Non-Solvable Ordinary Differential Equations With Applications

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Abstract: Different from the system in classical mathematics, a Smarandache system is a contradictory system in which an axiom behaves in at least two different ways within the same system, i.e., validated and invalided, or only invalided but in multiple distinct ways. Such systems exist extensively in the world, particularly, in our daily life. In this paper, we discuss such a kind of Smarandache system, i.e., non-solvable ordinary differential equation systems by a combinatorial approach, classify these systems and characterize their behaviors, particularly, the sum-stability and prod-stability of such linear and non-linear differential equations. Some applications of such systems to other sciences, such as those of globally controlling of infectious diseases, establishing dynamical equations of instable structure, particularly, the n-body problem and understanding global stability of matters with multilateral properties can be also found.

Key Words: Ordinary differential equation, general solution, $\vee$-solution, $\wedge$-solution, sum-stability, prod-stability, asymptotic behavior, Smarandache system, inherit graph, instable structure, dynamical equation, multilateral matter.

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§1. Introduction

Finding the exact solution of an equation system is a main but a difficult objective unless some special cases in classical mathematics. Contrary to this fact, *what is about the non-solvable case for an equation system?* In fact, such an equation system is nothing but a contradictory system, and characterized only by having no solution as a conclusion. But our world is overlap and hybrid. The number of non-solvable equations is much more than that of the solvable and such equation systems can be also applied for characterizing the behavior of things, which reflect the real appearances of things by that their complexity in our world. It should be noted that such non-solvable linear algebraic equation systems have been characterized recently by the author in the reference [7]. The main purpose of this paper is to characterize the behavior of such non-solvable ordinary differential equation systems.

Assume $m,\ n \geq 1$ to be integers in this paper. Let

$$\dot{X} = F(X) \qquad (DES^1)$$

be an autonomous differential equation with $F : \mathbf{R}^n \to \mathbf{R}^n$ and $F(\overline{0}) = 0$, particularly, let

$$\dot{X} = AX \qquad (LDES^1)$$

be a linear differential equation system and

$$x^{(n)} + a_1 x^{(n-1)} + \cdots + a_n x = 0 \qquad (LDE^n)$$

a linear differential equation of order n with

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \quad X = \begin{bmatrix} x_1(t) \\ x_2(t) \\ \cdots \\ x_n(t) \end{bmatrix} \quad \text{and} \quad F(t,X) = \begin{bmatrix} f_1(t,X) \\ f_2(t,X) \\ \cdots \\ f_n(t,X) \end{bmatrix},$$

where all $a_i,\ a_{ij},\ 1 \leq i,j \leq n$ are real numbers with

$$\dot{X} = (\dot{x}_1, \dot{x}_2, \cdots, \dot{x}_n)^T$$

and $f_i(t)$ is a continuous function on an interval $[a,b]$ for integers $0 \leq i \leq n$. The following result is well-known for the solutions of $(LDES^1)$ and (LDE^n) in references.

Theorem 1.1([13]) *If $F(X)$ is continuous in*

$$U(X_0): \ |t - t_0| \leq a, \quad \|X - X_0\| \leq b \ \ (a > 0,\ b > 0)$$

then there exists a solution $X(t)$ of differential equation (DES^1) in the interval $|t - t_0| \leq h$, where $h = \min\{a, b/M\}$, $M = \max\limits_{(t,X)\in U(t_0,X_0)} \|F(t,X)\|$.

Theorem 1.2([13]) *Let λ_i be the k_i-fold zero of the characteristic equation*

$$\det(A - \lambda I_{n\times n}) = |A - \lambda I_{n\times n}| = 0$$

or the characteristic equation

$$\lambda^n + a_1\lambda^{n-1} + \cdots + a_{n-1}\lambda + a_n = 0$$

with $k_1 + k_2 + \cdots + k_s = n$. Then the general solution of $(LDES^1)$ is

$$\sum_{i=1}^{n} c_i \overline{\beta}_i(t) e^{\alpha_i t},$$

where, c_i is a constant, $\overline{\beta}_i(t)$ is an n-dimensional vector consisting of polynomials in t determined as follows

$$\overline{\beta}_1(t) = \begin{bmatrix} t_{11} \\ t_{21} \\ \cdots \\ t_{n1} \end{bmatrix}$$

$$\overline{\beta}_2(t) = \begin{bmatrix} t_{11}t + t_{12} \\ t_{21}t + t_{22} \\ \cdots\cdots\cdots \\ t_{n1}t + t_{n2} \end{bmatrix}$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\overline{\beta}_{k_1}(t) = \begin{bmatrix} \frac{t_{11}}{(k_1-1)!}t^{k_1-1} + \frac{t_{12}}{(k_1-2)!}t^{k_1-2} + \cdots + t_{1k_1} \\ \frac{t_{21}}{(k_1-1)!}t^{k_1-1} + \frac{t_{22}}{(k_1-2)!}t^{k_1-2} + \cdots + t_{2k_1} \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \frac{t_{n1}}{(k_1-1)!}t^{k_1-1} + \frac{t_{n2}}{(k_1-2)!}t^{k_1-2} + \cdots + t_{nk_1} \end{bmatrix}$$

$$\overline{\beta}_{k_1+1}(t) = \begin{bmatrix} t_{1(k_1+1)} \\ t_{2(k_1+1)} \\ \cdots\cdots \\ t_{n(k_1+1)} \end{bmatrix}$$

$$\overline{\beta}_{k_1+2}(t) = \begin{bmatrix} t_{11}t + t_{12} \\ t_{21}t + t_{22} \\ \cdots\cdots\cdots \\ t_{n1}t + t_{n2} \end{bmatrix}$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\overline{\beta}_n(t) = \begin{bmatrix} \frac{t_{1(n-k_s+1)}}{(k_s-1)!}t^{k_s-1} + \frac{t_{1(n-k_s+2)}}{(k_s-2)!}t^{k_s-2} + \cdots + t_{1n} \\ \frac{t_{2(n-k_s+1)}}{(k_s-1)!}t^{k_s-1} + \frac{t_{2(n-k_s+2)}}{(k_s-2)!}t^{k_s-2} + \cdots + t_{2n} \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \frac{t_{n(n-k_s+1)}}{(k_s-1)!}t^{k_s-1} + \frac{t_{n(n-k_s+2)}}{(k_s-2)!}t^{k_s-2} + \cdots + t_{nn} \end{bmatrix}$$

with each t_{ij} *a real number for* $1 \leq i, j \leq n$ *such that* $\det([t_{ij}]_{n\times n}) \neq 0$,

$$\alpha_i = \begin{cases} \lambda_1, & if\ 1 \leq i \leq k_1; \\ \lambda_2, & if\ k_1 + 1 \leq i \leq k_2; \\ \cdots & \cdots\cdots\cdots\cdots\cdots\cdots; \\ \lambda_s, & if\ k_1 + k_2 + \cdots + k_{s-1} + 1 \leq i \leq n. \end{cases}$$

The general solution of linear differential equation (LDE^n) *is*

$$\sum_{i=1}^{s}(c_{i1}t^{k_i-1} + c_{i2}t^{k_i-2} + \cdots + c_{i(k_i-1)}t + c_{ik_i})e^{\lambda_i t},$$

with constants $c_{ij},\ 1 \leq i \leq s, 1 \leq j \leq k_i$.

Such a vector family $\overline{\beta}_i(t)e^{\alpha_i t},\ 1 \leq i \leq n$ of the differential equation system $(LDES^1)$ and a family $t^l e^{\lambda_i t},\ 1 \leq l \leq k_i, 1 \leq i \leq s$ of the linear differential equation (LDE^n) are called the *solution basis*, denoted by

$$\mathscr{B} = \{\ \overline{\beta}_i(t)e^{\alpha_i t} \mid 1 \leq i \leq n\ \} \ \text{ or }\ \mathscr{C} = \{\ t^l e^{\lambda_i t} \mid 1 \leq i \leq s, 1 \leq l \leq k_i\ \}.$$

We only consider autonomous differential systems in this paper. Theorem 1.2 implies that any linear differential equation system $(LDES^1)$ of first order and any differential equation (LDE^n) of order n with real coefficients are solvable. Thus a linear differential equation system of first order is non-solvable only if the number of equations is more than that of variables, and a differential equation system of order $n \geq 2$ is non-solvable only if the number of equations is more than 2. Generally, such a contradictory system, i.e., a Smarandache system [4]-[6] is defined following.

Definition 1.3([4]-[6]) *A rule* $\mathcal{R}$ *in a mathematical system* $(\Sigma; \mathcal{R})$ *is said to be Smarandachely denied if it behaves in at least two different ways within the same set* Σ, *i.e., validated and invalided, or only invalided but in multiple distinct ways.*

A Smarandache system $(\Sigma; \mathcal{R})$ *is a mathematical system which has at least one Smarandachely denied rule* $\mathcal{R}$.

Generally, let $(\Sigma_1;\mathcal{R}_1)$ $(\Sigma_2;\mathcal{R}_2)$, $\cdots$, $(\Sigma_m;\mathcal{R}_m)$ be mathematical systems, where $\mathcal{R}_i$ is a rule on Σ_i for integers $1\leq i\leq m$. If for two integers $i,j,\ 1\leq i,j\leq m$, $\Sigma_i\neq\Sigma_j$ or $\Sigma_i=\Sigma_j$ but $\mathcal{R}_i\neq\mathcal{R}_j$, then they are said to be *different*, otherwise, *identical*. We also know the conception of Smarandache multi-space defined following.

Definition 1.4([4]-[6]) *Let $(\Sigma_1;\mathcal{R}_1)$, $(\Sigma_2;\mathcal{R}_2)$, $\cdots$, $(\Sigma_m;\mathcal{R}_m)$ be $m\geq 2$ mathematical spaces, different two by two. A Smarandache multi-space $\widetilde{\Sigma}$ is a union $\bigcup\limits_{i=1}^{m}\Sigma_i$ with rules $\widetilde{\mathcal{R}}=\bigcup\limits_{i=1}^{m}\mathcal{R}_i$ on $\widetilde{\Sigma}$, i.e., the rule $\mathcal{R}_i$ on Σ_i for integers $1\leq i\leq m$, denoted by $\left(\widetilde{\Sigma};\widetilde{\mathcal{R}}\right)$.*

A Smarandache multi-space $\left(\widetilde{\Sigma};\widetilde{\mathcal{R}}\right)$ inherits a combinatorial structure, i.e., a vertex-edge labeled graph defined following.

Definition 1.5([4]-[6]) *Let $\left(\widetilde{\Sigma};\widetilde{\mathcal{R}}\right)$ be a Smarandache multi-space with $\widetilde{\Sigma}=\bigcup\limits_{i=1}^{m}\Sigma_i$ and $\widetilde{\mathcal{R}}=\bigcup\limits_{i=1}^{m}\mathcal{R}_i$. Its underlying graph $G\left[\widetilde{\Sigma},\widetilde{R}\right]$ is a labeled simple graph defined by*

$$V\left(G\left[\widetilde{\Sigma},\widetilde{R}\right]\right)=\{\Sigma_1,\Sigma_2,\cdots,\Sigma_m\},$$

$$E\left(G\left[\widetilde{\Sigma},\widetilde{R}\right]\right)=\{\ (\Sigma_i,\Sigma_j)\mid\Sigma_i\bigcap\Sigma_j\neq\emptyset,1\leq i,j\leq m\}$$

with an edge labeling

$$l^E:\ (\Sigma_i,\Sigma_j)\in E\left(G\left[\widetilde{S},\widetilde{R}\right]\right)\rightarrow l^E(\Sigma_i,\Sigma_j)=\varpi\left(\Sigma_i\bigcap\Sigma_j\right),$$

where ϖ is a characteristic on $\Sigma_i\bigcap\Sigma_j$ such that $\Sigma_i\bigcap\Sigma_j$ is isomorphic to $\Sigma_k\bigcap\Sigma_l$ if and only if $\varpi(\Sigma_i\bigcap\Sigma_j)=\varpi\left(\Sigma_k\bigcap\Sigma_l\right)$ for integers $1\leq i,j,k,l\leq m$.

Now for integers $m,\ n\geq 1$, let

$$\dot{X}=F_1(X),\ \dot{X}=F_2(X),\cdots,\dot{X}=F_m(X) \qquad (DES_m^1)$$

be a differential equation system with continuous $F_i:\mathbf{R}^n\rightarrow\mathbf{R}^n$ such that $F_i(\overline{0})=\overline{0}$, particularly, let

$$\dot{X}=A_1X,\cdots,\dot{X}=A_kX,\cdots,\dot{X}=A_mX \qquad (LDES_m^1)$$

be a linear ordinary differential equation system of first order and

$$\begin{cases} x^{(n)} + a_{11}^{[0]}x^{(n-1)} + \cdots + a_{1n}^{[0]}x = 0 \\ x^{(n)} + a_{21}^{[0]}x^{(n-1)} + \cdots + a_{2n}^{[0]}x = 0 \\ \cdots\cdots\cdots\cdots \\ x^{(n)} + a_{m1}^{[0]}x^{(n-1)} + \cdots + a_{mn}^{[0]}x = 0 \end{cases} \qquad (LDE_m^n)$$

a linear differential equation system of order n with

$$A_k = \begin{bmatrix} a_{11}^{[k]} & a_{12}^{[k]} & \cdots & a_{1n}^{[k]} \\ a_{21}^{[k]} & a_{22}^{[k]} & \cdots & a_{2n}^{[k]} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1}^{[k]} & a_{n2}^{[k]} & \cdots & a_{nn}^{[k]} \end{bmatrix} \quad \text{and} \quad X = \begin{bmatrix} x_1(t) \\ x_2(t) \\ \cdots \\ x_n(t) \end{bmatrix}$$

where each $a_{ij}^{[k]}$ is a real number for integers $0 \leq k \leq m,\ 1 \leq i, j \leq n$.

Definition 1.6 *An ordinary differential equation system* (DES_m^1) *or* $(LDES_m^1)$ *(or* (LDE_m^n)*) are called non-solvable if there are no function* $X(t)$ *(or* $x(t)$*) hold with* (DES_m^1) *or* $(LDES_m^1)$ *(or* (LDE_m^n)*) unless the constants.*

The main purpose of this paper is to find contradictory ordinary differential equation systems, characterize the non-solvable spaces of such differential equation systems. For such objective, we are needed to extend the conception of solution of linear differential equations in classical mathematics following.

Definition 1.7 *Let* S_i^0 *be the solution basis of the* i*th equation in* (DES_m^1). *The* $\vee$*-solvable,* $\wedge$*-solvable and non-solvable spaces of differential equation system* (DES_m^1) *are respectively defined by*

$$\bigcup_{i=1}^{m} S_i^0, \quad \bigcap_{i=1}^{m} S_i^0 \quad \textit{and} \quad \bigcup_{i=1}^{m} S_i^0 - \bigcap_{i=1}^{m} S_i^0,$$

where S_i^0 *is the solution space of the* i*th equation in* (DES_m^1).

According to Theorem 1.2, the general solution of the ith differential equation in $(LDES_m^1)$ or the ith differential equation system in (LDE_m^n) is a linear space spanned by the elements in the solution basis $\mathscr{B}_i$ or $\mathscr{C}_i$ for integers $1 \leq i \leq m$. Thus we can simplify the vertex-edge labeled graph $G\left[\widetilde{\sum}, \widetilde{R}\right]$ replaced each $\sum_i$ by the solution basis $\mathscr{B}_i$ (or $\mathscr{C}_i$) and $\sum_i \bigcap \sum_j$ by $\mathscr{B}_i \bigcap \mathscr{B}_j$ (or $\mathscr{C}_i \bigcap \mathscr{C}_j$) if $\mathscr{B}_i \bigcap \mathscr{B}_j \neq \emptyset$ (or $\mathscr{C}_i \bigcap \mathscr{C}_j \neq \emptyset$) for integers $1 \leq i, j \leq m$. Such a vertex-edge labeled graph is called the *basis graph* of $(LDES_m^1)$ $((LDE_m^n))$, denoted respectively by $G[LDES_m^1]$ or $G[LDE_m^n]$ and the underlying graph of $G[LDES_m^1]$ or $G[LDE_m^n]$, i.e., cleared away all labels on $G[LDES_m^1]$ or $G[LDE_m^n]$ are denoted by $\hat{G}[LDES_m^1]$ or $\hat{G}[LDE_m^n]$.

Notice that $\bigcap\limits_{i=1}^{m} S_i^0 = \bigcup\limits_{i=1}^{m} S_i^0$, i.e., the non-solvable space is empty only if $m = 1$ in $(LDEq)$. Thus $G[LDES^1] \simeq K_1$ or $G[LDE^n] \simeq K_1$ only if $m = 1$. But in general, the basis graph $G[LDES_m^1]$ or $G[LDE_m^n]$ is not trivial. For example, let $m = 4$ and $\mathscr{B}_1^0 = \{e^{\lambda_1 t}, e^{\lambda_2 t}, e^{\lambda_3 t}\}$, $\mathscr{B}_2^0 = \{e^{\lambda_3 t}, e^{\lambda_4 t}, e^{\lambda_5 t}\}$, $\mathscr{B}_3^0 = \{e^{\lambda_1 t}, e^{\lambda_3 t}, e^{\lambda_5 t}\}$ and $\mathscr{B}_4^0 = \{e^{\lambda_4 t}, e^{\lambda_5 t}, e^{\lambda_6 t}\}$, where $\lambda_i,\ 1 \leq i \leq 6$ are real numbers different two by two. Then its edge-labeled graph $G[LDES_m^1]$ or $G[LDE_m^n]$ is shown in Fig.1.1.

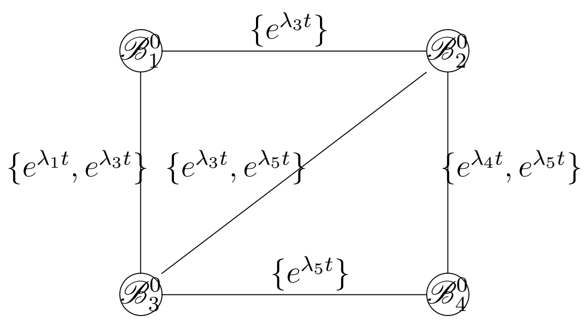

Fig.1.1

If some functions $F_i(X),\ 1 \leq i \leq m$ are non-linear in (DES_m^1), we can linearize these non-linear equations $\dot{X} = F_i(X)$ at the point $\overline{0}$, i.e., if

$$F_i(X) = F_i'(\overline{0})X + R_i(X),$$

where $F_i'(\overline{0})$ is an $n \times n$ matrix, we replace the ith equation $\dot{X} = F_i(X)$ by a linear differential equation

$$\dot{X} = F_i'(\overline{0})X$$

in (DES_m^1). Whence, we get a uniquely linear differential equation system $(LDES_m^1)$ from (DES_m^1) and its basis graph $G[LDES_m^1]$. Such a basis graph $G[LDES_m^1]$ of linearized differential equation system (DES_m^1) is defined to be the *linearized basis graph* of (DES_m^1) and denoted by $G[DES_m^1]$.

All of these notions will contribute to the characterizing of non-solvable differential equation systems. For terminologies and notations not mentioned here, we follow the [13] for differential equations, [2] for linear algebra, [3]-[6], [11]-[12] for graphs and Smarandache systems, and [1], [12] for mechanics.

§2. Non-Solvable Spaces of Linear Differential Equations

2.1 A Condition for Non-Solvable Linear Differential Equations

First, we know the following conclusion for non-solvable linear differential equation systems $(LDES_m^1)$ or (LDE_m^n).

Theorem 2.1 *The differential equation system* $(LDES_m^1)$ *is solvable if and only if*

$$(|A_1 - \lambda I_{n\times n}, |A_2 - \lambda I_{n\times n}|, \cdots, |A_m - \lambda I_{n\times n}|) \neq 1$$

i.e., $(LDEq)$ *is non-solvable if and only if*

$$(|A_1 - \lambda I_{n\times n}, |A_2 - \lambda I_{n\times n}|, \cdots, |A_m - \lambda I_{n\times n}|) = 1.$$

Similarly, the differential equation system (LDE_m^n) *is solvable if and only if*

$$(P_1(\lambda), P_2(\lambda), \cdots, P_m(\lambda)) \neq 1,$$

i.e., (LDE_m^n) *is non-solvable if and only if*

$$(P_1(\lambda), P_2(\lambda), \cdots, P_m(\lambda)) = 1,$$

where $P_i(\lambda) = \lambda^n + a_{i1}^{[0]}\lambda^{n-1} + \cdots + a_{i(n-1)}^{[0]}\lambda + a_{in}^{[0]}$ *for integers* $1 \leq i \leq m$.

Proof Let $\lambda_{i1}, \lambda_{i2}, \cdots, \lambda_{in}$ be the n solutions of equation $|A_i - \lambda I_{n\times n}| = 0$ and $\mathscr{B}_i$ the solution basis of ith differential equation in $(LDES_m^1)$ or (LDE_m^n) for integers $1 \leq i \leq m$. Clearly, if $(LDES_m^1)$ $((LDE_m^n))$ is solvable, then

$$\bigcap_{i=1}^{m} \mathscr{B}_i \neq \emptyset, \quad \text{i.e.,} \quad \bigcap_{i=1}^{m} \{\lambda_{i1}, \lambda_{i2}, \cdots, \lambda_{in}\} \neq \emptyset$$

by Definition 1.5 and Theorem 1.2. Choose $\lambda_0 \in \bigcap_{i=1}^{m} \{\lambda_{i1}, \lambda_{i2}, \cdots, \lambda_{in}\}$. Then $(\lambda - \lambda_0)$ is a common divisor of these polynomials $|A_1 - \lambda I_{n\times n}, |A_2 - \lambda I_{n\times n}|, \cdots, |A_m - \lambda I_{n\times n}|$. Thus

$$(|A_1 - \lambda I_{n\times n}, |A_2 - \lambda I_{n\times n}|, \cdots, |A_m - \lambda I_{n\times n}|) \neq 1.$$

Conversely, if

$$(|A_1 - \lambda I_{n\times n}, |A_2 - \lambda I_{n\times n}|, \cdots, |A_m - \lambda I_{n\times n}|) \neq 1,$$

let $(\lambda-\lambda_{01}),(\lambda-\lambda_{02}),\cdots,(\lambda-\lambda_{0l})$ be all the common divisors of polynomials $|A_1-\lambda I_{n\times n},|A_2-\lambda I_{n\times n}|,\cdots,|A_m-\lambda I_{n\times n}|$, where $\lambda_{0i}\neq\lambda_{0j}$ if $i\neq j$ for $1\leq i,j\leq l$. Then it is clear that

$$C_1e^{\lambda_{01}}+C_2e^{\lambda_{02}}+\cdots+C_le^{\lambda_{0l}}$$

is a solution of $(LEDq)$ $((LDE_m^n))$ for constants $C_1,C_2,\cdots,C_l$. □

For discussing the non-solvable space of a linear differential equation system $(LEDS_m^1)$ or (LDE_m^n) in details, we introduce the following conception.

Definition 2.2 *For two integers $1\leq i,j\leq m$, the differential equations*

$$\begin{cases}\dfrac{dX_i}{dt}=A_iX\\ \\ \dfrac{dX_j}{dt}=A_jX\end{cases}\qquad (LDES_{ij}^1)$$

in $(LDES_m^1)$ or

$$\begin{cases}x^{(n)}+a_{i1}^{[0]}x^{(n-1)}+\cdots+a_{in}^{[0]}x=0\\ x^{(n)}+a_{j1}^{[0]}x^{(n-1)}+\cdots+a_{jn}^{[0]}x=0\end{cases}\qquad (LDE_{ij}^n)$$

in (LDE_m^n) are parallel if $\mathscr{B}_i\bigcap\mathscr{B}_j=\emptyset$.

Then, the following conclusion is clear.

Theorem 2.3 *For two integers $1\leq i,j\leq m$, two differential equations $(LDES_{ij}^1)$ (or (LDE_{ij}^n)) are parallel if and only if*

$$(|A_i|-\lambda I_{n\times n},|A_j|-\lambda I_{n\times n})=1\quad (or\ (P_i(\lambda),P_j(\lambda))=1),$$

where $(f(x),g(x))$ is the least common divisor of $f(x)$ and $g(x)$, $P_k(\lambda)=\lambda^n+a_{k1}^{[0]}\lambda^{n-1}+\cdots+a_{k(n-1)}^{[0]}\lambda+a_{kn}^{[0]}$ for $k=i,j$.

Proof By definition, two differential equations $(LEDS_{ij}^1)$ in $(LDES_m^1)$ are parallel if and only if the characteristic equations

$$|A_i-\lambda I_{n\times n}|=0\quad\text{and}\quad|A_j-\lambda I_{n\times n}|=0$$

have no same roots. Thus the polynomials $|A_i|-\lambda I_{n\times n}$ and $|A_j|-\lambda I_{n\times n}$ are coprime, which means that

$$(|A_i-\lambda I_{n\times n},|A_j-\lambda I_{n\times n})=1.$$

Similarly, two differential equations (LED^n_{ij}) in (LDE^n_m) are parallel if and only if the characteristic equations $P_i(\lambda) = 0$ and $P_j(\lambda) = 0$ have no same roots, i.e., $(P_i(\lambda), P_j(\lambda)) = 1$. □

Let $f(x) = a_0x^m+a_1x^{m-1}+\cdots+a_{m-1}x+a_m,\ g(x) = b_0x^n+b_1x^{n-1}+\cdots+b_{n-1}x+b_n$ with roots $x_1, x_2, \cdots, x_m$ and $y_1, y_2, \cdots, y_n$, respectively. A *resultant* $R(f, g)$ of $f(x)$ and $g(x)$ is defined by

$$R(f,g) = a_0^m b_0^n \prod_{i,j}(x_i - y_j).$$

The following result is well-known in polynomial algebra.

Theorem 2.4 *Let* $f(x) = a_0x^m+a_1x^{m-1}+\cdots+a_{m-1}x+a_m,\ g(x) = b_0x^n+b_1x^{n-1}+\cdots+b_{n-1}x+b_n$ *with roots* $x_1, x_2, \cdots, x_m$ *and* $y_1, y_2, \cdots, y_n$, *respectively. Define a matrix*

$$V(f,g) = \begin{bmatrix} a_0 & a_1 & \cdots & a_m & 0 & \cdots & 0 & 0 \\ 0 & a_0 & a_1 & \cdots & a_m & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & 0 & 0 & a_0 & a_1 & \cdots & a_m \\ b_0 & b_1 & \cdots & b_n & 0 & \cdots & 0 & 0 \\ 0 & b_0 & b_1 & \cdots & b_n & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & 0 & 0 & b_0 & b_1 & \cdots & b_n \end{bmatrix}$$

Then

$$R(f,g) = \det V(f,g).$$

We get the following result immediately by Theorem 2.3.

Corollary 2.5 (1) *For two integers* $1 \leq i, j \leq m$, *two differential equations* $(LDES^1_{ij})$ *are parallel in* $(LDES^1_m)$ *if and only if*

$$R(|A_i - \lambda I_{n\times n}|, |A_j - \lambda I_{n\times n}|) \neq 0,$$

particularly, the homogenous equations

$$V(|A_i - \lambda I_{n\times n}|, |A_j - \lambda I_{n\times n}|)X = 0$$

have only solution $(\underbrace{0,0,\cdots,0}_{2n})^T$ *if* $|A_i-\lambda I_{n\times n}| = a_0\lambda^n + a_1\lambda^{n-1} + \cdots + a_{n-1}\lambda + a_n$ *and* $|A_j-\lambda I_{n\times n}| = b_0\lambda^n + b_1\lambda^{n-1} + \cdots + b_{n-1}\lambda + b_n$.

(2) *For two integers* $1 \le i,j \le m$, *two differential equations* (LDE^n_{ij}) *are parallel in* (LDE^n_m) *if and only if*

$$R(P_i(\lambda), P_j(\lambda)) \neq 0,$$

particularly, the homogenous equations $V(P_i(\lambda), P_j(\lambda))X = 0$ *have only solution* $(\underbrace{0,0,\cdots,0}_{2n})^T$.

Proof Clearly, $|A_i-\lambda I_{n\times n}|$ and $|A_j-\lambda I_{n\times n}|$ have no same roots if and only if

$$R(|A_i-\lambda I_{n\times n}|, |A_j-\lambda I_{n\times n}|) \neq 0,$$

which implies that the two differential equations $(LEDS^1_{ij})$ are parallel in $(LEDS^1_m)$ and the homogenous equations

$$V(|A_i-\lambda I_{n\times n}|, |A_j-\lambda I_{n\times n}|)X = 0$$

have only solution $(\underbrace{0,0,\cdots,0}_{2n})^T$. That is the conclusion (1). The proof for the conclusion (2) is similar. □

Applying Corollary 2.5, we can determine that an edge $(\mathscr{B}_i, \mathscr{B}_j)$ does not exist in $G[LDES^1_m]$ or $G[LDE^n_m]$ if and only if the ith differential equation is parallel with the jth differential equation in $(LDES^1_m)$ or (LDE^n_m). This fact enables one to know the following result on linear non-solvable differential equation systems.

Corollary 2.6 *A linear differential equation system* $(LDES^1_m)$ *or* (LDE^n_m) *is non-solvable if* $\hat{G}(LDES^1_m) \not\simeq K_m$ *or* $\hat{G}(LDE^n_m) \not\simeq K_m$ *for integers* $m, n > 1$.

2.2 Combinatorial Classification of Linear Differential Equations

There is a natural relation between linear differential equations and basis graphs shown in the following result.

Theorem 2.7 *Every linear homogeneous differential equation system* $(LDES^1_m)$ *(or* (LDE^n_m)*) uniquely determines a basis graph* $G[LDES^1_m]$ *(*$G[LDE^n_m]$*) inherited* *in $(LDES_m^1)$ (or in (LDE_m^n)). Conversely, every basis graph G uniquely determines a homogeneous differential equation system $(LDES_m^1)$ (or (LDE_m^n)) such that $G[LDES_m^1] \simeq G$ (or $G[LDE_m^n] \simeq G$).*

Proof By Definition 1.4, every linear homogeneous differential equation system $(LDES_m^1)$ or (LDE_m^n) inherits a basis graph $G[LDES_m^1]$ or $G[LDE_m^n]$, which is uniquely determined by $(LDES_m^1)$ or (LDE_m^n).

Now let G be a basis graph. For $\forall v \in V(G)$, let the basis $\mathscr{B}_v$ at the vertex v be $\mathscr{B}_v = \{\ \overline{\beta}_i(t)e^{\alpha_i t} \mid 1 \leq i \leq n_v\}$ with

$$\alpha_i = \begin{cases} \lambda_1, & if\ 1 \leq i \leq k_1; \\ \lambda_2, & if\ k_1+1 \leq i \leq k_2; \\ \cdots & \cdots\cdots\cdots\cdots\cdots\cdots; \\ \lambda_s, & if\ k_1+k_2+\cdots+k_{s-1}+1 \leq i \leq n_v \end{cases}$$

We construct a linear homogeneous differential equation $(LDES^1)$ associated at the vertex v. By Theorem 1.2, we know the matrix

$$T = \begin{bmatrix} t_{11} & t_{12} & \cdots & t_{1n_v} \\ t_{21} & t_{22} & \cdots & t_{2n_v} \\ \cdots & \cdots & \cdots & \cdots \\ t_{n_v1} & t_{n_v2} & \cdots & t_{n_vn_v} \end{bmatrix}$$

is non-degenerate. For an integer i, $1 \leq i \leq s$, let

$$J_i = \begin{bmatrix} \lambda_i & 1 & 0 & \cdots & 0 & 0 \\ 0 & \lambda_i & 1 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 0 & 0 & \lambda_i \end{bmatrix}$$

be a Jordan black of $k_i \times k_i$ and

$$A = T \begin{bmatrix} J_1 & & & O \\ & J_2 & & \\ & & \ddots & \\ O & & & J_s \end{bmatrix} T^{-1}.$$

Then we are easily know the solution basis of the linear differential equation system

$$\frac{dX}{dt} = AX \qquad (LDES^1)$$

with $X = [x_1(t), x_2(t), \cdots, x_{n_v}(t)]^T$ is nothing but $\mathscr{B}_v$ by Theorem 1.2. Notice that the Jordan black and the matrix T are uniquely determined by $\mathscr{B}_v$. Thus the linear homogeneous differential equation $(LDES^1)$ is uniquely determined by $\mathscr{B}_v$. It should be noted that this construction can be processed on each vertex $v \in V(G)$. We finally get a linear homogeneous differential equation system $(LDES_m^1)$, which is uniquely determined by the basis graph G.

Similarly, we construct the linear homogeneous differential equation system (LDE_m^n) for the basis graph G. In fact, for $\forall u \in V(G)$, let the basis $\mathscr{B}_u$ at the vertex u be $\mathscr{B}_u = \{\ t^l e^{\alpha_i t} \mid 1 \le i \le s, 1 \le l \le k_i\}$. Notice that λ_i should be a k_i-fold zero of the characteristic equation $P(\lambda) = 0$ with $k_1 + k_2 + \cdots + k_s = n$. Thus $P(\lambda_i) = P'(\lambda_i) = \cdots = P^{(k_i-1)}(\lambda_i) = 0$ but $P^{(k_i)}(\lambda_i) \neq 0$ for integers $1 \le i \le s$. Define a polynomial $P_u(\lambda)$ following

$$P_u(\lambda) = \prod_{i=1}^{s} (\lambda - \lambda_i)^{k_i}$$

associated with the vertex u. Let its expansion be

$$P_u(\lambda) = \lambda^n + a_{u1}\lambda^{n-1} + \cdots + a_{u(n-1)}\lambda + a_{un}.$$

Now we construct a linear homogeneous differential equation

$$x^{(n)} + a_{u1}x^{(n-1)} + \cdots + a_{u(n-1)}x' + a_{un}x = 0 \qquad (L^hDE^n)$$

associated with the vertex u. Then by Theorem 1.2 we know that the basis solution of (LDE^n) is just $\mathscr{C}_u$. Notices that such a linear homogeneous differential equation (LDE^n) is uniquely constructed. Processing this construction for every vertex $u \in V(G)$, we get a linear homogeneous differential equation system (LDE_m^n). This completes the proof. □

Example 2.8 Let (LDE_m^n) be the following linear homogeneous differential equation system

$$\begin{cases} \ddot{x} - 3\dot{x} + 2x = 0 & (1) \\ \ddot{x} - 5\dot{x} + 6x = 0 & (2) \\ \ddot{x} - 7\dot{x} + 12x = 0 & (3) \\ \ddot{x} - 9\dot{x} + 20x = 0 & (4) \\ \ddot{x} - 11\dot{x} + 30x = 0 & (5) \\ \ddot{x} - 7\dot{x} + 6x = 0 & (6) \end{cases}$$

where $\ddot{x} = \dfrac{d^2x}{dt^2}$ and $\dot{x} = \dfrac{dx}{dt}$. Then the solution basis of equations $(1) - (6)$ are respectively $\{e^t, e^{2t}\}$, $\{e^{2t}, e^{3t}\}$, $\{e^{3t}, e^{4t}\}$, $\{e^{4t}, e^{5t}\}$, $\{e^{5t}, e^{6t}\}$, $\{e^{6t}, e^t\}$ and its basis graph is shown in Fig.2.1.

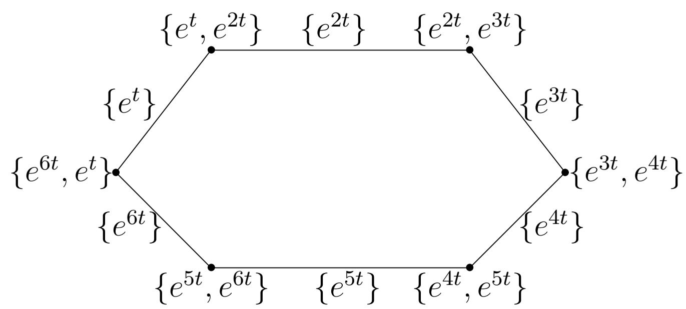

Fig.2.1 **The basis graph H**

Theorem 2.7 enables one to extend the conception of solution of linear differential equation to the following.

Definition 2.9 *A basis graph $G[LDES^1_m]$ (or $G[LDE^n_m]$) is called the graph solution of the linear homogeneous differential equation system $(LDES^1_m)$ (or (LDE^n_m)), abbreviated to G-solution.*

The following result is an immediately conclusion of Theorem 3.1 by definition.

Theorem 2.10 *Every linear homogeneous differential equation system $(LDES^1_m)$ (or (LDE^n_m)) has a unique G-solution, and for every basis graph H, there is a unique linear homogeneous differential equation system $(LDES^1_m)$ (or (LDE^n_m)) with G-solution H.*

Theorem 2.10 implies that one can classifies the linear homogeneous differential equation systems by those of basis graphs.

Definition 2.11 *Let $(LDES^1_m)$, $(LDES^1_m)'$ (or (LDE^n_m), $(LDE^n_m)'$) be two linear homogeneous differential equation systems with G-solutions H, H'. They are called combinatorially equivalent if there is an isomorphism $\varphi : H \rightarrow H'$, thus there is an isomorphism $\varphi : H \rightarrow H'$ of graph and labelings θ, τ on H and H' respectively such that $\varphi\theta(x) = \tau\varphi(x)$ for $\forall x \in V(H) \bigcup E(H)$, denoted by $(LDES^1_m) \overset{\varphi}{\simeq} (LDES^1_m)'$ (or $(LDE^n_m) \overset{\varphi}{\simeq} (LDE^n_m)'$).*

Example 2.12 Let $(LDE_m^n)'$ be the following linear homogeneous differential equation system

$$\begin{cases} \ddot{x}+3\dot{x}+2x=0 & (1) \\ \ddot{x}+5\dot{x}+6x=0 & (2) \\ \ddot{x}+7\dot{x}+12x=0 & (3) \\ \ddot{x}+9\dot{x}+20x=0 & (4) \\ \ddot{x}+11\dot{x}+30x=0 & (5) \\ \ddot{x}+7\dot{x}+6x=0 & (6) \end{cases}$$

Then its basis graph is shown in Fig.2.2 following.

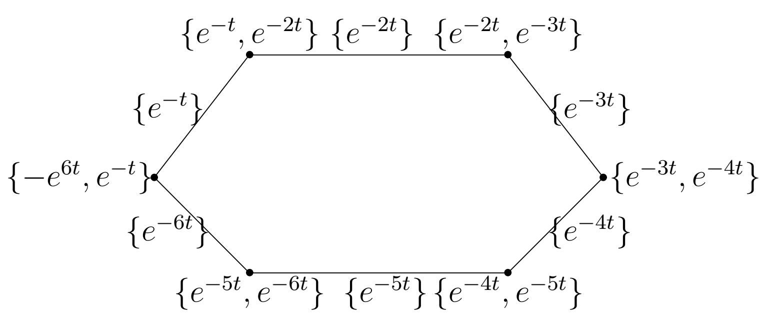

Fig.2.2 **The basis graph H'**

Let $\varphi : H \rightarrow H'$ be determined by $\varphi(\{e^{\lambda_i t}, e^{\lambda_j t}\}) = \{e^{-\lambda_i t}, e^{-\lambda_j t}\}$ and

$$\varphi(\{e^{\lambda_i t}, e^{\lambda_j t}\}\bigcap\{e^{\lambda_k t}, e^{\lambda_l t}\}) = \{e^{-\lambda_i t}, e^{-\lambda_j t}\}\bigcap\{e^{-\lambda_k t}, e^{-\lambda_l t}\}$$

for integers $1 \leq i, k \leq 6$ and $j = i+1 \equiv 6(\mathrm{mod}6)$, $l = k+1 \equiv 6(\mathrm{mod}6)$. Then it is clear that $H \overset{\varphi}{\simeq} H'$. Thus $(LDE_m^n)'$ is combinatorially equivalent to the linear homogeneous differential equation system (LDE_m^n) appeared in Example 2.8.

Definition 2.13 *Let G be a simple graph. A vertex-edge labeled graph $\theta : G \rightarrow \mathbb{Z}^+$ is called integral if $\theta(uv) \leq \min\{\theta(u), \theta(v)\}$ for $\forall uv \in E(G)$, denoted by G^{I_θ}.*

Let $G_1^{I_\theta}$ and $G_2^{I_\tau}$ be two integral labeled graphs. They are called identical if $G_1 \overset{\varphi}{\simeq} G_2$ and $\theta(x) = \tau(\varphi(x))$ for any graph isomorphism φ and $\forall x \in V(G_1)\bigcup E(G_1)$, denoted by $G_1^{I_\theta} = G_2^{I_\tau}$.

For example, these labeled graphs shown in Fig.2.3 are all integral on $K_4 - e$, but $G_1^{I_\theta} = G_2^{I_\tau}$, $G_1^{I_\theta} \neq G_3^{I_\sigma}$.

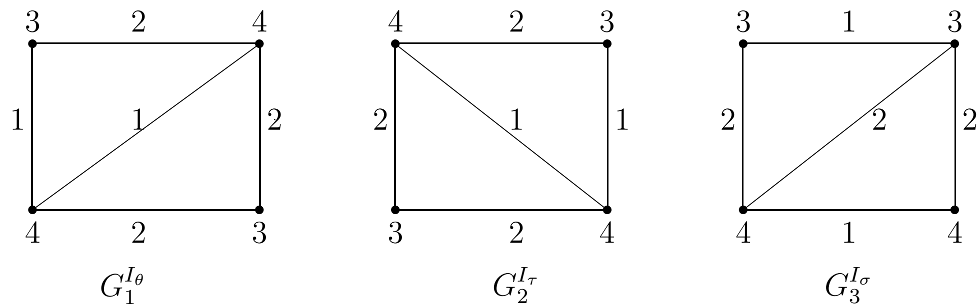

Fig.2.3

Let $G[LDES_m^1]$ ($G[LDE_m^n]$) be a basis graph of the linear homogeneous differential equation system $(LDES_m^1)$ (or (LDE_m^n)) labeled each $v \in V(G[LDES_m^1])$ (or $v \in V(G[LDE_m^n])$) by $\mathscr{B}_v$. We are easily get a vertex-edge labeled graph by relabeling $v \in V(G[LDES_m^1])$ (or $v \in V(G[LDE_m^n])$) by $|\mathscr{B}_v|$ and $uv \in E(G[LDES_m^1])$ (or $uv \in E(G[LDE_m^n])$) by $|\mathscr{B}_u \bigcap \mathscr{B}_v|$. Obviously, such a vertex-edge labeled graph is integral, and denoted by $G^I[LDES_m^1]$ (or $G^I[LDE_m^n]$). The following result completely characterizes combinatorially equivalent linear homogeneous differential equation systems.

Theorem 2.14 *Let $(LDES_m^1)$, $(LDES_m^1)'$ (or (LDE_m^n), $(LDE_m^n)'$) be two linear homogeneous differential equation systems with integral labeled graphs H, H'. Then $(LDES_m^1) \stackrel{\varphi}{\simeq} (LDES_m^1)'$ (or $(LDE_m^n) \stackrel{\varphi}{\simeq} (LDE_m^n)'$) if and only if $H = H'$.*

Proof Clearly, $H = H'$ if $(LDES_m^1) \stackrel{\varphi}{\simeq} (LDES_m^1)'$ (or $(LDE_m^n) \stackrel{\varphi}{\simeq} (LDE_m^n)'$) by definition. We prove the converse, i.e., if $H = H'$ then there must be $(LDES_m^1) \stackrel{\varphi}{\simeq} (LDES_m^1)'$ (or $(LDE_m^n) \stackrel{\varphi}{\simeq} (LDE_m^n)'$).

Notice that there is an objection between two finite sets S_1, S_2 if and only if $|S_1| = |S_2|$. Let τ be a $1-1$ mapping from $\mathscr{B}_v$ on basis graph $G[LDES_m^1]$ (or basis graph $G[LDE_m^n]$) to $\mathscr{B}_{v'}$ on basis graph $G[LDES_m^1]'$ (or basis graph $G[LDE_m^n]'$) for $v, v' \in V(H')$. Now if $H = H'$, we can easily extend the identical isomorphism id_H on graph H to a $1-1$ mapping $id_H^* : G[LDES_m^1] \to G[LDES_m^1]'$ (or $id_H^* : G[LDE_m^n] \to G[LDE_m^n]'$) with labelings $\theta : v \to \mathscr{B}_v$ and $\theta'_{v'} : v' \to \mathscr{B}_{v'}$ on $G[LDES_m^1]$, $G[LDES_m^1]'$ (or basis graphs $G[LDE_m^n]$, $G[LDE_m^n]'$). Then it is an immediately to check that $id_H^*\theta(x) = \theta'\tau(x)$ for $\forall x \in V(G[LDES_m^1]) \bigcup E(G[LDES_m^1])$ (or for $\forall x \in V(G[LDE_m^n]) \bigcup E(G[LDE_m^n])$). Thus id_H^* is an isomorphism between basis graphs $G[LDES_m^1]$ and $G[LDES_m^1]'$ (or $G[LDE_m^n]$ and $G[LDE_m^n]'$). Thus $(LDES_m^1) \overset{id_H^*}{\simeq} (LDES_m^1)'$ (or $(LDE_m^n) \overset{id_H^*}{\simeq} (LDE_m^n)'$). This completes the proof. □

According to Theorem 2.14, all linear homogeneous differential equation systems $(LDES_m^1)$ or (LDE_m^n) can be classified by G-solutions into the following classes:

Class 1. $\hat{G}[LDES_m^1] \simeq \overline{K}_m$ or $\hat{G}[LDE_m^n] \simeq \overline{K}_m$ for integers $m, n \geq 1$.

The G-solutions of differential equation systems are labeled by solution bases on $\overline{K}_m$ and any two linear differential equations in $(LDES_m^1)$ or (LDE_m^n) are parallel, which characterizes m isolated systems in this class.

For example, the following differential equation system

$$\begin{cases} \ddot{x} + 3\dot{x} + 2x = 0 \\ \ddot{x} - 5\dot{x} + 6x = 0 \\ \ddot{x} + 2\dot{x} - 3x = 0 \end{cases}$$

is of Class 1.

Class 2. $\hat{G}[LDES_m^1] \simeq K_m$ or $\hat{G}[LDE_m^n] \simeq K_m$ for integers $m, n \geq 1$.

The G-solutions of differential equation systems are labeled by solution bases on complete graphs K_m in this class. By Corollary 2.6, we know that $\hat{G}[LDES_m^1] \simeq K_m$ or $\hat{G}[LDE_m^n] \simeq K_m$ if $(LDES_m^1)$ or (LDE_m^n) is solvable. In fact, this implies that

$$\bigcap_{v \in V(K_m)} \mathscr{B}_v = \bigcap_{u,v \in V(K_m)} (\mathscr{B}_u \bigcap \mathscr{B}_v) \neq \emptyset.$$

Otherwise, $(LDES_m^1)$ or (LDE_m^n) is non-solvable.

For example, the underlying graphs of linear differential equation systems (A) and (B) in the following

$$(A) \quad \begin{cases} \ddot{x} - 3\dot{x} + 2x = 0 \\ \ddot{x} - x = 0 \\ \ddot{x} - 4\dot{x} + 3x = 0 \\ \ddot{x} + 2\dot{x} - 3x = 0 \end{cases} \qquad (B) \quad \begin{cases} \ddot{x} - 3\dot{x} + 2x = 0 \\ \ddot{x} - 5\dot{x} + 6x = 0 \\ \ddot{x} - 4\dot{x} + 3x = 0 \end{cases}$$

are respectively K_4, K_3. It is easily to know that (A) is solvable, but (B) is not.

Class 3. $\hat{G}[LDES_m^1] \simeq G$ or $\hat{G}[LDE_m^n] \simeq G$ with $|G| = m$ but $G \not\simeq K_m$, $\overline{K}_m$ for integers $m, n \geq 1$.

The G-solutions of differential equation systems are labeled by solution bases on G and all linear differential equation systems $(LDES_m^1)$ or (LDE_m^n) are non-solvable in this class, such as those shown in Example 2.12.

2.3 Stability of Linear Differential Equations

The following result on the initial problem of $(LDES^1)$ and (LDE^n) are well-known for differential equations.

Lemma 2.15([13]) *For $t \in [0, \infty)$, there is a unique solution $X(t)$ for the linear homogeneous differential equation system*

$$\frac{dX}{dt} = AX \qquad (L^hDES^1)$$

with $X(0) = X_0$ and a unique solution for

$$x^{(n)} + a_1 x^{(n-1)} + \cdots + a_n x = 0 \qquad (L^hDE^n)$$

with $x(0) = x_0, x'(0) = x_0', \cdots, x^{(n-1)}(0) = x_0^{(n-1)}$.

Applying Lemma 2.15, we get easily a conclusion on the G-solution of $(LDES_m^1)$ with $X_v(0) = X_0^v$ for $\forall v \in V(G)$ or (LDE_m^n) with $x(0) = x_0, x'(0) = x_0', \cdots, x^{(n-1)}(0) = x_0^{(n-1)}$ by Theorem 2.10 following.

Theorem 2.16 *For $t \in [0, \infty)$, there is a unique G-solution for a linear homogeneous differential equation systems $(LDES_m^1)$ with initial value $X_v(0)$ or (LDE_m^n) with initial values $x_v(0), x_v'(0), \cdots, x_v^{(n-1)}(0)$ for $\forall v \in V(G)$.*

For discussing the stability of linear homogeneous differential equations, we introduce the conceptions of *zero G-solution* and *equilibrium point* of that $(LDES_m^1)$ or (LDE_m^n) following.

Definition 2.17 *A G-solution of a linear differential equation system $(LDES_m^1)$ with initial value $X_v(0)$ or (LDE_m^n) with initial values $x_v(0), x_v'(0), \cdots, x_v^{(n-1)}(0)$ for $\forall v \in V(G)$ is called a zero G-solution if each label $\mathscr{B}_i$ of G is replaced by $(0, \cdots, 0)$ ($|\mathscr{B}_i|$ times) and $\mathscr{B}_i \bigcap \mathscr{B}_j$ by $(0, \cdots, 0)$ ($|\mathscr{B}_i \bigcap \mathscr{B}_j|$ times) for integers $1 \leq i, j \leq m$.*

Definition 2.18 *Let $dX/dt = A_v X$, $x^{(n)} + a_{v1} x^{(n-1)} + \cdots + a_{vn} x = 0$ be differential equations associated with vertex v and H a spanning subgraph of $G[LDES_m^1]$ (or*

$G[LDE_m^n]$). A point $X^* \in \mathbf{R}^n$ is called a H-equilibrium point if $A_vX^* = \overline{0}$ in $(LDES_m^1)$ with initial value $X_v(0)$ or $(X^*)^n + a_{v1}(X^*)^{n-1} + \cdots + a_{vn}X^* = \overline{0}$ in (LDE_m^n) with initial values $x_v(0), x_v'(0), \cdots, x_v^{(n-1)}(0)$ for $\forall v \in V(H)$.

We consider only two kind of stabilities on the zero G-solution of linear homogeneous differential equations in this section. One is the sum-stability. Another is the prod-stability.

2.3.1 Sum-Stability

Definition 2.19 *Let H be a spanning subgraph of $G[LDES_m^1]$ or $G[LDE_m^n]$ of the linear homogeneous differential equation systems $(LDES_m^1)$ with initial value $X_v(0)$ or (LDE_m^n) with initial values $x_v(0), x_v'(0), \cdots, x_v^{(n-1)}(0)$. Then $G[LDES_m^1]$ or $G[LDE_m^n]$ is called sum-stable or asymptotically sum-stable on H if for all solutions $Y_v(t)$, $v \in V(H)$ of the linear differential equations of $(LDES_m^1)$ or (LDE_m^n) with $|Y_v(0) - X_v(0)| < \delta_v$ exists for all $t \geq 0$, $|\sum\limits_{v\in V(H)} Y_v(t) - \sum\limits_{v\in V(H)} X_v(t)| < \varepsilon$, or furthermore, $\lim\limits_{t\to 0} |\sum\limits_{v\in V(H)} Y_v(t) - \sum\limits_{v\in V(H)} X_v(t)| = 0$.*

Clearly, an asymptotic sum-stability implies the sum-stability of that $G[LDES_m^1]$ or $G[LDE_m^n]$. The next result shows the relation of sum-stability with that of classical stability.

Theorem 2.20 *For a G-solution $G[LDES_m^1]$ of $(LDES_m^1)$ with initial value $X_v(0)$ (or $G[LDE_m^n]$ of (LDE_m^n) with initial values $x_v(0), x_v'(0), \cdots, x_v^{(n-1)}(0)$), let H be a spanning subgraph of $G[LDES_m^1]$ (or $G[LDE_m^n]$) and X^* an equilibrium point on subgraphs H. If $G[LDES_m^1]$ (or $G[LDE_m^n]$) is stable on any $\forall v \in V(H)$, then $G[LDES_m^1]$ (or $G[LDE_m^n]$) is sum-stable on H. Furthermore, if $G[LDES_m^1]$ (or $G[LDE_m^n]$) is asymptotically sum-stable for at least one vertex $v \in V(H)$, then $G[LDES_m^1]$ (or $G[LDE_m^n]$) is asymptotically sum-stable on H.*

Proof Notice that

$$|\sum_{v\in V(H)} p_vY_v(t) - \sum_{v\in V(H)} p_vX_v(t)| \leq \sum_{v\in V(H)} p_v|Y_v(t) - X_v(t)|$$

and

$$\lim_{t\to 0} |\sum_{v\in V(H)} p_vY_v(t) - \sum_{v\in V(H)} p_vX_v(t)| \leq \sum_{v\in V(H)} p_v \lim_{t\to 0} |Y_v(t) - X_v(t)|.$$

Then the conclusion on sum-stability follows. □

For linear homogenous differential equations $(LDES^1)$ (or (LDE^n)), the following result on stability of its solution $X(t) = \overline{0}$ (or $x(t) = 0$) is well-known.

Lemma 2.21 *Let $\gamma = \max\{\ \mathrm{Re}\lambda|\ |A - \lambda I_{n\times n}| = 0\}$. Then the stability of the trivial solution $X(t) = \overline{0}$ of linear homogenous differential equations $(LDES^1)$ (or $x(t) = 0$ of (LDE^n)) is determined as follows:*

(1) if $\gamma < 0$, then it is asymptotically stable;

(2) if $\gamma > 0$, then it is unstable;

(3) if $\gamma = 0$, then it is not asymptotically stable, and stable if and only if $m'(\lambda) = m(\lambda)$ for every λ with $\mathrm{Re}\lambda = 0$, where $m(\lambda)$ is the algebraic multiplicity and $m'(\lambda)$ the dimension of eigenspace of λ.

By Theorem 2.20 and Lemma 2.21, the following result on the stability of zero G-solution of $(LDES^1_m)$ and (LDE^n_m) is obtained.

Theorem 2.22 *A zero G-solution of linear homogenous differential equation systems $(LDES^1_m)$ (or (LDE^n_m)) is asymptotically sum-stable on a spanning subgraph H of $G[LDES^1_m]$ (or $G[LDE^n_m]$) if and only if $\mathrm{Re}\alpha_v < 0$ for each $\overline{\beta}_v(t)e^{\alpha_v t} \in \mathscr{B}_v$ in $(LDES^1)$ or $\mathrm{Re}\lambda_v < 0$ for each $t^{l_v}e^{\lambda_v t} \in \mathscr{C}_v$ in (LDE^n_m) hold for $\forall v \in V(H)$.*

Proof The sufficiency is an immediately conclusion of Theorem 2.20.

Conversely, if there is a vertex $v \in V(H)$ such that $\mathrm{Re}\alpha_v \geq 0$ for $\overline{\beta}_v(t)e^{\alpha_v t} \in \mathscr{B}_v$ in $(LDES^1)$ or $\mathrm{Re}\lambda_v \geq 0$ for $t^{l_v}e^{\lambda_v t} \in \mathscr{C}_v$ in (LDE^n_m), then we are easily knowing that

$$\lim_{t\to\infty} \overline{\beta}_v(t)e^{\alpha_v t} \to \infty$$

if $\alpha_v > 0$ or $\overline{\beta}_v(t) \neq$constant, and

$$\lim_{t\to\infty} t^{l_v}e^{\lambda_v t} \to \infty$$

if $\lambda_v > 0$ or $l_v > 0$, which implies that the zero G-solution of linear homogenous differential equation systems $(LDES^1)$ or (LDE^n) is not asymptotically sum-stable on H. □

The following result of Hurwitz on real number of eigenvalue of a characteristic polynomial is useful for determining the asymptotically stability of the zero G-solution of $(LDES_m^1)$ and (LDE_m^n).

Lemma 2.23 *Let $P(\lambda) = \lambda^n + a_1\lambda^{n-1} + \cdots + a_{n-1}\lambda + a_n$ be a polynomial with real coefficients a_i, $1 \le i \le n$ and*

$$\Delta_1 = |a_1|, \quad \Delta_2 = \begin{vmatrix} a_1 & 1 \\ a_3 & a_2 \end{vmatrix}, \cdots \Delta_n = \begin{vmatrix} a_1 & 1 & 0 & & \cdots & & 0 & \\ a_3 & a_2 & a_1 & 0 & & \cdots & & 0 \\ a_5 & a_4 & a_3 & a_2 & a_1 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & & & \cdots & & & & a_n \end{vmatrix}.$$

Then $Re\lambda < 0$ for all roots λ of $P(\lambda)$ if and only if $\Delta_i > 0$ for integers $1 \le i \le n$.

Thus, we get the following result by Theorem 2.22 and lemma 2.23.

Corollary 2.24 *Let $\Delta_1^v, \Delta_2^v, \cdots, \Delta_n^v$ be the associated determinants with characteristic polynomials determined in Lemma 4.8 for $\forall v \in V(G[LDES_m^1])$ or $V(G[LDE_m^n])$. Then for a spanning subgraph $H < G[LDES_m^1]$ or $G[LDE_m^n]$, the zero G-solutions of $(LDES_m^1)$ and (LDE_m^n) is asymptotically sum-stable on H if $\Delta_1^v > 0, \Delta_2^v > 0, \cdots, \Delta_n^v > 0$ for $\forall v \in V(H)$.*

Particularly, if $n = 2$, we are easily knowing that $Re\lambda < 0$ for all roots λ of $P(\lambda)$ if and only if $a_1 > 0$ and $a_2 > 0$ by Lemma 2.23. We get the following result.

Corollary 2.25 *Let $H < G[LDES_m^1]$ or $G[LDE_m^n]$ be a spanning subgraph. If the characteristic polynomials are $\lambda^2 + a_1^v\lambda + a_2^v$ for $v \in V(H)$ in $(LDES_m^1)$ (or $(L^hDE_m^2)$), then the zero G-solutions of $(LDES_m^1)$ and (LDE_m^2) is asymptotically sum-stable on H if $a_1^v > 0$, $a_2^v > 0$ for $\forall v \in V(H)$.*

2.3.2 Prod-Stability

Definition 2.26 *Let H be a spanning subgraph of $G[LDES_m^1]$ or $G[LDE_m^n]$ of the linear homogeneous differential equation systems $(LDES_m^1)$ with initial value $X_v(0)$ or (LDE_m^n) with initial values $x_v(0), x_v'(0), \cdots, x_v^{(n-1)}(0)$. Then $G[LDES_m^1]$ or $G[LDE_m^n]$ is called prod-stable or asymptotically prod-stable on H if for all solutions $Y_v(t)$, $v \in V(H)$ of the linear differential equations of $(LDES_m^1)$ or (LDE_m^n) with*

$|Y_v(0) - X_v(0)| < \delta_v$ *exists for all* $t \geq 0$, $|\prod\limits_{v\in V(H)} Y_v(t) - \prod\limits_{v\in V(H)} X_v(t)| < \varepsilon$, *or furthermore,* $\lim\limits_{t\to 0} |\prod\limits_{v\in V(H)} Y_v(t) - \prod\limits_{v\in V(H)} X_v(t)| = 0$.

We know the following result on the prod-stability of linear differential equation system $(LDES_m^1)$ and (LDE_m^n).

Theorem 2.27 *A zero G-solution of linear homogenous differential equation systems* $(LDES_m^1)$ *(or* (LDE_m^n)*) is asymptotically prod-stable on a spanning subgraph H of* $G[LDES_m^1]$ *(or* $G[LDE_m^n]$*) if and only if* $\sum\limits_{v\in V(H)} \mathrm{Re}\alpha_v < 0$ *for each* $\overline{\beta}_v(t)e^{\alpha_v t} \in \mathscr{B}_v$ *in* $(LDES^1)$ *or* $\sum\limits_{v\in V(H)} \mathrm{Re}\lambda_v < 0$ *for each* $t^{l_v}e^{\lambda_v t} \in \mathscr{C}_v$ *in* (LDE_m^n).

Proof Applying Theorem 1.2, we know that a solution $X_v(t)$ at the vertex v has the form

$$X_v(t) = \sum_{i=1}^{n} c_i\overline{\beta}_v(t)e^{\alpha_v t}.$$

Whence,

$$\begin{aligned}\left|\prod_{v\in V(H)} X_v(t)\right| &= \left|\prod_{v\in V(H)}\sum_{i=1}^{n} c_i\overline{\beta}_v(t)e^{\alpha_v t}\right| \\ &= \left|\sum_{i=1}^{n}\prod_{v\in V(H)} c_i\overline{\beta}_v(t)e^{\alpha_v t}\right| = \left|\sum_{i=1}^{n}\prod_{v\in V(H)} c_i\overline{\beta}_v(t)\right| e^{\sum\limits_{v\in V(H)}\alpha_v t}.\end{aligned}$$

Whence, the zero G-solution of homogenous $(LDES_m^1)$ (or (LDE_m^n)) is asymptotically sum-stable on subgraph H if and only if $\sum\limits_{v\in V(H)} \mathrm{Re}\alpha_v < 0$ for $\forall\overline{\beta}_v(t)e^{\alpha_v t} \in \mathscr{B}_v$ in $(LDES^1)$ or $\sum\limits_{v\in V(H)} \mathrm{Re}\lambda_v < 0$ for $\forall t^{l_v}e^{\lambda_v t} \in \mathscr{C}_v$ in (LDE_m^n). □

Applying Theorem 2.22, the following conclusion is a corollary of Theorem 2.27.

Corollary 2.28 *A zero G-solution of linear homogenous differential equation systems* $(LDES_m^1)$ *(or* (LDE_m^n)*) is asymptotically prod-stable if it is asymptotically sum-stable on a spanning subgraph H of* $G[LDES_m^1]$ *(or* $G[LDE_m^n]$*). Particularly, it is asymptotically prod-stable if the zero solution* $\overline{0}$ *is stable on* $\forall v \in V(H)$.

Example 2.29 Let a G-solution of $(LDES_m^1)$ or (LDE_m^n) be the basis graph shown in Fig.2.4, where $v_1 = \{e^{-2t}, e^{-3t}, e^{3t}\}$, $v_2 = \{e^{-3t}, e^{-4t}\}$, $v_3 = \{e^{-4t}, e^{-5t}, e^{3t}\}$, $v_4 =$

$\{e^{-5t}, e^{-6t}, e^{-8t}\}$, $v_5 = \{e^{-t}, e^{-6t}\}$, $v_6 = \{e^{-t}, e^{-2t}, e^{-8t}\}$. Then the zero G-solution is sum-stable on the triangle $v_4v_5v_6$, but it is not on the triangle $v_1v_2v_3$. In fact, it is prod-stable on the triangle $v_1v_2v_3$.

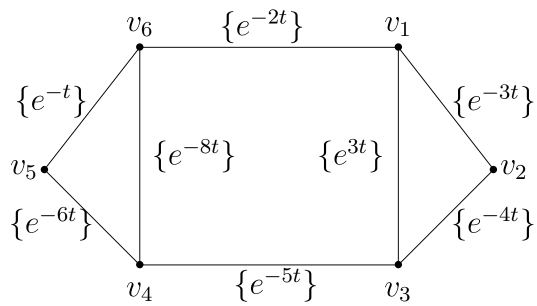

Fig.2.4 **A basis graph**

§3. Non-Solvable Spaces of Differential Equations

For differential equation system (DES_m^1), we consider the stability of its zero G-solution of linearized differential equation system $(LDES_m^1)$ in this section.

3.1 Stability of Non-Solvable Differential Equation

Definition 3.1 *Let H be a spanning subgraph of $G[DES_m^1]$ of the linearized differential equation systems (DES_m^1) with initial value $X_v(0)$. A point $X^* \in \mathbf{R}^n$ is called a H-equilibrium point of differential equation system (DES_m^1) if $f_v(X^*) = \overline{0}$ for $\forall v \in V(H)$.*

Clearly, $\overline{0}$ is a H-equilibrium point for any spanning subgraph H of $G[DES_m^1]$ by definition. Whence, its zero G-solution of linearized differential equation system $(LDES_m^1)$ is a solution of (DES_m^1).

Definition 3.2 *Let H be a spanning subgraph of $G[DES_m^1]$ of the linearized differential equation systems (DES_m^1) with initial value $X_v(0)$. Then $G[DES_m^1]$ is called sum-stable or asymptotically sum-stable on H if for all solutions $Y_v(t)$, $v \in V(H)$*

of (DES_m^1) *with* $\|Y_v(0) - X_v(0)\| < \delta_v$ *exists for all* $t \geq 0$,

$$\left\| \sum_{v \in V(H)} Y_v(t) - \sum_{v \in V(H)} X_v(t) \right\| < \varepsilon,$$

or furthermore,

$$\lim_{t \to 0} \left\| \sum_{v \in V(H)} Y_v(t) - \sum_{v \in V(H)} X_v(t) \right\| = 0,$$

and prod-stable or asymptotically prod-stable on H *if for all solutions* $Y_v(t)$, $v \in V(H)$ *of* (DES_m^1) *with* $\|Y_v(0) - X_v(0)\| < \delta_v$ *exists for all* $t \geq 0$,

$$\left\| \prod_{v \in V(H)} Y_v(t) - \prod_{v \in V(H)} X_v(t) \right\| < \varepsilon,$$

or furthermore,

$$\lim_{t \to 0} \left\| \prod_{v \in V(H)} Y_v(t) - \prod_{v \in V(H)} X_v(t) \right\| = 0.$$

Clearly, the asymptotically sum-stability or prod-stability implies respectively that the sum-stability or prod-stability.

Then we get the following result on the sum-stability and prod-stability of the zero G-solution of (DES_m^1).

Theorem 3.3 *For a G-solution $G[DES_m^1]$ of differential equation systems (DES_m^1) with initial value $X_v(0)$, let H_1, H_2 be spanning subgraphs of $G[DES_m^1]$. If the zero G-solution of (DES_m^1) is sum-stable or asymptotically sum-stable on H_1 and H_2, then the zero G-solution of (DES_m^1) is sum-stable or asymptotically sum-stable on $H_1 \bigcup H_2$.*

Similarly, if the zero G-solution of (DES_m^1) is prod-stable or asymptotically prod-stable on H_1 and $X_v(t)$ is bounded for $\forall v \in V(H_2)$, then the zero G-solution of (DES_m^1) is prod-stable or asymptotically prod-stable on $H_1 \bigcup H_2$.

Proof Notice that

$$\|X_1 + X_2\| \leq \|X_1\| + \|X_2\| \quad \text{and} \quad \|X_1 X_2\| \leq \|X_1\| \|X_2\|$$

in $\mathbf{R}^n$. We know that

$$\begin{aligned}\left\|\sum_{v\in V(H_1)\bigcup V(H_2)} X_v(t)\right\| &= \left\|\sum_{v\in V(H_1)} X_v(t) + \sum_{v\in V(H_2)} X_v(t)\right\| \\ &\le \left\|\sum_{v\in V(H_1)} X_v(t)\right\| + \left\|\sum_{v\in V(H_2)} X_v(t)\right\|\end{aligned}$$

and

$$\begin{aligned}\left\|\prod_{v\in V(H_1)\bigcup V(H_2)} X_v(t)\right\| &= \left\|\prod_{v\in V(H_1)} X_v(t) \prod_{v\in V(H_2)} X_v(t)\right\| \\ &\le \left\|\prod_{v\in V(H_1)} X_v(t)\right\| \left\|\prod_{v\in V(H_2)} X_v(t)\right\|.\end{aligned}$$

Whence,

$$\left\|\sum_{v\in V(H_1)\bigcup V(H_2)} X_v(t)\right\| \le \epsilon \quad \text{or} \quad \lim_{t\to 0}\left\|\sum_{v\in V(H_1)\bigcup V(H_2)} X_v(t)\right\| = 0$$

if $\epsilon = \epsilon_1 + \epsilon_2$ with

$$\left\|\sum_{v\in V(H_1)} X_v(t)\right\| \le \epsilon_1 \quad \text{and} \quad \left\|\sum_{v\in V(H_2)} X_v(t)\right\| \le \epsilon_2$$

or

$$\lim_{t\to 0}\left\|\sum_{v\in V(H_1)} X_v(t)\right\| = 0 \quad \text{and} \quad \lim_{t\to 0}\left\|\sum_{v\in V(H_2)} X_v(t)\right\| = 0.$$

This is the conclusion (1). For the conclusion (2), notice that

$$\left\|\prod_{v\in V(H_1)\bigcup V(H_2)} X_v(t)\right\| \le \left\|\prod_{v\in V(H_1)} X_v(t)\right\| \left\|\prod_{v\in V(H_2)} X_v(t)\right\| \le M\epsilon$$

if

$$\left\|\prod_{v\in V(H_1)} X_v(t)\right\| \le \epsilon \quad \text{and} \quad \left\|\prod_{v\in V(H_2)} X_v(t)\right\| \le M.$$

Consequently, the zero G-solution of (DES_m^1) is prod-stable or asymptotically prod-stable on $H_1\bigcup H_2$. □

Theorem 3.3 enables one to get the following conclusion which establishes the relation of stability of differential equations at vertices with that of sum-stability and prod-stability.

Corollary 3.4 *For a G-solution $G[DES_m^1]$ of differential equation system (DES_m^1) with initial value $X_v(0)$, let H be a spanning subgraph of $G[DES_m^1]$. If the zero solution is stable or asymptotically stable at each vertex $v \in V(H)$, then it is sum-stable, or asymptotically sum-stable and if the zero solution is stable or asymptotically stable in a vertex $u \in V(H)$ and $X_v(t)$ is bounded for $\forall v \in V(H) \setminus \{u\}$, then it is prod-stable, or asymptotically prod-stable on H.*

It should be noted that the converse of Theorem 3.3 is not always true. For example, let

$$\left\| \sum_{v \in V(H_1)} X_v(t) \right\| \leq a + \epsilon \quad \text{and} \quad \left\| \sum_{v \in V(H_2)} X_v(t) \right\| \leq -a + \epsilon.$$

Then the zero G-solution $G[DES_m^1]$ of differential equation system (DES_m^1) is not sum-stable on subgraphs H_1 and H_2, but

$$\left\| \sum_{v \in V(H_1 \bigcup H_2)} X_v(t) \right\| \leq \left\| \sum_{v \in V(H_1)} X_v(t) \right\| + \left\| \sum_{v \in V(H_2)} X_v(t) \right\| = \epsilon.$$

Thus the zero G-solution $G[DES_m^1]$ of differential equation system (DES_m^1) is sum-stable on subgraphs $H_1 \bigcup H_2$. Similarly, let

$$\left\| \prod_{v \in V(H_1)} X_v(t) \right\| \leq \frac{\epsilon}{t^r} \quad \text{and} \quad \left\| \sum_{v \in V(H_2)} X_v(t) \right\| \leq t^r$$

for a real number r. Then the zero G-solution $G[DES_m^1]$ of (DES_m^1) is not prod-stable on subgraphs H_1 and $X_v(t)$ is not bounded for $v \in V(H_2)$ if $r > 0$. However, it is prod-stable on subgraphs $H_1 \bigcup H_2$ for

$$\left\| \prod_{v \in V(H_1 \bigcup H_2)} X_v(t) \right\| \leq \left\| \prod_{v \in V(H_1)} X_v(t) \right\| \left\| \prod_{v \in V(H_2)} X_v(t) \right\| = \epsilon.$$

3.2 Linearized Differential Equations

Applying these conclusions on linear differential equation systems in the previous section, we can find conditions on $F_i(X),\ 1 \leq i \leq m$ for the sum-stability and prod-stability at $\overline{0}$ following. For this objective, we need the following useful result.

Lemma 3.5([13]) *Let $\dot{X} = AX + B(X)$ be a non-linear differential equation, where A is a constant $n \times n$ matrix and $\mathrm{Re}\lambda_i < 0$ for all eigenvalues λ_i of A and $B(X)$ is continuous defined on $t \geq 0,\ \|X\| \leq \alpha$ with*

$$\lim_{\|X\|\to 0} \frac{\|B(X)\|}{\|X\|} = 0.$$

Then there exist constants $c > 0,\ \beta > 0$ and $\delta,\ 0 < \delta < \alpha$ such that

$$\|X(0)\| \leq \varepsilon \leq \frac{\delta}{2c} \ \textit{ implies that } \ \|X(t)\| \leq c\varepsilon e^{-\beta t/2}.$$

Theorem 3.6 *Let (DES^1_m) be a non-linear differential equation system, H a spanning subgraph of $G[DES^1_m]$ and*

$$F_v(X) = F'_v\left(\overline{0}\right) X + R_v(X)$$

such that

$$\lim_{\|X\|\to \overline{0}} \frac{\|R_v(X)\|}{\|X\|} = 0$$

for $\forall v \in V(H)$. Then the zero G-solution of (DES^1_m) is asymptotically sum-stable or asymptotically prod-stable on H if $\mathrm{Re}\alpha_v < 0$ for each $\overline{\beta}_v(t)e^{\alpha_v t} \in \mathscr{B}_v,\ v \in V(H)$ in (DES^1_m).

Proof Define $c = \max\{c_v,\ v \in V(H)\}$, $\varepsilon = \min\{\varepsilon_v,\ v \in V(H)\}$ and $\beta = \min\{\beta_v,\ v \in V(H)\}$. Applying Lemma 3.5, we know that for $\forall v \in V(H)$,

$$\|X_v(0)\| \leq \varepsilon \leq \frac{\delta}{2c} \ \text{ implies that } \ \|X_v(t)\| \leq c\varepsilon e^{-\beta t/2}.$$

Whence,

$$\left\|\sum_{v\in V(H)} X_v(t)\right\| \leq \sum_{v\in V(H)} \|X_v(t)\| \leq |H| c\varepsilon e^{-\beta t/2}$$

$$\left\|\prod_{v\in V(H)} X_v(t)\right\| \leq \prod_{v\in V(H)} \|X_v(t)\| \leq c^{|H|}\varepsilon^{|H|} e^{-|H|\beta t/2}.$$

Consequently,

$$\lim_{t\to 0}\left\|\sum_{v\in V(H)} X_v(t)\right\| \to 0 \quad \text{and} \quad \lim_{t\to 0}\left\|\prod_{v\in V(H)} X_v(t)\right\| \to 0.$$

Thus the zero G-solution (DES_m^n) is asymptotically sum-stable or asymptotically prod-stable on H by definition. $\square$

3.3 Liapunov Functions on Graphs

We have know Liapunov functions associated with differential equations. Similarly, we introduce Liapunov functions for determining the sum-stability or prod-stability of (DES_m^1) following.

Definition 3.7 *Let (DES_m^1) be a differential equation system, $H < G[DES_m^1]$ a spanning subgraph and a H-equilibrium point X^* of (DES_m^1). A differentiable function $L : \mathscr{O} \to \mathbf{R}$ defined on an open subset $\mathscr{O} \subset \mathbf{R}^n$ is called a Liapunov sum-function on X^* for H if*

(1) $L(X^*) = 0$ *and* $L\left(\sum\limits_{v\in V(H)} X_v(t)\right) > 0$ *if* $\sum\limits_{v\in V(H)} X_v(t) \neq X^*$;

(2) $\dot{L}\left(\sum\limits_{v\in V(H)} X_v(t)\right) \leq 0$ *for* $\sum\limits_{v\in V(H)} X_v(t) \neq X^*$,

and a Liapunov prod-function on X^* for H if

(1) $L(X^*) = 0$ *and* $L\left(\prod\limits_{v\in V(H)} X_v(t)\right) > 0$ *if* $\prod\limits_{v\in V(H)} X_v(t) \neq X^*$;

(2) $\dot{L}\left(\prod\limits_{v\in V(H)} X_v(t)\right) \leq 0$ *for* $\prod\limits_{v\in V(H)} X_v(t) \neq X^*$.

Then, the following conclusions on the sum-stable and prod-stable of zero G-solutions of differential equations holds.

Theorem 3.8 *For a G-solution $G[DES_m^1]$ of a differential equation system (DES_m^1) with initial value $X_v(0)$, let H be a spanning subgraph of $G[DES_m^1]$ and X^* an equilibrium point of (DES_m^1) on H.*

(1) *If there is a Liapunov sum-function $L : \mathscr{O} \to \mathbf{R}$ on X^*, then the zero*

G-solution $G[DES_m^1]$ is sum-stable on X^* for H. Furthermore, if

$$\dot{L}\left(\sum_{v\in V(H)} X_v(t)\right) < 0$$

for $\sum\limits_{v\in V(H)} X_v(t) \neq X^*$, then the zero G-solution $G[DES_m^1]$ is asymptotically sum-stable on X^* for H.

(2) *If there is a Liapunov prod-function $L : \mathscr{O} \to \mathbf{R}$ on X^* for H, then the zero G-solution $G[DES_m^1]$ is prod-stable on X^* for H. Furthermore, if*

$$\dot{L}\left(\prod_{v\in V(H)} X_v(t)\right) < 0$$

for $\prod\limits_{v\in V(H)} X_v(t) \neq X^*$, then the zero G-solution $G[DES_m^1]$ is asymptotically prod-stable on X^* for H.

Proof Let $\epsilon > 0$ be a so small number that the closed ball $B_\epsilon(X^*)$ centered at X^* with radius ϵ lies entirely in $\mathscr{O}$ and ϖ the minimum value of L on the boundary of $B_\epsilon(X^*)$, i.e., the sphere $S_\epsilon(X^*)$. Clearly, $\varpi > 0$ by assumption. Define $U = \{X \in B_\epsilon(X^*)|L(X) < \varpi\}$. Notice that $X^* \in U$ and L is non-increasing on $\sum\limits_{v\in V(H)} X_v(t)$ by definition. Whence, there are no solutions $X_v(t),\ v \in V(H)$ starting in U such that $\sum\limits_{v\in V(H)} X_v(t)$ meet the sphere $S_\epsilon(X^*)$. Thus all solutions $X_v(t),\ v \in V(H)$ starting in U enable $\sum\limits_{v\in V(H)} X_v(t)$ included in ball $B_\epsilon(X^*)$. Consequently, the zero G-solution $G[DES_m^1]$ is sum-stable on H by definition.

Now assume that

$$\dot{L}\left(\sum_{v\in V(H)} X_v(t)\right) < 0$$

for $\sum\limits_{v\in V(H)} X_v(t) \neq X^*$. Thus L is strictly decreasing on $\sum\limits_{v\in V(H)} X_v(t)$. If $X_v(t),\ v \in V(H)$ are solutions starting in $U - X^*$ such that $\sum\limits_{v\in V(H)} X_v(t_n) \to Y^*$ for $n \to \infty$ with $Y^* \in B_\epsilon(X^*)$, then it must be $Y^* = X^*$. Otherwise, since

$$L\left(\sum_{v\in V(H)} X_v(t)\right) > L(Y^*)$$

by the assumption

$$\dot{L}\left(\sum_{v\in V(H)} X_v(t)\right) < 0$$

for all $\sum\limits_{v\in V(H)} X_v(t) \neq X^*$ and

$$L\left(\sum_{v\in V(H)} X_v(t_n)\right) \to L(Y^*)$$

by the continuity of L, if $Y^* \neq X^*$, let $Y_v(t), v \in V(H)$ be the solutions starting at Y^*. Then for any $\eta > 0$,

$$L\left(\sum_{v\in V(H)} Y_v(\eta)\right) < L(Y^*).$$

But then there is a contradiction

$$L\left(\sum_{v\in V(H)} X_v(t_n+\eta)\right) < L(Y^*)$$

yields by letting $Y_v(0) = \sum\limits_{v\in V(H)} X_v(t_n)$ for sufficiently large n. Thus, there must be $Y_v^* = X^*$. Whence, the zero G-solution $G[DES_m^1]$ is asymptotically sum-stable on H by definition. This is the conclusion (1).

Similarly, we can prove the conclusion (2). □

The following result shows the combination of Liapunov sum-functions or prod-functions.

Theorem 3.9 *For a G-solution $G[DES_m^1]$ of a differential equation system (DES_m^1) with initial value $X_v(0)$, let H_1, H_2 be spanning subgraphs of $G[DES_m^1]$, X^* an equilibrium point of (DES_m^1) on $H_1 \bigcup H_2$ and*

$$R^+(x,y) = \sum_{i\geq 0, j\geq 0} a_{i,j} x^i y^j$$

be a polynomial with $a_{i,j} \geq 0$ for integers $i,j \geq 0$. Then $R^+(L_1, L_2)$ is a Liapunov sum-function or Liapunov prod-function on X^* for $H_1 \bigcup H_2$ with conventions for *integers $i,j,k,l \geq 0$ that*

$$a_{ij}L_1^iL_2^j\left(\sum_{v\in V(H_1\bigcup V(H_2)}X_v(t)\right)+a_{kl}L_1^kL_2^l\left(\sum_{v\in V(H_1\bigcup V(H_2)}X_v(t)\right)$$
$$=a_{ij}L_1^i\left(\sum_{v\in V(H_1)}X_v(t)\right)L_2^j\left(\sum_{v\in V(H_2)}X_v(t)\right)$$
$$+a_{kl}L_1^k\left(\sum_{v\in V(H_1)}X_v(t)\right)L_2^l\left(\sum_{v\in V(H_2)}X_v(t)\right)$$

if L_1, L_2 are Liapunov sum-functions and

$$a_{ij}L_1^iL_2^j\left(\prod_{v\in V(H_1\bigcup V(H_2)}X_v(t)\right)+a_{kl}L_1^kL_2^l\left(\prod_{v\in V(H_1\bigcup V(H_2)}X_v(t)\right)$$
$$=a_{ij}L_1^i\left(\prod_{v\in V(H_1)}X_v(t)\right)L_2^j\left(\prod_{v\in V(H_2)}X_v(t)\right)$$
$$+a_{kl}L_1^k\left(\prod_{v\in V(H_1)}X_v(t)\right)L_2^l\left(\prod_{v\in V(H_2)}X_v(t)\right)$$

if L_1, L_2 are Liapunov prod-functions on X^* for H_1 and H_2, respectively. Particularly, if there is a Liapunov sum-function (Liapunov prod-function) L on H_1 and H_2, then L is also a Liapunov sum-function (Liapunov prod-function) on $H_1\bigcup H_2$.

Proof Notice that

$$\frac{d\left(a_{ij}L_1^iL_2^j\right)}{dt}=a_{ij}\left(iL_1^{i-1}\dot{L}_1L_2^j+jL_1^iL_1^{j-1}\dot{L}_2\right)$$

if $i,j\geq 1$. Whence,

$$a_{ij}L_1^iL_2^j\left(\sum_{v\in V(H_1\bigcup V(H_2)}X_v(t)\right)\geq 0$$

if

$$L_1\left(\sum_{v\in V(H_1)}X_v(t)\right)\geq 0 \quad\text{and}\quad L_2\left(\sum_{v\in V(H_2)}X_v(t)\right)\geq 0$$

and

$$\frac{d(a_{ij}L_1^iL_2^j)}{dt}\left(\sum_{v\in V(H_1\bigcup V(H_2)}X_v(t)\right)\leq 0$$

if

$$\dot{L}_1\left(\sum_{v\in V(H_1)} X_v(t)\right) \leq 0 \ \text{ and } \ \dot{L}_2\left(\sum_{v\in V(H_2)} X_v(t)\right) \leq 0.$$

Thus $R^+(L_1, L_2)$ is a Liapunov sum-function on X^* for $H_1 \bigcup H_2$.

Similarly, we can know that $R^+(L_1, L_2)$ is a Liapunov prod-function on X^* for $H_1 \bigcup H_2$ if L_1, L_2 are Liapunov prod-functions on X^* for H_1 and H_2. □

Theorem 3.9 enables one easily to get the stability of the zero G-solutions of (DES_m^1).

Corollary 3.10 *For a differential equation system (DES_m^1), let $H < G[DES_m^1]$ be a spanning subgraph. If L_v is a Liapunov function on vertex v for $\forall v \in V(H)$, then the functions*

$$L_S^H = \sum_{v\in V(H)} L_v \ \ \textit{and} \ \ L_P^H = \prod_{v\in V(H)} L_v$$

are respectively Liapunov sum-function and Liapunov prod-function on graph H. Particularly, if $L = L_v$ for $\forall v \in V(H)$, then L is both a Liapunov sum-function and a Liapunov prod-function on H.

Example 3.11 Let (DES_m^1) be determined by

$$\left\{\begin{array}{c} dx_1/dt = \lambda_{11}x_1 \\ dx_2/dt = \lambda_{12}x_2 \\ \cdots\cdots\cdots \\ dx_n/dt = \lambda_{1n}x_n \end{array}\right. \quad \left\{\begin{array}{c} dx_1/dt = \lambda_{21}x_1 \\ dx_2/dt = \lambda_{22}x_2 \\ \cdots\cdots\cdots \\ dx_n/dt = \lambda_{2n}x_n \end{array}\right. \quad \cdots \quad \left\{\begin{array}{c} dx_1/dt = \lambda_{n1}x_1 \\ dx_2/dt = \lambda_{n2}x_2 \\ \cdots\cdots\cdots \\ dx_n/dt = \lambda_{nn}x_n \end{array}\right.$$

where all $\lambda_{ij},\ 1 \leq i \leq m, 1 \leq j \leq n$ are real and $\lambda_{ij_1} \neq \lambda_{ij_2}$ if $j_1 \neq j_2$ for integers $1 \leq i \leq m$. Let $L = x_1^2 + x_2^2 + \cdots + x_n^2$. Then

$$\dot{L} = \lambda_{i1}x_1^2 + \lambda_{i2}x_2^2 + \cdots + \lambda_{in}x_n^2$$

for integers $1 \leq i \leq n$. Whence, it is a Liapunov function for the ith differential equation if $\lambda_{ij} < 0$ for integers $1 \leq j \leq n$. Now let $H < G[LDES_m^1]$ be a spanning subgraph of $G[LDES_m^1]$. Then L is both a Liapunov sum-function and a Liapunov prod-function on H if $\lambda_{vj} < 0$ for $\forall v \in V(H)$ by Corollaries 3.10.

Theorem 3.12 *Let $L : \mathscr{O} \to \mathbf{R}$ be a differentiable function with $L(\overline{0}) = 0$ and $L\left(\sum\limits_{v\in V(H)} X\right) > 0$ always holds in an area of its ϵ-neighborhood $U(\epsilon)$ of $\overline{0}$ for $\varepsilon > 0$,* *denoted by* $U^+(\overline{0},\varepsilon)$ *such area of* ε*-neighborhood of* $\overline{0}$ *with* $L\left(\sum\limits_{v\in V(H)} X\right) > 0$ *and* $H < G[DES^1_m]$ *be a spanning subgraph.*

(1) *If*

$$\left\|L\left(\sum_{v\in V(H)} X\right)\right\| \leq M$$

with M *a positive number and*

$$\dot{L}\left(\sum_{v\in V(H)} X\right) > 0$$

in $U^+(\overline{0},\epsilon)$*, and for* $\forall\epsilon > 0$*, there exists a positive number* c_1, c_2 *such that*

$$L\left(\sum_{v\in V(H)} X\right) \geq c_1 > 0 \ \ \textit{implies} \ \ \dot{L}\left(\sum_{v\in V(H)} X\right) \geq c_2 > 0,$$

then the zero G-solution $G[DES^1_m]$ *is not sum-stable on* H*. Such a function* $L : \mathscr{O} \to \mathbf{R}$ *is called a non-Liapunov sum-function on* H*.*

(2) *If*

$$\left\|L\left(\prod_{v\in V(H)} X\right)\right\| \leq N$$

with N *a positive number and*

$$\dot{L}\left(\prod_{v\in V(H)} X\right) > 0$$

in $U^+(\overline{0},\epsilon)$*, and for* $\forall\epsilon > 0$*, there exists positive numbers* d_1, d_2 *such that*

$$L\left(\prod_{v\in V(H)} X\right) \geq d_1 > 0 \ \ \textit{implies} \ \ \dot{L}\left(\prod_{v\in V(H)} X\right) \geq d_2 > 0,$$

then the zero G-solution $G[DES^1_m]$ *is not prod-stable on* H*. Such a function* $L : \mathscr{O} \to \mathbf{R}$ *is called a non-Liapunov prod-function on* H*.*

Proof Generally, if $\|L(X)\|$ is bounded and $\dot{L}(X) > 0$ in $U^+(\overline{0},\epsilon)$, and for $\forall\epsilon > 0$, there exists positive numbers c_1, c_2 such that if $L(X) \geq c_1 > 0$, then

$\dot{L}(X) \geq c_2 > 0$, we prove that there exists $t_1 > t_0$ such that $\|X(t_1, t_0)\| > \epsilon_0$ for a number $\epsilon_0 > 0$, where $X(t_1, t_0)$ denotes the solution of (DES^n_m) passing through $X(t_0)$. Otherwise, there must be $\|X(t_1, t_0)\| < \epsilon_0$ for $t \geq t_0$. By $\dot{L}(X) > 0$ we know that $L(X(t)) > L(X(t_0)) > 0$ for $t \geq t_0$. Combining this fact with the condition $\dot{L}(X) \geq c_2 > 0$, we get that

$$L(X(t)) = L(X(t_0)) + \int_{t_0}^{t} \frac{dL(X(s))}{ds} \geq L(X(t_0)) + c_2(t - t_0).$$

Thus $L(X(t)) \to +\infty$ if $t \to +\infty$, a contradiction to the assumption that $L(X)$ is bounded. Whence, there exists $t_1 > t_0$ such that $\|X(t_1, t_0)\| > \epsilon_0$.

Applying this conclusion, we immediately know that the zero G-solution $G[DES^1_m]$ is not sum-stable or prod-stable on H by conditions in (1) or (2). □

Similar to Theorem 3.9, we know results for non-Liapunov sum-function or prod-function by Theorem 3.12 following.

Theorem 3.13 *For a G-solution $G[DES^1_m]$ of a differential equation system (DES^1_m) with initial value $X_v(0)$, let H_1, H_2 be spanning subgraphs of $G[DES^1_m]$, $\overline{0}$ an equilibrium point of (DES^1_m) on $H_1 \bigcup H_2$. Then $R^+(L_1, L_2)$ is a non-Liapunov sum-function or non-Liapunov prod-function on $\overline{0}$ for $H_1 \bigcup H_2$ with conventions for*

$$a_{ij} L_1^i L_2^j \left(\sum_{v \in V(H_1 \bigcup V(H_2)} X_v(t) \right) + a_{kl} L_1^k L_2^l \left(\sum_{v \in V(H_1 \bigcup V(H_2)} X_v(t) \right)$$

and

$$a_{ij} L_1^i L_2^j \left(\prod_{v \in V(H_1 \bigcup V(H_2)} X_v(t) \right) + a_{kl} L_1^k L_2^l \left(\prod_{v \in V(H_1 \bigcup V(H_2)} X_v(t) \right)$$

the same as in Theorem 3.9 if L_1, L_2 are non-Liapunov sum-functions or non-Liapunov prod-functions on $\overline{0}$ for H_1 and H_2, respectively. Particularly, if there is a non-Liapunov sum-function (non-Liapunov prod-function) L on H_1 and H_2, then L is also a non-Liapunov sum-function (non-Liapunov prod-function) on $H_1 \bigcup H_2$.

Proof Similarly, we can show that $R^+(L_1, L_2)$ satisfies these conditions on $H_1 \bigcup H_2$ for non-Liapunov sum-functions or non-Liapunov prod-functions in Theorem 3.12 if L_1, L_2 are non-Liapunov sum-functions or non-Liapunov prod-functions on $\overline{0}$ for H_1 and H_2, respectively. Thus $R^+(L_1, L_2)$ is a non-Liapunov sum-function or non-Liapunov prod-function on $\overline{0}$. □

Corollary 3.14 *For a differential equation system (DES_m^1), let $H < G[DES_m^1]$ be a spanning subgraph. If L_v is a non-Liapunov function on vertex v for $\forall v \in V(H)$, then the functions*

$$L_S^H = \sum_{v\in V(H)} L_v \quad and \quad L_P^H = \prod_{v\in V(H)} L_v$$

are respectively non-Liapunov sum-function and non-Liapunov prod-function on graph H. Particularly, if $L = L_v$ for $\forall v \in V(H)$, then L is both a non-Liapunov sum-function and a non-Liapunov prod-function on H.

Example 3.15 Let (DES_m^1) be

$$\begin{cases} \dot{x}_1 = \lambda_1 x_1^2 - \lambda_1 x_2^2 \\ \dot{x}_2 = \dfrac{\lambda_1}{2} x_1 x_2 \end{cases} \quad \begin{cases} \dot{x}_2 = \lambda_2 x_1^2 - \lambda_2 x_2^2 \\ \dot{x}_2 = \dfrac{\lambda_2}{2} x_1 x_2 \end{cases} \quad \cdots \quad \begin{cases} \dot{x}_1 = \lambda_m x_1^2 - \lambda_m x_2^2 \\ \dot{x}_2 = \dfrac{\lambda_m}{2} x_1 x_2 \end{cases}$$

with constants $\lambda_i > 0$ for integers $1 \leq i \leq m$ and $L(x_1, x_2) = x_1^2 - 2x_2^2$. Then $\dot{L}(x_1, x_2) = 4\lambda_i x_1 L(x_1, x_2)$ for the i-th equation in (DES_m^1). Calculation shows that $L(x_1, x_2) > 0$ if $x_1 > \sqrt{2}x_2$ or $x_1 < -\sqrt{2}x_2$ and $\dot{L}(x_1, x_2) > 4c^{\frac{3}{2}}$ for $L(x_1, x_2) > c$ in the area of $L(x_1, x_2) > 0$. Applying Theorem 3.12, we know the zero solution of (DES_m^1) is not stable for the i-th equation for any integer $1 \leq i \leq m$. Applying Corollary 3.14, we know that L is a non-Liapunov sum-function and non-Liapunov prod-function on any spanning subgraph $H < G[DES_m^1]$.

§4. Non-Solvable Spaces of Shifted Differential Equations

The differential equation systems (DES_m^1) discussed in previous sections are all in a same Euclidean space $\mathbf{R}^n$. We consider the case that they are not in a same space $\mathbf{R}^n$, i.e., shifted differential equation systems in this section. These differential equation systems and their non-solvability are defined in the following.

Definition 4.1 *A shifted differential equation system $(SDES_m^1)$ is such a differential equation system*

$$\dot{X}_1 = F_1(X_1),\ \dot{X}_2 = F_2(X_2), \cdots, \dot{X}_m = F_m(X_m) \qquad (SDES_m^1)$$

with

$$X_1 = (x_1, x_2, \cdots, x_l, x_{1(l+1)}, x_{1(l+2)}, \cdots, x_{1n}),$$
$$X_2 = (x_1, x_2, \cdots, x_l, x_{2(l+1)}, x_{2(l+2)}, \cdots, x_{2n}),$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots,$$
$$X_m = (x_1, x_2, \cdots, x_l, x_{m(l+1)}, x_{m(l+2)}, \cdots, x_{mn}),$$

where $x_1, x_2, \cdots, x_l, x_{i(l+j)}, 1 \leq i \leq m, 1 \leq j \leq n - l$ *are distinct variables and* $F_s : \mathbf{R}^n \to \mathbf{R}^n$ *is continuous such that* $F_s(\overline{0}) = \overline{0}$ *for integers* $1 \leq s \leq m$.

A shifted differential equation system $(SDES^1_m)$ *is non-solvable if there are integers* $i, j,\ 1 \leq i, j \leq m$ *and an integer* $k,\ 1 \leq k \leq l$ *such that* $x_k^{[i]}(t) \neq x_k^{[j]}(t)$, *where* $x_k^{[i]}(t), x_k^{[j]}(t)$ *are solutions* $x_k(t)$ *of the* i*-th and* j*-th equations in* $(SDES^1_m)$, *respectively.*

The number $\dim(SDES^1_m)$ of variables $x_1, x_2, \cdots, x_l, x_{i(l+j)}, 1 \leq i \leq m, 1 \leq j \leq n - l$ in Definition 4.1 is uniquely determined by $(SDES^1_m)$, i.e., $\dim(SDES^1_m) = mn-(m-1)l$. For classifying and finding the stability of these differential equations, we similarly introduce the linearized basis graphs $G[SDES^1_m]$ of a shifted differential equation system to that of (DES^1_m), i.e., a vertex-edge labeled graph with

$$V(G[SDES^1_m]) = \{\mathscr{B}_i | 1 \leq i \leq m\},$$
$$E(G[SDES^1_m]) = \{(\mathscr{B}_i, \mathscr{B}_j) | \mathscr{B}_i \bigcap \mathscr{B}_j \neq \emptyset, 1 \leq i, j \leq m\},$$

where $\mathscr{B}_i$ is the solution basis of the i-th linearized differential equation $\dot{X}_i = F_i'(\overline{0})X_i$ for integers $1 \leq i \leq m$, called such a vertex-edge labeled graph $G[SDES^1_m]$ the *G-solution* of $(SDES^1_m)$ and its zero G-solution replaced $\mathscr{B}_i$ by $(0, \cdots, 0)$ ($|\mathscr{B}_i|$ times) and $\mathscr{B}_i \bigcap \mathscr{B}_j$ by $(0, \cdots, 0)$ ($|\mathscr{B}_i \bigcap \mathscr{B}_j|$ times) for integers $1 \leq i, j \leq m$.

Let $(LDES^1_m)$, $(LDES^1_m)'$ be linearized differential equation systems of shifted differential equation systems $(SDES^1_m)$ and $(SDES^1_m)$ with G-solutions H, H'. Similarly, they are called combinatorially equivalent if there is an isomorphism $\varphi : H \to H'$ of graph and labelings θ, τ on H and H' respectively such that $\varphi\theta(x) = \tau\varphi(x)$ for $\forall x \in V(H)\bigcup E(H)$, denoted by $(SDES^1_m) \overset{\varphi}{\simeq} (SDES^1_m)'$. Notice that if we remove these superfluous variables from $G[SDES^1_m]$, then we get nothing but the same vertex-edge labeled graph of $(LDES^1_m)$ in $\mathbf{R}^l$. Thus we can classify shifted differential similarly to $(LDES^1_m)$ in $\mathbf{R}^l$. The following result can be proved similarly to Theorem 2.14.

Theorem 4.2 *Let* $(LDES^1_m)$, $(LDES^1_m)'$ *be linearized differential equation systems* *of two shifted differential equation systems* $(SDES_m^1)$, $(SDES_m^1)'$ *with integral labeled graphs* H, H'. *Then* $(SDES_m^1) \stackrel{\varphi}{\simeq} (SDES_m^1)'$ *if and only if* $H = H'$.

The stability of these shifted differential equation systems $(SDES_m^1)$ is also similarly to that of (DES_m^1). For example, we know the results on the stability of $(SDES_m^1)$ similar to Theorems 2.22, 2.27 and 3.6 following.

Theorem 4.3 *Let* $(LDES_m^1)$ *be a shifted linear differential equation systems and* $H < G[LDES_m^1]$ *a spanning subgraph. A zero G-solution of* $(LDES_m^1)$ *is asymptotically sum-stable on* H *if and only if* $\mathrm{Re}\alpha_v < 0$ *for each* $\overline{\beta}_v(t)e^{\alpha_v t} \in \mathscr{B}_v$ *in* $(LDES^1)$ *hold for* $\forall v \in V(H)$ *and it is asymptotically prod-stable on* H *if and only if* $\sum\limits_{v\in V(H)} \mathrm{Re}\alpha_v < 0$ *for each* $\overline{\beta}_v(t)e^{\alpha_v t} \in \mathscr{B}_v$ *in* $(LDES^1)$.

Theorem 4.4 *Let* $(SDES_m^1)$ *be a shifted differential equation system,* $H < G[SDES_m^1]$ *a spanning subgraph and*

$$F_v(X) = F_v'\left(\overline{0}\right) X + R_v(X)$$

such that

$$\lim_{\|X\|\to\overline{0}} \frac{\|R_v(X)\|}{\|X\|} = 0$$

for $\forall v \in V(H)$. *Then the zero G-solution of* $(SDES_m^1)$ *is asymptotically sum-stable or asymptotically prod-stable on* H *if* $\mathrm{Re}\alpha_v < 0$ *for each* $\overline{\beta}_v(t)e^{\alpha_v t} \in \mathscr{B}_v$, $v \in V(H)$ *in* $(SDES_m^1)$.

For the Liapunov sum-function or Liapunov prod-function of a shifted differential equation system $(SDES_m^1)$, we choose it to be a differentiable function $L : \mathscr{O} \subset \mathbf{R}^{\dim(SDES_m^1)} \to \mathbf{R}$ with conditions in Definition 3.7 hold. Then we know the following result similar to Theorem 3.8.

Theorem 4.5 *For a G-solution* $G[SDES_m^1]$ *of a shifted differential equation system* $(SDES_m^1)$ *with initial value* $X_v(0)$, *let* H *be a spanning subgraph of* $G[DES_m^1]$ *and* X^* *an equilibrium point of* $(SDES_m^1)$ *on* H.

(1) *If there is a Liapunov sum-function* $L : \mathscr{O} \subset \mathbf{R}^{\dim(SDES_m^1)} \to \mathbf{R}$ *on* X^*, *then the zero G-solution* $G[SDES_m^1]$ *is sum-stable on* X^* *for* H, *and furthermore, if*

$$\dot{L}\left(\sum_{v\in V(H)} X_v(t)\right) < 0$$

for $\sum\limits_{v\in V(H)} X_v(t) \neq X^*$*, then the zero G-solution* $G[SDES^1_m]$ *is asymptotically sum-stable on* X^* *for* H*.*

(2) *If there is a Liapunov prod-function* $L : \mathscr{O} \subset \mathbf{R}^{\dim(SDES^1_m)} \to \mathbf{R}$ *on* X^* *for* H*, then the zero G-solution* $G[SDES^1_m]$ *is prod-stable on* X^* *for* H*, and furthermore, if*

$$\dot{L}\left(\prod_{v\in V(H)} X_v(t)\right) < 0$$

for $\prod\limits_{v\in V(H)} X_v(t) \neq X^*$*, then the zero G-solution* $G[SDES^1_m]$ *is asymptotically prod-stable on* X^* *for* H*.*

§5. Applications

5.1 Global Control of Infectious Diseases

An immediate application of non-solvable differential equations is the globally control of infectious diseases with more than one infectious virus in an area. Assume that there are three kind groups in persons at time t, i.e., infected $I(t)$, susceptible $S(t)$ and recovered $R(t)$, and the total population is constant in that area. We consider two cases of virus for infectious diseases:

Case 1 *There are* m *known virus* $\mathscr{V}_1, \mathscr{V}_2, \cdots, \mathscr{V}_m$ *with infected rate* k_i*, heal rate* h_i *for integers* $1 \leq i \leq m$ *and an person infected a virus* $\mathscr{V}_i$ *will never infects other viruses* $\mathscr{V}_j$ *for* $j \neq i$*.*

Case 2 *There are* m *varying* $\mathscr{V}_1, \mathscr{V}_2, \cdots, \mathscr{V}_m$ *from a virus* $\mathscr{V}$ *with infected rate* k_i*, heal rate* h_i *for integers* $1 \leq i \leq m$ *such as those shown in Fig.*5.1*.*

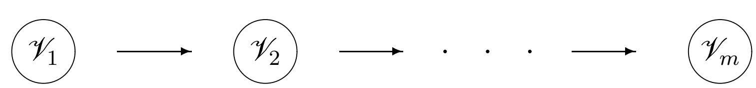

Fig.5.1

We are easily to establish a non-solvable differential model for the spread of infectious viruses by applying the SIR model of one infectious disease following:

$$\begin{cases} \dot{S} = -k_1 SI \\ \dot{I} = k_1 SI - h_1 I \\ \dot{R} = h_1 I \end{cases} \quad \begin{cases} \dot{S} = -k_2 SI \\ \dot{I} = k_2 SI - h_2 I \\ \dot{R} = h_2 I \end{cases} \quad \cdots \quad \begin{cases} \dot{S} = -k_m SI \\ \dot{I} = k_m SI - h_m I \\ \dot{R} = h_m I \end{cases} \qquad (DES_m^1)$$

Notice that the total population is constant by assumption, i.e., $S + I + R$ is constant. Thus we only need to consider the following simplified system

$$\begin{cases} \dot{S} = -k_1 SI \\ \dot{I} = k_1 SI - h_1 I \end{cases} \quad \begin{cases} \dot{S} = -k_2 SI \\ \dot{I} = k_2 SI - h_2 I \end{cases} \quad \cdots \quad \begin{cases} \dot{S} = -k_m SI \\ \dot{I} = k_m SI - h_m I \end{cases} \qquad (DES_m^1)$$

The equilibrium points of this system are $I = 0$, the S-axis with linearization at equilibrium points

$$\begin{cases} \dot{S} = -k_1 S \\ \dot{I} = k_1 S - h_1 \end{cases} \quad \begin{cases} \dot{S} = -k_2 S \\ \dot{I} = k_2 S - h_2 \end{cases} \quad \cdots \quad \begin{cases} \dot{S} = -k_m S \\ \dot{I} = k_m S - h_m \end{cases} \qquad (LDES_m^1)$$

Calculation shows that the eigenvalues of the ith equation are 0 and $k_i S - h_i$, which is negative, i.e., stable if $0 < S < h_i/k_i$ for integers $1 \le i \le m$. For any spanning subgraph $H < G[LDES_m^1]$, we know that its zero G-solution is asymptotically sum-stable on H if $0 < S < h_v/k_v$ for $v \in V(H)$ by Theorem 2.22, and it is asymptotically sum-stable on H if

$$\sum_{v \in V(H)} (k_v S - h_v) < 0 \qquad \text{i.e.,} \qquad 0 < S < \sum_{v \in V(H)} h_v \Bigg/ \sum_{v \in V(H)} k_v$$

by Theorem 2.27. Notice that if $I_i(t),\ S_i(t)$ are probability functions for infectious viruses $\mathscr{V}_i,\ 1 \le i \le m$ in an area, then $\prod_{i=1}^{m} I_i(t)$ and $\prod_{i=1}^{m} S_i(t)$ are just the probability functions for all these infectious viruses. This fact enables one to get the conclusion following for globally control of infectious diseases.

Conclusion 5.1 *For m infectious viruses $\mathscr{V}_1, \mathscr{V}_2, \cdots, \mathscr{V}_m$ in an area with infected rate k_i, heal rate h_i for integers $1 \le i \le m$, then they decline to 0 finally if*

$$0 < S < \sum_{i=1}^{m} h_i \Bigg/ \sum_{i=1}^{m} k_i \ ,$$

i.e., these infectious viruses are globally controlled. Particularly, they are globally controlled if each of them is controlled in this area.

5.2 Dynamical Equations of Instable Structure

There are two kind of engineering structures, i.e., stable and instable. An engineering structure is *instable* if its state moving further away and the equilibrium is upset after being moved slightly. For example, the structure (a) is engineering stable but (b) is not shown in Fig.5.2,

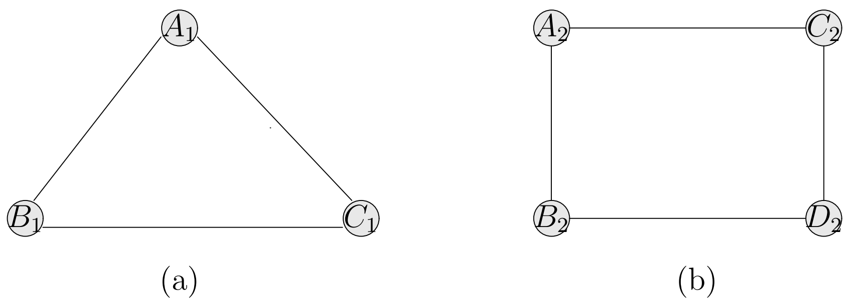

Fig.5.2

where each edge is a rigid body and each vertex denotes a hinged connection. The motion of a stable structure can be characterized similarly as a rigid body. But such a way can not be applied for instable structures for their internal deformations such as those shown in Fig.5.3.

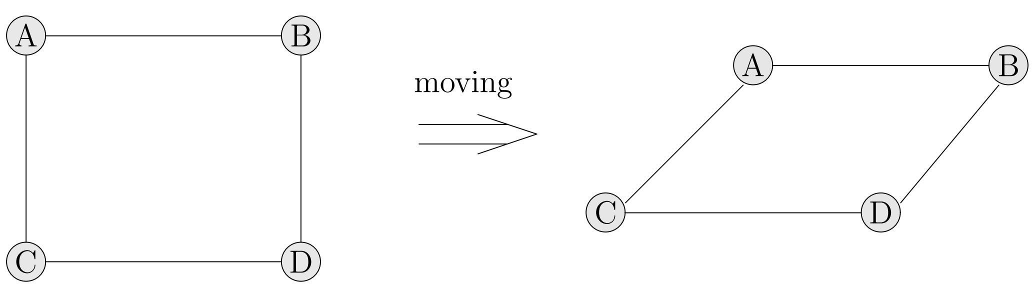

Fig.5.3

Furthermore, let $\mathscr{P}_1, \mathscr{P}_2, \cdots, \mathscr{P}_m$ be m particles in $\mathbf{R}^3$ with some relations, for instance, the gravitation between particles $\mathscr{P}_i$ and $\mathscr{P}_j$ for $1 \leq i, j \leq m$. Thus we get an instable structure underlying a graph G with

$$V(G) = \{\mathscr{P}_1, \mathscr{P}_2, \cdots, \mathscr{P}_m\};$$

$$E(G) = \{(\mathscr{P}_i, \mathscr{P}_j) | \text{there exists a relation between } \mathscr{P}_i \text{ and } \mathscr{P}_j\}.$$

For example, the underlying graph in Fig.5.4 is C_4. Assume the dynamical behavior of particle $\mathscr{P}_i$ at time t has been completely characterized by the differential equations $\dot{X} = F_i(X,t)$, where $X = (x_1, x_2, x_3)$. Then we get a non-solvable differential equation system

$$\dot{X} = F_i(X,t), \quad 1 \leq i \leq m$$

underlying the graph G. Particularly, if all differential equations are autonomous, i.e., depend on X alone, not on time t, we get a non-solvable autonomous differential equation system

$$\dot{X} = F_i(X), \quad 1 \leq i \leq m.$$

All of these differential equation systems particularly answer a question presented in [3] for establishing the graph dynamics, and if they satisfy conditions in Theorems 2.22, 2.27 or 3.6, then they are sum-stable or prod-stable. For example, let the motion equations of 4 members in Fig.5.3 be respectively

$$\text{AB}: \ddot{X}_{AB} = 0; \quad \text{CD}: \ddot{X}_{CD} = 0, \quad \text{AC}: \ddot{X}_{AC} = a_{AC}, \quad \text{BC}: \ddot{X}_{BC} = a_{BC},$$

where X_{AB}, X_{CD}, X_{AC} and X_{BC} denote central positions of members AB, CD, AC, BC and a_{AC}, a_{BC} are constants. Solving these equations enable one to get

$$X_{AB} = c_{AB}t + d_{AB}, \quad X_{AC} = a_{AC}t^2 + c_{AC}t + d_{AC},$$
$$X_{CD} = c_{CD}t + d_{CD}, \quad X_{BC} = a_{BC}t^2 + c_{BC}t + d_{BC},$$

where $c_{AB}, c_{AC}, c_{CD}, c_{BC}, d_{AB}, d_{AC}, d_{CD}, d_{BC}$ are constants. Thus we get a non-solvable differential equation system

$$\ddot{X} = 0; \quad \ddot{X} = 0, \quad \ddot{X} = a_{AC}, \quad \ddot{X} = a_{BC},$$

or a non-solvable algebraic equation system

$$X = c_{AB}t + d_{AB}, \quad X = a_{AC}t^2 + c_{AC}t + d_{AC},$$
$$X = c_{CD}t + d_{CD}, \quad X = a_{BC}t^2 + c_{BC}t + d_{BC}$$

for characterizing the behavior of the instable structure in Fig.5.3 if constants $c_{AB}, c_{AC}, c_{CD}, c_{BC}, d_{AB}, d_{AC}, d_{CD}, d_{BC}$ are different.

Now let $X_1, X_2, \cdots, X_m$ be the respectively positions in $\mathbf{R}^3$ with initial values $X_1^0, X_2^0, \cdots, X_m^0$, $\dot{X}_1^0$, $\dot{X}_2^0, \cdots, \dot{X}_m^0$ and $M_1, M_2, \cdots, M_m$ the masses of particles $\mathscr{P}_1, \mathscr{P}_2, \cdots, \mathscr{P}_m$. If $m = 2$, then from Newton's law of gravitation we get that

$$\ddot{X}_1 = GM_2 \frac{X_2 - X_1}{|X_2 - X_1|^3}, \qquad \ddot{X}_2 = GM_1 \frac{X_1 - X_2}{|X_1 - X_2|^3},$$

where G is the gravitational constant. Let $X = X_2 - X_1 = (x_1, x_2, x_3)$. Calculation shows that

$$\ddot{X} = -G\left(M_1 + M_2\right)\frac{X}{|X|^3}.$$

Such an equation can be completely solved by introducing the spherical polar coordinates

$$\begin{cases} x_1 = r\cos\phi\cos\theta \\ x_2 = r\cos\phi\cos\theta \\ x_3 = r\sin\theta \end{cases}$$

with $r \geq 0, 0 \leq \phi \leq \pi, 0 \leq \theta < 2\pi$, where $r = \|X\|$, $\phi = \angle Xoz, \theta = \angle X'ox$ with X' the projection of X in the plane xoy are parameters with $r = \alpha/(1 + \epsilon\cos\phi)$ hold for some constants α, ϵ. Whence,

$$X_1(t) = GM_2 \int \left(\int \frac{X}{|X|^3} dt \right) dt \ \text{ and } \ X_2(t) = -GM_1 \int \left(\int \frac{X}{|X|^3} dt \right) dt.$$

Notice the additivity of gravitation between particles. The gravitational action of particles $\mathscr{P}_1, \mathscr{P}_2, \cdots, \mathscr{P}_m$ on $\mathscr{P}$ can be regarded as the respective actions of $\mathscr{P}_1, \mathscr{P}_2, \cdots, \mathscr{P}_m$ on $\mathscr{P}$, such as those shown in Fig.5.4.

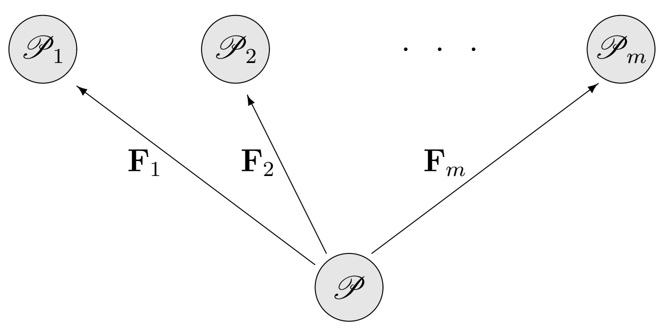

Fig.5.4

Thus we can establish the differential equations two by two, i.e., $\mathscr{P}_1$ acts on $\mathscr{P}$, $\mathscr{P}_2$

acts on $\mathscr{P}$, $\cdots$, $\mathscr{P}_m$ acts on $\mathscr{P}$ and get a non-solvable differential equation system

$$\ddot{X} = GM_i\frac{X_i - X}{|X_i - X|^3}, \quad \mathscr{P}_i \neq \mathscr{P}, \ 1 \leq i \leq m.$$

Fortunately, each of these differential equations in this system can be solved likewise that of $m = 2$. Not loss of generality, assume $\widehat{X}_i(t)$ to be the solution of the differential equation in the case of $\mathscr{P}_i \neq \mathscr{P}$, $1 \leq i \leq m$. Then

$$X(t) = \sum_{\mathscr{P}_i \neq \mathscr{P}} \widehat{X}_i(t) = G\sum_{\mathscr{P}_i \neq \mathscr{P}} M_i \int\left(\int \frac{X_i - X}{|X_i - X|^3}dt\right)dt$$

is nothing but the position of particle $\mathscr{P}$ at time t in $\mathbf{R}^3$ under the actions of $\mathscr{P}_i \neq \mathscr{P}$ for integers $1 \leq i \leq m$, i.e., its position can be characterized completely by the additivity of gravitational force.

5.3 Global Stability of Multilateral Matters

Usually, one determines the behavior of a matter by observing its appearances revealed before one's eyes. If a matter emerges more lateralities before one's eyes, for instance the different states of a multiple state matter. We have to establish different models, particularly, differential equations for understanding that matter. In fact, each of these differential equations can be solved but they are contradictory altogether, i.e., non-solvable in common meaning. Such a multilateral matter is *globally stable* if these differential equations are sum or prod-stable in all.

Concretely, let $S_1, S_2, \cdots, S_m$ be m lateral appearances of a matter $\mathscr{M}$ in $\mathbf{R}^3$ which are respectively characterized by differential equations

$$\dot{X}_i = H_i(X_i, t), \quad 1 \leq i \leq m,$$

where $X_i \in \mathbf{R}^3$, a 3-dimensional vector of surveying parameters for $S_i, 1 \leq i \leq m$. Thus we get a non-solvable differential equations

$$\dot{X} = H_i(X, t), \quad 1 \leq i \leq m \qquad (DES_m^1)$$

in $\mathbf{R}^3$. Noticing that all these equations characterize a same matter $\mathscr{M}$, there must be equilibrium points X^* for all these equations. Let

$$H_i(X, t) = H_i'(X^*)X + R_i(X^*),$$

where

$$H_i'(X^*) = \begin{bmatrix} h_{11}^{[i]} & h_{12}^{[i]} & \cdots & h_{1n}^{[i]} \\ h_{21}^{[i]} & h_{22}^{[i]} & \cdots & h_{2n}^{[i]} \\ \cdots & \cdots & \cdots & \cdots \\ h_{n1}^{[i]} & h_{n2}^{[i]} & \cdots & h_{nn}^{[i]} \end{bmatrix}$$

is an $n \times n$ matrix. Consider the non-solvable linear differential equation system

$$\dot{X} = H_i'(X^*)X, \quad 1 \leq i \leq m \qquad (LDES_m^1)$$

with a basis graph G. According to Theorem 3.6, if

$$\lim_{\|X\| \to X^*} \frac{\|R_i(X)\|}{\|X\|} = 0$$

for integers $1 \leq i \leq m$, then the G-solution of these differential equations is asymptotically sum-stable or asymptotically prod-stable on G if each $\mathrm{Re}\alpha_k^{[i]} < 0$ for all eigenvalues $\alpha_k^{[i]}$ of matrix $H_i'(X^*)$, $1 \leq i \leq m$. Thus we therefore determine the behavior of matter $\mathscr{M}$ is globally stable nearly enough X^*. Otherwise, if there exists such an equation which is not stable at the point X^*, then the matter $\mathscr{M}$ is not globally stable. By such a way, if we can determine these differential equations are stable in everywhere, then we can finally conclude that M is globally stable.

Conversely, let $\mathscr{M}$ be a globally stable matter characterized by a non-solvable differential equation

$$\dot{X} = H_i(X, t)$$

for its laterality S_i, $1 \leq i \leq m$. Then the differential equations

$$\dot{X} = H_i(X, t), \quad 1 \leq i \leq m \qquad (DES_m^1)$$

are sum-stable or prod-stable in all by definition. Consequently, we get a sum-stable or prod-stable non-solvable differential equation system.

Combining all of these previous discussions, we get an interesting conclusion following.

Conclusion 5.2 *Let $\mathscr{M}^{GS}, \overline{\mathscr{M}}^{GS}$ be respectively the sets of globally stable multilateral matters, non-stable multilateral matters characterized by non-solvable differential equation systems and $\mathscr{DE}, \overline{\mathscr{DE}}$ the sets of sum or prod-stable non-solvable differential* *equation systems, not sum or prod-stable non-solvable differential equation systems. then*

(1) $\forall \mathscr{M} \in \mathscr{M}^{GS} \Rightarrow \exists (DES_m^1) \in \mathscr{DE}$;

(2) $\forall \mathscr{M} \in \overline{\mathscr{M}}^{GS} \Rightarrow \exists (DES_m^1) \in \overline{\mathscr{DE}}$.

Particularly, let $\mathscr{M}$ *be a multiple state matter. If all of its states are stable, then* $\mathscr{M}$ *is globally stable. Otherwise, it is unstable.*

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