# NEW GENERAL FORMULAS FOR PYTHAGOREAN TRIPLES

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Abstract. This paper shows the General Formulas for Pythagorean triples that were derived from the differences of the sides of a right triangle. In addition, as computational proof, tables were made with a C++ script showing primitive Pythagorean triples and included as text files and screenshots. Furthermore, to enable readers to check and verify them, the C++ script which will interactively generate tables of Pythagorean triples from the computer console command line is attached. It can be run in Cling and ROOT C/C++ interpreters or compiled.

## 1. Introduction

I believe it would be best to proceed immediately to the problem at hand, i.e. to find general formulas for Pythagorean triples, rather than discuss many things and go in circles. So going straight to the point we will derive and prove the relevant theorem then derive from it the formulas given by the Greeks, Plato and Pythagoras. Then we will show the general formulas that generates Pythagorean triples. At this point, the pattern will manifest. Next we will provide tables from a C++ script to demonstrate validity. In conclusion, i attached the C++ script instead of typesetting verbatim so that the document will not look unnecessarily large.

## 2. Pythagorean Triples

**Definition.** If $a, b, c \in \mathbb{N}$ and $a < b < c$ or $b < a < c$ then a Pythagorean triple is a triple of natural numbers such that $a^2 + b^2 = c^2$. It is said to be primitive if $(a, b, c)$ is pairwise relatively prime and the parities of $a$ and $b$ are always opposites while $c$ is always odd.

**Theorem 1.** *If $(a, b, c)$ is a Pythagorean triple then there exists $\alpha, \beta, \gamma \in \mathbb{Z},\ k, n \in \mathbb{N}$ where $\alpha = b - a,\ \beta = c - a,\ \gamma = c - b,\ \beta = \alpha + \gamma,\ k = n,\ n = 1, 2, 3, \ldots$ such that*

$$
\begin{aligned}
(a - \gamma)^2 &= 2\gamma\beta \\
\gamma\beta &= 2k^2 \\
a &= \gamma + 2k \\
b &= \beta + 2k \\
c &= \gamma + \beta + 2k
\end{aligned}
$$

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**Proof.**

$$
\begin{aligned}
a^2 + b^2 &= c^2 \\
a^2 + (a+\alpha)^2 &= (a+\beta)^2 \\
a^2 + (a^2 + 2a\alpha + \alpha^2) &= a^2 + 2a\beta + \beta^2 \\
a^2 + 2a(\alpha - \beta) &= \beta^2 - \alpha^2 \\
a^2 - 2a(\beta - \alpha) &= (\beta + \alpha)(\beta - \alpha) \\
a^2 - 2\gamma a &= [(\alpha + \gamma) + \alpha]\gamma \\
(a-\gamma)^2 &= \gamma(2\alpha + \gamma) + \gamma^2 \\
(a-\gamma)^2 &= 2\gamma(\alpha + \gamma) \\
\therefore (a-\gamma)^2 &= 2\gamma\beta
\end{aligned}
$$

It is evident that $(a-\gamma)^2 = 2\gamma\beta$ has even parity and hence of the form $4k^2$ where $k, n \in \mathbb{N},\ k = n,\ n = 1, 2, 3, \ldots$ Thus $(a-\gamma)^2 = 4k^2$ and $\gamma\beta = 2k^2$. We now see that $a = \gamma + 2k$. Now $\beta - \alpha = (c-a)-(b-a) = c-b$. This is $\gamma$ thus $\alpha + \gamma = \beta$. From $b - a = \alpha$ we get $b = a + \alpha = \beta + 2k$ and since $c - b = \gamma$ we also get $c = b + \gamma = \gamma + \beta + 2k$. □

**Theorem 2.** *If $(a, b, c)$ is a Pythagorean triple then there exists $\alpha, \beta, \gamma \in \mathbb{Z},\ k, n \in \mathbb{N}$ where $\alpha = b - a,\ \beta = c - a,\ \gamma = c - b,\ \beta = \alpha + \gamma,\ (a-\gamma)^2 = 2\gamma\beta,\ \gamma\beta = 2k^2,\ k = n,\ n = 1, 2, 3, \ldots$ such that*

$$
\begin{aligned}
\gamma = 1,\ \beta = 2k^2,\ a &= 2k + 1 \\
b &= 2k(k+1) \\
c &= 2k^2 + 2k + 1 \\
\gamma = 2,\ \beta = k^2,\ a &= 2(k+1) \\
b &= k(k+2) \\
c &= k^2 + 2k + 2
\end{aligned}
$$

**Proof.** From Theorem1, we have $\beta = \dfrac{2k^2}{\gamma}$ and so Pythagorean triples are generated by

$$
a = \gamma + 2k,\ b = \frac{2k^2}{\gamma} + 2k,\ c = \gamma + \frac{2k^2}{\gamma} + 2k
$$

It is seen then that integral values can be obtained for $\gamma = 1, 2$. If $\gamma = 1$ then $\beta = 2k^2$ and $a = 2k + 1,\ b = 2k(k+1),\ c = 2k^2 + 2k + 1$.

If $\gamma = 2$ then $\beta = k^2$ and $a = 2(k+1),\ b = k(k+2),\ c = k^2 + 2k + 2$. □

**Corollary 2.1.** *If* $\gamma = 1, 2$ *then primitive Pythagorean triples are generated by*

$$\begin{aligned} \gamma = 1,\ \beta = 2n^2,\ a &= 2n + 1 \\ b &= 2n(n + 1) \\ c &= 2n^2 + 2n + 1 \\ \gamma = 2,\ \beta = n^2,\ a &= 4n \\ b &= 4n^2 - 1 \\ c &= 4n^2 + 1 \end{aligned}$$

**Proof.** From Theorem2, if $\gamma = 1,\ \beta = 2k^2$ then $a \nmid b,\ a \nmid c,\ b \nmid c$ and $\gcd(a, b, c) = 1$. Thus primitive Pythagorean triples can be found for $k = n$.

If $\gamma = 2,\ \beta = k^2$ then we need to consider when $k$ is even and when it is odd.

If $k$ is even let $k = 2n$ then $a = 2(2n+1),\ b = 4n(n+1),\ c = 2(2n^2+2n+1)$ thus $a \mid b,\ a \mid c,\ b \mid c$ and $\gcd(a, b, c) = 2$. Hence non-primitive Pythagorean triples are found when $k$ is even.

If $k$ is odd let $k = 2n - 1$ then $a = 4n,\ b = 4n^2 - 1,\ c = 4n^2 + 1$ thus $a \nmid b,\ a \nmid c,\ b \nmid c$ and $\gcd(a, b, c) = 1$. Hence primitive Pythagorean triples can be found when $k$ is odd.

From these results the corollary is proved. □

## 3. General formulas for Pythagorean Triples

**Theorem 3.** *If* $(a, b, c)$ *is a Pythagorean triple then there exists* $\alpha, \beta, \gamma \in \mathbb{Z},\ k, m, n \in \mathbb{N}$ *where* $\alpha = b - a,\ \beta = c - a,\ \gamma = c - b,\ beta = \alpha + \gamma,\ (a - \gamma)^2 = 2\gamma\beta,\ \gamma\beta = 2k^2,\ k = mn,\ m = 1, 2, 3, \dots,\ n = 1, 2, 3, \dots$ *such that*

$$\begin{aligned} \gamma = m^2,\ \beta = 2n^2,\ a &= m(m + 2n) \\ b &= 2n(n + m) \\ c &= m^2 + 2mn + 2n^2 \\ \gamma = 2m^2,\ \beta = n^2,\ a &= 2m(m + n) \\ b &= n(n + 2m) \\ c &= 2m^2 + 2mn + n^2 \end{aligned}$$

**Proof.** If $n \in \mathbb{N},\ n = 1, 2, 3, \dots$ then $n = 1(1), 1(2), 2(1), 3(1), 2(2), 5(1),2(3), \dots, 41(3), 31(4), \dots$ We see that $n$ can be expressed as the product of two natural numbers. Therefore if $k, m, n \in \mathbb{N}$ and $k = mn$ where $m = 1, 2, 3, \dots,\ n = 1, 2, 3, \dots$ then since $\gamma\beta = 2k^2$ we have $\gamma\beta = 2m^2n^2$.

At this point, we see that $(\gamma, \beta)$ is $\{(m^2, 2n^2), (2m^2, n^2)\}$. Thus by Theorem1 we get the general formulas. Observe that if $m = 1$ they become the formulas in Theorem 2. □

**Corollary 3.1.** *If* $\gamma = m^2$ *and* $\beta = 2n^2$ *then primitive Pythagorean triples are generated if* $m = 1$ *and* $m > 1$ *where* $m \neq n$*,* $m$ *is odd and* $\gcd(m, n) = 1$.

**Proof.** If $m = 1$ they become the formulas in Theorem 2. If $m > 1$ and if $q, t \in \mathbb{N},\ q = 1, 2, 3, \dots,\ t = 1, 2, 3, \dots,\ \gamma = m^2,\ m \neq n,\ m = t$ then let $m = 2t - 1$ and we have $a = (2t - 1)[(2t - 1) + 2n],\ b = 2n[n + (2t - 1)],\ c = (2t - 1)^2 + 2n[n + (2t - 1)],\ \gcd(a, b, c) = 1$. Now let $m = 2t$ and we have $a = 4(t)(t+n),\ b = 2(n)(2t+n),\ c = 2[2t^2+2tn+n^2],\ \gcd(a, b, c) = 2$.

If $n = qm$, then $a = 2(1+q)m^2,\ b = q(2+q)m^2,\ c = (2+2q+q^2)m^2$. We see that $gcd(a, b, c) = m^2$ therefore considering all of these, we conclude that primitive Pythagorean triples are found if $m = 1$ and if $m > 1,\ m \neq n$, $m$ is odd and $\gcd(m, n) = 1$. □

**Corollary 3.2.** *If $\gamma = 2m^2$ and $\beta = n^2$ then primitive Pythagorean triples are generated if $m = 1$ and $m > 1$ where $m \neq n$, $n$ is odd and* $\gcd(m, n) = 1$.

**Proof.** If $m = 1$ they become the formulas in Theorem 2. If $m > 1$ and if $q, t \in \mathbb{N}$, $q = 1, 2, 3, \ldots$ , $t = 1, 2, 3, \ldots$ , $\gamma = m^2$, $m \neq n$, $m = t$ then let $m = 2t - 1$ and we have $a = 2(2t-1)[(2t-1)+n]$, $b = n[n+2(2t-1)]$, $c = 2(2t-1)^2+n[n+2(2t-1)]$, $\gcd(a, b, c) = 1$. Now let $m = 2t$ and we have $a = 2(2t)[2t+n]$, $b = n[n+2(2t)]$, $c = 2(2t)^2+n[n+2(2t)]$, $\gcd(a, b, c) = 1$. Since both have $\gcd(a, b, c) = 1$ we consider $n$. A quick mental calculation will tell us that $\gcd(a, b, c) = 2$ if $n$ is even but $\gcd(a, b, c) = 1$ if $n$ is odd.

If $q \in \mathbb{N}$, $n = qm$, $q = 1, 2, 3, \ldots$ then $a = (2 + q)m^2$, $b = q(q + 2)m^2$, $c = (2 + 2q + q^2)m^2$. We see that $gcd(a,b,c) = m^2$ thus considering all of these we conclude that primitive Pythagorean triples are found if $m = 1$ and if $m > 1$, $m \neq n$, $n$ is odd and $\gcd(m, n) = 1$. □

**Corollary 3.3.** *If $a < b$ then $n > \left\lfloor \frac{m}{\sqrt{2}} \right\rfloor$ for $\gamma = m^2, \beta = 2n^2$ and $n > \left\lfloor m\sqrt{2} \right\rfloor$ for $\gamma = 2m^2, \beta = n^2$.*

**Proof.** *If $a < b$ then $\alpha > 0$ and so $\beta > \gamma$. Thus for $\gamma = m^2, \beta = 2n^2$ we have $m^2 > 2n^2$ which is $n > \left\lfloor \frac{m}{\sqrt{2}} \right\rfloor$. And for $\gamma = 2m^2, \beta = n^2$ we have $2m^2 > n^2$ which is $n > \left\lfloor m\sqrt{2} \right\rfloor$.*

## 4. Extra

**Theorem 4.** *If $(a, b, c)$ is a Pythagorean triple and $t \in \mathbb{N}$, $t = 3, 4, 5, \ldots$ then $a^t + b^t \neq c^t$.*

**Proof.** By Theorem 1 and the Binomial Theorem we have

$$a^t + b^t \neq c^t$$
$$(\gamma + 2k)^t + (\beta + 2k)^t \neq [(\gamma + \beta) + 2k]^t$$
$$\sum_{i=0}^{t} \binom{t}{i} (\gamma)^{(t-i)} (2k)^i + \sum_{i=0}^{t} \binom{t}{i} (\beta)^{(t-i)} (2k)^i \neq \sum_{i=0}^{t} \binom{t}{i} (\gamma + \beta)^{(t-i)} (2k)^i$$

Thus by Theorem 3

$\gamma = m^2$, $\beta = 2n^2$,

$$\sum_{i=0}^{t} \binom{t}{i} (m^2)^{(t-i)} (2mn)^i + \sum_{i=0}^{t} \binom{t}{i} (2n^2)^{(t-i)} (2mn)^i \neq \sum_{i=0}^{t} \binom{t}{i} (m^2 + 2n^2)^{(t-i)} (2mn)^i$$

$\gamma = 2m^2$, $\beta = n^2$,

$$\sum_{i=0}^{t} \binom{t}{i} (2m^2)^{(t-i)} (2mn)^i + \sum_{i=0}^{t} \binom{t}{i} (n^2)^{(t-i)} (2mn)^i \neq \sum_{i=0}^{t} \binom{t}{i} (2m^2 + n^2)^{(t-i)} (2mn)^i$$

where $i \in \mathbb{Z}$ and $k, m, n, t \in \mathbb{N}$, $k = mn$, $t = 3, 4, 5, \ldots$ , $m = 1, 2, 3, \ldots$ , $n = 1, 2, 3, \ldots$ □

**Corollary 4.1.** *If $(a, b, c)$ is a Pythagorean triple then $a^3 + b^3 \neq c^3$.*

**Proof.** From Theorem 4, if $t = 3$ then

$$\gamma = m^2,\ \beta = 2n^2$$
$$a^3 = m^6 + 6m^5n + 12m^4n^2 + 8m^3n^3$$
$$b^3 = 8n^6 + 24n^5m + 24n^4m^2 + 8n^3m^3$$
$$c^3 = m^6 + 6m^5n + 18m^4n^2 + 32m^3n^3 + 36m^2n^4 + 24mn^5 + 8n^6$$
$$\gamma = 2m^2,\ \beta = n^2$$
$$a^3 = 8m^6 + 24m^5n + 24m^4n^2 + 8m^3n^3$$
$$b^3 = n^6 + 6n^5m + 12n^4m^2 + 8n^3m^3$$
$$c^3 = 8m^6 + 24m^5n + 36m^4n^2 + 32m^3n^3 + 18m^2n^4 + 6mn^5 + n^6$$

Thus we have for $\gamma = m^2,\ \beta = 2n^2$

$$a^3 + b^3 = m^6 + 6m^5n + 12m^4n^2 + 16m^3n^3 + 24m^2n^4 + 24mn^5 + 8n^6$$
$$c^3 = m^6 + 6m^5n + 18m^4n^2 + 32m^3n^3 + 36m^2n^4 + 24mn^5 + 8n^6$$

and for $\gamma = 2m^2,\ \beta = n^2$

$$a^3 + b^3 = 8m^6 + 24m^5n + 24m^4n^2 + 16m^3n^3 + 12m^2n^4 + 6mn^5 + n^6$$
$$c^3 = 8m^6 + 24m^5n + 36m^4n^2 + 32m^3n^3 + 18m^2n^4 + 6mn^5 + n^6$$

We see that $a^3 + b^3 \neq c^3$ in both sets. The 3rd, 4th, and 5th terms differ. □

**Corollary 4.2.** *If $(a, b, c)$ is a Pythagorean triple then $a^4 + b^4 \neq c^4$.*

**Proof.** From Theorem 4, if $t = 4$ then

$$\gamma = m^2,\ \beta = 2n^2$$
$$a^4 = m^8 + 8m^7n + 32m^6n^2 + 32m^5n^3 + m^4n^4$$
$$b^4 = 16n^8 + 64n^7m + 128n^6m^2 + 64n^5m^3 + 16n^4m^4$$
$$c^4 = m^8 + 8m^7n + 32m^6n^2 + 80m^5n^3 + 136m^4n^4 + 160m^3n^5 + 128m^2n^6 + 64mn^7 + 16n^8$$
$$\gamma = 2m^2,\ \beta = n^2$$
$$a^4 = 16m^8 + 64m^7n + 128m^6n^2 + 64m^5n^3 + 16m^4n^4$$
$$b^4 = n^8 + 8n^7m + 32n^6m^2 + 32n^5m^3 + 16n^4m^4$$
$$c^4 = 16m^8 + 64m^7n + 128m^6n^2 + 160m^5n^3 + 136m^4n^4 + 80m^3n^5 + 32m^2n^6 + 8mn^7 + n^8$$

Thus we have for $\gamma = m^2,\ \beta = 2n^2$

$$a^4 + b^4 = m^8 + 8m^7n + 32m^6n^2 + 32m^5n^3 + 17m^4n^4 + 64m^3n^5 + 128m^2n^6 + 64mn^7 + 16n^8$$
$$c^4 = m^8 + 8m^7n + 32m^6n^2 + 80m^5n^3 + 136m^4n^4 + 160m^3n^5 + 128m^2n^6 + 64mn^7 + 16n^8$$

and for $\gamma = 2m^2,\ \beta = n^2$

$$a^4 + b^4 = 16^m8 + 64m^7n + 128m^6n^2 + 64m^5n^3 + 32m^4n^4 + 32m^3n^5 + 32m^2n^6 + 8mn^7 + n^8$$
$$c^4 = 16m^8 + 64m^7n + 128m^6n^2 + 160m^5n^3 + 136m^4n^4 + 80m^3n^5 + 32m^2n^6 + 8mn^7 + n^8$$

We also see that $a^4 + b^4 \neq c^4$ in both sets. The 4th, 5th, and 6th terms differ. □

## 5. Conclusion

We have found general formulas for Pythagorean triples and shown that they are valid. A glance between them and the formulas studied by the Greeks shows that the latter is a special case. It is also evident from these formulas that Pythagorean triples are infinite and grouped into two infinite sets.

As an extra, we found an application for the formulas. It was shown that they could be used to prove that $a^t + b^t \neq c^t$ for $t \geqslant 3$ if $(a, b, c)$ is a Pythagorean triple. Proofs for cases, $t = 3, 4$ were shown indicating that higher values for $t$ is also valid. The reason is that the terms around the middle of the binomial expansions will always differ for all $t$.

A C++ script that can be run in Cling and ROOT C/C++ interpreters is attached here instead of typesetting it verbatim.It is just a simple interactive command line interface program.

I also attached tables for $\gamma = 1, 2, 2(2)^2, 3^2, 77^2, 2(77^2)$ with $n$ up to 10000 here:

With hope that this humble work be of benefit to fellowmen, we conclude with these words:
Proverbs 3:13

> *Happy is the man that finds wisdom, and the man that gets understanding.*

Proverbs 9:10

> *The fear of the Lord is the beginning of knowledge: and the knowledge of the Holy One is understanding.*

## 6. Screenshots of C++ script

```
File  Edit  View  Terminal  Go  Help
******************************************************************************
                 A C++ script to generate Pythagorean triples
      Eduardo Calooy Roque / eddieboyroque@gmail.com / November 17, 2012
******************************************************************************
Set 1: m = 77, n > 54, Gamma = 5929
------------------------------------------------------------------------------
   n |      a ,      b ,      c |       Remarks
------------------------------------------------------------------------------
7000 |
7001 | 1084083 , 99106156 , 99112085 | Pythagorean , Primitive
7002 | 1084237 , 99134316 , 99140245 | Pythagorean , Primitive
7003 | 1084391 , 99162480 , 99168409 | Pythagorean , Primitive
7004 | 1084545 , 99190648 , 99196577 | Pythagorean , Primitive
7005 | 1084699 , 99218820 , 99224749 | Pythagorean , Primitive
7006 | 1084853 , 99246996 , 99252925 | Pythagorean , Primitive
7007 |
7008 | 1085161 , 99303360 , 99309289 | Pythagorean , Primitive
7009 | 1085315 , 99331548 , 99337477 | Pythagorean , Primitive
7010 | 1085469 , 99359740 , 99365669 | Pythagorean , Primitive
7011 | 1085623 , 99387936 , 99393865 | Pythagorean , Primitive
7012 | 1085777 , 99416136 , 99422065 | Pythagorean , Primitive
7013 | 1085931 , 99444340 , 99450269 | Pythagorean , Primitive
7014 |
7015 | 1086239 , 99500760 , 99506689 | Pythagorean , Primitive
7016 | 1086393 , 99528976 , 99534905 | Pythagorean , Primitive
7017 | 1086547 , 99557196 , 99563125 | Pythagorean , Primitive
7018 |
7019 | 1086855 , 99613648 , 99619577 | Pythagorean , Primitive
7020 | 1087009 , 99641880 , 99647809 | Pythagorean , Primitive
7021 |
7022 | 1087317 , 99698356 , 99704285 | Pythagorean , Primitive
7023 | 1087471 , 99726600 , 99732529 | Pythagorean , Primitive
7024 | 1087625 , 99754848 , 99760777 | Pythagorean , Primitive
7025 | 1087779 , 99783100 , 99789029 | Pythagorean , Primitive
7026 | 1087933 , 99811356 , 99817285 | Pythagorean , Primitive
7027 | 1088087 , 99839616 , 99845545 | Pythagorean , Primitive
7028 |
7029 |
Generate another table?(y/n):
```

Figure 1. $Set\ 1 : \gamma = m^2,\ \beta = 2n^2$

```
File  Edit  View  Terminal  Go  Help
******************************************************************************
                 A C++ script to generate Pythagorean triples
      Eduardo Calooy Roque / eddieboyroque@gmail.com / November 17, 2012
******************************************************************************
Set 2: m = 77, n > 108, Gamma = 11858
------------------------------------------------------------------------------
   n |      a ,      b ,      c |       Remarks
------------------------------------------------------------------------------
7000 |
7001 | 1090012 , 50092155 , 50104013 | Pythagorean , Primitive
7002 |
7003 | 1090320 , 50120471 , 50132329 | Pythagorean , Primitive
7004 |
7005 | 1090628 , 50148795 , 50160653 | Pythagorean , Primitive
7006 |
7007 |
7008 |
7009 | 1091244 , 50205467 , 50217325 | Pythagorean , Primitive
7010 |
7011 | 1091552 , 50233815 , 50245673 | Pythagorean , Primitive
7012 |
7013 | 1091860 , 50262171 , 50274029 | Pythagorean , Primitive
7014 |
7015 | 1092168 , 50290535 , 50302393 | Pythagorean , Primitive
7016 |
7017 | 1092476 , 50318907 , 50330765 | Pythagorean , Primitive
7018 |
7019 | 1092784 , 50347287 , 50359145 | Pythagorean , Primitive
7020 |
7021 |
7022 |
7023 | 1093400 , 50404071 , 50415929 | Pythagorean , Primitive
7024 |
7025 | 1093708 , 50432475 , 50444333 | Pythagorean , Primitive
7026 |
7027 | 1094016 , 50460887 , 50472745 | Pythagorean , Primitive
7028 |
7029 |
Generate another table?(y/n):
```

FIGURE 2. $Set\ 2 : \gamma = 2m^2,\ \beta = n^2$

## References

[1] James Tattersall: *Elementary Number Theory in Nine Chapters*, (1999)
[2] G. H. Hardy, E.M. Wright: *An Introduction to the Theory of Numbers*, 4th ed., (1960)
[3] M. Richardson: *College Algebra*, 3rd ed., (1966)
[4] Bjarne Stroustrup: *C++ Programming Language*, 3rd ed. (1997)
[5] Rene Brun, Fons Rademakers: *ROOT User's Guide*, (2007)
[6] Vassil Vassilev: *Cling The LLVM-based interpreter*, (2011)

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