theorem (Keisler, 1963) [2]. Let I be a set of power $\aleph_\alpha$. There is an ultrafilter D over I such that every ultraproduct $\Pi_D M_i$ is $\aleph_{\alpha+1}$ - saturated.

Observation 1. We assume that the reader is familiar with the concept of recursively saturated model, and the theorem on the existence of a recursively saturated model (for reference, see [2]).

Observation 2. We also note the following facts. Any model of Peano arithmetic which occurs as the integers in some model of non - standard analysis is recursively saturated. As emphasized in [2], "The results of this section can be readily extended to the case of an arbitrary countable language by modifying the notion of a recursively saturated model. A set S is said to be recursive relative to L if there is an algorithm which decides whether or not an arbitrary input belongs to S, but makes use of an oracle which will always correctly answer questions of the form - is φ a formula for L? Everything goes through with only minor changes when the notion of recursive saturation is replaced by recursive saturation relative to L". We will then work in an enlargement R* of standard analysis on R, where the integers represent a model of Peano arithmetic which will be countable and recursively saturated relative to a language L.

Observation 3. We notice that the statement from Littlewood's reformulation can be given with the range of the variable ε restricted to the set of rational numbers, and we will have a statement equivalent to Littlewood's reformulation.

Definitions. We write N for the natural numbers, and N* for the corresponding set in R*. We also write (R*) for the statement in Robin's reformulation in R*, in other words (R*) will be the statement:

"for any n $\in$ N* *and n > 5041 the relations* $\sigma(n)/n < e^{\gamma} \cdot \log(\log(n))$ *are all satisfied."*

We also write (L*) for the statement in Littlewood's reformulation in R*, in other words (L*) will be the statement:

"for every $\varepsilon > 0$, $\varepsilon \in R^*$ *there is a* $K > 0$, $K \in R^*$, *such that* $|M(x)| < K \cdot x^{(1/2 + \varepsilon)}$) *for every x in* N*"

As a notation, we must distinguish between (R*) - Robin's reformulation, and R* which is the enlargement of R. We also note that when we work in R*, all operations, relations, and functions are transferred to R* (but we will not always write <*, M*(x), σ*(n), and so on, it will be clear from the context).

Section 3. The main theorems.

Theorem 1. In the non - standard model of analysis described above R*, Littlewood's reformulation (L*) is a true statement.

Proof. Now we consider the statement (L*):

"for every $\varepsilon > 0$, $\varepsilon \in R^*$ *there is a* $K > 0$, $K \in R^*$, *such that* $|M(x)| < K \cdot x^{(1/2 + \varepsilon)}$) *for every x in* N*"

We consider the formulas $\varphi_{x,\varepsilon}(K)$, and by definition the formula $\varphi_{x,\varepsilon}(K)$ will mean ($|M(x)| < K \cdot x^{(1/2+\varepsilon)}$). We note here that K is a free variable (that is why $\varphi_{x,\varepsilon}(K)$ are formulas, not sentences). If we consider all the formulas $\varphi_{x,\varepsilon}(K)$, when $x \in N^*$, and $\varepsilon \in Q^*$ (see observation 3), we have a countable set of formulas. We consider the conjunction of all these formulas, and we write S(K), so by definition S(K) will mean:

$(\bigwedge_{x \text{ in } N^* \text{ and } \varepsilon \text{ in } R^*} \varphi_{x,\varepsilon}(K))$. (*).

We notice that if the sentence $(\exists K)(\bigwedge_{x \text{ in } N^* \, \varepsilon \text{ in } R^*} \varphi_{x,\varepsilon}(K))$ is satisfied, then (L*) is true in our model.

We notice that any finite conjunction of statements of (*), as described above, is satisfied in our model. The proof of this is based on the fact that any finite set of elements from **R*** has a maximum and a minimum, and the Archimedean property (its multiplicative form still holds in the nonstandard model). Any finite subset of statements from (*), as described above, involves a finite set of (extended) natural numbers $\{x_1, x_2, x_3, \ldots..x_p\}$ and a finite set of (extended) rational numbers $\{\varepsilon_1, \varepsilon_2, \varepsilon_3, \ldots..\varepsilon_p\}$. Among the values

$|M(x_1)|$, $|M(x_2)|$, $|M(x_3)|$, …… $|M(x_p)|$, there is an i such that $|M(x_i)|$ takes a maximum value (among the finite set of values above). Without limiting generality, we can consider $x_1 < x_2 < x_3 < \ldots < x_p$, and also $\varepsilon_1 < \varepsilon_2 < \varepsilon_3 < \ldots < \varepsilon_p$. Obviously, we can find a K such that ($|M(x_i)| < K \cdot x_1^{(1/2 + \varepsilon_1)}$), we just take $K > |M(x_i)|/x_1^{(1/2 + \varepsilon_1)}$. The inequality $|M(x_i)| < K \cdot x_1^{(1/2 + \varepsilon_1)}$) implies all the other inequalities involved in the finite subset of statements from (*) considered above. As a consequence, there is a K which is equal to the value for K found above such that the inequality $|M(x_i)| < K \cdot x_1^{(1/2 + \varepsilon_1)}$ is satisfied, and all the other formulas from the finite subset of statements from (*) (as chosen above) are satisfied.

We proved that for any S'(K) that contains only a finite conjunction of formulas of the form $\varphi_{x,\varepsilon}$ (K), the sentences:

∃K (S'(K)) is a true sentence.

From the saturation property (recursive saturation relative to L), we can conclude that the sentence $(\exists K)(\bigwedge_{x \text{ in } N^* \text{ and } \varepsilon \text{ in } R^*} \varphi_{x,\varepsilon}(K))$ is a true sentence. That means that in R* , (L*) is a true statement. **QED**.

Theorem 2. The equivalence (R*) ⇔ (L*) is true in R*.

Proof. In the standard model it is known that (R) ⇔ (L). We can transfer this known result to the enlargement R*. This means that (R*) ⇔ (L*) is true in R*. **QED**.

Theorem 3. Riemann's Hypothesis (RH) is true in the standard model.

Proof. We note that any counterexample to (R) in the standard model can also be considered a counterexample to (R*) in R*. This means that (R*) ⇒ (R). From theorem 1 we know that (L*) is true in R*. From theorem 2 we know that (R*) ⇔ (L*) is true in R*. In other words we have (L*) ⇔ (R*) ⇒ (R). That means that the statement from Robin's reformulation (R) is true in the standard model. It is known (in the standard model) that (L) ⇔ (RH) ⇔ (R). That means that the Riemann's Hypothesis is true in the standard model. **QED**.

Observation 4. I am grateful to Professor Feferman, Professor Haskell, Professor Scanlon and specially to Professor Keisler for their observations and suggestions (and the correction of many errors in the first versions of the article). Any other errors still present in this article (if any) belong to the author (Cristian Dumitrescu), but the observations of the model theory experts above corrected many errors present in the first versions of this article. I also emphasize that the current version of this proof has not yet passed the expert analysis, at this point. The main error that I made in the previous versions of this proof is when applying the $\aleph_{\alpha+1}$ - saturation property in a model of cardinality at least $\aleph_{\alpha+1}$.

Conclusion. The proof of the Riemann Hypothesis is based on the existence of a countable recursively saturated model relative to L, of the Peano system of axioms of arithmetic.

Appendix. In this appendix, we will briefly present some facts about model theory. Model theory is a combination of universal algebra and logic. We have a set L of symbols for operations, constants and relations, called a language.

Example. L = {+, ·, 0, 1, <}. The language L can be finite or countable. A model M for the language L is an object of the form M = < A, $+_M$, $\cdot_M$, 0_M, 1_M, $<_M$ >. A is a non - empty set, called the set of elements of M, and $+_M$ and $\cdot_M$, are binary operations on A×A into A, 0_M and 1_M are elements of A, and $<_M$ is a binary relation on A. Examples. The field of rationals < Q, +, ·, 0, 1, > is a model for the language {+, ·, 0, 1}. The ordered field < Q, +, ·, 0, 1, <, > is a model for the language {+, ·, 0, 1, <}.

Many facts about models can be expressed in first order logic. In addition to the operation, relation, and constant symbols of L, first order logic has an infinite list of variables, the equality symbol =, the connectives ∧ (and), ∨ (or), ¬ (not), and the quantifiers ∀ (for all), ∃ (there exists). Certain finite sequences of symbols are counted as terms, formulas, sentences. Every variable or constant is a term. If t, u are terms, so are t + u, t · u. If t and u are terms, then t = u, t < u, are formulas. If φ, ψ are formulas and v is a variable, then ¬φ, φ ∧ ψ, φ ∨ ψ, ∀v φ, ∃v φ are formulas. A sentence is a formula all of whose variables are bound by quantifiers. For example, the sentence $\forall x\ (x = 0 \vee \exists y\ (x \cdot y) = 1)$ states that every non - zero element has a right inverse. The central notion in model theory is that of a sentence φ being true in a model M. This relation between models and sentences is defined by induction on the subformulas of φ. For example, the sentence $\forall x\ (x = 0 \vee \exists y\ (x \cdot y) = 1)$ is true in the field of rationals, but not in the ring of integers. A set of sentences is called a theory. M is a model of a theory T, if for every sentence φ∈ T is true in M.

Examples. The theory of rings is the familiar finite list of ring axioms. The theory of real closed fields is a set of sentences, consisting of axioms for ordered fields, the axiom stating that every positive element has a square root, and for each odd n an axiom stating that every polynomial of degree n has a root. For each model M, the theory Th(M) is the set of all sentences true in M. Two classical theorems in model theory are the compactness theorem and the Lowenheim - Skolem - Tarski theorem.

The Compactness Theorem.[2][3] If every finite subset of a set of sentences has a model, then T has a model.

Lowenheim - Skolem - Tarski Theorem. [2] If T has at least one infinite model, then T has a model of every infinite cardinality.

Almost all the deeper results in model theory depend on the construction of a model.

The diagram of a language for M is obtained by adding to L a new constant symbol for each element of A. the elementary diagram of M, written as Diag(M), is the set of all sentences in the diagram language of M which are true in M. The difference between Th(M) and Diag(M) is that Diag(M) has new symbols for the elements of M, while Th(M) does not. There are many other concepts that are fundamental in model theory, like elementary chains, ultraproducts, saturation, but we will stop here with this brief introduction (saturation and the theorem on the existence of a saturated model is presented in section 2, and is fundamental in this work, and we wanted to take this brief introduction to the point where the reader can understand the concept of saturation, in particular, recursive saturation relative to L).

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