Difficulties of the set of natural numbers

Qiu Kui Zhang
Nanjing University of Information Science and Technology
210044 Nanjing, China
E-mail: zhangqk@nuist.edu.cn

Abstract

In this article some difficulties are deduced from the set of natural numbers. The demonstrated difficulties suggest that if the set of natural numbers exists it would conflict with the axiom of regularity. As a result, we have the conclusion that the class of natural numbers is not a set but a proper class.

Introduction

That all the natural numbers can be pooled together to form an infinite set is a fundamental hypothesis in mathematics and philosophy, which now is widely accepted by mathematicians and scientists from various disciplines. With this hypothesis mathematicians had systematically developed a theory of infinity, namely, set theory which had become the foundation of modern mathematics and science ever since. Although once this hypothesis was a controversial issue between different schools of mathematics and philosophy and some intuitionists object to it on the grounds that a collection of objects produced by an infinite process should not be regarded as a completed entity, they do not provide further evidence to prove that it will cause logical contradiction. And no contradiction resulting from this hypothesis had ever been reported. Today the debate has subsided and most scientists do not doubt about the validity of this hypothesis. However, in our recent study we have found some logical contradictions resulting from this hypothesis, which suggest the axiom of infinity is self-contradictory and conflicts with the axiom of regularity. So set theory is not as consistent as we had thought before. We anticipate our study to be a starting point for the establishment of a more sophisticated foundation theory to prevent mathematics and thus other sciences from contradiction.

1 The definition of natural numbers in set theory

In order to define natural numbers and study the set of natural numbers within the framework of set theory it is necessary to define a successor relation first [1].

Definition 1.1. The successor of a set x is the set $x^+ = x \cup \{x\}$.

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The notation '+' in above definition represents the successor operator which can be applied to any set to obtain its successor.

In set theory the first natural number 0 is defined with the empty set ϕ, then number 1 with the successor of 0 and so on. To make the expression more intuitively we usually use the more suggestive notation $n+1$ for n^+ when n is a number. So we have following inductive definition of natural numbers

Definition 1.2. The definition of natural numbers.

1. $0 = \phi$ is a natural number.

2. If n is a natural number, its successor $n+1$ is also a natural number.

3. Only sets obtained by application of 1 and 2 are natural numbers.

The first rule of above definition is the basis of the induction which defines the initial natural number 0, and the second rule is the inductive step which can be repeatedly applied to obtain other natural numbers. The third rule is the restriction clause. So we can assign each natural number a certain value of a particular set

$$0 = \phi, 1 = \phi^+, 2 = \phi^{++}, \cdots$$

Whether all the natural numbers can be pooled together to form a completed infinite entity i.e. a set is a critical issue in mathematics and philosophy. Around it two opposite concepts of infinity have been developed, which are potential infinity and actual infinity. The former regards the infinite series 0, 1, 2, ... is potentially endless and the process of adding more and more numbers can not be exhausted in principle, so it never can make a definite entity. The latter is based one the hypothesis that all natural numbers can form an actual, completed totality, namely, a set. That means the static set has already been completed and contained all natural numbers. Set theory is based on the notion of actual infinity that is clearly manifested in the axiom of infinity which postulates the existence of an inductive set and thus guarantees the existence of the set of natural numbers.

There is, however, an unnoticed difficulty behind the definition of natural numbers and the concept of the set of natural numbers. Notice that the third restriction clause in definition 1.2 does not limit the repetition of the inductive step to finite times only, so, literally, if the result of performing infinite times adding one operation to 0 exists it must be a natural number. And if the set of natural numbers exists, following the theory of ordinal numbers, it is exact the first transfinite ordinal number ω. Therefore, according to ordinal arithmetic $\omega = 0 + \omega = 0 + 1 \times \omega$, the result of performing ω times adding one operation to 0 exists and equals ω. That means number ω can be obtained by the application of once clause 1 and ω times clause 2 of definition 1.2. Consequently, following definition 1.2, ω is a natural number that leads to ω is a member of itself. However, this result obviously conflicts with the axiom of regularity which asserts no set can be a member of itself. And it also conflicts with the result of induction principle, for it is easy to prove that all natural numbers are finite sets with mathematical induction but ω is not a finite set. The proof with mathematical induction is as follow: 0 is a finite set; if n is a finite set $n + 1 = n \cup \{n\}$ is also a finite set; so all natural numbers are finite sets. Noticing above difficulty we make in-depth investigation into the set of natural numbers in the following sections.

2 Difficulties of the set of natural numbers

In set theory the axiom of infinity which postulates the existence of an inductive set guarantees the existence of the set of natural numbers.

The Axiom of Infinity. An inductive set exists [1].

Because N, the set of natural numbers, is the smallest inductive set, it is easy to prove its existence based on the axiom of infinity. Let C be an existing inductive set; then we justify the existence of N on the basis of the axiom of comprehension [1]

$$N = \{x \in C | x \in I \text{ for every inductive set } I\}.$$

That implies C exists and then N exists.

Usually set N can be expressed as an infinite list of natural numbers such as

$$N = \{0, 1, 2, \cdots\}. \tag{2.1}$$

or briefly as

$$N = \{x | n(x)\}. \tag{2.2}$$

where $n(x)$ ($n(x)$ represents $x \in I$ for every inductive set I) is the predicate that x is a natural number. However, this form of expression obviously uses the comprehension principle, which is thought to be the source of paradoxes in Cantor's naive set theory. Whether the using of comprehension principle here will result in contradiction is an interesting issue to us. And it is indeed the case, for we have found sufficient evidence to prove that the notion of the set of natural numbers is illogical and will lead to logical contradiction. Here we show our findings of a sequence of conflicts based on the question whether there is the greatest element in N. First let's consider a special collection S of all $x \in N$ with the property $P(x)$

$$S = \{x \in N | P(x)\}. \tag{2.3}$$

where the property $P(x)$ is $\forall y \in N(y \leq x)$ which means x is greater than or equal to all the elements of N. According to the axiom schema of comprehension [1], if N is a set, S is a definite set. Obviously, if S is an empty set the greatest element of N does not exist; if S is not an empty set it must contain the greatest element of N and thus the greatest element of N does exist. According to the law of excluded middle, for all x of N, x either has or does not have the property $P(x)$, so intuitively we have following method to obtain set S. That is we can deduct all N's elements without the property P from N and the remaining part is S. To do this we need to define an iterative process with transfinite recursion to recursively deduct all non-greatest elements of N. The iterative process can be implemented in this way. Choose two elements out of N, remove the smaller one that clearly does not have the property P and then return the bigger one to the remaining part. Repeat this procedure until there are no two elements left in the remaining part that can be further chosen to implement further deduction and this particular remaining part should be S. First let's define a Min function performed to two natural numbers to obtain the smaller one

$$Min(x, y) = \begin{cases} x & if \quad x \leq y \\ y & if \quad x > y \end{cases}$$

In set theory, it is obvious that the $Min(x, y)$ function can be implemented as the intersection of natural numbers x and y

$$Min(x, y) = x \cap y \tag{2.4}$$

Then according to the axiom of choice [1], there is a choice function f, defined on set $X = P(N) - \{\phi\}$ (where $P(N)$ is the power set of N, and $P(N) - \{\phi\}$ represents the difference of $P(N)$ and $\{\phi\}$), such that

$$\forall x(x \in X \to f(x) \in x)$$

where symbol $\to$ in above logical formula is the logical implication symbol. So we have following inductive definition.

Definition 2.1. For all ordinals $\alpha \in On$, recursively define following transfinite sequences A_α, B_α and a_α.

1. $A_\alpha = \{a_\beta | \beta < \alpha\}$.

2. $B_\alpha = N - A_\alpha$.

3. $a_\alpha = \begin{cases} Min(f(B_\alpha), f(B_\alpha - \{f(B_\alpha)\})) & if \quad Card(B_\alpha) > 1 \\ b & if \quad Card(B_\alpha) = 1 \\ c & if \quad Card(B_\alpha) = 0 \end{cases}$.

Where ordinal number α indicates a particular recursion step, A_α is the set that has already been deducted from N before the step α, B_α is the remaining part of N before the step α, a_α is the particular element of N that is deducted at the current step α if B_α contains more than one element, $Card(B_\alpha)$ stands for the cardinality of set B_α, $b = \{2\}$ and $c = \{2, 3\}$ are sets not belong to N.

Accordingly, we say the iterative process has already deducted all the elements of N before step α if $B_\alpha = \phi$ (the remaining part is already empty before step α is performed) and the iterative process can not deduct all the elements of N at step α if $B_{\alpha+1} \neq \phi$ (the remaining part is still not empty after step α is performed).

It is easy to obtain every elements of these transfinite sequences with definition 2.1. First it is obvious that $A_0 = \phi$ (before step 0 nothing is deducted), $B_0 = N$ (before step 0 the remaining part is exact N) and $a_0 = Min(f(N), f(N - \{f(N)\}))$. Second if we have obtained all a_β for $\beta < \alpha$, then we can obtain A_α, B_α and a_α with the three clauses of definition 2.1 respectively. So, in line with the principle of transfinite recursion, the transfinite sequences A_α, B_α and a_α exist.

According to the clause 3 of definition 2.1, the recursion steps can be classified into three classes corresponding to conditions $Card(B_\alpha) > 1$, $Card(B_\alpha) = 1$ and $Card(B_\alpha) = 0$. And it is easy to see that only if the step α satisfies condition $Card(B_\alpha) > 1$ does the iterative process deducts one element from N at step α; otherwise it deducts nothing from N at step α. And we have

Lemma 2.2. *The transfinite sequences have following properties.*

1. $\beta \leq \alpha \to A_\beta \subseteq A_\alpha \wedge B_\alpha \subseteq B_\beta$.

2. $\alpha \neq \beta \wedge Card(B_\alpha) > 1 \wedge Card(B_\beta > 1) \to B_\alpha \neq B_\beta$.

3. $\exists\gamma(Card(B_\gamma) = 1)$.

Proof. 1. Notice $\beta \leq \alpha$ and A_α's definition. Then we have
$\forall x(x \in A_\beta \to \exists\gamma(a_\gamma = x \wedge \gamma < \beta) \to \exists\gamma(a_\gamma = x \wedge \gamma < \alpha) \to x \in A_\alpha)$
$\to A_\beta \subseteq A_\alpha$
and
$A_\beta \subseteq A_\alpha \to \forall x(x \notin A_\alpha \to x \notin A_\beta)$
$\to \forall x(x \in N \wedge x \notin A_\alpha \to x \in N \wedge x \notin A_\beta)$
$\to \forall x(x \in N - A_\alpha \to x \in N - A_\beta)$
$\to \forall x(x \in B_\alpha \to x \in B_\beta)$
$\to B_\alpha \subseteq B_\beta$
So we obtain property 1.

2. If $\alpha \neq \beta$, then either $\alpha < \beta$ or $\beta < \alpha$. Let $\alpha < \beta$. Then
$Card(B_\alpha) > 1 \to a_\alpha = Min(f(B_\alpha), f(B_\alpha - \{f(B_\alpha)\})) \to a_\alpha \in B_\alpha$
Then noticing A_α's definition and $\alpha < \beta \to \alpha + 1 \leq \beta \to B_\beta \subseteq B_{\alpha+1}$, we have
$a_\alpha \in A_{\alpha+1} \to a_\alpha \notin N - A_{\alpha+1} \to a_\alpha \notin B_{\alpha+1} \to a_\alpha \notin B_\beta$
So considering above two cases: $a_\alpha \in B_\alpha$ and $a_\alpha \notin B_\beta$, we obtain
$B_\alpha \neq B_\beta$
For the same reason it is easy to prove if $\beta < \alpha$ then $B_\beta \neq B_\alpha$, so we have property 2 which indicates all B_α in the transfinite sequence are non-repeating when they satisfy $Card(B_\alpha) > 1$.

3. Let $B = \{B_\alpha | Card(B_\alpha) > 1\}$, so all the members of B are subsets of N. Therefore B is a subset of $P(N)$ that implies B is a set.
Then let $A = \{\alpha | Card(B_\alpha) > 1\}$. From property 2 of lemma 2.2 we know all B_α in the sequence are non-repeating when they satisfy $Card(B_\alpha) > 1$, so there is a one-to-one correspondence, $F : A \leftrightarrow B$ (ordinal α corresponds to B_α), between A and B. So A is a set also. Then for any ordinal numbers α and β, we have following logical derivation
$\alpha \in A \wedge \beta < \alpha \rightarrow Card(B_\alpha) > 1 \wedge B_\alpha \subseteq B_\beta \rightarrow Card(B_\beta) > 1 \rightarrow \beta \in A$
Therefore we obtain
$\forall\alpha\forall\beta(\beta < \alpha \wedge \alpha \in A \rightarrow \beta \in A)$
That indicates set A is an initial segment of ordinal and thus there is an ordinal number λ equals A
$A = \lambda = \{\alpha | \alpha < \lambda\}$
Observe that λ does not satisfy $\lambda < \lambda$. Then $\lambda \notin \lambda$ and thus $\lambda \notin A$. That implies ordinal number λ must not satisfy set A's condition, so $Card(B_\lambda) \ngtr 1$. Therefore there are only two cases, i.e., $Card(B_\lambda) = 1$ or $Card(B_\lambda) = 0$. Let $Card(B_\lambda) = 0$. Then
$B_\lambda = \phi$
So the iterative process can deduct all the elements of N that implies the iterative process has deducted all the elements of N before step λ (Observe A and λ's definitions; then we know only ordinal numbers α less than λ satisfy condition $Card(B_\alpha) > 1$. So the iterative process deducts N's element only at steps before λ and deduct nothing from N at step λ and steps after it. Therefore if the iterative process can deduct all the elements of N it must have deducted them before step λ).
On the other hand, notice that all the ordinal numbers α less than λ are λ's members; then we have
$\alpha < \lambda \rightarrow \alpha \in \lambda \rightarrow \alpha \in A \rightarrow Card(B_\alpha) > 1 \rightarrow Card(B_\alpha - \{a_\alpha\}) > 0 \rightarrow B_{\alpha+1} \neq \phi$
So we obtain
$\forall\alpha(\alpha < \lambda \rightarrow B_{\alpha+1} \neq \phi)$
that indicates at all steps before λ the iterative process can not deduct all the elements of N. So semantically the iterative process can not deduct all the elements of N before step λ that contradicts the previous conclusion. As a result the assumption $Card(B_\lambda) = 0$ is invalid and $Card(B_\lambda) = 1$ must hold. So we have found a particular ordinal number λ satisfies $Card(B_\lambda) = 1$.Then we obtain property 3.

□

Notice that every non-empty subset x of N has its least element. Let the choice function $f(x)$ choose the least element of x. So that

$$f(x) = \cap x. \tag{2.5}$$

and the equation in the clause 3 of definition 2.1 becomes

$$a_\alpha = \begin{cases} \cap B_\alpha & if \quad Card(B_\alpha) > 1 \\ b & if \quad Card(B_\alpha) = 1 \\ c & if \quad Card(B_\alpha) = 0 \end{cases} . \tag{2.6}$$

Proof. If $Card(B_\alpha) > 1$, then
$a_\alpha = Min(f(B_\alpha), f(B_\alpha - \{f(B_\alpha)\}))$
Observe Eq. 2.4 and 2.5. Then we have
$a_\alpha = f(B_\alpha) \cap f(B_\alpha - \{f(B_\alpha)\})$
$= (\cap B_\alpha) \cap (\cap(B_\alpha - \{f(B_\alpha)\}))$
$= (\cap B_\alpha) \cap (\cap(B_\alpha - \{\cap(B_\alpha)\}))$
$= \cap B_\alpha$
So we obtain Eq. (2.6). From it we know only under condition $Card(B_\alpha) > 1$ does the

recursion step generate a_α belongs to N, so if a_α belongs to N it must be generated by the first equation. Therefore we have

$$a_\alpha \in N \to a_\alpha = \cap B_\alpha. \tag{2.7}$$

□

And the transfinite sequences have the additional property

$$\forall x(x \in A_\alpha \cap N \wedge B_\alpha \neq \phi \to x \leq \cap B_\alpha). \tag{2.8}$$

Proof. Let $B_\alpha \neq \phi$ and $\beta < \alpha$, then from property 1 of lemma we know both B_α and B_β are non-empty sets of natural numbers and $B_\alpha \subseteq B_\beta$. So
$\cap B_\beta \leq \cap B_\alpha$
Above derivation can be expressed as formula (2.9) to facilitate following derivation

$$B_\alpha \neq \phi \wedge \beta < \alpha \to \cap B_\beta \leq \cap B_\alpha \tag{2.9}$$

Observe formula (2.7) and (2.9). Then we have
$\forall x(x \in A_\alpha \cap N \wedge B_\alpha \neq \phi$
$\to x \in A_\alpha \wedge x \in N \wedge B_\alpha \neq \phi$
$\to \exists\beta(\beta < \alpha \wedge a_\beta = x) \wedge x \in N \wedge B_\alpha \neq \phi$
$\to \exists\beta(\beta < \alpha \wedge a_\beta = x \wedge x \in N \wedge B_\alpha \neq \phi)$
$\to \exists\beta(\beta < \alpha \wedge a_\beta = x \wedge a_\beta \in N \wedge B_\alpha \neq \phi)$
$\to \exists\beta(\beta < \alpha \wedge a_\beta = x \wedge a_\beta = \cap B_\beta \wedge B_\alpha \neq \phi)$
$\to \exists\beta(a_\beta = x \wedge a_\beta = \cap B_\beta \wedge B_\alpha \neq \phi \wedge \beta < \alpha)$
$\to \exists\beta(a_\beta = x \wedge a_\beta = \cap B_\beta \wedge \cap B_\beta \leq \cap B_\alpha)$
$\to \exists\beta(a_\beta = x \wedge a_\beta \leq \cap B_\alpha)$
$\to x \leq \cap B_\alpha)$
Therefore, we obtain formula 2.8. □

As a result we have theorem 2.3.

Theorem 2.3. *The greatest element of N exists.*

Proof. From the property 3 of lemma 2.2 we know there is an ordinal number γ such that B_γ contains only one element z
$B_\gamma = \{z\}$
Considering definition 2.1, we have
$B_\gamma = N - A_\gamma \to B_\gamma \subseteq N \to z \in N$
and
$B_\gamma = B_\gamma \cap N = N - (A_\gamma \cap N) \to B_\gamma \cup (A_\gamma \cap N) = N$
Notice $B_\gamma = \{z\} \neq \phi$ and formula 2.8. Then we have
$\forall x(x \in A_\gamma \cap N \wedge B_\gamma \neq \phi \to x \leq \cap B_\gamma)$
$\to \forall x(x \in A_\gamma \cap N \to x \leq z)$
$\to \forall x(x \in A_\gamma \cap N \to x \leq z) \wedge \forall x(x \in B_\gamma \to x \leq z)$
$\to \forall x(x \in A_\gamma \cap N \vee x \in B_\gamma \to x \leq z)$
$\to \forall x(x \in (A_\gamma \cap N) \cup B_\gamma \to x \leq z)$
$\to \forall x(x \in N \to x \leq z)$
So z is greater than or equal to all the elements of N. Noticing $z \in N$, z is the greatest element of N. Therefore, the set S defined in Eq. 2.3 is not an empty set and equals $\{z\}$. □

Then we obtain theorem 2.4.

Theorem 2.4. *N is an element of itself.*

Proof. From set theory we have
$\forall x(x \in N \to x^+ \in N \wedge x \in x^+ \to x \in \cup N) \to N \subseteq \cup N$
As set N is transitive [1], we also have
$\forall x(x \in \cup N \to \exists y(y \in N \wedge x \in y) \to x \in N) \to \cup N \subseteq N$
Considering above two cases we obtain

$$\cup N = N. \tag{2.10}$$

Considering theorem 2.3 z is the greatest element of N and Eq. (2.10), we have
$\forall x(x \in z \to x \in z \wedge z \in N \to x \in \cup N \to x \in N) \to z \subseteq N$
on the other hand
$\forall x(x \in N \to x \in \cup N \to \exists y(y \in N \wedge x \in y \wedge y \leq z)$
$\to \exists y(y \in N \wedge x \in y \wedge y \subseteq z) \to x \in z)$
$\to N \subseteq z$
Therefore
$z = N$
and

$$N \in N. \tag{2.11}$$

□

However, the conclusion of formula (2.11) that N is the greatest element of itself not only conflicts with the common sense that there is no greatest natural number, but more severely it contradicts the axiom of regularity which asserts a set cannot be a member of itself [1]. And the latter is a serious conflict, because it leads to the conflict between the two axioms of set theory.

3 Discussion

Form the derivation of formula (2.11) we know that if set N exists it must have the greatest element, i.e., the greatest natural number, which in turn equals N itself. So if the set of natural numbers exists, it must also be the greatest natural number.

On the other hand, if the greatest natural number, z, exists, it must greater than all other natural numbers. According to the property that if natural numbers satisfy $n < m$ then $n \in m$, all other natural numbers should be members of z. And considering that z is already the greatest natural number, it cannot less than its successor $z + 1 = z \cup \{z\}$ and of course cannot greater than $z + 1$, so z just equals $z + 1$ that leads to $z \in z$. Therefore z not only contains all other natural numbers but also contains itself that indicates z contains all natural numbers. Noticing that natural number just contains natural numbers as its members, the greatest natural number, therefore, is exactly the set of natural numbers.

Considering above two cases we have the conclusion that the greatest natural number and the set of natural numbers are identical concepts. And both of them clearly conflict with the axiom of regularity, so both of them do not exist. Therefore it is improper to admit the existence of the set of natural numbers but deny the existence of the greatest natural number. Unfortunately, this is the current situation in set theory.

And here we cannot save the axiom of infinity by sacrificing the axiom of regularity. If we do so, Eq. (2.1) should be revised as following completed form to satisfy formula (2.11) regardless of the violation of regularity

$$N = \{0, 1, 2, \cdots, N\}. \tag{3.1}$$

This form of definition of N, however, is impredicative [2] and contains a vicious circle [3], from which we even can not determine the exact value of N since N appears in both sides of the definition. And what is more, without regularity we even cannot prevent

Mirimanoff's paradox [4]. Therefore this scheme is totally unacceptable, and the axiom of infinity should be excluded from set theory to keep the theory consistent.

Since the class of all natural numbers defined by the comprehension principle in Eq. (2.2) cannot be a set, in the light of NBG set theory [5], it should be a proper class. The essence of N is its incompleteness and non-substantiality. In other words N is too large to be any completed entity, and it just can be a dynamic class which is always under construction. Weyl had obviously seen the difference between completed entity and dynamic class, and deemed that blindly converting one into the other is the true source of our difficulties and antinomies, a source more fundamental than Russell's vicious circle principle indicated [6]. Our work has made it clear that the dynamic class N cannot be a set. Whether and how N can exist as a proper class is a question requiring further investigation. And we would like to study the nature of it in the future research.

If we do not regard N as a set but a proper class all the difficulties we encounter in this paper will be resolved. That is if N is not a set we cannot prove B and A are sets and then cannot obtain the property 3 of lemma 2.2, and therefore the proofs of theorem 2.3 and 2.4 are groundless.

4 Conclusion

The conflicts reveal that the axiom of infinity which guarantees the existence of the set of natural numbers does not consist with the axiom of regularity and the essence of the contradiction lies in the inductive definition of N. When we define the inductive collection $\{0, 1, 2, \cdots\}$ produced by the inductive adding one process is an infinite set N, we have already regarded it as a completed, static entity. But on the other hand, with regularity and the induction principle, the inductive construction of natural numbers still can step into the next step wherever it attains and produces a new natural number. So the completed state of the inductive construction does not exist that implies the infinite set N also does not have a completed form. How can an already existing entity possess the attribute that it does not have a completed form at the same time? This is the insidious logical fallacy deeply hiding behind the axiom of infinity.

In our point of view the inductive definition of natural numbers just could guarantee the existence of an infinite process, but it should not become the sufficient condition for that the infinite process can be finally done and thus produce an infinite totality, i.e., an infinite set. That is the misapprehension of infinity in the notion of actual infinity.

Since we have proved that the class of all natural numbers cannot be a set, the assertion made in the axiom of infinity that there is an inductive set is improper.

References

[1] K. Hrbacek, T. Jech, *Introduction to set theory (3nd edition).* (Marcel Dekker, New York, 1999). MR1697766(2000c:03001)

[2] H. Poincare, *Les mathematiques et la logique*, Revue de Metaphysique et de Morale., vol. 14, 1906, pp. 17-34, 294-317.

[3] B. Russell, *Les paradoxes de la logique*, Revue de Metaphysique et de Morale., vol. 14, 1906, pp. 627-650.

[4] R. Montague, *On the paradox of grounded classes*, Journal of symbolic logic., vol. 20, 1955, pp. 140. MR0071380(17,118h)

[5] I. T. Adamson, *A set theory workbook*, (Birkhauser Boston, Cambridge, 1998). M-R1481967

[6] H. Weyl, *Mathematics and logic: A brief survey serving as a preface to a review of The Philosophy of Bertrand Russell*, American Mathematical Monthly., vol. 53, 1946, pp. 2-13.