Product of distributions applied to Discrete Differential Geometry

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Abstract

A method for dealing with the product of step discontinuities and Dirac delta functions, related each other by a continuous function, is proposed. The proposed method is similar, for many aspects, to the Colombeau theory but different in the formalism and the perspective.

The method is extended to the product of more general step discontinuous distributions and to the product of distributions in a multidimensional case. A space extension of generalised functions, in which product of step and delta functions is commutative and associative, is constructed.

A standard method, for applying the above defined product of distributions to polyhedron vertices, is analysed and the method is applied to a special case where the famous defect angle formula, for the discrete curvature of polyhedra, is derived using the tools of tensor calculus.

Key Words: distribution theory, product of distributions, discrete differential geometry.

1 Introduction

Products of distributions are quite common in several fields of both mathematics and physics. Examples arise naturally in quantum field theory, gravitation and, in general, every time the problem under study is modelled by means of differential equations, which solutions are non-smooth functions (see [1]). An important issue, related to product of distributions, is the fact that the product, in the general case, is not associative, issue known as the Schwartz impossibility result (see [1] §1.3) and that only the product between a smooth function and a distribution is well defined.

Discrete differential geometry is a rather new field of mathematics which borrows concepts and ideas from both differential geometry and discrete mathematics. Main applications are concerned with the discrete version of several classical concepts of differential geometry such as discrete curvature, minimal surfaces, geodesics coordinates, minimal paths, surfaces of constant curvature, curvature line parametrisation and the discrete version of continuous functionals

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(see [2]). At the moment, discrete differential geometry uses many tools of discrete mathematics while the classical tools of differential geometry (e.g. tensor calculus) are difficult to be applied. This leads to an ambiguous definition of the various operators (see [3]) which are instead well defined in the continuous counterpart of the theory.

In this paper, we propose a method for evaluating the product of step discontinuous functions and Dirac delta functions, related each other by a continuous function. Moreover, the method is applied to a special class of non differentiable varieties for which, the classical idea of curvature, together with all tools of differential geometry, needs to be redefined in terms of distribution functions. In particular, the class of varieties analysed is the one composed of a collection of several Riemannian varieties glued in such a way the final surface is not differentiable on the resulting edges and vertices. In this case, it is possible to show that vertices and edges carry a concentrated discrete curvature which gives a contribution to the total curvature of the surface, contribution that has to be taken into account in order for the Gauss-Bonnet theorem to work.

For vertices, an important results was already known since the time of Descartes which proved, in the first half of the 17th century, its defect angle theorem for polyhedra. That idea, using the modern concept of curvature and applied to the class of surfaces defined above, can be stated by saying that the discrete total curvature of a vertex is equal to 2π minus the sum of the angles between edges.

For edges, using the Gauss-Bonnet theorem, it is easy to see that the discrete curvature carried by an edge L_{ij} is given by:

$$k_{L_{ij}} = \int_{L_{ij}} (k_{g_i} + k_{g_j}) ds \tag{1}$$

where k_{g_i} and k_{g_j} are the geodesic curvatures, evaluated on the edge L_{ij}, of the two variety S_i and S_j for which L_{ij} is the boundary. If the surface is differentiable on L_{ij}, then k_{g_i} and k_{g_j} are opposite and the integral vanishes. If the surface is not differentiable, the integral (1) gives in general a finite result witch corresponds to the discrete curvature concentrated on L_{ij} and $(k_{g_i} + k_{g_j})$ is the discrete curvature for unit length of the surface on L_{ij}.

This kind of surfaces, characterised by a step discontinuous metric, are typical of problems ranging from theoretical physics up to computer graphics, where the usual way to proceed is to brake down the problem and to define boundary condition (with conserved quantities) in order to keep the whole problem definition consistent (see [4]) or to use methods of discrete mathematics to define the relevant operators (see [3]). The approach proposed in this paper is to use the tensor calculus where all the derivatives are performed according to the rules of distributions and to use the above mentioned method to evaluate the products of step discontinuities and Dirac delta functions present in the coefficients of the various differential quantities.

2 Product of steps and delta functions

Proposition 1. *Let $g(x)$ be a function discontinuous in 0 and defined as follows:*

$$g(x) = \begin{cases} a & for\, x < 0 \\ b & for\, x > 0 \end{cases} \tag{2}$$

with $a, b \in \mathbb{R}$, and let $f(x)$ be any function continuous in $A \supseteq [a, b]$ (or $[b, a]$ if $b < a$). Also let $(b - a)\delta(x)$ be the derivative of $g(x)$. Then:

$$f(g(x))\delta(x) = \frac{1}{b-a}\left(\int_a^b f(x)dx\right)\delta(x) \tag{3}$$

where the product $(b-a)f(g(x))\delta(x)$ has to be intended as the $\lim_{n\to\infty} f(g_n)g'_n$ for any sequence g_n such that, $\lim_{n\to\infty} g_n = g$ and $\forall n$, the g_n have value in $B \subseteq A$.

Proof.[1] The prove is given for $a < b$ only, changes to the proof, for the case $b < a$, are trivial. First, we write down an useful equality. Let $h(x)$ be a function having the following characteristics:

$$\begin{array}{l} 1)\ h(x)\ is\ continuous\ \forall x \in \mathbb{R} \\ 2)\ \lim_{x\to-\infty} h(x) = a \\ 3)\ \lim_{x\to+\infty} h(x) = b \end{array} \tag{4}$$

we have:

$$\int_{-\infty}^{+\infty} f(h(x))h'(x)dx = \int_{-\infty}^{+\infty} \frac{d}{dx}F(h(x))dx = F(b) - F(a) \tag{5}$$

where $F(x)$ is the primitive of $f(x)$.

Note that $F(h(-\infty)) = F(a)$ and $F(h(+\infty)) = F(b)$ has been used. It is easy to see that the (5) is independent from the function h(x) since it is depending only on $F(x)$, a and b.

Then, let $g_n(x)$ be a sequence of locally integrable functions (inducing regular distributions, the same symbol $g_n(x)$ will be used for both the functions and the induced regular distributions) having the characteristics (4) and having in addition the following characteristics:

$$\begin{array}{l} 1)\ g_n(x)\ is\ monotone\ \forall x \in R \\ 2)\ g_n(x)\ constant \Rightarrow g_n^{'}(x) = 0\ \forall x \notin [-\frac{1}{n}, \frac{1}{n}] \\ 3)\ \lim_{n\to\infty} g_n(x) = g(x) \Rightarrow \lim_{n\to\infty} g'_n(x) = (b-a)\delta(x) \end{array} \tag{6}$$

Let also $\phi(x)$ be a test function. Since $g_n^{'}(x)$ vanishes outside the interval $[-\frac{1}{n}, \frac{1}{n}]$ and taking into account the (5) it is possible to write:

$$\left|\int_{-\infty}^{+\infty} f(g_n(x))g'_n(x)\phi(x)dx - [F(b) - F(a)]\phi(0)\right|$$

[1] The proof will be given for a restricted class of sequences, composed of functions having characteristics (6), and not for any sequence, as stated by the proposition. However, the (6) define quite general and nice functions to be used for constructing sequences which have, as a limit, step functions. For the above reason, the given partial proof does not lead to any limitation for practical applications. The full prof of proposition 1, which is a little more involved but not conceptually difficult, will not be given in this paper.

$$= \left| \int_{-1/n}^{1/n} f(g_n(x)) g_n'(x) [\phi(x) - \phi(0)] dx \right|$$

$$\leq \int_{-1/n}^{1/n} |f(g_n(x)) g_n'(x)| \, |\phi(x) - \phi(0)| dx \tag{7}$$

Since ϕ is a test function, it is continuous at $x = 0$. By definition of continuity, given any $\epsilon > 0$, it is possible to find $\delta > 0$ such that whenever $|x| < \delta$, $|\phi(x) - \phi(0)| < \epsilon$. So if we choose any $n > \frac{1}{\delta}$,

$$\int_{-1/n}^{1/n} |f(g_n(x)) g_n'(x)| \, |\phi(x) - \phi(0)| dx \leq \epsilon \int_{-1/n}^{1/n} |f(g_n(x)) g_n'(x)| dx \tag{8}$$

Given the (6), for which $g_n'(x) \geq 0$, the (5) and writing $f(x) = f(x)_+ - f(x)_-$ as the sum of its positive and negative part (both continuous) we have:

$$\int_{-1/n}^{1/n} |f(g_n(x)) g_n'(x)| dx$$

$$= \int_{-1/n}^{1/n} f_+(g_n(x)) g_n'(x) dx + \int_{-1/n}^{1/n} f_-(g_n(x)) g_n'(x) dx = M > 0 \tag{9}$$

where M is independent from $g_n(x)$. Given the (8) and the (9) we have:

$$\left| \int_{-1/n}^{1/n} f(g_n(x)) g_n'(x) \phi(x) dx - [F(b) - F(a)] \phi(0) \right| \leq \epsilon M \tag{10}$$

This proves that:

$$\lim_{n \to \infty} \int_{-\infty}^{+\infty} f(g_n(x)) g_n'(x) \phi(x) dx = [F(b) - F(a)] \phi(0) \tag{11}$$

Now, if we call $(b - a) f(g(x)) \delta(x)$ the limit of the sequence of distributions $f(g_n(x)) g_n'(x)$, the (11) proves the following:

- the limit exists
- the limit is a Dirac delta function
- the amplitude of the delta function is given by the (3) □

Proposition 1 can be further generalised as for the following lemma:

Lemma. *Let $s(x)$ be a locally integrable function with a step discontinuity in x_0 with $s(x_0^-) = a$, $s(x_0^+) = b$ where $a, b \in \mathbb{R}$ and let $f(x)$ be any function continuous in $A \supseteq [a, b]$ (or $[b, a]$ if $b < a$). Also let $(b - a)\delta(x - x_0)$ be the derivative of $s(x)$ at x_0. Then:*

$$f(s(x)) \delta(x - x_0) = \frac{1}{b - a} \left(\int_a^b f(x) dx \right) \delta(x - x_0) \tag{12}$$

where the product has to be intended as for proposition 1.

Proof. The prove is given for $a < b$ only, changes to the proof, for the case $b < a$, are trivial. Any locally integrable function $s(x)$, with a step discontinuity in x_0, can always be decomposed in the sum $s(x) = \tilde{s}(x)+g(x)$ where $\tilde{s}(x)$ is a function continuous in x_0 and $g(x)$ is defined as follows:

$$g(x) = \begin{cases} a & for\, x < x_0 \\ b & for\, x > x_0 \end{cases} \tag{13}$$

In this case, let $s_n(x) = \tilde{s}_n(x) + g_n(x)$ be any succession of distributions with $g_n(x)$ constant for $x \notin [x_0 - 1/n, x_0 + 1/n]$, $\tilde{s}(x) = 0$ for $x \in [x_0 - 1/n, x_0 + 1/n]$ and with $\lim_{n\to\infty} \tilde{s}_n(x) = \tilde{s}(x)$ and $\lim_{n\to\infty} g_n(x) = g(x)$ (note that the limit of $s_n(x)$, for n that goes to infinity, is $s(x)$). Let also $\phi(x)$ be a test function. We have:

$$\int_{-\infty}^{+\infty} f(\,\tilde{s}_n(x) + g_n(x)\,)[\tilde{s}_n^{'}(x) + g_n'(x)]\phi(x)dx$$

$$= \int_{-\infty}^{+\infty} f(s_n(x))\tilde{s}_n^{'}(x)\phi(x)dx + \int_{x_0-1/n}^{x_0+1/n} f(g_n(x))g_n^{'}(x)\phi(x)dx \tag{14}$$

Now, if we take the limit for n that goes to infinity, clearly the first term of the (14) is the regular distribution induced by $f(s(x))\tilde{s}^{'}(x)$ (note that $\tilde{s}_n^{'}(x) = s_n^{'}(x)$ for $x \neq x_0$). Moreover we already know how to treat the second term, which is simply the (11) shifted to a coordinate x_0. We have therefore:

$$f(s(x))s^{'}(x - x_0) = f(s(x))\tilde{s}^{'}(x) + (b - a)f(g(x))\delta(x - x_0) \tag{15}$$

Finally, from the (15) we can easily prove the lemma. □

Note that, even in the case where $F(a) = F(b)$ and therefore there is no step in the discontinuity, proposition 1 and its lemma are essential to evaluate the product of the discontinuity with a related delta function. For example, is easy to show that $sign^2(x)\delta(x) = \frac{1}{3}\delta(x)$.

3 The multidimensional case

Proposition 2. *Let $g_1(x)$ and $g_2(y)$ be two functions discontinuous in 0 and defined as follows:*

$$g_1(x) = \begin{cases} a & for\, x < 0 \\ b & for\, x > 0 \end{cases} \tag{16}$$

$$g_2(y) = \begin{cases} c & for\, y < 0 \\ d & for\, y > 0 \end{cases} \tag{17}$$

with $a, b, c, d \in \mathbb{R}$ and let $f(x, y)$ be any function continuous in $A \supseteq [a, b] \times [c, d]$ (if $b < a$ and/or $d < c$ the definition of A has to be changed accordingly). Also let $(b - a)(c - d)\delta(x, y)$ be the product of the derivatives of $g_1(x)$ and $g_2(y)$. Then:

$$f(g_1(x), g_2(y))\delta(x, y) = \frac{1}{(b - a)(d - c)} \left(\int_c^d dy \int_a^b f(x, y)dx \right) \delta(x, y) \tag{18}$$

where the product $(b-a)(c-d)f(g_1(x), g_2(y))\delta(x,y)$ *has to be intended as the* $\lim_{n\to\infty} f(g_{1n}, g_{2n})g'_{1n}g'_{2n}$ *for any pair of sequences* g_{1n},g_{2n} *such that,* $\lim_{n\to\infty} g_{1n} = g_1$*,* $\lim_{n\to\infty} g_{2n} = g_2$ *and* $\forall n$*,* g_{1n} *has value in* B_1*,* g_{2n} *has value in* B_2 *and* $B_1 \times B_2 \subseteq A$*.*

Obviously, we can interchange the roles of x and y since we may integrate first with respect of y and then with respect of x. Note that the discontinuity $f(g_1(x), g_2(y))$ addressed by this proposition is not the most general step discontinuity we may have in two dimensions.

As for proposition 1, in order to prove the above proposition, we first need to prove an useful equality. Let $h_1(x), h_2(y)$ be two functions which have the following characteristics:

$$\begin{array}{l} 1)\ h_1(x), h_2(y)\ \textit{are continuous}\ \forall x, y \in \mathbb{R} \\ 2)\ \lim_{x\to-\infty} h_1(x) = a;\ \lim_{x\to+\infty} h_1(x) = b \\ 3)\ \lim_{y\to-\infty} h_2(y) = c;\ \lim_{y\to+\infty} h_2(y) = d \end{array} \tag{19}$$

and let F(x,y) be a function such that $F_{xy} = F_{yx} = f(x,y)$. We have:

$$\int_{-\infty}^{+\infty} dy \int_{-\infty}^{+\infty} f(h_1(x), h_2(y))h_1^{'}(x)h_2^{'}(y)dx$$

$$= \int_{-\infty}^{+\infty} dy \frac{\partial}{\partial y} \int_{-\infty}^{+\infty} \frac{\partial}{\partial x} F(h_1(x), h_2(y))dx \tag{20}$$

where, to prove the (20), we have taken the symbol $\frac{\partial}{\partial y}$ inside the integral (for the linearity of integrals) and applied the definition of $F(x,y)$. It is easy to see that the (20) is independent from the $h_{1,2}$ and is equal to $F(b,d) - F(a,d) - F(b,c) + F(a,c)$. From this point on, it is possible to prove proposition 2 by following similar steps to the ones used for proving proposition 1. As we did for the monodimensional case (se lemma of proposition 1), a generalization to locally integrable functions, may also be given in this case.

Note that proposition 2 gives a clear path on the possible way to generalise the idea of products of step discontinuities and delta functions to the case with as many dimensions as we like.

4 Further discussions on product of distribution

We are mainly interested in step discontinuities and Dirac delta functions. Each discontinuity of this kind can be defined by means of the limit of an infinite number of successions of distributions, all different each other and having, in a distributional sense, the same limit. For the purpose of this paper, we define the structure of such discontinuities to be the specific succession we use to define them. In general, since we want a distribution to be as generic as possible, we never define its own structure and we leave it indeterminate. However, the concept of structure of a discontinuity is essential to this paper as will be clear shortly.

Moreover, every time we define the product of distributions in a point x_0, where the distributions are discontinuous, we always want the discontinuities to have each other structure related by a well known law so that, if the structure of one distribution in x_0, which is unknown to us, changes, the structure of all other distributions in the same point will change accordingly.

Since we never want to define the structures of the distributions, we are mostly interested in products of distributions, like the ones of proposition 1 and 2, which work regardless their underlying structures. This is why in proposition 1 and 2 we define the product as the limit of any possible succession (i.e. structure) and we want this limit to be independent from it.

The idea that a particular distribution may have an infinite number of different structures is very similar to the notion of associated distributions present in the Colombeau theory (see [1] §3.2), where the product make sense if it is independent from the particular representative of the involved generalised functions (see [1] §3.1).

As a final remark, we state that even though proposition 1 and 2 require the function f to be continuous in, at least, respectively $A = [a, b]$ and $A = [a, b] \times [c, d]$, the two propositions work also if f goes to infinity, in a finite number of points in A, provided that f is integrable in the same set. Once again, this should be proved formally. However, it easy to see that, in this case, f can be defined as the limit of a succession of function f_n all continuous in A. Since the (5) and (20) hold for each n, they hold also for f and proposition 1 and 2 are still valid. An important example, where we use integrable functions f, which go to infinity in a point of the integration set, is shown in paragraph 6 and 7 of this paper.

5 D^e space, commutativity and associativity.

So far, we have focused our attention only on the structure of step discontinuities and the way they are modified (by composition with a continuous function f). When it comes to Dirac delta functions, it is possible to show that they change their own structure by means of multiplication by step discontinuous functions. Let us consider the discontinuous function $f(g(x))$ where $f \in C^1$ and g is a step discontinuous function defined as in (2). We have:

$$Df(g(x)) = \frac{1}{b-a} f'(g(x))\delta(x) \tag{21}$$

from which we see that by multiplying a delta function having structure $g(x)$ (i.e. derivative of a step discontinuous function $g(x)$) by $f'(g(x))$ we get a delta function with structure $f(g(x))$.

We have seen that, in a product of distributions, if we change the structure of a term we get a different result. In order to overcame this limitation, we want now to extend the space of distributions D' by adding to it, as separate generalised functions, additional elements representing any possible structure needed for describing product of step and delta functions.

In the previous paragraphs, since we never define the structure of a distribution, we have always considered two identical functions (e.g. two Heaviside functions) different when they had different origin in our calculations. It is like to have a separate D' space for each function. In this case a step discontinuous and a delta functions lies in the same D' space only if they are really the same element, which has been differentiated and multiplied by a step function from one side, and composed by a continuous function on the other side.

We will drop this requirement now. For simplicity, we will assume that all step discontinuous and delta functions, we are dealing with, have the same structures (i.e they are all related to the same Heaviside function). Note that this is analogous to what is done in Colombeau theory. From this new point of view, the function f, which before was used to relate distribution structures, became now the structure itself of the distribution.
We will use the following notation:

$$\begin{array}{ll} u_{[f(x)]} = f(g(x)) & step\, function\, with\, structure\, f \\ \delta_{[f'(x)]} = f'(g(x))\delta(x) & delta\, function\, with\, structure\, f \end{array} \tag{22}$$

Where $u_{[f(x)]}$ and $\delta_{[f(x)]}$ are not normalised (i.e they may have amplitude different from 1) and $u_{[x]} = u(x) \in D'$, $\delta_{[1]} = \delta(x) \in D'$. We are now ready to define our space extension of distribution D^e as follows:

$$D^e = \{d: \; d \in D' \; or \; d = u_{[f(x)]} \; or \; d = \delta_{[f'(x)]} \; , \; for \; any \; f \in C^1 \; in \; [0,1]\} \tag{23}$$

Obviously $D^e \supset D'$. Note that by defining D^e we have done something analogous to the Colombeau theory where an space extension is build adding to D', among others, the generalised functions $u^n(x)$ as separate functions. The new thing here is that we explicitly add to D' also the generalised functions $\delta_{[f'(x)]}$.
In the new space of distributions D^e we define the multiplication as follows:

$$u_{[f_1]}u_{[f_2]} \cdot \; \ldots \; \cdot u_{[f_n]}\delta_{[f_{n+1}]} = \delta_{[f_1 f_2 \cdot \; \ldots \; \cdot f_n f_{n+1}]} \tag{24}$$

Finally we define a projector operator P which project any distribution $d \in D^e$ on the space D'. For step discontinuous functions the way P works is trivial (e.g. $u^2(x)$ goes to $u(x)$). For delta functions, we apply the theory developed in paragraph 2 and by using proposition 1 we have:

$$P(\delta_{[f_1 f_2 \cdot \; \ldots \; \cdot f_n f_{n+1}]}) = \left(\int_0^1 f_1 f_2 \cdot \; \ldots \; \cdot f_n f_{n+1} dx \right) \delta(x) \in D' \tag{25}$$

Note that the (24) and (25) provide a well defined product of distribution in D^e which is fully coherent with the theory developed in the previous paragraphs.

We are now in a position to discuss commutativity and associativity in D^e. We simply note that, in the product defined by the (24) and (25), commutativity and associativity is ensured by commutativity and associativity of the f_i functions.

Let us make an example. Consider the product of distributions $sign^2(x)\delta(x)$ (compare with [5] §1.1 ex. iii). By using proposition 1 we find easily that:

$$sign^2(x)\delta(x) = \frac{1}{3}\delta(x) \tag{26}$$

Let us check associativity by using, once again proposition 1:

$$sign^2(x)\delta(x) = sign(x)[sign(x)\delta(x)] = sign(x) \cdot 0 = 0 \tag{27}$$

we conclude that in D' our product is not associative. Let us see what happen in D^e:

$$sign(x)[sign(x)\delta(x)] = sign(x)\delta_{[(2x-1)\cdot 1]} = sign(x)[\delta_{[2x]} - \delta_{[1]}] \tag{28}$$

In D', $\delta_{[1]} = \delta$ and $P(\delta_{[2x]}) = \delta$. However, in D^e they are separate distributions and they do not cancell each other. We have eventually:

$$sign^2(x)\delta(x) = P(\delta_{[(2x-1)^2]}) = \frac{1}{3}\delta(x) \tag{29}$$

6 Metrics for a polyhedron vertex

The product of distributions, defined in the paragraphs above, may be applied to a number of fields of both physics and mathematics where the product of step discontinuity and Dirac delta function arise naturally from the theory. Among all, we have decided to focus our attention to applications related to differential geometry and, in particular, to the evaluation of the curvature for those varieties, described in the introduction, having step discontinuous metric.

As mentioned in the introduction, this kind of variety may have discrete curvature concentrated on edges and vertices. In both cases, Christoffel symbols, Riemann and Ricci tensors, curvature as well as a number of different differential operators, may only be expressed by means of product of step and delta functions. In this case, the relationship between the structures of the step discontinuities and the delta functions codify the geometrical aspects of the non-differentiable point of the surface and proposition 1 (for edges) and proposition 2 (for vertices) turn up to be very useful in finding an expression for the differential quantity of interest

As an example, in this paragraph we will show a convenient and standard way to define a step discontinuous metric for vertices of polyhedra with 3 or 4 concurrent edges, which are very common in many applications, and in paragraph 7 we will show how to use these metrics to evaluate the curvature of that polyhedron in the vertices. Even thought this paragraph is focused on curvatures, the same method can be applied to evaluate any kind of differential parameters and operators (e.g. Laplace-Beltrami operators).

Before we proceed, we need to introduce a definition. For the purpose of this paper, we will call a 2d-step function any function defined as follows:

$$s(x_1, x_2) = \begin{cases} r_1 & for\, x_1 > 0, x_2 > 0 \\ r_2 & for\, x_1 < 0, x_2 > 0 \\ r_3 & for\, x_1 < 0, x_2 < 0 \\ r_4 & for\, x_1 > 0, x_2 < 0 \end{cases} \tag{30}$$

where $r_i \in \mathbb{R}$ and $s(x_1, x_2)$ is not defined on the axis (x_1, x_2). Any function of the kind (30) can always be expressed in the form:

$$s(x_1, x_2) = s_0 + s_1(x_1)s_2(x_2) \tag{31}$$

where $s_0 \in \mathbb{R}$ and s_1, s_2 are defined as follows:

$$s_1(x) = \begin{cases} a & for\, x_1 < 0 \\ b & for\, x_1 > 0 \end{cases} \tag{32}$$

$$s_2(y) = \begin{cases} c & for\, x_2 < 0 \\ d & for\, x_2 > 0 \end{cases} \tag{33}$$

and where there is always one degree of freedom in the parameters (s_0, a, b, c, d). Conversely any function of the form (31) is always a 2d-step function.

Now, let V be a vertex of a polyhedron whit 4 edges and angles between edges α, β, γ and η. Let also S be the surface composed of the vertex, the 4 edges and the relevant 4 faces. We can always open S on a (x_1, x_2) plane by stretching each face by a different amount so that each of the 4 edges lies on one of the semi-axes of the plane. By doing so, we basically map each face of S to a specific sector of the plane (x_1, x_2). It is easy to see that the metric of S is:

$$g_{ij} = \begin{pmatrix} 1 & s(x_1, x_2) \\ s(x_1, x_2) & 1 \end{pmatrix} \tag{34}$$

where $s(x_1, x_2)$ is a 2d-step function for which the amplitude, in each sector of the (x_1, x_2) plane, is a function of one of the angles $\alpha, \beta, \gamma, \eta$ and the parameters (s_1, a, b, c, d) are defined as follows:

$$s(x_1, x_2) = \begin{cases} cos(\alpha) = s_0 + bd & for\ x_1 > 0, x_2 > 0 \\ -cos(\beta) = s_0 + ad & for\ x_1 < 0, x_2 > 0 \\ cos(\gamma) = s_0 + ac & for\ x_1 < 0, x_2 < 0 \\ -cos(\eta) = s_0 + bc & for\ x_1 > 0, x_2 < 0 \end{cases} \tag{35}$$

The (35) define at the same time $s(x_1, x_2)$ and the equation to determine its parameters. The minus signs in the (35) is to take into account that we are in a sector with one of the two dx_i negative and therefore the angle to consider in the metrics is the one between dx_1 and dx_2 positive which is equal to π minus the angle of the relevant polyhedron face for that sector. Since $cos(\pi - x) = -cos(x)$ a minus sign is needed.

As far as vertices with 3 concurrent edges are concerned, we can apply the same procedure by adding a 4th face with angle between edges equal to ϵ and then take the limit for $\epsilon \to 0$. This is equivalent to cut the surface along one of the edges, open the surface on the plane so that each face corresponds to a sector of the axis (x_1, x_2) while the 4th sector remains uncovered and, finally, assign a null metric to that sector (i.e. $s(x_1, x_2) = 1$). This obviously will lead to an infinity inverse metric in the sector. This is not a problem since we are mainly interested in evaluating the curvature in the discontinuity and not the curvature on the surface (which we know to vanish).

An infinity inverse metric will lead to a function $f(x, y)$, of proposition 2 above, which is continuous in $A =]a, b[\times]c, d[$ and that goes to infinity in one of the point of the border of A (the one related to the null metric). Since proposition 2 works also for function which have discontinuities where $f(x, y)$ goes to infinity in A, as long as the function is integrable in the same set, this is not really an issue.

7 Vertex curvature and the defect angle formula

Given the metric of a vertex defined as for the previous paragraph, we will see now how to evaluate its curvature by means of proposition 2. To do that, we will evaluate all the classical differential parameters, and eventually the curvature, as distributions. First of all we evaluate the $g^{i,j}$. From the (34) we have:

$$g^{ij} = \frac{1}{1 - s^2} \begin{pmatrix} 1 & -s \\ -s & 1 \end{pmatrix} \tag{36}$$

The derivatives of the metric are:

$$\Delta_1 = \frac{\partial g_{12}}{\partial x_1} = \frac{\partial g_{21}}{\partial x_1} = (b-a)\delta(x_1)s_2(x_2) \tag{37}$$

$$\Delta_2 = \frac{\partial g_{12}}{\partial x_2} = \frac{\partial g_{21}}{\partial x_2} = (d-c)s_1(x_1)\delta(x_2) \tag{38}$$

all other derivatives vanish. We proceed by evaluating the Christoffel symbol of the first kind. We have:

$$\Gamma_{112} = \frac{1}{2}(-0 + \Delta_1 + \Delta_1) = (b-a)\delta(x_1)s_2(x_2) \tag{39}$$

$$\Gamma_{221} = \frac{1}{2}(-0 + \Delta_2 + \Delta_2) = (d-c)s_1(x_1)\delta(x_2) \tag{40}$$

all other coefficients of the Christoffel symbol of the first kind vanish. For our purpose we need to evaluate only one of the coefficients of the Christoffel symbol of the second kind:

$$\Gamma_{22}^2 = g^{21}\Gamma_{221} + g^{22}\Gamma_{222} = -\frac{(d-c)s}{1-s^2}s_1(x_1)\delta(x_2) \tag{41}$$

We have now all the elements we need to evaluate the Riemann tensor:

$$R_{1212} = \frac{(b-a)(d-c)}{1-s^2}(1-s^2+s\,s_1s_2)\delta(x_1,x_2) \tag{42}$$

for surfaces and given the Riemann tensor, a classical formula for evaluating the curvature is the following:

$$k = \frac{R_{1212}}{g_{11}g_{22}-g_{12}g_{21}} = \frac{R_{1212}}{1-s^2} \tag{43}$$

as expected the curvature is a Dirac delta function in (0,0). The total curvature can be evaluated by integrating the curvature on S:

$$\begin{aligned} k_T &= \iint_S k\sqrt{1-s^2}dx_1dx_2 = \iint_S R_{1212}\frac{\sqrt{1-s^2}}{1-s^2}dx_1dx_2 \\ &= (b-a)(d-c)\iint_S (1-s^2+s\,s_1s_2)(1-s^2)^{-\frac{3}{2}}\delta(x_1,x_2)dx_1dx_2 \end{aligned} \tag{44}$$

since the integrand is impulsive, it is clear that the total curvature is equal to the amplitude of the impulse, which can be evaluated using proposition 2. We have:

$$s_1(x_1) = x;\ s_2(x_2) = y;\ s(x_1,x_2) = s_0 + xy; \tag{45}$$

by using the (45) in the (18) we get the final expression for the total curvature:

$$k_T = \int_a^b dy \int_c^d (1-s_0^2-s_0xy)\left[1-s_0^2-2s_0xy-x^2y^2\right]^{-\frac{3}{2}}dx \tag{46}$$

integrating, first with respect of x and then with respect of y, we obtain the primitive $F(x,y)$:

$$F(x,y) = \arctan\left(\frac{s_0+xy}{\sqrt{1-(s_0+xy)^2}}\right) \tag{47}$$

Let us see how to use the (47) by checking, for example, the value of $F(x, y)$ in (b, d). Given the (35) we have:

$$F(b,d) = \arctan\left(\frac{s_0 + bd}{\sqrt{1-(s_0+bd)^2}}\right) = \arctan\left(\frac{\cos\alpha}{\sin\alpha}\right) = \frac{\pi}{2} - \alpha \tag{48}$$

where we have used the plus sign of the square root. The minus sign corresponds to the case where we swap all the signs in the (35). This is equivalent to choosing a different mapping, between faces and sectors, of the surface on (x_1, x_2). From the (46) we evaluate our final results:

$$k_T = F(b,d) - F(a,d) - F(b,c) + F(a,c) = 2\pi - \alpha - \beta - \gamma - \eta \tag{49}$$

which is, as expected, the defect angle formula. It is remarkable that, by means of proposition 2, we have derived the defect angle formula, in a non-differentiable point, by using the tools of differential geometry.

Taking the limit for one of the angles going to zero, we get the example, mentioned at the end of the previous paragraph, of a null metric and an infinite inverse metric in a sector. As anticipated above, in this case the function $f(x, y)$ of proposition 2 goes to infinity (compare with the integrand of (46) above) in a point of the integration set. However, the function is still integrable as clearly shown by the (47) where the primitive is finite in the same point.

8 Conclusions

Although the proposed product of distributions does not introduce anything new with respect of the theory developed by Colombeau (which, by the way, has a wider range of applications), the formalism and the viewpoint from which our theory has been developed is very different. On one hand, in its range of applicability, the difference between our method and the Colombeau theory is only formal and the underlying essence of the two theories is the same (see appendix). On the other hand, our formalism allow us to define, in a very natural way, products of distributions, which have complex relationships between each other structure, just by defining the function f of proposition 1 and 2.

Of course this kind of problems can be addressed also using Colombeau formalism where, however, the approach is not so straightforward. Maybe this is the reason way, in many fields of physics and mathematics, like for example the discrete differential geometry, the product of distributions, which may have a wide range of applications, is not so popular.

Appendix

Proof of the (3) using Colombeau coefficients

There are many ways to show that the product of distributions proposed in this paper is equivalent to the Colombeau theory. Let us prove, for example, the (3) by using Colombeau coefficients. For simplicity, the proof will be given for $g(x) = u(x)$, the Heaviside function, and for $f \in C^\infty$.

Proof. Colombeau coefficients are defined as follows (see [1] §3.3):

$$u^n(x)\delta(x) = \frac{1}{n+1}\delta(x) \tag{50}$$

we have:

$$f(u(x))\delta(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} u^n(x)\delta(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!(n+1)}\delta(x) \tag{51}$$

where we have used the (50). With the substitution $k = n + 1$ we have:

$$f(u(x))\delta(x) = \sum_{k=1}^{\infty} \frac{f^{(k-1)}(0)}{k!}\delta(x) = \sum_{k=1}^{\infty} \frac{F^{(k)}(0)}{k!}\delta(x) \tag{52}$$

where F is the primitive of f. We have eventually:

$$f(u(x))\delta(x) = \left[-F(0) + \sum_{k=0}^{\infty} \frac{F^{(k)}(0)}{k!}(1)^k\right]\delta(x) = [F(1) - F(0)]\delta(x) \tag{53}$$

□

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