Law of Sums of the Squares of Areas, Volumes and Hyper Volumes of Regular Polytopes from Clifford polyvectors

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Abstract

Inspired by the recent sums of the squares law obtained by [1] we derive the law of the sums of the squares of the areas, volumes and hyper-volumes associated with the faces, cells and hyper-cells of regular polytopes in diverse dimensions after using Clifford algebraic methods [3].

1 Introduction

The sums of squares law derived by [1] states that the ratio of the sums of the squares of the edge-lengths of a regular polytope, before and after their orthogonal projection from D to lower D'-dimensions, is given by the ratio of dimensions D'/D. Such sums of squares law was a direct consequence of the Schur orthogonality formula and the edge-transitive property of the point group G that allows to break the order of the group $|G|$ into an integer multiple of the number edges $|G| = kE$, and to recast any inner product of edges $< e_b, e_a >$ in the form $< R_b(e_a), e_a >$, where $R_b(e_a)$ is the linear map (represented as a $D \times D$ matrix) which sends edge e_a into edge e_b.

The Schur orthogonality relations [2] (see Appendix) among the matrix components $R^{(g)}_{nm}$ and associated with the real-valued D-dimensional irreducible representation of the point group G are given by

$$\sum_g^{|G|} R^{(g)}_{nm} \; R^{(g)}_{n'm'} \;=\; \delta_{nn'} \; \delta_{mm'} \; \frac{|G|}{dim \; V} \;=\; \delta_{nn'} \; \delta_{mm'} \; \frac{|G|}{D} \qquad (1a)$$

where one has dim $V = D$. In particular

$$\sum_g^{|G|} R^{(g)}_{n1} \; R^{(g)}_{n'1} \;=\; \delta_{nn'} \; \delta_{11} \; \frac{|G|}{D} \qquad (1b)$$

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where one sums over the number of group elements g denoted by $|G|$. If one takes the trace over the n, n' indices but now with the *restriction* that n, n' range from 1 to D' (since we are projecting down to the subspace S of D'-dim), and if instead of summing over all the group elements from 1 to $|G|$, one sums only over the E group elements associated with one representative of each one of the respective E cosets gH corresponding to the (normal) subgroup H of G that leaves fixed a chosen edge, then one gets in the right hand side of (1b)

$$\sum_{n=1}^{n=D'} \sum_{1}^{E} R_{n1}\ R_{n1} \ = \ \frac{E\ D'}{D} \tag{2}$$

If the lengths $L_a = 1$ of all the edges of the regular polytope (polyhedra) in D-dim are normalized to *unity*, for convenience , then $\sum_{a=1}^{E}(L_a)^2 = E$, hence from eq-(2), after dividing both sides by $E = \sum_{a=1}^{E}(L_a)^2$, one arrives at

$$\frac{1}{\sum_{a=1}^{E}(L_a)^2} \sum_{a=1}^{E} \sum_{n=1}^{n=D'} (R_{n1}^{(a1)})^2 \ = \ \frac{1}{\sum_{a=1}^{E}(L_a)^2} \sum_{a=1}^{E} \ (L'_a)^2 \ = \ \frac{D'}{D} \tag{3}$$

which is the sum of squares law described by [1] in a nutshell.

This can be seen by choosing the reference edge $\vec{e}^{(1)}$ to point along the direction of the first coordinate axis such that the D-components of this fiducial edge are $\vec{e}^{(1)} = (1, 0, 0, ..., 0) \Rightarrow e_1^{(1)} = 1$. If $\mathbf{R}^{(a1)}(e^{(1)})$ represents the linear map (matrix) which sends edge $\vec{e}^{(1)}$ to edge $\vec{e}^{(a)}$, after projecting $\mathbf{R}^{(a1)}(\vec{e}^{(1)})$ onto the subspace S of D'-dimensions, and spanned by the unit basis vectors $\vec{u}_n$, where the values of n are now restricted to be $n = 1, 2, 3, ...D'$, one finds that the latter matrix $\mathbf{R}^{(a1)}$has for non-vanishing components the following $\mathbf{R}_{n1}^{(a1)}$ ones.

Therefore, the square of the orthogonal projection of each edge of unit length onto the subspace S of D'-dimensions is given by

$$(L'_a)^2 \ = \ \sum_{n=1}^{n=D'} \ (< \vec{e}^{(a)},\ \vec{u}_n >)^2 \ = \ \sum_{n=1}^{n=D'} \ (< \mathbf{R}^{(a1)}\vec{e}^{(1)},\ \vec{u}_n >)^2 \ =$$

$$\sum_{n=1}^{n=D'} \ (< \ (\mathbf{R}_{n1}^{(a1)}\ e_1^{(1)})\ \vec{u}_n,\ \vec{u}_n \ >)^2 \ = \ \sum_{n=1}^{n=D'} \ (R_{n1}^{(a1)})^2 \tag{4}$$

hence, after inserting eq-(4) into the left hand side of eq-(3) it yields the sum of length squares law, see [1] for further details.

One can generalize the sums of length squares result of [1] to the sums of areas, volumes, hyper-volumes squared by recurring to generalized transformations involving bi-vector area coordinates $x^{\mu_1\mu_2} = -x^{\mu_2\mu_1}$; tri-vector volume coordinates $x^{\mu_1\mu_2\mu_3} = -x^{\mu_2\mu_1\mu_3}$, ..., and poly-vector coordinates $x^{\mu_1\mu_2\cdots\mu_p}$. The index p ranges from 1 to D. The generalized coordinates in Clifford spaces [3] are associated with a Clifford-valued $\mathbf{X} = X^M\Gamma_M$ coordinate in D-dimensions and corresponding to a Clifford algebra $Cl(D)$. For example, in $D = 4$ dimensions one has that $\mathbf{X}$ can be expanded in a Clifford basis as

$$\mathbf{X} \ = \ s\,\mathbf{1} + x^{\mu}\,\gamma_{\mu} + x^{\mu\nu}\,\gamma_{\mu}\wedge\gamma_{\nu} + x^{\mu\nu\rho}\,\gamma_{\mu}\wedge\gamma_{\nu}\wedge\gamma_{\rho} + x^{\mu\nu\rho\tau}\,\gamma_{\mu}\wedge\gamma_{\nu}\wedge\gamma_{\rho}\wedge\gamma_{\tau} \quad (5)$$

It is very natural to construct group representations of face-transitive, cell-transitive, hyper-cell-transitive groups in terms of generalized matrices (not to be confused with hyper-matrices) with polyvector-valued indices. And as such, one can extend the Schur orthogonality relations to generalized matrices. The Schur orthogonality relation among the *generalized* matrix components $R^{(g)}_{[n_1 n_2]\ [m_1 m_2]}$ involving bi-vector valued indices and associated with the real-valued $\frac{D(D-1)}{2}$-dimensional irreducible representation of the face-transitive group G' , which maps any face of the regular polytope into another congruent face, is given by a generalization of the formula (1b)

$$\sum_{g}^{|G'|} R^{(g)}_{[n_1 n_2]\ [12]}\ R^{(g)}_{[n'_1 n'_2]\ [12]} \ = \ \delta_{[n_1 n_2]\ [n'_1 n'_2]}\ \delta_{[12]\ [12]}\ \frac{|G'|}{D(D-1)/2} \quad (6)$$

If instead of edge-transitivity one assumes now $face$-transitivity, then $R^{(a1)}_{[n_1 n_2]\ [12]}$ are the generalized matrices entries corresponding to the generalized transformations (poly-rotations, for example) which sends the fiducial face $f^{(1)}$, pointing along the bivector basis direction $e^{[12]} = e^1 \wedge e^2 = \gamma^1 \wedge \gamma^2$, onto any other face $f^{(a)}$ (another bi-vector) whose components expansion in terms of the bi-vector basis elements is $f^{(a)} = \sum_1^{D(D-1)/2} f^{(a)}_{[n_1 n_2]} e^{[n_1 n_2]}$.

The double-index notation in (6) just represents the bi-vector coordinates character of the areas. Because there are $D(D-1)/2$ independent bi-vectors in D-dimensions, this explains the presence of the factor $D(D-1)/2$ in the denominator of eq-(6) and which coincides with the dim V of the real-valued irreducible representation of the face-transitive group G'. The generalized Kronecker deltas are

$$\delta_{[n_1 n_2]\ [n'_1 n'_2]} \ = \ \delta_{n_1 n'_1}\delta_{n_2 n'_2} \ - \ \delta_{n_1 n'_2}\delta_{n_2 n'_1}$$

The extension of the Schur orthogonality relations for polyvectors is rigorously analyzed in the Appendix. In D, D'-dimensions there are $D(D-1)/2, D'(D'-1)/2$ bi-vector coordinates, respectively. Summing (tracing) over the double indices $[n_1 n_2]$, restricting the summation over the bi-vector indices from 1 to $D'(D'-1)/2$, and summing over the F group elements associated with one representative of each one of the respective F cosets gH' corresponding to the (normal) subgroup H' of G' that leaves fixed a chosen face in eq- (6), gives

$$\sum_{[n_1 n_2]=1}^{D'(D'-1)/2}\ \sum_{1}^{F} R_{[n_1 n_2]\ [12]}\ R_{[n_1 n_2]\ [12]} \ = \ \frac{D'(D'-1)}{2}\ \frac{F}{D(D-1)/2} \quad (7)$$

after following similar arguments as above, eq-(7) leads to the sums of areas squared law ($A = 1 \Rightarrow \sum_{faces}(A)^2 = F$)

$$\frac{\sum_{faces}\ (A')^2}{\sum_{faces}(A)^2} = \frac{D'(D'-1)}{D(D-1)} \tag{8}$$

For example, lets take regular polyhedra in $D = 3$ and project down onto $D' = 2$. Taking a cube with 6 sides of unit area, upon an orthogonal projection onto the base comprised of one of its square faces , one has two squares of unit area (the top and bottom), and four faces of zero area (the four side faces, the four walls, yield a zero projection). Then the ratio of the sums of the areas squared will be $\frac{2}{6} = \frac{1}{3} = \frac{2(2-1)}{3(3-1)}$, which agrees with eq-(8).

In the case of a regular tetrahedron, one has upon an orthogonal projection onto the equilateral triangular base, the following areas : the base triangle of area equal to *unity*. Each one of the remaining 3 projections of the other 3 equilateral triangles will have an area equal to $\frac{1}{3}$ of the area of the base triangle. Their projections are 3 congruent triangles which fit inside the base triangle without any gaps. So the ratio of the sums of the areas squared is

$$\frac{1+3(\frac{1}{3})^2}{4} = \frac{1+\frac{1}{3}}{4} = \frac{\frac{4}{3}}{4} = \frac{1}{3} = \frac{2(2-1)}{3(3-1)} = \frac{D'(D'-1)}{D(D-1)} \tag{9}$$

In the case of an octahedron involving 8 equilateral triangular faces of *unit* areas, one has that the edge length L must be such that $Area = L^2\frac{\sqrt{3}}{4} = 1 \Rightarrow L^2 = \frac{4}{\sqrt{3}}$. The projection onto the equatorial square section gives 2 copies of 4 triangles which fit inside the square of area $L^2 = \frac{4}{\sqrt{3}}$. So each triangle will have an equal area of $\frac{1}{4}L^2 = \frac{1}{\sqrt{3}}$. The sum of the unit areas squared of the 8 faces of the octahedron is 8. Thus the sums of the projected areas squared is then

$$2\ \times 4\ [\ \frac{1}{4}\ L^2\ ]^2\ =\ 8\ \frac{1}{3}\ \Rightarrow$$

$$\frac{\sum_{faces}\ (A')^2}{\sum_{faces}(A)^2} = \frac{\frac{8}{3}}{8} = \frac{1}{3} = \frac{D'(D'-1)}{D(D-1)} \tag{10}$$

In the case of a dodecahedron, with 12 pentagonal faces of unit area, one has after the orthogonal projection onto a pentagonal base of unit area, the following : The top and bottom base give an area squared equal to $1 + 1 = 2$. The remaining 10 faces are oriented such that the angles formed by the normals to their faces with the vertical axis are given by $\alpha = arcos\frac{1}{\sqrt{5}} = 63.4349 = 180 - 116.5651$ degrees. The angle 116.5651 degrees is the dihedral angle among the pentagonal faces [4]. Therefore, the squared of the 10 projected areas yield a net value of $10\ cos^2\alpha = \frac{10}{5} = 2$. Hence the sum of the projection of the 12 areas squared is then $2 + 2 = 4$, and such that the ratio of the areas is $\frac{4}{12} = \frac{1}{3}$ as expected.

Finally in the icosahedron case, one has 20 equilateral triangular faces than can be grouped into 5 top faces, 10 middle faces and 5 lower faces. If the areas of the 20 equilateral triangles are normalized to *unity* their edges must have for length L such that $Area = L^2 \frac{\sqrt{3}}{4} = 1 \Rightarrow L^2 = \frac{4}{\sqrt{3}}$. The projection of the 5 top and 5 bottom faces onto two pentagonal horizontal cross sections yield a total of 10 congruent triangular regions which fit inside the two pentagons. The areas of the pentagonal horizontal cross section are

$$Area(pentagon) \ = \ 5 \ \frac{L^2}{4} \ \frac{1}{tan(\pi/5)} \ = \ 5 \ \frac{\frac{4}{\sqrt{3}}}{4} \ \frac{1}{tan(\pi/5)} \ = \ \frac{5}{\sqrt{3} \ tan(\pi/5)} \quad (11)$$

so the projection of the 5 bottom and 5 top triangles will *each* have an area equal to 1/5-th the area of the above pentagon : $\frac{1}{\sqrt{3} \ tan(\pi/5)}$. So the sums of the projections of these areas-squared will be

$$10 \ (\frac{1}{\sqrt{3} \ tan(\pi/5)})^2 \ = \ 10 \ \frac{1}{3 \ tan^2(\pi/5)} \ = \ 6.314 \quad (12)$$

We are left with the projections of the 10 triangles in the middle region. Their normals make an angle with respect to the vertical axis (orthogonal to the pentagonal horizontal cross sections) given by $\beta = 79.19$ degrees. The supplement angle is $180 - 79.19 = 100.81$ degrees. The angle formed by the normals to the 5 top and 5 bottom triangles with the vertical axis is

$$\alpha \ = \ arcos(\frac{1}{\sqrt{3} \ tan(\pi/5)}) \ = \ 37.372 \ degrees \quad (13)$$

Therefore the net angle is $100.81 + 37.372 = 138.182$ degrees which coincides with the value of the dihedral angle of the icosahedron [4].

Therefore the sums of the squares of the projected areas of the 10 triangles of the middle region of the icosahedron are

$$10 \ cos^2(\beta) \ = \ 10 \ cos^2(79.19) \ = \ 10 \ \times \ 0.0352 \ = \ 0.352 \quad (14)$$

Adding then eqs(12, 14) leads to a net value of $0.352 + 6.314 = 6.666 \sim \frac{20}{3}$. Therefore the ratio of the areas is $\frac{20}{3 \times 20} = \frac{1}{3}$ as expected. An *exact* result would be obtained if one writes all the expressions in terms of the Golden Mean (in terms of quantities involving the $\sqrt{5}$). This completes our arguments for the 5 Platonic solids.

In the case of 3-dim cells and for a cell-transitive group G'' one has for the Schur orthogonality relation the following generalization of eq-(6) to the case of generalized matrices $R^{(g)}_{[n_1 n_2 n_3] \ [m_1 m_2 m_3]}$ with trivector-valued indices

$$\sum_g^{|G''|} R^{(g)}_{[n_1 n_2 n_3] \ [123]} \ R^{(g)}_{[n'_1 n'_2 n'_3] \ [123]} \ = \ \delta_{[n_1 n_2 n_3] \ [n'_1 n'_2 n'_3]} \ \delta_{[123] \ [123]} \ \frac{|G''|}{D(D-1)(D-2)/3!} \quad (15)$$

The triple-index notation in (15) just represents the tri-vector coordinates character of the volumes. Because there are $D(D-1)(D-2)/3!$ independent tri-vectors in D-dimensions, this explains the presence of the factor $D(D-1)(D-2)/3!$ in the denominator of eq-(15) and which coincides with the dim V of the real-valued irreducible representation of the 3-dim *cell*-transitive group G''. The generalized Kronecker deltas in (15) are defined by the determinant

$$\delta_{[n_1 n_2 n_3]\ [n'_1 n'_2 n'_3]} \ = \ det \left(\begin{array}{ccc} \delta_{n_1 n'_1} & \ldots & \delta_{n_1 n'_3} \\ \delta_{n_2 n'_1} & \ldots & \delta_{n_2 n'_3} \\ \delta_{n_3 n'_1} & \ldots & \delta_{n_3 n'_3} \end{array} \right) \tag{16}$$

Repeating the same argument as before, if instead of edge/face transitivity one assumes now *cell*-transitivity, one arrives at

$$\frac{\sum_{cells}\ (V')^2}{\sum_{cells} (V)^2} \ = \ \frac{D'(D'-1)(D'-2)}{D(D-1)(D-2)} \tag{17}$$

and in general for higher dimensional cells, p-dimensional hyper-volumes , one has

$$\frac{\sum_{hypercells}\ (\mathcal{V}')^2}{\sum_{hypercells} (\mathcal{V})^2} \ = \ \frac{D'(D'-1)(D'-2)\ .....\ (D'-p+1)}{D(D-1)(D-2)\ .....\ (D-p+1)} \tag{18}$$

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Appendix : Schur Orthogonality Relations for Polyvectors

In $D = 3$ there is a one-to-one correspondence between vectors $A = A^i e_i = A^i \gamma_i$ and bivectors $B = B_{jk} e^{jk} = B_{jk} e^j \wedge e^k = B_{jk} \gamma^j \wedge \gamma^k$. In particular they are duals of each other : $A^i = \epsilon^{ijk} B_{jk}$. Therefore, one can rotate (transform) the bivectors using the generalized matrices with multi-indices such that $B'_{[j_1 j_2]} = \mathbf{R}_{[j_1 j_2]\ [k_1 k_2]} B_{[k_1 k_2]}$ (recurring to Einstein's summation convention for repeated indices), or one can rotate (transform) their dual vectors (the normals to the planes associated to the bivectors) such that $A'_j = R_{jk} A_k$. The one-to-one correspondence between the matrices and generalized matrices' components is

$$\mathbf{R}_{[12]\ [12]} \leftrightarrow R_{33}, \ \ \mathbf{R}_{[13]\ [13]} \leftrightarrow R_{22}, \ \ \mathbf{R}_{[23]\ [23]} \leftrightarrow R_{11}$$

$$\mathbf{R}_{[12]\ [13]} \leftrightarrow -R_{32}, \ \mathbf{R}_{[12]\ [23]} \leftrightarrow R_{31}, \ \ \mathbf{R}_{[23]\ [12]} \leftrightarrow R_{13}, \ .... \tag{A.1}$$

The face-transitive group G' in this case will map the normal vector of one face into the normal vector of another face; which is equivalent to mapping the bivector of one face into the corresponding bivector of another face of the polyhedron. Therefore, in $D = 3$ one can implement the Schur orthogonality relations involving unitary irreducible representations of the face-transitive group G' , either by using the ordinary 3×3 matrices R_{jk}, or by using their associated generalized matrices $\mathbf{R}_{[j_1 j_2]\ [k_1 k_2]}$ with bivector indices in (A.1), as follows

$$\sum_g^{|G'|} R^{(g)}_{nm}\ R^{*(g)}_{n'm'} \ = \ \delta_{nn'}\ \delta_{mm'}\ \frac{|G'|}{dim\ V} \ \longleftrightarrow$$

$$\sum_g^{|G'|} R^{(g)}_{[n_1 n_2]\ [m_1 m_2]}\ R^{*(g)}_{[n'_1 n'_2]\ [m'_1 m'_2]} \ = \ \delta_{[n_1 n_2]\ [n'_1 n'_2]}\ \delta_{[m_1 m_2]\ [m'_1 m'_2]}\ \frac{|G'|}{dim\ V} \quad (A.2)$$

where in our case described above one has in $D = 3$ that the dim $V = 3$.

In higher dimensions, because there are more than one normal vector to a given surface, the correct way to proceed is as follows. If the symmetry group is edge-transitive (face-transitive) it implies that it maps an edge (face) into another edge (face). Given two adjacent (incident) and directed edges $\vec{e}^{(1)}, \vec{e}^{(2)}$ at a vertex $\mathbf{v}$ of a face of a regular polytope in $D > 3$, the bivector $\vec{e}^{(1)} \wedge \vec{e}^{(2)}$ associated with its face has for magnitude a value proportional to its area. In the case of the cube in $D = 3$, the proportionality constant for a square face is unity. In the case of a pentagonal face (dodecahedron) the proportionality constant is $\frac{4cos(3\pi/5)}{5}$.

If there is a symmetry group G' which is face-transitive it means that one can map a bivector $\vec{e}^{(1)} \wedge \vec{e}^{(2)}$ to another bivector $\vec{e}^{(\alpha)} \wedge \vec{e}^{(\alpha+1)}$ lying on another face. The indices $\alpha, \alpha+1$ denote two other adjacent (incident) edges at a vertex in another face of the regular polytope. Furthermore, if the point group G of the polytope is edge-transitive one has that

$$\vec{e}^{(\alpha)} \ = \ R^{(\alpha,1)}_{n_1 m_1}\ \vec{e}^{(1)}_{m_1}\ e^{n_1}, \quad \vec{e}^{(\alpha+1)} \ = \ R^{(\alpha+1,2)}_{n_2 m_2}\ \vec{e}^{(2)}_{m_2}\ e^{n_2} \qquad (A.3)$$

where once again we employ the Einstein summation convention : a summation of indices is performed over the repeated indices. Given that

$$\vec{e}^{(2)} \ = \ R^{(2,1)}_{n_3 m_3}\ \vec{e}^{(1)}_{m_3}\ e^{n_3} \qquad (A.4)$$

one can rewrite

$$\vec{e}^{(\alpha+1)} \ = \ R^{(\alpha+1,2)}_{n_2 m_2}\ \vec{e}^{(2)}_{m_2}\ e^{n_2} \ = \ R^{(\alpha+1,2)}_{n_2 m_2}\ R^{(2,1)}_{m_2 m_3}\ \vec{e}^{(1)}_{m_3}\ e^{n_2} \ =$$

$$R^{(\alpha+1,1)}_{n_2 m_3}\ \vec{e}^{(1)}_{m_3}\ e^{n_2}\ , \quad since\ \ R^{(\alpha+1,1)}_{n_2 m_3} \ = \ R^{(\alpha+1,2)}_{n_2 m_2}\ R^{(2,1)}_{m_2 m_3} \qquad (A.5)$$

resulting from the group composition law and definition of a group representation $R(g)R(g') = R(g \cdot g')$. From eqs-(A.3-A.5) we infer that the bivector $\vec{e}^{(\alpha)} \wedge \vec{e}^{(\alpha+1)}$ can be expressed as

$$\vec{e}^{(\alpha)} \wedge \vec{e}^{(\alpha+1)} \ = \ [\ R^{(\alpha,1)}_{n_1 m_1}\ \vec{e}^{(1)}_{m_1}\ e^{n_1}\ ]\ \wedge\ [\ R^{(\alpha+1,1)}_{n_2 m_3}\ \vec{e}^{(1)}_{m_3}\ e^{n_2}\ ] \qquad (A.6)$$

Choosing the first coordinate axis to coincide with the direction of $\vec{e}^{(1)}$ implies that its only nonvanishing component is $\vec{e}_1^{(1)} = 1$, so the bivector in (A.6) becomes

$$\vec{e}^{(\alpha)} \wedge \vec{e}^{(\alpha+1)} \ = \ [\ R^{(\alpha,1)}_{n_1 1}\ e^{n_1}\ ] \wedge [\ R^{(\alpha+1,1)}_{n_2 1}\ e^{n_2}\ ] \ = \ [\ R^{(\alpha,1)}_{n_1 1}\ R^{(\alpha+1,1)}_{n_2 1}\ ]\ e^{n_1} \wedge e^{n_2} \ =$$

$$\frac{1}{2}\ [\ R^{(\alpha,1)}_{n_1 1}\ R^{(\alpha+1,1)}_{n_2 1}\ -\ R^{(\alpha,1)}_{n_2 1}\ R^{(\alpha+1,1)}_{n_1 1}\ ]\ e^{[n_1 n_2]}, \quad e^{[n_1 n_2]}\ \equiv\ e^{n_1}\ \wedge\ e^{n_2} \quad (A.7)$$

In the last line of (A.7) we have antisymmetrized the expression with respect to the indices n_1, n_2 due to the antisymmetry property of the bivector indices. The contribution of the symmetric terms is zero. Hence, from (A.7) we can finally read-off the components of the generalized matrix $\mathbf{R}^{(a1)}$, comprised of *bivector* indices, in terms of the ordinary matrix components with vector indices and representing the map which sends the face $f^{(1)}$ of unit area into another face $f^{(a)}$ of unit area

$$\mathbf{R}^{(a1)}_{[n_1 n_2]\ [12]}\ =\ \frac{\kappa}{2}\ [\ R^{(\alpha,1)}_{n_1 1}\ R^{(\alpha+1,1)}_{n_2 1}\ -\ R^{(\alpha,1)}_{n_2 1}\ R^{(\alpha+1,1)}_{n_1 1}\ ] \Rightarrow$$

$$f^{(a)}\ =\ \mathbf{R}^{(a1)}_{[n_1 n_2]\ [12]}\ f^{(1)}_{[12]}\ e^{[n_1 n_2]}\ =\ \mathbf{R}^{(a1)}_{[n_1 n_2]\ [12]}\ e^{[n_1 n_2]} \qquad (A.8)$$

since $f^{(1)}_{[12]} = 1$, after orienting the face $f^{(1)}$ along the bivector $e^{[12]}$, and where κ is a numerical constant whose value depends on the shape of the faces; i.e. squares, triangles, pentagons. For example, $\kappa = 1$ for squares, $\kappa = \frac{4cos(3\pi/5)}{5}$ for pentagons, ... In (A.8) there is a correlation between the edge index α enumeration and the face index a enumeration of the polytope. Correlation which can always be found in any polytope. In the case of the cube, there are 3 incident edges at a common vertex within each one of the 6 faces (planes). The independent bivectors are e^{12}, e^{13}, e^{23} and correspond to the 3 plane classes (orientations) of the cube in 3-dimensions.

Concluding, the Schur orthogonality relations of the face-transitive group G', for unitary irreducible representations of dimension $D(D-1)/2$, given in terms of generalized matrices $\mathbf{R}^{(a1)}$ and whose components are expressed in terms of bivector indices $\mathbf{R}^{(a1)}_{[n_1 n_2]\ [12]}$ as displayed explicitly in eq-(A.8), are given by

$$\sum_g^{|G'|} R^{(g)}_{[n_1 n_2]\ [m_1 m_2]}\ R^{*(g)}_{[n'_1 n'_2]\ [m'_1 m'_2]}\ =\ \delta_{[n_1 n_2]\ [n'_1 n'_2]}\ \delta_{[m_1 m_2]\ [m'_1 m'_2]}\ \frac{|G'|}{dim\ V} \quad (A.9)$$

with dim $V = D(D-1)/2$. In the case of real-valued representations one has $R^{*(g)}_{[n'_1 n'_2]\ [m'_1 m'_2]} = R^{(g)}_{[n'_1 n'_2]\ [m'_1 m'_2]}$. One can repeat the above arguments for the higher grade polyvectors corresponding to volumes, hypervolumes, ... leading to eqs-(17,18) for cell-transitive and hyper-cell-transitive groups.

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